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ROTATING FLUID AT HIGH ROSSBY NUMBER DRIVEN BY A SURFACE STRESS: EXISTENCE AND CONVERGENCE

Thierry Colin* and Pierre Fabrie
Mathématiques Appliquées de Bordeaux, CNRS ERS 123 et Université Bordeaux 1
351 cours de la libération, 33405 Talence Cedex, France

(Submitted by: Roger Temam)

Abstract. We consider the 3-D Navier-Stokes equations with Coriolis force of order $\frac{1}{\varepsilon}$ and vanishing vertical viscosity of order ε. For suitable initial data, we prove some long-time existence results. Moreover, we obtain convergence as ε goes to 0 to the 2-D Navier-Stokes equations. We deal with periodic boundary conditions and nonhomogeneous stress. In this case, we compute and justify the corrector.

1. Introduction and setting of the problem. The aim of this work is to study the following 3-D Navier-Stokes equations:

$$\begin{aligned} &U_t + U\cdot\nabla U + \frac{1}{\varepsilon}k\times U + \frac{1}{\varepsilon}\nabla p - \varepsilon U_{zz} - \Delta_{x,y}U = 0,\\ &\nabla\cdot U = 0,\\ &U(t=0) = U_0, \end{aligned} \tag{1.1}$$

where U is the 3-D velocity vector field, p the pressure and k the unit vertical vector $k=(0,0,1)^t$. The operator $\Delta_{x,y}$ denotes the 2-D Laplace operator $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

This model occurs in geophysical fluid dynamics. It is introduced in order to study large-scale motions in the ocean. Of course, one has to add some boundary conditions. Before we make precise these conditions, let us make some comments on the physical meaning of the model. This system is a simplified model for the primitive equations of the atmosphere that have been studied in [9]. See also [6] for a detailed study of gravity waves in geophysical systems. In what follows, one writes $U=(u_1,u_2,u_3)^t$.

1.1. Justification of the model and formal asymptotic expansion. The motivation of the introduction of this model is to characterize large-scale motions at high Rossby number with the presence of friction at the surface.

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After a dimensionalization, the model reads ([10, page 201])

$$U_t + U \cdot \nabla U + \frac{1}{\varepsilon} k \times U + \frac{1}{\varepsilon}\nabla p - \alpha U_{zz} - \Delta_{x,y} U = 0.$$

The parameter α is a priori small. We will now explain why this parameter α has to be taken equal to ε . This fact is linked with the boundary conditions that we make precise now. For simplicity, we work with $(x, y) \in \mathbf{T}^2$, where $\mathbf{T}^2$ denotes the 2-D torus and with $z \in [0, 1]$. The boundary conditions are

$$\text{at} \quad z = 0 : u_3 = 0, \; \frac{\partial}{\partial z}(u_1, u_2) = 0, \tag{1.2}$$

$$\text{at} \quad z = 1 : u_3 = 0, \; \frac{\partial}{\partial z}(u_1 u_2) = \beta\tau(x, y), \tag{1.3}$$

where τ is a 2-D vector field whose amplitude is $O(1)$. The relationships (1.2) are those used in [8], while (1.3) corresponds to the rigid lid assumption with an applied stress. The aim of the following formal expansion is to make precise the order of magnitude α and β with respect to ε in order to obtain a correct description of these large-scale motions.

First of all, the scaling $\frac{1}{\varepsilon} k \times U$ corresponds to a large Rossby number and is quite natural. We therefore rescale the pressure field in $\frac{1}{\varepsilon}\nabla p$ in order to cancel the Coriolis acceleration by the horizontal part of the gradient of the pressure. At this step, one can begin the expansion.

At the leading order, one gets if $(u_1^0, u_2^0, u_3^0, p^0)$ denotes the formal limit

$$-u_2^0 = -\frac{\partial p^0}{\partial x}, \tag{1.4}$$

$$u_1^0 = -\frac{\partial p^0}{\partial y}, \tag{1.5}$$

$$0 = -\frac{\partial p^0}{\partial z}, \tag{1.6}$$

$$\frac{\partial u_1^0}{\partial x} + \frac{\partial u_2^0}{\partial y} + \frac{\partial u_3^0}{\partial z} = 0. \tag{1.7}$$

Equation (1.6) implies that p^0 does not depend on z. Therefore (1.4) and (1.5) give that u_1^0 and u_2^0 do not depend on z either. On the other hand (1.4) and (1.5) show that $\frac{\partial u_1^0}{\partial x} + \frac{\partial u_2^0}{\partial y} = 0$ and therefore (1.7) leads to the fact that u_3^0 does not depend on z. In order to obtain the equation satisfied by (u_1^0, u_2^0, u_3^0), we need to introduce the second-order expansion, i.e.,

$$u_1 = u_1^0 + \varepsilon u_1^1, \quad u_2 = u_2^0 + \varepsilon u_2^1, \quad u_3 = u_3^0 + \varepsilon u_3^1, \quad p = p^0 + \varepsilon p^1.$$

One gets as $(u_1^0, u_2^0, u_3^0, p^0)$ do not depend on z

$$\frac{\partial u_1^0}{\partial t} + u_1^0 \frac{\partial u_1^0}{\partial x} + u_2^0 \frac{\partial u_1^0}{\partial y} - u_2^1 = -\frac{\partial p^1}{\partial x} + \Delta_{x,y} u_1^0, \tag{1.8}$$

$$\frac{\partial u_2^0}{\partial t} + u_1^0 \frac{\partial u_2^0}{\partial x} + u_2^0 \frac{\partial u_2^0}{\partial y} + u_1^1 = -\frac{\partial p^1}{\partial y} + \Delta_{x,y} u_2^0, \tag{1.9}$$

$$\frac{\partial u_3^0}{\partial t} + u_1^0 \frac{\partial u_3^0}{\partial x} + u_2^0 \frac{\partial u_3^0}{\partial y} = -\frac{\partial p^1}{\partial z} + \Delta_{x,y} u_3^0, \tag{1.10}$$

$$\frac{\partial u_1^1}{\partial x} + \frac{\partial u_2^1}{\partial y} + \frac{\partial u_3^1}{\partial z} = 0. \tag{1.11}$$

Note that at this step, we have not used the sizes of α and β.

Computing $\frac{\partial(1.8)}{\partial y} - \frac{\partial(1.9)}{\partial x}$ yields

$$\frac{\partial \zeta_0}{\partial t} + \begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix} \cdot \nabla_{x,y} \zeta_0 - \Delta_{x,y} \zeta_0 = \left(\frac{\partial u_1^1}{\partial x} + \frac{\partial u_2^1}{\partial y}\right), \tag{1.12}$$

where

$$\zeta_0 = \frac{\partial u_1^0}{\partial y} - \frac{\partial u_2^0}{\partial x}.$$

Using (1.11) in (1.12) yields

$$\frac{\partial \zeta_0}{\partial t} + \begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix} \cdot \nabla_{x,y} \zeta_0 - \Delta_{x,y} \zeta_0 = -\frac{\partial u_3^1}{\partial z}. \tag{1.13}$$

Since ζ_0 and (u_1^0, u_2^0) do not depend on z, it is clear that $\frac{\partial u_3^1}{\partial z}$ is a constant function in z, so that $z \to u_3^1(z)$ is affine. It follows that in order to compute $\frac{\partial u_3^1}{\partial z}$, we just have to know the correct boundary values for u_3^1. Of course, since $(u_1^0, u_2^0, u_3^0, p^0)$ do not depend on z, we can not use (1.2) and (1.3) for this vector field. The only way, in order to obtain a coherent model, is to introduce a boundary layer.

We first deal with $z = 1$. Denoting by l the size of the boundary layer and by $(\tilde{u}_1^0, \tilde{u}_2^0, \tilde{u}_3^0, \tilde{p}^0)(x, y, \zeta; t) = (u_1^0, u_2^0, u_3^0, p^0)(x, y, \frac{1-z}{l}; t)$, we get at the leading order

$$-\tilde{u}_2^0 = -\frac{\partial \tilde{p}^0}{\partial x} + \frac{\varepsilon\alpha}{l^2} \frac{\partial^2 \tilde{u}_1^0}{\partial \zeta^2}, \tag{1.14}$$

$$\tilde{u}_1^0 = -\frac{\partial \tilde{p}^0}{\partial y} + \frac{\varepsilon\alpha}{l^2} \frac{\partial^2 \tilde{u}_2^0}{\partial \zeta^2}, \tag{1.15}$$

$$0 = \frac{\partial \tilde{p}^0}{\partial \zeta} + \frac{\varepsilon\alpha}{l^2} \frac{\partial^2 \tilde{u}_3^0}{\partial \zeta^2}, \tag{1.16}$$

$$\frac{\partial \tilde{u}_1^0}{\partial x} + \frac{\partial \tilde{u}_2^0}{\partial y} - \frac{1}{l} \frac{\partial(\tilde{u}_3^0 + \varepsilon \tilde{u}_3^1)}{\partial \zeta} = 0. \tag{1.17}$$

In order to achieve a cancellation between the Coriolis force, the horizontal pressure gradient and the viscous force in the boundary layer, (1.14) and (1.15) impose $\frac{\varepsilon\alpha}{l^2} = \mathcal{O}(1)$ (see [10, p. 212]) and for the sake of simplicity, we take

$$\frac{\varepsilon\alpha}{l^2} = 1. \tag{1.18}$$

It follows that at the leading order, (1.16) becomes $\frac{\partial\tilde{p}^0}{\partial\zeta} = 0$, and therefore $\tilde{p}^0$ is equal to its value p^0 outside the boundary layer thanks to the matching principle ([12]), which states that $\lim_{\zeta\to+\infty}\tilde{p}^0(\zeta)$ has to be equal to $\lim_{z\to 0} p^0(z)$.

On the other hand, (1.17) implies that $\frac{\partial\tilde{u}_3^0}{\partial\zeta} = 0$, and since $\tilde{u}_3^0(\zeta = 0) = 0$, one gets $\tilde{u}_3^0(\zeta) = 0$. The matching principle again implies that $u_3^0(z) \equiv 0$, since u_3^0 does not depend on z.

We next rewrite the continuity equation (1.17):

$$\frac{\partial\tilde{u}_1^0}{\partial x} + \frac{\partial\tilde{u}_2^0}{\partial y} - \frac{\varepsilon}{l}\frac{\partial\tilde{u}_3^1}{\partial\zeta} = 0. \tag{1.19}$$

Moreover, the boundary conditions on $\tilde{u}_1^0$ and $\tilde{u}_2^0$ are

$$\frac{\partial(\tilde{u}_1^0, \tilde{u}_2^0)}{\partial\zeta}(\zeta = 0) = -l\beta\tau.$$

Therefore, in order to impose that the fluid is driven by the stress at its surface, one takes (see [10, page 235])

$$l\beta = 1, \tag{1.20}$$

and the boundary condition becomes

$$\frac{\partial(\tilde{u}_1^0, \tilde{u}_2^0)}{\partial\zeta}(\zeta = 0) = -\tau. \tag{1.21}$$

We can now solve (1.14)–(1.15) and (1.21) with (1.18) and (1.20) and we get

$$\begin{aligned}\begin{pmatrix}\tilde{u}_1^0\\ \tilde{u}_2^0\end{pmatrix}(x, y, \zeta; t) = \begin{pmatrix}u_1^0\\ u_2^0\end{pmatrix}(x, y; t)\\ + e^{-\frac{\zeta}{\sqrt{2}}}\big(\tau\cos\big(\frac{\zeta}{\sqrt{2}} + \frac{\pi}{4}\big) - L\tau\cos\big(\frac{\zeta}{\sqrt{2}} - \frac{\pi}{4}\big)\big),\end{aligned}$$

where $L\begin{pmatrix}f_1\\ f_2\end{pmatrix} = \begin{pmatrix}-f_2\\ f_1\end{pmatrix}$.

Let us now recall that we want to find the correct boundary value for u_3^1 and since the goal of this expansion is to see how the stress at the surface can drive the flow, the driving term in (1.3) must be $-\frac{\partial u_3^1}{\partial z}$ and has to be computed from (1.19).

We have therefore to take $\frac{\varepsilon}{l} = O(1)$ in order to obtain nontrivial information from (1.19); for simplicity, we take

$$\frac{\varepsilon}{l} = 1. \tag{1.22}$$

Relationships (1.18)–(1.20) and (1.22) imply that both ε in (1.1) are the same and that the boundary conditions (1.3) in $z = 1$ are

$$\frac{\partial(u_1, u_2)}{\partial z}|_{z=1} = \frac{\tau}{\varepsilon}.$$

These assumptions ensure that we deal with a viscous flow at high Rossby number, driven by a stress applied at its surface.

We can now end the computation of $\frac{\partial u_3^1}{\partial z}|_{z=1}$: We use (1.19) with (1.22) and get

$$\frac{\partial \tilde{u}_3^1}{\partial \zeta} = \frac{\partial \tilde{u}_1^0}{\partial x} + \frac{\partial \tilde{u}_2^0}{\partial y},$$

and an explicit computation yields

$$\tilde{u}_3^1 = \operatorname{curl} \tau \big(e^{-\frac{\zeta}{\sqrt{2}}} \cos(\frac{\zeta}{\sqrt{2}}) - 1\big),$$

so that

$$\lim_{\zeta \to +\infty} \tilde{u}_3^1 = -\operatorname{curl} \tau \equiv \frac{\partial \tau_2}{\partial x} - \frac{\partial \tau_1}{\partial y},$$

where $\tau = (\tau_1, \tau_2)$.

All these calculations show that there is no boundary layer at $z = 0$. Equation (1.13) therefore becomes

$$\frac{\partial \zeta_0}{\partial t} + \begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix} \cdot \nabla_{x,y} \zeta_0 - \Delta \zeta_0 = \operatorname{curl} \tau. \tag{1.23}$$

Coming back to (u_1^0, u_2^0) , (1.23) means that (u_1^0, u_2^0) satisfy the 2-D Navier-Stokes equations:

$$\frac{\partial}{\partial t} \begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix} + \begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix} \cdot \nabla_{x,y} \begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix} - \Delta_{x,y} \begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix} + \nabla q = \tau. \tag{1.24}$$

The remainder of this paper is devoted to the mathematical justification of the above asymptotic expansion.

1.2. Statement of the results. For the mathematical justification, we will begin by the purely periodic case, i.e., $(x, y, z) \in \mathbf{T}^3$, and then we treat the physical case.

The periodic case is academic, and if we replace $\varepsilon\frac{\partial^2}{\partial z^2}$ by $\frac{\partial^2}{\partial z^2}$, one gets instead of (1.1)

$$\begin{gathered}\frac{\partial U}{\partial t}+U\cdot\nabla U+\frac{1}{\varepsilon}k\times U-\Delta_{x,y,z}U+\frac{1}{\varepsilon}\nabla p=0,\\ \nabla\cdot U=0.\end{gathered}\tag{1.25}$$

This problem has been studied by Grenier ([7]) and Babin and all ([1]). In these works, the authors show that the solution of (1.25) can be split in two parts, $\bar{U}+U_{osc}$, and they give the system of equations satisfied asymptotically by $\bar{U}$ and U_{osc}. More precisely, they introduce

$$V^\varepsilon=e^{-\frac{tP\mathbf{L}}{\varepsilon}}U^\varepsilon\equiv\mathcal{L}(-\frac{t}{\varepsilon})U^\varepsilon,$$

where $\mathbf{L}$ denotes the operator $f\to k\times f$ and P is the projector on the divergence-free vector fields in $\left(L^2(\mathbf{T}^3)\right)^3$. One can then write the P.D.E. satisfied by V^ε:

$$\frac{\partial V^\varepsilon}{\partial t}+\mathcal{L}(-\frac{t}{\varepsilon})P\nabla\cdot\left(\mathcal{L}(\frac{t}{\varepsilon})V^\varepsilon\otimes\mathcal{L}(\frac{t}{\varepsilon})V^\varepsilon\right)-\Delta V^\varepsilon=0.\tag{1.26}$$

Standard energy estimates give some bounds for U^ε (and hence for V^ε) in $L^\infty(\mathbb{R}^+;L^2)\cap L^2(\mathbb{R}^+;H^1)$ while (1.26) gives a bound for $\partial_t V^\varepsilon$ which allows one to obtain strong convergence in $L^2(\mathbb{R}^+;L^2)$ for example. However, this method does not work for (1.1) since we cannot obtain, because of the term $\varepsilon\frac{\partial^2}{\partial z^2}$, a uniform bound in ε of U^ε in $L^2(\mathbb{R}^+;H^1)$, and therefore we do not recover some strong convergence of the equivalent of V^ε in $L^2(\mathbb{R}^+;L^2)$.

Moreover this method (as noticed in [7]) is inoperant in the case of nonperiodic boundary conditions in the z variable. We therefore choose another strategy, which is that of Chemin ([3]): we force the oscillatory part of U to be small at $t=0$ and show that this property propagates through time. In the periodic case we will be able to do that directly, while for the physical boundary data, the boundary layer has to be introduced as a corrector for the solution in order to achieve this goal. This technique will enable us to prove the existence of global strong solutions to (1.1) in the periodic case, while in the physical case we show that the oscillatory part remains small on a time interval of length $K\ln(\frac{1}{\varepsilon})$.

We now make precise how we split the solution. Since the formal expansion shows that the limit solution does not depend on z, we introduce the vertical average of the unknowns; for $j=1,2$ we denote by $\bar{u}_j=\int_0^1 u_j(z)\,dz\equiv ⨍ u_j$ and $\bar{p}=\int_0^1 p(z)\,dz\equiv ⨍ p$.

The oscillatory part is then $\tilde{u}_j=u_j-\bar{u}_j$ for $j=1,2$ and $\tilde{p}=p-\bar{p}$. Since the above formal expansion gives that $u_3^0\equiv 0$, we also will impose that u_3 is small. Therefore denoting by $\bar{U}=(\bar{u}_1,\bar{u}_2)$ and $\tilde{U}=(\tilde{u}_1,\tilde{u}_2,u_3)$ and $\tilde{U}_2=(\tilde{u}_1,\tilde{u}_2)$, we integrate the two first equations of (1.1) with respect to z and we get in the periodic case

$$\frac{\partial \bar{U}}{\partial t} + \bar{U} \cdot \nabla_{x,y} \bar{U} + \nabla_{x,y} \cdot \big(\fint \tilde{U}_2 \otimes \tilde{U}_2 \big) + \frac{1}{\varepsilon} L \bar{U} + \frac{1}{\varepsilon} \nabla \bar{p} - \Delta_{x,y} \bar{U} = 0, \quad (1.27)$$

$$\nabla_{x,y} \cdot \bar{U} = 0, \quad (1.28)$$

$$\bar{U}(t=0) = \fint (u_{01}, u_{02})^t. \quad (1.29)$$

Subtracting (1.27) from (1.1) leads to

$$\tilde{U}_t + \tilde{U} \cdot \nabla_{x,y} \tilde{U} - \begin{pmatrix} \nabla_{x,y} \cdot (\fint \tilde{U}_2 \otimes \tilde{U}_2) \\ 0 \end{pmatrix} + \bar{U} \cdot \nabla_{x,y} \tilde{U} + \begin{pmatrix} \tilde{U}_2 \cdot \nabla_{x,y} \bar{U} \\ 0 \end{pmatrix}$$

$$+ \frac{1}{\varepsilon} k \times \tilde{U} - \varepsilon \frac{\partial^2 \tilde{U}}{\partial z^2} - \Delta_{x,y} \tilde{U} + \frac{1}{\varepsilon} \nabla \tilde{p} = 0, \quad (1.30)$$

$$\nabla_{x,y,z} \cdot \tilde{U} = 0, \quad (1.31)$$

$$\tilde{U}(t=0) = \tilde{U}_0 = U - \begin{pmatrix} \bar{U} \\ 0 \end{pmatrix}, \quad (1.32)$$

where we have used that $\fint \tilde{U}_2 = 0$. We next impose that $\tilde{U}_0$ is small and the main result that we obtain is the following.

Theorem. *Let us suppose that $\bar{U}_0$ is given in $H^1(\mathbf{T}^2)$; then there exists a constant D (which depends only on $|\bar{U}_0|_{H^1(\mathbf{T}^2)}$), $\varepsilon_0 > 0$ and K such that for any $\varepsilon \le \varepsilon_0$, if $|\tilde{U}_0^\varepsilon|^2_{H^1(\mathbf{T}^3)} \le D\varepsilon^\beta$ with $\beta \ge 3/2$, then (1.25)–(1.30) has a unique global strong solution satisfying*

$$|\nabla_{x,y,z} \tilde{U}^\varepsilon(t)|^2_{L^2} + \int_0^t |D^2_\varepsilon \tilde{U}^\varepsilon|^2_{L^2} \, d\tau \le K \varepsilon^\beta,$$

$$|\tilde{U}^\varepsilon(t)|^2_{L^2} + \int_0^t \big(|\nabla_{x,y} \tilde{U}^\varepsilon|^2_{L^2} + \varepsilon |\frac{\partial \tilde{U}^\varepsilon}{\partial z}|^2_{L^2} \big) \, d\tau \le K \varepsilon^\beta$$

and

$$|\nabla \bar{U}^\varepsilon(t)|^2_{L^2} + \int_0^t |\Delta \bar{U}^\varepsilon|^2_{L^2} \, d\tau \le K,$$

where $|D^2_\varepsilon f|^2_{L^2} = \int f^2_{xx} + f^2_{yy} + \varepsilon f^2_{zz} + 2 f^2_{xy} + (1+\varepsilon)(f^2_{xz} + f^2_{yz})$. Moreover, $\bar{U}^\varepsilon \to \bar{V}$ in $L^p(\mathbb{R}^+; H^1)$ strongly for all $2 \le p \le +\infty$, and in $L^2(\mathbb{R}^+; H^2)$ strongly, where $\bar{V}$ is the solution of the 2-D periodic Navier-Stokes equations:

$$\bar{V}_t + \bar{V} \cdot \nabla_{x,y} \bar{V} + \nabla \bar{q} - \Delta_{x,y} \bar{V} = 0, \quad \nabla_{x,y} . \bar{V} = 0 \text{ and } \bar{V}(0) = \bar{U}_0.$$

This result is proved in the next section (Theorem 2). Note that we also prove a theorem for weak solutions: $\bar{U}_0 \in L^2, \tilde{U}_0 \in L^2$ (Theorem 1).

For the case of physical boundary conditions, we use the same splitting, but due to the condition at $z = 1$, (1.27) and (1.30) become

$$\bar{U}_t + \bar{U} \cdot \nabla_{x,y} \bar{U} + \nabla_{x,y} \cdot (\fint \tilde{U}_2 \otimes \tilde{U}_2) + \frac{1}{\varepsilon} L\bar{U} + \frac{1}{\varepsilon} \nabla \bar{p} - \Delta_{x,y} \bar{U} = \tau, \tag{1.33}$$

and

$$\tilde{U}_t + \tilde{U} \cdot \nabla_{x,y,z} \tilde{U} - \begin{pmatrix} \nabla_{x,y} \cdot (\fint \tilde{U}_2 \otimes \tilde{U}_2) \\ 0 \end{pmatrix} + \bar{U} \cdot \nabla_{x,y} \tilde{U} + \begin{pmatrix} \tilde{U}_2 \cdot \nabla_{x,y} \bar{U} \\ 0 \end{pmatrix}$$
$$+ \frac{1}{\varepsilon} k \times \tilde{U} + \frac{1}{\varepsilon} \nabla \tilde{p} - \varepsilon \frac{\partial^2}{\partial z^2} \tilde{U} - \Delta_{x,y} \tilde{U} = \begin{pmatrix} \tau \\ 0 \end{pmatrix}. \tag{1.34}$$

We see that the averaging process introduces a source term in (1.31). The solution to this equation therefore converges formally to

$$\bar{V}_t + \bar{V}.\nabla_{x,y} \bar{V} + \nabla \bar{q} - \Delta_{x,y} \bar{V} = \tau,$$

since $\tilde{U}_2$ will be forced to be small and $\frac{1}{\varepsilon} L\bar{U}$ is a gradient because $\bar{U}$ is a 2-D divergence-free vector field.

On the other hand, there also exists a source term in (1.34); it will therefore not be possible to show directly that $\tilde{U}$ is small. We will need to introduce the boundary layer computed in the first part of this introduction. Namely, let

$$R_2^\star = e^{-(\frac{1-z}{\varepsilon\sqrt{2}})} \big(\tau \cos(\frac{1-z}{\varepsilon\sqrt{2}} + \frac{\pi}{4}) - L\tau \cos(\frac{1-z}{\varepsilon\sqrt{2}} - \frac{\pi}{4})\big),$$

and $\tilde{R}^* = \begin{pmatrix} R_2^* \\ 0 \end{pmatrix}$. The main result gives (Theorem 4):

Theorem. *Let $\bar{U}_0$ be given in $H^3(\mathbf{T}^2)$ and $\tau \in W^{4,\infty}(\mathbf{T}^2)$ such that $\nabla_{x,y} \cdot \tau = 0$. Let $\tilde{U}_0 \in L^2(\mathbf{T}^2 \times (0,1))$; then for any $\beta \leq 2$ such that $|\tilde{U}_0 - \tilde{R}^*|^2_{L^2} \leq B\varepsilon^\beta$, there exists K depending only on $|\bar{U}_0|_{H^3}$, $|\tau|_{W^{4,\infty}}$ and B such that for every $\alpha < \beta$, there exists a time $T_\varepsilon \geq \frac{\beta-\alpha}{K} \ln(\frac{1}{\varepsilon})$ and $\varepsilon_0 > 0$ such that $\forall \varepsilon \leq \varepsilon_0$, the solution of* (1.33)–(1.34) *given by the Galerkin approximation satisfies*:

$$|\bar{U}^\varepsilon - \bar{W}|^2_{L^\infty(0,T_\varepsilon;L^2) \cap L^2(0,T_\varepsilon;H^1)} \leq K\varepsilon^\alpha,$$

and $\forall T \leq T_\varepsilon$

$$|(\tilde{U}_\varepsilon - \tilde{R}^*)(T)|^2_{L^2} + \int_0^T (|\nabla_{x,y}(\tilde{U}_\varepsilon - \tilde{R}^*)|^2_{L^2} + \varepsilon |\frac{\partial}{\partial z}(\tilde{U}_\varepsilon - \tilde{R}^*)|^2_{L^2})\, dt \leq K\varepsilon^\alpha,$$

where $\bar{W}$ is the solution of

$$\bar{W}_t + \bar{W} \cdot \nabla_{x,y} \bar{W} - \Delta_{x,y} \bar{W} + \nabla \bar{q} = \tau,$$

$$\nabla_{x,y} \cdot \bar{W} = 0, \quad and \quad \bar{W}(t=0) = \bar{U}_0.$$

This result is proved in the last section. Note that in Section 3 we give the same type of theorem only assuming

$$|\tilde{U}_0|^2_{L^2} \leq \beta\varepsilon^\beta \text{ with } \beta \leq 1,$$

i.e., without "well-preparing" the initial data.

Note also that in all these theorems, there are no temporal oscillations since our strategy precisely consists in "killing" these oscillations when we impose that $|\tilde{U}_0| << 1$.

The results of this paper were announced in [5].

2. Periodic case. In this section, we deal with the three-dimensional Navier-Stokes equations with periodic boundary conditions:

$$U_t + U \cdot \nabla U + \frac{1}{\varepsilon} k \times U - \varepsilon \frac{\partial^2}{\partial z^2} U - \Delta_{x,y} U + \frac{1}{\varepsilon} \nabla p = 0, \tag{2.1}$$

$$\nabla \cdot U = 0, \tag{2.2}$$

$$U(t=0) = U_0. \tag{2.3}$$

Recall that k is the unit vertical vector and that $\Delta_{x,y}$ is the 2-D Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Let us denote, as usual, by $\mathbf{H}$ the following functions space:

$$\mathbf{H} = \left\{ V \in (L^2(\mathbf{T}^3))^3 / \nabla \cdot V = 0 \text{ and } \int_{\mathbf{T}^3} V = 0 \right\},$$

where $\mathbf{T}^3$ is the 3-dimensional torus in the x, y, z variables, and

$$\mathbf{V} = \mathbf{H} \cap (H^1(\mathbf{T}^3))^3.$$

Concerning system (2.1)–(2.3), we have the classical result ([11]):

Theorem. i) *Weak solutions*: *Let U_0 be given in $\mathbf{H}$, then there exists at least one global solution to* (2.1)–(2.3) *satisfying*

$$U \in L^\infty(\mathbb{R}^+; \mathbf{H}) \cap L^2(\mathbb{R}^+; \mathbf{V}).$$

ii) *Strong solutions*: *Suppose that $U_0 \in \mathbf{V}$; then there exists a constant C independent of ε and a time $T_\varepsilon \geq \frac{C\varepsilon^4}{|U_0|_{\mathbf{V}}}$ and a unique solution U to* (2.1)–(2.3) *belonging to* $\mathcal{C}([0,T]; \mathbf{V}) \cap L^2([0,T]; \mathbf{V} \cap H^2(\mathbf{T}^3))$ $\forall T < T_\varepsilon$.

Remark 1. i) Without loss of generality, one takes the viscosity coefficient of $\Delta_{x,y}$ equal to 1. It is clear that one could take any other constant.

ii) The Coriolis term $\frac{1}{\varepsilon}k \times U$ in (2.1) is skew-symmetric, and therefore does not contribute to the energy estimates. Therefore the limitation on T_ε comes from the vanishing vertical viscosity.

iii) In order to improve the previous result, it is sufficient to obtain a priori estimates independent of ε for ε small enough.

2.1. Splitting. As explained in the introduction, the solution U of (2.1)–(2.3) tends to become 2-dimensional. It is therefore natural to introduce the vertical average of the unknowns. For $j = 1, 2$, we denote $\bar{u}_j = \int_0^1 u_j(z)\,dz \equiv \fint u_j$, and $\bar{p} = \int_0^1 p(z)\,dz$. We then need to take $\tilde{u}_j = u_j - \bar{u}_j$ for $j = 1, 2$ and $\tilde{p} = p - \bar{p}$. Moreover, we denote by $\bar{U} = (\bar{u}_1, \bar{u}_2)$, $\tilde{U} = (\tilde{u}_1, \tilde{u}_2, u_3)$ and $\tilde{U}_2 = (\tilde{u}_1, \tilde{u}_2)$.

In this case, $\tilde{U}$ takes into account the oscillatory part of U; we are going to force $\tilde{U}$ to be small. This kind of decomposition was used by Chemin in [3] in the context of the quasigeostrophic approximation.

Averaging the two first equations of (2.1), we get the system governing the evolution of $\bar{U}$:

$$\bar{U}_t + \bar{U}\cdot\nabla_{x,y}\bar{U} + \nabla_{x,y}\cdot\left(\fint \tilde{U}_2\otimes\tilde{U}_2\right) + \frac{1}{\varepsilon}L\bar{U} + \frac{1}{\varepsilon}\nabla\bar{p} - \Delta_{x,y}\bar{U} = 0, \tag{2.4}$$

$$\nabla_{x,y}\cdot\bar{U} = 0, \tag{2.5}$$

$$\bar{U}(t=0) = \fint (u_{01}, u_{02})^t, \tag{2.6}$$

where $L\bar{U} = (-\bar{u}_2, \bar{u}_1)^t$.

Remark 2. In order to obtain these equations, we used the periodicity along the vertical axis and the facts that $\nabla_{x,y}\cdot\bar{U} = 0$, $\fint \tilde{U}_2 = 0$.

Now, by subtraction of (2.4) from (2.1), one has

$$\tilde{U}_t + \tilde{U}\cdot\nabla_{x,y,z}\tilde{U} - \begin{pmatrix}\nabla_{x,y}\cdot(\fint\tilde{U}_2\otimes\tilde{U}_2)\\ 0\end{pmatrix} + \bar{U}\cdot\nabla_{x,y}\tilde{U}$$
$$+ \begin{pmatrix}\tilde{U}_2\cdot\nabla_{x,y}\bar{U}\\ 0\end{pmatrix} + \frac{1}{\varepsilon}k\times\tilde{U} - \varepsilon\frac{\partial^2\tilde{U}}{\partial z^2} - \Delta_{x,y}\tilde{U} + \frac{1}{\varepsilon}\nabla\tilde{p} = 0. \tag{2.7}$$

$$\nabla_{x,y,z}\cdot\tilde{U} = 0, \tag{2.8}$$

$$\tilde{U}(t=0) = \tilde{U}_0. \tag{2.9}$$

As said above, we use this decomposition to perform the limit $\varepsilon \to 0$.

2.2. Weak solutions. The main result of this subsection is:

Theorem 1. *For $|\bar{U}(0)|^2_{L^2} \le \mu$ and $|\tilde{U}(0)|^2_{L^2} \le \mu\varepsilon^\alpha$, $\alpha > 0$, there exists an intrinsic constant C and $\varepsilon_0 > 0$ such that, if $C\ \mu^2 e^{C(1+\mu)}\varepsilon^{2\alpha} \le 1/2$ and $\varepsilon \le \varepsilon_0$, then any* *weak solution given by the Galerkin approximation satisfies*:

$$i) \qquad |\bar{U}_\varepsilon(t)|^2_{L^2} + \int_0^t |\nabla_{x,y}\bar{U}_\varepsilon(\tau)|^2_{L^2}\, d\tau \leq \mu(\varepsilon^\alpha + 1),$$

$$ii) \qquad |\tilde{U}_\varepsilon(t)|^2_{L^2} + \int_0^t \varepsilon|\frac{\partial \tilde{U}_\varepsilon}{\partial z}|^2_{L^2} + |\nabla_{x,y}\tilde{U}_\varepsilon|^2_{L^2} \leq \mu e^{C(1+\mu)}\varepsilon^\alpha.$$

From this result, we deduce

Corollary 1. *Let $\bar{V}$ be the solution of the 2-D Navier-Stokes equations with periodic boundary conditions in x, y variables*:

$$\bar{V}_t + \bar{V}\cdot\nabla_{x,y}\bar{V} - \Delta_{x,y}\bar{V} + \nabla\bar{q} = 0,$$
$$\nabla_{x,y}\cdot\bar{V} = 0, \quad \bar{V}(t=0) = \bar{U}_0.$$

The solution $\bar{U}_\varepsilon$ to (2.4)–(2.6) *converges, as ε goes to zero, to $\bar{V}$ in $L^p(\mathbb{R}^+; L^2)$ strongly for any p, $2 \leq p \leq +\infty$ and in $L^2(\mathbb{R}^+; H^1)$ strongly.*

Remark 3. In the sequel, all the constants which are independent of ε will be denoted by C and can change from one line to another.

Proof of Theorem 1. Since we deal with weak solutions, we would have to work on the Galerkin approximation of U. Nevertheless, we shall present the computations directly on equations (2.4)–(2.5) and (2.7)–(2.8).

Taking the L^2 inner product of (1.4) with $\bar{U}$ leads to

$$\frac{1}{2}\frac{d}{dt}|\bar{U}|^2_{L^2} + |\nabla_{x,y}\bar{U}|^2_{L^2} = -\int \big(⨍ \tilde{U}_2 \otimes \tilde{U}_2\big) : \nabla_{x,y}\bar{U}\, dx\, dy. \tag{2.10}$$

In order to estimate the right-hand side of (2.10), we need the following lemma, that we will use all along in this paper.

Lemma 1. *For any f and g given in $H^1(\mathbf{T}^3)$, one has*:

$$|⨍ fg - \int_{x,y,z} fg|_{L^2(\mathbf{T}^2)} \leq C\big(|f|_{L^2(\mathbf{T}^3)}|\nabla_{x,y}g|_{L^2(\mathbf{T}^3)} + |g|_{L^2(\mathbf{T}^3)}|\nabla_{x,y}f|_{L^2(\mathbf{T}^3)}\big). \tag{2.11}$$

Proof of Lemma 1. According to the classical Sobolev imbedding, one obtains

$$|⨍ fg - \int_{x,y,z} fg|_{L^2(\mathbf{T}^2)} \leq C|⨍ fg - \int_{x,y,z} fg|_{W^{1,1}(\mathbf{T}^2)}.$$

On the other hand, Poincaré's inequality for zero average functions of $W^{1,1}(\mathbf{T}^2)$ leads to

$$|⨍ fg - \int_{x,y,z} fg|_{L^2(\mathbf{T}^2)} \leq C|\nabla_{x,y}\big(⨍ fg - \int_{x,y,z} fg\big)|_{L^1(\mathbf{T}^2)},$$

from which (2.11) follows. □

Going back to (2.10), one has

$$\frac{1}{2}\frac{d}{dt}|\bar{U}|_{L^2}^2 + |\nabla_{x,y}\bar{U}|_{L^2}^2 = -\int\Big\{ \fint \big(\tilde{U}_2\otimes\tilde{U}_2\big) - \int_{\mathbf{T}^3}\tilde{U}_2\otimes\tilde{U}_2\Big\} : \nabla_{x,y}\bar{U}\,dx\,dy$$
$$- \int\Big(\int_{\mathbf{T}^3}\tilde{U}_2\otimes\tilde{U}_2\Big) : \nabla_{x,y}\bar{U}\,dx\,dy.$$

As this last term vanishes, one gets, according to Lemma 1,

$$\frac{1}{2}\frac{d}{dt}|\bar{U}|_{L^2}^2 + |\nabla_{x,y}\bar{U}|_{L^2}^2 \le C|\tilde{U}|_{L^2}|\nabla_{x,y}\tilde{U}|_{L^2}|\nabla_{x,y}\bar{U}|_{L^2}. \tag{2.12}$$

We now have to obtain a similar estimate for $\tilde{U}$. This is performed by taking the L^2 inner product of (2.7) with $\tilde{U}$:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\tilde{U}|_{L^2}^2 + \varepsilon\Big|\frac{\partial\tilde{U}}{\partial z}\Big|_{L^2}^2 + |\nabla_{x,y}\tilde{U}|_{L^2}^2\\ &= -\int\big(\tilde{U}_2\cdot\nabla_{x,y}\bar{U}\big)\tilde{U}_2 - \int\Big(\fint\tilde{U}_2\otimes\tilde{U}_2\Big) : \nabla_{x,y}\tilde{U}_2 - \int\big(\bar{U}\cdot\nabla_{x,y}\tilde{U}\big)\tilde{U}.\end{aligned} \tag{2.13}$$

As $\fint\tilde{U}_2 = 0$ and $\nabla_{x,y}\cdot\bar{U} = 0$, the two last terms of the right-hand side of (2.12) are zero and Lemma 1 implies

$$\frac{1}{2}\frac{d}{dt}|\tilde{U}|_{L^2}^2 + \varepsilon\Big|\frac{\partial\tilde{U}}{\partial z}\Big|_{L^2}^2 + |\nabla_{x,y}\tilde{U}|_{L^2}^2 \le C|\tilde{U}_2|_{L^2}|\nabla_{x,y}\tilde{U}_2|_{L^2}|\nabla_{x,y}\bar{U}|_{L^2}. \tag{2.14}$$

Remark 4. The main interest of Lemma 1 is to estimate the nonlinear terms in (2.12) without using the vertical derivative of $\tilde{U}$ that we would not be able to control by the left-hand side of (2.12) as ε goes to zero.

We now go on proving Theorem 1 by using a dynamical argument. We suppose

$$|\tilde{U}(t)|_{L^2}^2 \le A\varepsilon^\alpha. \tag{2.15}$$

One first absorbs the term $|\nabla_{x,y}\bar{U}|_{L^2}$ of the right-hand side of (2.12) in the left-hand side in the following way. We first apply Young's inequality to obtain

$$\frac{1}{2}\frac{d}{dt}|\bar{U}|_{L^2}^2 + |\nabla_{x,y}\bar{U}|_{L^2}^2 \le C|\tilde{U}|_{L^2}^2|\nabla_{x,y}\tilde{U}|_{L^2}^2 + \frac{1}{2}|\nabla_{x,y}\bar{U}|_{L^2}^2,$$

which yields, according to (2.15),

$$\frac{d}{dt}|\bar{U}|_{L^2}^2 + |\nabla_{x,y}\bar{U}|_{L^2}^2 \le CA\varepsilon^\alpha|\nabla_{x,y}\tilde{U}|_{L^2}^2. \tag{2.16}$$

The same manipulation on (2.14) leads to the estimate

$$\frac{d}{dt}|\tilde{U}|^2_{L^2} + \varepsilon|\frac{\partial \tilde{U}}{\partial z}|^2_{L^2} + |\nabla_{x,y}\tilde{U}|^2_{L^2} \leq CA\varepsilon^\alpha |\nabla_{x,y}\bar{U}|^2_{L^2}. \tag{2.17}$$

By integration of (2.16) from 0 to t, one gets

$$|\bar{U}(t)|^2_{L^2} + \int_0^t |\nabla_{x,y}\bar{U}|^2_{L^2}\, d\tau \leq |\bar{U}(0)|^2_{L^2} + CA\varepsilon^\alpha \int_0^t |\nabla_{x,y}\tilde{U}|^2_{L^2}\, d\tau. \tag{2.18}$$

We do the same with (2.17) to obtain

$$|\tilde{U}(t)|^2_{L^2} + \int_0^t \big(\varepsilon|\frac{\partial \tilde{U}}{\partial z}|^2_{L^2} + |\nabla_{x,y}\tilde{U}|^2_{L^2}\big)\, d\tau \leq |\tilde{U}(0)|^2_{L^2} + CA\varepsilon^\alpha \int_0^t |\nabla_{x,y}\bar{U}|^2_{L^2}\, d\tau. \tag{2.19}$$

Plugging the bound of $\int_0^t |\nabla_{x,y}\tilde{U}|^2_{L^2}\, d\tau$ given by (2.19) in equation (2.18) yields

$$\begin{aligned}|\bar{U}(t)|^2_{L^2} &+ \int_0^t |\nabla_{x,y}\bar{U}|^2_{L^2}\, d\tau \\ &\leq |\bar{U}(0)|^2_{L^2} + C^2A^2\varepsilon^{2\alpha} \int_0^t |\nabla_{x,y}\bar{U}|^2_{L^2}\, d\tau + CA\varepsilon^\alpha |\tilde{U}(0)|^2_{L^2}.\end{aligned}$$

If we suppose that

$$C^2A^2\varepsilon^{2\alpha} \leq \frac{1}{2}, \tag{2.20}$$

the above inequality gives

$$|\bar{U}(t)|^2_{L^2} + \frac{1}{2}\int_0^t |\nabla_{x,y}\bar{U}|^2_{L^2}\, d\tau \leq \frac{1}{\sqrt{2}}|\tilde{U}(0)|^2_{L^2} + |\bar{U}(0)|^2_{L^2}. \tag{2.21}$$

This is exactly the first estimate of Theorem 1.

Absorbing the term $|\nabla_{x,y}\tilde{U}|_{L^2}$ of the right-hand side of (2.14) in the left-hand side leads to

$$\frac{d}{dt}|\tilde{U}|^2_{L^2} + \varepsilon|\frac{\partial \tilde{U}}{\partial z}|^2_{L^2} + |\nabla_{x,y}\tilde{U}|^2_{L^2} \leq C|\tilde{U}|^2_{L^2}|\nabla_{x,y}\bar{U}|^2_{L^2},$$

which gives by Gronwall's lemma

$$|\tilde{U}(t)|^2_{L^2} + \int_0^t \big(\varepsilon|\frac{\partial \tilde{U}}{\partial z}|^2_{L^2} + |\nabla_{x,y}\tilde{U}|^2_{L^2}\big)\, d\tau \leq |\tilde{U}(0)|^2_{L^2} e^{C\int_0^t |\nabla_{x,y}\bar{U}|^2_{L^2}\, d\tau}.$$

Using the estimate (2.21), we finally obtain

$$|\tilde{U}(t)|^2_{L^2} + \int_0^t \big(\varepsilon|\frac{\partial \tilde{U}}{\partial z}|^2_{L^2} + |\nabla_{x,y}\tilde{U}|^2_{L^2}\big) d\tau \leq |\tilde{U}(0)|^2_{L^2} e^{C(1+\mu)}. \tag{2.22}$$

We still have to check (2.15), i.e., $\mu\varepsilon^\alpha e^{C(1+\mu)} \leq A\varepsilon^\alpha$, so that (2.15) is true as soon as $A = \mu e^{C(1+\mu)}$ and $C^2\mu^2 e^{2C(1+\mu)}\varepsilon^{2\alpha} \leq \frac{1}{2}$. This ends the proof of Theorem 1. $\square$

We now deal with Corollary 1.

We first remark that $\frac{\partial \bar{U}^\varepsilon}{\partial t}$ is bounded in $L^2(\mathbb{R}^+; \mathbf{V}_2')$, where

$$\mathbf{V}_2 = \Big\{\bar{V} \in H^1(\mathbf{T}^2), \nabla_{x,y} \cdot \bar{V} = 0,\ \int_{\mathbf{T}^2} \bar{V} = 0\Big\}.$$

Aubin's lemma therefore implies that $(\bar{U}^\varepsilon)_\varepsilon$ is compact in $L^p(0,T; L^2(\mathbf{T}^2))$ for every finite p and finite T.

Since, according to Lemma 1 and Theorem 1,

$$\int_0^T |\fint \tilde{U}_2 \otimes \tilde{U}_2|_{L^2(\mathbf{T}^2)}\, dt \leq C\varepsilon^{2\alpha},$$

and as $L\bar{U}$ is a gradient, the equation satisfied by the limit $\bar{V}$ of any subsequence $(\bar{U}^{\varepsilon_k})_k$ is

$$\begin{aligned} &\bar{V}_t + \bar{V} \cdot \nabla \bar{V} - \Delta_{x,y} \bar{V} - \nabla \bar{q} = 0, \\ &\nabla_{x,y} \cdot \bar{V} = 0, \quad \bar{V}(t=0) = \bar{U}_0. \end{aligned} \tag{2.23}$$

Moreover, since the solution of (2.23) is unique, all the sequence $(\bar{U}^\varepsilon)_\varepsilon$ converges to $\bar{V}$. In order to obtain the convergence in $L^p(\mathbb{R}^+; L^2)$ for any finite p, it is sufficient to prove it for $p = 2$ as we have an L^∞ estimate. Applying Poincaré's inequality to (2.16), one has

$$\frac{d}{dt}|\bar{U}|_{L^2}^2 + \gamma|\bar{U}|_{L^2}^2 \leq C\varepsilon^\alpha |\nabla_{x,y}\tilde{U}|_{L^2}^2,$$

which becomes after time integration

$$|\bar{U}(t)|_{L^2}^2 \leq e^{-\gamma t}|\bar{U}(0)|_{L^2}^2 + C\varepsilon^\alpha \int_0^t e^{-\gamma(t-s)}|\nabla_{x,y}\tilde{U}(s)|^2\, ds.$$

Now, Fubini's theorem implies

$$\int_T^{+\infty} |\bar{U}(t)|_{L^2}^2\, dt \leq \frac{1}{\gamma}e^{-\gamma T}|\bar{U}(0)|_{L^2}^2 + \frac{C\varepsilon^\alpha}{\gamma}\int_T^{+\infty} |\nabla_{x,y}\tilde{U}(s)|^2\, ds,$$

that is,

$$\int_T^{+\infty} |\bar{U}(t)|_{L^2}^2\, dt \leq \frac{1}{\gamma}e^{-\gamma T}|\bar{U}(0)|_{L^2}^2 + C\varepsilon^{2\alpha}.$$

On the other hand,

$$\int_T^{+\infty} |\bar{V}(t)|_{L^2}^2\, dt \leq \frac{1}{\gamma}e^{-\gamma T}|\bar{U}(0)|_{L^2}^2.$$

It is therefore clear that

$$\int_0^{+\infty} |\bar{U}^\varepsilon(t) - \bar{V}(t)|_{L^2}^2 \, dt \to 0, \quad \text{as} \quad \varepsilon \to 0,$$

thereby proving Corollary 1 for $2 \le p < +\infty$.

Now, we deal with $p = +\infty$. Let us write the equation satisfied by $\bar{W} = \bar{U} - \bar{V}$:

$$\bar{W}_t + \bar{U} \cdot \nabla \bar{W} + \bar{W} \cdot \nabla \bar{V} + \nabla_{x,y} \cdot \big(\fint \tilde{U}_2 \otimes \tilde{U}_2 \big) + \frac{1}{\varepsilon} L \bar{U} + \nabla (\frac{1}{\varepsilon} \bar{p} - \bar{q}) - \Delta_{x,y} \bar{W} = 0,$$

which gives by taking the L^2 inner product with $\bar{W}$:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\bar{W}|_{L^2}^2 + |\nabla_{x,y} \bar{W}|^2 &\le \Big| \int \bar{W} \cdot \nabla \bar{V} \bar{W} \Big| + \Big| \int \big(\fint \tilde{U}_2 \otimes \tilde{U}_2 \big) : \nabla_{x,y} \bar{W} \Big|, \\ &\le C \big(|\bar{W}|_{L^2} |\nabla_{x,y} \bar{W}|_{L^2} |\nabla_{x,y} \bar{V}|_{L^2} + |\nabla_{x,y} \bar{W}|_{L^2} |\tilde{U}_2|_{L^2} |\nabla_{x,y} \tilde{U}_2|_{L^2} \big). \end{aligned}$$

That is, absorbing the terms $|\nabla_{x,y} \bar{W}|_{L^2}$,

$$\frac{d}{dt} |\bar{W}|_{L^2}^2 + |\nabla_{x,y} \bar{W}|_{L^2}^2 \le C |\bar{W}|_{L^2}^2 |\nabla \bar{V}|_{L^2}^2 + |\tilde{U}_2|_{L^2}^2 |\nabla_{x,y} \tilde{U}_2|_{L^2}^2.$$

After integration, this gives, since $\bar{W}(0) = 0$,

$$\begin{aligned} |\bar{W}(t)|_{L^2}^2 + \int_0^t |\nabla_{x,y} \bar{W}|_{L^2}^2 &\le \int_0^t e^{C \int_s^t |\nabla_{x,y} \bar{V}|_{L^2}^2 \, d\tau} |\tilde{U}_2(s)|_{L^2}^2 |\nabla_{x,y} \tilde{U}_2(s)|_{L^2}^2 \, ds, \\ &\le A^2 \varepsilon^{2\alpha} e^C, \end{aligned}$$

since $\int_0^{+\infty} |\nabla \bar{V}|_{L^2}^2 \, d\tau \le C$ and thanks to Theorem 1. □

2.3. Strong solutions. The goal of this subsection is to extend the previous theorem to strong solutions.

Theorem 2. *Let us suppose that $\bar{U}_0$ is given in $H^1(\mathbf{T}^2)$; then there exists a constant D which depends only on $|\bar{U}_0|_{H^1(\mathbf{T}^2)}$, $\varepsilon_0 > 0$ and K such that for any $\varepsilon \le \varepsilon_0$, if $|\tilde{U}_0^\varepsilon|_{H^1}^2 \le D \varepsilon^\beta$ with $\beta \ge \frac{3}{2}$, then* (2.4)–(2.6) *and* (2.7)–(2.9) *has a unique global strong solution satisfying*

$$|\nabla_{x,y,z} \tilde{U}^\varepsilon(t)|_{L^2}^2 + \int_0^t |D_\varepsilon^2 \tilde{U}^\varepsilon|_{L^2}^2 \, d\tau \le K \varepsilon^\beta,$$

$$|\tilde{U}^\varepsilon(t)|_{L^2}^2 + \int_0^t \big(|\nabla_{x,y} \tilde{U}^\varepsilon|_{L^2}^2 + \varepsilon \big| \frac{\partial \tilde{U}^\varepsilon}{\partial z} \big|_{L^2}^2 \big) \, d\tau \le K \varepsilon^\beta$$

$$|\nabla \bar{U}^\varepsilon(t)|_{L^2}^2 + \int_0^t |\Delta \bar{U}^\varepsilon|_{L^2}^2 \, d\tau \le K,$$

where

$$|D^2_\varepsilon f|^2_{L^2} = \int \{f^2_{xx} + f^2_{yy} + \varepsilon f^2_{zz} + 2f^2_{xy} + (1+\varepsilon)(f^2_{xz} + f^2_{yz})\}.$$

Moreover, $\bar{U}^\varepsilon \to \bar{V}$ in $L^p(\mathbb{R}^+; H^1)$ strongly for all $2 \le p \le +\infty$ and in $L^2(\mathbb{R}^+; H^2)$ strongly, where $\bar{V}$ is the solution of 2-D Navier-Stokes equations introduced in Corollary 1.

Proof. We follow the same lines as for Theorem 1. We first need an estimate on $\bar{U}$ which follows from taking the L^2 inner product of (2.24) with $\Delta_{x,y}\bar{U}$. Namely,

$$\frac{1}{2}\frac{d}{dt}|\nabla \bar{U}|^2_{L^2} + |\Delta \bar{U}|^2_{L^2} = \int_{\mathbf{T}^2} \bar{U}\cdot\nabla\bar{U}\Delta\bar{U} + \int_{\mathbf{T}^2} \nabla_{x,y}\cdot\big(\fint \tilde{U}_2\otimes\tilde{U}_2\big)\Delta\bar{U}.$$

Moreover, in 2-D case with periodic boundary conditions, one has

$$\int_{\mathbf{T}^2} \bar{U}\cdot\nabla\bar{U}\Delta\bar{U} = 0;$$

see [11]. Therefore, one obtains

$$\frac{1}{2}\frac{d}{dt}|\nabla \bar{U}|^2_{L^2} + |\Delta \bar{U}|^2_{L^2} = \int \nabla_{x,y}\cdot\big(\fint \tilde{U}_2\otimes\tilde{U}_2\big)\Delta\bar{U}. \tag{2.24}$$

In order to estimate the right-hand side of (2.24), we have to deal with terms of the form $\fint(\partial_x\tilde{u}_i)\tilde{u}_j$ or $\fint(\partial_y\tilde{u}_i)\tilde{u}_j$ for $i,j = 1$ or 2. This is done using Lemma 1 in the following way:

$$\begin{aligned}\int_{\mathbf{T}^2}\fint(\partial_x\tilde{u}_i\tilde{u}_j)\Delta\bar{U} &= \int_{\mathbf{T}^2}\big(\fint \partial_x\tilde{u}_i\tilde{u}_j - \int_{\mathbf{T}^3}\partial_x\tilde{u}_i\tilde{u}_j\big)\Delta\bar{U}\\ &\le C\big(|\nabla_{x,y}\partial_x\tilde{u}_i|_{L^2}|\tilde{u}_j|_{L^2} + |\partial_x\tilde{u}_i|_{L^2}|\nabla_{x,y}\tilde{u}_j|_{L^2}\big)|\Delta\bar{U}|_{L^2}.\end{aligned}$$

Hence (2.24) becomes

$$\frac{d}{dt}|\nabla \bar{U}|^2_{L^2} + |\Delta \bar{U}|^2_{L^2} \le C\big(|\Delta_{x,y}\tilde{U}_2|^2_{L^2}|\tilde{U}_2|^2_{L^2} + |\nabla_{x,y}\tilde{U}_2|^4_{L^2}\big). \tag{2.25}$$

We now deal with $\tilde{U}$ by multiplying (2.7) by $\Delta_{x,y,z}\tilde{U}$. Let us introduce $\Delta_\varepsilon = \varepsilon\frac{\partial^2}{\partial z^2} + \Delta_{x,y}$. We emphasize that we do not multiply (2.7) by Δ_ε but by $\Delta_{x,y,z}\tilde{U}$ which allows us to obtain an estimate independent of ε of $\frac{\partial^2\tilde{U}}{\partial x\partial z}$, $\frac{\partial^2\tilde{U}}{\partial y\partial z}$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla\tilde{U}|^2_{L^2} + |D^2_\varepsilon\tilde{U}|^2_{L^2} &= \int\tilde{U}\cdot\nabla\tilde{U}\Delta_{x,y,z}\tilde{U} - \int\nabla_{x,y}\cdot\big(\fint\tilde{U}_2\otimes\tilde{U}_2\big)\Delta_{x,y,z}\tilde{U}_2\\ &\quad + \int\bar{U}\cdot\nabla_{x,y}\tilde{U}\Delta_{x,y,z}\tilde{U} + \int\tilde{U}_2\cdot\nabla_{x,y}\bar{U}\Delta_{x,y,z}\tilde{U},\end{aligned} \tag{2.26}$$

since $|D_\varepsilon^2\tilde{U}|_{L^2}^2 = \int \Delta_\varepsilon \tilde{U}\Delta_{x,y,z}\tilde{U}$. Moreover, we note that, since $\fint \Delta_{x,y,z}\tilde{U}_2 = 0$, the term $\int \nabla_{x,y} \cdot \big(\fint \tilde{U}_2 \otimes \tilde{U}_2\big)\Delta_{x,y,z}\tilde{U}_2$ vanishes. We still have to estimate the other terms of the right-hand side of (2.26):

- The term $\int \tilde{U}\cdot\nabla\tilde{U}\Delta\tilde{U}$ has to be split in the following way:

$$|\int \tilde{U}\cdot\nabla\tilde{U}\Delta\tilde{U}| \le |\tilde{U}_2|_{L^6}|\nabla_{x,y}\tilde{U}|_{L^3}|\Delta\tilde{U}|_{L^2} + |\tilde{u}_3|_{L^6}|\frac{\partial\tilde{U}}{\partial z}|_{L^3}|\Delta\tilde{U}|_{L^2} = a+b.$$

Since one has $\int_{\mathbf{T}^3}\tilde{U} = 0$, Poincaré's inequality leads to $|\tilde{u}_i|_{L^6} \le C|\nabla\tilde{u}_i|_{L^2}$ for $i=1$ to 3, and $|\nabla_{x,y}\tilde{U}|_{L^3} \le C|\nabla_{x,y}\tilde{U}|_{L^2}^{1/2}|\nabla_{x,y,z}\nabla_{x,y}\tilde{U}|_{L^2}^{1/2}$. Therefore,

$$a \le C|\nabla\tilde{U}|_{L^2}|\nabla_{x,y}\tilde{U}|_{L^2}^{1/2}|D_\varepsilon^2\tilde{U}|_{L^2}^{1/2}|\Delta_{x,y,z}\tilde{U}|_{L^2}$$

since $|\nabla_{x,y,z}\nabla_{x,y}\tilde{U}|_{L^2}$ is controlled by $|D_\varepsilon^2\tilde{U}|_{L^2}$.

Note that we kept the term $|\nabla_{x,y}\tilde{U}|_{L^2}$ which lies in $L^2(\mathbb{R}^+)$ by Theorem 1. This will be useful in the sequel in order to obtain global estimates.

Moreover, $|\Delta_{x,y,z}\tilde{U}|_{L^2} \le \frac{C}{\sqrt{\varepsilon}}|D_\varepsilon^2\tilde{U}|_{L^2}$ and therefore

$$a \le \frac{C}{\sqrt{\varepsilon}}|\nabla\tilde{U}|_{L^2}|\nabla_{x,y}\tilde{U}|_{L^2}^{1/2}|D_\varepsilon^2\tilde{U}|_{L^2}^{3/2}.$$

The same kind of manipulation for b leads to

$$b \le |\nabla\tilde{u}_3|_{L^2}^{3/2}|\nabla\frac{\partial\tilde{U}}{\partial z}|_{L^2}^{1/2}|\Delta\tilde{U}|_{L^2} \le \frac{C}{\varepsilon^{3/4}}|\nabla\tilde{u}_3|_{L^2}^{3/2}|D_\varepsilon^2\tilde{U}|_{L^2}^{3/2}.$$

We remark that $|\nabla\tilde{u}_3|_{L^2}^{3/2} \le C|\nabla_{x,y}\tilde{U}|_{L^2}^{3/2}$ since $\nabla_{x,y,z}\cdot\tilde{U} = 0$; this implies

$$b \le \frac{C}{\varepsilon^{3/4}}|\nabla_{x,y}\tilde{U}|_{L^2}^{3/2}|D_\varepsilon^2\tilde{U}|_{L^2}^{3/2}.$$

Finally, one has

$$\int \tilde{U}\cdot\nabla\tilde{U}\Delta\tilde{U}| \le \frac{C}{\varepsilon^{3/4}}|\nabla\tilde{U}|_{L^2}|\nabla_{x,y}\tilde{U}|_{L^2}^{1/2}|D_\varepsilon^2\tilde{U}|_{L^2}^{3/2}. \tag{2.27}$$

- The term $\int \bar{U}\cdot\nabla_{x,y}\tilde{U}\Delta\tilde{U}$ has also to be split in two parts:

$$\int \bar{U}\cdot\nabla_{x,y}\tilde{U}\Delta\tilde{U} = \int \bar{U}\cdot\nabla_{x,y}\tilde{U}\Delta_{x,y}\tilde{U} + \int \bar{U}\cdot\nabla_{x,y}\tilde{U}\frac{\partial^2\tilde{U}}{\partial z^2}. \tag{2.28}$$

The first term of the right-hand side of (2.28) is easily bounded by

$$|\int \bar{U}\cdot\nabla_{x,y}\tilde{U}\Delta_{x,y}\tilde{U}| \le C|\nabla\bar{U}|_{L^2}^{1/2}|\Delta\bar{U}|_{L^2}^{1/2}|\nabla\tilde{U}|_{L^2}|D_\varepsilon^2\tilde{U}|_{L^2}. \tag{2.29}$$

The second one is more difficult; we integrate by parts with respect to z in order to take into account that there is no z derivative in $\bar{U}\cdot\nabla_{x,y}\tilde{U}$:

$$|\int \bar{U}\cdot\nabla_{x,y}\tilde{U}\frac{\partial^2\tilde{U}}{\partial z^2}| = |\int \bar{U}\cdot(\nabla_{x,y}\frac{\partial\tilde{U}}{\partial z})\frac{\partial\tilde{U}}{\partial z}| \le |\bar{U}|_{L^\infty}|\nabla_{x,y}\frac{\partial\tilde{U}}{\partial z}|_{L^2}|\frac{\partial\tilde{U}}{\partial z}|_{L^2},$$
$$\le C|\nabla\bar{U}|_{L^2}^{1/2}|\Delta\bar{U}|_{L^2}^{1/2}|\nabla\tilde{U}|_{L^2}|D_\varepsilon^2\tilde{U}|_{L^2},$$

which yields with (2.28) and (2.29)

$$|\int \bar{U}\cdot\nabla_{x,y}\tilde{U}\Delta\tilde{U}| \le C|\nabla\bar{U}|_{L^2}^{1/2}|\Delta\bar{U}|_{L^2}^{1/2}|\nabla\tilde{U}|_{L^2}|D_\varepsilon^2\tilde{U}|_{L^2}. \tag{2.30}$$

- The term

$$\int \tilde{U}_2\cdot\nabla_{x,y}\bar{U}\Delta\tilde{U}_2 = \int \tilde{U}_2\cdot\nabla_{x,y}\bar{U}\Delta_{x,y}\tilde{U}_2 + \int \tilde{U}_2\cdot\nabla_{x,y}\bar{U}\frac{\partial^2\tilde{U}_2}{\partial z^2} = c + d.$$

The term c is easily controlled by

$$c \le |\Delta_{x,y}\tilde{U}|_{L^2}|\tilde{U}_2|_{L^6}|\nabla_{x,y}\bar{U}|_{L^3};$$

that is,

$$c \le |\Delta_{x,y}\tilde{U}|_{L^2}|\nabla\tilde{U}|_{L^2}|\nabla_{x,y}\bar{U}|_{L^2}^{1/2}|\Delta_{x,y}\bar{U}|_{L^2}^{1/2}. \tag{2.31}$$

- The term d has to be integrated by parts as for (2.28):

$$-\int \tilde{U}_2\cdot\nabla_{x,y}\bar{U}\frac{\partial^2\tilde{U}_2}{\partial z^2} = \int \frac{\partial\tilde{U}_2}{\partial z}\cdot\nabla_{x,y}\bar{U}\frac{\partial\tilde{U}_2}{\partial z}.$$

In order to use the fact that $|\bar{U}|_{L^\infty}$ is bounded, we integrate by parts in the x,y variables. This gives terms of the following form:

$$\int \partial_{x,z}^2\tilde{u}_i\frac{\partial\tilde{u}_j}{\partial z}\bar{U} \quad\text{and}\quad \int \partial_{y,z}^2\tilde{u}_i\frac{\partial\tilde{u}_j}{\partial z}\bar{U}$$

for $i,j = 1$ or 2; they are controlled by $|D_\varepsilon^2\tilde{U}_2|_{L^2}|\nabla\tilde{U}_2||\bar{U}|_{L^\infty}$.

Finally with (2.31), one gets

$$|\int \tilde{U}_2\cdot\nabla_{x,y}\bar{U}\Delta\tilde{U}_2| \le C|D_\varepsilon^2\tilde{U}|_{L^2}|\nabla\tilde{U}|_{L^2}|\nabla_{x,y}\bar{U}|_{L^2}^{1/2}|\Delta\bar{U}|_{L^2}^{1/2}. \tag{2.32}$$

Plugging (2.27), (2.30) and (2.32) in (2.26) yields

$$\frac{1}{2}\frac{d}{dt}|\nabla\tilde{U}|_{L^2}^2 + |D_\varepsilon^2\tilde{U}|_{L^2}^2 \le \frac{C}{\varepsilon^{3/4}}|\nabla\tilde{U}|_{L^2}|\nabla_{x,y}\tilde{U}|_{L^2}^{1/2}|D_\varepsilon^2\tilde{U}|_{L^2}^{3/2}$$
$$+ C|\nabla\bar{U}|_{L^2}^{1/2}|\Delta\bar{U}|_{L^2}^{1/2}|\nabla\tilde{U}|_{L^2}|D_\varepsilon^2\tilde{U}|_{L^2}. \tag{2.33}$$

Absorbing $|D_\varepsilon^2\tilde{U}|_{L^2}$ from the right-hand side of (2.33), one has

$$\frac{d}{dt}|\nabla\tilde{U}|_{L^2}^2 + |D_\varepsilon^2\tilde{U}|_{L^2}^2 \le \frac{C}{\varepsilon^3}|\nabla\tilde{U}|_{L^2}^4|\nabla_{x,y}\tilde{U}|_{L^2}^2 + C|\nabla\bar{U}|_{L^2}|\Delta\bar{U}|_{L^2}|\nabla\tilde{U}|_{L^2}^2. \tag{2.34}$$

We then apply the same dynamical argument as for Theorem 1, assuming that

$$|\nabla\tilde{U}|_{L^2}^2 \le B\varepsilon^\beta. \tag{2.35}$$

Plugging this bound in (2.25), one has with Poincaré's inequality applied to $\tilde{U}_2$

$$\frac{d}{dt}|\nabla\bar{U}|_{L^2}^2 + |\Delta\bar{U}|_{L^2}^2 \le C\big(|\Delta_{x,y}\tilde{U}_2|_{L^2}^2 + |\nabla_{x,y}\tilde{U}_2|_{L^2}^2\big)B\varepsilon^\beta.$$

After time integration, it follows that

$$|\nabla\bar{U}(t)|_{L^2}^2 + \int_0^t |\Delta\bar{U}|_{L^2}^2\,d\tau \le |\nabla\bar{U}(0)|_{L^2}^2 + B\varepsilon^\beta\int_0^t\big(|\Delta_{x,y}\tilde{U}_2|_{L^2}^2 + |\nabla_{x,y}\tilde{U}_2|_{L^2}^2\big)\,d\tau. \tag{2.36}$$

The same manipulation on (2.34) gives according to (2.35)

$$\begin{aligned}|\nabla\tilde{U}(t)|_{L^2}^2 &+ \int_0^t |D_\varepsilon^2\tilde{U}|_{L^2}^2\,d\tau \\ &\le |\nabla\tilde{U}(0)|_{L^2}^2 + C\varepsilon^{2\beta-3}B^2\int_0^t|\nabla_{x,y}\tilde{U}|_{L^2}^2\,d\tau + C\varepsilon^\beta B\int_0^t|\Delta\bar{U}|_{L^2}^2\,d\tau.\end{aligned}$$

Now, remark that Theorem 1 gives

$$\int_0^t |\nabla_{x,y}\tilde{U}|_{L^2}^2\,d\tau \le C\varepsilon\beta.$$

This is why we needed to keep terms in $|\nabla_{x,y}\tilde{U}|_{L^2}$ in (2.27).

Equation (2.36) becomes

$$\begin{aligned}|\nabla\bar{U}|_{L^2}^2 + \int_0^t |\Delta\bar{U}|_{L^2}^2\,d\tau &\le |\nabla\bar{U}(0)|_{L^2}^2 + BC\varepsilon^{2\beta} + B\varepsilon^\beta|\nabla\tilde{U}(0)|_{L^2}^2 \\ &\quad + CB^2\varepsilon^{2\beta}\int_0^t|\Delta\bar{U}|_{L^2}^2\,d\tau + B^3\varepsilon^{4\beta-3}C.\end{aligned} \tag{2.37}$$

Taking

$$CB^2\varepsilon^{2\beta} \le \frac{1}{2}, \tag{2.38}$$

equation (2.37) implies

$$|\nabla\bar{U}(t)|_{L^2}^2 + \frac{1}{2}\int_0^t|\Delta\bar{U}|_{L^2}^2\,d\tau \le |\nabla\bar{U}(0)|_{L^2}^2 + C, \tag{2.39}$$

as soon as $4\beta - 3 \geq 0$. Plugging (2.35) in (2.34) yields

$$\frac{d}{dt}|\nabla\tilde{U}|_{L^2}^2 + |D_\varepsilon^2\tilde{U}|_{L^2}^2 \leq C\varepsilon^{\beta-3}B|\nabla\tilde{U}|_{L^2}^2|\nabla_{x,y}\tilde{U}|_{L^2}^2 + C|\Delta\bar{U}|_{L^2}^2|\nabla\tilde{U}|_{L^2}^2.$$

By Gronwall's lemma, one has

$$\begin{aligned}|\nabla\tilde{U}|_{L^2}^2 + \int_0^t |D_\varepsilon^2\tilde{U}|_{L^2}^2\,d\tau &\leq |\nabla\tilde{U}(0)|_{L^2}^2 e^{\int_0^t C(\varepsilon^{\beta-3}B|\nabla_{x,y}\tilde{U}|_{L^2}^2 + |\Delta\bar{U}|_{L^2}^2)\,d\tau} \\ &\leq |\nabla\tilde{U}(0)|_{L^2}^2 e^{\{C\varepsilon^{2\beta-3}B + C(|\nabla\bar{U}(0)|_{L^2}^2+1)\}},\end{aligned}$$

thanks to (2.39) and Theorem 1.

Therefore, if $2\beta - 3 \geq 0$, (2.35) is fulfilled as soon as $|\nabla\tilde{U}(0)|_{L^2}^2 \leq \lambda\varepsilon^\beta$ with $\lambda e^{\{CB + C\left(|\nabla\bar{U}(0)|_{L^2}^2+1\right)\}} \leq B$, thereby proving the first part of Theorem 2. The convergence part follows exactly the same lines as for Corollary 1.

3. Boundary value problem.

3.1. Boundary value problem: first-order expansion. We now deal with equations (2.1)–(2.3) with $(x, y) \in \mathbf{T}^2$ and $z \in [0, 1]$. In all the sequel of this paper, the boundary conditions are the following:

$$\text{at } z = 1: \quad u_3 = 0 \text{ and } \frac{\partial u_1}{\partial z} = \frac{\tau_1}{\varepsilon}, \quad \frac{\partial u_2}{\partial z} = \frac{\tau_2}{\varepsilon}, \tag{3.1}$$

$$\text{at } z = 0: \quad u_3 = 0 \text{ and } \frac{\partial u_1}{\partial z} = \frac{\partial u_2}{\partial z} = 0. \tag{3.2}$$

The conditions at $z = 0$ are those considered in [8] at the bottom of the ocean. At $z = 1$, the conditions correspond to the rigid lid assumption with a stress due to the wind; see [10]. Here $\tau = (\tau_1, \tau_2)(x, y)$ is a divergence-free vector field such that $\int_{\mathbf{T}^2} \tau = 0$. For the sake of simplicity, we take τ independent of time t. The splitting introduced in the periodic case now reads with (3.1)–(3.2):

$$\begin{gathered}\bar{U}_t + \bar{U}\cdot\nabla_{x,y}\bar{U} + \nabla_{x,y}\cdot\Big(\fint \tilde{U}_2\otimes\tilde{U}_2\Big) + \frac{1}{\varepsilon}L\bar{U} + \frac{\nabla\bar{p}}{\varepsilon} - \Delta_{x,y}\bar{U} = \tau, \\ \nabla_{x,y}\cdot\bar{U} = 0, \quad \bar{U}(t=0) = \bar{U}_0.\end{gathered} \tag{3.3}$$

Subtracting (3.3) from (3.1) gives the equation for $\tilde{U}$:

$$\begin{gathered}\tilde{U}_t + \tilde{U}\cdot\nabla_{x,y,z}\tilde{U} - \begin{pmatrix}\nabla_{x,y}\cdot\big(\fint \tilde{U}_2\otimes\tilde{U}_2\big)\\ 0\end{pmatrix} + \bar{U}\cdot\nabla_{x,y}\tilde{U} + \begin{pmatrix}\tilde{U}_2.\nabla_{x,y}\bar{U}\\ 0\end{pmatrix} \\ + \frac{1}{\varepsilon}k\times\tilde{U} + \frac{\nabla\tilde{p}}{\varepsilon} - \varepsilon\frac{\partial^2\tilde{U}}{\partial z^2} - \Delta_{x,y}\tilde{U} = -\begin{pmatrix}\tau\\ 0\end{pmatrix}, \\ \nabla_{x,y,z}\cdot\tilde{U} = 0, \quad \tilde{U}(t=0) = \tilde{U}_0.\end{gathered} \tag{3.4}$$

The main result of this section is:

Theorem 3. *Let $\bar{U}_0 \in L^2(\mathbf{T}^2), \tilde{U}_0 \in L^2(\mathbf{T}^2 \times (0,1))$ and $\tau \in W^{2,\infty}(\mathbf{T}^2)$ such that $\nabla_{x,y} \cdot \tau = 0$. Let $\alpha \leq 1$ and $|\tilde{U}_0|^2_{L^2} \leq A\varepsilon^\alpha$; then there exists a constant K which depends only on $|\tau|_{W^{2,\infty}}, |\bar{U}_0|_{L^2}$ and A such that for all $\beta < \alpha$, there exists*

$$T_\varepsilon \geq \frac{\alpha - \beta}{K} \ln(\frac{1}{\varepsilon})$$

and ε_0 such that for any $\varepsilon \leq \varepsilon_0$ one has

$$|\tilde{U}^\varepsilon|_{L^\infty(0,T_\varepsilon;L^2)} \leq K\varepsilon^{\beta/2}, \quad \int_0^{T_\varepsilon} \big(|\nabla_{x,y}\tilde{U}^\varepsilon|^2_{L^2} + \varepsilon|\frac{\partial}{\partial z}\tilde{U}^\varepsilon|^2_{L^2}\big)\, dt \leq K\varepsilon^\beta \tag{3.5}$$

$$|\bar{U}^\varepsilon - \bar{W}|_{L^\infty(0,T_\varepsilon;L^2)\cap L^2(0,T_\varepsilon;H^1)} \leq K\varepsilon^{\beta/2}, \tag{3.6}$$

where $\bar{W}$ is the solution to

$$\bar{W}_t + \bar{W} \cdot \nabla_{x,y}\bar{W} - \Delta_{x,y}\bar{W} + \nabla\bar{q} = \tau, \quad \nabla_{x,y} \cdot \bar{W} = 0, \quad \bar{W}(t=0) = \bar{U}_0. \tag{3.7}$$

Remark 5. As explained in the introduction, the boundary data at $z = 1$ leads to a source term in the limit Navier-Stokes equations.

A direct energy estimate of equation (2.1) with boundary conditions (3.1)–(3.2) yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int |U|^2 + \int |\nabla_{x,y}U|^2 + \varepsilon|\frac{\partial U}{\partial z}|^2 &= \big(\int\int \tau \cdot U_2 \, dx\, dy\big)\big|_{z=1} \\ &\leq |\tau|_{L^2}\big(|U|_{L^2} + |\nabla_{x,y}U|_{L^2} + |\frac{\partial U}{\partial z}|_{L^2}\big). \end{aligned}$$

Therefore, the bound obtained from the above inequality blows-up as $\varepsilon \to 0$, even on a finite time interval $[0,T]$.

We will see during the proof that the introduction of the corrector gives some source terms whose L^2 size is of order $\sqrt{\varepsilon}$; this explains why $\alpha \leq 1$ in Theorem 3.

In the sequel ∇_ε will denote the differential operator $\big(\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \sqrt{\varepsilon}\frac{\partial}{\partial z}\big)$.

Proof. The method is the same as for the periodic case, but here it is not possible to show directly that $\tilde{U}$ is small since there is a source term in (3.4). We need therefore to introduce a corrector to eliminate this source term and to obtain homogeneous boundary data. We first search the corrector R_2^* for $\tilde{U}_2$ as the solution to

$$\frac{LR_2^*}{\varepsilon} - \varepsilon\frac{\partial^2 R_2^*}{\partial z^2} = 0, \quad \frac{\partial R_2^*}{\partial z} = \frac{\tau}{\varepsilon} \ \text{ at } z = 1. \tag{3.8}$$

One finds

$$R_2^* = e^{-\frac{(1-z)}{\varepsilon\sqrt{2}}}\big[\tau\cos(\frac{1-z}{\varepsilon\sqrt{2}} + \frac{\pi}{4}) - L\tau\cos(\frac{1-z}{\varepsilon\sqrt{2}} - \frac{\pi}{4})\big].$$

Obviously, R_2^* does not satisfy the correct boundary condition at $z = 0$, but this is not a serious problem since $\frac{\partial R_2^*}{\partial z}|_{z=0}$ is of order $\frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon\sqrt{2}}}$. We now find the third component of this corrector by imposing that it is divergence-free; one obtains

$$\varepsilon \operatorname{curl} \tau \, e^{-\frac{1-z}{\varepsilon\sqrt{2}}} \cos(\frac{1-z}{\varepsilon\sqrt{2}}) - \varepsilon \operatorname{curl} \tau.$$

It is clear that this expression does not vanish at $z = 0$ because of the term $-\varepsilon \operatorname{curl} \tau$. We replace this term by $-\varepsilon z \operatorname{curl} \tau$. The value of the third component of the corrector is then

$$r_3^* - \varepsilon z \operatorname{curl} \tau \equiv \varepsilon \operatorname{curl} \tau \, e^{-\frac{(1-z)}{\varepsilon\sqrt{2}}} \cos(\frac{1-z}{\varepsilon\sqrt{2}}) - \varepsilon z \operatorname{curl} \tau.$$

It is now clear that $\big(R_2^*, r_3^* - \varepsilon z \operatorname{curl} \tau\big)$ is not divergence-free; we therefore add to R_2^* the quantity $\varepsilon L\tau$ and the principal part of the corrector becomes

$$\begin{pmatrix} \varepsilon L\tau + R_2^* \\ -\varepsilon z \operatorname{curl} \tau + r_3^* \end{pmatrix}.$$

We still have to fix up the boundary value at $z = 0$ and we moreover impose that the complete corrector has a vanishing mean value in z. More precisely, one obtains

$$\tilde{U} = Q + F \equiv \begin{pmatrix} Q_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} \varepsilon L\tau + R_2^* + \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon\sqrt{2}}} R_2 \\ -\varepsilon z \operatorname{curl} \tau + r_3^* + \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon\sqrt{2}}} r_3 \end{pmatrix}, \tag{3.9}$$

with

$$Q_2 = (q_1, q_2), \quad R_2 = (r_1, r_2), \quad R_2^* = (r_1^*, r_2^*), \quad F = (F_2, f_3),$$

where the different terms are given by

$$\begin{aligned} R_2 &= \varepsilon^2 L\tau \cos(\frac{1}{\varepsilon\sqrt{2}}) \frac{\partial \varphi_2}{\partial z} - \varepsilon\tau \sin(\frac{1}{\varepsilon\sqrt{2}}) \varphi_3(z) \\ &\quad + \big(\sin(\frac{1}{\varepsilon\sqrt{2}}) L\tau - \cos(\frac{1}{\varepsilon\sqrt{2}}) \tau\big) \varphi_1(z), \\ r_3 &= -\big(\varepsilon^2 \operatorname{curl} \tau \cos(\frac{1}{\varepsilon\sqrt{2}}) \varphi_2(z) - \operatorname{curl} \tau \sin(\frac{1}{\varepsilon\sqrt{2}}) \int_z^1 \varphi_1(s)\, ds\big), \\ R_2^* &= e^{-\frac{(1-z)}{\varepsilon\sqrt{2}}} \big(\tau \cos\big(\frac{1-z}{\varepsilon\sqrt{2}} + \frac{\pi}{4}\big) - L\tau \cos\big(\frac{1-z}{\varepsilon\sqrt{2}} - \frac{\pi}{4}\big)\big), \\ r_3^* &= \varepsilon e^{-\frac{(1-z)}{\varepsilon\sqrt{2}}} \operatorname{curl} \tau \cos(\frac{1-z}{\varepsilon\sqrt{2}}). \end{aligned}$$

The functions $\varphi_1, \varphi_2, \varphi_3$ are smooth and satisfy

- The support of φ_1 is included in $[0, 1/2]$,

$$\frac{\partial \varphi_1}{\partial z}(z=0) = 1 \quad \text{and} \quad \int_0^1 \varphi_1(z)\, dz = 0.$$

- The support of φ_2 is included in $[0, 1/2], \varphi_2 \equiv 1$ on $[0, \frac{1}{8}]$.
- The support of φ_3 is included in $[\frac{1}{4}, \frac{3}{4}]$, and $\int_0^1 \varphi_3(z)\, dz = 1$.

Note that one has

$$\nabla_{x,y,z} \cdot (\varepsilon L\tau, -\varepsilon z \operatorname{curl} \tau) = 0,$$
$$\nabla_{x,y,z} \cdot R = 0, \quad \nabla_{x,y,z} \cdot R^* = 0 \quad \text{and} \quad ⨍ F_2 = 0,$$
$$\text{at} \quad z = 0, \quad \frac{\partial F_2}{\partial z} = 0, \quad f_3 = 0,$$
$$\text{at} \quad z = 1, \quad \frac{\partial F_2}{\partial z} = \frac{\tau}{\varepsilon}, \quad f_3 = 0.$$

The conditions obtained on Q are therefore

$$\left.\begin{array}{l} \text{at} \quad z = 0 : \dfrac{\partial Q_2}{\partial z} = 0, \quad q_3 = 0, \\ \text{at} \quad z = 1 : \dfrac{\partial Q_2}{\partial z} = 0, \quad q_3 = 0, \\ \nabla \cdot Q = 0, ⨍ Q_2 = 0. \end{array}\right\} \tag{3.10}$$

Note that R^* is just the computation of the Ekman boundary layer (see [10]). The term $(-\varepsilon z \operatorname{curl} \tau)|_{z=1}$ is called the Ekman pumping. We now write the equations satisfied by $\bar{U}$ and Q.

Equation (3.3) becomes with (3.9)

$$\begin{aligned} &\bar{U}_t + \bar{U} \cdot \nabla_{x,y} \bar{U} + \nabla_{x,y} \cdot ⨍ Q_2 \otimes Q_2 + \nabla_{x,y} \cdot ⨍ Q_2 \otimes F_2 \\ &+ \nabla_{x,y} \cdot ⨍ F_2 \otimes Q_2 + \nabla_{x,y} \cdot ⨍ F_2 \otimes F_2 + \frac{1}{\varepsilon} L\bar{U} - \Delta_{x,y} \bar{U} + \frac{1}{\varepsilon} \nabla \bar{p} = \tau, \end{aligned} \tag{3.11}$$

while equation (3.4) leads to

$$\begin{aligned} Q_t &+ Q \cdot \nabla Q - \Delta_{x,y} Q - \varepsilon \frac{\partial^2 Q}{\partial z^2} + \frac{1}{\varepsilon} \nabla \tilde{p} = -F \cdot \nabla Q - Q \cdot \nabla F - F \cdot \nabla F \\ &- \bar{U} \cdot \nabla_{x,y} Q - \bar{U} \cdot \nabla_{x,y} F - \begin{pmatrix} Q_2 \cdot \nabla_{x,y} \bar{U} \\ 0 \end{pmatrix} - \begin{pmatrix} F_2 \cdot \nabla_{x,y} \bar{U} \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} \nabla_{x,y} \cdot (⨍ \tilde{U}_2 \otimes \tilde{U}_2) \\ 0 \end{pmatrix} - \frac{1}{\varepsilon} \begin{pmatrix} LQ_2 \\ 0 \end{pmatrix} - \begin{pmatrix} L^2 \tau \\ 0 \end{pmatrix} - \frac{1}{\varepsilon^2} e^{-\frac{1}{\varepsilon\sqrt{2}}} \begin{pmatrix} LR_2 \\ 0 \end{pmatrix} \\ &- \frac{1}{\varepsilon} \begin{pmatrix} LR_2^* \\ 0 \end{pmatrix} + e^{\frac{-1}{\varepsilon\sqrt{2}}} \frac{\partial^2 R}{\partial z^2} + \varepsilon \frac{\partial^2}{\partial z^2} \begin{pmatrix} R_2^* \\ r_3^* \end{pmatrix} - \begin{pmatrix} \tau \\ 0 \end{pmatrix} + \Delta_{x,y} F. \end{aligned} \tag{3.12}$$

Let us note that $L^2 = -\mathrm{Id}$, therefore the source terms in (3.12) vanish and thanks to (3.8) one has

$$-\frac{LR_2^*}{\varepsilon} + \varepsilon \frac{\partial^2 {R_2}^*}{\partial z^2} = 0.$$

Multiplying (3.12) by Q and integrating yields with $⨍ Q_2 = 0$, $\nabla_{x,y} \cdot \bar{U} = 0$ and $\nabla_{x,y,z} \cdot F = 0$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|Q|_{L^2}^2 + |\nabla_{x,y} Q|_{L^2}^2 + \varepsilon |\frac{\partial Q}{\partial z}|_{L^2}^2 &= -\int Q \cdot \nabla F Q - \int F \cdot \nabla F Q \\ - \int \bar{U} \cdot \nabla_{x,y} F Q - \int Q_2 \cdot \nabla_{x,y} \bar{U} Q_2 &- \int F \cdot \nabla_{x,y} \bar{U} Q_2 \\ -\frac{1}{\varepsilon^2} e^{-\frac{1}{\varepsilon\sqrt{2}}} \int L R_2 Q_2 + e^{-\frac{1}{\varepsilon\sqrt{2}}} \int \frac{\partial^2 R}{\partial z^2} Q &+ \varepsilon \int \frac{\partial^2}{\partial z^2} r_3^* q_3 + \int \Delta_{x,y} F Q. \end{aligned} \tag{3.13}$$

Let us now estimate each term of the right-hand side of (3.13).

• The term $\int Q \cdot \nabla F Q$ has to be split in the following way:

$$\begin{aligned} \int Q \cdot \nabla F Q &= \int Q_2 \cdot \nabla_{x,y} F_2 Q_2 + \int q_3 \frac{\partial F_2}{\partial z} Q_2 + \int Q \cdot \nabla_{x,y,z} f_3 q_3 \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{3.14}$$

* We easily have

$$|I_1| + |I_3| \le |Q_2|_{L^2}^2 |\nabla_{x,y} F_2|_{L^\infty} + |Q|_{L^2}^2 |\nabla_{x,y,z} f_3|_{L^\infty} \le C|Q|_{L^2}^2, \tag{3.15}$$

by the explicit value of F.

* For I_2, it is more subtle: we first integrate by parts with respect to z:

$$I_2 = -\int \frac{\partial q_3}{\partial z}(F_2 - \varepsilon L\tau) Q_2 - \int q_3 (F_2 - \varepsilon L\tau) \frac{\partial Q_2}{\partial z} = J_1 + J_2. \tag{3.16}$$

Then

$$|J_1| \le |\frac{\partial q_3}{\partial z}|_{L^2} |(F_2 - \varepsilon L\tau)|_{L^\infty} |Q_2|_{L^2} \le C |\nabla_{x,y} Q|_{L^2} |Q|_{L^2}, \tag{3.17}$$

since $\nabla_{x,y,z} \cdot Q = 0$. On the other hand

$$J_2 = -\int\int\int_0^1 (F_2 - \varepsilon L\tau) \frac{\partial Q_2}{\partial z} \int_z^1 \frac{\partial q_3}{\partial z}(s)\, ds\, dz\, dx\, dy,$$

where we have used that $q_3(z=1) = 0$. Fubini's theorem then yields

$$J_2 = -\int\int\int_0^1 \frac{\partial q_3}{\partial z}(s) \Big(\int_0^s (F_2 - \varepsilon L\tau) \frac{\partial Q_2}{\partial z}\, dz \Big)\, ds\, dx\, dy.$$

We notice that

$$|\int_0^s (F_2 - \varepsilon L\tau)\frac{\partial Q_2}{\partial z}\,dz| \le \big(\int_0^s (F_2 - \varepsilon L\tau)^2\,dz\big)^{1/2}\big(\int_0^1 (\frac{\partial Q_2}{\partial z})^2\,dz\big)^{1/2}.$$

By the exact expression of F_2, one has

$$\big(\int_0^s (F_2 - \varepsilon L\tau)^2\,dz\big)^{1/2} \le C\sqrt{\varepsilon}(e^{\frac{-\sqrt{2}}{\varepsilon}(1-s)} - e^{-\frac{\sqrt{2}}{\varepsilon}})^{1/2},$$

therefore

$$|J_2| \le C\int\int\int_0^1 |\frac{\partial q_3}{\partial z}(s)|\sqrt{\varepsilon}\big(e^{\frac{-\sqrt{2}}{\varepsilon}(1-s)} - e^{\frac{-\sqrt{2}}{\varepsilon}}\big)^{1/2}\big(\int_0^1 (\frac{\partial Q_2}{\partial z})^2\,dz\big)^{1/2}dx\,dy\,ds.$$

Cauchy-Schwarz's inequality for $\int_0^1 ds$ gives

$$\begin{aligned}|J_2| &\le C\sqrt{\varepsilon}\int\int\Big\{\big(\int_0^1 |\frac{\partial q_3}{\partial z}|^2\,ds\big)^{1/2}\big(\int_0^1 (e^{-\frac{\sqrt{2}}{\varepsilon}(1-s)} - e^{-\frac{\sqrt{2}}{\varepsilon}})\,ds\big)^{1/2} \\ &\quad \times \big(\int_0^1 (\frac{\partial Q_2}{\partial z})^2\,dz\big)^{1/2}\Big\}dx\,dy \\ &\le C\varepsilon|\frac{\partial q_3}{\partial z}|_{L^2}|\frac{\partial Q_2}{\partial z}|_{L^2} \le C\sqrt{\varepsilon}|\nabla_{x,y}Q|_{L^2}|\nabla_\varepsilon Q|_{L^2},\end{aligned} \tag{3.18}$$

since

$$\frac{\partial q_3}{\partial z} = -\nabla_{x,y}\cdot Q_2.$$

Plugging (3.17) and (3.18) in (3.16) gives

$$|I_2| \le C|\nabla_{x,y}Q|_{L^2}|Q|_{L^2} + C\sqrt{\varepsilon}|\nabla_{x,y}Q|_{L^2}|\nabla_\varepsilon Q|_{L^2},$$

which leads to, with (3.15),

$$|\int Q\cdot\nabla FQ| \le C\big(|Q|_{L^2}^2 + |\nabla_{x,y}Q|_{L^2}|Q|_{L^2} + \sqrt{\varepsilon}|\nabla_{x,y}Q|_{L^2}|\nabla_\varepsilon Q|_{L^2}\big). \tag{3.19}$$

• The term

$$\begin{aligned}\int F\cdot\nabla FQ &= \int F_2\cdot\nabla_{x,y}FQ + \int f_3\frac{\partial F}{\partial z}Q \\ &\le \big(|F_2|_{L^\infty}|\nabla_{x,y}F|_{L^2}|Q|_{L^2} + |f_3|_{L^\infty}|\frac{\partial F}{\partial z}|_{L^2}|Q|_{L^2}\big) \le C\sqrt{\varepsilon}|Q|_{L^2},\end{aligned} \tag{3.20}$$

since $|F_2|_{L^\infty} \le C$, $|\nabla_{x,y}F|_{L^2} \le C\sqrt{\varepsilon}$, $|f_3|_{L^\infty} \le C\varepsilon$, $|\frac{\partial F}{\partial z}|_{L^2} \le \frac{C}{\sqrt{\varepsilon}}$.

• The term $\int \bar{U}\cdot\nabla_{x,y}FQ$ is bounded by

$$|\int \bar{U}\cdot\nabla_{x,y}FQ| \le C|\bar{U}|_{L^2}|\fint \nabla_{x,y}FQ|_{W^{1,1}(\mathbf{T}^2)},$$

since $\bar{U}$ does not depend on z. Therefore,

$$\begin{aligned} &|\int \bar{U}\cdot\nabla_{x,y}FQ| \\ &\le C|\bar{U}|_{L^2}\big(|\nabla_{x,y}F|_{L^2}|Q|_{L^2} + |D^2_{x,y}F|_{L^2}|Q|_{L^2} + |\nabla_{x,y}F|_{L^2}|\nabla_{x,y}Q|_{L^2}\big) \\ &\le C|\bar{U}|_{L^2}\sqrt{\varepsilon}\big(|Q|_{L^2} + |\nabla_{x,y}Q|_{L^2}\big), \end{aligned} \tag{3.21}$$

since $|\nabla_{x,y}F|_{L^2}, |D^2_{x,y}F|_{L^2} \le C\sqrt{\varepsilon}$.

• The term $\int Q_2\cdot\nabla_{x,y}\bar{U}Q_2$ is controlled as in the periodic case by

$$|\int Q_2\cdot\nabla_{x,y}\bar{U}Q_2| \le C|\nabla_{x,y}\bar{U}|_{L^2}|Q|_{L^2}|\nabla_{x,y}Q|_{L^2}. \tag{3.22}$$

• The term $\int F\cdot\nabla_{x,y}\bar{U}Q_2$ is treated as in (3.21):

$$|\int F\nabla_{x,y}\bar{U}Q_2| \le C|\nabla_{x,y}\bar{U}|_{L^2}\sqrt{\varepsilon}\big(|Q|_{L^2} + |\nabla_{x,y}Q|_{L^2}\big). \tag{3.23}$$

• The following term is easy:

$$|\frac{-1}{\varepsilon^2}e^{\frac{-1}{\varepsilon\sqrt{2}}}\int LR_2Q_2 + e^{\frac{-1}{\varepsilon\sqrt{2}}}\int \frac{\partial^2 R}{\partial z^2}Q| \le C|Q|_{L^2}\varepsilon^2. \tag{3.24}$$

• The term $\varepsilon\int \frac{\partial^2 r_3^*}{\partial z^2}q_3$ is first integrated by parts:

$$\varepsilon|\int \frac{\partial^2 r_3^*}{\partial z^2}q_3| = \varepsilon|\int \frac{\partial r_3^*}{\partial z}\frac{\partial q_3}{\partial z}| \le \varepsilon|\frac{\partial r_3^*}{\partial z}|_{L^2}|\frac{\partial q_3}{\partial z}|_{L^2} \le \varepsilon|\nabla_{x,y}R_2^*|_{L^2}|\nabla_{x,y}Q_2|_{L^2},$$

since $\nabla_{x,y,z}\cdot R^* = 0$ and $\nabla_{x,y,z}\cdot Q = 0$, therefore

$$|\varepsilon\int \frac{\partial^2 r_3^*}{\partial z^2}q_3| \le \varepsilon^{3/2}C|\nabla_{x,y}Q|_{L^2}. \tag{3.25}$$

• The last term

$$\int \nabla_{x,y}FQ \ \text{ is bounded by } \ \sqrt{\varepsilon}|Q|_{L^2}. \tag{3.26}$$

Plugging (3.19), (3.20), (3.21), (3.26) in (3.13) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|Q|^2_{L^2} + |\nabla_{x,y}Q|^2_{L^2} + \varepsilon|\frac{\partial Q}{\partial z}|^2_{L^2} \\
&\leq C\big\{|Q|^2_{L^2} + |\nabla_{x,y}Q|_{L^2}|Q|_{L^2} + \sqrt{\varepsilon}|\nabla_{x,y}Q|_{L^2}|\nabla_\varepsilon Q|_{L^2} + \sqrt{\varepsilon}|Q|_{L^2} \\
&\quad +\sqrt{\varepsilon}\big(|\bar U|_{L^2} + |\nabla_{x,y}\bar U|_{L^2}\big)\big(|Q|_{L^2} + |\nabla_{x,y}Q|_{L^2}\big) \\
&\quad +|\nabla_{x,y}\bar U|_{L^2}|Q|_{L^2}|\nabla_{x,y}Q|_{L^2} + \varepsilon^{3/2}|\nabla_{x,y}Q|_{L^2}\big\}.
\end{aligned} \tag{3.27}
$$

Absorbing the terms $|\nabla_\varepsilon Q|_{L^2}$ of the right-hand side of (3.27) into the left-hand side, one gets

$$
\frac{d}{dt}|Q|^2_{L^2} + |\nabla_\varepsilon Q|^2_{L^2} \leq C\{|Q|^2_{L^2} + \varepsilon|\nabla\bar U|^2_{L^2} + |\nabla\bar U|^2_{L^2}|Q|^2_{L^2} + \varepsilon\}. \tag{3.28}
$$

In order to estimate $\bar U$, we take the L^2 inner product of (3.11) with $\bar U$ to get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\bar U|^2_{L^2} + |\nabla_{x,y}\bar U|^2_{L^2} &\leq C|\nabla_{x,y}\bar U|_{L^2}\Big(\Big|⨍ Q_2\otimes Q_2 - \int_{\mathbf{T}^3} Q_2\otimes Q_2\Big|_{W^{1,1}} \\
&+\Big|⨍ Q_2\otimes F_2 - \int_{\mathbf{T}^3} Q_2\otimes F_2\Big|_{W^{1,1}} + \Big|⨍ F_2\otimes Q_2 - \int_{\mathbf{T}^3} F_2\otimes Q_2\Big|_{W^{1,1}} \\
&+\Big|⨍ F_2\otimes F_2 - \int_{\mathbf{T}^3} F_2\otimes F_2\Big|_{W^{1,1}}\Big) + |\tau|_{L^2}|\bar U|_{L^2};
\end{aligned}
$$

as in the periodic case, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\bar U|^2_{L^2} + |\nabla\bar U|^2_{L^2} \\
&\leq C|\nabla_{x,y}\bar U|_{L^2}\big(|Q_2|_{L^2}|\nabla_{x,y}Q_2|_{L^2} + \sqrt{\varepsilon}(|Q_2|_{L^2} + |\nabla_{x,y}Q_2|_{L^2}) + \varepsilon\big) + |\tau|_{L^2}|\bar U|_{L^2},
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\frac{d}{dt}|\bar U|^2_{L^2} + |\nabla\bar U|^2_{L^2} \\
&\leq C\big\{|Q_2|^2_{L^2}|\nabla_{x,y}Q_2|^2_{L^2} + \varepsilon\big(|Q_2|^2_{L^2} + |\nabla_{x,y}Q_2|^2_{L^2}\big) + \varepsilon^2 + |\tau|^2\big\}.
\end{aligned} \tag{3.29}
$$

As in the previous section, we use a dynamical argument by supposing that

$$
|Q|^2_{L^2} \leq B\varepsilon^\beta. \tag{3.30}
$$

Plugging this estimate in (3.29), one gets

$$
\begin{aligned}
|\bar U|^2_{L^2} + \int_0^t |\nabla\bar U|^2_{L^2} &\leq |\bar U(0)|^2_{L^2} + C(B\varepsilon^\beta + \varepsilon)\int_0^t |\nabla_{x,y}Q_2|^2_{L^2} \\
&+t\big(\varepsilon^2 + |\tau|^2 + B\varepsilon^{\beta+1}\big).
\end{aligned} \tag{3.31}
$$

Integrating (3.28) using (3.30) yields

$$|Q|_{L^2}^2 + \int_0^t |\nabla_\varepsilon Q|_{L^2}^2 \leq |Q(0)|_{L^2}^2 + C\{(\varepsilon + B\varepsilon^\beta)t + (\varepsilon + B\varepsilon^\beta)\int_0^t |\nabla \bar{U}|_{L^2}^2\}.$$

Plugging this last estimate in (3.31), one obtains

$$|\bar{U}|_{L^2}^2 + \int_0^t |\nabla \bar{U}|_{L^2}^2 \leq |\bar{U}(0)|_{L^2}^2 + t(\varepsilon^2 + |\tau|^2 + B\varepsilon^{\beta+1}) + C^2(B\varepsilon^\beta + \varepsilon)^2 t$$
$$+C^2(\varepsilon + B\varepsilon^\beta)^2 \int_0^t |\nabla \bar{U}|_{L^2}^2 + C(B\varepsilon^\beta + \varepsilon)|Q(0)|_{L^2}^2.$$

We now choose ε such that $C^2(\varepsilon + B\varepsilon^\beta)^2 \leq \frac{1}{2}$ in order to get

$$|\bar{U}|_{L^2}^2 + \frac{1}{2}\int_0^t |\nabla \bar{U}|_{L^2}^2 \leq |\bar{U}(0)|_{L^2}^2 + Ct + C|Q(0)|_{L^2}^2. \tag{3.32}$$

The Gronwall lemma applied to (3.28) yields

$$|Q(t)|_{L^2}^2 + \int_0^t |\nabla_\varepsilon Q|_{L^2}^2 \, d\tau$$
$$\leq |Q(0)|_{L^2}^2 e^{C\int_0^t (1+|\nabla \bar{U}|_{L^2}^2)\, d\tau} + C\varepsilon \int_0^t e^{C\int_s^t (1+|\nabla \bar{U}|_{L^2}^2)\, d\tau}(1 + |\nabla \bar{U}|_{L^2}^2)\, ds;$$

according to (3.32), we get

$$|Q(t)|_{L^2}^2 + \int_0^t |\nabla_\varepsilon Q|_{L^2}^2 \, d\tau$$
$$\leq |Q(0)|_{L^2}^2 e^{C(t+|\bar{U}_0|_{L^2}^2+|Q(0)|_{L^2}^2)} + C\varepsilon e^{C(t+|\bar{U}_0|_{L^2}^2+|Q(0)|_{L^2}^2)}(t + |\bar{U}_0|_{L^2}^2 + |Q(0)|_{L^2}^2).$$

This last expression is simplified in the following way:

$$|Q(t)|_{L^2}^2 + \int_0^t |\nabla_\varepsilon Q|_{L^2}^2 d\tau \leq |Q(0)|_{L^2}^2 e^{C(1+t)} + \varepsilon e^{C(1+t)}(1+t),$$

that is (possibly changing the value of C in $e^{C(1+t)}$),

$$|Q(t)|_{L^2}^2 + \int_0^t |\nabla_\varepsilon Q|_{L^2}^2 \, d\tau \leq (|Q(0)|_{L^2}^2 + \varepsilon)e^{C'(1+t)},$$

where this constant C' depends on $|\bar{U}_0|_{L^2}$.

We assume that $|Q(0)|_{L^2}^2 \leq A\varepsilon^\alpha$, with $\beta < \alpha \leq 1$, and (3.30) is satisfied on $[0, T_\varepsilon]$ as soon as $T_\varepsilon = \frac{\alpha-\beta}{C} \ln \frac{1}{\varepsilon}$. In order to obtain (3.6), we subtract (3.7) from (3.11):

$$\frac{\partial}{\partial t}(\bar{U} - \bar{W}) - \Delta(\bar{U} - \bar{W}) + (\bar{U} - \bar{W}) \cdot \nabla \bar{W} + \bar{U} \cdot \nabla(\bar{U} - \bar{W})$$
$$+\nabla(\frac{\bar{p}}{\varepsilon} - \bar{q}) + \frac{L\bar{U}}{\varepsilon} = -\nabla_{x,y} \cdot \big(\fint \tilde{U}_2 \otimes \tilde{U}_2\big).$$

Taking the L^2-inner product with $(\bar{U} - \bar{W})$ yields

$$\frac{1}{2}\frac{d}{dt}|\bar{U} - \bar{W}|_{L^2}^2 + |\nabla(\bar{U} - \bar{W})|_{L^2}^2$$
$$\leq \int (\bar{U} - \bar{W})^2 |\nabla \bar{W}| + |\int \big(\fint \tilde{U}_2 \otimes \tilde{U}_2\big) : \nabla(\bar{U} - \bar{W})|,$$

since $L\bar{U}$ is a gradient.

As one has

$$|\fint \tilde{U}_2 \otimes \tilde{U}_2 - \int_{\mathbf{T}^3} \tilde{U}_2 \otimes \tilde{U}_2|_{L^2(\mathbf{T}^2)} \leq C|\tilde{U}_2|_{L^2} |\nabla_{x,y}\tilde{U}_2|_{L^2},$$

one gets

$$\frac{1}{2}\frac{d}{dt}|\bar{U} - \bar{W}|_{L^2}^2 + |\nabla(\bar{U} - \bar{W})|_{L^2}^2$$
$$\leq C|\nabla \bar{W}|_{L^2} |\bar{U} - \bar{W}|_{L^2} |\nabla(\bar{U} - \bar{W})|_{L^2} + C|\tilde{U}_2|_{L^2} |\nabla_{x,y}\tilde{U}_2|_{L^2} |\nabla(\bar{U} - \bar{W})|_{L^2};$$

absorbing the terms $|\nabla(\bar{U} - \bar{W})|_{L^2}$, one obtains

$$\frac{d}{dt}|\bar{U} - \bar{W}|_{L^2}^2 + |\nabla(\bar{U} - \bar{W})|_{L^2}^2 \tag{3.33}$$
$$\leq C|\nabla \bar{W}|_{L^2}^2 |\bar{U} - \bar{W}|_{L^2}^2 + C|\tilde{U}_2|_{L^2}^2 |\nabla_{x,y}\tilde{U}_2|_{L^2}^2.$$

Applying Gronwall's lemma to this inequality gives

$$|\bar{U} - \bar{W}|_{L^2}^2 + \int_0^t |\nabla(\bar{U} - \bar{W})|_{L^2}^2 \, d\tau \leq C \int_0^t e^{C\int_s^t |\nabla \bar{W}|_{L^2}^2 \, d\tau} |\tilde{U}_2|_{L^2}^2 |\nabla_{x,y}\tilde{U}_2|_{L^2}^2 \, ds,$$
$$\leq Ce^{C\int_0^t |\nabla \bar{W}|_{L^2}^2 \, d\tau} B\varepsilon^\beta \int_0^t |\nabla_{x,y}\tilde{U}_2|_{L^2}^2 \, ds, \leq Ce^{C\int_0^t |\nabla \bar{W}|_{L^2}^2 d\tau} B^2 \varepsilon^{2\beta},$$

as long as $t \leq T_\varepsilon$. We finally obtain

$$|\bar{U} - \bar{W}|_{L^2}^2 + \int_0^t |\nabla(\bar{U} - \bar{W})|_{L^2}^2 d\tau \leq Ce^{Ct} B^2 \varepsilon^{2\beta}.$$

Let us now recall that $T_\varepsilon = \frac{\alpha-\beta}{C'} \ln \frac{1}{\varepsilon}$, one gets, possibly increasing the value of C',

$$|\bar{U} - \bar{W}|_{L^2}^2 + \int_0^t |\nabla(\bar{U} - \bar{W})|_{L^2}^2 d\tau \leq CB\varepsilon^\beta,$$

which ends the proof of Theorem 3.

Remark 6. As noticed in Remark 5, we cannot improve the estimates (3.5) and (3.6) since the source terms are of order $\sqrt{\varepsilon}$. In order to obtain a precise result, we have to introduce the corrector at a higher order; this is the aim of the next section.

3.2. Boundary value problem: second-order expansion. The aim of this section is to make precise the asymptotic expansion given in the previous part. Indeed we show, denoting by $\tilde{R}^* = \begin{pmatrix} R_2^* \\ 0 \end{pmatrix}$,

Theorem 4. *Let $\bar{U}_0$ be given in $H^3(\mathbf{T}^2)$ and $\bar{U}_0 \in L^2(\mathbf{T}^2 \times (0;1))$. For $\tau \in W^{4,\infty}(\mathbf{T}^2)$ such that $\nabla \cdot \tau = 0$, for any $\beta \leq 2$ such that $|\tilde{U}_0 - \tilde{R}^*|_{L^2}^2 \leq B\varepsilon^\beta$, there exists a constant κ depending only on $|\bar{U}_0|_{H^3}, |\tau|_{W^{4,\infty}}$ and B such that for every $\alpha < \beta$, there exists a time $T_\varepsilon \geq \frac{\beta-\alpha}{\kappa} \ln(\frac{1}{\varepsilon})$ and $\varepsilon_0 > 0$ such that $\forall \varepsilon \leq \varepsilon_0$, the solution to* (3.3)–(3.4) *given by the Galerkin approximation satisfies*

$$|\bar{U}^\varepsilon - \bar{W}|_{L^\infty(0,T_\varepsilon;L^2) \cap L^2(0,T_\varepsilon;H^1)}^2 \leq \kappa\varepsilon^\alpha,$$

and $\forall T \leq T_\varepsilon$

$$|(\tilde{U}_\varepsilon - \tilde{R}^*)(T)|_{L^2}^2 + \int_0^T |\nabla_{x,y}(\tilde{U}_\varepsilon - \tilde{R}^*)|_{L^2}^2 + \varepsilon |\frac{\partial}{\partial z}(\tilde{U}_\varepsilon - \tilde{R}^*)|_{L^2}^2 \, dt \leq \kappa\varepsilon^\alpha,$$

where $\bar{W}$ is the solution to (3.7).

Remark 7. This result enables us to obtain a better rate of convergence for $\bar{U}$: $|\bar{U}^\varepsilon - \bar{W}| \leq \kappa\varepsilon^{\alpha/2}$ for any $\alpha < 2$. This rate is (almost) the one used in the introduction for the formal expansion. It is done through the utilization of R_2^*.

Proof of the theorem. As noticed in Remark 6, in order to improve the rate of convergence, we have to expand $\tilde{U}$ to higher order. Namely, let

$$\tilde{U} = \begin{pmatrix} Q_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} \varepsilon L\tau + R_2^* + \frac{1}{\varepsilon} e^{\frac{-1}{\varepsilon\sqrt{2}}} R_2 \\ -\varepsilon z \operatorname{curl} \tau + r_3^* + \frac{1}{\varepsilon} e^{\frac{-1}{\varepsilon\sqrt{2}}} r_3 \end{pmatrix} + \varepsilon \begin{pmatrix} S_2^* + \varepsilon S_2 \\ s_3^* + \varepsilon s_3 \end{pmatrix} \equiv Q + F + \varepsilon G, \quad (3.34)$$

where $\nabla_{x,y,z} \cdot S^* = 0$ and $\nabla_{x,y,z} \cdot S = 0$. We now make precise how to obtain S^* and S. In this direction let us recall that $\bar{W}$ denotes the solution to

$$\bar{W}_t + \bar{W} \cdot \nabla_{x,y} \bar{W} - \Delta_{x,y} \bar{W} + \nabla \bar{q} = \tau, \quad \nabla \cdot \bar{W} = 0, \; \bar{W}(t=0) = \bar{U}(t=0) = \bar{U}_0.$$

The vector field S_2^* is by definition the solution to

$$\varepsilon^2 \frac{\partial^2 S_2^*}{\partial z^2} - L S_2^* = \bar{W} \cdot \nabla R_2^* + R_2^* \cdot \nabla \bar{W} + R_2^* \cdot \nabla R_2^* + \big(r_3^* - \varepsilon z \,\mathrm{curl}\, \tau\big) \frac{\partial R_2^*}{\partial z} - \Delta_{x,y} R_2^* \tag{3.35}$$

and satisfies $\frac{\partial}{\partial z} S_2^*|_{z=1} = 0$ with S_2^* bounded as $\varepsilon \to 0$. This term S_2^* is the resolution of the boundary layer at $z = 1$ at the order ε. Some tedious computations give that S_2^* is a sum of terms of the form

$$e^{\frac{(1-z)}{\sqrt{2}\varepsilon}} \big((\alpha_1 + \frac{1-z}{\varepsilon\sqrt{2}} \alpha_2) \cos\big(\frac{1-z}{\varepsilon\sqrt{2}}\big) + (\beta_1 + \frac{1-z}{\varepsilon\sqrt{2}} \beta_2) \sin\big(\frac{1-z}{\varepsilon\sqrt{2}}\big)\big) + \gamma e^{\frac{-(1-z)\sqrt{2}}{\varepsilon}},$$

where the coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ and γ depend only on τ and $\bar{W}$. We do not make precise the global expression of these functions. We obtain s_3^* by solving the continuity equation:

$$\frac{\partial s_3^*}{\partial z} = -\nabla_{x,y} \cdot S_2^*.$$

As in the previous part, we introduce S_2 and s_3 in order to ensure the boundary conditions at $z = 0$ and $\fint G_2 \, dz = 0$. Finally we have the following estimates on G:

$$\begin{aligned} &|\nabla_{x,y}^\alpha \frac{\partial^\beta S_2^*}{\partial z^\beta}|_{L^2} \le K \varepsilon^{1/2-\beta}, \ |\nabla_{x,y}^\alpha \frac{\partial^\beta}{\partial z^\beta} S_2|_{L^2} \le K\varepsilon, \\ &|\nabla_{x,y}^\alpha \frac{\partial^\beta s_3^*}{\partial z^\beta}|_{L^2} \le K \varepsilon^{3/2-\beta}, \ |\nabla_{x,y}^\alpha \frac{\partial^\beta}{\partial z^\beta} s_3|_{L^2} \le K\varepsilon, \end{aligned} \tag{3.36}$$

while Q still satisfies (3.10); i.e.,

$$\begin{aligned} &\fint Q_2 \, dz = 0; \quad \nabla_{x,y,z} \cdot Q = 0, \\ &\text{at} \ \ z = 0: \quad \frac{\partial Q_2}{\partial z} = 0, \quad q_3 = 0, \\ &\text{at} \ \ z = 1: \quad \frac{\partial Q_2}{\partial z} = 0, \quad q_3 = 0. \end{aligned}$$

We now give the equations governing the evolution of $\bar{U} - \bar{W}$:

$$\begin{aligned} &\frac{\partial}{\partial t}(\bar{U} - \bar{W}) + \bar{U} \cdot \nabla_{x,y}(\bar{U} - \bar{W}) + (\bar{U} - \bar{W}) \cdot \nabla_{x,y} \bar{W} \\ &-\Delta_{x,y}(\bar{U} - \bar{W}) + \nabla\big(\frac{\bar{p}}{\varepsilon} - \bar{q}\big) + \frac{1}{\varepsilon} L\bar{U} = -\nabla_{x,y} \cdot \big(\fint Q_2 \otimes Q_2\big) \\ &-\nabla_{x,y} \cdot \big(\fint (F_2 + \varepsilon G_2) \otimes Q_2\big) - \nabla_{x,y} \cdot \big(\fint Q_2 \otimes (F_2 + \varepsilon G_2)\big) \\ &-\nabla_{x,y} \cdot \big(\fint (F_2 + \varepsilon G_2) \otimes (F_2 + \varepsilon G_2)\big), \end{aligned} \tag{3.37}$$

with $\nabla\cdot(\bar U-\bar W)=0,\quad (\bar U-\bar W)|_{t=0}=0$. Plugging (3.34) in equation (3.4) yields the equation satisfied by Q, namely,

$$\begin{aligned}
&\frac{\partial Q}{\partial t}-\Delta_{x,y}Q-\varepsilon\frac{\partial^2 Q}{\partial z^2}+Q\cdot\nabla Q+\frac{\nabla\tilde p}{\varepsilon}=-(F+\varepsilon G)\cdot\nabla Q\\
&-Q\cdot\nabla(F+\varepsilon G)-\Big(\begin{pmatrix}\varepsilon L\tau\\ -\varepsilon z\,\mathrm{curl}\,\tau\end{pmatrix}+\frac{1}{\varepsilon}e^{\frac{-1}{\sqrt{2}\varepsilon}}R+\varepsilon G\Big)\cdot\nabla(F+\varepsilon G)\\
&-R^*\cdot\nabla\Big[\begin{pmatrix}\varepsilon L\tau\\ -\varepsilon z\,\mathrm{curl}\,\tau\end{pmatrix}+\frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}R+\varepsilon G\Big]-R^*\cdot\nabla R^*\\
&-\bar U\cdot\nabla_{x,y}\Big[\begin{pmatrix}\varepsilon L\tau\\ -\varepsilon z\,\mathrm{curl}\,\tau\end{pmatrix}+\frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}R+\varepsilon G\Big]-\bar U\cdot\nabla_{x,y}R^*-\bar U\cdot\nabla_{x,y}Q\\
&-\begin{pmatrix}Q_2\cdot\nabla_{x,y}\bar U\\ 0\end{pmatrix}-\begin{pmatrix}(\varepsilon L\tau+\frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}R_2+\varepsilon G_2)\cdot\nabla_{x,y}\bar U\\ 0\end{pmatrix}-\begin{pmatrix}R_2^*\cdot\nabla_{x,y}\bar U\\ 0\end{pmatrix}\\
&-\begin{pmatrix}\nabla_{x,y}\cdot(\fint\tilde U_2\otimes\tilde U_2)\\ 0\end{pmatrix}-\frac{1}{\varepsilon}\begin{pmatrix}LQ_2\\ 0\end{pmatrix}-\frac{1}{\varepsilon^2}e^{\frac{-1}{\varepsilon\sqrt{2}}}\begin{pmatrix}LR_2\\ 0\end{pmatrix}-\varepsilon\begin{pmatrix}LS_2\\ 0\end{pmatrix}\\
&-\frac{1}{\varepsilon}\begin{pmatrix}LR_2^*\\ 0\end{pmatrix}-\begin{pmatrix}LS_2^*\\ 0\end{pmatrix}+e^{\frac{-1}{\varepsilon\sqrt{2}}}\frac{\partial^2 R}{\partial z^2}+\varepsilon\frac{\partial^2 R^*}{\partial z^2}+\varepsilon^2\frac{\partial^2 S^*}{\partial z^2}\\
&+\varepsilon^3\frac{\partial^2 S}{\partial z^2}+\Delta_{x,y}\Big(\begin{pmatrix}\varepsilon L\tau\\ -\varepsilon z\,\mathrm{curl}\,\tau\end{pmatrix}+\frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}R+\varepsilon G\Big)+\Delta_{x,y}R^*-\varepsilon\frac{\partial G}{\partial t}.
\end{aligned}$$

We simplify this equation using (3.8) and (3.1) and we get

$$\begin{aligned}
&\frac{\partial Q}{\partial t}-\Delta_{x,y}Q-\varepsilon\frac{\partial^2 Q}{\partial z^2}+\frac{\nabla\tilde p}{\varepsilon}=-Q\cdot\nabla Q-(F+\varepsilon G)\cdot\nabla Q-Q\cdot\nabla(F+\varepsilon G)\\
&-\big(\varepsilon L\tau+\frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}R_2+\varepsilon G_2\big)\cdot\nabla_{x,y}(F+\varepsilon G)\\
&-\big(-\varepsilon z\ \mathrm{curl}\,\tau+\frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}r_3+\varepsilon g_3\big)\frac{\partial}{\partial z}\begin{pmatrix}\varepsilon L\tau+\frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}R_2+\varepsilon G_2\\ f_3+\varepsilon g_3\end{pmatrix}\qquad(3.38)
\end{aligned}$$

$$\begin{aligned}
&-R^*\cdot\nabla\Big[\begin{pmatrix}\varepsilon L\tau\\ -\varepsilon z\,\mathrm{curl}\,\tau\end{pmatrix}+\frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}R+\varepsilon G\Big]\\
&-\big(\frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}r_3+\varepsilon g_3\big)\frac{\partial}{\partial z}\begin{pmatrix}R_2^*\\ 0\end{pmatrix}-\begin{pmatrix}0\\ R^*\cdot\nabla r_3^*\end{pmatrix}\\
&+\bar U\cdot\nabla_{x,y}\Big(\begin{pmatrix}\varepsilon L\tau\\ -\varepsilon z\,\mathrm{curl}\,\tau\end{pmatrix}+\frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}R+\varepsilon G\Big)-(\bar U-\bar W)\cdot\nabla_{x,y}R_2^*\\
&-\begin{pmatrix}0\\ \bar U\cdot\nabla_{x,y}r_3^*\end{pmatrix}-\bar U\cdot\nabla_{x,y}Q-\begin{pmatrix}Q_2\cdot\nabla_{x,y}\bar U\\ 0\end{pmatrix}\\
&-\begin{pmatrix}(\varepsilon L\tau+\frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}R_2+\varepsilon G_2)\cdot\nabla_{x,y}\bar U\\ 0\end{pmatrix}-\begin{pmatrix}R_2^*\cdot\nabla_{x,y}(\bar U-\bar W)\\ 0\end{pmatrix}
\end{aligned}$$

$$-\begin{pmatrix} \nabla_{x,y}\cdot \fint \tilde U_2\otimes \tilde U_2 \\ 0\end{pmatrix} - \frac{1}{\varepsilon}\begin{pmatrix} LQ_2 \\ 0\end{pmatrix} - \frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt2}}\begin{pmatrix} LR_2 \\ 0\end{pmatrix}$$
$$-\varepsilon\begin{pmatrix} LS_2 \\ 0\end{pmatrix} + e^{\frac{-1}{\varepsilon\sqrt2}}\frac{\partial^2 R}{\partial z^2} + \begin{pmatrix} 0 \\ \varepsilon\frac{\partial^2 r_3^*}{\partial z^2}\end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon^2\frac{\partial^2 s_3^*}{\partial z^2}\end{pmatrix}$$
$$+\varepsilon^3\frac{\partial^2 S}{\partial z^2} + \Delta_{x,y}\Big(\begin{pmatrix} \varepsilon L\tau \\ -\varepsilon z\,\mathrm{curl}\,\tau\end{pmatrix} + \frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt2}}R + \varepsilon G + \begin{pmatrix} 0 \\ r_3^*\end{pmatrix}\Big) - \varepsilon\frac{\partial G}{\partial t}.$$

We now write the energy estimate for Q by multiplying (3.38) by Q. Each term of the right-hand side is now bounded in the following way. These bounds are very similar to that in the previous part except that the source terms of L^2 size $\sqrt{\varepsilon}$ were canceled by the second-order corrector. We now give the list of these bounds:

- $\int Q\cdot\nabla QQ = 0$ since Q is divergence-free.
- $\int (F+\varepsilon G)\cdot\nabla QQ = 0$ since $(F+\varepsilon G)$ is also divergence-free.
- $|\int Q\cdot\nabla(F+\varepsilon G)Q|$ is estimated as in (3.19) by

$$\Big|\int Q\cdot\nabla(F+\varepsilon G)Q\Big| \le C\big(|Q|_{L^2}^2 + |\nabla_{x,y}Q|_{L^2}|Q|_{L^2} + \sqrt{\varepsilon}|\nabla_{x,y}Q|_{L^2}|\nabla_\varepsilon Q|_{L^2}\big). \quad (3.39)$$

- The term

$$\int (\varepsilon L\tau + \frac{1}{\varepsilon}e^{\frac{-1}{\sqrt{2}\varepsilon}}R_2 + \varepsilon G_2)\cdot\nabla_{x,y}(F+\varepsilon G)Q|,$$
$$\le C\varepsilon|\nabla_{x,y}(F+\varepsilon G)|_{L^2}|Q|_{L^2} \le C\varepsilon^{3/2}|Q|_{L^2}, \quad (3.40)$$

since $|\varepsilon L\tau + \frac{1}{\varepsilon}e^{\frac{-1}{\sqrt{2}\varepsilon}}R_2 + \varepsilon G_2|_{L^\infty} \le C\varepsilon$.

- In the same way, one has

$$\Big|\int(-\varepsilon z\,\mathrm{curl}\,\tau + \frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon\sqrt2}}r_3 + \varepsilon g_3)\frac{\partial}{\partial z}\begin{pmatrix} \varepsilon L\tau + \frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon\sqrt2}}R_2 + \varepsilon G_2 \\ f_3 + \varepsilon g_3\end{pmatrix}Q\Big| \le C\varepsilon^{3/2}|Q|_{L^2}, \quad (3.41)$$

since $|\frac{\partial f_3}{\partial z} + \varepsilon\frac{\partial g_3}{\partial z}|_{L^2} \le C\varepsilon^{1/2}$ and $|\frac{\partial G_2}{\partial z}|_{L^2} \le C\varepsilon^{-1/2}$.

- The term

$$\int R^*\cdot\nabla\Big[\begin{pmatrix} \varepsilon L\tau \\ -\varepsilon z\,\mathrm{curl}\,\tau\end{pmatrix} + \frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt2}}R + \varepsilon G\Big]Q$$

has to be split into

$$\int R_2^*\cdot\nabla_{x,y}\Big[\begin{pmatrix} \varepsilon L\tau \\ -\varepsilon z\,\mathrm{curl}\,\tau\end{pmatrix} + \frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt2}}R + \varepsilon G\Big]Q$$
$$+\int r_3^*\cdot\frac{\partial}{\partial z}\Big[\begin{pmatrix} \varepsilon L\tau \\ -\varepsilon z\,\mathrm{curl}\,\tau\end{pmatrix} + \frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt2}}R + \varepsilon G\Big]Q.$$

The first part is bounded by $C\varepsilon|R_2^*|_{L^2}|Q|_{L^2} \leq C\varepsilon^{3/2}|Q|_{L^2}$. The second part is smaller since $|r_3^*|_{L^2} \leq C\varepsilon^{3/2}$. Therefore

$$|\int R^* \cdot \nabla\Big[\begin{pmatrix} \varepsilon L\tau \\ -\varepsilon z \operatorname{curl}\tau \end{pmatrix} + \frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}R + \varepsilon G\Big]Q\big| \leq C\varepsilon^{3/2}|Q|_{L^2}. \tag{3.42}$$

- The term

$$\big|\int\big(\frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}r_3 + \varepsilon g_3\big)\frac{\partial}{\partial z}R_2^* Q_2\big| \leq C\varepsilon^2|\frac{\partial}{\partial z}R_2^*|_{L^2}|Q|_{L^2} \leq C\varepsilon^{3/2}|Q|_{L^2}. \tag{3.43}$$

- The following one is

$$\begin{aligned} \big|\int R^* \cdot \nabla r_3^* Q\big| &\leq \big|\int R_2^* \cdot \nabla_{x,y} r_3^* Q\big| + \big|\int r_3^* \frac{\partial r_3^*}{\partial z}Q\big| \\ &\leq C\varepsilon|R_2^*|_{L^2}|Q|_{L^2} + C\varepsilon|\frac{\partial r_3^*}{\partial z}|_{L^2}|Q|_{L^2} \leq C\varepsilon^{3/2}|Q|_{L^2}. \end{aligned} \tag{3.44}$$

- In the term

$$\big|\int \bar{U}\cdot\nabla_{x,y}\Big(\begin{pmatrix} \varepsilon L\tau \\ -\varepsilon z \operatorname{curl}\tau \end{pmatrix} + \frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}R + \varepsilon G\Big)Q\big|,$$

we replace $\bar{U}$ by $\bar{U} - \bar{W} + \bar{W}$ to get

$$\big|\int \bar{U}\cdot\nabla_{x,y}\Big(\begin{pmatrix} \varepsilon L\tau \\ -\varepsilon z \operatorname{curl}\tau \end{pmatrix} + \frac{1}{\varepsilon}e^{\frac{-1}{\varepsilon\sqrt{2}}}R + \varepsilon G\Big)Q\big| \leq C\varepsilon(|\bar{U} - \bar{W}|_{L^2} + |\bar{W}|_{L^2})|Q|_{L^2}. \tag{3.45}$$

- In order to estimate

$$\big|\int(\bar{U} - \bar{W})\cdot\nabla_{x,y}R_2^* Q_2\big|,$$

we use Lemma 1:

$$\begin{aligned} |\int(\bar{U} - \bar{W})\cdot\nabla_{x,y}R_2^* Q_2| &\leq C|\bar{U} - \bar{W}|_{L^2}|\textstyle\fint \nabla_{x,y}R_2^* Q_2|_{W^{1,1}(\mathbf{T}^2)} \\ \leq C|\bar{U} - \bar{W}|_{L^2}(|\nabla_{x,y}R_2^*|_{L^2}|\nabla_{x,y}Q_2|_{L^2} &+ |Q_2|_{L^2}|\nabla_{x,y}^2 R_2^*|_{L^2} + |\nabla_{x,y}R_2^*|_{L^2}|Q_2|_{L^2}) \\ &\leq C|\bar{U} - \bar{W}|_{L^2}\sqrt{\varepsilon}(|\nabla_{x,y}Q_2|_{L^2} + |Q_2|_{L^2}). \end{aligned} \tag{3.46}$$

- The term $|\int \bar{U}\cdot\nabla_{x,y}r_3^* Q_2|$ is estimated directly by

$$|\int \bar{U}\cdot\nabla_{x,y}r_3^* Q_2| \leq C\varepsilon(|\bar{U} - \bar{W}|_{L^2} + |\bar{W}|_{L^2})|Q|_{L^2}. \tag{3.47}$$

- Since $\bar{U}$ is divergence-free, one has $\int \bar{U} \cdot \nabla_{x,y} QQ = 0$.
- Again using Lemma 1, one gets

$$
\begin{aligned}
|\int Q_2 \cdot \nabla_{x,y} \bar{U} Q_2| &\le |\int Q_2 \cdot \nabla_{x,y} (\bar{U} - \bar{W}) Q_2| + |\int Q_2 \cdot \nabla_{x,y} \bar{W} Q_2| \\
&\le |\nabla_{x,y} (\bar{U} - \bar{W})|_{L^2} | \fint Q_2 Q_2 |_{W^{1,1}} + |\nabla_{x,y} \bar{W}|_{L^\infty} |Q_2|^2_{L^2} \\
&\le C |\nabla_{x,y} (\bar{U} - \bar{W})|_{L^2} |Q_2|_{L^2} |\nabla_{x,y} Q_2|_{L^2} + C |Q_2|^2_{L^2}.
\end{aligned}
\tag{3.48}
$$

- Directly:

$$
|\int (\varepsilon L \tau + \frac{1}{\varepsilon} e^{\frac{-1}{\varepsilon \sqrt{2}}} R_2 + \varepsilon G_2) \cdot \nabla_{x,y} \bar{U} Q_2| \le C \varepsilon (|\nabla(\bar{U} - \bar{W})|_{L^2} + |\nabla \bar{W}|_{L^2}) |Q_2|_{L^2}.
\tag{3.49}
$$

- The next one: $|\int R_2^* \cdot \nabla_{x,y} (\bar{U} - \bar{W}) Q_2|$ is estimated thanks to Lemma 1:

$$
\begin{aligned}
|\int R_2^* \cdot \nabla_{x,y} (\bar{U} - \bar{W}) Q_2| &\le C |\nabla_{x,y} (\bar{U} - \bar{W})|_{L^2} | \fint R_2^* Q_2 |_{W^{1,1}(\mathbf{T}^2)} \\
&\le C \sqrt{\varepsilon} |\nabla_{x,y} (\bar{U} - \bar{W})|_{L^2} (|Q_2|_{L^2} + |\nabla_{x,y} Q_2|_{L^2}).
\end{aligned}
\tag{3.50}
$$

- Since $\fint Q_2 = 0$, one has

$$
\int \nabla_{x,y} \cdot (\fint \tilde{U}_2 \otimes \tilde{U}_2) Q_2 = 0.
$$

The following terms are estimated directly without comment:

$$
-\frac{1}{\varepsilon} \int L Q_2 Q_2 = 0,
$$

$$
|\int \frac{1}{\varepsilon} e^{\frac{-1}{\varepsilon \sqrt{2}}} L R_2 Q_2| \le C \varepsilon^4 |Q|_{L^2},
\tag{3.51}
$$

$$
|\int \varepsilon L S_2 Q_2| \le C \varepsilon^{3/2} |Q_2|_{L^2},
\tag{3.52}
$$

$$
|\int e^{\frac{-1}{\varepsilon \sqrt{2}}} \frac{\partial^2 R}{\partial z^2} Q| \le C \varepsilon^4 |Q|_{L^2}.
\tag{3.53}
$$

As in (3.25), one gets

$$
|\int \varepsilon \frac{\partial^2 r_3^*}{\partial z^2} q_3| \le C \varepsilon^{3/2} |\nabla_{x,y} Q|_{L^2},
\tag{3.54}
$$

and

$$|\int \varepsilon^2 \frac{\partial^2 s_3^*}{\partial z^2} q_3| \le C\varepsilon^{5/2} |\nabla_{x,y} Q|_{L^2}, \tag{3.55}$$

$$|\int \varepsilon^3 \frac{\partial^2 S}{\partial z^2} Q| \le C\varepsilon^{3/2} |Q|_{L^2}. \tag{3.56}$$

Moreover,

$$|\int \Delta_{x,y}\Big(\begin{pmatrix} \varepsilon L\tau \\ -\varepsilon z \operatorname{curl} \tau \end{pmatrix} + \frac{1}{\varepsilon} e^{\frac{-1}{\varepsilon\sqrt{2}}} R + \varepsilon G + \begin{pmatrix} 0 \\ r_3^* \end{pmatrix} \Big) Q| \le C\varepsilon |Q|_{L^2}. \tag{3.57}$$

The last term:

$$|\int \varepsilon \frac{\partial G}{\partial t} Q| \le C\varepsilon^{3/2} |Q|_{L^2}. \tag{3.58}$$

Summing estimates (3.39)–(3.58), one obtains after absorbing the terms $|\nabla_\varepsilon Q|_{L^2}^2$:

$$\begin{aligned} &\frac{d}{dt}|Q|_{L^2}^2 + |\nabla_{x,y} Q|_{L^2}^2 + \varepsilon |\frac{\partial Q}{\partial z}|_{L^2}^2 \\ &\le C\big\{ |Q|_{L^2}^2 + \varepsilon(|\bar U - \bar W|_{L^2}^2 + |\nabla_{x,y}(\bar U - \bar W)|_{L^2}^2) |\nabla_{x,y}(\bar U - \bar W)|_{L^2}^2 |Q_2|_{L^2}^2 + \varepsilon^2 \big\}. \end{aligned} \tag{3.59}$$

In order to obtain the energy estimate on $\bar U - \bar W$, let us recall (3.33):

$$\frac{d}{dt}|\bar U - \bar W|_{L^2}^2 + |\nabla(\bar U - \bar W)|_{L^2}^2 \le C|\nabla \bar W|_{L^2}^2 |\bar U - \bar W|_{L^2}^2 + C|\tilde U_2|_{L^2}^2 |\nabla_{x,y} \tilde U_2|_{L^2}^2.$$

Replacing $\tilde U_2$ by $Q_2 + F_2 + \varepsilon G_2$, one gets:

$$\begin{aligned} &\frac{d}{dt}|\bar U - \bar W|_{L^2}^2 + |\nabla(\bar U - \bar W)|_{L^2}^2 \\ &\le C(|\bar U - \bar W|_{L^2}^2 + |Q_2|_{L^2}^2 |\nabla_{x,y} Q_2|_{L^2}^2 + \varepsilon(|Q_2|_{L^2}^2 + |\nabla_{x,y} Q_2|_{L^2}^2) + \varepsilon^2). \end{aligned} \tag{3.60}$$

Inequalities (3.59) and (3.60) are the equivalents of (3.28) and (3.33) except that the source term in (3.59) is ε^2 instead of ε in (3.28). Therefore, the end of the proof follows the same lines as for Theorem 3 and we omit it.

Remark 8. i) We are not able to prove some results for strong solutions since we can not estimate correctly the term $\int q_3 \frac{\partial F_2}{\partial z} \Delta Q$ which would be the equivalent of I_2 in (3.14). Therefore, one can not obtain the equivalent of (3.19).

ii) We are also unable to deal with the case of homogeneous Dirichlet boundary conditions. Indeed, in this case the corrector would involve $\bar U$ and is for $U = 0$ at $z = 0$:

$$\begin{pmatrix} -\bar U(e^{-\frac{z}{\varepsilon\sqrt{2}}} \cos(\frac{z}{\varepsilon\sqrt{2}}) - 1) - L\bar U e^{-\frac{z}{\varepsilon\sqrt{2}}} \sin(\frac{z}{\varepsilon\sqrt{2}}) \\ -\operatorname{curl} \bar U \varepsilon((e^{-\frac{z}{\varepsilon\sqrt{2}}} \cos(\frac{z}{\varepsilon\sqrt{2}}) - 1) \end{pmatrix}.$$

We see that the third component of this corrector is less regular that the two first ones, and therefore, plugging this corrector in the equations does not enable us to obtain some bounds.

We could, on the other hand, use $\bar{W}$ instead of $\bar{U}$ as above, but in this case, the boundary condition on Q at $z = 0$ is not zero.

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