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ALMOST-PERIODIC OSCILLATIONS OF MONOTONE SECOND-ORDER SYSTEMS

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Abstract. We study the bounded and the a.p. (almost-periodic) solutions of forced second order systems with monotone fields and a linear damping term. A special class of such systems is the class of the second order Lagrangian systems with convex Lagrangians. We provide results of existence and uniqueness, we study the dependence of the bounded and a.p. solutions on the bounded and a.p. forcing terms, and finally we treat the case where an additional small nonlinear damping term is present in the equation.

1. Introduction. The numerical space $\mathbb{R}^N$ is endowed with its standard inner product $x \cdot y := \sum_{k=1}^{N} x_k y_k$, and $|\cdot|$ denotes the associated Euclidean norm.

From functions $F : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $b : \mathbb{R} \to \mathbb{R}$, $B : \mathbb{R} \to \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$ and $e : \mathbb{R} \to \mathbb{R}^N$, we build the following forced second order ordinary differential equation:

$$\ddot{x}(t) + [b(t)I + B(t)]\dot{x}(t) - F(t, x(t)) = e(t), \tag{1}$$

where $F(t, \cdot)$ is Minty-monotone ([13, Section 11]) on $\mathbb{R}^N$, and I is the identity operator on $\mathbb{R}^N$. A special class of such systems is the class of the following Lagrangian systems:

$$\ddot{x}(t) - V_x(t, x(t)) = e(t), \tag{2}$$

where $V : \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$, with $V(t, \cdot)$ convex differentiable for each $t \in \mathbb{R}$; $V_x(t, x)$ denotes the gradient of the function $V(t, \cdot)$ with respect to the standard inner product of $\mathbb{R}^N$.

We successively assume that e, b, B, and $F(\cdot, x)$ are bounded on $\mathbb{R}$, and that e, b, and B are a.p. (Bohr almost periodic) ([12, Chapter VI]) and F is a.p. in t

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uniformly for $x \in \mathbb{R}^N$ in the sense given in the Yoshizawa book ([21, page 6]). We shall make precise these definitions in the next section.

Our aim is to study the bounded solutions and the a.p. solutions of the equation (1), notably to establish existence results. Recently, Berger and Chen ([3, 4]) have built a variational method to study the a.p. solutions of the equation

$$\ddot{x}(t) - \Psi'(x(t)) = e(t), \tag{3}$$

where Ψ' is the gradient of a convex function $\Psi : \mathbb{R}^N \longrightarrow \mathbb{R}$. The equation (3) is a special case of the equations (2) and (1). In this paper we give some improvements and generalizations to the results of Berger and Chen.

We also consider the presence of a small nonlinear damping term:

$$\ddot{x}(t) + [b(t)I + B(t)]\dot{x}(t) - F(t, x(t)) + \epsilon G(t, x(t), \dot{x}(t)) = e(t). \tag{4}$$

In Section 3, we provide existence results about bounded and a.p. solutions of the equation (1). In Section 4, we study the question of the dependence on the a.p. (respectively bounded) forcing term of the a.p. (respectively bounded) solutions of the equation (1). In Section 5, we give an existence result about the bounded and a.p. solutions of the system (4) for small values of the parameter ε.

Since the equations (2) and (3) are in the framework of Lagrangian systems with convex Lagrangians, we indicate that the question of the a.p. solutions of autonomous convex Lagrangian systems, without forcing term, is studied in [5, 7] (and in references therein) by using a variational approach, the so-called Calculus of Variations in Mean. This way provides results about the structure of the set of the a.p. solutions of such systems. A related viewpoint is used to study the structure of the a.p. solutions of a (convex) nonlinear evolution equation in [11].

The question of the a.p. solutions of convex Lagrangian Systems, in presence of an a.p. forcing term, is considered in [8] by using an extension of Calculus of Variations in Mean on a Hilbert space of Besicovitch a.p. functions which looks like a Sobolev space. The results obtained by this way are theorems of existence of weak (in a new sense) a.p. solutions. These results about weak solutions provide density results in terms of usual solutions. In the convex case, an improvement of this method is the work [2], where Azé and Boutaib introduce new spaces of Besicovitch a.p. functions which look like Orlicz spaces; they obtain also results about weak a.p. solutions. In [9] some of the results of [5, 7] are generalized to be applied to economic models.

2. Notations and assumptions. We denote by E a real or complex Banach space. $BC^0(\mathbb{R}, E)$ denotes the space of the continuous bounded functions from $\mathbb{R}$ to E, and $AP^0(E)$ denotes the space of the Bohr a.p. functions from $\mathbb{R}$ to E. When k is a nonnegative integer, $BC^k(\mathbb{R}, E)$ (respectively $AP^k(E)$) is the space of the functions in $BC^0(\mathbb{R}, E) \cap C^k(\mathbb{R}, E)$ (respectively $AP^0(E) \cap C^k(\mathbb{R}, E)$) such that all their derivatives, up to order k, are bounded (respectively a.p.) functions.

When $u \in BC^0(\mathbb{R}, E)$, we set $\|u\|_\infty := \sup_{t\in\mathbb{R}} \|u(t)\|_E$; when $u \in BC^1(\mathbb{R}, E)$, we set $\|u\|_{C^1} := \|u\|_\infty + \|\dot{u}\|_\infty$; and when $u \in BC^2(\mathbb{R}, E)$, we set $\|u\|_{C^2} := \|u\|_\infty + \|\dot{u}\|_\infty + \|\ddot{u}\|_\infty$. We denote by $\mathcal{P}_c(E)$ the set of the compact subsets of E.

For a mapping $f : \mathbb{R} \times \mathbb{R}^N \longrightarrow E$, we consider the three following properties:

(U$_1$) $f \in C^0(\mathbb{R} \times \mathbb{R}^N, E)$.
(U$_2$) $\forall x \in \mathbb{R}^N$, $f(\cdot, x) \in BC^0(\mathbb{R}, E)$.
(U$_3$) $\forall K \in \mathcal{P}_c(\mathbb{R}^N)$, $\sup_{x\in K} \sup_{t\in\mathbb{R}} \|f(t, x)\|_E < +\infty$.
(U$_4$) $\forall K \in \mathcal{P}_c(E)$, $\forall \varepsilon > 0$, $\exists \eta = \eta(K, \varepsilon) > 0$, $\forall x \in K$, $\forall y \in K$,
$|y - x| \le \eta \Longrightarrow \sup_{t\in\mathbb{R}} \|f(t, y) - f(t, x)\|_E \le \varepsilon$.

We define two mappings spaces, $\mathcal{U}_0(\mathbb{R} \times \mathbb{R}^N, E)$ and $\mathcal{U}(\mathbb{R} \times \mathbb{R}^N, E)$, in the following manner:

$$f \in \mathcal{U}_0(\mathbb{R} \times \mathbb{R}^N, E) \Longleftrightarrow f \text{ satisfies (U}_1\text{) and (U}_3\text{).}$$
$$f \in \mathcal{U}(\mathbb{R} \times \mathbb{R}^N, E) \Longleftrightarrow f \text{ satisfies (U}_1\text{), (U}_2\text{) and (U}_4\text{).}$$

We note that (U$_3$) implies (U$_2$), and that (U$_2$) and (U$_4$) imply (U$_3$). Consequently the next sentence is valid: $\mathcal{U}(\mathbb{R} \times \mathbb{R}^N, E) \subset \mathcal{U}_0(\mathbb{R} \times \mathbb{R}^N, E)$.

Following [21, page 6], $f \in C^0(\mathbb{R} \times \mathbb{R}^N, E)$ is so-called a.p. in t uniformly for $x \in \mathbb{R}^N$ when:

$$\begin{cases} \forall K \in \mathcal{P}_c(\mathbb{R}^N), \ \forall \varepsilon > 0, \ \exists \ell = \ell(K, \varepsilon) > 0, \ \forall \alpha \in \mathbb{R}, \\ \exists \tau = \tau(\alpha, K, \varepsilon) \in [\alpha, \alpha + \ell), \ \sup_{t\in\mathbb{R}} \sup_{x\in K} \|f(t + \tau, x) - f(t, x)\|_E \le \varepsilon. \end{cases} \tag{5}$$

When $e \in AP^0(E)$ and $f \in C^0(\mathbb{R} \times \mathbb{R}^N, E)$ are a.p. in t uniformly for $x \in \mathbb{R}^n$, we denote respectively by Mod(e) and Mod(f) the module of frequencies of e and f ([14, Chapter 4], [21, Chapter I, Section 2]).

Now we give the list of the assumptions what we use in the present work:

(M) $\exists c_* \in (0, +\infty)$, $\forall t \in \mathbb{R}$, $\forall x, y \in \mathbb{R}^N$,
$\big(F(t, y) - F(t, x) - \frac{1}{4}B(t)B^*(t)(y - x)\big) \cdot (y - x) \ge c_*|y - x|^2$.

Here $B^*(t)$ denotes the transpose matrix of $B(t)$. For linear two-point boundary value problems a similar condition was introduced by Picard in the scalar case and by Hartman and Wintner in the vector case (see [16], Ch. XII, Th. 3.3).

(B$_1$) $e \in BC^0(\mathbb{R}, \mathbb{R}^N)$.
(B$_2$) $F \in \mathcal{U}_0(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$.
(B$_3$) $F \in \mathcal{U}(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$.
(B$_4$) For every $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, the partial differential $F_x(t, x)$ exists and $F_x \in \mathcal{U}(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$.
(B$_5$) $b \in BC^0(\mathbb{R}, \mathbb{R})$.
(B$_6$) $B \in BC^0(\mathbb{R}, \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N))$.

(A_1) $e \in AP^0(\mathbb{R}^N)$.
(A_2) F is a.p. in t uniformly for $x \in \mathbb{R}^N$.
(A_3) For every $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, the partial differential $F_x(t,x)$ exists and F_x is a.p. in t uniformly for $x \in \mathbb{R}^N$.
(A_4) $b \in AP^0(\mathbb{R})$.
(A_5) $B \in AP^0(\mathcal{L}(\mathbb{R}^N, \mathbb{R}^N))$.

Comments. We note that (A_1) $\Longrightarrow$ (B_1), (A_2) $\Longrightarrow$ (B_3) $\Longrightarrow$ (B_2), (A_3) $\Longrightarrow$ (B_4), (A_4) $\Longrightarrow$ (B_5), and (A_5) $\Longrightarrow$ (B_6). The implication (A_2) $\Longrightarrow$ (B_3) will be proven in Section 4, Lemma 5 (i).

3. Existence results. Let E be a Banach space with norm $\|\cdot\|_E$, $J \subset \mathbb{R}$ an interval with length greater than or equal to two. For $y \in BC^0(J,E)$, write $\|y\|_J = \sup_{t \in J} \|y(t)\|_E$.

Lemma 1. *If $y \in C^2(J,E) \cap BC^0(J,E)$ and $\ddot{y} \in BC^0(J,E)$, then we have $\dot{y} \in BC^0(J,E)$ and*

$$\|\dot{y}\|_J \le 2\sqrt{\|y\|_J}\sqrt{\|\ddot{y}\|_J}.$$

This lemma of Landau is established in [14, Lemma 5.5] and in [15] for scalar functions, but it is easy to extend its proof to vector valued functions, using the Taylor formula with integral term. We shall need the following slight extension of Lemma 1.

Lemma 2. *Let $A \in C^0(\mathbb{R}, \mathcal{L}(E,E))$ and $y \in C^2(\mathbb{R},E)$ be such that*

$$\|y\|_\infty \le \alpha, \ \|\ddot{y} + A(\cdot)\dot{y}\|_\infty \le \beta, \ \|A\|_\infty \le \gamma.$$

Then $\|\dot{y}\|_\infty \le \delta$, where δ is the largest root (δ is nonnegative) of the equation

$$x^2 - 4\alpha\gamma x - 4\alpha\beta = 0.$$

Proof. Let $J \subset \mathbb{R}$ be an interval of length greater than or equal to two. By Lemma 1 and assumptions, we have

$$\|\dot{y}\|_J \le 2\sqrt{\alpha}\sqrt{\|\ddot{y}\|_J} \le 2\sqrt{\alpha}\sqrt{\beta + \gamma\|\dot{y}\|_J},$$

and the result follows easily, as the largest root of the algebraic equation does not depend upon J.

Lemma 3. *Let $\alpha > 0$, $\beta \in \mathbb{R}$ and $\gamma \in BC^0(\mathbb{R},\mathbb{R})$. If $r \in BC^2(\mathbb{R},\mathbb{R})$ satisfies the differential inequality*

$$\ddot{r}(t) \ge \alpha r(t) + \gamma(t)\dot{r}(t) - \beta$$

for all $t \in \mathbb{R}$, then $\sup_{t \in \mathbb{R}} r(t) \le \dfrac{\beta}{\alpha}$.

Proof. We have $\sup_{t\in\mathbb{R}} r(t) = \max\big(\sup_{t\geq 0} r(t), \sup_{t\leq 0} r(t)\big) < +\infty$. We can assume that we have $\sup_{t\in\mathbb{R}} r(t) = \sup_{t\geq 0} r(t)$, else we consider $q(t) := r(-t)$. We denote by S the real-valued function defined on $(0, +\infty)$ by the following formula:

$$S(T) := \sup_{0\leq t\leq T} r(t). \tag{6}$$

The function S is monotonically nondecreasing on $(0, +\infty)$, therefore we have

$$\lim_{T\to+\infty} S(T) = \sup_{T>0} S(T) = \sup_{t\geq 0} r(t) = \sup_{t\in\mathbb{R}} r(t). \tag{7}$$

Since r is continuous on the compact set $[0, T]$, we have

$$\exists t_T \in [0, T] \ \text{ such } \ \text{ that } \ r(t_T) = \sup_{0\leq t\leq T} r(t).$$

Case 1: there exists $T_0 > 0$ such that, for every $T \geq T_0$, we have $t_T = T$.

For every $T \geq T_0$, we have $S(T) = r(T)$, and therefore, by using (7) we have

$$\lim_{t\to+\infty} r(t) = \sup_{t\in\mathbb{R}} r(t) < +\infty. \tag{8}$$

We also have

$$\lim_{t\to+\infty} \dot{r}(t) = 0; \tag{9}$$

if not, we can find $\varepsilon_0 > 0$ and a real sequence $(t_n)_n$ such that

$$\lim_{n\to+\infty} t_n = +\infty,$$

and for every $n \in \mathbb{N}$, we have

$$|\dot{r}(t_n)| \geq \varepsilon_0. \tag{10}$$

Since $r \in BC^2(\mathbb{R}, \mathbb{R})$, $\dot{r}$ is a Lipschitzian function; i.e., there exists $A > 0$, such that, for every $\tau_1, \tau_2 \in \mathbb{R}$, we have

$$|\dot{r}(\tau_2) - \dot{r}(\tau_1)| \leq A\, |\tau_2 - \tau_1|,$$

therefore it follows that

$$|\dot{r}(\tau_2)| \geq |\dot{r}(\tau_1)| - A\, |\tau_2 - \tau_1|. \tag{11}$$

We set $s_n := t_n + \frac{\varepsilon_0}{2A}$; then there exists c_n such that $t_n < c_n < s_n$, and

$$|r(s_n) - r(t_n)| = |\dot{r}(c_n)| \frac{\varepsilon_0}{2A}. \tag{12}$$

By using the relation (11) with $\tau_1 = t_n$ and $\tau_2 = c_n$, we obtain

$$|\dot{r}(c_n)| \geq |\dot{r}(t_n)| - A|c_n - t_n| \geq |\dot{r}(t_n)| - A|s_n - t_n| = |\dot{r}(t_n)| - \frac{\varepsilon_0}{2},$$

and by using (10) we obtain

$$|\dot{r}(c_n)| \geq \frac{\varepsilon_0}{2},$$

and the relation (12) implies

$$|r(s_n) - r(t_n)| \geq \frac{\varepsilon_0^2}{4A} > 0;$$

which provides a contradiction with both the relations: (8) and

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} t_n = +\infty.$$

Therefore the relation (9) is verified.

Now we want to prove the following relation:

$$\liminf_{t\to+\infty} \ddot{r}(t) \leq 0. \tag{13}$$

If this relation is false, we have

$$\liminf_{t\to+\infty} \ddot{r}(t) = \varepsilon_0 > 0,$$

therefore there exists $T_1 > 0$ such that, for every $t \geq T_1$, we have $\ddot{r}(t) \geq \frac{\varepsilon_0}{2}$, and consequently, for every $t \geq T_1$, we have

$$r(t) = r(T_1) + \dot{r}(T_1)(t - T_1) + \frac{1}{2}\ddot{r}(c)(t - T_1)^2$$

with $T_1 < c < t$. From this last relation we deduce the following inequality:

$$r(t) \geq r(T_1) + \dot{r}(T_1)(t - T_1) + \frac{\varepsilon_0}{4}(t - T_1)^2.$$

From the equality

$$\lim_{t\to+\infty} r(T_1) + \dot{r}(T_1)(t - T_1) + \frac{\varepsilon_0}{4}(t - T_1)^2 = +\infty,$$

we obtain $\lim_{t\to+\infty} r(t) = +\infty$; that contradicts the boundedness of r on $\mathbb{R}$, which justifies the relation (13).

By using the relations (8), (9), (13), the boundedness of γ on $\mathbb{R}$, and the following inequality:

$$\ddot{r}(t) \geq \alpha r(t) + \gamma(t)\dot{r}(t) - \beta,$$

we obtain

$$\alpha \sup_{t\in\mathbb{R}} r(t) - \beta \leq 0,$$

which implies: $\sup_{t\in\mathbb{R}} r(t) \leq \dfrac{\alpha}{\beta}$.

Case 2: For every $T_0 > 0$, there exists $T \geq T_0$ such that $t_T \neq T$.

And so there exists a sequence $(T_n)_n$ such that

$$\lim_{n\to+\infty} T_n = +\infty \text{ and } 0 \leq t_{T_n} < T_n,$$

and consequently, by using the definition of t_T: $r(t_T) = \sup_{0\leq t\leq T} r(t)$ with $0 \leq t_T \leq T$, we have

$$\lim_{n\to+\infty} r(t_{T_n}) = \lim_{n\to+\infty} \sup_{0\leq t\leq T_n} r(t) = \sup_{t\geq 0} r(t) = \sup_{t\in\mathbb{R}} r(t). \tag{14}$$

If there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $0 < t_{T_n} < T_n$, then

$$\dot{r}(t_{T_n}) = 0 \tag{15}$$

$$\ddot{r}(t_{T_n}) \leq 0. \tag{16}$$

By using the relations (14), (15), (16) and the following inequality:

$$\ddot{r}(t_{T_n}) \geq \alpha r(t_{T_n}) + \gamma(t_{T_n})\dot{r}(t_{T_n}) - \beta,$$

we obtain

$$\alpha \sup_{t\in\mathbb{R}} r(t) - \beta \leq 0,$$

that implies: $\sup_{t\in\mathbb{R}} r(t) \leq \dfrac{\alpha}{\beta}$.

If there exists an infinite set of $n \in \mathbb{N}$ such that $t_{T_n} = 0$, we can exhibit a subsequence of $(t_{T_n})_n$, denote by $(t^*_{T_n})_n$, such that, for every $n \in \mathbb{N}$, we have $t^*_{T_n} = 0$. By using the relation (14), we obtain $r(0) = \sup_{t\in\mathbb{R}} r(t)$, therefore

$$\dot{r}(0) = 0 \text{ and } \ddot{r}(0) \leq 0.$$

By using the following inequality

$$\ddot{r}(0) \geq \alpha r(0) + \gamma(0)\dot{r}(0) - \beta$$

we obtain $\sup_{t\in\mathbb{R}} r(t) \leq \frac{\alpha}{\beta}$.

Proposition 1. *We assume the conditions* (M), (B_1), (B_2), (B_5) *and* (B_6) *fulfilled. Then, for each* $T > 0$*, there exists* $u_T \in C^2(\mathbb{R}, \mathbb{R}^N)$ *such that:*

i) u_T *is a solution of the equation* (1) *on* $[-T, T]$.

ii) u_T *is* $4T$*-periodic.*

iii) $\forall t \in \mathbb{R}, |u_T(t)| \leq c_*^{-1} \sup_{t\in\mathbb{R}} |F(t, 0) + e(t)|$.

iv) $\exists c_1 \in (0, +\infty), \forall T > 0, \forall t \in \mathbb{R}, |\dot{u}_T(t)| \leq c_1$.

Proof. We fix $T > 0$. We define $X : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ as follows:

$$X(t, x) := \begin{cases} F(t, x) + e(t) & \text{if } t \in [-T, T] \\ F(2T - t, x) + e(2T - t) & \text{if } t \in [T, 3T] \end{cases}$$

and $X(t + 4kT, x) = X(t, x)$ for every $(t, x) \in [-T, 3T] \times \mathbb{R}^N$ and $k \in \mathbb{Z}$. Since $([-T+kT, T+kT] \times \mathbb{R}^N)_{k\in\mathbb{Z}}$ is a locally finite closed covering of $\mathbb{R} \times \mathbb{R}^N$, and since X is continuous on each subset of this covering, we can assert that $X \in C^0(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^N)$ ([20, pages 19–20]). Also we see that, for every $x \in \mathbb{R}^N$, $X(\cdot, x)$ is $4T$-periodic. We define $C : \mathbb{R} \to \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$ as follows:

$$C(t) := \begin{cases} b(t)I + B(t) & \text{if } t \in [-T, T] \\ b(2T - t)I + B(2T - t) & \text{if } t \in [T, 3T] \end{cases}$$

and $C(t + 4kT) = C(t)$ for all $t \in [-T, 3T]$ and all $k \in \mathbb{Z}$. So C is continuous and $4T$-periodic.

Now we want to use a Leray-Schauder approach to study the $4T$-periodic solutions of the following second-order differential equation:

$$\ddot{x}(t) + C(t)\dot{x}(t) = X(t, x(t)). \tag{X}$$

We embed (X) into the family of differential equations

$$\ddot{x}(t) + C(t)\dot{x}(t) = (1 - \lambda)cx(t) + \lambda X(t, x(t)), \quad \lambda \in [0, 1], \tag{X_λ}$$

where $c > 0$ is sufficiently large so that

$$c|v|^2 \geq \frac{1}{4}\left|B^*(t)v\right|^2 + c_*|v|^2,$$

for all $t \in \mathbb{R}$ and all $v \in \mathbb{R}^N$. By Theorem IV.5 of [18], with $X = Z$ the Banach space of continuous $4T$-periodic functions from $\mathbb{R}$ to $\mathbb{R}^N$, $Lx = \ddot{x} + C(\cdot)\dot{x}$, $Ax = cx$ and $Nx = X(\cdot, x)$, the existence of a $4T$-periodic solution for (X) will follow from the existence of a bound independent of λ for the set of possible $4T$-periodic solutions of (X_λ). Let u be a possible $4T$-periodic solution of (X_λ) for some $\lambda \in [0, 1]$, and let

$$v(t) = \frac{u(t) \cdot u(t)}{2} = \frac{|u(t)|^2}{2}.$$

Then,

$$\dot{v}(t) = u(t) \cdot \dot{u}(t),$$

$$\ddot{v}(t) = |\dot{u}(t)|^2 + u(t) \cdot \ddot{u}(t) = |\dot{u}(t)|^2 + u(t) \cdot [(1-\lambda)cu(t) + \lambda X(t, u(t)) - C(t)\dot{u}(t)].$$

Let $\tau \in [-T, 3T]$ be such that

$$v(\tau) = \max_{t \in [-T,3T]} v(t) = \max_{t \in \mathbb{R}} v(t).$$

Then

$$u(\tau) \cdot \dot{u}(\tau) = 0,$$

and, if $\tau \in [-T, T]$,

$$\begin{aligned} 0 \geq \ddot{v}(\tau) &= |\dot{u}(\tau)|^2 - B^*(\tau)u(\tau) \cdot \dot{u}(\tau) + (1-\lambda)c|u(\tau)|^2 \\ &+ \lambda u(\tau) \cdot [F(\tau, u(\tau)) - F(\tau, 0)] + \lambda u(\tau) \cdot [F(\tau, 0) + e(\tau)], \end{aligned}$$

although, if $\tau \in [T, 3T]$,

$$\begin{aligned} 0 \geq \ddot{v}(\tau) = |\dot{u}(\tau)|^2 - B^*(2T-\tau)u(\tau) \cdot \dot{u}(\tau) + (1-\lambda)c|u(\tau)|^2 \\ +\lambda \cdot [F(2T-\tau, u(\tau)) - F(2T-\tau, 0)] + \lambda u(\tau) \cdot [F(2T-\tau, 0) + e(2T-\tau)]. \end{aligned}$$

Using assumption (M) and the condition upon c, we get, if $\tau \in [-T, T]$,

$$\begin{aligned} 0 &\geq \left|\dot{u}(\tau) - \frac{B^*(\tau)u(\tau)}{2}\right|^2 - \left|\frac{B^*(\tau)u(\tau)}{2}\right|^2 + (1-\lambda)c|u(\tau)|^2 \\ &\quad + \lambda u(\tau) \cdot [F(\tau, u(\tau)) - F(\tau, 0)] + \lambda u(\tau) \cdot [F(\tau, 0) + e(\tau)] \\ &\geq \left|\dot{u}(\tau) - \frac{B^*(\tau)u(\tau)}{2}\right|^2 - \left|\frac{B^*(\tau)u(\tau)}{2}\right|^2 \\ &\quad + (1-\lambda)\left|\frac{B^*(\tau)u(\tau)}{2}\right|^2 + (1-\lambda)c_*|u(\tau)|^2 \\ &\quad + \lambda u(\tau) \cdot [F(\tau, u(\tau)) - F(\tau, 0)] + \lambda u(\tau) \cdot [F(\tau, 0) + e(\tau)] \\ &\geq (1-\lambda)c_*|u(\tau)|^2 + \lambda u(\tau) \cdot \big[F(\tau, u(\tau)) - F(\tau, 0) \\ &\quad - \frac{B(\tau)B^*(\tau)u(\tau)}{4}\big] - \|F(\cdot, 0) + e\|_\infty |u(\tau)| \\ &\geq c_*|u(\tau)|^2 - \|F(\cdot, 0) + e\|_\infty |u(\tau)|, \end{aligned}$$

and similarly for $\tau \in [T, 3T]$. Consequently, we have

$$\|u\|_\infty = |u(\tau)| \leq c_*^{-1} \|F(\cdot, 0) + e\|_\infty := \alpha,$$

which is the required a priori bound. If we call u_T such a solution, then, by construction of X, u_T is $4T$-periodic and satisfies condition (iii). Now we have, for all $t \in [-T, 3T]$ (and hence, by periodicity, for all $t \in \mathbb{R}$),

$$|\ddot{u}_T(t) + C(t)\dot{u}_T(t)| = |X(t, u_T(t))| \leq \sup_{|v| \leq \alpha} \sup_{t \in \mathbb{R}} |F(t, v) + e(t)| := \beta,$$

so that α and β do not depend upon T. From Lemma 2, we deduce that, for all $t \in [-T, 3T]$ and hence, by periodicity, for all $t \in \mathbb{R}$, one has $|\dot{u}_T(t)| \leq c_1$, where c_1 does not depend upon T. Since $X(t, x) = F(t, x) + e(t)$ when $t \in [-T, T]$, u_T is a solution of equation (1) on $[-T, T]$.

Theorem 1.

i) *Under the conditions* (M), (B_1), (B_2), (B_5) *and* (B_6)*, there exists a unique*

$$u \in C^2(\mathbb{R}, \mathbb{R}^N) \cap BC^0(\mathbb{R}, \mathbb{R}^N)$$

which is a solution of equation (1) *on* $\mathbb{R}$*. Moreover, we have*

$$u \in BC^2(\mathbb{R}, \mathbb{R}^N).$$

ii) *Under the conditions* (M), (A_1), (A_2), (A_4) *and* (A_5)*, there exists a unique*

$$u \in C^2(\mathbb{R}, \mathbb{R}^N) \cap AP^0(\mathbb{R}^N)$$

which is a solution of equation (1) *on* $\mathbb{R}$*. Moreover, we have* $u \in AP^2(\mathbb{R}^N)$*, and, if* b *and* B *are constant, we have* $Mod(u) \subset Mod(F + e)$*.*

Proof. i) By using Proposition 1 and Lemma 8.1 of [17, page 159], we can assert that there exists $u \in C^2(\mathbb{R}, \mathbb{R}^N) \cap BC^0(\mathbb{R}, \mathbb{R}^N)$, a solution of equation (1) on $\mathbb{R}$. Then, by using Lemma 2, this solution belongs to $BC^2(\mathbb{R}, \mathbb{R}^N)$. To prove the uniqueness, we consider $v \in C^2(\mathbb{R}, \mathbb{R}^N) \cap BC^0(\mathbb{R}, \mathbb{R}^N)$ a solution of equation (1) on $\mathbb{R}$, and we set $r(t) := \frac{1}{2}|u(t) - v(t)|^2$. Since $u, v \in BC^2(\mathbb{R}, \mathbb{R}^N)$, the function $r \in BC^2(\mathbb{R}, \mathbb{R})$. Moreover, we have

$$\begin{aligned}
&\ddot{r}(t) = |\dot{u}(t) - \dot{v}(t)|^2 + (u(t) - v(t)) \cdot (\ddot{u}(t) - \ddot{v}(t)) \\
&= |\dot{u}(t) - \dot{v}(t)|^2 - (B^*(t)(u(t) - v(t))) \cdot (\dot{u}(t) - \dot{v}(t)) \\
&\quad - b(t)(u(t) - v(t)) \cdot (\dot{u}(t) - \dot{v}(t)) + (u(t) - v(t)) \cdot (F(t, u(t)) - F(t, v(t))) \\
&\geq \Big|\dot{u}(t) - \dot{v}(t) - \frac{B^*(t)(u(t) - v(t))}{2}\Big|^2 \\
&\quad + \Big(F(t, u(t)) - F(t, v(t)) - \frac{B(t)B^*(t)}{4}(u(t) - v(t))\Big) \cdot (u(t) - v(t)) \\
&\quad - b(t)(u(t) - v(t)) \cdot (\dot{u}(t) - \dot{v}(t)) \geq 2c_* r(t) - b(t)\dot{r}(t).
\end{aligned}$$

Consequently, from Lemma 3, we have $\sup_{t\in\mathbb{R}} r(t) \leq 0$. Hence $r(t) = 0$ for all $t \in \mathbb{R}$ and $u = v$.

ii) We set $f(t,x,y) := -(b(t)I + B(t))y + F(t,x) + e(t)$. The hull of f is denoted by $H(f)$ ([21, page 17]). We recall that $g \in H(f)$ means that there exists a real sequence $(r_k)_k$ such that, for every $K \in \mathcal{P}_c(\mathbb{R}^N)$, we have

$$\sup_{t\in\mathbb{R}} \sup_{x\in K} |f(t+r_k,x,y) - g(t,x,y)| \longrightarrow 0 \ (k \to +\infty).$$

By using the Bochner theorem ([12, page 156]) and Theorem 2.2 of [21, page 10], we can say that there exists a subsequence $(s_k)_k$ of $(r_k)_k$, there exist $e_1, b_1 \in AP^0(\mathbb{R}^N)$, $B_1 \in AP^0(\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N))$, and there exists a mapping $F_1 : \mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$ which is a.p. in t uniformly for $x \in \mathbb{R}^N$, such that, for every $K \in \mathcal{P}_c(\mathbb{R}^N)$, we have

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} |e(t+s_k) - e_1(t)| \longrightarrow 0 \ \ (k\to+\infty),\\
&\sup_{t\in\mathbb{R}} |b(t+s_k) - b_1(t)| \longrightarrow 0 \ \ (k\to+\infty),\\
&\sup_{t\in\mathbb{R}} \|B(t+s_k) - B_1(t)\|_{\mathcal{L}} \longrightarrow 0 \ \ (k\to+\infty),\\
&\sup_{t\in\mathbb{R}} \sup_{x\in K} |F(t+s_k,x) - F_1(t,x)| \longrightarrow 0 \ \ (k\to+\infty).
\end{aligned}$$

Then, we have

$$\sup_{t\in\mathbb{R}} \sup_{y\in K} |(b(t+s_k)I + B(t+s_k))y - (b_1(t)I + B_1(t))y| \longrightarrow 0 \ \ (k\to+\infty),$$

and consequently $g(t,x,y) := -(b_1(t)I+B_1(t))y+F_1(t,x)+e_1(t)$. Since the compact convergence implies the pointwise convergence, for every $t \in \mathbb{R}, x_1 \in \mathbb{R}^N, x_2 \in \mathbb{R}^N$, we have

$$\big(F_1(t,x_2) - F_1(t,x_1) - \frac{1}{4}B_1(t)B_1^*(t)(x_2-x_1)\big)\cdot(x_2-x_1) \geq c_*|x_2-x_1|^2.$$

Since (A_1) $\Longrightarrow$ (B_1), (A_2) $\Longrightarrow$ (B_2), (A_4) $\Longrightarrow$ (B_5), and (A_5) $\Longrightarrow$ (B_6), we can use the assertion (i) on the following differential equation

$$\ddot{x}(t) + (b_1(t)I + B_1(t))\dot{x}(t) - F_1(t,x(t)) = e_1(t).$$

Therefore, there exists a constant $\Gamma \geq 0$ such that, for each $g \in H(f)$, the differential equation $\ddot{x} = g(t,x,\dot{x})$ possesses a unique bounded (by Γ for the C^1-norm) solution in $BC^2(\mathbb{R},\mathbb{R}^N)$.

Consequently, we can use Theorem 10.1 of [14, page 170] and we obtain that there exists $u \in AP^2(\mathbb{R}^N)$ a solution of the equation (1) on $\mathbb{R}$, with $\mathrm{Mod}(u) \subset$

Mod(f). Moreover, if b and B are constant, we have Mod$(u) \subset$ Mod$(F+e)$. The uniqueness results from the uniqueness in assertion (i).

Remarks on Assumption (M). 1. Let us consider the following linear system, whose form is motivated by a problem of gyroscopic stabilization,

$$\ddot{x} + B\dot{x} - x = 0, \tag{G}$$

where $x = \mathrm{col}(x_1, x_2)$ and, for some $\omega > 0$,

$$B = \begin{pmatrix} 0 & \omega^2 \\ -\omega^2 & 0 \end{pmatrix}.$$

The conditions for the existence of a unique almost-periodic solution (here the trivial one) given by Theorem 1 are fulfilled if the matrix $\left(1 - \frac{\omega^4}{4}\right)I$ is positive definite, i.e., if the condition $\omega < \sqrt{2}$ holds. Now the characteristic equation associated to system (G) is easily found to be

$$\lambda^4 + (\omega^4 - 2)\lambda^2 + 1 = 0,$$

and (G) will have nontrivial almost-periodic solutions if and only if the corresponding second-order equation,

$$\mu^2 + (\omega^4 - 2)\mu + 1 = 0,$$

has negative roots, which is the case if and only if $\omega \geq \sqrt{2}$. Thus Assumption (M) is sharp in this situation.

2. Assumption (M) is in particular satisfied if

$$(F(t,y) - F(t,x)) \cdot (y - x) \geq c|y - x|^2$$

for all $t \in \mathbb{R},\ x, y \in \mathbb{R}^N$ and some $c > \frac{\|B\|_\infty^2}{4}$.

Comments. 1. Add the possibility, with respect to Berger and Chen, of having some linear dissipation.

2. Here we explain why, for the equation (3), Theorem 1 (ii) provides an improvement on Theorem 4 and Theorem 6 of Berger and Chen in [4].

In [4], we have $\Psi(x) := \frac{1}{2}Ax \cdot x + U(x)$, where A is a symmetric positive-definite matrix, where $U \in C^2(\mathbb{R}^N, \mathbb{R})$ satisfies:$U''(x)$ is semi-positive definite. Therefore there exists $c_* > 0$ (the smallest eigenvalue of A) such that $A\xi \cdot \xi \geq c_*|\xi|^2$, for every $\xi \in \mathbb{R}^N$. Furthermore, U necessarily is convex, and therefore its gradient U' is monotone. Consequently, for every $x, y \in \mathbb{R}^N$, we have

$$\begin{aligned}(\Psi'(y) - \Psi'(x)) \cdot (y - x) &= A(y - x) \cdot (y - x) + (U'(y) - U'(x)) \cdot (y - x) \\ &\geq c_*|y - x|^2 + 0.\end{aligned}$$

We see that our hypotheses of Theorem 1 (ii) are satisfied by Ψ.

In [4], the authors use the following additional condition:

$$\begin{cases} \exists M > 0, \ \ \forall x = (x_1, ..., x_N), y = (y_1, ..., y_N) \in \mathbb{R}^N, \\ (\forall i = 1, ..., N, \ \ M \leq |x_i| \leq |y_i|) \Longrightarrow (U(x) \leq U(y)), \end{cases}$$

but we have no need of this additional condition to prove Theorem 2.

Moreover Theorem 1 is valid for the equation (2). Indicate that to treat the equation (2) we can replace the proof of Proposition 1 by a variational method in the spirit of the methods presented in [17, Chapter 1].

3. When b and B are constant and when the forcing term e is T-periodic and when $F(\cdot, x)$ is T-periodic (for each $x \in \mathbb{R}^N$), then $\mathrm{Mod}(F + e) = 2\pi T^{-1}\mathbb{Z}$, and consequently the unique a.p. solution of the equation (1) necessarily is T-periodic. When e and $F(\cdot, x)$ are periodic with noncommensurable periods, we can simply say that there exists a unique a.p. solution of the equation (1).

In the next statement we treat the linear case.

Corollary. *Let A, $B \in BC^0(\mathbb{R}, \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N))$, $b \in BC^0(\mathbb{R}, \mathbb{R})$. We assume the following condition fulfilled:*

$$\exists c_* \in (0, +\infty), \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^N, \ \big(A(t) - \frac{B(t)B^*(t)}{4}\big)x \cdot x \geq c_* |x|^2.$$

Then the two following assertions hold.

i) *For each $e \in BC^0(\mathbb{R}, \mathbb{R}^N)$, there exists a unique*

$$u \in C^2(\mathbb{R}, \mathbb{R}^N) \cap BC^0(\mathbb{R}, \mathbb{R}^N)$$

which is a solution on $\mathbb{R}$ of the following ordinary differential equation:

$$\ddot{u}(t) + [b(t)I + B(t)]\dot{u}(t) - A(t)u(t) = e(t).$$

Moreover, we have $u \in BC^2(\mathbb{R}, \mathbb{R}^N)$.

ii) *If in addition we assume that A, $B \in AP^0(\mathcal{L}(\mathbb{R}^N, \mathbb{R}^N))$, and $b \in AP^0(\mathbb{R})$, then, for each $e \in AP^0(\mathbb{R}^N)$, this unique bounded solution u belongs to $AP^2(\mathbb{R}^N)$ and, if b and B are constant,* $\mathrm{Mod}(u) \subset \mathrm{Mod}(A + e)$.

Proof. We use Theorem 1 with the mapping $F(t, x) = A(t)x$.

Remarks. 1. In the corollary, we have no need of the symmetry of $A(t)$ and $B(t)$.

2. The assumption of the corollary is in particular satisfied if $A(t)v \cdot v \geq c|v|^2$ for all $t \in \mathbb{R}$, $v \in \mathbb{R}^N$ and some $c > \frac{\|B\|_\infty^2}{4}$.

4. Dependence results. In this section, we study the relations of continuity and differentiability between the bounded (respectively a.p.) forcing term e and the bounded (respectively a.p.) solution of the equation (1). We begin to establish several lemmas.

Lemma 4. *Let E be a Banach space, and $f \in \mathcal{U}(\mathbb{R} \times \mathbb{R}^N, E)$. Then the Nemytski operator $\mathcal{N}_f(u) := f(\cdot, u(\cdot))$ is well-defined and continuous from $(BC^0(\mathbb{R}, \mathbb{R}^N), \|\cdot\|_\infty)$ to $(BC^0(\mathbb{R}, E), \|\cdot\|_\infty)$.*

Proof. We fix $u \in BC^0(\mathbb{R}, \mathbb{R}^N)$. By using (U_3), we see that $f(\cdot, u(\cdot))$ is bounded on $\mathbb{R}$ since the closure of $u(\mathbb{R})$ is compact in $\mathbb{R}^N$, and $f(\cdot, u(\cdot))$ is continuous as a composition of continuous mappings.

We arbitrarily fix $\varepsilon > 0$. Since the closure of $u(\mathbb{R})$ is compact in $\mathbb{R}^N$, we take $r > 0$ and we obtain that $K := \{\xi \in \mathbb{R}^N \ : \ \operatorname{dist}(\xi, u(\mathbb{R})) \le r\}$ is compact. Taking $\beta_\varepsilon := \min\{r, \eta(K, \varepsilon)\}$, it follows from (U_4) that:

$$\sup_{t \in \mathbb{R}} \|f(t, u(t)) - f(t, v(t))\|_E \le \varepsilon \ \text{ when } \ \|u - v\|_\infty \le \beta_\varepsilon.$$

Lemma 5. *Let E be a Banach space, and $f : \mathbb{R} \times \mathbb{R}^N \longrightarrow E$, $(t, x) \longmapsto f(t, x)$, be a mapping which is a.p. in t uniformly for $x \in \mathbb{R}^N$ (cf. Section 1). Then we have*

i) *$f \in \mathcal{U}(\mathbb{R} \times \mathbb{R}^N, E)$.*

ii) *For each $u \in AP^0(\mathbb{R}^N)$, the function $t \mapsto f(t, u(t))$ belongs to $AP^0(E)$; i.e., the Nemytski operator $\mathcal{N}_f(u) := f(\cdot, u(\cdot))$ is well-defined from $AP^0(\mathbb{R}^N)$ to $AP^0(E)$.*

iii) *$\mathcal{N}_f$ is continuous from $(AP^0(\mathbb{R}^N), \|\cdot\|_\infty)$ to $(AP^0(E), \|\cdot\|_\infty)$.*

Proof. i) The conditions (U_1) and (U_3) are obviously satisfied. To prove (U_4), we reason by contradiction. Suppose that non-(U_4) is true, and so there exists $\varepsilon_0 > 0$, there exists $K_0 \in \mathcal{P}_c(\mathbb{R}^n)$, there exist sequences $(x_n)_n$, $(y_n)_n$, $(t_n)_n$ respectively with values in K, K, and $\mathbb{R}$ such that, for every positive integer n,

$$|x_n - y_n| \le \frac{1}{n}, \quad \text{and} \quad |f(t_n, x_n) - f(t_n, y_n)| > \varepsilon_0.$$

In the definition (5) of almost periodicity, taking $\varepsilon = \frac{\varepsilon_0}{5}$ and denoting $\ell := \ell(K_0, \frac{\varepsilon_0}{5})$, for each $\alpha = -t_n$, there exists $\tau_n \in [-t_n, -t_n + \ell)$ such that:

$$\sup_{t \in \mathbb{R}} \sup_{\xi \in K_0} |f(t + \tau_n, \xi) - f(t, \xi)| \le \frac{\varepsilon_0}{5}.$$

Therefore, for every positive integer n, we have:

$$|f(t_n + \tau_n, x_n) - f(t_n, x_n)| \le \frac{\varepsilon_0}{5},$$
$$|f(t_n + \tau_n, y_n) - f(t_n, y_n)| \le \frac{\varepsilon_0}{5}.$$

We note that $s_n := t_n + \tau_n \in [0, \ell]$. And so the three sequences $(s_n)_n$, $(x_n)_n$, $(y_n)_n$ take their values in compact sets, respectively $[0, \ell]$, K_0, and K_0. Then, by using the Bolzano-Weierstrass theorem, there exists a monotonically increasing mapping $n \longmapsto k_n$, from $\mathbb{N}$ to $\mathbb{N}$, there exist $\hat{s} \in [0, \ell]$ and $\hat{x} \in K$, such that $s_{k_n} \to \hat{s}$, $x_{k_n} \to \hat{x}$, $y_{k_n} \to \hat{x}$ when $n \to +\infty$. Since f is continuous, we have:

$$\Delta_n := |f(s_{k_n}, x_{k_n}) - f(\hat{s}, \hat{x})| \longrightarrow 0 \ \ (n \to +\infty),$$
$$\Gamma_n := |f(s_{k_n}, y_{k_n}) - f(\hat{s}, \hat{x})| \longrightarrow 0 \ \ (n \to +\infty),$$

and consequently, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Longrightarrow \Delta_n \leq \frac{\varepsilon_0}{5}$ and $\Gamma_n \leq \frac{\varepsilon_0}{5}$. Finally, for $n \geq n_0$, we have

$$\begin{aligned} \varepsilon_0 &< |f(t_{k_n}, x_{k_n}) - f(t_{k_n}, y_{k_n})| \\ &\leq |f(t_{k_n}, x_{k_n}) - f(s_{k_n}, x_{k_n})| + |f(s_{k_n}, x_{k_n}) - f(\hat{s}, \hat{x})| \\ &\quad + |f(\hat{s}, \hat{x}) - f(s_{k_n}, y_{k_n})| + |f(s_{k_n}, y_{k_n}) - f(t_{k_n}, y_{k_n})| \\ &\leq \frac{\varepsilon_0}{5} + \Delta_n + \Gamma_n + \frac{\varepsilon_0}{5} \leq \frac{4}{5}\varepsilon_0 < \varepsilon_0; \end{aligned}$$

that is the contradiction.

ii) cf. [21, page 17].

iii) Since $AP^0(\mathbb{R}^N) \subset BC^0(\mathbb{R}, \mathbb{R}^N)$, and since (i) holds, we can use Lemma 4 and assert that the restriction $\mathcal{N}_f$ is continuous from $AP^0(\mathbb{R}^N)$ to $BC^0(\mathbb{R}, E)$. And since $\mathcal{N}_f(AP^0(\mathbb{R}^N)) \subset AP^0(E)$, we have proven the sentence.

Proposition 2. *We assume the conditions* (M), (B_2), (B_5), *and* (B_6) *fulfilled. Let* $g_1, g_2 \in BC^0(\mathbb{R}, \mathbb{R}^N)$, $\tau \in \mathbb{R}$, *and let* $u, v \in C^2(\mathbb{R}, \mathbb{R}^N) \cap BC^0(\mathbb{R}, \mathbb{R}^N)$ *which verify, for every* $t \in \mathbb{R}$, *the following equations:*

$$\begin{cases} \ddot{u}(t) + [b(t)I + B(t)]\dot{u}(t) - F(t, u(t)) = g_1(t) \\ \ddot{v}(t) + [b(t)I + B(t)]\dot{v}(t) - F(t, v(t)) = g_2(t). \end{cases}$$

Then we have

$$\sup_{t \in \mathbb{R}} |u(t) - v(t)| \leq c_*^{-1} \sup_{t \in \mathbb{R}} |g_1(t) - g_2(t)|.$$

Proof. We set

$$\varrho(t) := \frac{1}{2}|u(t) - v(t)|^2, \quad \Delta(t) := g_1(t) - g_2(t), \quad p(t) := \Delta(t) \cdot (u(t) - v(t)).$$

Then we have

$$\begin{aligned} \ddot{\varrho}(t) &= |\dot{u}(t) - \dot{v}(t)|^2 + (u(t) - v(t)) \cdot (\ddot{u}(t) - \ddot{v}(t)) \\ &= |\dot{u}(t) - \dot{v}(t)|^2 - (u(t) - v(t)) \cdot (b(t)I + B(t))(\dot{u}(t) - \dot{v}(t)) \\ &\quad + (u(t) - v(t)) \cdot (F(t, u(t)) + g_1(t) - F(t, v(t)) - g_2(t)) \\ &= |\dot{u}(t) - \dot{v}(t)|^2 - (B^*(t)(u(t) - v(t))) \cdot (\dot{u}(t) - \dot{v}(t)) \\ &\quad - b(t)(u(t) - v(t)) \cdot (\dot{u}(t) - \dot{v}(t)) + (u(t) - v(t)) \cdot \\ &\quad \cdot (F(t, u(t)) - F(t, v(t))) + (u(t) - v(t)) \cdot \Delta(t) \end{aligned}$$

$$= |\dot{u}(t) - \dot{v}(t) - \frac{1}{2}B^*(t)(u(t) - v(t))|^2$$
$$+ \big(F(t, u(t)) - F(t, v(t)) - \frac{1}{4}B(t)B^*(t)(u(t) - v(t))\big)\cdot$$
$$\cdot (u(t) - v(t)) - b(t)\dot{\varrho}(t) + p(t) \geq 2c_*\varrho(t) - b(t)\dot{\varrho}(t) + p(t).$$

And so we can use Lemma 3, with $\alpha = 2c_*$, $r = \varrho$, $\gamma(t) = -b(t)$, and $\beta = -\sup_{t\in\mathbb{R}} |p(t)|$, and we obtain

$$\sup_{t\in\mathbb{R}} \varrho(t) \leq (2c_*)^{-1} \sup_{t\in\mathbb{R}} |p(t)|;$$

i.e.,

$$\frac{1}{2}\|u - v\|_\infty^2 \leq \frac{1}{2c_*}\|\Delta \cdot (u - v)\|_\infty \leq \frac{1}{2c_*}\|\Delta\|_\infty.\|u - v\|_\infty,$$

which implies the announced inequality. □

Now we can state the theorem about continuous dependence.

Theorem 2. *We assume the conditions* (M), (B_3), (B_5) *and* (B_6) *fulfilled. For each* $e \in BC^0(\mathbb{R}, \mathbb{R}^N)$*, we denote by* $\mathcal{X}_{[e]}$ *the unique solution in* $BC^2(\mathbb{R}, \mathbb{R}^N)$ *of the equation* $\ddot{x} + (b(t)I + B(t))\dot{x} - F(t, x) = e$ *(cf. Theorem* 1*), and we denote by* $\mathcal{X} : e \longmapsto \mathcal{X}_{[e]}$ *the operator from* $BC^0(\mathbb{R}, \mathbb{R}^N)$ *in* $BC^2(\mathbb{R}, \mathbb{R}^N)$*.*

Then the following assertions hold:

i) *For every* $e_1, e_2 \in BC^0(\mathbb{R}, \mathbb{R}^N)$*,*

$$\|\mathcal{X}_{[e_1]} - \mathcal{X}_{[e_2]}\|_\infty \leq c_*^{-1}\|e_1 - e_2\|_\infty.$$

ii) *The operator* $\mathcal{X}$ *is an homeomorphism between* $(BC^0(\mathbb{R}, \mathbb{R}^N), \|\cdot\|_\infty)$ *and* $(BC^2(\mathbb{R}, \mathbb{R}^N), \|\cdot\|_{C^2})$

iii) *If in addition we assume* (A_2), (A_4) *and* (A_5) *fulfilled (instead of* (B_3), (B_5) *and* (B_6)*), the operator* $\mathcal{X}$ *is a homeomorphism between the spaces* $(AP^0(\mathbb{R}^N), \|.\|_\infty)$ *and* $(AP^2(\mathbb{R}^N), \|.\|_{C^2})$*.*

Proof. i) It is a consequence of Proposition 2, taking $g_1 = e_1$, $g_2 = e_2$, $u = \mathcal{X}_{[e_1]}$, and $v = \mathcal{X}_{[e_2]}$.

ii) To abridge the writing, we denote by BC^k the space $BC^k(\mathbb{R}, \mathbb{R}^N)$. By the assertion (i), we have:

$$\mathcal{X} \text{ is continuous from } (BC^0, \|\cdot\|_\infty) \text{ in } (BC^0, \|\cdot\|_\infty). \tag{17}$$

We set

$$v(t) := \mathcal{X}_{[e_2]} - \mathcal{X}_{[e_1]}, \qquad C(t) := b(t)I + B(t),$$
$$X_1(t, x) := F(t, x) + e_1(t), \quad X_2(t, x) := F(t, x) + e_2(t).$$

Then, we have

$$\ddot{v}(t) + C(t)\dot{v}(t) = X_2(t, \mathcal{X}_{[e_2]}) - X_1(t, \mathcal{X}_{[e_1]}).$$

We set

$$\begin{cases} \alpha := \|\mathcal{X}_{[e_2]} - \mathcal{X}_{[e_1]}\|_\infty \\ \beta := \|\mathcal{N}_F(\mathcal{X}_{[e_2]}) - \mathcal{N}_F(\mathcal{X}_{[e_1]})\|_\infty + \|e_2 - e_1\|_\infty \\ \gamma := \|C\|_\infty. \end{cases}$$

We have $\alpha < +\infty$, $\beta < +\infty$ (Lemma 4) and $\gamma < +\infty$ (Assumptions (B_5) and (B_6)). We note that we have $\|v\|_\infty \leq \alpha$, $\|\ddot{v} + C(\cdot)\dot{v}\|_\infty \leq \beta$, and $\|C\|_\infty \leq \gamma$, then we can use Lemma 2 and assert that

$$\|\dot{v}\|_\infty \leq 2\alpha(\gamma + \sqrt{\gamma^2 + \beta^2}) \leq 4\alpha(\gamma + \beta).$$

Consequently we have

$$\|\dot{\mathcal{X}}_{[e_2]} - \dot{\mathcal{X}}_{[e_1]}\|_\infty \leq$$
$$4\|\mathcal{X}_{[e_2]} - \mathcal{X}_{[e_1]}\|_\infty \big(\|C\|_\infty + \|\mathcal{N}_F(\mathcal{X}_{[e_2]}) - \mathcal{N}_F(\mathcal{X}_{[e_1]})\|_\infty + \|e_2 - e_1\|_\infty\big).$$

Then by using Lemma 4 and (17), we obtain that

$$e \longmapsto \dot{\mathcal{X}}_{[e]} \text{ is continuous from } (BC^0, \|\cdot\|_\infty) \text{ in } (BC^0, \|\cdot\|_\infty). \tag{18}$$

We have

$$\ddot{\mathcal{X}}_{[e]} = \mathcal{K}(\cdot, \dot{\mathcal{X}}_{[e]}) + F(\cdot, \mathcal{X}_{[e]}) + e = \mathcal{N}_\mathcal{K}(\dot{\mathcal{X}}_{[e]}) + \mathcal{N}_F(\mathcal{X}_{[e]}) + e,$$

with $\mathcal{K}(t, x) := -C(t)x$. Then by using Lemma 4, (17) and (18), we obtain

$$e \longmapsto \ddot{\mathcal{X}}_{[e]} \text{ is continuous from } (BC^0, \|\cdot\|_\infty) \text{ in } (BC^0, \|\cdot\|_\infty). \tag{19}$$

Finally, the assertions (17), (18), (19) imply:

$$\mathcal{X} : (BC^0, \|\cdot\|_\infty) \longrightarrow (BC^2, \|\cdot\|_{C^2}) \text{ is continuous.}$$

The existence and the uniqueness provided by Theorem 1 imply that $\mathcal{X}$ is a bijection between BC^0 and BC^2. Its inverse operator is the following nonlinear differential operator:

$$\mathcal{T}(x) := \ddot{x} - \mathcal{K}(\cdot, \dot{x}) - F(\cdot, x) = \frac{d^2}{dt^2}x - \mathcal{N}_\mathcal{K}(In_1(x)) - \mathcal{N}_F(In_2(x)), \tag{20}$$

where $In_1 : BC^1 \longrightarrow BC^0$, $In_1(x) := x$, and $In_1 : BC^2 \longrightarrow BC^0$, $In_2(x) := x$.

By the definition of the norms $\|\cdot\|_{C^1}$ and $\|\cdot\|_{C^2}$, In_1, In_2 and $\frac{d^2}{dt^2}$ are continuous (linear) operators, and by Lemma 4, $\mathcal{N}_F$ and $\mathcal{N}_\mathcal{K}$ are also continuous, therefore $\mathcal{T}$ is continuous, and $\mathcal{X}$ is a homeomorphism.

iii) By using Theorem 1 (ii) we have $\mathcal{X}(AP^0(\mathbb{R}^N)) \subset AP^2(\mathbb{R}^N))$, and by using Lemma 5, we have $\mathcal{T}(AP^2(\mathbb{R}^N)) \subset AP^0(\mathbb{R}^N))$, thus (iii) is a consequence of (ii). □

Now in order to study the differentiable dependence, we begin to establish the differentiability of the Nemytski operators.

Lemma 6. *Let E be a Banach space, and $f \in \mathcal{U}(\mathbb{R} \times \mathbb{R}^N, E)$. We assume that, for every $t \in \mathbb{R}$, the partial mapping $f(t,\cdot)$ is Fréchet differentiable on $\mathbb{R}^N$, and also that $f_x \in \mathcal{U}(\mathbb{R} \times \mathbb{R}^N, \mathcal{L}(\mathbb{R}^N, E))$.*

Then the Nemytski operator $\mathcal{N}_f : BC^0(\mathbb{R}, \mathbb{R}^N) \longrightarrow BC^0(\mathbb{R}, E)$ is of class C^1, and for every $u, h \in BC^0(\mathbb{R}, \mathbb{R}^N)$, we have:

$$(\mathcal{N}_f)'(u) \cdot h = [t \longmapsto f_x(t, u(t) \cdot h(t)].$$

Proof. Let u and K be defined in the same manner as in the proof of Lemma 4. We arbitrarily fix an $\varepsilon > 0$.

Since f_x satisfies the assertion (U_4), there exists $\eta_\varepsilon \in (0, r]$ such that, for every $x, y \in K$,

$$|x - y| \leq \eta_\varepsilon \Longrightarrow (\forall t \in \mathbb{R},\ \ \|f_x(t,x) - f_x(t,y)\| \leq \varepsilon).$$

We take $h \in BC^0(\mathbb{R}^N)$ such that $\|h\|_\infty \leq \eta_\varepsilon$. By using the mean value inequality [1, p. 144], for every $t \in \mathbb{R}$, we obtain:

$$\begin{aligned} &\|f(t, u(t) + h(t)) - f(t, u(t)) - f_x(t, u(t)) \cdot h(t)\| \\ &\quad \leq \big(\sup\big\{\|f_x(t, u(t)) - f_x(t, \xi)\| : \xi \in (u(t), u(t) + h(t))\big\}\big)|h(t)|. \end{aligned}$$

For every $\xi \in (u(t), u(t) + h(t))$, we have $|\xi| \leq |h(t)| \leq \eta_\varepsilon$, therefore

$$\|f_x(t, u(t)) - f_x(t, \xi)\| \leq \varepsilon,$$

and consequently

$$\|f(t, u(t) + h(t)) - f(t, u(t)) - f_x(t, u(t)) \cdot h(t)\| \leq \varepsilon |h(t)|,$$

which implies

$$\|\mathcal{N}_f(u + h) - \mathcal{N}_f(u) - f_x(\cdot, u) \cdot h\|_\infty \leq \varepsilon \|h\|_\infty.$$

That proves the differentiability of $\mathcal{N}_f$ and the validity of the announced formula of the differential of $\mathcal{N}_f$.

The continuity of $(\mathcal{N}_f)'$ results from the continuity of the Nemytski operator $\mathcal{N}_{f_x}$ provided by Lemma 4 in which we replace f by f_x.

Lemma 7. *Let E be a Banach space, and $f : \mathbb{R} \times \mathbb{R}^N \longrightarrow E$, $(t, x) \longmapsto f(t, x)$, be a mapping which is a.p. in t uniformly for $x \in \mathbb{R}^N$ (cf. Section 1). We also assume that for every $t \in \mathbb{R}$, $f(t, \cdot)$ is Fréchet differentiable on $\mathbb{R}^N$, and that the partial differential f_x is a.p. in t uniformly for $x \in \mathbb{R}^N$. Then the Nemytski operator $\mathcal{N}_f$ is of class C^1 from $AP^0(\mathbb{R}^N)$ to $AP^0(E)$, and for every $u, h \in AP^0(\mathbb{R}^N)$, we have: $(\mathcal{N}_f)'(u) \cdot h = [t \longmapsto f_x(t, u(t)) \cdot h(t)]$.*

Proof. By using Lemma 5 (i), since f and f_x are a.p. in t uniformly for $x \in \mathbb{R}^N$, we have $f \in \mathcal{U}(\mathbb{R} \times \mathbb{R}^N, E)$ and $f_x \in \mathcal{U}(\mathbb{R} \times \mathbb{R}^N, \mathcal{L}(\mathbb{R}^N, E))$. Thus Lemma 7 is a consequence of Lemma 6. $\square$

In the autonomous case, i.e., when f does not depend on t, the study of the continuity and the differentiability of the Nemytski operators on the spaces BC^0 is treated in [9, Section 2], and on the space $AP^0(\mathbb{R}^N)$ is treated in [6, page 19]. Lemma 7 is stated in [11, page 997].

Theorem 3. *We assume the conditions* (M), (B_3), (B_4), (B_5) *and* (B_6) *fulfilled. We use the notations of Theorem* 2. *Then the following assertions hold:*

i) $\mathcal{X} : (BC^0(\mathbb{R}, \mathbb{R}^N), \|\cdot\|_\infty) \longrightarrow (BC^2(\mathbb{R}, \mathbb{R}^N), \|\cdot\|_{C^2})$ *is a* C^1*-diffeomorphism.*

ii) *If in addition we assume the conditions* (A_2), (A_3), (A_4) *and* (A_5) *fulfilled* (*instead of* (B_3), (B_4), (B_5) *and* (B_6)), *then* $\mathcal{X}$ *is a* C^1*-diffeomorphism between the space* $(AP^0(\mathbb{R}, \mathbb{R}^N), \|\cdot\|_\infty)$ *and the space* $(AP^2(\mathbb{R}, \mathbb{R}^N), \|\cdot\|_{C^2})$.

Proof. i) To abridge the writing, we denote by BC^p the space $BC^p(\mathbb{R}, \mathbb{R}^N)$. We consider the operator $\mathcal{T}$ defined in the formula (20). Since $\frac{d^2}{dt^2}$ and In_2 are linear continuous operators from BC^2 in BC^0, and since In_1 is a linear continuous operator from BC^1 in BC^0, they are of class C^1.

By using Lemma 6, with $f = F$ and $f = \mathcal{K}$, we see that $\mathcal{N}_F$ and $\mathcal{N}_\mathcal{K}$ are of class C^1. And so, $\mathcal{T}$ is of class C^1 as a composition of C^1-operators, and by using the linearity of the differentiation and the chain rule, we obtain, for every $u, h \in BC^2$,

$$\mathcal{T}'(u) \cdot h = [t \longmapsto \ddot{h}(t) + (b(t)I + B(t))\dot{h}(t) - F_x(t, u(t))h(t)]. \tag{21}$$

And so, when we fix $u \in BC^2$, for $k \in BC^0$ and $h \in BC^2$, to study the the equation $\mathcal{T}'(u) \cdot h = k$ (where h is the unknown) is equivalent to studying the bounded solutions of the following linear second-order differential the equation:

$$\ddot{h}(t) + (b(t)I + B(t))\dot{h}(t) - F_x(t, u(t))h(t) = k(t). \tag{22}$$

We set $A(t) := F_x(t, u(t))$ and we use the corollary of Section 3. And so there exists a unique solution $h \in BC^2$ of the equation (22). Translating this result in terms of $\mathcal{T}'$, we obtain:

$$\mathcal{T}'(u) : BC^2 \longrightarrow BC^0 \ \text{ is } \ \text{an } \ \text{isomorphism} , \tag{23}$$

where isomorphism means isomorphism of topological vector spaces.

Finally, $\mathcal{T} : BC^2 \longrightarrow BC^0$ is an homeomorphism of class C^1 such that, for every $u \in BC^2$, $\mathcal{T}'(u)$ is invertible, therefore $\mathcal{T}$ is a C^1-diffeomorphism ([10, page 55]), and since $\mathcal{X} = \mathcal{T}^{-1}$ we have proven the assertion (i).

ii) It is a consequence of (i), of Theorem 2, and of Lemma 7.

5. Perturbation by a damping term. In this section, we consider the equation (4); i.e., we introduce a damping term in the equation (1). By using the implicit function theorem and Theorem 1, we show the existence of bounded solutions and a.p. solutions when the perturbation parameter ε is small enough. We introduce the mapping

$$G : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad (t, x, v) \longmapsto G(t, x, v).$$

About G we formulate the next list of conditions:

(D_1) $G \in \mathcal{U}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R}^N)$, and for every $(t,x,y)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$, the partial mapping $G(t,\cdot,\cdot)$ is Fréchet differentiable at (x,v), its partial differentials are denoted by $G_x(t,x,v)$ and $G_{\dot{x}}(t,x,v)$, and $G_x, G_{\dot{x}} \in \mathcal{U}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R}^{N^2})$.

(D_2) G is a.p. in t uniformly for $(x,v)\in\mathbb{R}^N\times\mathbb{R}^N$, and for every $(t,x,y)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$, the partial mapping $G(t,\cdot,\cdot)$ is Fréchet differentiable at (x,v), and G_x and $G_{\dot{x}}$ are a.p. in t uniformly for $(x,v)\in\mathbb{R}^N\times\mathbb{R}^N$.

For each $\varepsilon\in\mathbb{R}$, we consider the following second-order differential equation:

$$\ddot{x}(t)+[b(t)I+B(t)]\dot{x}(t)-F(t,x(t))+\varepsilon G(t,x(t),\dot{x}(t))=e(t). \qquad (4)_\varepsilon$$

Theorem 4. *We assume the conditions* (M), (B_1), (B_3), (B_4), (B_5), (B_6) *and* (D_1) *fulfilled. Then the following assertions hold:*

i) *There exists $\varepsilon_0>0$ such that for each $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, there exists a solution $x_\varepsilon\in BC^2(\mathbb{R},\mathbb{R}^N)$ of the equation $(4)_\varepsilon$.*

ii) *If, in addition, we assume the conditions* (A_1), (A_2), (A_3), (A_4), (A_5) *and* (D_2) *fulfilled* (*instead of* (B_1), (B_3), (B_4), (B_5), (B_6) *and* (D_1)), *then there exists $\varepsilon_0>0$ such that for each $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, there exists a solution $x_\varepsilon\in AP^2(\mathbb{R}^N)$ of the equation $(4)_\varepsilon$.*

Proof. i) By the hypotheses, we can consider the nonlinear operator:

$$\Phi: BC^2(\mathbb{R},\mathbb{R}^N)\times\mathbb{R}\longrightarrow BC^0(\mathbb{R},\mathbb{R}^N)$$
$$\Phi(u,\varepsilon):=\ddot{u}+[b(\cdot)I+B(\cdot)]\dot{u}-F(\cdot,u)+\varepsilon G(\cdot,u,\dot{u})+e.$$

We define the linear operator

$$j: BC^2(\mathbb{R},\mathbb{R}^N)\longrightarrow BC^0(\mathbb{R},\mathbb{R}^N),\quad j(u):=(u,\dot{u}).$$

We easily see that j is continuous since $BC^2(\mathbb{R},\mathbb{R}^N)$ is endowed with the norm $\|\cdot\|_{C^2}$, and consequently j is of class C^1.

We consider the Nemytski operator built on G,

$$\mathcal{N}_G: BC^0(\mathbb{R},\mathbb{R}^N)\times BC^0(\mathbb{R},\mathbb{R}^N)=BC^0(\mathbb{R},\mathbb{R}^N\times\mathbb{R}^N)\longrightarrow AP^0(\mathbb{R}^N),$$
$$\mathcal{N}_G(u,v):=[t\longmapsto G(t,u(t),v(t))].$$

Under (D_1), we can apply Lemma 7 on G and assert that $\mathcal{N}_G$ is of class C^1.

By using the operator $\mathcal{T}$ defined by the formula (20), we can express Φ in the following form:

$$\Phi(u,\varepsilon)=\mathcal{T}(u)+\varepsilon\mathcal{N}_G\circ j(u)+e. \qquad (24)$$

By using Lemma 7 and the formula (21), we see that, for every $\varepsilon \in \mathbb{R}$, $\Phi(\cdot, \varepsilon)$ is of class C^1, and that, for every $h \in BC^2(\mathbb{R}, \mathbb{R}^N)$, we have:

$$\Phi_u(u, \varepsilon) \cdot h = \mathcal{T}'(u) \cdot h + \varepsilon \mathcal{N}'_G(j(u)) \cdot j(h), \tag{25}$$

therefore Φ_u is continuous on $BC^2(\mathbb{R}, \mathbb{R}^N) \times \mathbb{R}$.

On the other hand, we calculate $\Phi_\varepsilon(u, \varepsilon) = \mathcal{N}_G \circ j(u)$, and consequently Φ_ε is also continuous on $BC^2(\mathbb{R}, \mathbb{R}^N) \times \mathbb{R}$. Then, by using a usual result of differential calculus about the partial differentials, we have proven:

$$\Phi \in C^1(BC^2(\mathbb{R}, \mathbb{R}^N) \times \mathbb{R}, BC^0(\mathbb{R}, \mathbb{R}^N)). \tag{26}$$

Since e is fixed in $BC^0(\mathbb{R}, \mathbb{R}^N)$, by using Theorem 1 (i), we can assert that there exists $u_0 \in BC^2(\mathbb{R}, \mathbb{R}^N)$ a solution of the equation $(4)_0 = (1)$ on $\mathbb{R}$; i.e., we have:

$$\Phi(u_0, 0) = 0. \tag{27}$$

By the formula (25), we have $\Phi_u(u_0, 0) \cdot h = \mathcal{T}'(u_0) \cdot h$, and by the sentence (23), $\mathcal{T}'(u_0)$ is an isomorphism between $BC^2(\mathbb{R}, \mathbb{R}^N)$ and $BC^0(\mathbb{R}, \mathbb{R}^N)$, therefore we have:

$$\Phi_u(u_0, 0) \ \text{is} \quad \text{invertible}. \tag{28}$$

And so, with (26), (27), (28), we can use the implicit function theorem ([10, page 61]) and assert that there exists $\varepsilon_0 > 0$, there exists a C^1-mapping $\varepsilon \longmapsto x_\varepsilon$, from $(-\varepsilon_0, \varepsilon_0)$ to $BC^2(\mathbb{R}, \mathbb{R}^N)$ such that, for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we have $\Phi(x_\varepsilon, \varepsilon) = 0$; i.e., x_ε is a solution of the equation $(4)_\varepsilon$.

ii) The reasoning is similar to that of assertion (i).

Added in Proof. After this paper was accepted we learned about the manuscript "Forced systems with almost periodic and quasiperiodic forcing term" by C. Carminati, showing by a variational method that, in the special case of (3), the Berger-Chen growth condition described in Comments 2 is superfluous. So our Theorem 1 also extends Carminati's result to system (1).

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