Advances in Differential Equations Volume 2, Number 5, September 1997, pp. 779–810.

THE PERTURBED RIEMANN PROBLEM FOR A BALANCE LAW

Corrado Mascia
Dipartimento di Matematica "G. Castelnuovo," Università degli Studi di Roma "La Sapienza"
P.le Aldo Moro 5, I–00185 Roma, Italy

Carlo Sinestrari
Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata"
Via della Ricerca Scientifica , I–00133 Roma, Italy

(Submitted by: J.A. Goldstein)

Abstract. We study the asymptotic behaviour of the bounded solutions of a hyperbolic conservation law with source term

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = g(u(x,t)), \quad x \in \mathbb{R}, t \geq 0,$$

where the flux f is convex and the source term g has simple zeros. We assume that the initial value coincides outside a compact set with an initial value of Riemann type. We prove that the solutions converge in general to a sequence of travelling waves delimited by two shock waves. Some of the travelling waves are smooth and connect two consecutive zeros of the source term, while the remaining are discontinuous and oscillate around an unstable zero of the source. However, we prove that for a generic class of initial data the asymptotic profile contains only travelling waves of the first type. We also analyze the rate of convergence of the solutions to the asymptotic profile.

1. Introduction. Equations of the form

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = g(u(x,t)), \qquad x \in \mathbb{R}, t \geq 0 \tag{1.1}$$

are usually called *conservation laws with source* or *balance laws* and may be regarded as model equations for many physical problems, including combustion ([1]), semiconductors ([17]) and the flow of gas in a duct of variable size ([13]). Recently various researchers have studied the asymptotic behaviour of solutions of (1.1) with periodic initial data or with data with compact support. Rather than studying a specific physical model, these authors intended to classify the asymptotic states of the solutions to (1.1) for a fairly general choice of the functions f and g. The results obtained show interesting differences with the conservative case (i.e., the case $g \equiv 0$). Let us consider for instance the case of periodic initial data. It was proved in [5], [15], [20] that the solutions converge either to a constant state (as in

Accepted for publication November 1996.
AMS Subject Classifications: 35L65.

the $g \equiv 0$ case) or to a discontinuous periodic travelling wave, oscillating around an unstable zero of g (i.e., an unstable equilibrium point for the o.d.e. $\dot{u} = g(u)$). The case when the initial datum has compact support and $g(0) = 0$ was studied in [21]. It was shown there that the asymptotic profile of the solution is given in some regions by smooth profiles connecting two consecutive zeros of g and in the remaining regions by discontinuous travelling waves of the same form as in the periodic case. More detailed results in the case of a dissipative g (e.g. $g(u) = -u^p$) and data with compact support have been obtained in [19], [7], [8]. In all the papers we have mentioned it is assumed that the function f is convex. The structure of the solutions of (1.1) in the case of a nonconvex f is more difficult to analyze; the available results concern only particular choices of g (as in [11], [14]) or of initial data (as in [22]).

The aim of the present paper is to carry on the above analysis to the case of the so-called *perturbed Riemann data*, which have the form

$$u_0(x) = \begin{cases} u_l & x < -L \\ \phi(x) & x \in [-L, L] \\ u_r & x > L, \end{cases} \tag{1.2}$$

where $L > 0$, u_l, u_r are constants and $\phi \in [-L, L] \to \mathbb{R}$ is any function of bounded variation. Such data are a natural generalization of the classical Riemann data (or unperturbed Riemann data), which are defined as in (1.2) with $L = 0$, and thus are given by two constant states u_l and u_r. Observe that compactly supported initial data are a special case of (1.2), namely, they correspond to the choice $u_l = u_r = 0$. We assume that the functions f and g in (1.1) are smooth, that f is convex and that the zeros of g are simple. The main goal of our analysis is to describe the asymptotic behaviour of the solutions and to investigate to which extent this behaviour can be reconstructed by studying the corresponding unperturbed Riemann problem.

To explain the motivation of this study we recall some results when $g \equiv 0$. In this case, the solution of the unperturbed Riemann problem has a very simple structure, thanks to the invariance of the problem with respect to the transformation $(x, t) \to (\lambda x, \lambda t)$ with $\lambda > 0$. On the other hand, the patterns exhibited by the solutions of the Riemann problem, called *shock waves* and *rarefaction waves*, are the building blocks for the asymptotic profile of the solutions with more general initial data. In particular, it has been shown in [12] and [2] that the asymptotic profile of the solution to the perturbed Riemann problem is given by the solution of the corresponding unperturbed problem.

We cannot expect the same results to hold also in the case of a balance law. To begin with, the equation is no longer invariant with respect to the above-mentioned family of transformations. The unperturbed Riemann problem can be still solved by elementary methods, but the solution has not such a simple structure as in the conservative case. Moreover, the solutions of the perturbed Riemann problem may exhibit patterns which do not appear in the unperturbed case, like the above-mentioned oscillating profiles in the case of an initial value with compact support.

However, the results of this paper show that, even in the presence of a source term, the behaviour of the solutions of both the perturbed and the unperturbed Riemann problem can be described in a fairly detailed way. Moreover, there are more analogies with the conservative case than one would expect. To explain this, we give a sketch of our results.

In Section 2 we consider the unperturbed Riemann problem. As already mentioned, the solutions are qualitatively similar to the conservative case, but their expression is considerably more involved, especially in the case of a rarefaction wave. Nevertheless, the picture becomes again very simple if one is only interested in the large-time behaviour. In fact we prove that rarefaction waves converge with an exponential rate to a sequence of smooth travelling waves. Each of these waves connects two consecutive zeros of g and its expression can be found by solving a simple ordinary differential equation.

Sections 3 and 4 are devoted to the study of the asymptotic profile of the solutions of the perturbed Riemann problem (1.1)–(1.2). We consider two cases separately. The first case is when the perturbation in the initial data disappears in a finite time, and thus the solution reduces to a shock wave. It is worth noting that when $g \equiv 0$ this behaviour occurs if and only if $u_l > u_r$ (see [12], [2]), while when $g \not\equiv 0$ this condition is necessary but no longer sufficient. The other case is when the perturbation remains for all positive times. Then the large-time behaviour can be analyzed with techniques similar to those of [21] when solutions have compact support. It turns out that the asymptotic profile of the solutions is given by travelling waves of two types: the smooth ones already found in the unperturbed problem, and the discontinuous oscillating waves which appear also in the periodic case. We then study the rate of convergence of the solutions to the asymptotic profile. We give a sufficient condition on the initial datum under which the convergence is exponential. On the other hand, we show that there exist data such that the rate is arbitrarily slow.

The results of Sections 3 and 4 show that the solution of a perturbed Riemann problem may develop discontinuous oscillatory patterns which do not appear in the unperturbed case. In Section 5 we prove that such patterns appear only for a small class of data; namely, there exists a generic set of initial values (with respect to the L^1 topology) giving rise to a solution whose asymptotic profile is given only by smooth waves. This property agrees with the results of [16] about the instability of the travelling waves described above, and shows that the unperturbed and perturbed Riemann problem have a similar behaviour for most of the initial data.

Let us finally remark that the perturbed Riemann problem has been studied also in the case of systems of conservation laws. Of course, here it is necessary to make stronger assumptions on the source term. For instance, there are many results concerning the case when g represents a damping or relaxation term (see for instance [9]). These kinds of terms have a dissipative character. To our knowledge, no results are available for systems with more general functions g, in particular for systems with nondissipative source.

2. The unperturbed Riemann problem. The objects of our study are the qualitative properties of the solutions of the Cauchy problem

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = g(u(x,t)), \qquad x \in \mathbb{R}, t \geq 0, \tag{2.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{2.2}$$

We make the following assumptions on f, g and u_0.

$$\begin{aligned} &f \in C^2(\mathbb{R}), \quad f' \text{ is strictly increasing,} \\ &f''(v) > 0 \text{ for any } v \text{ such that } g(v) = 0, \end{aligned} \tag{H1}$$

$$\begin{aligned} &g \in C^1(\mathbb{R}), \quad g'(v) \neq 0 \text{ for any } v \text{ such that } g(v) = 0, \\ &\text{there exists } M_0 > 0 \text{ such that } v\, g(v) < 0 \text{ for } |v| > M_0, \end{aligned} \tag{H2}$$

$$u_0 \in BV(\mathbb{R}). \tag{H3}$$

Moreover, we suppose that there exists $L \geq 0$ such that

$$u_0(x) = u_l \text{ for } x < -L, \quad u_0(x) = u_r \text{ for } x > L, \tag{H4}$$

for some constants u_l, u_r.

We say that $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is a solution of our problem if

(i) $u \in L^\infty(\mathbb{R} \times [0,T]) \cap C([0,T], L^1_{loc}(\mathbb{R}))$ for any $T > 0$;

(ii) u solves (2.1)–(2.2) in the sense of distributions;

(iii) $u(\cdot,t)$ belongs to $BV(\mathbb{R})$ for every t and the *entropy condition*

$$u(x+,t) \leq u(x-,t)$$

holds for all $x \in \mathbb{R}, t > 0$ (with $u(x+,t)$, $u(x-,t)$ we denote the one-sided limits of $u(\cdot,t)$).

It is known (see [10], [24], [6]) that, under the above assumptions, there exists a unique solution of problem (2.1)–(2.2). We assume that it is normalized in such a way that it is continuous from the left, i.e., that $u(x,t) = u(x-,t)$, for any $x \in \mathbb{R}$, $t \in (0,\infty)$.

From standard comparison results (see [10]) and hypothesis (H2) it follows easily that the solution u is uniformly bounded in $\mathbb{R} \times \mathbb{R}_+$.

In this section we discuss the properties of the solution of the unperturbed Riemann problem, that is, when we have $L = 0$ in (H4) and thus the initial value of our equation is

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0. \end{cases} \tag{2.3}$$

We first recall the results when $g \equiv 0$ (see e.g. [24] or [6] for more details). In this case, due to the invariance of the problem under the transformation $(x,t) \to (\lambda x, \lambda t)$ for $\lambda > 0$, one is led to consider functions of the form $u(x,t) = \phi(x/t)$. It turns out that the solution of the Riemann problem is given by a *shock wave* in the case $u_l > u_r$

$$u(x,t) = \begin{cases} u_l & \dfrac{x}{t} < \dfrac{f(u_l) - f(u_r)}{u_l - u_r} \\ u_r & \dfrac{x}{t} > \dfrac{f(u_l) - f(u_r)}{u_l - u_r}, \end{cases} \tag{2.4}$$

and by a *rarefaction wave* when $u_l \leq u_r$

$$u(x,t) = \begin{cases} u_l & x < f'(u_l)t \\ (f')^{-1}\left(\dfrac{x}{t}\right) & f'(u_l) \leq \dfrac{x}{t} \leq f'(u_r) \\ u_r & x > f'(u_r)t. \end{cases} \tag{2.5}$$

When the source term g is present in the equation, the solution of the Riemann problem is in general no longer self-similar and must be obtained by other methods. For the reader's convenience we recall the construction of [21], [22], based on the method of characteristics. To this purpose we first introduce some notations which will be useful also in the sequel of the paper. First we set

$$\sigma(u,v) = \begin{cases} \dfrac{f(u) - f(v)}{u - v} & \text{if } u \neq v \\ f'(u) & \text{if } u = v \end{cases} \tag{2.6}$$

for $u, v \in \mathbb{R}$.

We denote by $\mathcal{Z}(g)$ the set of the zeros of g. By assumption (H2) such a set is finite, and we can label its elements by $w_1 < w_2 < \cdots < w_N$. Then N is odd, and $g'(w_i) < 0$ for i odd, $g'(w_i) > 0$ for i even. We set

$$\mathcal{Z}_a(g) = \{w \in \mathcal{Z}(g) : g'(w) < 0\}, \quad \mathcal{Z}_r(g) = \{w \in \mathcal{Z}(g) : g'(w) > 0\}, \tag{2.7}$$

where the subscripts a and r refer to the attractive or repulsive character of the value w with respect to the o.d.e. $\dot{v} = g(v)$. We also define

$$Z_+(v) = \min(\{w > v : w \in \mathcal{Z}(g)\} \cup \{+\infty\})$$
$$Z_-(v) = \max(\{w < v : w \in \mathcal{Z}(g)\} \cup \{-\infty\})$$

and set

$$\alpha_g = \min\{|g'(w)| : w \in \mathcal{Z}(g)\}. \tag{2.8}$$

Then $\alpha_g > 0$ by assumption (H2).

To simplify our presentation, we assume that g' is uniformly bounded (this is not restrictive, since we know that our solution is bounded).

Definition 2.1. For any $v \in \mathbb{R}$, let $W(v, \cdot)$ denote the solution of

$$\begin{cases} \partial_t W(v,t) = g(W(v,t)) & t \in \mathbb{R} \\ W(v,0) = v. \end{cases} \tag{2.9}$$

Let us set, for $v \in \mathbb{R}$, $t \geq 0$,

$$F_+(v,t) = \int_0^t f'(W(v,s))\, ds, \quad F(v,t) = \int_0^t f'(W(v,-s))\, ds.$$

For any i even, $1 < i < N$, let $\phi_i : \mathbb{R} \to (w_{i-1}, w_{i+1})$ be the solution of

$$\phi_i' = \begin{cases} \dfrac{g(\phi_i)}{f'(\phi_i) - f'(w_i)} & \phi_i \neq w_i \\ \dfrac{g'(w_i)}{f''(w_i)} & \phi_i = w_i \end{cases}$$

satisfying the condition $\phi_i(0) = w_i$. We also set

$$\alpha(v) = \lim_{t \to -\infty} W(v,t), \quad \omega(v) = \lim_{t \to +\infty} W(v,t)$$

and define $z_r = \omega(u_r)$, $z_l = \omega(u_l)$ (these limits exist by Lemma 2.2(i)–(ii) below).

The following properties are consequences of the definitions and of assumptions (H1)–(H2) (see [20]).

Lemma 2.2.

(i) *If* $g(v) = 0$ *then* $W(v,t) \equiv v$ *and* $F_+(v,t) = F(v,t) = f'(v)t$.

(ii) *If* $g(v) > 0$ (*respectively* $g(v) < 0$), *then* $W(v,\cdot)$ *is strictly increasing from* $Z_-(v)$ *to* $Z_+(v)$ (*respectively is decreasing from* $Z_+(v)$ *to* $Z_-(v)$). *In both cases* F, F_+ *satisfy*

$$F(v,t) = F_+(W(v,-t),t). \tag{2.10}$$

(iii) *Given* $u_0 \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}$, *the functions*

$$x(t) = x_0 + F_+(u_0,t), \quad u(t) = W(u_0,t) \tag{2.11}$$

are the solutions of the characteristic system associated with equation (2.1)

$$\begin{cases} x'(t) = f'(u(t)) \\ u'(t) = g(u(t)) \end{cases} \tag{2.12}$$

with initial data $x(0) = x_0$, $u(0) = u_0$.

(iv) W *and* F *are of class* C^1 *and satisfy*

$$\partial_v W(v,t) = \begin{cases} \dfrac{g(W(v,t))}{g(v)} & \text{if } v \notin \mathcal{Z}(g) \\ e^{g'(v)t} & \text{if } v \in \mathcal{Z}(g), \end{cases} \tag{2.13}$$

$$\partial_t F(v,t) = f'(W(v,-t)), \tag{2.14}$$

$$\partial_v F(v,t) = \begin{cases} \dfrac{f'(v) - f'(W(v,-t))}{g(v)} & \text{if } v \notin \mathcal{Z}(g) \\ \dfrac{f''(v)}{g'(v)}\big(1 - e^{-g'(v)t}\big) & \text{if } v \in \mathcal{Z}(g). \end{cases} \tag{2.15}$$

(v) *For any* i *even,* $1 < i < N$, ϕ_i *is well defined and satisfies*

$$\int_{w_i}^{\phi_i(\xi)} \frac{f'(v) - f'(w_i)}{g(v)}\, dv = \xi, \qquad \xi \in \mathbb{R}. \tag{2.16}$$

Moreover, $\phi_i(x - f'(w_i)t)$ *is a solution of equation* (2.1).

As a consequence of (2.15) we obtain

Corollary 2.3. *Given* $v_1 < v_2$ *and* $T > 0$ *there exists* $C > 0$ *such that*

$$F(v'',t) - F(v',t) \geq C(v'' - v'), \quad v_1 \leq v' \leq v'' \leq v_2, \quad t \geq T.$$

Definition 2.4. Let the function $\tilde{u}(x,t)$ be defined by the equality $F(\tilde{u}(x,t),t) = x$ for any $t > 0$, $x \in (f'(-\infty)t, f'(+\infty)t)$.

Proposition 2.5. *The function* $\tilde{u}$ *is well defined and satisfies equation* (2.1) *in the classical sense.*

Proof. From Corollary 2.3 we deduce that $\tilde{u}$ is well defined and smooth. Moreover F satisfies, by Lemma 2.2(iv),

$$\partial_t F(v,t) + g(v)\partial_v F(v,t) = f'(v).$$

From the definition of $\tilde{u}$ it follows

$$\partial_v F \partial_x \tilde{u} = 1, \qquad \partial_v F \partial_t \tilde{u} + \partial_t F = 0.$$

Hence we have, at any point of differentiability of $\tilde{u}$,

$$\begin{aligned} 0 &= \partial_x \tilde{u}(\partial_v F \partial_t \tilde{u} + \partial_t F) = \partial_t \tilde{u} + \partial_x \tilde{u} \partial_t F \\ &= \partial_t \tilde{u} + f'(\tilde{u}) \partial_x \tilde{u} - g(\tilde{u}) = \partial_t \tilde{u} + \partial_x f(\tilde{u}) - g(\tilde{u}). \quad \square \end{aligned}$$

Now we turn back to the Riemann problem for a balance law. When $u_l > u_r$ the solution is given by a shock wave as in the case $g \equiv 0$: the only difference is that the two states evolve according to the o.d.e. $u' = g(u)$ and consequently the shock curve is not a straight line in general. The case $u_l < u_r$ is less obvious, since we have to replace in some way the function $(f')^{-1}(x/t)$ which appears in the case $g \equiv 0$. We then consider a family of characteristics starting at the origin. These characteristics are parametrized by the values lying in $[u_l, u_r]$, which are taken as initial values for the solution along the characteristics. This means that we solve system (2.12) with initial values $x(0) = 0$, $u(0) = u_0$ for $u_0 \in [u_l, u_r]$. Such a procedure is quite natural if one thinks of the Riemann data as the limit of smooth monotone initial values. Then we eliminate the parameter u_0 and obtain an equality involving u, x and t that gives the solution. By (2.11)(ii) we have $u_0 = W(u, -t)$; substituting in (2.11)(i) and using (2.10) we deduce

$$x = F_+(W(u, -t), t) = F(u, t),$$

which is the equality defining $\tilde{u}$. Thus, in considering a balance law the function $\tilde{u}(x, t)$ of Definition 2.4 plays the role of the function $(f')^{-1}(x/t)$ when the source term is absent. The existence of such a function is due to the convexity of f which ensures the monotonicity of F with respect to v. When f is not convex F is no longer monotone, and the solutions of the Riemann problem have a much more complex structure (see [22]).

The above discussion leads us to state the following result.

Theorem 2.6. *If $u_l > u_r$ the solution of problem* (2.1)–(2.3) *is*

$$u(x,t) = \begin{cases} W(u_l, t) & x \le \gamma(t) \\ W(u_r, t) & x > \gamma(t), \end{cases} \tag{2.17}$$

where

$$\gamma(t) = \int_0^t \sigma(W(u_l, s), W(u_r, s))\, ds.$$

If $u_l \le u_r$ the solution of problem (2.1)–(2.3) *is*

$$u(x,t) = \begin{cases} W(u_l, t) & x \le F_+(u_l, t) \\ \tilde{u}(x,t) & F_+(u_l, t) \le x \le F_+(u_r, t) \\ W(u_r, t) & x > F_+(u_r, t). \end{cases} \tag{2.18}$$

Proof. The assertion follows from well-known properties of entropy solutions (see for instance [6]) and from our previous results. In fact, the function in (2.17) has a discontinuity along the curve γ where both the Rankine–Hugoniot condition and the entropy admissibility condition are satisfied. The function in (2.18) is continuous except at the origin by (2.10), is differentiable almost everywhere and satisfies the equation at the points of differentiability by Proposition 2.5. □

In the remainder of the section we study the asymptotic profile of the solution of the Riemann problem given in (2.17) and (2.18). In the case of a shock wave this is easily done. In fact, we obtain from the definitions that the function defined in (2.17) satisfies, for any $\alpha < \alpha_g$ (see (2.8) and Definition 2.1),

$$u(x,t) = \begin{cases} z_l + o(e^{-\alpha t}) & x \leq \gamma(t) \\ z_r + o(e^{-\alpha t}) & x > \gamma(t), \end{cases} \tag{2.19}$$

where

$$\gamma(t) = \sigma(z_l, z_r)t + O(1). \tag{2.20}$$

Let us now consider the case of a rarefaction wave. The content of the next theorem is that the function $\tilde{u}$ converges to a superposition of the travelling waves ϕ_i introduced in Definition 2.1. This will allow us to obtain an asymptotic representation of the function given in (2.18) which does not require the knowledge of $\tilde{u}$.

Theorem 2.7. *Let w_i, w_{i+1} be consecutive zeros of g, with i even, and let be ϕ_i as in Definition* 2.1. *Then we have*

$$\lim_{t\to\infty} e^{\alpha t}(\sup\{|\tilde{u}(x,t) - \phi_i(x - f'(w_i)t)| : x \in [f'(w_i)t, f'(w_{i+1})t]\}) = 0 \tag{2.21}$$

for any $\alpha < \frac{1}{2}\alpha_g$.

We need a preliminary result.

Lemma 2.8. *Let w_i, w_{i+1} be as in Theorem* 2.7. *Then, for any*

$$0 < \alpha < \frac{1}{2}\min\{g'(w_i), -g'(w_{i+1})\}$$

we have

$$\lim_{t\to\infty} e^{\alpha t}(w_{i+1} - W(w_i + e^{-\alpha t}, t)) = 0.$$

Proof. Let us fix β_1, β_2 such that $\alpha < \beta_1 < \beta_2 < \frac{1}{2}\min\{g'(w_i), -g'(w_{i+1})\}$. Then there exists $\delta > 0$ such that

$$\begin{aligned} g(v) &> 2\beta_2(v - w_i) && v \in (w_i, w_i + \delta) \\ g(v) &> 2\beta_2(w_{i+1} - v) && v \in (w_{i+1} - \delta, w_{i+1}). \end{aligned}$$

Therefore

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t}\int_{w_i+e^{-\beta_1 t}}^{w_{i+1}-e^{-\beta_1 t}} \frac{dv}{g(v)}\\
&=\limsup_{t\to\infty} \frac{1}{t}\Big\{\int_{w_i+e^{-\beta_1 t}}^{w_i+\delta} \frac{dv}{g(v)} + \int_{w_i+\delta}^{w_{i+1}-\delta} \frac{dv}{g(v)} + \int_{w_{i+1}-\delta}^{w_{i+1}-e^{-\beta_1 t}} \frac{dv}{g(v)}\Big\}\\
&\le \limsup_{t\to\infty} \frac{1}{t}\Big\{\int_{w_i+e^{-\beta_1 t}}^{w_i+\delta} \frac{dv}{2\beta_2(v-w_i)} + \int_{w_{i+1}-\delta}^{w_{i+1}-e^{-\beta_1 t}} \frac{dv}{2\beta_2(w_{i+1}-v)}\Big\}\\
&=\limsup_{t\to\infty} \frac{1}{\beta_2 t}\int_{e^{-\beta_1 t}}^{\delta} \frac{dv}{v} = \lim_{t\to\infty}\frac{1}{\beta_2 t}(\ln\delta+\beta_1 t) = \frac{\beta_1}{\beta_2} < 1.
\end{aligned}$$

It follows that, for t large enough,

$$\int_{w_i+e^{-\beta_1 t}}^{w_{i+1}-e^{-\beta_1 t}} \frac{dv}{g(v)} < t,$$

which is equivalent to $W(w_i+e^{-\beta_1 t},t) > w_{i+1}-e^{-\beta_1 t}$. Therefore

$$w_{i+1} - W(w_i+e^{-\alpha t},t) < w_{i+1} - W(w_i+e^{-\beta_1 t},t) < e^{-\beta_1 t}$$

for t large enough. We conclude

$$\begin{aligned}
&\lim_{t\to\infty} e^{\alpha t}(w_{i+1}-W(w_i+e^{-\alpha t},t))\\
&=\lim_{t\to\infty} e^{(\alpha-\beta_1)t}e^{\beta_1 t}(w_{i+1}-W(w_i+e^{-\alpha t},t)) = 0. \quad \square
\end{aligned}$$

Proof of Theorem 2.7. Let us consider (x,t) with $t>0$, $x\in[f'(w_i)t, f'(w_{i+1})t]$. Since by Lemma 2.2(i) $F(w_i,t)=f'(w_i)t$ and $F(w_{i+1},t)=f'(w_{i+1})t$, we have $\tilde u(x,t)\in[w_i,w_{i+1}]$. Furthermore we deduce from (2.15)

$$x - f'(w_i)t = \int_{w_i}^{\tilde u(x,t)} \partial_v F(v,t)\,dv = \int_{w_i}^{\tilde u(x,t)} \frac{f'(v)-f'(W(v,-t))}{g(v)}\,dv.$$

Comparing with (2.16) and using the monotonicity of f' we obtain

$$w_i \le \phi_i(x-f'(w_i)t) \le \tilde u(x,t) \le w_{i+1}, \tag{2.22}$$

$$\int_{w_i}^{\phi_i(x-f'(w_i)t)} \frac{f'(W(v,-t))-f'(w_i)}{g(v)}\,dv = \int_{\phi_i(x-f'(w_i)t)}^{\tilde u(x,t)} \frac{f'(v)-f'(W(v,-t))}{g(v)}\,dv.$$

This equality can be rewritten, setting $z = W(v, -t)$ in the integral in the left-hand side and using (2.13), as

$$\int_{w_i}^{W(\phi_i(x-f'(w_i)t),-t)} \frac{f'(z)-f'(w_i)}{g(z)}\,dz = F(\tilde{u}(x,t),t) - F(\phi_i(x-f'(w_i)t),t). \quad (2.23)$$

By Corollary 2.3 there exists $C > 0$ such that

$$v'' - v' \le C(F(v'',t) - F(v',t)), \qquad \text{for all } w_i \le v' \le v'' \le w_{i+1}, \quad t \ge 1. \quad (2.24)$$

Now let us take $\beta \in (\alpha, \frac{1}{2}\min\{g'(w_i), -g'(w_{i+1})\})$. If $x \in [f'(w_i)t, f'(w_{i+1})t]$ is such that $\phi_i(x - f'(w_i)t) > w_{i+1} - e^{-\beta t}$, then we have, by (2.22), $|\phi_i(x - f'(w_i)t) - \tilde{u}(x,t)| < e^{-\beta t}$. Thus, to prove estimate (2.21) it is enough to consider points (x,t) such that $\phi_i(x - f'(w_i)t) \le w_{i+1} - e^{-\beta t}$. By Lemma 2.8, if t is large enough we have at such points

$$W(\phi_i(x - f'(w_i)t), -t) \le w_i + e^{-\beta t}. \quad (2.25)$$

By (2.23), (2.24) and (2.25) if we let $M = 1 + f''(w_i)/g'(w_i)$ we have, for t large enough and x such that $\phi_i(x - f'(w_i)t) \le w_{i+1} - e^{-\beta t}$,

$$0 \le \tilde{u}(x,t) - \phi_i(x - f'(w_i)t) \le CMe^{-\beta t},$$

and estimate (2.21) follows. □

Analogously, we can prove that in the regions $f'(w_i)t \le x \le f'(w_{i+1})t$ with i odd the function $\tilde{u}$ is asymptotic to $\phi_{i+1}(x - f'(w_{i+1})t)$. As a consequence, we obtain the following corollary concerning the asymptotic behaviour of the Riemann problem in the case $u_l < u_r$.

Corollary 2.9. *Suppose $u_l < u_r$. Let $j(i) = i$ if i is even, $j(i) = i + 1$ if i is odd. Then the solution of* (2.1)–(2.3) *satisfies*

$$u(x,t) = \begin{cases} z_l + o(e^{-\alpha t}) & \text{if } x \le f'(z_l)t, \\ \phi_{j(i)}(x - f'(w_{j(i)})t) + o(e^{-\alpha t}) & \text{if } f'(w_i)t < x \le f'(w_{i+1})t \\ & \text{and } z_l \le w_i < w_{i+1} \le z_r, \\ z_r + o(e^{-\alpha t}) & \text{if } x > f'(z_r)t \end{cases}$$

for any $\alpha < \alpha_g/2$.

In the next sections we turn our attention to more general initial data, our aim being to check to what extent the asymptotic profile of the solutions admits a description like the one found above for the Riemann problem.

3. Properties of the interfaces. We extend now our analysis to the perturbed Riemann problem; that is, we assume no longer $L = 0$ in condition (H4). The hyperbolic character of equation (2.1) and the assumption that the initial value is constant outside a compact interval imply that the solution is also constant outside a compact interval, the constants evolving in time according to the o.d.e. $\dot{u} = g(u)$. More precisely, there exist two Lipschitz curves $x = \gamma_l(t)$ and $x = \gamma_r(t)$ such that $u(x,t) = W(u_l, t)$ for $x < \gamma_l(t)$ and $u(x,t) = W(u_r, t)$ for $x > \gamma_r(t)$. In this section we focus our attention on the properties of these curves, which we call the *interfaces* of the solution, in analogy with the case when u has compact support in x. One possible case is that the interfaces meet at a certain time T_0. Then for $t \geq T_0$ the solution is given by two constant states divided by a shock wave. We will discuss how this behaviour is related with the choice of u_l, u_r in the initial value. The interesting case is when the interfaces do not meet. Then it is possible to characterize the behaviour of the solution u along the interfaces, and to extend to our case the results proved in [21] for solutions with compact support (see Corollary 3.12).

An important role in our analysis is played by the theory of generalized characteristics (see [2], [3]). We recall that a (generalized) characteristic associated with the solution u of (2.1)–(2.2) is a Lipschitz curve $\zeta : [a,b] \to \mathbb{R}$ which satisfies

$$\zeta'(s) \in [f'(u(\zeta(s)+,s)), f'(u(\zeta(s),s))] \qquad s \in [a,b] \text{ a.e.}$$

Examples of such curves are classical characteristics and shock curves. Actually, it turns out that any characteristic propagates either with classical or with shock speed almost everywhere. In general there exists more than one characteristic passing through a given point $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ and any such characteristic is confined between a maximal and a minimal one. We assume in the following that the reader is familiar with the results of [3]; here we only recall two properties which will be repeatedly used in the following.

Theorem 3.1. *Let $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ be given, with $t > 0$. Then there exists a unique forward characteristic starting at (x,t). Consequently, if ζ_1 and ζ_2 are characteristics and $\zeta_1(t_0) = \zeta_2(t_0)$ for some $t_0 > 0$, then $\zeta_1(t) = \zeta_2(t)$ for any $t \geq t_0$.*

Theorem 3.2. *From any point $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ starts at least one classical backward characteristic. More precisely, let $(v(\cdot), y(\cdot)) = (\, v(x,t;\cdot), y(x,t;\cdot)\,)$ denote the solution of the system*

$$\begin{cases} y'(s) = f'(v(s)) \\ v'(s) = g(v(s)) \end{cases} \qquad s \in [0,t] \tag{3.1}$$

with terminal condition

$$\begin{cases} y(t) = x \\ v(t) = u(x,t). \end{cases}$$

Then

$$u(y(s),s) = u(y(s)+,s) = v(s) \qquad s \in (0,t)\,, \tag{3.2}$$

$$u_0(y(0)) \le v(0) \le u_0(y(0)+). \tag{3.3}$$

Recalling Definition 2.1 we find

$$y(x,t;s) = x - F(u(x,t),t-s), \quad v(x,t;s) = W(u(x,t),s-t).$$

Let us set $\eta(x,t) = y(x,t;0)$. Then we have

$$\eta(x,t) = x - F(u(x,t),t). \tag{3.4}$$

Furthermore, we easily deduce from Theorems 3.1 and 3.2 the following properties.

Corollary 3.3.

(i) *The function $\eta(\cdot,t)$ is increasing for any $t > 0$.*

(ii) *If $u(x,t) = w_i$ for some $w_i \in \mathcal{Z}(g)$, then $\eta(x,t) = x - f'(w_i)t$ and in addition*

$$u_0(x - f'(w_i)t) \le w_i \le u_0(x - f'(w_i)t+).$$

We can now give a precise definition of the curves which we have previously called the interfaces of u.

Definition 3.4. We set

$$L_- = \max\{x \in \mathbb{R} : u_0(y) = u_l \ \forall y < x\},$$
$$L_+ = \min\{x \in \mathbb{R} : u_0(y) = u_r \ \forall y > x\}.$$

Let γ_l be the minimal forward characteristic through $(L_-,0)$, and γ_r the maximal forward characteristic through $(L_+,0)$. We also define

$$v_l(t) = u(\gamma_l(t)+,t), \quad v_r(t) = u(\gamma_r(t),t), \quad l(t) = \eta(\gamma_r(t),t). \tag{3.5}$$

Proposition 3.5. *Let $T_0 = \sup\{t > 0 : \gamma_l(t) < \gamma_r(t)\}$. Then*

(i) $\gamma_l(t) = \max\{x \in \mathbb{R} \ : \ u(y,t) = W(u_l,t) \ \forall y < x\}, \quad t \in [0,T_0];$
$\gamma_r(t) = \min\{x \in \mathbb{R} \ : \ u(y,t) = W(u_r,t) \ \forall y > x\}, \quad t \in [0,T_0];$
$\gamma_l'(t) = \sigma(v_l(t),W(u_l,t)), \quad \gamma_r'(t) = \sigma(v_r(t),W(u_r,t)), \quad t \ge 0.$

(ii) *There exist $T_l^*, T_r^* \in [0,T_0]$ such that*

$$\gamma_l(t) \equiv L_- + F_+(u_l,t), \quad v_l(t) \equiv W(u_l,t), \quad t \in [0,T_l^*);$$

$$\gamma_r(t) \equiv L_+ + F_+(u_r,t), \quad v_r(t) \equiv W(u_r,t), \quad t \in [0,T_r^*);$$

$$v_l(t) < W(u_l,t), \quad t \in (T_l^*,T_0) \ a.e., \quad v_r(t) > W(u_r,t), \quad t \in (T_r^*,T_0) \ a.e.$$

(iii) *$l(\cdot)$ is decreasing and satisfies*

$$L_- \le l(t) \le L_+ \qquad t \in [0,T_0).$$

Proof. It is analogous to the proof of Proposition 4.2 in [21].

It is convenient to distinguish two cases. We say that the solution is of type (I) if the interfaces meet (i.e., $T_0 < \infty$), and of type (II) if the interfaces do not meet.

Let us first consider the structure of a solution of type (I). By Theorem 3.1, we have $\gamma_l(t) \equiv \gamma_r(t)$ for $t \geq T_0$. Therefore, for $t \geq T_0$ the solution is given by

$$u(x,t) = \begin{cases} W(u_l,t) & x \leq \gamma_l(t) \\ W(u_r,t) & x > \gamma_l(t), \end{cases}$$

where

$$\gamma_l(t) = \gamma_r(t) = \gamma_r(T_0) + \int_{T_0}^{t} \sigma(W(u_l,s), W(u_r,s))\, ds.$$

Thus, up to a translation in x, u coincides for $t \geq T_0$ with the solution of the unperturbed Riemann problem given in (2.17).

It is interesting to have conditions on the initial data which ensure a priori that the solution is of type (I) or of type (II). The following result provides a partial answer.

Theorem 3.6. *Let u be the solution of problem* (2.1)–(2.2).

(i) *If u is of type* (I)*, then $u_l > u_r$.*
(ii) *If $z_l > z_r$ (see Definition* 2.1*) and $u_l \geq u_0(x) \geq u_r$ for any $x \in \mathbb{R}$ then u is of type* (I).

Proof. Suppose $u_l \leq u_r$. Then Proposition 3.5(i)–(ii) implies that $\gamma_r(t) - \gamma_l(t)$ is an increasing function of t in $[0, T_0)$ and that the solution cannot be of type (I). This proves part (i) of the Theorem.

Suppose now that $u_l \geq u_0(x) \geq u_r$ for any $x \in \mathbb{R}$. Then, by standard comparison results (see [10]) we obtain

$$W(u_l,t) \geq u(x,t) \geq W(u_r,t), \qquad x \in \mathbb{R}, t \geq 0.$$

From Proposition 3.5(ii) it follows that $T_l^* = T_r^* = T_0$ and that $\gamma_l(t) \equiv L_- + F_+(u_l,t)$, $\gamma_r(t) \equiv L_+ + F_+(u_r,t)$ for any $t \in [0, T_0]$. Since, as $t \to \infty$,

$$\frac{F_+(u_l,t)}{t} \to f'(z_l), \qquad \frac{F_+(u_r,t)}{t} \to f'(z_r),$$

it follows that T_0 is finite if $z_l > z_r$. $\square$

We recall that in the $g \equiv 0$ case the solution of the perturbed Riemann problem is of type (I) if and only if $u_l > u_r$. In our case the solution may be of type (II) even if $z_l > z_r$ as the next example shows.

Example 3.7. Let us take $f(u) = u^2/2$ and $g(u) = -(u+1)u(u-2)$ in equation (2.1). We define $\psi : \mathbb{R} \to (-1, 2)$ to be the solution of

$$\psi'(x) = -(\psi(x)+1)(\psi(x)-2), \quad \psi(0) = 0. \tag{3.6}$$

Then ψ is strictly increasing and $\psi(x) \to 2$ as $x \to \infty$. In addition, ψ is a stationary solution to (2.1). Let us fix $L_0 > 0$ such that $\psi(L_0) > 3/2$. We take as initial value for our equation

$$u_0(x) = \begin{cases} 0 & x \le 0 \\ \psi(x) & 0 < x \le L_0 \\ -1 & x > L_0. \end{cases}$$

In this case we have $u_l = z_l = 0$, $u_r = z_r = -1$, $L_- = 0$, $L_+ = L_0$. The solution, at least for small times, has the form

$$u(x,t) = \begin{cases} 0 & x \le 0 \\ \psi(x) & 0 < x \le \gamma_r(t) \\ -1 & x > \gamma_r(t), \end{cases} \tag{3.7}$$

where γ_r is the solution of

$$\gamma_r'(t) = \frac{1}{2}(\psi(\gamma_r(t)) - 1), \quad \gamma_r(0) = L_0.$$

Our choice of L_0 implies that $\gamma_r'(t) > 1/4$ whenever $\gamma_r(t) \ge L_0$. Since $\gamma_r(0) = L_0$, it follows that γ_r is increasing and $\gamma_r(t) \to \infty$ as $t \to \infty$. In particular $\gamma_r(t) > 0$ for any $t > 0$, and the solution of our problem has the form (3.7) for any positive time and is of type (II). Let us also observe that $v_r(t) = \psi(\gamma_r(t)) \to 2$ as $t \to \infty$. □

The above example shows that the hypothesis $u_l \ge u_0 \ge u_r$ in Theorem 3.6(ii) is necessary in general. However, after having analyzed case (II), we shall see that in certain cases it can be removed (see Example 3.13).

We leave now case (I), where the study of the asymptotic behaviour of the solution is trivial, and we assume from now on that $\gamma_l(t) < \gamma_r(t)$ for any $t > 0$.

Theorem 3.8. *There exists the limit of $v_r(t)$ as $t \to \infty$.*

Proof. Suppose to the contrary $v' < v''$, where $v' = \liminf_{t\to\infty} v_r(t)$, $v'' = \limsup_{t\to\infty} v_r(t)$. Then we can fix $\bar{v} \in (v', v'')$ such that $\sigma(\bar{v}, z_r) \ne f'(w_i)$ for all $w_i \in \mathcal{Z}(g)$. From (3.4) and (3.5) we deduce, by setting $x = \gamma_r(t)$,

$$\gamma_r(t) = l(t) + F(v_r(t), t). \tag{3.8}$$

By Proposition 3.5(iii) $l(t) \to l_+$ as $t \to \infty$ for some $l_+ \in [L_-, L_+]$. We claim that there exists $\varepsilon > 0$ such that, for t large enough,

$$F(v' + \varepsilon, t) + l(t) < F(\bar{v}, t) + l_+ < F(v'' - \varepsilon, t) + l(t). \tag{3.9}$$

In fact, if $v' + \varepsilon < \bar{v}$, the function $t \to F(\bar{v}, t) - F(v' + \varepsilon, t)$ is bounded from below by a positive constant by Corollary 2.3. Thus, it is eventually larger than $l(t) - l_+$, which tends to zero by definition. This proves the first inequality in (3.9); the other is checked analogously.

From (3.8) and (3.9) we deduce that the two curves $x = \gamma_r(t)$ and $x = l_+ + F(\bar{v}, t)$ intersect themselves infinitely many times. More precisely, we can find an increasing sequence t_n such that $t_n \to \infty$ and

$$\begin{aligned} \gamma_r(t_n) < F(\bar{v}, t_n) + l_+ \quad & \text{if } n \text{ is odd} \\ \gamma_r(t_n) > F(\bar{v}, t_n) + l_+ \quad & \text{if } n \text{ is even.} \end{aligned}$$

By the Lipschitz continuity of γ_r we can find a sequence s_n such that $t_{n-1} < s_n < t_n$, γ_r is differentiable at s_n and

$$|\gamma_r(s_n) - F(\bar{v}, s_n) - l_+| \to 0 \text{ as } n \to \infty, \tag{3.10}$$

$$\begin{cases} \gamma_r'(s_n) \le F_t(\bar{v}, s_n) & \text{if } n \text{ is odd} \\ \gamma_r'(s_n) \ge F_t(\bar{v}, s_n) & \text{if } n \text{ is even.} \end{cases} \tag{3.11}$$

From (3.8) and (3.10) we deduce

$$\lim_{n\to\infty} F(v_r(s_n), s_n) - F(\bar{v}, s_n) = 0. \tag{3.12}$$

By formula (2.15), $F_v(v, t)$ is positive and increasing in t for any v. Thus (3.12) implies that $v_r(s_n) \to \bar{v}$ as $n \to \infty$. By Proposition 3.5(i)

$$\lim_{n\to\infty} \gamma_r'(s_n) = \lim_{n\to\infty} \sigma(v_r(s_n), W(u_r, s_n)) = \sigma(\bar{v}, z_r).$$

From (3.11) and (2.14) we deduce

$$\sigma(\bar{v}, z_r) = \lim_{n\to\infty} \gamma_r'(s_n) = \lim_{n\to\infty} F_t(\bar{v}, s_n) = \lim_{n\to\infty} f'(W(\bar{v}, -s_n)) = f'(\alpha(\bar{v})).$$

Since $\alpha(\bar{v}) \in \mathcal{Z}(g)$, this is in contradiction with our choice of $\bar{v}$. Therefore v' and v'' coincide. $\square$

We set $v_{r,\infty} = \lim_{t\to\infty} v_r(t)$. We have $v_{r,\infty} \ge z_r$ by Proposition 3.5(ii). In order to characterize the possible values of $v_{r,\infty}$ we introduce some notations.

Definition 3.9. Given $w \in \mathcal{Z}_a(g)$, $w > z_r$, we say that w belongs to $\mathcal{S}_+(z_r)$ if $\sigma(w, z_r) > f'(Z_-(w))$. Similarly, if $w \in \mathcal{Z}_a(g)$, $w < z_l$, we say that w belongs to $\mathcal{S}_-(z_l)$ if $\sigma(w, z_l) < f'(Z_+(w))$.

By the monotonicity of σ we deduce that, given $w \in \mathcal{S}_+(z_r)$, there exists a unique value $v^* = v^*(w) \in [Z_-(w), w)$ such that $f'(Z_-(w)) = \sigma(v^*, z_r)$. Analogously, given $w \in \mathcal{S}_-(z_l)$, there exists a unique value $v^* = v^*(w) \in (w, Z_+(w)]$ such that $f'(Z_+(w)) = \sigma(v^*, z_l)$.

Theorem 3.10. *If $v_{r,\infty} \notin \mathcal{Z}_a(g)$ and $v_{r,\infty} > z_r$ then $v_{r,\infty} = v^*(w)$ for some $w \in \mathcal{S}_+(z_r)$.*

Proof. From (3.8) we deduce

$$\frac{\gamma_r(t)}{t} = \frac{l(t)}{t} + \frac{F(v_r(t),t)}{t}. \tag{3.13}$$

Let us consider the behaviour of the terms in the above equality as $t \to \infty$. Since by Proposition 3.5 $\gamma_r'(t) = \sigma(v_r(t), W(u_r,t))$ for $t > 0$ almost everywhere and $l(t)$ is bounded, we obtain

$$\lim_{t\to\infty} \frac{\gamma_r(t)}{t} = \sigma(v_{r,\infty}, z_r), \qquad \lim_{t\to\infty} \frac{l(t)}{t} = 0.$$

On the other hand, since we assume that $v_{r,\infty}$ is not an attractor, we have by (2.14)

$$\lim_{t\to\infty} F_t(v,t) = \lim_{t\to\infty} f'(W(v,-t)) = f'(\alpha(v_{r,\infty})),$$

uniformly for v in a neighbourhood of $v_{r,\infty}$. It follows

$$\lim_{t\to\infty} \frac{F(v_r(t),t)}{t} = f'(\alpha(v_{r,\infty})).$$

By (3.13) we deduce, letting $t \to \infty$,

$$\sigma(v_{r,\infty}, z_r) = f'(\alpha(v_{r,\infty})). \tag{3.14}$$

Since $v_{r,\infty} > z_r$ and f is strictly convex, we have $\sigma(v_{r,\infty}, z_r) < f'(v_{r,\infty}) \le f'(v)$ for any $v \ge v_{r,\infty}$. Then (3.14) implies that $z_r < \alpha(v_{r,\infty}) < v_{r,\infty}$. Therefore, if we set $\bar{w} = Z_+(\alpha(v_{r,\infty}))$ we have $\bar{w} \in \mathcal{S}_+(z_r)$ and $v_{r,\infty} = v^*(\bar{w})$. $\square$

Theorem 3.11. *If $v_{r,\infty} \in \mathcal{Z}_a(g)$ and $v_{r,\infty} > z_r$, then $v_{r,\infty} \in \mathcal{S}_+(z_r)$.*

Proof. By (3.8) we have, for t large enough,

$$\gamma_r(t) > l(t) + F(Z_-(v_{r,\infty}),t) = l(t) + f'(Z_-(v_{r,\infty}))t.$$

Dividing by t and letting $t \to \infty$ we obtain

$$\sigma(v_{r,\infty}, z_r) \ge f'(Z_-(v_{r,\infty})). \tag{3.15}$$

To conclude we need to prove that the inequality above is strict. Let us suppose instead that $\sigma(v_{r,\infty}, z_r) = f'(Z_-(v_{r,\infty}))$. Then we have, as $t \to \infty$,

$$\gamma_r'(t) \to \sigma(v_{r,\infty}, z_r) = f'(Z_-(v_{r,\infty})) < f'(v_{r,\infty}),$$

which implies

$$\gamma_r(t) < l(t) + f'(v_{r,\infty})t = l(t) + F(v_{r,\infty}, t)$$

for t large enough. Comparing with (3.8) and using the monotonicity of $F(\cdot, t)$ we obtain that $v_r(t)$ is eventually smaller than $v_{r,\infty}$.

Thus, for large t almost everywhere

$$\begin{aligned}\gamma_r'(t) &= \sigma(v_r(t), W(u_r, t)) < \sigma(v_{r,\infty}, W(u_r, t)) \\ &= \sigma(v_{r,\infty}, z_r) + \frac{\sigma(z_r, W(u_r, t)) - \sigma(v_{r,\infty}, W(u_r, t))}{v_{r,\infty} - z_r}(z_r - W(u_r, t)) \\ &= \sigma(v_{r,\infty}, z_r) + O(|W(u_r, t) - z_r|) = \sigma(v_{r,\infty}, z_r) + O(e^{-at}) \\ &= f'(Z_-(v_{r,\infty})) + O(e^{-at}),\end{aligned}$$

for any $a \in (0, \alpha_g)$. This implies that $\gamma_r(t) - f'(Z_-(v_{r,\infty}))t$ is bounded from above by some constant M. On the other hand, since

$$\lim_{t\to\infty} F(v_{r,\infty}, t) - F(Z_-(v_{r,\infty}), t) = \lim_{t\to\infty} [f'(v_{r,\infty}) - f'(Z_-(v_{r,\infty}))]t = \infty$$

we can find, by (2.15), $T > 0$ and $\varepsilon > 0$ such that

$$F(v_{r,\infty} - \varepsilon, t) - F(Z_-(v_{r,\infty}), t) > M - l_+ + 1$$

for any $t \geq T$. We deduce from (3.8)

$$\begin{aligned}&\limsup_{t\to\infty}\ \gamma_r(t) - f'(Z_-(v_{r,\infty}))t = \\ &\limsup_{t\to\infty}\ l(t) + F(v_r(t), t) - F(Z_-(v_{r,\infty}), t) \geq M + 1,\end{aligned}$$

contradicting the definition of M. Therefore inequality (3.15) is strict. $\square$

By using the same method we can prove that the left interface γ_l satisfies properties analogous to Theorems 3.8, 3.10 and 3.11. We summarize these results in the following Corollary.

Corollary 3.12. *Let u be the solution of problem* (2.1)–(2.2) *and suppose that the interfaces γ_l and γ_r (see Definition* 3.4) *do not meet. Then the value of u along the right interface tends to a limit $v_{r,\infty}$ as $t \to \infty$, and we have either $v_{r,\infty} = z_r$, or $v_{r,\infty} \in \mathcal{S}_+(z_r)$, or $v_{r,\infty} = v^*(w)$ for some $w \in \mathcal{S}_+(z_r)$ (see Definition* 3.9). *Similarly, u converges along the left interface to some value $v_{l,\infty}$, where either $v_{l,\infty} = z_l$, or $v_{l,\infty} \in \mathcal{S}_-(z_l)$, or $v_{l,\infty} = v^*(w)$ for some $w \in \mathcal{S}_-(z_l)$.*

From Proposition 3.5 we deduce

$$\gamma_r(t) = \sigma(v_{r,\infty}, z_r)t + o(t), \qquad \gamma_l(t) = \sigma(v_{l,\infty}, z_l)t + o(t), \tag{3.16}$$

which implies $\sigma(v_{r,\infty}, z_r) \geq \sigma(v_{l,\infty}, z_l)$. Since we have $f'(v_{r,\infty}) \geq \sigma(v_{r,\infty}, z_r)$ and $f'(v_{l,\infty}) \leq \sigma(v_{l,\infty}, z_l)$, we obtain $v_{l,\infty} \leq v_{r,\infty}$.

With the help of the above results we can give in some cases a sufficient condition on the initial value which ensures that the solution of (2.1)–(2.2) is of type (I).

Example 3.13. Let us consider equation (2.1) with $f(u) = u^2/2$ and $g(u) = -(u+1)u(u-a)$, where a is a positive constant. We choose a perturbed Riemann initial value with $u_l = 0$, $u_r = -1$. According to Definition 3.9, we have

$$\mathcal{S}_-(z_l) = \mathcal{S}_-(0) = \{-1\},$$
$$\mathcal{S}_+(z_r) = \mathcal{S}_+(-1) = \{a\} \text{ if } a > 1, \qquad \mathcal{S}_+(z_r) = \emptyset \text{ if } a \leq 1.$$

Example 3.7 shows that when $a = 2$ there exist solutions of type (II). On the other hand, it is easy to see that if $a \leq 1$ the solution is of type (I) for any choice of perturbed Riemann data satisfying the above conditions. In fact we obtain from Corollary 3.12 that if $a \leq 1$ any solution of type (II) satisfies $v_{r,\infty} = -1$ and $v_{l,\infty} = -1$ or $v_{l,\infty} = 0$. Then (3.16) implies that the interfaces meet, and that the solution is of type (I). $\square$

4. Asymptotic profile of the solution. In this section we provide a description of the asymptotic profile of the solution of problem (2.1)–(2.2). Following the procedure of [21] for solutions with compact support, we divide the half-plane $\mathbb{R} \times \mathbb{R}_+$ into a finite number of subregions and study each of them separately. It turns out that in any of these regions the solution tends either to a constant state, or to a travelling wave. The results are collected in Theorem 4.5.

Throughout the section we exclude the trivial case when the interfaces meet. We begin by introducing the following sets.

Definition 4.1. For any $w_i \in \mathcal{Z}(g)$, we associate with the solution u of problem (2.1)–(2.2) the set

$$\mathcal{I}(w_i) = \{x \in [L_-, L_+] : u(x + f'(w_i)t, t) = w_i \ \forall t > 0\}.$$

We say that $w_i \in \mathcal{Z}(g)$ belongs to $\mathcal{W}$ if the set $\mathcal{I}(w_i)$ is nonempty. We also define $\mathcal{W}_a = \mathcal{W} \cap \mathcal{Z}_a(g)$ and $\mathcal{W}_r = \mathcal{W} \cap \mathcal{Z}_r(g)$ (see (2.7)).

In general it is not possible to determine the sets $\mathcal{I}(w_i)$ without knowing the solution. However, we can deduce from Corollary 3.3(ii) the following useful inclusion.

$$\mathcal{I}(w_i) \subset \{x \in [L_-, L_+] \,:\, u_0(x) \leq w_i \leq u_0(x+)\}. \tag{4.1}$$

Moreover, the following properties can be proved (see Proposition 5.2 in [21]).

Proposition 4.2. *Either $\mathcal{W}$ is empty or $\mathcal{W} = \{w_i : i_l \leq i \leq i_r\}$ for some $1 \leq i_l \leq i_r \leq N$. The set $\mathcal{I}(w_i)$ is closed for any $i = i_l, \ldots, i_r$. Moreover, if $i < j$, then $x_i \leq x_j$, for any $x_i \in \mathcal{I}(w_i), x_j \in \mathcal{I}(w_j)$.*

In contrast to the case of solution with compact support, for the perturbed Riemann problem the set $\mathcal{W}$ can be empty (for instance, if the range of the initial value lies in an interval containing no zeros of g). The following result shows that in this case the solution converges to a constant state.

Proposition 4.3. *Suppose that $\mathcal{W}_r$ is empty. Then there exists $\bar{w} \in \mathcal{Z}_a(g)$ such that $u(\cdot,t) \to \bar{w}$ as $t \to \infty$ uniformly in x.*

Proof. Let us suppose that u does not converge uniformly to any stable zero of g. We claim that there exists $w_i \in \mathcal{Z}_r(g)$ such that

$$w_i \in [\inf u(\cdot,t), \sup u(\cdot,t)] \text{ for all } t > 0. \tag{4.2}$$

This follows by a standard comparison argument with the o.d.e. $\dot{u} = g(u)$. In fact, if no such $w_i \in \mathcal{Z}_r(g)$ exists, we obtain that there is $T > 0$ such that the range of $u(\cdot,T)$ lies in a closed interval containing no unstable zeros of g. Then u converges uniformly to some stable zero of g, contradicting our assumption.

We now want to show that $\mathcal{I}(w_i)$ is nonempty. We recall a sufficient condition for this property (see [21]). If there exist two sequences (x_n,t_n) and (y_n,t_n) such that $\gamma_l(t_n) \le x_n \le y_n \le \gamma_r(t_n)$, $t_n \to \infty$, and $u(x_n+,t_n) \le w_i \le u(y_n,t_n)$, then $\mathcal{I}(w_i)$ is nonempty. From (4.2) we deduce that either this sufficient condition is satisfied or there exists $T > 0$ and a shock curve $\gamma : [T,\infty) \to \mathbb{R}$ such that

$$u(x,t) > w_i \text{ if } x \le \gamma(t), \qquad u(x,t) < w_i \text{ if } x > \gamma(t)$$

for any $t \ge T$. By Theorem 3.1, either $\gamma_l(t) < \gamma(t) < \gamma_r(t)$ for any $t > T$ or γ coincides eventually with one of the two interfaces. We claim that in both cases the solution turns out to be of type (I), contradicting our assumption that the interfaces do not meet. Let us consider for instance the first case (the other is treated similarly). We have $v_{l,\infty} \ge w_i \ge v_{r,\infty}$. Moreover, since w_i is an unstable zero, we have $z_l > w_i > z_r$ and thus $\sigma(z_l, v_{l,\infty}) > \sigma(z_r, v_{r,\infty})$. By (3.16) the solution is of type (I). The contradiction shows that $w_i \in \mathcal{W}_r$. $\square$

We assume from now on that $\mathcal{W}$ is nonempty and introduce the following notations.

Definition 4.4. For $i = i_l, i_l+1, \ldots, i_r$ we set

$$\underline{x}_i = \min \mathcal{I}(w_i), \quad \overline{x}_i = \max \mathcal{I}(w_i);$$
$$\underline{\zeta}_i(t) = \underline{x}_i + f'(w_i)t, \quad \overline{\zeta}_i(t) = \overline{x}_i + f'(w_i)t.$$

We divide the half plane $\mathbb{R} \times \mathbb{R}_+$ into the following regions:

$$S = \{(x,t) : x \notin (\gamma_l(t), \gamma_r(t)]\};$$
$$R_i = \{(x,t) : \overline{\zeta}_i(t) < x \le \underline{\zeta}_{i+1}(t)\} \quad i = i_l, \ldots, i_r - 1;$$
$$R_l = \{(x,t) : \gamma_l(t) < x \le \underline{\zeta}_{i_l}(t)\}, \quad R_r = \{(x,t) : \overline{\zeta}_{i_r}(t) < x \le \gamma_r(t)\};$$
$$P_i = \{(x,t) : \underline{\zeta}_i(t) < x \le \overline{\zeta}_i(t)\} \quad i = i_l, \ldots, i_r.$$

In the region S the solution is equal to $W(u_l, t)$ or to $W(u_r, t)$ by Proposition 3.5(i). We observe that the strips P_i may be empty (in the case $\overline{x}_i = \underline{x}_i$). Similarly, the region R_l (respectively R_r) is empty if $T_l^* = \infty$ and $v_l \in \mathcal{Z}(g)$ (respectively $T_r^* = \infty$ and $v_r \in \mathcal{Z}(g)$). The remaining regions instead are nonempty. We can describe the asymptotic behaviour of u in the following way.

Theorem 4.5. *Let u be the solutions of problem* (2.1)–(2.2). *Suppose that the interfaces of u do not meet, and that u does not converge uniformly to a constant state. Then, as $t \to \infty$, u satisfies the following properties*:

(i) *In the regions R_i, with i even* (*respectively i odd*), *u converges uniformly to the travelling wave $\phi_i(x - \overline{x}_i - f'(w_i)t)$* (*respectively $\phi_{i+1}(x - \underline{x}_{i+1} - f'(w_{i+1})t)$*), *where ϕ_i is given by Definition* 2.1.

(ii) *In the region R_l the solution converges uniformly to w_{i_l}* (*if i_l is odd*) *or to the travelling wave $\phi_{i_l}(x - \underline{x}_{i_l} - f'(w_{i_l})t)$* (*if i_l is even*). *Similarly, in R_r there is convergence either to w_{i_r} or to $\phi_{i_r}(x - \overline{x}_{i_r} - f'(w_{i_r})t)$* (*if i_r is even*).

(iii) *In the regions P_i with i odd the solution converges uniformly to w_i, while if i is even it converges almost everywhere to a discontinuous travelling wave oscillating around w_i.*

The above theorem will be a consequence of the following Theorems 4.6, 4.10 and 4.11, which also contain results about the rate of convergence of the solution to the asymptotic profile. We begin with the regions of the form R_i with i even (the case when i is odd is treated in a completely similar way).

Theorem 4.6. *For any $w_i \in \mathcal{W}_r$ we have, as $t \to \infty$,*

$$u(x,t) = \phi_i(x - \overline{x}_i - f'(w_i)t) + o(1) \qquad (x,t) \in R_i.$$

Moreover, if

$$\liminf_{x \to \overline{x}_i^+} \frac{u_0(x) - w_i}{x - \overline{x}_i} > 0,$$

then we have, as $t \to \infty$,

$$\sup_{x \in [\overline{\zeta}_i(t), \underline{\zeta}_{i+1}(t)]} |u(x,t) - \phi_i(x - \overline{x}_i - f'(w_i)t)| = o(e^{-\alpha t})$$

for any $\alpha < \alpha_g/2$.

We need some preliminary results. In the following lemmas it is always assumed that $w_i \in \mathcal{W}_r$ and $\alpha < \alpha_g/2$.

Lemma 4.7. *Given $d > 0$ we have*

$$w_{i+1} - o(e^{-\alpha t}) \leq \phi_i(x - f'(w_i)t) \leq \tilde{u}(x,t) \leq w_{i+1} + o(e^{-\alpha t})$$

for any $x \in [f'(w_{i+1})t, f'(w_{i+1})t + d]$.

Proof. Let $x \in [f'(w_{i+1})t, f'(w_{i+1})t + d]$. We have

$$\phi_i(f'(w_{i+1})t - f'(w_i)t) \le \phi_i(x - f'(w_i)t) < w_{i+1}. \tag{4.3}$$

Furthermore, by Theorem 2.7,

$$\phi_i(f'(w_{i+1})t - f'(w_i)t) \ge \tilde{u}(f'(w_{i+1})t, t) - o(e^{-\alpha t}) = w_{i+1} - o(e^{-\alpha t}). \tag{4.4}$$

On the other hand we have, again by Theorem 2.7,

$$\begin{aligned} w_{i+1} &\le \tilde{u}(x,t) \le \phi_{i+2}(x - f'(w_{i+2})t) + o(e^{-\alpha t}) \\ &\le \phi_{i+2}\big((f'(w_{i+1}) - f'(w_{i+2}))t'\big) + o(e^{-\alpha t}) \\ &\le \tilde{u}\big(f'(w_{i+1})t', t'\big) + o(e^{-\alpha t}) + o(e^{-\alpha t'}) = w_{i+1} + o(e^{-\alpha t}), \end{aligned} \tag{4.5}$$

where we have set

$$t' = t + \frac{d}{f'(w_{i+1}) - f'(w_{i+2})}.$$

From inequalities (4.3)–(4.4)–(4.5) the assertion follows. □

From Theorem 2.7 and Lemma 4.7 we obtain

$$|\phi_i(x - \overline{x}_i - f'(w_i)t) - \tilde{u}(x - \overline{x}_i, t)| = o(e^{-\alpha t}), \qquad (x,t) \in R_i. \tag{4.6}$$

Let us define

$$\overline{R}_i = \{(x,t) \in R_i : u(x,t) \le W(w_i + e^{-\alpha t}, t)\}.$$

Lemma 4.8. *We have*

$$|u(x,t) - \phi_i(x - \overline{x}_i - f'(w_i)t)| \le o(e^{-\alpha t}), \qquad (x,t) \in R_i \setminus \overline{R}_i.$$

Proof. We write equality (3.4) in the equivalent way

$$u(x,t) = \tilde{u}(x - \eta(x,t), t). \tag{4.7}$$

From Corollary 3.3 we deduce

$$\eta(x,t) \in [\overline{x}_i, \underline{x}_{i+1}], \qquad (x,t) \in R_i. \tag{4.8}$$

The two above relations imply $u(x,t) \le \tilde{u}(x - \overline{x}_i, t)$ in R_i. On the other hand, by Lemma 2.8, we have

$$u(x,t) \ge w_{i+1} - o(e^{-\alpha t}) \ge \phi_i(x - \overline{x}_i - f'(w_i)t) - o(e^{-\alpha t})$$

for any $(x,t) \in R_i \setminus \overline{R}_i$. We conclude by (4.6). □

Lemma 4.9. *We have*

$$\lim_{t\to\infty}(\sup\{\eta(x,t) : (x,t) \in \bar{R}_i\}) = \overline{x}_i.$$

Proof. The result can be proved by a technique which is standard in the theory of the generalized characteristics (see e.g. [2, Lemma 10.2], [20, Proposition 4.8]). We suppose the assertion is not true. Then, by (4.8) there exists $\varepsilon > 0$ and a sequence $\{(x_n, t_n)\} \subset R_i$ with $t_n \to \infty$, $W(u(x_n, t_n), -t_n) \le w_i + e^{-\alpha t_n}$ and $\underline{x}_{i+1} \ge \eta(x_n, t_n) \ge \overline{x}_i + \varepsilon$. Let ξ_n be a backward classical characteristics starting at (x_n, t_n). It can be checked that the sequence ξ_n converges uniformly to a classical characteristic of the form $\xi(t) = \overline{x} + f'(w_i)t$, with $\overline{x} \ge \overline{x}_i + \varepsilon$, thus contradicting the maximality of $\overline{x}_i$. We omit the details. □

Proof of Theorem 4.6. By estimate (4.6) it is enough to prove the result with $\phi_i(x - f'(w_i)t - \overline{x}_i)$ replaced by $\tilde{u}(x - \overline{x}_i, t)$. Moreover, by Lemma 4.8 we only need to consider the points $(x,t) \in \bar{R}_i$. We observe that equalities (4.7) and (4.8) imply

$$\tilde{u}(x - \overline{x}_i, t) - \omega_t(\eta(x,t) - \overline{x}_i) \le u(x,t) \le \tilde{u}(x - \overline{x}_i, t), \tag{4.9}$$

where ω_t is the modulus of continuity of $\tilde{u}(\cdot, t)$ in $[f'(w_i)t, f'(w_{i+1})t + \underline{x}_{i+1} - \overline{x}_i]$. By Corollary 2.3, there exists $C > 0$ such that $\omega_t(h) \le Ch$ for t large enough. Then the first assertion of the theorem follows from Lemma 4.9.

Let us now assume that there exists $l > 0$ and a right neighbourhood of $\overline{x}_i$ where $u_0(x) \ge w_i + l(x - \overline{x}_i)$. Since we are considering points in $\bar{R}_i$, we have by (3.3)

$$u_0(\eta(x,t)) \le W(u(x,t), -t) \le w_i + e^{-\alpha t}.$$

In addition, Lemma 4.9 implies that $\eta(x,t) \to \overline{x}_i$ as $t \to \infty$. Thus, we obtain that $\eta(x,t) - \overline{x}_i \le l^{-1}e^{-\alpha t}$ for t large enough. We conclude by (4.9) and the estimate $\omega_t(h) \le Ch$. □

We next describe the behaviour of u in the region R_r (a similar statement can be proved about the behaviour in R_l).

Theorem 4.10.

(i) *If $w_{i_r} \in \mathcal{W}_r$, then $u(x,t) = \phi_{i_r}(x - \overline{x}_{i_r} - f'(w_{i_r})t) + o(1)$, $(x,t) \in R_r$. Moreover, if*

$$\liminf_{x\to\overline{x}_{i_r}^+} \frac{u_0(x) - w_{i_r}}{x - \overline{x}_{i_r}} > 0,$$

then in the above estimate we have $o(1) = o(e^{-\alpha t})$ for any $\alpha < \alpha_g/2$.

(ii) *If $w_{i_r} \in \mathcal{W}_a$, then $w_{i_r} = v_{r,\infty} = z_r$ and $u(x,t) = z_r + o(e^{-\alpha t})$, $(x,t) \in R_r$ for any $\alpha \le \alpha_g/2$.*

Proof. We give only a sketch of the proof, which is almost entirely similar to the proof of Theorem 4.6. The inclusion corresponding to (4.8) for the region R_r is $\eta(x,t) \in [\overline{x}_{i_r}, L_+]$, $(x,t) \in R_r$. Thus (4.7) implies

$$\tilde{u}(x - L_+, t) \leq u(x,t) \leq \tilde{u}(x - \overline{x}_{i_r}, t), \qquad (x,t) \in R_r. \tag{4.10}$$

If $w_{i_r} \in \mathcal{W}_r$, we can proceed as in the proof of Theorem 4.6 and obtain part (i) of the Theorem.

Let us consider the case $w_{i_r} \in \mathcal{W}_a$. We claim that $v_{r,\infty} \leq w_{i_r}$. In fact, if $v_{r,\infty} > w_{i_r}$ Corollary 3.12 implies $v_{r,\infty} > w_{i_r+1}$. Then we can choose a sequence t_n such that $t_n \to \infty$ and define $x_n = \overline{x}_{i_r} + f'(w_{i_r})t_n$, $y_n = \gamma_r(t_n)$. The sequences (x_n, t_n) and (y_n, t_n) satisfy the criterion recalled in the proof of Proposition 4.3 and we can deduce that $\mathcal{I}(w_{i_r+1})$ is not empty. But this contradicts the maximality of w_{i_r}. Therefore $v_{r,\infty} \leq w_{i_r}$.

Since by Proposition 3.5(ii) $z_r \leq v_{r,\infty}$, we have $\sigma(z_r, v_{r,\infty}) \leq f'(w_{i_r})$, and the inequality is strict unless $z_r = v_{r,\infty} = w_{i_r}$. If the inequality is strict, then (3.16) implies that the two characteristics $x = \gamma_r(t)$ and $x = \overline{x}_{i_r} + f'(w_{i_r})t$ eventually intersect. But this contradicts well-known properties of generalized characteristics. Therefore we have $z_r = v_{r,\infty} = w_{i_r}$, and $\gamma_r(t) = f'(w_{i_r})t + o(t)$. Now we can argue as in Lemma 4.7 to prove that

$$\tilde{u}(x - L_+, t) - w_{i_r} = \tilde{u}(x - \overline{x}_{i_r}, t) - w_{i_r} = o(e^{-\alpha t}), \qquad (x,t) \in R_r.$$

We can conclude by inequality (4.10). □

The behaviour of the solutions in the regions P_i, with $i = i_l, \ldots, i_r$ can be recovered from the results of [5], [14] and [20] about periodic solutions of (2.1). For any $i = i_l, \ldots, i_r$ the set $[\underline{x}_i, \overline{x}_i] \setminus \mathcal{I}(w_i)$ is open by Proposition 4.2 and therefore can be decomposed into a family, at most countable, of open disjoint intervals (a^i_j, b^i_j), $j \geq 1$. If i is even, it can be checked that there exists a unique value $\eta^i_j \in (a^i_j, b^i_j)$ such that

$$\sigma(\phi_i(\eta^i_j - a^i_j), \phi_i(\eta^i_j - b^i_j)) = f'(w_i).$$

We define

$$\tilde{\phi}_i(x) = \begin{cases} w_i & x \in \mathcal{I}(w_i) \\ \phi_i(x - a^i_j) & x \in (a^i_j, \eta^i_j] \\ \phi_i(x - b^i_j) & x \in (\eta^i_j, b^i_j). \end{cases}$$

It is easily checked that $\tilde{\phi}_i(x - f'(w_i)t)$ is an entropy solution of equation (2.1) in P_i. Unlike the travelling wave $\phi_i(x - f'(w_i)t)$, it is discontinuous and oscillates around w_i. It turns out that any travelling wave solution of (2.1) in the strip P_i assuming the value w_i along the half-lines which delimit P_i is of the above form for some open subset $\cup(a_j, b_j) \subset [\underline{x}_i, \overline{x}_i]$.

The following result holds (see [5], [14] and [20]).

Theorem 4.11. *If i is odd, we have $u(x,t) = w_i + o(e^{-\alpha t})$, $(x,t) \in P_i$, for any $\alpha \in (0, -g'(w_i))$. If i is even, $u(x,t)$ tends to $\tilde{\phi}_i(x - f'(w_i)t)$; namely, we have*

$$\lim_{t\to\infty} u(x + f'(w_i)t, t) = \tilde{\phi}_i(x), \qquad x \in [\underline{x}_i, \overline{x}_i], x \neq \eta^i_j.$$

The above result completes the proof of Theorem 4.5. We conclude this section by considering again the question of the rate of convergence of the solution to the travelling waves in the regions R_i. In Theorem 4.6 we proved that this rate is exponential provided the initial value satisfies a suitable condition around the point $\overline{x}_i$. We now show that, without such an assumption, the convergence of the solution may be arbitrarily slow.

Theorem 4.12. *Let $w_i \in \mathcal{Z}_r(g)$ and let a nonincreasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be given, with $\omega(t) \to 0$ as $t \to \infty$. Then there exists an initial value u_0 satisfying* (H3), (H4) *such that $w_i \in \mathcal{W}$ and*

$$\sup\{|u(x,t) - \phi(x - f'(w_i)t - \overline{x}_i)| \ : \ x \in [\overline{\zeta}_i(t), \underline{\zeta}_{i+1}(t)]\} > \omega(t) \tag{4.11}$$

for any t large enough.

Proof. To simplify notations we set $w_i = a$, $w_{i+1} = b$, $\phi_i = \phi$ and we assume $f'(a) = 0$. It is not restrictive to assume $\omega(t) \geq e^{-g'(a)t/2}$. We define

$$u_0(x) = \begin{cases} a & x \leq 0 \\ h(x) & 0 \leq x \leq 1 \\ b & x > 1, \end{cases} \tag{4.12}$$

where h is a smooth strictly increasing function satisfying $h(0) = a$, $h(1) = b$ and

$$h(x) = a + e^{-2g'(a)\omega^{-1}(x^2)}$$

for x in a right neighbourhood of zero. Since u_0 is continuous and increasing, the solution of problem (2.1)–(2.2) can be found by the classical method of characteristics. For any t we have $u(x,t) \equiv a$ for $x \leq 0$, $u(x,t) \equiv b$ for $x \geq 1 + f'(b)t$, while $u(\cdot, t)$ increases continuously from a to b in the interval $[0, 1 + f'(b)t]$. Therefore, $\overline{x}_i = 0$, $\underline{x}_{i+1} = 1$, and

$$R_i = \{(x,t) : 0 \leq x \leq 1 + f'(b)t\}.$$

In addition, by Theorem 4.6, $u(x,t)$ converges to $\phi(x)$ in R_i as $t \to \infty$.

Let us fix $c \in (a,b)$. For t large enough we have

$$e^{-2g'(a)t} < W(c, -t) - a < e^{-g'(a)t/2}. \tag{4.13}$$

For any $t > 0$ there exists a unique value $\gamma(t) \in [0, 1+f'(b)t]$ such that $u(\gamma(t), t) = c$. We find, by (3.4) and the definitions of ϕ, F, W,

$$\begin{aligned} \int_c^{\phi(\gamma(t))} \frac{f'(v)}{g(v)}\,dv &= \int_a^{\phi(\gamma(t))} \frac{f'(v)}{g(v)}\,dv + \int_c^{W(c,-t)} \frac{f'(v)}{g(v)}\,dv + \int_{W(c,-t)}^{a} \frac{f'(v)}{g(v)}\,dv \\ &= \gamma(t) - F(c,t) - \int_a^{W(c,-t)} \frac{f'(v)}{g(v)}\,dv \\ &= \eta(\gamma(t), t) - \int_a^{W(c,-t)} \frac{f'(v)}{g(v)}\,dv. \end{aligned} \tag{4.14}$$

Moreover, $h(\eta(\gamma(t), t)) = W(c, -t)$ by (3.3). Let us choose $M > 0$ such that

$$\frac{f'(v)}{g(v)} < M, \qquad v \in [a, c+\delta],$$

for some fixed $\delta \in (0, b-c)$. Since u converges to ϕ, we have $\phi(\gamma(t)) < u(\gamma(t), t) + \delta = c + \delta$ for t large enough. We obtain from (4.14) and (4.13)

$$\begin{aligned} |\phi(\gamma(t)) - c| &\geq \frac{1}{M}\Big|\int_c^{\phi(\gamma(t))} \frac{f'(v)}{g(v)}\,dv\Big| \\ &\geq \frac{1}{M}\big\{h^{-1}(W(c,-t)) - M(W(c,-t) - a)\big\} \\ &\geq \frac{1}{M} h^{-1}(W(c,-t)) - e^{-g'(a)t/2}. \end{aligned} \tag{4.15}$$

From the definition of h we deduce, for small positive v,

$$h^{-1}(v) = \sqrt{\omega\big(-\tfrac{\ln(v-a)}{2g'(a)}\big)}.$$

Then (4.13) implies

$$h^{-1}(W(c,-t)) > h^{-1}(a + e^{-2g'(a)t}) = \sqrt{\omega(t)} \geq e^{-g'(a)t/4}.$$

We deduce from (4.15), for t large enough,

$$\begin{aligned} \sup |u(x,t) - \phi(x)| &\geq |c - \phi(\gamma(t))| \\ &\geq (2M)^{-1}\big(\sqrt{\omega(t)} + e^{-g'(a)t/4}\big) - e^{-g'(a)t/2} > \omega(t). \quad \square \end{aligned}$$

A similar result holds in the regions P_i with i even. We have seen in Theorem 4.11 that the solution converges pointwise to a discontinuous travelling wave; since the solution is bounded, there is also convergence in the L^1 norm. However, it follows from the results in [23] that, given a function ω as in Theorem 4.12, there exist solutions of (2.1)–(2.2) approaching their limiting profile in the L^1-norm in the regions P_i with a rate slower than ω.

5. Generic behaviour of solutions. In the previous section we have shown that the solution to the perturbed Riemann problem (2.1)–(2.2) converges, as time goes to infinity, to a sequence of travelling waves bordered by the two curves γ_l and γ_r, which are possibly shock curves. The striking difference with the unperturbed case is that in general the asymptotic profile contains some discontinuous travelling waves: we have seen in fact that in the regions denoted by P_i, with i even, the solution develops oscillations around the value w_i. However, since these regions can be empty, not all the solutions exhibit such a behaviour.

The goal of this section is to show that the solution of (2.1)–(2.2) develops no oscillating patterns for "most" of the initial data, namely for a set of data which is generic with respect to the L^1-topology. Given u_l, u_r and L, let us denote by $\mathcal{L}(u_l, u_r, L)$ the set of initial data which satisfy (H3), (H4) for some $L \geq 0$ and by $\mathcal{L}^0(u_l, u_r, L)$ the subset of the data giving rise to a solution such that the regions P_i with i even are empty. We prove in Theorem 5.2 that $\mathcal{L}^0(u_l, u_r, L)$ is dense in $\mathcal{L}(u_l, u_r, L)$. Then, in Theorem 5.5 we prove that $\mathcal{L}^0(u_l, u_r, L)$ is the countable intersection of open subsets of $\mathcal{L}(u_l, u_r, L)$. Heuristically, these results can be motivated with the unstable character of the homoclinic travelling waves of (2.1) given by oscillations around a zero of $\mathcal{Z}_r(g)$ (see [16]).

Definition 5.1. We say that $w_i \in \mathcal{W}_r$ is in $\mathcal{W}_r^*(u_0)$ if $\underline{x}_i \neq \overline{x}_i$ ($\underline{x}_i$ and $\overline{x}_i$ as in Definition 4.4).

Theorem 5.2. *Let $u_0 \in \mathcal{L}(u_l, u_r, L)$. For any $\varepsilon > 0$, there exists $u_0^\varepsilon \in \mathcal{L}(u_l, u_r, L)$ such that*

(i) *$meas(supp(u_0^\varepsilon - u_0)) < \varepsilon$;*
(ii) *$\|u_0^\varepsilon - u_0\|_{L^1(\mathbb{R})} < \varepsilon$;*
(iii) *$\mathcal{W}_r^*(u_0^\varepsilon) = \varnothing$.*

To prove the result we need some preliminary Lemmas. We assume that the reader is familiar with the structure of the travelling-wave solutions of (2.1) (see the remarks given before Theorem 4.9 or the general analysis in [16]).

Lemma 5.3. *Let $w_i \in \mathcal{Z}_r(g)$. Let $\psi_1, \psi_2 : [x_0, x_1] \to \mathbb{R}$ be given such that the travelling waves $\psi_1(x - f'(w_i)t)$ and $\psi_2(x - f'(w_i)t)$ are solutions of* (2.1). *Set*

$$\overline{x}_1 = \inf\{x \in (x_0, x_1) : \psi_1(x) = w_i\}, \quad \overline{x}_2 = \inf\{x \in (x_0, x_1) : \psi_2(x) = w_i\}$$

(*if one of the above sets is empty, we define the infimum to be ∞*). *Suppose that*

(i) *$\psi_1(x_0) = \psi_2(x_0) = w_i$;*
(ii) *$\psi_1(x) \leq \psi_2(x)$ for $x \in [x_0, x_1]$;*
(iii) *$\overline{x}_1 \neq \overline{x}_2$.*

Then $\psi_2(x) = \phi_i(x - x_0)$ *for any* $x \in [x_0, x_1]$ (ϕ_i *as in Definition* 2.1).

Proof. If ψ_1 coincides with $\phi_i(\cdot - x_0)$ in $x \in [x_0, x_1]$, then (ii) implies that the same is true for ψ_2. Thus, we may assume $\psi_1 \not\equiv \phi_i(\cdot - x_0)$, which implies that $\overline{x}_1 \neq \infty$. We first consider the case $\overline{x}_1 > x_0$. Then there exists $\eta_1 \in (x_0, \overline{x}_1)$ such that

$$\psi_1(x) = \begin{cases} \phi_i(x - x_0) & x \in [x_0, \eta_1) \\ \phi_i(x - \overline{x}_1) & x \in (\eta_1, \overline{x}_1]. \end{cases} \tag{5.1}$$

Suppose $\overline{x}_2 \neq \infty$. Since $\psi_2 \geq \psi_1 > w_i$ in (x_0, η_1), we have $\overline{x}_2 \neq x_0$. Then there exists $\eta_2 \in (x_0, \overline{x}_2)$ such that

$$\psi_2(x) = \begin{cases} \phi_i(x - x_0) & x \in [x_0, \eta_2) \\ \phi_i(x - \overline{x}_2) & x \in (\eta_2, \overline{x}_2]. \end{cases} \tag{5.2}$$

By assumption (ii), this implies $\eta_2 \geq \eta_1$ and $\overline{x}_2 \leq \overline{x}_1$. The second inequality is strict by (iii). Then we have

$$\psi_1(\eta_1-) = \phi_i(\eta_1 - x_0) \leq \phi_i(\eta_2 - x_0) = \psi(\eta_2-),$$

$$\psi_1(\eta_1+) = \phi_i(\eta_1 - \overline{x}_1) < \phi_i(\eta_2 - \overline{x}_2) = \psi(\eta_2+).$$

This is impossible, since $\sigma(\psi_h(\eta_h-), \psi_h(\eta_h+)) = f'(w_i)$, for $h = 1, 2$ and σ is strictly increasing in both arguments. The contradiction proves that $\overline{x}_2 = \infty$, and so $\psi_2 \equiv \phi(\cdot - x_0)$.

Now consider the case $\overline{x}_1 = x_0$. By (iii) we have $\overline{x}_2 > x_0$. If $\overline{x}_2 \neq \infty$, then there exists as before $\eta_2 \in (x_0, \overline{x}_2)$ such that (5.2) holds. By the definition of $\overline{x}_1$, there exists $x' \in (x_0, \eta_2)$ such that $\psi_1(x') = w_i$. Furthermore, by (ii), we have $\psi_1 \leq \psi_2 < w_i$ in $(\eta_2, \overline{x}_2)$. Thus there is at least one point of discontinuity for ψ_1 in $(x', \eta_2]$; let us call η_1 the supremum of those points. Then ψ_1 is continuous in $(\eta_1, \overline{x}_2]$, and so

$$\psi_1(\eta_1+) \leq \psi_1(\eta_2+) \leq \psi_2(\eta_2+).$$

On the other hand

$$\psi_1(\eta_1-) \leq \phi_i(\eta_1 - x') < \phi_i(\eta_2 - x_0) = \psi_2(\eta_2-).$$

As in the case previously considered, we find a contradiction. Thus the assertion is proved also if $\overline{x}_1 = x_0$. $\square$

Lemma 5.4. *Let* u, v *be the solutions of* (2.1) *corresponding to the initial data* u_0 *and* v_0 *respectively. Suppose that* $u_0 \leq v_0$, *and that there exists* $\bar{y} \in \mathbb{R}$ *such that* $u_0(x) = v_0(x)$ *for any* $x < \bar{y}$. *Then*

$$u(x, t) = v(x, t) \qquad t \geq 0, \ x \leq \zeta(t),$$

where ζ is the maximal forward characteristic associated to u starting from the point $(\bar{y}, 0)$.

Proof. Since $u_0 \leq v_0$, we have $u \leq v$, and in particular $u(\zeta(t), t) \leq v(\zeta(t), t)$. Then we can apply Lemma 2.2.10 in [18] to the region lying to the left of ζ and obtain the assertion. $\square$

Proof of Theorem 5.2. Let w_i be the smallest zero belonging to $\mathcal{W}_r^*(u_0)$. It is enough to find an initial datum v_0 that satisfies (i), (ii) and such that $w_j \notin \mathcal{W}_r^*(v_0)$ for $j \leq i$. Then, the datum u_0^ε is obtained iterating the procedure a finite number of times.

We construct v_0 in the following way. Define $\overline{x} = \inf\{x > \underline{x}_i : x \in \mathcal{I}(w_i)\}$. We set

$$v_0(x) = \begin{cases} u_0(x) & x \leq \underline{x}_i,\ x > \underline{x}_i + \delta, \\ \max\{u_0(x), w_i + \delta\} & \underline{x}_i < x \leq \underline{x}_i + \delta \end{cases}$$

if $\overline{x} = \underline{x}_i$, while if $\overline{x} > \underline{x}_i$ we define

$$v_0(x) = \begin{cases} u_0(x) & x \leq \overline{x} - \delta,\ x > \overline{x} + \delta, \\ \max\{u_0(x), w_i + \delta\} & \overline{x} - \delta < x \leq \overline{x} + \delta. \end{cases}$$

Here δ is some positive constant, which we choose suitably small in order to satisfy properties (i) and (ii). In the case $\overline{x} > \overline{x}_i$ we also require $\delta < \overline{x} - \overline{x}_i$. Let v be the solution of (2.1) with initial value v_0 and let $\underline{y}_j$ and $\overline{y}_j$ be defined as $\underline{x}_j$ and $\overline{x}_j$ with u replaced by v for any $j = 1, \ldots, N$.

We have $v \geq u$ and, by Lemma 5.4, $u(x,t) = v(x,t),\ t \geq 0,\quad x \leq \underline{x}_i + f'(w_i)t$. It follows

$$\underline{x}_j = \underline{y}_j = \overline{x}_j = \overline{y}_j, \quad j = 1, \ldots, i-1, \quad \underline{x}_i = \underline{y}_i,\ \overline{x}_i \geq \overline{y}_i.$$

Let us suppose $\overline{y}_i > \underline{y}_i$. We denote by ψ_1 and ψ_2 the travelling waves giving the asymptotic profiles of u and v respectively in the strip

$$\{(x,t) : x - f'(w_i)t \in [\underline{y}_i, \overline{y}_i]\}.$$

By the definition of v_0 and by (4.1) we deduce that $\psi_2(x) > w_i$ for $x \in (\underline{x}_i, \underline{x}_i + \delta)$ in the case $\overline{x} = \underline{x}_i$ and that $\psi_2(\overline{x}) > w_i$ in the case $\overline{x} > \underline{x}_i$. This shows that we can apply Lemma 5.3 and obtain that ψ_2 is continuous and strictly increasing in $[\underline{y}_i, \overline{y}_i]$. This is a contradiction, since $\psi_2(\overline{y}_i) = w_i$. It follows that $\underline{y}_i = \overline{y}_i$. Thus $w_j \notin \mathcal{W}_r^*(v_0)$ for $j \leq i$, which was the desired property. $\square$

In the following we measure the oscillations in the asymptotic profile of a solution u corresponding to an initial value u_0 by the quantity

$$\Theta(u_0) = \sum_{w_i \in W_r^*(u_0)} (\overline{x}_i - \underline{x}_i),$$

with $W_r^*(u_0)$ given by Definition 5.1.

Theorem 5.5. *For any given $u_l, u_r \in \mathbb{R}$, $L > 0$ and $\alpha > 0$, the set $\{u_0 \in \mathcal{L}(u_l, u_r, L) : \Theta(u_0) < \alpha\}$ is an open subset of $\mathcal{L}(u_l, u_r, L)$ in the L^1-topology.*

To prove Theorem 5.5, we need a preliminary result.

Proposition 5.6. *Let $w_i \in \mathcal{Z}_r(g)$ and let $u_0 \in \mathcal{L}(u_l, u_r, L)$. For any $\sigma > 0$ there exists $\delta > 0$ with the following property: given $v_0 \in \mathcal{L}(u_l, u_r, L)$ such that $w_i \in \mathcal{W}_r^*(v_0)$ and $\|u_0 - v_0\|_{L^1} < \delta$, we have*

$$\underline{y}_i > \underline{x}_i - \sigma, \quad \overline{y}_i < \overline{x}_i + \sigma$$

(where $\underline{y}_i$, $\overline{y}_i$ are defined as $\underline{x}_i$ and $\overline{x}_i$ with u_0 replaced by v_0).

Proof. To simplify notations we omit the subscript i and we assume $f'(w) = 0$. We prove the part of the statement concerning $\underline{x}, \underline{y}$ (the other is analogous).

Let u and v be the solutions corresponding to the initial data u_0 and v_0. We only need to consider the case $\underline{y} \leq \underline{x}$. We first observe that, by (4.7) and Corollary 3.3,

$$v(x,t) \geq \tilde{u}(x - \underline{y}, t), \qquad x \leq \underline{y},\ t > 0. \tag{5.3}$$

Moreover, by Theorem 4.5(i) we have, for any $a > 0$,

$$\lim_{t \to +\infty} u(x,t) = \phi(x - \underline{x}) \quad \text{uniformly for } x \in [\underline{x} - a, \underline{x}], \tag{5.4}$$

where ϕ is the smooth travelling wave associated with w as in Definition 2.2.

Let us define, for $d \geq 0$,

$$I(d) = \int_{-1}^{0} [\phi(x) - \phi(x - d)]\, dx.$$

The function I is strictly increasing with respect to d and $I(0) = 0$. In addition, by Theorem 2.7 and by (5.4),

$$I(d) = \lim_{t \to +\infty} \int_{-1}^{0} [\tilde{u}(x,t) - u(x + \underline{x} - d, t)]\, dx,$$

the limit being uniform for $d \in [0, \underline{x} + L]$. Now we choose $T = T(u_0) > 0$ such that

$$\int_{-1}^{0} [\tilde{u}(x,T) - u(x + \underline{y}, T)]\, dx > \frac{1}{2} I(\underline{x} - \underline{y}). \tag{5.5}$$

By continuous dependence in L^1-norm for a solution of (2.1)–(2.2), for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, T) > 0$ such that if $\|u_0 - v_0\|_{L^1} < \delta$, we have

$$\|v(\cdot, T) - u(\cdot, T)\|_{L^1} < \varepsilon.$$

Given $\sigma > 0$, we define $\varepsilon = I(\sigma)/2$. Then, if δ is chosen as above and $\|u_0 - v_0\|_{L^1} < \delta$, we have by (5.3) and (5.5)

$$\begin{aligned}\varepsilon > \|v(\cdot, T) - u(\cdot, T)\|_{L^1} &\geq \int_{\underline{y}-1}^{\underline{y}} [v(x,T) - u(x,T)]\, dx \\ &\geq \int_{\underline{y}-1}^{\underline{y}} [\tilde{u}(x - \underline{y}, T) - u(x,T)]\, dx > \frac{1}{2} I(\underline{x} - \underline{y}),\end{aligned}$$

and thus $\underline{x} - \underline{y} < \sigma$. □

Proof of Theorem 5.5. Given $\alpha > 0$, let u_0 be such that $\Theta(u_0) < \alpha$. We take $\sigma < (\alpha - \Theta(u_0))/2N$ and choose $\delta > 0$ as in Proposition 5.6 (we can choose δ independent of $w_i \in \mathcal{Z}_r(g)$, since this is a finite set). Given $v_0 \in \mathcal{L}(u_l, u_r, L)$ such that $\|u_0 - v_0\|_{L^1} < \delta$, we define $\underline{y}_i, \overline{y}_i$ as $\underline{x}_i, \overline{x}_i$ with u_0 replaced by v_0. By Proposition 5.6 we have, for any $w_i \in \mathcal{W}_r^*(v_0)$,

$$\underline{x}_i - \underline{y}_i < \sigma, \quad \overline{y}_i - \overline{x}_i < \sigma,$$

and therefore

$$\Theta(v_0) = \sum_{i=1}^{N} (\overline{y}_i - \underline{y}_i) < \Theta(u_0) + 2N\sigma < \alpha. \quad \square$$

REFERENCES

[1] J. Bebernes and D. Eberly, *Mathematical problems from combustion theory*, Applied Mathematical Sciences vol. 83, Springer-Verlag, New York, 1989.

[2] C.M. Dafermos, *Characteristics in hyperbolic conservation laws*, Nonlinear analysis and mechanics Vol. 1 (Knops R.J., ed.), Research Notes in Math. No. 17, Pitman, Boston, 1977, pp. 1–58.

[3] C.M. Dafermos, *Generalized characteristics and the structure of solutions of hyperbolic conservation laws*, Indiana Univ. Math. J. **26** (1977), 1097–1119.

[4] A.F. Filippov, *Differential equations with discontinuous right-hand side*, Mat. Sb. **51** (1960), 99–128.

[5] H. Fan and J.K. Hale, *Large-time behavior in inhomogeneous conservation laws*, Arch. Rat. Mech. Anal. **125** (1993), 201–216.

[6] E. Godlewski and P. Raviart, *Hyperbolic Systems of Conservation Laws*, Mathématiques et applicationes, n.3/4, S.M.A.I., Ellipses, Paris, 1991.

[7] F.R. Guarguaglini, *On a class of balance laws with absorption: support properties and finite time extinction*, J. Math. Anal. Appl. **188** (1994), 500–520.

[8] F.R. Guarguaglini, *Singular solutions and asymptotic behaviour for a class of conservation laws with absorption*, Comm. Partial Differential Equations **20** (1995), 1395–1425.

[9] L. Hsiao, *Hyperbolic systems with dissipation*, Qualitative aspects and applications of nonlinear evolution equations (T.T. Li, P. De Mottoni, eds.), World Scientific Publishing, Singapore, 1991, pp. 86–106.

[10] S.N. Kruzkov, *First order quasilinear equations in several independent variables*, Math. USSR Sbornik **10** (1970), 217–243.

[11] C.Z. Li and T.P. Liu, *Asymptotic states for hyperbolic conservation laws with a moving source*, Adv. Appl. Math. **4** (1983), 353–379.

[12] T.P. Liu, *Invariants and asymptotic behavior of solutions of a conservation law*, Proc. Amer. Math. Soc. **71** (1978), 227–231.

[13] T.P. Liu, *Nonlinear resonance for quasilinear hyperbolic equation*, J. Math. Phys. **28:11** (1987), 2593–2602.

[14] A.N. Lyberopoulos, *Large time structure of solutions of scalar conservation laws without convexity in the presence of a linear source field*, J. Diff. Eq. **99** (1992), 342–380.

[15] A.N. Lyberopoulos, *A Poincaré–Bendixson theorem for scalar balance laws*, Proc. Roy. Soc. Edinburgh **124A** (1994), 589–607.

[16] C. Mascia, *Travelling wave solutions for a balance law*, Proc. Roy. Soc. Edinburgh (to appear).

[17] J.D. Murray, *On the Gunn effect and other physical examples of perturbed conservation equations*, J. Fluid Mech. **44** (1970), 315–346.

[18] R. Natalini, C. Sinestrari, and A. Tesei, *Incomplete blow-up of solutions of quasilinear hyperbolic balance laws*, Arch. Rat. Mech. Anal. **135** (1996), 259–296.

[19] R. Natalini and A. Tesei, *On a class of perturbed conservation laws*, Adv. Appl. Math. **13** (1992), 429–453.

[20] C. Sinestrari, *Large time behaviour of solutions of balance laws with periodic initial data*, NoDEA **2** (1995), 111–131.

[21] C. Sinestrari, *Asymptotic profile of solutions of conservation laws with source*, Differential and Integral Equations **9** (1996), 499–525.

[22] C. Sinestrari, *The Riemann problem for an inhomogeneous conservation law without convexity*, SIAM J. Math. Anal. (to appear).

[23] C. Sinestrari, *Instability of discontinuous travelling waves for hyperbolic conservation laws*, J. Differential Equat. (to appear).

[24] J. Smoller, *Shock waves and reaction-diffusion equations*, Springer-Verlag, New York, 1983.