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INFINITELY MANY SOLUTIONS OF WEAKLY COUPLED SUPERLINEAR SYSTEMS

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Abstract. We study weakly coupled superlinear systems with different boundary conditions. The boundary conditions that we consider are the Sturm-Liouville and the three-point boundary conditions. We first prove a continuation theorem based on coincidence degree and explain how to compute the degree for some simple scalar differential equations. Then we apply those results to prove the existence of infinitely many solutions to the weakly coupled system.

This paper is devoted to the study of weakly coupled systems of equations of the type

$$\begin{aligned} &x''(t) + g(x(t)) = p(t, x(t), x'(t)) \\ &l_{i,1}(x_i) = A_i \quad l_{i,2}(x_i) = B_i, \end{aligned}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $g(x) = (g_1(x_1), \dots, g_n(x_n))$, $p : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is continuous and $l_{i,j} : C^1(I, \mathbb{R}) \to \mathbb{R}$ $(1 \le i \le n, j = 1, 2)$ are linear and continuous. We will study multiplicity results for different boundary conditions. The two types of conditions that we will study here are the *three-point boundary conditions*

$$l_1(u) = u(0) \quad \text{and} \quad l_2(u) = u(\eta) - u(1),$$

where $\eta \in (0, 1)$ is a fixed constant and the *Sturm-Liouville boundary conditions*

$$l_1(u) = -\sin\delta\, u(0) + \cos\delta\, u'(0) \quad \text{and} \quad l_2(u) = -\sin\gamma\, u(1) + \cos\gamma\, u'(1).$$

The three-point boundary conditions, which can be found, for example, in [5], can be seen as a discretisation of the conditions $u(0) = 0$, $u'(1) = 0$. When $\eta \to 1$, we will see that the degree associated to the three-point boundary conditions is similar to the one associated to the given two-point boundary conditions.

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The Sturm-Liouville boundary conditions are a generalization of *Dirichlet* conditions ($\delta = \gamma = \frac{\pi}{2}$) and of *Neumann* boundary conditions ($\delta = \gamma = 0$).

The main hypothesis we will make is that g_i ($1 \leq i \leq n$) is superlinear; i.e.,

$$\lim_{x \to \infty} \frac{g_i(x)}{x} = +\infty.$$

Even for a single equation, the results presented here are new for the three-point boundary conditions (see also [6]). The study of the Sturm-Liouville boundary conditions for a single equation has been made in [1]. For this case, we generalize the results to perturbations with growth more than linear.

Existence results for weakly coupled systems have been already obtained by A. Capietto, J. Mawhin and F. Zanolin ([2]) for periodic boundary conditions. We will introduce a continuation theorem similar to those of [1] and [2] but adapted for systems.

1. Continuation theorem. Let $\mathcal{X}$ and $\mathcal{Z}$ be real normed vector spaces, $\mathcal{L} : \operatorname{dom} \mathcal{L} \subset \mathcal{X} \to \mathcal{Z}$ be a linear Fredholm operator of index zero and $I = [0, 1]$. And let $\mathcal{N} : \mathcal{X} \times I \to \mathcal{Z}$ be a $\mathcal{L}$-completely continuous operator. We consider the equation

$$\mathcal{L}u = \mathcal{N}(u, \lambda) \tag{1}$$

for $(u, \lambda) \in \operatorname{dom} \mathcal{L} \times I$. Denote by Σ^* the set of solutions of this equation; i.e.,

$$\Sigma^* = \{(u, \lambda) \in \operatorname{dom} \mathcal{L} \times I : \mathcal{L}u = \mathcal{N}(u, \lambda)\}.$$

From now on, we use the following conventions. For $\mathcal{O} \subset \mathcal{X} \times I$ and $\lambda \in I$, we denote by $\mathcal{O}_\lambda$ the section $\{x \in \mathcal{X} : (x, \lambda) \in \mathcal{O}\}$. If $\mathcal{V}$ is an open set of X (possibly unbounded), such that $\Sigma = \Sigma^*_\lambda \cap \overline{\mathcal{V}}$ is compact and $\Sigma \subset \mathcal{V}$ (i.e., there is no solution on $\partial\mathcal{V}$), there exists $\mathcal{U}$, an open bounded set, such that $\Sigma \subset \mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{V}$. On $\mathcal{U}$, the coincidence degree is defined (see [4] or [8] for definitions and notations). Moreover if we take two open sets $\mathcal{U}$ and $\mathcal{U}'$ having this property, by the excision property, the degree is equal on those two sets. We denote it

$$D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}(., \lambda), \mathcal{V}) = D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}(., \lambda), \mathcal{U}).$$

For vectors $a, b \in \mathbb{R}^n$ such that $a_i \leq b_i$, we denote by $[a, b]$ the cube $\times_{i=1}^n [a_i, b_i]$.

Theorem 1. (Continuation) *Let $\mathcal{O}$ be an open set of $\mathcal{X} \times I$ and $\Sigma = \Sigma^* \cap \overline{\mathcal{O}}$. Suppose that the following conditions are satisfied:*

(i1) *Σ_0 is bounded in $\mathcal{X}$ and $\Sigma_0 \subset \mathcal{O}_0$,*
(i2) *$D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}(., 0), \mathcal{O}_0) \neq 0$ and*
(i3) *there exists $\phi : \mathcal{X} \times I \to \mathbb{R}^n$ continuous on $\mathcal{X} \times I$ and proper on Σ.*

Denote $\phi_i^- = \inf\{\phi_i(u,0) : u \in \Sigma_0\}$ *and* $\phi_i^+ = \sup\{\phi_i(u,0) : u \in \Sigma_0\}$*. If there exist constants* c^- *and* $c^+ \in \mathbb{R}^n$ *with* $c_i^- < \phi_i^- \leq \phi_i^+ < c_i^+$ *such that*

$$\forall (u,\lambda) \in (\operatorname{dom}\mathcal{L} \times (0,1)) \cap \mathcal{O} \cap \Sigma : \quad \phi(u,\lambda) \notin \partial[c^-, c^+]$$

and

$$\forall (u,\lambda) \in (\operatorname{dom}\mathcal{L} \times (0,1)) \cap \partial\mathcal{O} \cap \Sigma : \quad \phi(u,\lambda) \notin [c_-, c_+],$$

then the equation

$$\mathcal{L}u = \mathcal{N}(u,1)$$

admits at least one solution on $\operatorname{dom} L \cap \overline{\mathcal{O}}_1$*.*

Proof. Let $\mathcal{A} = \phi^{-1}((c_-, c_+)) \cap \mathcal{O}$ be an open set and $\tilde{\Sigma} = \phi^{-1}([c_-, c_+]) \cap \Sigma \cap \bar{\mathcal{O}}$, a compact. Then we have $\mathcal{A} \cap \Sigma \subset \tilde{\Sigma}$.

If $Lu = N(u,1)$ has no solution in $\overline{\mathcal{O}}_1$ then $\phi(\Sigma \cap \mathcal{O}) \cap \partial[c^-, c^+] = \emptyset$ and $\phi(\Sigma \cap \partial\mathcal{O}) \cap [c_-, c_+] = \emptyset$. In this case, $\tilde{\Sigma} \subset \mathcal{A}$.

As $\tilde{\Sigma}$ is compact and $\mathcal{A}$ open, there exists $\mathcal{B}$ an open bounded set of $X \times I$ such that $\tilde{\Sigma} \subset \mathcal{B} \subset \bar{\mathcal{B}} \subset \mathcal{A}$. We show that there is no solution on $\partial\mathcal{B}$. If $(x,\lambda) \in \Sigma \cap \partial\mathcal{B}$, then $(x,\lambda) \in \Sigma \cap \mathcal{A}$ and so $(x,\lambda) \in \tilde{\Sigma}$. But as $(x,\lambda) \in \partial\mathcal{B}$, $x \notin \mathcal{B}$ and then $(x,\lambda) \notin \tilde{\Sigma}$, a contradiction.

We can then make a homotopy on $\mathcal{B}$ and

$$D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}(.,0), \mathcal{B}_0) = D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}(.,1), \mathcal{B}_1).$$

As by hypothesis there is no solution in $\mathcal{B}_1$, this last degree is zero. But by definition of c_- and c_+, $\Sigma_0 \subset \mathcal{A}_0$, so $\Sigma_0 \subset \tilde{\Sigma}_0 \subset \mathcal{B}_0 \subset \mathcal{O}_0$. By the hypothesis (i2), $D_L(L - N(.,0), \mathcal{B}_0) \neq 0$, a contradiction. $\square$

Let us write a corollary of Theorem 1 for which the hypotheses are easier to check in applications.

Theorem 2. *Let* Ω *be an open set of* $\mathcal{X} \times I$ *such that there exists* $R > 0$ *with* $\Sigma^* \cap \partial\Omega \setminus B(R) = \emptyset$*. Let* $\mathcal{X} = \oplus_{i-1}^n \mathcal{X}_i$*. Suppose that* $\phi : \mathcal{X} \times I \to \mathbb{R}^n$ *is continuous and that* $(c_k)_{k\in\mathbb{N}}$ *is a sequence of* $\mathbb{R}^n$ *such that all of its components are increasing and unbounded and such that*

(i4) $(\forall (u,\lambda) \in \Sigma^* \cap \Omega : \|u_i\| > R)(\forall k \in \mathbb{N}) : \phi_i(u,\lambda) \neq (c_k)_i$,
(i5) $\phi^{-1}([0, c_k)) \cap \Sigma^* \cap \Omega$ *is bounded for all* $k \in \mathbb{N}$.

Let $k_+ \in \mathbb{N}$ *be such that for all* $1 \leq i \leq n$,

$$(c_{k_+})_i > \sup\{\phi_i(u,\lambda) : (u,\lambda) \in \Sigma^* \cap \Omega \text{ and } \|u_i\| \leq R\}.$$

For $K \in \mathbb{N}^n$ *such that* $K_i > k_+$*, denote* $\mathcal{O}^K = \phi^{-1}(\times_{i=1}^n ((c_{K_i})_i, (c_{K_i+1})_i)) \cap \Omega$*. Then* $D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}(.,\lambda), \mathcal{O}^K{}_\lambda)$ *exists and for all* $K \in \mathbb{N}^n$ *such that* $K_i > k_+$ *such that* $D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}(.,0), \mathcal{O}^K{}_0) \neq 0$*, the equation*

$$\mathcal{L}u = \mathcal{N}(u,1)$$

admits at least one solution $u_K \in \overline{\mathcal{O}^K}_1$.

Proof. Let $\Sigma^K = \Sigma^* \cap \overline{\mathcal{O}^K}$ and $I_K = \times_{i=1}^n((c_{K(i)})_i, (c_{K(i)+1})_i)$. As

$$\Sigma^* \cap \phi^{-1}(I_K) \cap \Omega = \Sigma^* \cap \phi^{-1}(\overline{I_K}) \cap \overline{\Omega},$$

${\Sigma^K}_\lambda \subset {\mathcal{O}^K}_\lambda$ and by (i5) and the local compactness of Σ^*, Σ^K is compact. Then one can compute $D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}(., \lambda), {\mathcal{O}^K}_\lambda)$ for all λ. We still have to verify that ϕ is proper on Σ. Let W be a compact set of $\mathbb{R}^n$; then

$$\phi^{-1}(W) \cap \Sigma \subset ((B[R] \times I) \cup (\cup_{K:\overline{I_K} \cap W \neq \emptyset} \phi^{-1}(\overline{I_K}))) \cap \Sigma$$

so is a closed set in a compact one. This shows that ϕ is proper on Σ.

We take $c_- = ((c_{K_1})_1, \dots, (c_{K_n})_n)$ and $c_+ = ((c_{K_1+1})_1, \dots, (c_{K_n+1})_n)$. Then on $\mathcal{O}^K \cap \Sigma$, $\phi(u, \lambda) \notin \partial I_K$ and $\partial \mathcal{O}^K \cap \Sigma = \emptyset$. Using the previous theorem, the equation has at least one solution in $\operatorname{dom} \mathcal{L} \cap \overline{\mathcal{O}^K}_1$.

2. Computation of the degree. In this section, we compute the degree associated to some simple scalar equations with three-point and Sturm-Liouville boundary conditions.

Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz mapping such that $g(x)x > 0$ for all $x \in \mathbb{R}_0$. We consider the boundary value problem for the scalar autonomous equation

$$x''(t) + g(x(t)) = 0 \tag{2}$$

with boundary conditions

$$\langle v, (x(0), x'(0)) \rangle = 0, \quad l(x) = 0, \tag{3}$$

where $v \in \mathbb{R}^2 \setminus \{0\}$ and $l : C^1(I, \mathbb{R}^n) \to \mathbb{R}$ is a continuous linear functional. This equation can be written as an equivalent system in the phase plane $(x, y) = (x, x')$ by

$$z' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -g(x) \end{pmatrix} = f(z). \tag{4}$$

This boundary value problem can be written as a coincidence equation. Let $X = C(I, \mathbb{R}^2)$, the space of continuous functions with norm $\|z\|_0 = \sup_{t \in I} |z(t)|$, $Z = C(I, \mathbb{R}^2) \times \mathbb{R}^2$ and $\operatorname{dom} L = C^1(I, \mathbb{R}^2)$. Let

$$L : \operatorname{dom} L \subset X \to Z; z \mapsto (z', 0).$$

Then L is a Fredholm operator of index zero. Effectively, $\ker L = \{z : I \to \mathbb{R}^2 : z(t) = A\}$ and $\operatorname{im} L = C(I, \mathbb{R}^2) \times \{0\}$. Let $N : X \to Z$ be defined by

$$N(z) = \big(f \circ z, \langle v, z(0) \rangle, l(z)\big).$$

This operator is L-completely continuous. So the problem (2)–(3) can be written as

$$Lz = N(z).$$

We see that the solutions of the system conserve the energy $H(x,y) = G(x) + \frac{y^2}{2}$, where $G(x) = \int_0^x g(s)\,ds$. We note that $\sigma(t,u)$ is the solution of the Cauchy problem associated to (4) with initial conditions $(x(0), y(0)) = u$.

Let $v^\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} v$ be orthogonal to v and $P^+ = \{z \in \mathbb{R}^2 : \langle v^\perp, z\rangle > 0\}$. For all open bounded sets $E \subset \mathbb{R}$, the sets

$$\Omega(E) = \{z \in C(I, \mathbb{R}^2) : (\forall t \in I) : H(z(t)) \in E\}$$

and

$$\Omega^+(E) = \{z \in \Omega(E) : u(0) \in P^+\}$$

are a open bounded sets of C.

Suppose now that E is such that the problem (4)–(3) has no solution on $\partial\Omega^+(E)$; then the degree

$$D_L(L - N, \Omega^+(E))$$

is well defined for L and N given above.

We will use a duality theorem due to M.A. Krasnosels'kiĭ and P.P. Zabreĭko to prove that this degree is the same as the one of a two-dimensional map.

We consider the operator $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathcal{U}(z) = (z_1 + \langle v, z\rangle, z_2 + l(\sigma(., z)))$$

whose fixed points coincide with initial values of the solutions of (4)–(3).

Theorem 3. *If E is such that $D_L(L - N, \Omega^+(E))$ is defined, then*

$$D_L(L - N, \Omega^+(E)) = \deg_B(I - \mathcal{U}, H^{-1}(E) \cap P^+).$$

Remark. If the degree is defined on $\Omega^+(E)$, then $0 \notin \bar{E}$ and $0 \notin \overline{\Omega^+(E)}$. By definition of P^+, $0 \notin \operatorname{int}(\Omega^+(E))$ and by definition of the degree, 0, which is a solution of the equation, doesn't belong to $\partial\Omega^+(E)$.

Proof. By definition of D_L,

$$D_L(L - N, \Omega^+(E)) = \deg(I - \mathcal{M}, \Omega^+(E)),$$

where

$$\mathcal{M}(z)(t) = (P + J^{-1}QN + K_{PQ}N)(z)(t)$$

with $P : X \to X; z \mapsto z(0)$, $Q : Z \to Z; (z,c) \mapsto (0,c)$, $J : \ker L \to \operatorname{im} Q : A \mapsto (0,A)$ and $K_{PQ} = \big(L_{|\,\mathrm{dom}\, L \cap \ker P}\big)^{-1}(I-Q)$. So we have

$$\mathcal{M}(z)(t) = z(0) + (\langle v, z(0)\rangle, l(z)) + \int_0^t f(z(s))\, ds.$$

The sets $\Omega^+(E)$ and $H^{-1}(E) \cap P^+$ have a common core with respect to (4)–(3). Effectively, z is a solution of (4)–(3) with $z \in \Omega^+(E)$ if and only if $z(t) \in H^{-1}(E)$ for all $t \in I$ and $z(0) \in P^+$. But as $H(z(t))$ is constant, this is equivalent to $z(0) \in H^{-1}(E) \cap P^+$. Moreover if $\mathcal{U}$ has a fixed point u on $\partial(H^{-1}(E) \cap P^+)$, then $z = \sigma(.,u)$ is a solution of (4)–(3) with $z(0) \in \partial(H^{-1}(E) \cap P^+)$ and by conservation of the energy H, $z \in \partial\Omega^+(E)$.

Then by the duality theorem [7, Theorem 29.4],

$$D_L(L-N, \Omega^+(E)) = \deg_B\,(I - \mathcal{U}, H^{-1}(E) \cap P^+).$$

Now for two types of boundary conditions, we will prove that this last degree is the same as the one of a one-dimensional map. For this we will use some "times" associated to the rotation of the solution around the origin.

We denote by $c_+(\alpha)$ the abscissa of the intersection in the right half plane of the x-axis and the energy level $\alpha^2/2$ and $c_-(\alpha)$ the abscissa of the intersection in the left half plane. This means that $G(c_\pm(\alpha)) = \alpha^2/2$ and $\pm c_\pm(\alpha) \geq 0$.

The "time" required by a solution of energy level $\frac{\alpha^2}{2}$ to pass from the abscissa x to the abscissa $c_+(\alpha)$ is given by

$$\tau_+(\alpha, x) = \int_x^{c_+(\alpha)} \frac{1}{\sqrt{\alpha^2 - 2G(s)}} ds$$

and to pass from the abscissa $c_-(\alpha)$ to the abscissa x,

$$\tau_-(\alpha, x) = \int_{c_-(\alpha)}^x \frac{1}{\sqrt{\alpha^2 - 2G(s)}} ds.$$

2.1. Sturm-Liouville boundary conditions. In this section we compute the degree associated to the equation (2) with the boundary conditions

$$\begin{aligned} -\sin\delta\, x(0) + \cos\delta\, x'(0) &= 0 \\ -\sin\gamma\, x(1) + \cos\gamma\, x'(1) &= 0 \end{aligned} \tag{5}$$

where $\gamma, \delta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$.

Let D and A be the lines of equation $-\sin\delta\, x + \cos\delta\, y = 0$ and $-\sin\gamma\, x + \cos\gamma\, y = 0$. We note respectively by $A^\pm(\alpha)$ and $D^\pm(\alpha)$ the abscissa of the intersection of ∂G^α with A and D, i.e., the numbers such that

$$2\cos^2\delta\, G(D^\pm(\alpha)) + \sin^2\delta\,(D^\pm(\alpha))^2 = \cos^2\delta\,\alpha^2$$

with $\pm D^{\pm}(\alpha) \geq 0$ and

$$2\cos^2\gamma\, G(A^{\pm}(\alpha)) + \sin^2\gamma\,(A^{\pm}(\alpha))^2 = \cos^2\gamma\,\alpha^2$$

with $\pm A^{\pm}(\alpha) \geq 0$. Then we can define the "times" to pass from the lines A and D to the axis Y :

$$\begin{array}{rl} \text{if } \delta \geq 0 & \tau_D^{\pm}(\alpha) = \tau_{\pm}(\alpha, D^{\pm}(\alpha)) + \tau_{\pm}(\alpha, 0) \\ \delta < 0 & \tau_D^{\pm}(\alpha) = \tau_{\pm}(\alpha, 0) - \tau_{\pm}(\alpha, D^{\pm}(\alpha)) \\ \text{if } \gamma \geq 0 & \tau_A^{\pm}(\alpha) = \tau_{\pm}(\alpha, A^{\pm}(\alpha)) + \tau_{\pm}(\alpha, 0) \\ \gamma < 0 & \tau_A^{\pm}(\alpha) = \tau_{\pm}(\alpha, 0) - \tau_{\pm}(\alpha, A^{\pm}(\alpha)). \end{array}$$

Finally, we define the times which will be used for the computation of the degree, i.e., the times to make a half turn (from A to A) and to pass from D to A :

$$\begin{array}{rl} \text{if } \tau_D^{+}(\alpha) \geq \tau_A^{+}(\alpha) & \tau^{\pm}(\alpha) = \tau_A^{\pm}(\alpha) + 2\tau_{\mp}(\alpha, 0) - \tau_A^{\mp}(\alpha), \\ & \tau_1^{+}(\alpha) = \tau_D^{+}(\alpha) - \tau_A^{+}(\alpha), \\ \text{and if } \tau_D^{+}(\alpha) < \tau_A^{+}(\alpha) & \tau^{\pm}(\alpha) = \tau_A^{\mp}(\alpha) + 2\tau_{\pm}(\alpha, 0) - \tau_A^{\pm}(\alpha), \\ & \tau_1^{+}(\alpha) = \tau_D^{+}(\alpha) + 2\tau_{-}(\alpha, 0) - \tau_A^{-}(\alpha). \end{array}$$

The solution $u(., \alpha) = \sigma(., D^{+}(\alpha), \tan\delta\, D^{+}(\alpha))$ ($\sigma(., 0, \alpha)$ if $\delta = \frac{\pi}{2}$) will be a solution of (2)–(5) if and only if there exists $n \in \mathbb{N}$ such that

$$\tau_1^{+}(\alpha) + n\tau^{+}(\alpha) + n\tau^{-}(\alpha) = 1$$

or

$$\tau_1^{+}(\alpha) + (n+1)\tau^{+}(\alpha) + n\tau^{-}(\alpha) = 1.$$

Let

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : z > x \geq 0, y > 0 \text{ and there exists } n \in \mathbb{N} \\ \text{such that } x + ny + nz = 1 \text{ or } x + (n+1)y + nz = 1\}.$$

With this writing, the problem (2)–(5) has a solution of energy level $\frac{\alpha^2}{2}$ if and only if the triple $(\tau_1^{+}(\alpha), \tau^{+}(\alpha), \tau^{-}(\alpha)) \in \mathcal{S}$.

Theorem 4. *If $\alpha > \bar{\alpha} > 0$ are such that*

$$(\tau_1^{+}(\alpha), \tau^{+}(\alpha), \tau^{-}(\alpha)) \notin \mathcal{S} \quad \textit{and} \quad (\tau_1^{+}(\bar{\alpha}), \tau^{+}(\bar{\alpha}), \tau^{-}(\bar{\alpha})) \notin \mathcal{S},$$

then

$$D_L(L - N, \Omega^{+}((\bar{\alpha}, \alpha))) = \deg_B(\phi, D_{\bar{\alpha},\alpha}),$$

where $D_{\bar{\alpha},\alpha} = \{\xi \in \mathbb{R} : T(0,\xi) \in H^{-1}((\bar{\alpha},\alpha))\}$ *with* $T = \begin{pmatrix} -\sin\delta & \cos\delta \\ \cos\delta & \sin\delta \end{pmatrix}$ *and* $\phi : \mathbb{R} \to \mathbb{R}$ *is defined by*

$$\phi(\xi) = -\sin\gamma\,\sigma\big(1, T(0,\xi)\big) + \cos\gamma\,\sigma'\big(1, T(0,\xi)\big).$$

Proof. We know that

$$\begin{aligned} D_L(L-N, \Omega^+((\bar{\alpha},\alpha))) &= \deg_B\,(I - \mathcal{U}, H^{-1}((\bar{\alpha},\alpha)) \cap P^+) \\ &= \deg_B\,(\mathcal{U} - I, H^{-1}((\bar{\alpha},\alpha)) \cap P^+). \end{aligned}$$

Let $\tilde{U} = (\mathcal{U} - I) \circ T$ and $B = T^{-1}(H^{-1}((\bar{\alpha},\alpha)) \cap P^+)$. By the product formula

$$\deg_B\,(\tilde{U}, B) = \deg_B\,(\mathcal{U} - I, H^{-1}((\bar{\alpha},\alpha)) \cap P^+),$$

and by the choice of T,

$$\tilde{U}(z) = (z_1, -\sin\gamma\,\sigma(1,Tz) + \cos\gamma\,\sigma'(1,Tz)).$$

Let $\mathcal{R} = \{(z_1,z_2) \in \mathbb{R}^2 : |z_1| < R \text{ and } z_2 \in D_{\bar{\alpha},\alpha}\}$ for some $R > 0$. We show that

$$\deg_B\,(\tilde{U}, B) = \deg_B\,(\tilde{U}, \mathcal{R}).$$

Indeed, if $\tilde{U}(z) = 0$, then $z_1 = 0$. But, by construction, $B \cap OY = \mathcal{R} \cap OY = D_{\bar{\alpha},\alpha}$, so by excision, the degree on these two sets must be the same.

Then we use invariance by homotopy to compute this last degree. Let $h(z,\lambda) = (z_1, \tilde{U}_2(\lambda z_1, z_2))$, so that $h(.,1) = \tilde{U}$ and $h(.,0) = I_{\mathbb{R}} \times \phi$. Moreover, $h(z,\lambda) = 0$ if and only if $h(z,1) = 0$. By the choice of $\mathcal{R}$, $h(z,\lambda) \neq 0$ for all $\lambda \in I$ and $z \in \partial\mathcal{R}$. Hence

$$\begin{aligned} \deg_B\,(\tilde{U}, \mathcal{R}) &= \deg_B\,(I_{\mathbb{R}} \times \phi, (-R,R) \times D_{\bar{\alpha},\alpha}) \\ &= \deg_B\,(I_{\mathbb{R}}, (-R,R))\deg_B\,(\phi, D_{\bar{\alpha},\alpha}) = \deg_B\,(\phi, D_{\bar{\alpha},\alpha}). \end{aligned}$$

We introduce subsets of $\mathbb{R}^3$ that will be useful to express the degree. Let

$$\begin{aligned} \mathcal{S}_{2n+1} = \{&(x,y,z) \in \mathbb{R}^3 : z > x \geq 0, y > 0, \\ &x + ny + nz < 1 \text{ and } x + (n+1)y + nz > 1\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_{2n} = \{&(x,y,z) \in \mathbb{R}^3 : z > x \geq 0, y > 0, \\ &x + ny + nz > 1 \text{ and } x + ny + (n-1)z < 1\}. \end{aligned}$$

Then we have the following theorem.

Theorem 5. *There exists* $\sigma \in \{-1, 1\}$ *such that if*

$$(\tau_1^+(\alpha), \tau^+(\alpha), \tau^-(\alpha)) \in \mathcal{S}_m \text{ and } (\tau_1^+(\bar\alpha), \tau^+(\bar\alpha), \tau^-(\bar\alpha)) \in \mathcal{S}_n$$

for some m *and* $n \in \mathbb{N}$*, then*

$$D_L(L - N, \Omega^+((\bar\alpha, \alpha))) = \sigma \frac{(-1)^m - (-1)^n}{2}.$$

Proof. It is sufficient to show that $\deg_B(\phi, D_{\bar\alpha,\alpha}) = \sigma \frac{(-1)^m - (-1)^n}{2}$. As $D_{\bar\alpha,\alpha}$ is an interval, that we denote by $(l_{\bar\alpha}, l_\alpha)$, the degree of ϕ is given by

$$\deg_B(\phi, D_{\bar\alpha,\alpha}) = \frac{\operatorname{sgn}\phi(l_\alpha) - \operatorname{sgn}\phi(l_{\bar\alpha})}{2}.$$

Let $z^\alpha = T(0, l_\alpha)$ and $\sigma = \operatorname{sgn}(-\sin\gamma\, {z^\alpha}_1 + \cos\gamma\, {z^\alpha}_2) = \operatorname{sgn}(\delta - \gamma)$.

In $\mathcal{S}_k$, the solution makes at least $k-1$ half turns after its first contact with A and at most k. So if k is even, the point $(u(1, z^\alpha), u'(1, z^\alpha))$ will be on the same side of A as z^α and $\operatorname{sgn}\phi(l_\alpha) = \sigma$. Similarly if k is odd, $\operatorname{sgn}\phi(l_\alpha) = -\sigma$.

So we have that

$$\deg_B(\phi, D_{\bar\alpha,\alpha}) = \sigma \frac{(-1)^m - (-1)^n}{2}.$$

2.2. Three-point boundary conditions. In this section we compute the degree associated to the equation (2) with the boundary conditions

$$x(0) = 0 \quad x(\eta) - x(1) = 0, \tag{6}$$

where $\eta \in (0, 1)$ is a fixed constant.

Then we can define the "times" needed to make the first quarter of lap ($\tau_0^\pm$) and a half lap (τ) :

$$\tau(\alpha) = \tau_+(\alpha, 0) + \tau_-(\alpha, 0), \quad \tau_0^\pm(\alpha) = \tau_\pm(\alpha, 0).$$

Then $u(., \alpha) = \sigma(., (0, \alpha))$ with $\alpha > 0$ will be a solution of (2)–(6) if and only if there exists $k \in \mathbb{N}$ such that $\tau_0^+(\alpha) + k\tau(\alpha) = \frac{1+\eta}{2}$ or $2k\tau(\alpha) = 1 - \eta$. Effectively, in the first case, at the time $\frac{1+\eta}{2}$, which is the middle of $[\eta, 1]$, the solution is on the axis X (the positions at η and at 1 are symmetric with respect to X). In the second case, the solution makes exactly k laps around the origin in the interval of time $[\eta, 1]$.

Let

$$\mathcal{S} = \{(x, y) \in (\mathbb{R}^+)^2 : x \le y \text{ and } (\exists k \in \mathbb{N}) : x + ky = \frac{1+\eta}{2} \text{ or } 2ky = 1 - \eta\}.$$

With those notations, $u(., \alpha)$ satisfies the boundary conditions if and only if

$$(\tau_0^+(\alpha), \tau(\alpha)) \in \mathcal{S}.$$

We divide $(\mathbb{R}^+)^2 \setminus \mathcal{S}$ into its connected components. Let

$$\mathcal{S}_{n,k} = \{(x, y) \in (\mathbb{R}^+)^2 : x \le y \text{ and}$$
$$x + (k-1)y < \frac{1+\eta}{2} < x + ky \text{ and } 2ny < 1 - \eta < 2(n+1)y\}.$$

On some sets, the degree associated to the problem can be evaluated by the computation of the degree of a function from $\mathbb{R}$ to $\mathbb{R}$.

Theorem 6. *If $\alpha > \bar{\alpha} > 0$ is such that $(\tau_0^+(\alpha), \tau(\alpha)) \notin \mathcal{S}$ and $(\tau_0^+(\bar{\alpha}), \tau(\bar{\alpha})) \notin \mathcal{S}$, then*

$$D_L(L - N, \Omega^+((\bar{\alpha}, \alpha))) = \deg_B(\phi, (\bar{\alpha}, \alpha)),$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is defined by $\phi(x) = \sigma(\eta, (0, x)) - \sigma(1, (0, x))$.

Proof. By the choice of v,

$$(\mathcal{U} - I)(z) = (z_1, \sigma(\eta, z) - \sigma(1, z)).$$

Let $\mathcal{R} = (-R, R) \times (\bar{\alpha}, \alpha)$ for some $R > 0$. We show that

$$\deg_B(\mathcal{U} - I, H^{-1}((\bar{\alpha}, \alpha)) \cap P^+) = \deg_B(\mathcal{U} - I, \mathcal{R}).$$

Indeed, if $(\mathcal{U} - I)(z) = 0$, then $z_1 = 0$. But, by construction, $H^{-1}((\bar{\alpha}, \alpha)) \cap P^+ \cap OY = \mathcal{R} \cap OY = (\bar{\alpha}, \alpha)$. So by excision, the degree on these two sets must be the same. But

$$\begin{aligned}\deg_B(\mathcal{U} - I, \mathcal{R}) &= \deg_B(I_{\mathbb{R}} \times \phi, (-R, R) \times (\bar{\alpha}, \alpha)) \\ &= \deg_B(I_{\mathbb{R}}, (-R, R)) \deg_B(\phi, (\bar{\alpha}, \alpha)) = \deg_B(\phi, (\bar{\alpha}, \alpha)).\end{aligned}$$

Using the times τ_0^+ and τ one has the following theorem.

Theorem 7. *Suppose that $\alpha > \bar{\alpha} > 0$ is such that $(\tau_0^+(\alpha), \tau(\alpha)) \in \mathcal{S}_{k,n}$ and $(\tau_0^+(\bar{\alpha}), \tau(\bar{\alpha})) \in \mathcal{S}_{\bar{k},\bar{n}}$; then*

$$D_L(L - N, \Omega^+((\bar{\alpha}, \alpha))) = \frac{(-1)^{n+k+1} - (-1)^{\bar{n}+\bar{k}+1}}{2}.$$

Proof. We will compute the degree of ϕ. For this it is sufficient to know the sign of ϕ on the boundary of $(\bar{\alpha}, \alpha)$.

Suppose that k and n are even. As $(\tau_0^+(\alpha), \tau(\alpha)) \in \mathcal{S}_{k,n}$, then $u'(\frac{1+\eta}{2}, \alpha) > 0$. Let $n\pi + \psi$ $(0 < \psi < 2\pi)$ be the angle covered during the interval of time $[\frac{1+\eta}{2}, 1]$ and $n\pi + \phi$ $(0 < \phi < 2\pi)$ be the angle covered during the interval $[\eta, \frac{1+\eta}{2}]$. Then $\psi + \phi < 2\pi$.

If $u(\eta) \leq u(\frac{1+\eta}{2}) \leq u(1)$, then $\phi(\alpha) < 0$.

If $u(1) < u(\frac{1+\eta}{2})$, then for some $\frac{1+\eta}{2} + n\tau(\alpha) < \tilde{t} < 1$, $u(\tilde{t}) = u(\frac{1+\eta}{2})$. The angle ξ covered in the time $[\tilde{t}, 1]$ is the angle covered in the time $[\frac{1+\eta}{2} - (1 - \tilde{t}), \frac{1+\eta}{2}]$. Then the angle covered in the time $[\eta, \frac{1+\eta}{2}]$ is between $n\pi + \xi$ and $n\pi + (2\pi - \psi)$. This proves that in this case also $\phi(\alpha) < 0$.

If $u(\eta) > u(\frac{1+\eta}{2})$, then for some $\eta < \tilde{t} < \frac{1+\eta}{2} + n\tau(\alpha)$, $u(\tilde{t}) = u(\frac{1+\eta}{2})$. The angle ξ covered in the time $[\eta, \tilde{t}]$ is the angle covered in the time $[\frac{1+\eta}{2}, \frac{1+\eta}{2} + (\tilde{t} - \eta)]$. Then the angle covered in the time $[\frac{1+\eta}{2}, 1]$ is between $n\pi + \xi$ and $n\pi + (2\pi - \phi)$. And once more $\phi(\alpha) < 0$. The other cases can be treated similarly. So we obtain

$$\operatorname{sgn} \phi(\alpha) = (-1)^{n+k+1}.$$

Remark. Suppose that g is odd (and thus $\tau_0^+ = \tau_0^-$). If η tends to 0, for α fixed, the second inequality that defines $\mathcal{S}_{n,k}$ is satisfied with $n = 0$. So we have that $(\tau_0^+(\alpha), \tau(\alpha)) \in \mathcal{S}_{0,k}$ if

$$(2k-1)\tau_0^+(\alpha) < 1 < (2k+1)\tau_0^+(\alpha).$$

With the Sturm-Liouville boundary conditions $u(0) = 0$ and $u'(1) = 0$, we have $\tau_1^+(\alpha) = \frac{1}{2}\tau^+(\alpha) = \frac{1}{2}\tau^-(\alpha)$. So $(\tau_0^+(\alpha), \tau(\alpha)) \in \mathcal{S}_{0,k}$ if and only if $(\tau_1^+(\alpha), \tau^+(\alpha), \tau^-(\alpha)) \in \mathcal{S}_k$. Then when η tends to 0, the degree associated to the three-point boundary conditions is equal to the one associated to the given two-point boundary conditions.

We finish this section with a theorem which will be useful in applying the theorem we have just proved to some more general sets. For this we define

$$S = \{z \in C^1(I, \mathbb{R}^2) : z \text{ is a solution of (4)}\}$$

and $\tilde{H} : S \to \mathbb{R}, z \mapsto H(z(0))$.

Theorem 8. *Let $\phi : X \to \mathbb{R}$ be continuous and $c_1, c_2 \in \mathbb{R}$ be such that $\phi(\Sigma^* \cap P^+) \cap \{c_1, c_2\} = \emptyset$. Let $E = \tilde{H}(\phi^{-1}((c_1, c_2)))$. If for $i = 1, 2$ and for any $I \subset \mathbb{R}$ such that $\phi(\partial \tilde{H}^{-1}(I) \cap \Sigma^* \cap P^+) = \{c_i\}$, one has $D_L(L - N, \Omega^+(I)) = 0$, then, if E is bounded,*

$$D_L(L - N, \phi^{-1}((c_1, c_2)) \cap P^+) = D_L(L - N, \Omega^+(E)) = D_L(L - N, \Omega^+(\operatorname{co}(E))).$$

Proof. The first equality is a direct consequence of the definition of E and the excision property.

For the second equality, let $\operatorname{co}(E) = E \cup (\cup_{\lambda \in \Lambda} F_\lambda)$ where F_λ are the connected components of $\operatorname{co}(E) \setminus E$. We have that $\Omega^+(F_\lambda) \cap \Sigma^* \neq \emptyset$ only for a finite number of $\lambda \in \Lambda$. If not, there exists, for $n \in \mathbb{N}$, $x_n \in \Omega^+(F_{\lambda_n}) \cap \Sigma^*$ with $\lambda_n \neq \lambda_m$. As $\Sigma^* \cap \Omega^+(\operatorname{co}(E))$ is precompact, passing eventually to a subsequence, we have $x_n \to x \in \Sigma^*$. If $x \in \Omega^+(E)$, there exists n_0 such that for all $n \geq n_0$, $x_n \in \Omega^+(E)$; this is a contradiction with the definition of x_n. If $x \in \Omega^+(F_{\lambda_0})$, as $x \notin \Omega^+(\partial F_{\lambda_0})$, then there exists n_0 such that for all $n \geq n_0$, $x_n \in \Omega^+(F_{\lambda_0})$. So $\lambda_n = \lambda_0 = \lambda_{n+1}$, a contradiction.

Now write $\operatorname{co}(E) = E \cup (\cup_{i=1}^n \bar{I}_i) \cup F$. For $1 \leq i \leq n$, $\phi(\partial \tilde{H}^{-1}(I_i) \cap \Sigma^* \cap P^+) \subset \{c_1, c_2\}$. As $\tilde{H}^{-1}(I_i)$ is connected, if there exists $x_1, x_2 \in \partial \tilde{H}^{-1}(I_i)$ such that $\phi(x_j) = c_j$ $(j = 1, 2)$, then for all $c_1 < c < c_2$, there exists $x \in \tilde{H}^{-1}(I_i)$ such that $\phi(x) = c$. Then $H(x) \in E$, a contradiction. Then using the addition-excision property of the degree, we have

$$\begin{aligned} D_L(L-N, \Omega^+(E)) &= D_L(L-N, \Omega^+(E)) + \sum_{i=1}^{n} D_L(L-N, \Omega^+(I_i)) \\ &= D_L(L-N, \Omega^+(E \cup \cup_{i=1}^{n} \bar{I}_i \cup F)) = D_L(L-N, \Omega^+(\mathrm{co}(E))). \end{aligned}$$

3. Superlinear weakly coupled systems. Now using the continuation theorem and the computation of the degree, we can prove the existence of infinitely many solutions to the system

$$x''(t) + g(x(t)) = p(t, x(t), x'(t)) \tag{7}$$

$$\begin{aligned} &l_{1,1}(x_1) = 0,\, l_{1,2}(x_1) = 0, \dots, l_{m,1}(x_m) = 0,\, l_{m,2}(x_m) = 0 \\ &l_{m+1,1}(x_{m+1}) = A_{m+1},\, l_{m+1,2}(x_{m+1}) = B_{m+1}, \dots, l_{n,2}(x_n) = B_n, \end{aligned} \tag{8}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $g(x) = (g_1(x_1), \dots, g_n(x_n))$ and $p : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is continuous.

The boundary conditions are three-point boundary conditions or Sturm-Liouville boundary conditions; i.e., there exists $m \in \mathbb{N}$ with $1 \le m \le n$, $\delta_i \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\gamma_i \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ $(m+1 \le i \le n)$ such that for all $1 \le i \le m$

$$l_{i,1}(x) = x(0) \quad \text{and} \quad l_{i,2}(x) = x(\eta) - x(1) \tag{9}$$

and for all $m+1 \le j \le n$

$$\begin{aligned} l_{j,1}(x) &= -\sin\delta_j\, x(0) + \cos\delta_j\, x'(0) \\ l_{j,2}(x) &= -\sin\gamma_j\, x(1) + \cos\gamma_j\, x'(1). \end{aligned} \tag{10}$$

The main hypothesis on g and p are that g is superlinear; i.e.,

$$\forall i = 1, \dots, n \qquad \lim_{x \to \infty} \frac{g_i(x)}{x} = +\infty \tag{11}$$

and p satisfied a growth restriction:

$$\begin{aligned} &\text{for } 1 \le i \le m \quad \|p_i\|_0 = \sup_{[0,1] \times \mathbb{R}^{2n}} |p_i(t, x, y)| < +\infty \\ &\text{for } m+1 \le i \le n \quad |p_i(t, x, y)| \le a_i h_i(x_i) + b_i |y_i| + c, \end{aligned} \tag{12}$$

where h_i is such that there exists $\mu \in (0, 1)$ such that

$$a_i h_i(u) \le (1-\mu) g_i(u) \quad \text{and} \quad 0 \le h_i(u) \le \sqrt{G_i(u)}. \tag{13}$$

To simplify the writing, we will suppose that, for $1 \le i \le n$

$$|p_i(t, x, y)| \le a_i h_i(x_i) + b_i|y_i| + c$$

with, for $1 \le i \le m$, $a_i = 0$ and $b_i = 0$.

To avoid some technical problems, we suppose that g is differentiable and $|g_i(u)| \ge |u|$. Thanks to the superlinearity of g, this condition is satisfied for $|u|$ sufficiently large. So we can replace g by a function that satisfies this condition everywhere and put the difference (which is bounded) in p. Similarly, we can approximate g by a function which is differentiable and put the difference into p.

We will study the problem (7)–(8) through the homotopy

$$x''(t) + g(x(t)) = \lambda p(t, x(t), x'(t)) \tag{14}$$

$$\begin{aligned} &l_{1,1}(x_1) = 0,\, l_{1,2}(x_1) = 0, \dots, l_{m,1}(x_m) = 0,\, l_{m,2}(x_m) = 0 \\ &l_{m+1,1}(x_{m+1}) = \lambda A_{m+1},\, l_{m+1,2}(x_{m+1}) = \lambda B_{m+1}, \dots, l_{n,2}(x_n) = \lambda B_n. \end{aligned} \tag{15}$$

We can write this system as a first-order system

$$z'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ -g(x(t)) + \lambda p(t, x(t), y(t)) \end{pmatrix} = f(t, z(t), \lambda). \tag{16}$$

Let $\mathcal{X} = C(I, \mathbb{R}^{2n})$, the space of continuous functions with norm $\|z\|_0 = \sup_{t \in I} |z(t)|$, $\mathcal{Z} = C(I, \mathbb{R}^{2n}) \times \mathbb{R}^{2n}$ and $\operatorname{dom} \mathcal{L} = C^1(I, \mathbb{R}^{2n})$. Let

$$\mathcal{L} : \operatorname{dom} \mathcal{L} \subset \mathcal{X} \to \mathcal{Z}; z \mapsto (z', 0).$$

Then $\mathcal{L}$ is a Fredholm operator of index zero. Effectively, $\ker \mathcal{L} = \{z : I \to \mathbb{R}^n : z(t) = A\}$ and $\operatorname{im} \mathcal{L} = C(I, \mathbb{R}^n) \times \{0\}$. Let $\mathcal{N} : \mathcal{X} \times I \to \mathcal{Z}$ be defined by

$$\mathcal{N}(z, \lambda)(t) = \big(f(t, z(t), \lambda), l_{1,1}(u), \dots, l_{n,2}(u) - \lambda B_n\big).$$

This operator is $\mathcal{L}$-completely continuous. So the problem (14)–(15) can be written as

$$\mathcal{L}x = \mathcal{N}(x, \lambda).$$

We give now a lemma that is valid for all the solutions of the system

$$z'(t) = f(t, z(t), \lambda) \tag{17}$$

with $z \in \mathbb{R}^{2n}$, so certainly for the solutions of the equation (14).

Lemma 1. *Let $V \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ be such that $\lim_{z\to\infty} |V(z)| = +\infty$. If there exist constants $K \geq 0$ and $d > 0$ such that for all $t \in I$ and $u \in \mathbb{R}^{2n}$ with $|u| > K$,*

$$|V'_{x_i}(u)f_i(t,u,\lambda) + V'_{y_i}(u)f_{n+i}(t,u,\lambda)| \leq d|V(u)|$$

then for all $R_1 \geq 0$, there exists $R_2(K, R_1, d, V) \geq R_1$ such that for all (x, λ) solution of (17) *with*

$$\min_{t\in I} |(x_i(t), y_i(t))| \leq R_1,$$

one has $\|(x_i, y_i)\|_0 \leq R_2$.

Proof. We show it for $R_1 > K$. Put $W(z) = \ln |V(z)|$. Let (z, λ) be a solution of (17) such that $\min_{t\in I} |(x_i(t), y_i(t))| \leq R_1$. If $\|(x_i, y_i)\|_0 \leq R_1$, take $R_2 = R_1$. Otherwise, take t_1 such that $|(x_i(t_1), y_i(t_1))| = R_1$ and t_0 such that $|(x_i(t_0), y_i(t_0))| = \|(x_i, y_i)\|_0$ with $|(x_i(t), y_i(t))| > R_1$ between t_0 and t_1. Let $w(t) = W_\lambda(0, \dots, x_i(t), 0, \dots, y_i(t), \dots, 0)$. Then,

$$\begin{aligned}
|w(t_0)| &= \big|w(t_1) + \int_{t_1}^{t_0} w'(s)ds\big| \leq |w(t_1)| + \big|\int_{t_1}^{t_0} |w'(s)|ds\big| \\
&\leq |w(t_1)| + \big|\int_{t_1}^{t_0} \big|\frac{V'_{x_i}(z(s))x'_i(s) + V'_{y_i}(z(s))y'_i(s)}{V(z(s))}\big|ds\big| \\
&\leq |w(t_1)| + |t_1 - t_0|d \leq c_2 + d
\end{aligned}$$

with $c_2 = \sup\{W(z) : |z| = R_1\}$. As $\lim_{z\to\infty} W(z) = +\infty$ there exists R_2 such that, for all $\lambda \in I$ and $|z| \geq R_2$, $W(z) > c_2 + d$. So, $\|(x_i, y_i)\|_0 \leq R_2$.

Lemma 2. *Suppose that g satisfies* (11), *$|g_i(x)| \geq |x|$ and p satisfies* (12). *For each $R_1 > 0$ there exists $R_2 \geq R_1$ such that for any (x, λ) solutions of* (14)–(15) *satisfying $\min_{t\in I} |(x_i(t), x'_i(t))| \leq R_1$ we have $\|x_i\|_1 \leq R_2$.*

Proof. Let $G_i(x) = \int_0^x g_i(s)\,ds$ and

$$V(x, y) = \sum_{i=1}^{n} \big(G_i(x_i) + \frac{{y_i}^2}{2}\big).$$

Then $\lim_{z\to\infty} V(z) = +\infty$. We show that there exist constants $d > 0$ and $K > 0$ such that for $|(x, y)| > K$, one has that $|y_i p_i(t, x, y)| \leq dV(x, y)$. Indeed, for $|(x, y)|$ large enough,

$$\begin{aligned}
&|y_i p_i(t, x, y)| \leq |y_i|(a_i h_i(x_i) + b_i|y_i| + c) \\
&\leq \tfrac{1}{2}{a_i}^2 h_i^2(x_i) + (b_i + \tfrac{1}{2})y_i^2 + c|y_i| \leq 2(b_i + 1 + {a_i}^2)(G_i(x_i) + \frac{y_i^2}{2}) \leq dV(x, y).
\end{aligned}$$

We can apply the previous lemma with $f(t, z, \lambda) = (z_2, -g(z_1) + \lambda p(t, z))$. $\square$

We prove now that, for each component, the revolution around the origin is fast for solutions of large norm.

Lemma 3. *Suppose that g satisfies (11), $|g_i(u)| \geq |u|$ and p satisfies (12). For each $c > 0$ and for each $N > 0$ there exists $R_1(N) > 0$ such that for all $1 \leq i \leq n$ and (x, λ) solution of (14)–(15) with $\min_{t\in[0,c]} |(x_i(t), x_i'(t))| \geq R_1$ one has $\theta_i(0) - \theta_i(c) \geq N$.*

Proof. By superlinearity of g_i and the growth condition on p, for all K, there exists $c_K > 0$ such that for all x and y,

$$\begin{aligned}
&x(g_i(x) - \lambda p_i(t, x, y)) + y^2\\
&\quad\geq xg_i(x) - |x_i a_i h(x_i)| - b_i|x_i||y_i| - c|x_i| + y^2\\
&\quad\geq \mu x_i g_i(x_i) - b_i|x_i||y_i| - c|x_i| + y^2\\
&\quad\geq Kx_i^2 - c_K - b_i|x_i||y_i| - c|x_i| + y^2\\
&\quad\geq Kx_i^2 - c_K - \frac{b_i}{2}(b_i x_i^2 + \frac{y_i^2}{b_i}) - c|x_i| + y^2\\
&\quad\geq \frac{y_i^2}{2} + (K - \frac{b_i^2}{2})x_i^2 - c|(x_i, y_i)| - c_K = \frac{K'}{2}x_i^2 + \frac{y_i^2}{2} - c|(x_i, y_i)| - c_K,
\end{aligned}$$

where $K' = 2(K - \frac{b_i^2}{2})$. If (x, λ) is a solution of (14)–(15) such that

$$\min_{t\in[0,c]} |(x_i(t), x_i'(t))| \geq \max\{\sqrt{c_K}{K'}^{\frac{1}{4}}, \gamma {K'}^{\frac{1}{2}}\},$$

then

$$-\theta_i'(t) = \frac{x_i(t)(g_i(x_i(t)) - \lambda p_i(t, x, x')) + {x_i'}^2(t)}{|(x_i(t), x_i'(t))|^2} \geq K'\Big(\frac{x_i(t)^2 + \frac{{x_i'}^2(t)}{K'}}{|(x_i(t), x_i'(t))|^2}\Big) - 2{K'}^{\frac{-1}{2}}.$$

If we let $\Theta(x, y) = \frac{x^2 + \frac{y^2}{K'}}{|(x,y)|^2}$ and $\sigma = \min_{(x,y)\in\mathbb{R}^2} \Theta(x, y) = \frac{1}{K'}$, then

$$-\int_{\theta_i(0)}^{\theta_i(c)} \frac{1}{\Theta(\cos\theta, \sin\theta)}\, d\theta = \int_0^c \frac{-\theta_i'(t)}{\Theta(\cos\theta_i(t), \sin\theta_i(t))}\, dt \geq cK' - 2c{K'}^{-\frac{1}{2}}\frac{1}{\sigma}.$$

Computing directly the first integral, we show that this integral is smaller than $(\theta_i(0) - \theta_i(c) + 2\pi)\sqrt{K'}$. So we have

$$\theta_i(0) - \theta_i(c) \geq c\sqrt{K'} - 2c - 2\pi = \alpha(K).$$

For N fixed, we take K big enough to be sure that $\alpha(K) \geq N$. So if

$$\min_{t\in[0,c]} |(x_i(t), x_i'(t))| \geq \max\{c_K {K'}^{\frac{1}{4}}, \gamma {K'}^{\frac{1}{2}}\},$$

then $\theta_i(0) - \theta_i(c) \geq N$. $\square$

Let $\delta : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\delta(x,y) = \min\big\{1, \frac{1}{x^2+y^2}\big\}.$$

We now define, for $\kappa \in [0,1]$ and $1 \le i \le n$, the continuous functional φ_i^κ on $C^1(I,\mathbb{R}^n) \times I$ by

$$\varphi_i^\kappa(x,\lambda) = \Big|\int_0^\kappa \big[x_i'(t)^2 + x_i(t)(g_i(x_i(t)) - \lambda p_i(t,x(t),x'(t)))\big]\delta(x_i(t),x_i'(t))\,dt\Big|.$$

If (x,λ) is a solution of (14) such that $x_i^2(t) + {x_i'}^2(t) \ge 1$ for all $t \in [0,\kappa]$, we get

$$\varphi_i^\kappa(x,\lambda) = \Big|\int_0^\kappa \frac{x_i'(t)^2 - x_i(t)x_i''(t)}{x_i^2(t)+{x_i'}^2(t)}\,dt\Big| = \Big|\int_0^\kappa \frac{d}{dt}\,\arctan\big(\frac{x_i(t)}{x_i'(t)}\big)\,dt\Big|.$$

Thus, for a solution of (14) such that $\min_{t\in[0,\kappa]} |(x_i(t),x_i'(t))| \ge 1$, it turns out that $\varphi_i^\kappa(x,\lambda)$ is the angle covered by the solution during the interval of time $[0,\kappa]$.

Lemma 4. *There exists $\bar{R}$ such that for all $m+1 \le i \le n$ and all solutions (x,λ) of* (14)–(15) *with $\|x_i\|_1 \ge \bar{R}$, there exists $k \in \mathbb{N}$ such that*

$$\phi_i^1(x,\lambda) \in [\delta_i - \gamma_i + k\pi - \frac{\pi}{4}, \delta_i - \gamma_i + k\pi + \frac{\pi}{4}].$$

Proof. As x satisfies the boundary conditions, we know that $(x_i(0),x_i'(0))$ is in the strip $S(\delta_i,A_i) = \{(x,y) \in \mathbb{R}^2 : |-\sin\delta_i\,x + \cos\delta_i\,y| < |A_i|\}$ and $(x_i(1),x_i'(1)) \in S(\gamma_i,B_i)$. We can take M large enough to have that $S(\delta_i,A_i) \setminus B(M)$ is included in the cone $C(\delta_i) = \{z \in \mathbb{R}^2 : \langle z, (\cos\delta_i,\sin\delta_i)\rangle/|z| \ge \cos\frac{\pi}{8}\}$ and $(S(\gamma_i,B_i)\setminus B(M)) \subset C(\gamma_i)$. So if we take $\bar{R} \ge R_2(M)$, where R_2 is given by Lemma 2, we have

$$\phi_i^1(x,\lambda) \in [\delta_i - \gamma_i + k\pi - \frac{\pi}{4}, \delta_i - \gamma_i + k\pi + \frac{\pi}{4}].$$

3.1. Time-map and behavior of the rotation around the origin. To use the continuation theorem, in the case of three-point boundary conditions, we will need fine estimates on φ_i^1 and φ_i^η $(1 \le i \le m)$. For this, we will use fine estimates on the time-map developed in [3]. We will make also restrictions on the growth of g_i.

To simplify the notation in this section, we will denote by g any of the functions g_i $(1 \le i \le m)$.

The estimates on φ^1 and φ^η presented here are similar to the ones presented in [3]. The theorems are stated here in a more "quantitative" version in opposition to the "qualitative" version. The lemmas for which we need a more quantitative version are Lemma 5.3, Lemma 5.4 and Lemma 3.4 of [3].

First we need an estimation on the energy. To this end, we define

$$l(t) = \sqrt{H(x_i(t),x_i'(t))}.$$

Lemma 5. *There exists $E > 0$ and $\tilde{R} > 0$ such that if u is a solution of* (14) *such that* $\max u_i > \tilde{R}$, *then for all $t, s \in I$,*

$$|l(t) - l(s)| \leq E.$$

Proof. Take $\tilde{R} = R_2(\|p\|_0)$ where R_2 is given by Lemma 2. So $l(t) \neq 0$. Then

$$|l'(t)| = \Big|\frac{u_i'(t)u_i''(t) + g_i(u_i(t))u_i'(t)}{l(t)}\Big| \leq \|p_i\|_0 \frac{|u_i'(t)|}{|l(t)|} \leq \sqrt{2}\|p_i\|_0.$$

So we can take $E = \sqrt{2}\|p_i\|_0$.

Lemma 6 (5.3). *Assume that g is strictly increasing. Then for all $0 < y < z$,*

$$\frac{\sqrt{2(z-y)}}{\sqrt{g(z)}} \leq \int_y^z \frac{du}{\sqrt{2(G(z) - G(u))}} \leq \frac{\sqrt{2(z-y)}}{\sqrt{g(y)}}.$$

Proof. Let $0 \leq y \leq u < z$. Then

$$G(z) - G(u) = \int_u^z g(\xi)\, d\xi \geq \min_{u \leq \xi \leq z} g(\xi)(z-u) = g(u)(z-u) \geq g(y)(z-u)$$

and $G(z) - G(u) \leq g(z)(z-u)$. So

$$\frac{\sqrt{2(z-y)}}{\sqrt{g(z)}} = \int_y^z \frac{du}{\sqrt{2g(z)(z-u)}} \leq \int_y^z \frac{du}{\sqrt{2(G(z) - G(u))}}$$

and

$$\int_y^z \frac{du}{\sqrt{2(G(z) - G(u))}} \leq \int_y^z \frac{du}{\sqrt{2g(y)(z-u)}} = \frac{\sqrt{2(z-y)}}{\sqrt{g(y)}}.$$

We will suppose that g_i $(1 \leq i \leq m)$ is strictly increasing and

$$(\forall c > 0)(\forall A, B : AB > 0 \text{ and } |\sqrt{G_i(B)} - \sqrt{G_i(A)}| < c) : |B - A| < c. \tag{18}$$

Remark. This condition is satisfied for functions such that $(\sqrt{G_i(x)})' \geq 1$. And in particular this condition is satisfied for $g_i(x) = x|x|^{\alpha-1}$ (slightly modified near the origin) for $\alpha > 1$.

Lemma 7 (5.4). *Assume that g is strictly increasing and satisfies* (18) *and let $K>0$ be a fixed constant. Then for $0<y<z$ such that $|\sqrt{2G(z)}-\sqrt{2G(y)}|<K$ and $\sqrt{2G(y)}>K$, one has*

$$\tau_0^+(\sqrt{2G(z)})-\frac{\sqrt{2K}}{\sqrt{g(z-K)}}\le\tau_0^+(\sqrt{2G(y)})\le\big(1+\frac{K}{H}\big)\tau_0^+(\sqrt{2G(z)})+\epsilon_2(z),$$

where $H=\sqrt{K}\sqrt[4]{2G(z)-K}$ and

$$\epsilon_2(z)=\sqrt{2H}\big(\frac{1}{\sqrt{|g(z-(H+K))|}}-\frac{1}{\sqrt{|g(z)|}}\big).$$

Proof. Let $H>K$. Take x such that $0<x<y<z$ and $\sqrt{2G(y)}-\sqrt{2G(x)}=H$. Then $\sqrt{2G(z)}-\sqrt{2G(x)}\le H+K$ and by (18), $z-(H+K)\le x<y<z$.

Let, for $0\le u\le x$,

$$\psi(u)=\frac{G(y)-G(u)}{G(z)-G(u)}.$$

Using the fact that G is increasing, by differentiating, we can prove that ψ is nonincreasing and therefore, for $u\in[0,x]$,

$$\frac{G(y)-G(u)}{G(z)-G(u)}\ge\frac{G(y)-G(x)}{G(z)-G(x)}.$$

But

$$\begin{aligned}\frac{G(y)-G(x)}{G(z)-G(x)}&=\frac{(\sqrt{G(y)}-\sqrt{G(x)})(\sqrt{G(y)}+\sqrt{G(x)})}{(\sqrt{G(z)}-\sqrt{G(x)})(\sqrt{G(z)}+\sqrt{G(x)})}\\&\ge\frac{H}{H+K}\frac{2\sqrt{G(x)}}{2\sqrt{G(z)}}\ge\frac{H}{H+K}\frac{\sqrt{2G(z)}-(H+K)}{\sqrt{2G(z)}}.\end{aligned}$$

So, as for K fixed, $\frac{\sqrt{2G(z)}-(H+K)}{\sqrt{2G(z)}}=\frac{H}{H+K}$ for $H=\sqrt{K}\sqrt[4]{2G(z)-K}$,

$$\frac{G(y)-G(x)}{G(z)-G(x)}\ge\big(\frac{H}{H+K}\big)^2.$$

Then for all $y<z$ satisfying the hypothesis and $0\le u\le x<y$,

$$G(y)-G(u)\ge\big(\frac{H}{H+K}\big)^2(G(z)-G(u)).$$

Now

$$\int_0^x \frac{du}{\sqrt{2(G(y)-G(u))}} \le \int_0^x \frac{du}{\frac{H}{H+K}\sqrt{2(G(z)-G(u))}}$$
$$\le \big(1+\frac{K}{H}\big)\int_0^x \frac{du}{\sqrt{2(G(z)-G(u))}}.$$

To simplify the writing, let $\tau_z = \tau_0^+(\sqrt{2G(z)})$. Then passing to the time-maps τ, we have

$$\tau_y - \int_x^y \frac{du}{\sqrt{2(G(y)-G(u))}} \le \big(1+\frac{K}{H}\big)\tau_z - \big(1+\frac{K}{H}\big)\int_x^z \frac{du}{\sqrt{2(G(z)-G(u))}}.$$

Now as $z-(H+K) \le x < y < z$ and using Lemma 6, we have

$$\tau_y \le \big(1+\frac{K}{H}\big)\tau_z - \int_x^z \frac{du}{\sqrt{2(G(z)-G(u))}} + \int_x^y \frac{du}{\sqrt{2(G(y)-G(u))}}$$
$$\le \big(1+\frac{K}{H}\big)\tau_z + \sqrt{2H}\Big(\frac{1}{\sqrt{g(z-(H+K))}} - \frac{1}{\sqrt{g(z)}}\Big).$$

On the other hand,

$$\tau_y = \int_0^y \frac{du}{\sqrt{2(G(y)-G(u))}} \ge \int_0^y \frac{du}{\sqrt{2(G(z)-G(u))}}$$
$$= \tau_z - \int_y^z \frac{du}{\sqrt{2(G(z)-G(u))}}.$$

Then by Lemma 6,

$$\tau_y \le \tau_z - \frac{\sqrt{2K}}{\sqrt{g(z-K)}}.$$

Symmetrically, we have also the following lemma.

Lemma 8. *Assume that g is strictly increasing and satisfies* (18) *and let $K>0$ be a fixed constant. Then for $z<y<0$ such that $|\sqrt{2G(z)}-\sqrt{2G(y)}|<K$ and $\sqrt{2G(y)}>K$, one has*

$$\tau_0^-(\sqrt{2G(z)}) - \frac{\sqrt{2K}}{\sqrt{|g(z+K)|}} \le \tau_0^-(\sqrt{2G(y)}) \le \big(1+\frac{K}{H}\big)\tau_0^-(\sqrt{2G(z)}) + \epsilon_2(z).$$

To simplify the notation, we will denote by u any of the components x_i ($1\le i\le m$) of a solution x of (14)–(15). If $\max u$ is sufficiently large, $(u(t),u'(t))\ne 0$ and so all zeros of u are simple. We denote them by $0=t_0<t_1<\cdots<t_k\le 1$. If we take u with $\max u > R_2(\|p\|_0)$ and s such that $u'(s)=0$, then

$$u''(s)u(s) = (-g(u(s))+p)u(s) \le (-g(u(s))+\|p\|_0)u(s) < 0.$$

So there exists a unique point $s_i \in (t_i,t_{i+1})$ such that $u'(s_i)=0$.

Lemma 9 (3.4). *Assume that g_i is strictly increasing and satisfies* (18) *and let x be a solution of* (14)–(15) *such that $\max x_i = M$ with $M > \tilde{R}$. Then, for j even, one has*

$$(\tau_0^+(\sqrt{2G(M)}) - \epsilon_2(M))\frac{H}{H+K} - \frac{\tilde{K}}{\sqrt{g(M-4E)}} < s_j - t_j, t_{j+1} - s_j$$
$$< \big(1 + \frac{K}{H}\big)\tau_0^+(\sqrt{2G(M)}) + \epsilon_2(M) + \frac{\tilde{K}}{\sqrt{g(M-4E)}}$$

and for j odd, one has

$$(\tau_0^-(\sqrt{2G(M)}) - \epsilon_2(M'))\frac{H}{H+K} - \frac{\tilde{K}}{\sqrt{-g(M'+4E)}} < s_j - t_j, t_{j+1} - s_j$$
$$< \big(1 + \frac{K}{H}\big)\tau_0^-(\sqrt{2G(M)}) + \epsilon_2(M') + \frac{\tilde{K}}{\sqrt{-g(M'+4E)}}$$

where $M' < 0$ is such that $G(M) = G(M')$.

Proof. We prove this theorem in the case where j is even ($u(s_j) > 0$). Take $M \geq R_2$ and apply Lemma 5, so that for all $t \in I$

$$\sqrt{2G(M)} - E \leq \sqrt{{u'}^2(t) + 2G(u(t))} \leq \sqrt{2G(M)} + E.$$

There exists $0 < A < B$ such that

$$\sqrt{2G(A)} = \sqrt{2G(M)} - 2E < \sqrt{2G(M)} + 2E = \sqrt{2G(B)}.$$

Then

$$\sqrt{2G(B)} - \sqrt{2G(A)} = 4E$$

and by (18), $B - A \leq 4E$.

Note also that

$$2G(A) < {u'}^2(t) + 2G(u(t)) < 2G(B) \tag{19}$$

and

$$A < u(s_j) < B.$$

Let α_j, β_j with $t_j < \alpha_j < s_j < \beta_j < t_{j+1}$ be such that $u(\alpha_j) = u(\beta_j) = A$. Observe that $u(t) > A$ for $\alpha_j < t < \beta_j$. Hence for $t \in [\alpha_j, s_j]$,

$$u'(t) = u'(t) - u'(s_j) = -\int_{s_j}^{t} g(u(s))\, ds + \int_{s_j}^{t} \lambda p_i(t, u(s), u'(s))\, ds$$
$$\geq \int_{t}^{s_j} g(A)\, ds - \int_{t}^{s_j} \|p_i\|_0\, ds \geq (s_j - t)g(A) - \|p_i\|_0$$

and for $t \in [s_j, \beta_j]$,

$$-u'(t) \geq \int_{s_j}^{t} g(A)\, ds - \int_{s_j}^{t} \|p_i\|_0\, ds \geq (t - s_j)g(A) - \|p_i\|_0.$$

Integrating the inequality over $[\alpha_j, s_j]$ gives

$$4E \geq u(s_j) - u(\alpha_j) \geq \frac{1}{2}(s_j - \alpha_j)^2 g(A) - \|p_i\|_0$$

so that

$$(s_j - \alpha_j)^2 \leq \frac{8E + 2\|p_i\|_0}{g(A)}.$$

Similarly, integrating over $[s_j, \beta_j]$ gives

$$(\beta_j - s_j)^2 \leq \frac{8E + 2\|p_i\|_0}{g(A)}.$$

Take $t \in [t_j, t_{j+1}]$. By (19), one has

$$\sqrt{2(G(A) - G(u(t))} < |u'(t)| < \sqrt{2(G(B) - G(u(t))}.$$

Observe also that $G(u(t)) < G(A)$ for $t \in [t_j, \alpha_j) \cup (\beta_j, t_{j+1}]$. If $t_j \leq t \leq \alpha_j$, then $u'(t) > 0$ and one has

$$\int_{t_j}^{\alpha_j} \frac{u'(t)}{\sqrt{2(G(A) - G(u(t)))}}\, dt > \alpha_j - t_j > \int_{t_j}^{\alpha_j} \frac{u'(t)}{\sqrt{2(G(B) - G(u(t)))}}\, dt$$

and

$$\int_0^A \frac{du}{\sqrt{2(G(A) - G(u))}} > \alpha_j - t_j > \int_0^A \frac{du}{\sqrt{2(G(B) - G(u))}}.$$

If $\beta_j \leq t \leq t_{j+1}$, then $u'(t) < 0$ and a similar computation gives

$$\int_0^A \frac{du}{\sqrt{2(G(A) - G(u))}} > t_{j+1} - \beta_j > \int_0^A \frac{du}{\sqrt{2(G(B) - G(u))}}.$$

So we have

$$\begin{aligned} t_{j+1} - s_j &= t_{j+1} - \beta_j + \beta_j - s_j \\ &\leq \int_0^A \frac{du}{\sqrt{2(G(A) - G(u))}} + \frac{\sqrt{8E + 2\|p\|_0}}{\sqrt{g(A)}} = \tau_0^+(\sqrt{2G(A)}) + \frac{\tilde{K}}{\sqrt{g(A)}} \\ &\leq \big(1 + \frac{K}{H(M)}\big)\tau_0^+(\sqrt{2G(M)}) + \epsilon_2(M) + \frac{\tilde{K}}{\sqrt{g(A)}} \end{aligned}$$

and

$$s_j - t_j \le \big(1 + \frac{K}{H(M)}\big)\tau_0^+(\sqrt{2G(M)}) + \epsilon_2(M) + \frac{\tilde{K}}{\sqrt{g(A)}}.$$

On the other hand

$$\begin{aligned} t_{j+1} - s_j &\ge \int_0^A \frac{du}{\sqrt{2(G(B)-G(u))}} \\ &\ge \tau_0^+(\sqrt{2G(B)}) - \int_A^B \frac{du}{\sqrt{2(G(B)-G(u))}} \ge \tau_0^+(\sqrt{2G(B)}) - \frac{\sqrt{2(B-A)}}{\sqrt{g(A)}} \\ &\ge \tau_0^+(\sqrt{2G(B)}) - \frac{\tilde{K}}{\sqrt{g(A)}} \ge \frac{\tau_0^+(\sqrt{2G(M)}) - \epsilon_2(B)}{1+\frac{K}{H(B)}} - \frac{\tilde{K}}{\sqrt{g(A)}} \end{aligned}$$

and

$$s_j - t_j \ge \frac{\tau_0^+(\sqrt{2G(M)}) - \epsilon_2(B)}{1+\frac{K}{H(B)}} - \frac{\tilde{K}}{\sqrt{g(A)}}.$$

Lemma 10. *Assume that g is increasing. Then for $M > 0$ (respectively $M < 0$),*

$$\tau_0^+(\sqrt{2G(M)}) \ge \frac{M}{\sqrt{2G(M)}} \quad \big(\textit{respectively } \tau_0^-(\sqrt{2G(M)}) \ge \frac{|M|}{\sqrt{2G(M)}}\big).$$

Proof. As $g(s) > 0$ for $s > 0$ then $G(s) > 0$ and

$$\tau_0^+(\sqrt{2G(M)}) = \int_0^M \frac{du}{\sqrt{2(G(M)-G(u))}} \ge \int_0^M \frac{du}{\sqrt{2G(M)}} = \frac{M}{\sqrt{2G(M)}}.$$

We impose a technical condition on g_i $(1 \le i \le m)$ to insure certain regularity on the behavior of the solutions:

$$\lim_{M\to+\infty} \frac{\sqrt{G_i(M)}}{M}\Big(\frac{\epsilon_2(M)\sqrt{G_i(M)}}{M} + \frac{\sqrt{G_i(M)}}{M\sqrt{g_i(M)}} + \frac{1}{\sqrt[4]{G_i(M)}}\Big) = 0. \tag{20}$$

Remark. For $g(x) = x^\alpha$, this condition is satisfied for $1 < \alpha < 2$.

Then we have the following theorem.

Theorem 9. *Assume that g is differentiable, satisfies* (11)*, that g_i $(1 \le i \le m)$ are odd and satisfy* (18) *and* (20)*. There exists $R_3 > 0$ such that if k and l are two integers such that $\eta(\frac{\pi}{2} + k\pi) \in ((l+\epsilon)\pi, (l+1-\epsilon)\pi)$ and (x, λ) is a solution of* (14)–(15) *such that $\max x_i = M > R_3$ and $\phi_i^1(x,\lambda) = \frac{\pi}{2} + k\pi$ then*

$$\phi_i^\eta(x,\lambda) \in [l\pi, (l+1)\pi].$$

Proof. Let (x,λ) be a solution of (14)–(15) such that $\max x_i = M$. There exists m_M such that $\phi_i^\eta(x) \in [m_M\frac{\pi}{2}, (m_M+1)\frac{\pi}{2}]$. We abbreviate $\tau_0^+(\sqrt{2G_i(M)})$ by τ_M. Note that as g_i is odd, $\tau_0^+(\alpha) = \tau_0^-(\alpha)$.

Now using Lemma 9 for $i = 1$ to $2k$ and $i = 1$ to m_M, one has

$$m_M\big((\tau_M - \epsilon_2(M))\frac{H}{H+K} - \frac{\tilde{K}}{\sqrt{g(M-4E)}}\big) < \eta < (m_M+1)\big((1+\frac{K}{H})\tau_M + \epsilon_2(M)\big)$$

and

$$(2k+1)\big((\tau_M - \epsilon_2(M))\frac{H}{H+K} - \frac{\tilde{K}}{\sqrt{g(M-4E)}}\big) < 1 < (2k+1)\big((1+\frac{K}{H})\tau_M + \epsilon_2(M)\big).$$

Those inequalities can be written has

$$m_M(1-\epsilon_1(M))\tau_M < \eta < (m_M+1)(1+\epsilon_1(M))\tau_M \tag{21}$$

and

$$(2k+1)(1-\epsilon_1(M))\tau_M < 1 < (2k+1)(1+\epsilon_1(M))\tau_M \tag{22}$$

if ϵ_1 is such that

$$\epsilon_1(M) \geq \frac{\epsilon_2(M)\sqrt{2G(M)}}{M} + \frac{\tilde{K}\sqrt{2G(M)}}{M\sqrt{g(M-4E)}} + \frac{\sqrt{K}}{\sqrt[4]{2G(M)} - \sqrt{K}}.$$

But $l + \epsilon < \frac{\eta}{2}(2k+1) < l+1-\epsilon$, so

$$2\frac{l+\epsilon}{2k+1} < \eta < 2\frac{l+1-\epsilon}{2k+1}. \tag{23}$$

Now dividing the second inequality of (21) by the first of (22) and combining with (23), one has

$$2\frac{l+\epsilon}{2k+1} < \eta < \frac{(m_M+1)(1+\epsilon_1(M))\tau_M}{(2k+1)(1-\epsilon_1(M))\tau_M}.$$

So $l+\epsilon < \frac{1}{2}(m_M+1) + (m_M+1)\frac{2\epsilon_1}{1-\epsilon_1(M)}$. If in (21) we take $\epsilon_1(M) < 1/2$, we have $\frac{1}{2}m_M\tau_M < \eta$ and thus $m_M < \frac{2\eta}{\tau_M}$. Now by (20), we can take R_3 such that for $M > R_3$,

$$(m_M+1)\frac{\epsilon_1}{1-\epsilon_1} < \frac{2\eta\epsilon_1}{\tau_M(1-\epsilon_1)} + \frac{\epsilon_1}{1-\epsilon_1} \leq \frac{2\eta\epsilon_1(M)\sqrt{2G(M)}}{M}\frac{1}{1-\epsilon_1} + \frac{\epsilon_1}{1-\epsilon_1} < \epsilon.$$

Then $l < \frac{1}{2}(m_M+1)$. Similarly, using the first inequality of (21) and the second of (22), one has

$$2\frac{l+1-\epsilon}{2k+1} > \eta > \frac{m_M(1-\epsilon_1(M))\tau_M}{(2k+1)(1+\epsilon_1(M))\tau_M}.$$

So $l+1-\epsilon > 1/2m_M - m_M\frac{\epsilon_1}{1+\epsilon_1}$. Using (20) once more, we have that for $M > R_3$, $m_M\frac{\epsilon_1}{1+\epsilon_1} < \epsilon$ and then $l+1 > \frac{1}{2}m_M$. Then when m_M is even, $l = \frac{m_M}{2}$ and when m_M is even, $l = \frac{m_M-1}{2}$. And in both cases,

$$\phi_i^\eta(x,\lambda) \in \big[\frac{m_M}{2}\pi, \frac{m_M+1}{2}\pi\big] \subset [l\pi, (l+1)\pi].$$

3.2. Multiplicity results. We will also use a theorem of approximation of irrational numbers by rational numbers of special type.

Theorem 10. *Let $\rho \in \mathbb{R}\setminus\mathbb{Q}$ and r_1, $r_2 \in \{0,1,2,3\}$. Then for all $\epsilon > 0$ and $p \in \mathbb{N}$, there exist $n, m \in \mathbb{N}$ with $n = r_1 \pmod 4$, $m = r_2 \pmod 4$, $n > p$ and $m > p$ such that*

$$|n\rho - m| < \epsilon.$$

Proof. Let $\bar{p} \geq \max\{p, \rho p + 4\}$ be such that $\bar{p} = r_1 \pmod 4$. We define $\alpha : \mathbb{N} \to [0,4)$ by $\alpha(j) = \rho\bar{p} + 4\rho j \pmod 4$. As ρ is irrational, this sequence is dense in $[0,4)$ (Kronecker). So for all $\epsilon > 0$, there exists $j \in \mathbb{N}$ such that $|\alpha(j) - r_2| \leq \epsilon$. Taking $n = \bar{p} + 4j$ and $m = \rho\bar{p} + 4\rho j - \alpha(j) + r_2$, we have the result.

Remark. The particular value 4 is not important in this theorem. This number can be replace by any positive integer. We state the theorem in this form because we use it with the number 4 in Theorem 11.

As a consequence of Lemma 2, we have also the following lemma.

Lemma 11. *For each $M > 0$, there exists $R > M$ such that for all (x, λ) solution of (14)–(15) satisfying $\max_{t\in I} x_i(t) \leq M$ we have $\|x_i\|_1 \leq R$.*

Proof. If the maximum is reached at $x_i(0) = 0$, then $\phi_i^1(u,\lambda) \leq \pi$ and $\|x_i\|_1 \leq R_2(R_1(\pi))$. If the maximum is reached at $x_i(1)$, it is reached at $x_i(\eta) = x_i(1)$. So when the maximum is not reached at $x_i(0)$, it is reached at an interior point $\bar{t}$ and

$$\min_{t\in I} |(x_i(t), x_i'(t))| \leq |(x_i(\bar{t}), x_i'(\bar{t}))| = x_i(\bar{t}) = \max_{t\in I} x_i(t) \leq M.$$

So by Lemma 2, $\|x_i\|_1 \leq R_2(M)$. $\square$

We are now ready to prove our final result.

Theorem 11. *Assume that g is differentiable, satisfies (11), that g_i $(1 \leq i \leq m)$ are odd and satisfy (18) and (20) and that $p : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is continuous and satisfies (12). Then there exist infinitely many solutions x_K of (7)–(8) with $\phi_i^1(x_K, 1) \in ((c_{K(i)})_i, (c_{K(i)+1})_i)$.*

Proof. We will use Theorem 2 with $\Omega = \{(x,\lambda) \in X \times I : x_i(0) \in P_i^+\}$.

For $m+1 \leq j \leq n$, we define $(c_k)_j = \delta_j + \gamma_j + k\pi - \frac{\pi}{2}$. If $\phi_j^1(x,\lambda) = \delta_j + \gamma_j + k\pi - \frac{\pi}{2}$, then, by Lemma 4, x_j doesn't satisfy (15).

Now we define $(c_k)_j$ for $1 \le j \le m$. In the case where η is irrational, take two strictly increasing sequences n_k and m_k. The numbers $n_{2k'}$ and $m_{2k'}$ are given by Theorem 10 with $\rho = \frac{1}{\eta}$, $r_1 = 1$, $r_2 = 3$, $\epsilon = \frac{1}{2}$ and $p = n_{2k'-1}$. The numbers $n_{2k'+1}$ and $m_{2k'+1}$ are given by Theorem 10 with $\rho = \frac{1}{\eta}$, $r_1 = 3$, $r_2 = 1$, $\epsilon = \frac{1}{2}$ and $p = n_{2k'}$. Then we define $(c_k)_j = \frac{\pi}{2} m_k$.

Suppose that x is a solution of (14) with $l_{1,j}(x_j) = 0$. Suppose that $\phi_j^1(x,\lambda) = \frac{\pi}{2} m_{2j} = \frac{\pi}{2}(4k+3) = \pi(2k+\frac{3}{2})$. By Theorem 10, $|(4l+1)\frac{1}{\eta} - (4k+3)| < \frac{1}{2}$ and so

$$\eta\big(\frac{\pi}{2} + (2k+1)\pi\big) \in \big((2l + \frac{1}{2} - \frac{\eta}{4})\pi, (2l+1-(\frac{1}{2}-\frac{\eta}{4}))\pi\big).$$

Then using Theorem 9, $\phi_j^\eta(x,\lambda) \in [2l\pi, (2l+1)\pi]$. This proves that $x_j(\eta)$ and $x_j(1)$ have opposite signs and thus that x_j doesn't satisfy (15).

In the case where η is rational, $\eta = \frac{p}{q}$ for p and q relatively prime integers. As $p \ne q$, there exists $i \in \mathbb{N}$ such that $p = 2^i k + r$ and $q = 2^i l + r$ with $k, l, r \in R$, $0 \le r \le 2^i$ and the parity of k and l different. Then

$$\begin{aligned}(l + \frac{1}{2})\eta &= (l + \frac{r}{2^i} + \frac{1}{2} - \frac{r}{2^i})\frac{2^i k + r}{2^i l + r} \\ &= \frac{2^i k + r}{2^i} + (\frac{1}{2} - \frac{r}{2^i})\eta = k + \frac{r}{2^i}(1-\eta) + \frac{1}{2}\eta.\end{aligned}$$

So $k < (l+\frac{1}{2})\eta < k+1$.

If $i = 0$, then we take $(c_{2k})_j = ((2k+1)q - \frac{1}{2})\pi$ and $(c_{2k+1})_j = ((2k+1)q + \frac{1}{2})\pi$. If $\phi_j^1(x,\lambda) = (c_k)_j$, then as $\eta(c_k)_j = ((2k+1)p \pm \frac{\eta}{2})\pi$, by Theorem 9, $\phi_j^\eta(x,\lambda) \in [((2k+1)p-1)\pi, (2k+1)p\pi]$ when k is even and $\phi_j^\eta(x,\lambda) \in [(2k+1)p\pi, ((2k+1)p+1)\pi]$ when k is odd. This proves that $x_j(\eta)$ and $x_j(1)$ have not the same sign.

If $i \ge 1$, we take $(c_s)_j = (sq + l + \frac{1}{2})\pi$. So if $\phi_j^1(x,\lambda) = (c_s)_j$, as $\eta(c_s)_j = (sp + (l + \frac{1}{2})\eta)\pi \in ((sp+k)\pi, (sp+k+1)\pi)$, by Theorem 9, $\phi_j^\eta(x,\lambda) \in [(sp+k)\pi, (sp+k+1)\pi]$. As the parity of k and l are different, the sign of $x_j(\eta)$ and of $x_j(1)$ are different.

So in both cases, using Lemma 11 and Lemma 4, (i_4) is satisfied with $R = \max\{R(R_3), \bar{R}\}$ where R_3 is given by Theorem 9.

For condition (i_5), suppose that $(x,\lambda) \in \Sigma^*$ and $\phi_j^1(x,\lambda) < (c_k)_j$. Then by Lemma 3, $\min_{t\in I} |(x_j(t), x_j'(t))| \le R_1((c_k)_j)$. Thus, by Lemma 2,

$$\|x_j\|_1 \le R_2(R_1((c_k)_j)).$$

This proves that $\phi^{-1}([0,c_k]) \cap \Sigma^* \cap \Omega$ is bounded.

We still have to compute the degree

$$D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}(.,0), {\mathcal{O}^K}_0).$$

For $\lambda = 0$, the equations are uncoupled, and so

$$D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}(.,0), {\mathcal{O}^K}_0) = \prod_{i=1}^{n} D_L(L - N, {\mathcal{O}^K}_0 \cap \mathcal{X}_i),$$

where $\mathcal{X}_i = \{z \in X : z_j(t) = 0 \text{ for } j \neq i \text{ and } j \neq n+i\} \sim \mathcal{X}$.

But we have already computed the degree associated to scalar equations. Moreover, by Theorem 8,

$$\begin{aligned} D_L(L-N, \mathcal{O}^K{}_0 \cap \mathcal{X}_i) &= D_L(L-N, \phi_i^{-1}(((c_{K(i)})_i, (c_{K(i)+1})_i)) \cap P_i^+) \\ &= D_L(L-N, \Omega^+(E)) = D_L(L-N, \Omega^+(\mathrm{co}(E))) \\ &= D_L(L-N, \Omega^+((\bar{\alpha}, \alpha))). \end{aligned}$$

For $m+1 \leq j \leq n$, in the case where $\delta_i + \gamma_i \geq 0$, $\bar{\alpha}$ is such that $(\tau_1^+(\bar{\alpha}), \tau^+(\bar{\alpha}), \tau^-(\bar{\alpha})) \in \mathcal{S}_{K_i}$ and α is such that $(\tau_1^+(\alpha), \tau^+(\alpha), \tau^-(\alpha)) \in \mathcal{S}_{K_i+1}$. In the case where $\delta_i + \gamma_i < 0$, $\bar{\alpha}$ is such that $(\tau_1^+(\bar{\alpha}), \tau^+(\bar{\alpha}), \tau^-(\bar{\alpha})) \in \mathcal{S}_{K_i-1}$ and α is such that $(\tau_1^+(\alpha), \tau^+(\alpha), \tau^-(\alpha)) \in \mathcal{S}_{K_i}$. So in both cases, we have, by Theorem 5,

$$D_L(L-N, \Omega^+((\bar{\alpha}, \alpha))) \neq 0.$$

Now for $1 \leq j \leq m$, for (x, λ) solution of (14) with $\lambda = 0$ such that $l_{j,1}(x_j) = 0$ and $x_j \in \tilde{H}^{-1}(\alpha)$, $\phi_j^1(x,0) = (4k+3)\frac{\pi}{2}$ or $(4k+1)\frac{\pi}{2}$, depending on the parity of K_i. In each case, if $\phi_j^1(x,0) = (4k+3)\frac{\pi}{2}$, then $\phi_j^\eta(x,0) \in [4l\frac{\pi}{2}, (4l+2)\frac{\pi}{2}]$ and if $\phi_j^1(x,0) = (4k+1)\frac{\pi}{2}$, then $\phi_j^\eta(x,0) \in [(4l+2)\frac{\pi}{2}, (4l+4)\frac{\pi}{2}]$. In the first case, $(\tau_0^+(\alpha), \tau(\alpha)) \in \mathcal{S}_{k-l,k+l+1}$ and in the second case, $(\tau_0^+(\alpha), \tau(\alpha)) \in \mathcal{S}_{k-l-1,k+l+1}$. So in all cases, we have by Theorem 7, $D_L(L-N, \Omega^+((\bar{\alpha}, \alpha))) \neq 0$.

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