Advances in Differential Equations Volume 3, Number 4, July 1998, pp. 595–624.

SOLVABILITY CONDITIONS FOR SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS AT RESONANCE WITH BOUNDED AND UNBOUNDED NONLINEAR TERMS

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Abstract. In this paper we generalize and improve some well-known solvability conditions for semilinear elliptic boundary value problems at resonance within the context of Besov and Triebel-Lizorkin spaces. In particular we will show that many such solvability conditions can be viewed as special cases of a single generalized Landesman-Lazer condition. Our methods apply an adaptation of Leray-Schauder degree ideas to quasi-Banach spaces.

1. Introduction. Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, let A, defined by

$$(Au)(x) = \sum_{|\alpha|\leq 2} a_\alpha(x) D^\alpha u(x),\, x \in \Omega, \quad a_\alpha \in C^\infty(\bar{\Omega})\, \forall \mid \alpha \mid \leq 2,$$

be a uniformly elliptic differential operator of order 2 in Ω, and let B, defined by

$$(Bu)(y) = \sum_{|\alpha|\leq d} b_\alpha(y) D^\alpha u(y),\, y \in \partial\Omega, \quad b_\alpha \in C^\infty(\partial\Omega)\, \forall \mid \alpha \mid \leq d \leq 1,$$

be a boundary operator of order d such that $\{A, B\}$ is regular elliptic. For example we can let

$$Au = -\Delta u \quad \text{and} \quad Bu = u_{|\partial\Omega}.$$

Received for publication January 1998.
*Supported partly by Deutsche Forschungsgemeinschaft, Tr 374/1-1.
AMS Subject Classifications: 34, 35.

We suppose that λ is a real eigenvalue associated with $\{A, B\}$, and note that the properties of $\{A, B\}$ imply that the corresponding eigenspace is finite-dimensional and consists of real-valued $\mathbb{C}^\infty$-functions. We are looking for solutions in $\mathbb{F}^s_{p,q}(\Omega)$ and $\mathbb{B}^s_{p,q}(\Omega)$, Triebel-Lizorkin and Besov spaces, respectively, of the following semilinear boundary value problem

$$Au - \lambda u + g(u) = f \quad \text{in} \quad \Omega, \qquad Bu = 0 \quad \text{on} \quad \partial\Omega, \tag{1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a $\mathbb{C}^\infty$-function satisfying a linear growth condition, and f is a given data function belonging to a suitably chosen function space of Besov or Triebel-Lizorkin type. In this paper we formulate some solvability conditions of Landesman-Lazer type for such boundary value problems. The purpose is to show that many well-known existence results of semilinear elliptic boundary value problems, with perturbations that satisfy bounded, sublinear or linear growth conditions, can be derived from abstract existence theorems similar to those in S. Robinson [27] and A. Rumbos [29]. In that sense our paper is a generalization of the recent work by S. Robinson and E.M. Landesman [28] within the framework of the two scales of function spaces of Besov and Triebel-Lizorkin type, which cover many well-known spaces, for example classical Sobolev spaces, Bessel-potential spaces, Hardy spaces, classical Besov spaces and Hölder-Zygmund spaces. It should also be noted that in some cases our results significantly sharpen those of [28], particularly in the case of strongly resonant problems involving bounded and sublinear nonlinear terms. (See Theorem 4 and Remark 11.)

2. Spaces of Besov and Triebel-Lizorkin type.

2.1. Definitions and preparations. For our purpose Besov spaces, $B^s_{p,q}(\mathbb{R}^n)$, and Triebel-Lizorkin spaces, $F^s_{p,q}(\mathbb{R}^n)$, are of special interest. We start by listing some basic material. These types of function spaces are studied systematically in H. Triebel [33], [34], and we refer the reader to these works for complete proofs and definitions. Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of rapidly decreasing functions on the euclidean n-space $\mathbb{R}^n$, and let $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ be its topological dual. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ with

$$\text{supp } \phi \subset \{\xi \in \mathbb{R}^n, |\xi| \le 2\}, \quad \phi(\xi) = 1 \ \text{ if } |\xi| \le 1.$$

Let $j \in \mathbb{N}$ and

$$\phi_j(x) = \phi(2^{-j}\xi) - \phi(2^{-j+1}\xi), \, \xi \in \mathbb{R}^n.$$

Then we have

$$\text{supp } \phi_j \subset \{\xi \in \mathbb{R}^n,\, 2^{j-1} \leq |\xi| \leq 2^{j+1}\},\, j \in \mathbb{N},$$

and with $\phi_0 = \phi$,

$$\sum_{j=0}^{\infty} \phi_j(x) = 1 \quad \text{if } \xi \in \mathbb{R}^n.$$

We call these a dyadic resolution of unity.

Definition 1. Let $s \in \mathbb{R}, 0 < p \leq \infty, 0 < q \leq \infty$. (i) Then

$$B^s_{p,q}(\mathbb{R}^n) = \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : \\ \|f|B^s_{p,q}(\mathbb{R}^n)\| = \Big(\sum_{j=0}^{\infty} 2^{jsq} \big(\int_{\mathbb{R}^n} \mid \mathcal{F}^{-1}\phi_j \mathcal{F} f(x) \mid^p dx \big)^{\frac{q}{p}} \Big)^{\frac{1}{q}} < \infty \Big\}.$$

(ii) Let $p < \infty$. Then

$$F^s_{p,q}(\mathbb{R}^n) = \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : \\ \|f|F^s_{p,q}(\mathbb{R}^n)\| = \Big(\int_{\mathbb{R}^n} \big(\sum_{j=0}^{\infty} 2^{jsq} \mid \mathcal{F}^{-1}\phi_j \mathcal{F} f(x) \mid^q \big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}} < \infty \Big\}$$

(with obvious modification for $p = \infty$ and/or $q = \infty$).

Here $\mathcal{F}$ stands for the Fourier transform in $\mathcal{S}'$ and $\mathcal{F}^{-1}$ for its inverse.

$$\|f|B^s_{p,q}(\mathbb{R}^n)\| \quad \text{and} \quad \|f|F^s_{p,q}(\mathbb{R}^n)\|$$

are quasi-norms (norms if $p, q \geq 1$). The spaces $F^s_{p,q}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ become Banach spaces for $p, q \geq 1$ and quasi-Banach spaces otherwise. They are independent of the generating function ϕ with the above properties (equivalent quasi-norms).

We have the following continuous embeddings:

$$\mathcal{S} \hookrightarrow B^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}' \text{ and } \mathcal{S} \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'.$$

Remark 1. The two scales of spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ cover many well-known function spaces. We give some special examples, for details we refer to H. Triebel [33].

(i) $F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n), 1 < p < \infty$ (Lebesgue spaces),

(ii) $F^s_{p,2}(\mathbb{R}^n) = W^s_p(\mathbb{R}^n), s$ a natural number, $1 < p < \infty$ (Sobolev spaces),
(iii) $F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n), 1 < p < \infty, s \in \mathbb{R}$ (fractional Sobolev spaces),
(iv) $F^0_{p,2}(\mathbb{R}^n) = h_p(\mathbb{R}^n), 0 < p < \infty$ (inhomogeneous Hardy spaces),
(v) $B^s_{p,q}(\mathbb{R}^n), s > 0, 1 < p < \infty, 1 \le q \le \infty$ (classical Besov spaces),
(vi) $B^s_{\infty,\infty}(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n), s > 0$ (Hölder- Zygmund spaces), and
(vii) $B^s_{2,2}(\mathbb{R}^n) = F^s_{2,2}(\mathbb{R}^n) = H^s_2(\mathbb{R}^n)$.

For dealing with boundary value problems it is useful to define Besov and Triebel-Lizorkin spaces on domains. Let Ω be a bounded C^∞- domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. Then one can introduce the spaces $B^s_{p,q}(\partial\Omega)$ and $F^s_{p,q}(\partial\Omega)$ by a standard procedure via local charts, see H. Triebel [33, 3.2]. The spaces $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$ are then defined by the restriction method, as follows. If $g \in \mathcal{S}'$, then the restriction of g to Ω is an element of $\mathcal{D}'(\Omega)$ which will be denoted by $g_{\,|\Omega}$. We recall that $\mathcal{D}(\Omega)$ is the collection of all complex-valued infinitely differentiable functions f in $\mathbb{R}^n$ with supp $f \subset \Omega$ and $\mathcal{D}'(\Omega)$ is the dual space, respectively.

Definition 2.

(i) Let $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Then

$$B^s_{p,q}(\Omega) = \{f \in \mathcal{D}'(\Omega) : \exists g \in B^s_{p,q}(\mathbb{R}^n) \text{ with } g_{\,|\Omega} = f\}, \text{ and}$$

$$\|f \mid B^s_{p,q}(\Omega)\| = \inf \|g \mid B^s_{p,q}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in B^s_{p,q}(\mathbb{R}^n)$ with $g_{\,|\Omega} = f$.

(ii) Let $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Then

$$F^s_{p,q}(\Omega) = \{f \in \mathcal{D}'(\Omega) : \exists g \in F^s_{p,q}(\mathbb{R}^n) \text{ with } g_{|\Omega} = f\}, \text{ and}$$

$$\|f \mid F^s_{p,q}(\Omega)\| = \inf \|g \mid F^s_{p,q}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in F^s_{p,q}(\mathbb{R}^n)$ in the above sense.

Convention: For a simpler exposition throughout the paper, only *additional* restrictions on p, q, and s will be given. Thus if an assertion contains no assumptions on p, q, or s, then it holds for all admissible values in Definition 2/2, above. In particular, if not otherwise stated, we always assume that $p < \infty$ for $F^s_{p,q}(\Omega)$ spaces.

The following continuous embeddings are important for our considerations, see H. Triebel [33, Theorem 3.3.1]. $C(\bar{\Omega})$ denotes, as usual, the space of all continuous functions on $\bar{\Omega}$.

$$B^s_{p,q}(\Omega) \hookrightarrow C(\bar{\Omega}) \;\text{ if }\; s > \frac{n}{p} \;\text{ or }\; s = \frac{n}{p} \text{ and } 0 < q \leq 1, \tag{1}$$

$$F^s_{p,q}(\Omega) \hookrightarrow C(\bar{\Omega}) \;\text{ if } s > \frac{n}{p} \text{ or } s = \frac{n}{p} \text{ and } 0 < p \leq 1. \tag{2}$$

Let $-\infty < s_1 < s_0 < \infty$. Then

$$B^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow B^{s_1}_{p_1,q_1}(\Omega) \text{ if } s_0 - \frac{n}{p_0} > s_1 - \frac{n}{p_1}. \tag{3}$$

Let $-\infty < s_1 < s_0 < \infty$. Then

$$F^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow F^{s_1}_{p_1,q_1}(\Omega) \text{ if } s_0 - \frac{n}{p_0} \geq s_1 - \frac{n}{p_1}. \tag{4}$$

Furthermore, we have

$$B^0_{\infty,1}(\Omega) \hookrightarrow L_\infty(\Omega) \hookrightarrow B^0_{\infty,\infty}(\Omega). \tag{5}$$

For proofs we refer to H. Triebel [33, Theorem 3.3.1].

2.2. Regular elliptic boundary value problems. Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, and let ν represent the unit outward normal with respect to the boundary. It is well known that for

$$s > s_0 := \frac{1}{p} + \max\big(0, (n-1)(\frac{1}{p} - 1)\big) \tag{1}$$

the spaces $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$ admit traces on $\partial\Omega$. We refer to H.Triebel [Theorem 3.3.3]. These boundary values can be characterized in a similar way, namely as $B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)$ and $B^{s-\frac{1}{p}}_{p,p}(\partial\Omega)$, respectively. If $s > s_0$ and if $k = 0, 1, \ldots$, then the trace operator, Tr, defined by

$$Trf = \{f_{|\partial\Omega}, \frac{\partial f}{\partial\nu}_{|\partial\Omega}, \ldots, \frac{\partial^k f}{\partial\nu^k}_{|\partial\Omega}\},$$

is a linear and continuous map

$$\text{from } B^{s+k}_{p,q}(\Omega) \text{ onto } \prod_{j=0}^{k} B^{s-j-1/p+k}_{p,q}(\partial\Omega)$$

and

$$\text{from } F^{s+k}_{p,q}(\Omega) \text{ onto } \prod_{j=0}^{k} B^{s-j-1/p+k}_{p,p}(\partial\Omega).$$

We consider regular elliptic boundary value problems in the sense of S. Agmon, A. Douglis and L. Nirenberg [1]. Given A and B, as before, and a real scalar λ we consider

$$\begin{aligned} (Au-\lambda u)(x) &= f(x) \quad \text{if } x\in\Omega, \\ (Bu)(y) &= 0 \quad \text{if } y\in\partial\Omega. \end{aligned} \tag{2}$$

In J. Franke, T. Runst [12] the following results were proved (see also H. Triebel [33, Chapter 4] in the case of Besov spaces).

Proposition 1. *Let $\{A-\lambda, B\}$ be the above system and let $s > s_0 + d$.*

(i) *If $u \in F^s_{p,q}(\Omega)\cup B^s_{p,q}(\Omega)$ with $Au-\lambda u = 0, Bu = 0$, then $u\in C^\infty(\bar\Omega)$. Hence $\{A-\lambda, B\}$ has a finite-dimensional kernel*

$$\ker\{A-\lambda, B\} = \{u\in C^\infty(\bar\Omega) : Au-\lambda u = 0, Bu = 0\},$$

independent of s, p, and q.

(ii) *If $\ker\{A-\lambda, B\}$ is trivial, then $\{A-\lambda, B\}$ yields an isomorphic map*

$$\text{from } B^s_{p,q}(\Omega) \text{ onto } B^{s-2}_{p,q}(\Omega)\times B^{s-d-1/p}_{p,q}(\partial\Omega)$$

and

$$\text{from } F^s_{p,q}(\Omega) \text{ onto } F^{s-2}_{p,q}(\Omega)\times B^{s-d-1/p}_{p,p}(\partial\Omega).$$

(iii) (*general homogeneous boundary conditions*)
Let $\{A^*-\lambda, B^*\}$ be the formal adjoint of $\{A-\lambda, B\}$ with respect to Green's formula. Then $\{A^*-\lambda, B^*\}$ is regular elliptic, too. Let $\{g_1, \dots, g_M\}$ be an orthonormal basis of $\ker\{A-\lambda, B\}\subset C^\infty(\bar\Omega)$, and let $\{f_1,\dots,f_N\}$ be an orthonormal basis of $\ker\{A^*-\lambda, B^*\}\subset$

$C^\infty(\bar{\Omega})$. (Both with respect to the standard L_2 inner product.) Then P, defined by

$$Pu = u - \sum_{j=1}^{N} f_j \int_\Omega u \cdot \overline{f_j},$$

is a projection in $B^t_{p,q}(\Omega)$ for any $t > s_0 + d - 2$. Also, there exists an unique bounded (right-inverse) operator $K : B^{s-2}_{p,q}(\Omega) \to \{u \in B^s_{p,q}(\Omega) : Bu = 0, \int_\Omega u\overline{g_j}dx = 0, j = 1, \dots, M\}$ with $(A-\lambda)Kf = Pf$. Thus $(A-\lambda)u = f, Bu = 0$ is solvable in $B^s_{p,q}(\Omega)$ if and only if $Pf = f$, i.e. the image is given by

$$im\{A-\lambda, B\} = \{f \in B^{s-2}_{p,q}(\Omega) : Pf = f\}.$$

The image has a closed range with

$$\dim(coker\{A-\lambda, B\}) = \dim(\ker\{A^* - \lambda, B^*\}).$$

Hence, the operator $\{A-\lambda, B\}$ is a Fredholm operator with

$$ind\{A-\lambda, B\} = \dim(\ker\{A-\lambda, B\}) - \dim(\ker\{A^* - \lambda, B^*\}).$$

If $Pf = f$ holds, then the general solution is given by

$$u = Kf + \sum_{j=1}^{M} \alpha_j g_j, \quad \alpha_j \in \mathbf{C}, j = 1, \dots, M.$$

A corresponding result holds for the spaces $F^s_{p,q}(\Omega)$.

(iv) *For the subspaces with vanishing trace in the above sense we write (for admissible s, p, q):*

$$F^s_{p,q,0}(\Omega) = \{u \in F^s_{p,q}(\Omega) : u_{|\partial\Omega} = 0\} \tag{3}$$

and

$$B^s_{p,q,0}(\Omega) = \{u \in B^s_{p,q}(\Omega) : u_{|\partial\Omega} = 0\}, \tag{4}$$

respectively. Note that the Laplacian, Δ, yields an isomorphic map from $B^s_{p,q,0}(\Omega)$ onto $B^{s-2}_{p,q}(\Omega)$.

A corresponding result holds for $F^s_{p,q,0}(\Omega)$.

In the following, $\mathbb{X}$ denotes the real part of X, for exact definition see J. Franke and T. Runst [10]. The following maximum principle, which is true under less restrictive hypotheses on the coefficients of A, is well-known, we refer to M.H.Protter and H.F.Weinberger [26, Theorem 2.3/6] and S. Fučík [14, Chapter 34].

Theorem 1. *Let $\mu \in \mathbb{R}^+$, and $u \in \mathbb{C}^{2+\alpha}(\Omega)$ with*

$$Au + \mu u \geq 0 \text{ in } \Omega, \quad u_{|\partial\Omega} = 0.$$

Then $u \geq 0$ in $\bar{\Omega}$.

This result was generalized by J.-M. Bony [5] for $u \in W_p^2(\Omega)$, where $p > n$, and in J. Franke and T. Runst [11] this maximum principle was extended to more general function spaces. Later on, we apply this result to the investigation of semilinear elliptic boundary value problems with unbounded nonlinear terms.

As usual, let λ_1 be the principle eigenvalue of the homogeneous Dirichlet problem, $A_{|\overset{o}{\mathbb{B}}{}^1_{2,2}(\Omega)}$ ($\overset{o}{B}{}^s_{p,q}(\Omega)$ denotes the completion of $D(\Omega)$ in $B^s_{p,q}(\Omega)$). Then $\lambda_1 > 0$. Furthermore, there exists an (unique) positive eigenfunction $\varphi_1 \in \mathbb{C}^\infty(\bar{\Omega})$ corresponding to λ_1 with $\int_\Omega \varphi_1^2(x)dx = 1$.

Definition 1. A distribution $f \in \mathbb{D}'(\Omega)$ is said to be non-negative if and only if $f(\varphi) \geq 0$ for any $\varphi \in \mathbb{D}(\Omega)$ with $\varphi \geq 0$.

The following result was proved in J. Franke and T. Runst [11].

Proposition 2. *Let $v \in \bigcup_{\varepsilon>0} \mathbb{B}^\varepsilon_{\infty,\infty}(\Omega)$ and $\mu > -\lambda_1$. If $v_{|\partial\Omega} = 0$ and $Av + \mu v \geq 0$ (in the sense of distributions), then $v \geq 0$.*

For the proof of the next result one uses Proposition 1 and standard arguments given in M.H. Protter and H.F. Weinberger [26, Theorem 5].

Proposition 3. *Let $v \in \bigcup_{\varepsilon>0} \mathbb{B}^{1+\varepsilon}_{\infty,\infty}(\Omega)$, and let $\mu > -\lambda_1$ be a constant. If $(A+\mu)v \geq 0$ in Ω, $v \geq 0$ in Ω and $v \not\equiv 0$, then $v(x) > 0$ in Ω for all $x \in \Omega$, and for any $x_0 \in \partial\Omega$ for which $v(x_0) = 0$ it follows that $\frac{\partial v}{\partial \nu}(x_0) < 0$, where ν denotes the outward unit normal to $\partial\Omega$ at x_0.*

Remark. We note that any such function v belongs to $\mathbb{C}^1(\bar{\Omega})$, see (2.1/1).

2.3. Nonlinear superposition operators. We formulate some results concerning mapping properties of nonlinear operators which may be found in T. Runst [30]. We refer also to W. Sickel [32]. As above, the real part of the spaces $B^s_{p,q}(\mathbb{R}^n)$, etc., is denoted by $\mathbb{B}^s_{p,q}(\mathbb{R}^n), \ldots$. Let Ω be a bounded smooth domain in $\mathbb{R}^n$.

Lemma 1. *Let $g \in \mathbb{C}^\infty(\mathbb{R})$ and $s > n \cdot \max(0, \dfrac{1}{p} - 1)$.*

(i) *Then there exists a positive constant c_g such that*

$$\| g(u) \mid B^s_{p,q}(\Omega) \| \leq c_g \| u \mid B^s_{p,q}(\Omega) \| (1 + \| u \mid L_\infty(\Omega) \|^{\max(0,s-1)}) \qquad (1)$$

holds for all $u \in \mathbb{B}^s_{p,q}(\Omega) \cap \mathbb{L}_\infty(\Omega)$. *Furthermore, the map* $u \to g(u)$ *is continuous from* $\mathbb{B}^s_{p,q}(\Omega) \cap \mathbb{L}_\infty(\Omega)$ *into* $\mathbb{B}^s_{p,q}(\Omega)$.

(ii) *Then there exists a positive constant* c_g *such that*

$$\| g(u) \mid F^s_{p,q}(\Omega) \| \le c_g \| u \mid F^s_{p,q}(\Omega) \| (1 + \| u \mid L_\infty(\Omega) \|^{\max(0,s-1)}) \tag{2}$$

holds for all $u \in \mathbb{F}^s_{p,q}(\Omega) \cap \mathbb{L}_\infty(\Omega)$. *Furthermore, the map* $u \to g(u)$ *is continuous from* $\mathbb{F}^s_{p,q}(\Omega) \cap \mathbb{L}_\infty(\Omega)$ *into* $\mathbb{F}^s_{p,q}(\Omega)$.

The following result is a consequence of (1), (2), and the fact that the embeddings $F^{s+\varepsilon}_{p,q}(\Omega) \hookrightarrow\hookrightarrow F^s_{p,q}(\Omega)$ and $B^{s+\varepsilon}_{p,q}(\Omega) \hookrightarrow\hookrightarrow B^s_{p,q}(\Omega)$ are compact if $\varepsilon > 0$ (see H. Triebel [35]). We refer also to Edmunds and Triebel [9].

Corollary 1. *Let* $g \in \mathbb{C}^\infty(\mathbb{R}), s > n \cdot \max(0, \frac{1}{p} - 1)$, *and* $\varepsilon > 0$. *Then* $u \to g(u)$ *defines a completely continuous map* (*i.e. the map is continuous and compact*) *from* $\mathbb{B}^{s+\varepsilon}_{p,q}(\Omega) \cap \mathbb{L}_\infty(\Omega)$ *into* $\mathbb{B}^s_{p,q}(\Omega)$ *and from* $\mathbb{F}^{s+\varepsilon}_{p,q}(\Omega) \cap \mathbb{L}_\infty(\Omega)$ *into* $\mathbb{F}^s_{p,q}(\Omega)$.

3. Solvability conditions for semilinear elliptic boundary value problems. In the next subsections, we consider solvability conditions for semilinear elliptic boundary value problems of second order. We need only qualitative properties of the function spaces, the considered differential operators, and the nonlinearities, so it will be clear that the problems are only prototypes standing for wider classes of differential operators and nonlinearities in the equations and in the boundary conditions.

3.1. Bounded and sublinear nonlinearities. In this subsection we derive solvability conditions for the boundary value problem (1/1) with the additional assumptions that $\ker\{A - \lambda, B\} = \ker\{A^* - \lambda, B^*\}$ and $\lim_{|t| \to \infty} \frac{g(t)}{t} = 0$. Several of our theorems will require that g is bounded.

Let K and P be the right-inverse and projection maps, respectively, defined in the previous section and let $Q = I - P$. If $s > s_0 + d$, then, based upon our previous discussion, we can conclude that (1/1) has a solution $u_0 \in F^s_{p,q}(\Omega)$ ($u_0 \in B^s_{p,q}(\Omega)$) if and only if the bifurcation system

$$v = -KPg(v + w) + KPf \tag{1}$$

$$Qg(v + w) = Qf \tag{2}$$

has a solution $u_0 = v_0 + w_0$, where $v_0 = Pu_0$ and $w_0 = Qu_0$.

Now we define a family of completely continuous maps

$$T_\tau : \begin{cases} F^s_{p,q}(\Omega) \\ B^s_{p,q}(\Omega) \end{cases} \to \begin{cases} F^s_{p,q}(\Omega) \\ B^s_{p,q}(\Omega), \end{cases}$$

$0 \le \tau \le 1$, by

$$T_\tau = \tau\{Qu - Q(g[u] - f) + KP(f - g[u])\}. \tag{3}$$

Then $T_\tau u = u, 0 \le \tau \le 1$, is equivalent to

$$v = \tau KP[f - g(v + w)] \tag{4}$$

$$w = \tau w - \tau Q[g(v + w) - f]. \tag{5}$$

In particular, $T_1 u = u$ is equivalent to (1/1).

For our existence proofs we will apply the Leray-Schauder theory. Note that the function spaces considered here are in general only quasi-Banach spaces. In J. Franke and T. Runst [10], it was shown that the spaces of Besov and Triebel-Lizorkin type are admissible in the sense of V. Klee [18]. Therefore, one can introduce a generalized Leray-Schauder degree, denoted in the following by d_{LS}, having the basic properties of the classical Leray-Schauder degree (normalization, existence of solutions, additivity, homotopy invariance, index formula for compact linear operators). We refer to [10] and L. Päivärinta and T. Runst [24]. To get a priori estimates with respect to function spaces of type $B^s_{p,q}$ and $F^s_{p,q}$, respectively, the crucial step is to prove L_∞-estimates.

Lemma 1. *Suppose that $s > n/p$ and that the set of all solutions $u = u_\tau = v + w$ of $T_\tau u = u$, $0 \le \tau \le 1$, is bounded in the topology of $L_\infty(\Omega)$, i.e. there is a positive number $c > 0$ with $\| u|L_\infty \| < c$.*

(i) *Let $f \in \mathbb{F}^{s-2}_{p,q}(\Omega)$. Then (1/1) has at least one solution $u \in \mathbb{F}^s_{p,q}(\Omega)$.*

(ii) *Let $f \in \mathbb{B}^{s-2}_{p,q}(\Omega)$. Then (1/1) has at least one solution $u \in \mathbb{B}^s_{p,q}(\Omega)$.*

Proof. We prove (i). The proof of (ii) is almost the same. Let $0 < \varepsilon < 2$ be small enough such that $s - \varepsilon > n/p$. Applying the results from 2.3 and $\| u|L_\infty \| < c$ we get

$$\|g(u)|F^{s-2}_{p,q}\| \le c_1(1 + \|u|F^{s-\varepsilon}_{p,q}\|). \tag{6}$$

Now the embedding $L_\infty(\Omega) \hookrightarrow F^0_{p,2}(\Omega)$ and the known inequality

$$\|v|F^{\theta s_0+(1-\theta)s_1}_{p,q}\| \le C\ \|v|F^{s_0}_{p,q}\|^\theta \|v|F^{s_1}_{p,q}\|^{1-\theta}, \quad 0<\theta<1,$$

show

$$\|u|F^s_{p,q}\| \le c_2(1+\|u|F^s_{p,q}\|^\theta). \tag{7}$$

We conclude the a priori estimate

$$\|u|F^s_{p,q}\| \le c_3.$$

Now an application of the generalized Leray-Schauder theory within the framework of the admissible quasi-Banach spaces of Besov and Triebel-Lizorkin type shows the existence of a solution $u \in \mathbb{F}^s_{p,q}(\Omega)$ of $T_1 u = u$. Consequently, u is a solution of (1/1). The proof is finished.

Remark. Note that in the case $d=0$ the condition with respect to f is the following: $f \in \mathbb{F}^{s-2}_{p,q}(\Omega)$, $s > n/p$, and $f \in \mathbb{B}^{s-2}_{p,q}(\Omega)$, $s>n/p$, respectively.

We are now ready for the first theorem.

Theorem 1. *Suppose that $s > \max(n/p, 1/p+d)$ and $t > d-2$.*

(i) *Let $f \in \mathbb{F}^{s-2}_{p,q}(\Omega) \cap B^t_{\infty,\infty}(\Omega)$, and suppose that the following generalized Landesman-Lazer condition,* (GL1)*, is satisfied:*

(GL1): *If $\{u_n\}^\infty_{n=1} \subset \mathbb{F}^s_{p,q}(\Omega)$ such that $\|\, u_n|L_\infty \,\| \to \infty$ and $\frac{\|\, v_n|L_\infty\,\|}{\|\, w_n|L_\infty\,\|} \to 0$, then there is an $N>0$ such that*

$$\int_\Omega [g(u_n)-f]w_n \ge 0 \quad \textit{for all} \ \ n \ge N. \tag{8}$$

Then (1/1) *has a solution $u \in \mathbb{F}^s_{p,q}(\Omega)$.*

(ii) *Let $f \in \mathbb{B}^{s-2}_{p,q}(\Omega) \cap B^t_{\infty,\infty}(\Omega)$. If condition* (GL1)*, with respect to $\mathbb{B}^s_{p,q}(\Omega)$, is satisfied, then* (1/1) *has a solution $u \in \mathbb{B}^s_{p,q}(\Omega)$.*

Proof. We prove (i), the proof of (ii) is almost the same. For $u \in \mathbb{F}^s_{p,q}(\Omega)$ we write $u = v + w = Pu + Qu$. We must study the solvability of

$$v = \tau KP[f - g(v+w)] \tag{9}$$

$$w = \tau w - \tau Q[g(v+w)-f], \tag{10}$$

and by the previous lemma we must show that there is a positive number, c, such that

$$\| u|L_\infty \| < c$$

holds for all $\tau \in [0,1]$. Observe that for $\tau = 0$ there is only the trivial solution, and if there is a solution for $\tau = 1$, then we are done. Thus it is sufficient to establish an L_∞ a priori bound on the set of solutions to the family of equations (9) and (10), where $0 < \tau < 1$. Assume the contrary, so there is a sequence of approximate solutions $\{u_n\}_{n=1}^\infty \subset \mathbb{F}^s_{p,q}(\Omega)$ such that $\| u_n|L_\infty \| \to \infty$ with a corresponding sequence $\{\tau_n\}_{n=1}^\infty \subset (0,1)$. By our assumptions, there exists a constant $d < l < 2$ such that $f \in \mathbb{B}^{l-2}_{\infty,\infty}(\Omega)$. Also, by our sublinear growth condition on g, we can conclude that for every $\varepsilon > 0$ there exists a $c_\varepsilon > 0$ such that

$$\| v|L_\infty \| \le c_\varepsilon + \varepsilon \| u|L_\infty \|. \tag{11}$$

It follows that $\frac{\| v_n|L_\infty \|}{\| w_n|L_\infty \|} \to 0$. Thus using (GL1) and the assumption $\ker\{A - \lambda, B\} = \ker\{A^* - \lambda, B^*\}$ we get

$$\int_\Omega Q[g(u_n) - f]w_n = \int_\Omega [g(u_n) - f]Qw_n = \int_\Omega [g(u_n) - f]w_n \ge 0. \tag{12}$$

However, (10) implies

$$\int_\Omega w_n w_n = \tau_n \int_\Omega w_n w_n - \tau_n \int_\Omega Q[g(v_n + w_n) - f]w_n \le \tau_n \int_\Omega w_n w_n, \tag{13}$$

where $0 < \tau_n < 1$. This yields a contradiction. Hence there is a positive number $c_1 > 0$ such that $\| w|L_\infty \| < c_1$ holds for all $0 < \tau < 1$. Hence, there exists a positive constant c such that

$$\| u|L_\infty \| < c. \tag{14}$$

Remark 1. Because of the fact that equality is admissible in (GL1), one obtains, with $g \equiv 0$, the Fredholm Alternative for linear operators as a special case, see 2.2.

Remark 2. It is clear from the proof above that if g is bounded, then we can modify (GL1) by replacing $\frac{\| v_n|L_\infty \|}{\| w_n|L_\infty \|} \to 0$ with $\| v_n|L_\infty \| < c$ for some $c > 0$. More generally, if $g(t) < \gamma(t) \forall t$, then $\| v_n|L_\infty \| \le c\gamma(\| u_n|L_\infty \|)$ for some $c > 0$.

In what follows, we give several applications. In the first we prove solvability based upon a standard form of the Landesman-Lazer condition.

Theorem 2. *Suppose that* $s > \max(n/p, 1/p + d)$ *and* $t > d - 2$. *Let* g *be bounded and let*

$$\limsup_{t\to-\infty} g(t) = g(-\infty) \quad \textit{and} \quad \liminf_{t\to+\infty} g(t) = g(+\infty).$$

Suppose

$$g(+\infty)\int_{\Omega^+} w + g(-\infty)\int_{\Omega^-} w > \int_\Omega fw \tag{L}$$

for all $w \in \ker\ \{A - \lambda, B\} \setminus \{0\}$, *where* $\Omega^+ = \{x \in \Omega : w(x) > 0\}$ *and* $\Omega^- = \{x \in \Omega : w(x) < 0\}$.

(i) *If* $f \in \mathbb{F}^{s-2}_{p,q}(\Omega) \cap B^t_{\infty,\infty}(\Omega)$, *then* (1/1) *has a solution* $u \in \mathbb{F}^s_{p,q}(\Omega)$.

(ii) *If* $f \in \mathbb{B}^{s-2}_{p,q}(\Omega) \cap B^t_{\infty,\infty}(\Omega)$, *then* (1/1) *has a solution* $u \in \mathbb{B}^s_{p,q}(\Omega)$.

Proof. A complete proof may be found in J. Franke and T. Runst [10]. By Theorem 1 , we need only check that condition (L) implies condition (GL1). So let $\{u_n\}_{n=1}^\infty \subset \mathbb{F}^s_{p,q}(\Omega)$ such that $\| u_n|L_\infty \| \to \infty$ and $\frac{\| v_n|L_\infty \|}{\| w_n|L_\infty \|} \to 0$. Since g is bounded, we may assume that $\| v_n|L_\infty \| < c$ for some $c > 0$. (See Remark 2 following Theorem 1.) Notice that $\{\frac{w_n}{\| w_n|L_\infty \|}\}_{n=1}^\infty$ is bounded in the finite dimensional space $\ker\{A-\lambda, B\}$, and so we may assume that this sequence converges uniformly to some $w \in \ker\{A - \lambda, B\}$. Moreover, it is clear that w is nontrivial and $\{\frac{w_n}{\| w_n|L_\infty \|}\}_{n=1}^\infty$ converges uniformly to w as well. Thus $u_n(x) \to \infty$ in Ω^+ and $u_n(x) \to -\infty$ in Ω^-. Since the integrands below are clearly bounded we may apply Fatou's Lemma and then condition (L) to get

$$\begin{aligned}\liminf_{n\to\infty} \int_\Omega (g(u_n) - f)\frac{w_n}{\| w_n|L_\infty \|} \\ \geq g(+\infty)\int_{\Omega^+} w + g(-\infty)\int_{\Omega^-} w - \int_\Omega fw > 0.\end{aligned} \tag{15}$$

Thus

$$\liminf_{n\to\infty} \int_\Omega [g(u_n) - f]w_n = \infty, \tag{16}$$

and (GL1) is clearly satisfied.

Remark 3. An easy consequence of this result is that in the case when $\{A-\lambda, B\}$ is invertible and not necessarily selfadjoint our semilinear boundary value problem (1/1) has at least one solution in $B^s_{p,q}(\Omega)$ $(F^s_{p,q}(\Omega))$ if s, p, q and t satisfies the conditions of Theorem 2. Therefore, one obtains the "Fredholm Alternative for nonlinear operators". This was probably first formulated independently by J. Nečas [22] and S.I. Pokhozhaev [25]. The result is the following, see also E. Zeidler [36, Chapter 29]:

If the linear equation $Lu = 0$ has only the trivial solution $u = 0$, then the nonlinear equation $Lu + Nu = f$ has at least one solution for arbitrary right-hand side f if N is at most sublinear. If $Lu = 0$ has a nontrivial solution $u \neq 0$, then the equation $Lu + Nu = f$ has only a solution u, if f satisfies certain conditions of solvability with respect to $\ker L$.

Remark 4. Let all assumptions of Theorem 2 be satisfied. It is not hard to check that if

$$g(-\infty) < g(t) < g(+\infty) \text{ for all } t \in \mathbb{R},$$

then (L) is also necessary for the solvability of (1/1).

Remark 5. Theorem 2 extends a classical result due to E.M.Landesman and A.C. Lazer [19]. There are many generalizations, for example if there exist functions h^+ and h^- in $\mathbb{C}^\infty(\bar{\Omega})$ such that

$$\limsup_{\xi \to -\infty} g(x,\xi) = h^-(x), \quad \liminf_{\xi \to +\infty} g(x,\xi) = h^+(x)$$

for all $(x,\xi) \in \bar{\Omega} \times \mathbb{R}$. Then one substitutes the solvability condition (L) by (L′) defined by

$$\int_{\Omega^+} h^+ w + \int_{\Omega^-} h^- w > \int_\Omega f w \tag{L′}$$

for $w \in \ker\{A - \lambda, B\} \setminus \{0\}$.

Remark 6. The question arises "What happens if we are looking for non-smooth solutions of (1/1)?" In this case, the situation within the framework of spaces of Besov or Triebel-Lizorkin type is quite different and more complicated. We refer to a recent paper by J. Johnsen and T. Runst [17].

Remark 7. Observe that in the previous theorem (GL1) is satisfied with room to spare. Also, it is easy to see that condition (L) does not apply to nonlinearities where $g(+\infty) = g(-\infty)$, for example $g(t) = \frac{t}{1+t^2}$. The following theorem addresses both of these facts and generalizes a result in [28].

Theorem 3. *Let* $s > \max(n/p, 1/p + d)$ *and* $t > d - 2$*, and let* $g \in \mathbb{C}^\infty(\mathbb{R})$ *be a bounded function such that* $\liminf_{t\to\infty} tg(t) = G(+\infty) > -\infty$*, and* $\liminf_{t\to-\infty} tg(t) = G(-\infty) > -\infty$*. Also suppose that* $\int_\Omega fw = 0$ *for all* $w \in \ker\{A - \lambda, B\}$*.*

(i) *Let* $f \in \mathbb{F}^{s-2}_{p,q}(\Omega) \cap B^t_{\infty,\infty}(\Omega)$*. If*

$$|\Omega^+|G(+\infty) + |\Omega^-|G(-\infty) > 0 \tag{17}$$

for all $w \in \ker\,\{A - \lambda, B\} \setminus \{0\}$*, where* Ω^+ *and* Ω^- *are defined as in Theorem* 2*, then* (1/1) *has a solution* $u \in \mathbb{F}^s_{p,q}(\Omega)$*.*

(ii) *Let* $f \in \mathbb{B}^{s-2}_{p,q}(\Omega) \cap B^t_{\infty,\infty}(\Omega)$*. If* (17) *holds for all* $w \in \ker\{A-\lambda, B\} \setminus \{0\}$*, then* (1/1) *has a solution* $u \in \mathbb{B}^s_{p,q}(\Omega)$*.*

Proof. We prove (i). We must show that condition (GL1) is satisfied. As in Theorem 2 let $\{u_n\}_{n=1}^\infty \subset F^s_{p,q}(\Omega)$ with $\| u_n|L_\infty \| \to \infty$ and $\| v_n|L_\infty \| < c$. Since $\frac{v_n}{\| w_n|L_\infty \|} \to 0$, uniformly, and $\{\frac{w_n}{\| w_n|L_\infty \|}\}_{n=1}^\infty$ is bounded in the finite dimensional space $\ker\,\{A - \lambda, B\}$, we may assume without loss of generality that

$$\frac{u_n}{\| w_n|L_\infty \|} \to w \in \ker\,\{A - \lambda, B\} \quad \text{as} \quad n \to \infty, \tag{18}$$

uniformly. Moreover, this w is nontrivial, and, by our assumptions with respect to $\{A, B\}$, we obtain that $\ker\{A - \lambda, B\}$ satisfies the unique continuation property in the sense that the only function $w \in \ker L$ vanishing on a set of positive Lebesgue measure in Ω is $w = 0$, see S. Mizohata [21, p.368]. It follows that $u_n(x) \to \pm\infty$ a.e. in Ω. By hypothesis we know that $tg(t)$ is bounded below for $t \in \mathbb{R}$. Hence, $g(u_n)w_n = g(u_n)(u_n - v_n) = g(u_n)u_n - g(u_n)v_n$ is bounded below with

$$\liminf_{n\to\infty} g(u_n(x))w_n(x) = \liminf_{n\to\infty} g(u_n(x))u_n(x)(1 - \frac{v_n(x)}{u_n(x)}) = G(+\infty)$$

in Ω^+ and, similarly,

$$\liminf_{n\to\infty} g(u_n(x))w_n(x) = G(-\infty)$$

in Ω^-. Our condition on f implies that

$$\int_\Omega [g(u_n) - f]w_n = \int_\Omega g(u_n)w_n. \tag{19}$$

Therefore an application of Fatou's Lemma and (17) shows that

$$\liminf_{n\to\infty} \int_\Omega [g(u_n) - f]w_n > 0, \tag{20}$$

which implies (GL1).

Remark 8. It is easy to see that $g(t) = \frac{t}{1+t^2}$ satisfies the conditions of this theorem, with $G(+\infty) = G(-\infty) = 1$. Also, if resonance occurs at higher eigenvalues where eigenfunctions change sign, it is possible that $G(+\infty)$ and $G(-\infty)$ can have opposite signs. A simple example to consider is the differential equation

$$u''(t) + 4u(t) + g(t) + \sin(t) = 0$$

with

$$u(0) = u(\pi) = 0$$

and with

$$g(t) = (\frac{\pi}{4} + \arctan(t))(\frac{t}{1+t^2}).$$

Here we have $G(+\infty) = \frac{3\pi}{4}$, $G(-\infty) = -\frac{\pi}{4}$, and $|\Omega_+| = |\Omega_-| = \frac{\pi}{2}$, so it is easy to check that Theorem 3 applies. Finally, it is a simple matter to generalize theorems such as this to nonautonomous nonlinear terms $g(x,t)$, which require generalized solvability conditions such as

$$\int_{\Omega^+} G(x,+\infty) + \int_{\Omega^-} G(x,-\infty) > 0.$$

Since this type of generalization adds more notation than insight we omit it.

Boundary value problems with functions g satisfying $g(-\infty) = g(+\infty) = 0$ were investigated by A. Ambrosetti and G. Mancini [4], S. Fučík and M. Krbec [15], P. Hess [16] and J. Nečas [23, Theorem 3.5.20] within the framework of Sobolev and Hölder spaces.

Now we consider an application to semilinear boundary value problems with sublinear nonlinearities. This theorem generalizes a result of J. Franke and T. Runst [11, Theorem 3.3/1], and it extends the result of Theorem 3 for certain types of nonlinearities.

Theorem 4. *Let $s > \max(n/p, 1/p + d)$ and $t > d - 2$, let $g \in \mathbb{C}^\infty(\mathbb{R})$, and let $|g(t)| < \gamma(|t|)\forall t$, where $\gamma(t)$ is a monotone and positive function with*

$$\frac{\gamma(t)}{t} \to 0 \quad \mathit{if} \quad t \to +\infty. \tag{21}$$

(i) *Let $f \in \mathbb{F}^{s-2}_{p,q}(\Omega) \cap \mathbb{B}^t_{\infty,\infty}(\Omega)$, and let $f^0 := Qf$. If there exists $\delta > 0$ such that*

$$\liminf_{t\to\pm\infty} \left(\frac{(g(t) - f^0)t)}{\gamma(\gamma(|t|))\gamma(|t|)} \right) \geq \delta, \text{ uniformly in } x,$$

then $(1/1)$ *has a solution $u \in \mathbb{F}^s_{p,q}(\Omega)$.*

(ii) *Let $f \in \mathbb{B}^{s-2}_{p,q}(\Omega) \cap \mathbb{B}^t_{\infty,\infty}(\Omega)$, and let $f^0 := Qf$. If there exists $\delta > 0$ such that*

$$\liminf_{t\to\pm\infty} \left(\frac{(g(t) - f^0)t)}{\gamma(\gamma(|t|))\gamma(|t|)} \right) \geq \delta, \text{ uniformly in } x,$$

then $(1/1)$ *has a solution $u \in \mathbb{B}^s_{p,q}(\Omega)$.*

Proof. As before we show that this result is a corollary of Theorem 1. We will prove (i). The proof of (ii) is similar.

Let $\| u_n|L_\infty \| \to \infty$ and $\frac{\| v_n|L_\infty \|}{\| w_n|L_\infty \|} \to 0$. Without loss of generality we assume that $\frac{u_n}{\| w_n|L_\infty \|}$ and $\frac{w_n}{\| w_n|L_\infty \|}$ converge uniformly to $w \in Ker\{A - \lambda, B\}/\{0\}$, and that $\| v_n|L_\infty \| \leq c\gamma(\| u_n|L_\infty \|)$ for some $c > 0$ (See Remark 2 above). For later reference we note that w satisfies the unique continuation property, i.e. that $|\{x : w(x) = 0\}| = 0$.

Let $r > 0$ such that $(g(t) - f^0)t \geq 0 \quad \forall|t| \geq r$, and define $\Omega_n = \{x : |u_n(x)| \geq r, \frac{w_n(x)}{u_n(x)} \geq \frac{1}{2}\}$. For $x \in \Omega_n$ we have

$$\begin{aligned}&\frac{(g(u_n) - f^0)w_n}{\gamma(\gamma(\| u_n|L_\infty \|))\gamma(\| u_n|L_\infty \|)}\\ &\qquad = \Big(\frac{(g(u_n) - f^0)u_n}{\gamma(\gamma(|u_n|))\gamma(|u_n|)}\Big)\Big(\frac{w_n}{u_n}\Big)\Big(\frac{\gamma(\gamma(|u_n|))\gamma(|u_n|)}{\gamma(\gamma(\| u_n|L_\infty \|))\gamma(\| u_n|L_\infty \|)}\Big),\end{aligned}$$

which is nonnegative. Moreover, for a.e. such x we know that $\frac{u_n(x)}{\| u_n|L_\infty \|} \to w(x) \neq 0$, and similarly for $\frac{w_n(x)}{\| u_n|L_\infty \|}$, so

$$\frac{w_n}{u_n} \to 1, \text{ and } \frac{\gamma(\gamma(|u_n|))\gamma(|u_n|)}{\gamma(\gamma(\| u_n|L_\infty \|))\gamma(\| u_n|L_\infty \|)} \to 1,$$

where the last limit makes strong use of the fact that γ is monotone and sublinear. Finally, the characteristic function χ_{Ω_n} satisfies $\chi_{\Omega_n} \to 1$ for a.e. $x \in \Omega$, since $w(x) \neq 0$ a.e. and $\frac{w_n(x)}{u_n(x)} \to 1$ a.e. Hence we can apply Fatou's lemma to get

$$\liminf_{n\to\infty} \big(\frac{1}{\gamma(\gamma(\| u_n|L_\infty \|))\gamma(\| u_n|L_\infty \|)}\big) \int_{\Omega_n} (g(u_n) - f^0)w_n \geq \delta.$$

For $x \in \Omega/\Omega_n$ we have that $|u_n| \leq 2|v_n| \leq 2\| v_n|L_\infty \| \leq 2c\gamma(\| u_n|L_\infty \|)$, and an identical statement for $|w_n|$. Similarly, $|g(u_n)| \leq \gamma(2c\gamma(\| u_n|L_\infty \|))$. Using the monotonicity and sublinearity of γ we see that

$$\frac{\gamma(2c\gamma(\| u_n|L_\infty \|))\gamma(2c\| u_n|L_\infty \|)}{\gamma(\gamma(\| u_n|L_\infty \|))\gamma(\| u_n|L_\infty \|)} \to 1.$$

It follows that $\frac{(g(u_n)-f^0)w_n}{\gamma(\gamma(\| u_n|L_\infty \|))\gamma(\| u_n|L_\infty \|)}$ is bounded in Ω/Ω_n. Moreover, the characteristic function $\chi_{\Omega/\Omega_n} = 1 - \chi_{\Omega_n}$ converges to 0 a.e.. Hence,

$$\lim_{n\to\infty} \big(\frac{1}{\gamma(\gamma(\| u_n|L_\infty \|))\gamma(\| u_n|L_\infty \|)}\big) \int_{\Omega/\Omega_n} (g(u_n) - f^0)w_n = 0.$$

The estimates above imply that GL(1) is satisfied.

Remark 9. In the case where $\{A-\lambda, B\}$ is invertible it is not hard to check the existence of solutions of (1/1) in $B^s_{p,q}(\Omega)$ $(F^s_{p,q}(\Omega))$ if the real-valued smooth function g satisfies only a sublinear growth condition, and if s, p, q, t and γ satisfy the conditions of Theorem 3 ($\{A - \lambda, B\}$ is not necessarily selfadjoint). Hence one obtains results of the "Fredholm Alternative for nonlinear operators".

Remark 10. If $d = 0$ in Theorem 2 and Theorem 3, respectively, then f has only to satisfy the condition

$$f \in \mathbb{F}^{s-2}_{p,q}(\Omega) \quad (f \in \mathbb{B}^{s-2}_{p,q}(\Omega)),\ s > n/p.$$

Furthermore, it is not hard to construct examples with bounded asymptote for $t \to -\infty$ and with sublinear behaviour in the sense of Theorem 4 as $t \to +\infty$, and vice versa.

Remark 11. In order to understand the implications of Theorem 4 it is useful to consider a few simple examples where $|g(t)| < |t|^r$ for $0 \leq r < 1$.

If $r = 0$, then g is bounded and the conclusion of Theorem 4 becomes a similar, but somewhat restricted, version of Theorem 2 for $f^0 \neq 0$ and of Theorem 3 for $f^0 = 0$. If $r = \frac{1}{2}$ and $f^0 = 0$, then Theorem 4 can be seen as an extension of Theorem 3 to the sublinear growth case where the decay rate of g is controlled by bounding $g(t)t^{\frac{1}{4}}$ from below rather than $g(t)t$. If $r = \frac{3}{4}$, then Theorem 4 requires g to grow as fast as $t^{\frac{5}{16}}$ as $t \to \infty$. This demonstrates some improvement over the result in [11], where the bound on growth from below had to be comparable to the bound on growth from above, i.e. $\delta\gamma(t) \leq g(t) \leq \gamma(t)$.

3.2. Nonlinearities of linear growth. In this subsection we consider the boundary value problem (1/1) with $Bu = u_{|\partial\Omega}$, i.e. Dirichlet boundary conditions, and $g(t)$ satisfying a linear growth condition. We formulate a generalized solvability condition in the case when resonance occurs at the principle eigenvalue, $\lambda_1 > 0$. We know that $\ker\{A - \lambda_1, B\}$ and $\ker\{A^* - \lambda_1, B^*\}$ are one-dimensional and are spanned by smooth eigenfunctions, denoted by $\varphi_1 \in \mathbb{C}^\infty(\bar{\Omega})$ and $\varphi_1^* \in \mathbb{C}^\infty(\bar{\Omega})$, respectively, with

$$\varphi_1 > 0 \text{ in } \Omega,\ \varphi_1^* > 0 \text{ in } \Omega, \quad \frac{\partial \varphi_1}{\partial \nu} < c < 0 \text{ on } \partial\Omega,\ \frac{\partial \varphi_1^*}{\partial \nu} < c < 0 \text{ on } \partial\Omega,$$

where ν denotes the outward normal, and

$$\int_\Omega \varphi_1^2 = \int_\Omega \varphi_1^{*2} = 1.$$

Furthermore, for every $u \in \mathbb{C}^1(\bar{\Omega})$, with $u = 0$ on $\partial\Omega$, there are constants α, $\beta \in \mathbb{R}$ with

$$\alpha\varphi_1(x) \leq u(x) \leq \beta\varphi_1(x), \text{ and } \quad \alpha\varphi_1^*(x) \leq u(x) \leq \beta\varphi_1^*(x)$$

for all $x \in \Omega$.

To begin with, let us consider the following two point boundary value problem

$$u''(x) + u(x) + g[u(x)] = f(x), x \in (0,\pi), \quad u(0) = u(\pi) = 0, \tag{1}$$

where $f \in L_1(0,\pi)$ and g is a bounded real-valued smooth function defined on $\mathbb{R}$. If the Landesman-Lazer condition

$$g(-\infty)\int_0^\pi \sin x < \int_0^\pi f(x)\sin x dx < g(+\infty)\int_0^\pi \sin x \tag{L$_*$}$$

holds, where $g(\pm\infty)$ have the same meaning as before, then it follows from Subsection 3.1 that (1) has a solution. This result has been generalized to allow unbounded g and infinite values of $g(-\infty)$ and $g(+\infty)$, provided that the growth of the function g at infinity is restricted in some sense, see S. Fučík [13], M. Schechter, J. Shapiro and M. Snow [31], L. Cesari and R. Kannan [6] and S. Ahmad [2]. In S. Ahmad [2], it was proved that if g satisfies a linear growth condition of the type

$$|g(t)| \leq c_1 + c_2\ |t|,$$

where $c_1 > 0$ and $0 < c_2 < 3$, then (1) is solvable provided that (L_*) holds. Since the boundary value problem

$$u''(x) + u(x) + 3u(x) = \sin 2x, x \in (0,\pi), \quad u(0) = u(\pi) = 0,$$

has no solution, the condition $c_2 < 3$ is sharp (note that (L_*) is satisfied). It follows from the proof of S. Ahmad [2] that $c_2 < 3$ and (L_*) are sufficient for the solvability of (1). Observe that $\lambda_2 - \lambda_1 = 3$, where λ_1 and λ_2 are the first and second eigenvalue, respectively, of

$$u''(x) + \lambda u(x) = 0, \quad x \in (0,\pi), \quad u(0) = u(\pi) = 0,$$

i.e., the distance between λ_1 and λ_2 limits the the linear growth of the nonlinearity g, see also P. Drábek [8]. The n-dimensional analogue of this assertion was proved by S. Ahmad [3]. In a recent work [28], the result was extended to a more general weak solvability condition.

In this subsection, we shall be concerned with the n-dimensional case of (1) within the framework of function spaces of Besov or Triebel-Lizorkin type.

The following results are important for our further considerations.

Lemma 1. *Let λ_1 be the principle eigenvalue of $\{A, B\}$ and let $c \leq \lambda_1$. Then there exists a positive constant d, $d > \lambda_1$, such that if $q \in C(\bar{\Omega})$ satisfies*

$$c \leq q(x) \leq d \text{ in } \Omega, \tag{2}$$

and $v \in \bigcup_{\varepsilon > 0} \mathbb{B}^{1+\varepsilon}_{\infty,\infty}(\Omega)$ for which

$$(Av)(x) = q(x)v(x) \text{ in } \Omega, \quad v_{|\partial\Omega} = 0, \tag{3}$$

and $v \not\equiv 0$*, then either* $v(x) > 0$ *for all* $x \in \Omega$ *and* $\frac{\partial v}{\partial \nu} < 0$ *on* $\partial\Omega$*, or* $v(x) < 0$ *for all* $x \in \Omega$ *and* $\frac{\partial v}{\partial \nu} > 0$ *on* $\partial\Omega$*.*

Proof. Let $v \in \bigcup_{\varepsilon>0} \mathbb{B}^{1+\varepsilon}_{\infty,\infty}(\Omega)$ be a solution of (3) such that $v \not\equiv 0$ and $v \geq 0$ in Ω. If μ is a positive number large enough such that

$$\mu + q(x) > 0 \quad \text{for all} \quad x \in \Omega,$$

then

$$(A + \mu)[v(x)] \geq 0 \quad \text{for} \quad x \in \Omega.$$

Now the assertion follows from Proposition 2.2/1 and Proposition 2.2/2, and the maximum principle arguments used in the proof of S. Ahmad [3, Lemma 2.2]. Similarly, if v is a solution of (3) with $v \not\equiv 0$ and $v \leq 0$ in Ω, then $v < 0$ and $\frac{\partial v}{\partial \nu} > 0$ on $\partial\Omega$.

Remark 1. S. Ahmad [3] proved a corresponding result for more general functions q and non-selfadjoint differential operators A with respect to $W^2_p(\Omega)$, $n < p < \infty$. For our purposes, as well as for Ahmad's result, the following corollary suffices.

Corollary 1. *Let all assumptionsof Lemma* 1 *be satisfied with the additional restriction that* $q(x) \geq \lambda_1$ *in* Ω*, i.e.* $c = \lambda_1$*. Let* $v \in \bigcup_{\varepsilon>0} \mathbb{B}^{1+\varepsilon}_{\infty,\infty}(\Omega)$ *be a solution of* (3)*. Then* $v \in \ker\{A - \lambda_1, B\}$*.*

Proof. Applying Lemma 1 the proof can be done similiarly to S.B. Robinson and E.M. Landesman [28]. By Lemma 1 we may conclude that either $v \equiv 0$, $v(x) > 0$ for all $x \in \Omega$ and $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega$, or $v(x) < 0$ for all $x \in \Omega$ and $\frac{\partial v}{\partial \nu} > 0$ on $\partial\Omega$. If $v \equiv 0$, then we are finished. Now we assume that $v > 0$ in Ω. The other case can be treated similarly. We choose $k > 0$ small enough such that $v - k\varphi_1 > 0$ for all $x \in \Omega$. Let $\bar{k}$ be the supremum of all such k. Now we define a function z by $z = v - \bar{k}\varphi_1$. Then $z \geq 0$ in Ω and $\frac{\partial z}{\partial \nu} \leq 0$ on $\partial\Omega$. The definition of $\bar{k}$ shows there is either a point $x^* \in \Omega$ such that $z(x^*) = 0$, or a point $x^* \in \partial\Omega$ with $\frac{\partial z}{\partial \nu}(x^*) = 0$. Finally,

$$(Az)(x) = \lambda_1 z(x) + [q(x) - \lambda_1]v(x) \geq 0 \text{ in } \Omega, \quad z_{|\partial\Omega} = 0,$$

so the maximum principle implies that $z \equiv 0$. The corollary is proved. □

For given $c \leq \lambda_1$, let $d^*(c)$ be the supremum of all numbers d, $d > \lambda_1$, such that if $q \in C(\bar{\Omega})$ satisfies the inequality (2), then Lemma 1 holds. Note that $c_1 < c_2$ implies $d^*(c_1) \leq d^*(c_2)$.

Now we consider the case when $\{A, B\}$ is selfadjoint. Then for any c, $c \leq \lambda_1$, one gets

$$d^*(c) = \lambda_2. \tag{4}$$

We list some known results concerning eigenvalue problems with indefinite weight functions. Consider the following eigenvalue problem

$$(Av)(x) = \mu q(x)v(x) = 0 \text{ in } \Omega \quad v_{|\partial\Omega} = 0, \tag{P$_q$}$$

where A is selfadjoint, and let $q \in \mathbb{C}(\bar{\Omega})$. Furthermore, q is positive on a set with nonzero Lebesgue measure. Then the following results hold.

Proposition 1.

(i) *If q is nonpositive in Ω, there are no positive eigenvalues μ of* (P_q)*. In the other case, the positive eigenvalues of* (P_q) *form a nondecreasing sequence $\{\mu_k\}_{k=0}^{\infty}$ tending to $+\infty$.*

(ii) *The first positive eigenvalue μ_1 is simple, and the corresponding eigenfunction is strictly positive* (*negative*) *on Ω.*

(iii) *The k-th eigenvalue $\mu_k(q)$ is a nonincreasing function of q: If q_1 and q_2 are two continuous functions and q_1 is positive on a set of positive Lebesgue measure, $q_2 \geq q_1$ on Ω, and $q_1 \not\equiv q_2$ on Ω, then $\mu_k(q_2) < \mu_k(q_1)$ for $k \geq 1$.*

A proof for more general functions q may be found in A. Manes and A.M. Micheletti [20]. That is a generalization of well-known considerations of eigenvalue problems with positive weight functions, see R. Courant and D. Hilbert [7]. Now we are in position to prove (4).

Lemma 2. *Let $0 < \lambda_1 < \lambda_2 \leq \ldots$ denote the eigenvalues, each appearing as often in the sequence as its multiplicity, of*

$$Au = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{5}$$

Then for any c, $c \leq \lambda_1$, we have $d^*(c) = \lambda_2$.

Proof. Let $u_2 \in \mathbb{C}^{\infty}(\bar{\Omega})$ be a nontrivial eigenfunction of (5) corresponding to $\lambda = \lambda_2$. Then a classical result shows that u_2 has to change the sign on Ω. This implies $d^*(c) \leq \lambda_2$. Now we suppose that d is an arbitrary number satisfying $\lambda_1 < d < \lambda_2$, q is a bounded continuous function with

$$c < q(x) \leq d \text{ in } \Omega,$$

and that $u \in \bigcup_{\varepsilon>0} \mathbb{B}^{1+\varepsilon}_{\infty,\infty}(\Omega)$ is a nontrivial solution of (3). Since $\mu = 1$ is a positive eigenvalue of (P_q), this implies that q is positive on a set of positive Lebesgue measure and $\mu_k(q) = 1$ for some $k \geq 1$. It holds

$$\mu_k(\lambda_2) = \frac{\lambda_k}{\lambda_2} \text{ for } k \geq 1.$$

By our assumption $q(x) \leq d < \lambda_2$ in Ω, we can conclude that $1 = \mu_2(\lambda_2) < \mu_2(q)$ and $\mu_1(q) = 1$. Hence the corresponding (nontrivial) eigenfunction u is strictly positive (negative) in Ω. Now we choose a positive constant γ with $\gamma + q(x) > 0$ in Ω. We have

$$(A + \gamma)[u(x)] = [\gamma + q(x)]u(x).$$

Hence either u or $-u$ satisfies the hypotheses of Lemma 1. This shows $d^*(c) \geq \lambda_2$. $\square$

Our aim is to prove a generalized solvability condition of Landesman-Lazer type, where the nonlinearity is of linear growth. We follow the ideas of S. Ahmad [3, Theorem 3.1] and S. Robinson and E.M. Landesman [28]. Let $\{A, B\}$, $\lambda_1 > 0$, φ_1, and φ_1^* be the same as above.

Theorem 1. *Let $s > n/p$ and $\rho > -1$. Let $0 < b < d^*(\lambda_1)$ and let $g : \mathbb{R} \to \mathbb{R}$ be a $\mathbb{C}^\infty$-function such that*

$$0 \leq \liminf_{|t|\to\infty} \frac{g(t)}{t} \leq \limsup_{|t|\to\infty} \frac{g(t)}{t} \leq b.$$

(i) *Let $f \in \mathbb{F}^{s-2}_{p,q}(\Omega) \cap \mathbb{B}^{\rho}_{\infty,\infty}(\Omega)$. Then* (1/1)*, with $\lambda = \lambda_1$, has a solution $u \in \mathbb{F}^s_{p,q,0}(\Omega)$ provided the following solvability condition* (GL2) *is satisfied.*

(GL2): *Let $\{u_n\}_{n=1}^\infty \subset \mathbb{F}^s_{p,q}(\Omega)$ such that $\| u_n | L_\infty \| \to \infty$. If $\frac{u_n}{\| u_n | L_\infty \|} \to \varphi = \pm\varphi_1$ in the $C^1(\bar{\Omega})$ norm, then there is an $N > 0$ such that*

$$\mathit{sign}\ (\varphi) \int_\Omega [g(u_n) - f]\varphi_1^* \geq 0 \quad \text{for all } n \geq N. \tag{6}$$

(ii) *Let $f \in \mathbb{B}^{s-2}_{p,q}(\Omega) \cap \mathbb{B}^{\rho}_{\infty,\infty}(\Omega)$. Then* (1/1)*, with $\lambda = \lambda_1$, has a solution $u \in \mathbb{B}^s_{p,q,0}(\Omega)$ provided* (GL2)*, with respect to $\mathbb{B}^s_{p,q}(\Omega)$ is satisfied.*

Proof. We prove (i). The proof of (ii) is almost the same. From the assumptions we can conclude the existence of a number $k > 0$ be such that $\lambda_1 + k \leq b$ and $\lambda_1 + k$ is not an eigenvalue of (5).

For $\tau \in [0,1]$ we define the family of boundary value problems

$$(Au)(x) = (\lambda_1 + \tau k)u(x) + (1-\tau)(g[u(x)] - f(x)) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{P$_\tau$}$$

We prove that there exists a positive number R such that if u_τ is a solution of (P_τ), then

$$\|u_\tau | L_\infty\| \leq R, \tag{7}$$

where R is independent of $\tau \in [0,1]$. As in Section 3.1 it suffices to consider $\tau \in (0,1)$. Assume the contrary. Then there exist a sequence $\{\tau_n\}_{n=1}^{\infty} \subset (0,1)$ and a sequence of functions $\{u_n\}_{n=1}^{\infty} \subset \mathbb{F}^s_{p,q}(\Omega)$ such that u_n satisfies (P_{τ_n}) and $\|u_n | L_\infty\| \to \infty$ as $n \to \infty$. Without loss of generality we may assume that $\|u_n | L_\infty\| > 0$ for all $n \in \mathbb{N}$. Let w_n be given by

$$w_n(x) = \frac{u_n(x)}{\|u_n | L_\infty\|}, \quad n = 1, 2, \dots,.$$

Then we obtain

$$(Aw_n)(x) = q_n(x) - f_n(x) \text{ in } \Omega, \quad w_n = 0 \text{ on } \partial\Omega, \tag{8}$$

where

$$q_n(x) = (\lambda_1 + \tau_n k)w_n(x) + (1-\tau_n)\frac{g(u_n(x))}{\|u_n | L_\infty\|},$$
$$f_n(x) = -(1-\tau_n)\frac{f(x)}{\|u_n | L_\infty\|}.$$

We may assume $\tau_n \to \tau \in [0,1]$. The linear growth restriction on g and the mapping properties show that the $B^\rho_{\infty,\infty}$-norm of the right hand side of (8) is bounded independently of n. (We have $\|f_n | B^\rho_{\infty,\infty}\| \leq c_1$ and $\|q_n | B^\rho_{\infty,\infty}\| \leq c'\|q_n | L_\infty\| \leq c_2$.) Hence for some $K > 0$ one gets $\|Aw_n | B^\rho_{\infty,\infty}\| < K$ for $n = 1, 2, \dots,$. Applying the mapping properties of A, see 2.2, and compactness results, see 2.3, it follows that $w_n \to w$ as $n \to \infty$ in $\mathbb{C}^1(\bar{\Omega})$, where $\|w | L_\infty\| = 1$. Clearly, $f_n \to 0$ in $\mathbb{B}^\rho_{\infty,\infty}(\Omega)$, $\rho > -1$. Also, it is easy to see that without loss of generality $\frac{g(u_n)}{\|u_n | L_\infty\|}$ converges weakly in L_p for any $1 < p < \infty$, and by writing $\frac{g(u_n)}{\|u_n|L_\infty\|} = \frac{g(u_n)}{u_n}\frac{u_n}{\|u_n|L_\infty\|}$ it follows that $\frac{g(u_n)}{\|u_n|L_\infty\|} \rightharpoonup G(x)w(x)$, where G is a bounded measurable function satisfying $0 \le G(x) \le b$. Thus $q_n \rightharpoonup q(x)w(x)$, where $q(x) = \lambda_1 + \tau k + (1-\tau)G(x)$, which satisfies $\lambda_1 \le q(x) \le \lambda_1 + b$. Let $\phi \in \mathbb{D}(\Omega)$. Then (8) yields

$$\int_\Omega w_n(A^*\phi) = \int_\Omega (Aw_n)\phi = \int_\Omega q_n\phi - \int_\Omega f_n\phi,$$

and hence, because of $\int_\Omega f_n\phi \to 0$ as $n \to \infty$,

$$\int_\Omega w(A^*\phi) = \int_\Omega qw\phi. \tag{9}$$

Now (9) shows

$$(Aw)(x) = q(x)w(x) \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

Since $\|w|L_\infty\| = 1$ it follows from Corollary 1 that $w = \pm\varphi_1$ and so condition (GL2) applies. Because of the definition of w_n, we may assume without loss of generality that for $n \ge 1$ the function u_n is either strictly positive and $\lim_{n\to\infty} u_n(x) = +\infty$ for all $x \in \Omega$, or strictly negative and $\lim_{n\to\infty} u_n(x) = -\infty$ for all $x \in \Omega$. We suppose that the first alternative is true. The other case can be considered similarly. If we compute the L_2 inner product of (P_{τ_n}) with φ_1^* and simplify, we get

$$0 = \tau_n k \int_\Omega u_n\varphi_1^* + (1-\tau_n)\int_\Omega (g(u_n) - f)\varphi_1^* \tag{10}$$

It follows that

$$0 > \int_\Omega (g(u_n) - f)\varphi_1^*, \tag{11}$$

which contradicts (GL2).

A careful look at our arguments reveals that an a priori bound has been established for $\tau \in (0,1)$ and that it is trivial to include the case $\tau = 0$. However, it is possible that the set of solutions corresponding to $\tau = 1$ is unbounded, as it is in the linear case where $g \equiv 0$ and $f \in \ker\{A^* - \lambda_1, B^*\}^\perp$. Thus we are left with the possibilities that there are infinitely many solutions, and the proof is done, or that there is an a priori bound on the set of solutions for all $\tau \in [0,1]$. We assume the latter and proceed. As in Lemma 3.1 we can conclude the a priori estimate

$$\|u|F^s_{p,q}\| \le c$$

for all solutions of (P_τ) when $\tau \in [0,1]$. By the definition of the number k we know that the linear map $T = id_{F^s_{p,q}} - (\lambda_1 + k)A$ is invertible. Since λ_1 is simple, we get by the index formula of compact linear operators

$$d_{LS}(u - h(0,\cdot), B_{2c}, 0) = d_{LS}(u - h(1,\cdot), B_{2c}, 0) = -1,$$

where $h(t,u)$ is the operator which assigns to each $u \in \mathbb{F}^s_{p,q}(\Omega)$ and $t \in [0,1]$ the unique solution $w \in \mathbb{F}^s_{p,q}(\Omega)$ of the problem

$$(Aw)(x) = (\lambda_1 + tk)u(x) + (1-t)g[u(x)] - f(x) \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

In the usual way, we get that

$$h : [0,1] \times \mathbb{F}^s_{p,q}(\Omega) \to \mathbb{F}^s_{p,q}(\Omega)$$

is a completely continuous homotopy. Now $d_{LS}(u - h(0,\cdot), B_{2c}, 0) \neq 0$ implies the solvability of (1/1). The proof is finished.

Remark 2. Consider the following condition, (L*), given by

$$g(-\infty)\int_\Omega \varphi_1^* < \int_\Omega \varphi_1^* f < g(+\infty)\int_\Omega \varphi_1^*, \tag{L*}$$

where $g(\pm\infty)$ are defined as before. Moreover, let $d_0 > \lambda_1$ denote the supremum of the numbers $d^*(c)$ for all $c \le \lambda_1$, and assume that there exists $r_0 > 0$ and $b_0 \in (0, d_0 - \lambda_1)$ such that

$$\frac{g(t)}{t} \le b_0 \text{ if } |t| \ge r_0. \tag{12}$$

It is not hard to check that these conditions, which are used in [3], imply (GL2). Notice that $d_0 = d^*(\lambda_1)$, and that the lower bound $\liminf_{|t|\to\infty} \frac{g(t)}{t} \ge 0$ is implicit in (L*), but not in (GL2). If $\{A, B\}$ is selfadjoint, then we have $d_0 = \lambda_2$.

The next result shows that in some cases (L*) is also necessary for the solvability of (1/1).

Corollary 2. *If* $\lim_{t\to\pm\infty} g(t) = g(\pm\infty)$ *exist or are infinite, and* $g(-\infty) < g(t) < g(+\infty)$ *for all* $t \in \mathbb{R}$*, then* (L^*) *is also necessary for solvability.*

Remark 3. Note that the growth condition

$$\limsup_{|t|\to\infty} \frac{g(t)}{t} < \lambda_2 - \lambda_1$$

cannot be improved in the selfadjoint case. This follows from the fact that

$$(Au)(x) = \lambda_2 u(x) - f(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

is solvable if and only if

$$\int_\Omega f\varphi_2 = 0$$

for every eigenfunction φ_2 of A corresponding to λ_2. Now we choose $g(t) = (\lambda_2 - \lambda_1)t$.

As in Subsection 3.1 one can give examples for which the set of the right hand sides f may be empty. Applying the methods of S.B. Robinson and E.M. Landesman [28] one can prove the following theorem, which is an analog of Theorem 3.1/3.

Theorem 2. *Let* $s > n/p$ *and* $\rho > -1$*, and let* $g \in \mathbb{C}^\infty(\mathbb{R})$ *be the smooth function from Theorem* 1 *which satisfies the following additional properties.*

(a) $\liminf_{t\to\infty} tg(t) = G(+\infty) \in \mathbb{R}$ *and* $\liminf_{t\to-\infty} tg(t) = G(-\infty) \in \mathbb{R}$ *exist.*

(b) *Let*

$$G(\pm\infty) > 0. \tag{13}$$

Furthermore, we suppose that

(c) $\int_\Omega fw^* = 0$ *for all* $w^* \in \ker\{A^* - \lambda_1, B^*\}$.

(i) *Let* $f \in \mathbb{F}^{s-2}_{p,q}(\Omega) \cap B^\rho_{\infty,\infty}(\Omega)$*. Then* (1/1) *has a solution* $u \in \mathbb{F}^s_{p,q}(\Omega)$.

(ii) *Let* $f \in \mathbb{B}^{s-2}_{p,q}(\Omega) \cap B^\rho_{\infty,\infty}(\Omega)$*. Then* (1/1) *has a solution* $u \in \mathbb{B}^s_{p,q}(\Omega)$.

Proof. The proof is a copy of S. Robinson and E.M. Landesman [28, Theorem 4]. As in Theorem 1 we may assume that $\| u_n | L_\infty \| \to \infty$ and $\frac{u_n}{\| u_n | L_\infty \|} \to \varphi = \pm\varphi_1$. Let $\varphi = \varphi_1$. Then

$$\text{sign } (\varphi) \int_\Omega [g(u_n) - f]\varphi_1^* = \frac{1}{\| u_n | L_\infty \|} \int_\Omega g(u_n) u_n \Big(\frac{\varphi_1^*}{u_n / \| u_n | L_\infty \|} \Big). \tag{14}$$

Applying the arguments of [28] we claim that there are positive numbers k_1, k_2, and N such that

$$k_1 \leq \frac{\varphi_1^*}{\frac{u_n}{\| u_n | L_\infty \|}} \leq k_2 \quad \text{for } n > N. \tag{15}$$

Finally, the properties of g imply that the integrand of (14) is L_∞ bounded below. An application of Fatou's Lemma shows that

$$\liminf_{n\to\infty} \| u_n | L_\infty \| \int_\Omega g(u_n)\varphi_1^* \geq G(+\infty) \int_\Omega \frac{\varphi_1^*}{\varphi_1} > 0. \tag{16}$$

Here we used the properties of φ_1 and φ_1^* and the fact that $\varphi_1 = \varphi_1^*$ on $\partial\Omega$. Now (16) shows that (GL2) is satisfied. The proof is finished.

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