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EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS TO CERTAIN DIFFERENTIAL SYSTEMS

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Abstract. In this article we study existence and uniqueness of positive solutions for elliptic systems of the form

$$\begin{aligned} -\Delta v =& f(x,u) \quad \text{in} \quad \Omega \\ -\Delta u =& v^{\beta} \quad \text{in} \quad \Omega, \end{aligned}$$

with Dirichlet boundary condition on a bounded smooth domain in $\mathbb{R}^N$. The nonlinearity f is assumed to have a sub-β growth with $\beta > 0$, that in case $f(x,u) = u^{\alpha}, \alpha > 0$, corresponds to $\alpha\beta < 1$.

The results are also valid for a larger class of systems, including some infinite dimensional Hamiltonian Systems.

1. Introduction. In recent years a lot of attention has been given to the study of the nonlinear elliptic system

$$\begin{aligned} -\Delta v = u^{\alpha} \quad & \text{in} \quad \Omega \\ -\Delta u = v^{\beta} \quad & \text{in} \quad \Omega \\ u = v = 0 \quad & \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

and several of its generalizations. Here $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. We mention the work of Clément, de Figueiredo and Mitidieri [5], [6], Hulshof and van der Vorst [16], de Figueiredo and Felmer [12], Clément and van der Vorst [9], and others, for the existence of positive solutions for the superlinear, subcritical case. In the quasilinear case we mention the work of Clément, Manásevich and Mitidieri [10].

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The sublinear case of (1.1), that here corresponds to $\alpha\beta < 1$, has been less studied. It is the purpose of this paper to obtain conditions for the existence and uniqueness of positive solutions for (1.1) and generalizations.

Following a recent work of Clément, Felmer and Mitidieri [7] and [8], we isolate v in the second equation to obtain the fourth order scalar equation

$$\begin{aligned} -\Delta(-\Delta u)^{1/\beta} &= u^\alpha \quad \text{in} \quad \Omega \\ u = -\Delta u &= 0 \quad \text{in} \quad \partial\Omega. \end{aligned}$$

This equation has a variational structure, and its solutions can be obtained through minimization arguments. We let $J : W^{2,1+\beta^*}(\Omega) \cap W_0^{1,1+\beta^*}(\Omega) \to \mathbb{R}$ be a functional defined as

$$J(u) = \frac{1}{1+\beta^*}\int_\Omega |-\Delta u|^{1+\beta^*} - \frac{1}{1+\alpha}\int_\Omega |u_+|^{1+\alpha},$$

where we use $\beta^* = 1/\beta$ for notational convenience. The sublinearity assumption $\alpha\beta < 1$ allows to prove that J is coercive and that it takes negative values. Thus J possesses a nontrivial critical point, that gives rise to a solution to the equation and then to system (1.1).

In Section §2 we follow this approach to obtain existence results in a very general setting. We consider a functional $I : B \to \mathbb{R}$ defined as

$$I(u) = \frac{1}{1+\beta^*}\int_\Omega |Au|^{1+\beta^*} - \int_\Omega F(x,u)dx,$$

where B is a Banach space compactly contained in $L^{1+\beta^*}(\Omega)$, A is an isomorphism $A : B \to L^{1+\beta^*}(\Omega)$ and the function F is a primitive of the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$. This setting was considered in [7] and [8] in the study of superlinear problems. In what follows we will denote by $||\cdot||_{1+\beta^*}$ the norm in $L^{1+\beta^*}(\Omega)$.

Our existence condition will be given in terms of certain generalized eigenvalues determined by the asymptotic behavior of the nonlinearity f. We consider the functions $a_{\pm 0}$ and $a_{\pm\infty}$ defined as

$$a_{\pm 0}(x) = \liminf_{u\to 0^\pm} \frac{f(x,u)}{u^{\beta^*}} \quad \text{and} \quad a_{\pm\infty}(x) = \limsup_{u\to\pm\infty} \frac{f(x,u)}{u^{\beta^*}}. \tag{1.2}$$

In Section §2, we prove our main existence result that states that the functional I possesses nontrivial critical points when

(E) $\qquad \lambda(A, a_{+0}, a_{-0}) < 0 \quad \text{and} \quad \lambda(A, a_{+\infty}, a_{-\infty}) > 0,$

where the generalized eigenvalues $\lambda(A,a,b)$ are defined by

$$\lambda(A,a,b) = \inf_{\|Au\|_{1+\beta^*}=1} \int_\Omega |Au|^{1+\beta^*} - a(x)|u^+|^{\beta^*+1} - b(x)|u^-|^{\beta^*+1} dx. \quad (1.3)$$

What is interesting is that, by particularizing somehow the operator A and the nonlinearity f, we can prove that condition (E) above is also necessary. We do this in Section §4.

In studying the semilinear elliptic equation

$$-\Delta u = f(x,u), \quad (1.4)$$

with sublinear nonlinearity, Brezis and Oswald in [4], introduced the asymptotic condition (E) on f to assure existence of nontrivial positive solutions. This work was generalized by Díaz and Saa [11] to a quasilinear version of (1.4). Later, Felmer, Manásevich and de Thélin in [13] considered certain quasilinear systems obtaining similar results. More recently Fleckinger, Hernández and de Thélin in [14], and Fleckinger-Pellé and Takác in [15] obtained further results for quasilinear systems. Our results can be considered as generalization of the work in [11] to higher order quasilinear operators, or as generalization to systems, often called Hamiltonian, of results in [13], [14] and [15].

Starting from the results on existence of critical points obtained in Section §2, we derive in Section §3 existence theorems for the associated general systems. Imposing additional positivity hypotheses on the operator A and the function f, we obtain existence of positive solutions. Our results apply, in particular, to existence of positive solutions of elliptic systems of the form

$$\begin{aligned} -\Delta v &= f(x,u) \quad \text{in} \quad \Omega \\ -\Delta u &= v^\beta \quad \text{in} \quad \Omega \\ u = v &= 0 \quad \text{in} \quad \partial\Omega, \end{aligned} \quad (1.5)$$

and to existence of periodic positive solutions for a class of infinite dimensional Hamiltonian systems like

$$\begin{aligned} \frac{\partial v}{\partial t} &= \Delta v + f(t,x,u) \quad \text{in} \quad (-T,T)\times\Omega \\ \frac{\partial u}{\partial t} &= -\Delta u - v^\beta \qquad \text{in} \quad (-T,T)\times\Omega \\ u(t,x) = v(t,x) &= 0 \quad \text{in} \quad \mathbb{R}\times\partial\Omega, \\ u(-T,x) = u(T,x), \quad v(-T,x) &= u(T,x) \quad \text{in} \quad \Omega. \end{aligned} \quad (1.6)$$

This class of systems has been recently considered in the work of Clément, Felmer and Mitidieri [7], [8] and Barbú [1].

In Section §5 we discuss the uniqueness problem. When the nonlinearity f satisfies a monotonicity condition, uniqueness results have been proved for the semilinear equation (1.4) by Brezis and Oswald, based on an idea of Benguria, see [2] and [3]. Also, extension to some quasilinear operator of second order of the p-laplacian type has been obtained in [11], and for corresponding systems by [13], [14] and [15].

In this paper we prove a uniqueness result for positive solutions to system (1.5) and for positive periodic solutions of (1.6). See Theorem 5.1 and 5.2. Easy extensions can be given to systems of this nature, but with variable coefficients.

In view of the known results, and the form of the functional I, it seems reasonable to expect a uniqueness result for a general differential system of the form considered in Section §3. However difficulties arise quickly since general operators do not have rules to deal with powers. We believe that our results can be further generalized, but we do not have an appropriate characterization of the class of operators to be considered.

2. A general existence result: Sufficient conditions. In this section we study the existence of critical points for the general functional I defined in the Introduction. The existence theorem we prove in this section will provide existence of solutions of general systems as we will discuss in Section §3.

We start introducing the setting in which we will study the critical point problem. Let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain, and let $(B, ||\cdot||)$ be a real Banach space satisfying

(B1) *B is compactly contained in $L^{1+\beta^*}(\Omega)$, for $\beta^* > 0$ and*
(B2) *$L^\infty(\Omega) \cap B$ is dense in B.*

Next we consider a linear operator A satisfying

(A1) *$A : B \to L^{1+\beta^*}(\Omega)$ is an isomorphism.*

We observe that, in view of (A1), we can use $\|Au\|_{1+\beta^*}$ as an equivalent norm in B. We will study critical points of the functional I defined in the Introduction, which can also be written as

$$I(u) = \frac{1}{1+\beta^*}||Au||_{1+\beta^*}^{1+\beta^*} - \int_\Omega F(x,u)dx,$$

where $F(x,u) = \int_0^u f(x,t)dt$. We assume the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying the following growth conditions:

(f1) *There is a constant* $C > 0$ *such that*

$$f(x,u)u \leq C(|u|^{1+\beta^*} + |u|) \quad \forall u \in \mathbb{R}, \quad x \in \Omega \textit{ a.e.}$$

(f2) *For all* $\delta > 0$*, there is* $C_\delta > 0$ *such that*

$$f(x,u)u \geq -C_\delta |u|^{1+\beta^*} \quad \forall u \in [-\delta, \delta], \quad x \in \Omega \textit{ a.e.}$$

We observe that from (f1) we have

$$F(x,u) \leq C\left(|u|^{1+\beta^*} + 1\right), \tag{2.2}$$

for a certain constant C. Thus $I(u)$ is well defined in B with values in $(-\infty, \infty]$.

In studying critical points of I, a key role is played by the asymptotic behavior of f, both in zero and infinity. To take this into account we consider the functions a_{+0}, a_{-0}, $a_{+\infty}$ and $a_{-\infty}$ defined in (1.2). It follows from the hypotheses (f1) and (f2) that for a constant C

$$-C \leq a_{\pm 0}(x) \quad \text{and} \quad a_{\pm\infty}(x) \leq C \qquad \text{for all} \quad x \in \Omega \textit{ a.e.} \tag{2.3}$$

This estimates implies that the generalized eigenvalues defined in (1.3) satisfy $\lambda(A, a_{+0}, a_{-0}) \in [-\infty, \infty)$ and $\lambda(A, a_{+\infty}, a_{-\infty}) \in (-\infty, \infty]$.

Now we can state the main theorem of this section

Theorem 2.1. *Under the general hypotheses* **(B1)**, **(B2)**, **(A1)** *and assuming* f *satisfies* **(f1)** *and* **(f2)**, *the condition* **(E)** *implies that the functional* I *has a non trivial critical point, characterized as a global minimum.*

Remark 2.1. We want to observe that our definition of generalized eigenvalue slightly differs from that in the work of Brezis and Oswald [4] and in [11], [13]. The present form is more suitable for treating our general problem.

In order to prove Theorem 2.1 we follow the strategy considered in [4]. First we show that the functional I is coercive and weakly lower semi continuous. Then, since I is bounded below, it possesses a minimum. Next, to make sure this critical point is not trivial, we show that I takes negative values.

Lemma 2.1. *The functional $I : B \to (-\infty, \infty]$ is coercive.*

Proof. Assume I is not coercive. Then there exists $\{u_n\} \subset B$ and $C \in \mathbb{R}$ so that $|Au_n|_{1+\beta^*} \to \infty$ and $I(u_n) \le C$. From here, using (2.2) we find that

$$||Au_n||_{1+\beta^*}^{1+\beta^*} \le C \int_\Omega \left(|u_n|^{1+\beta^*} + 1\right). \tag{2.4}$$

Let us define $t_n = ||u_n||_{1+\beta^*}$ and $v_n = u_n/t_n$, then from (2.4) we have that $t_n \to \infty$ and $|v_n|$ is bounded. Then, up to a subsequence, $v_n \rightharpoonup v$ in B and $v_n \to v$ in $L^{1+\beta^*}(\Omega)$ after (B1); thus $||v||_{1+\beta^*} = 1$. Moreover there is $h \in L^{1+\beta^*}(\Omega)$ such that

$$v_n \to v \quad \text{pointwise} \quad \text{and} \quad |v_n| \le h. \tag{2.5}$$

Since $t_n v_n = u_n$ we have

$$\begin{aligned} \int_\Omega F(x,u_n)dx = &\int_{\{v\le 0\}} F(x,t_n v_n^+)dx + \int_{\{v\ge 0\}} F(x,t_n v_n^-)dx \\ &+ \int_{\{v>0\}} F(x,t_n v_n^+)dx + \int_{\{v<0\}} F(x,t_n v_n^-)dx. \end{aligned} \tag{2.6}$$

First we see that, using (2.2) and (2.5), we have

$$\int_{\{v\ge 0\}} \frac{F(x,t_n v_n^-)dx}{t_n^{1+\beta^*}}dx + \int_{\{v\le 0\}} \frac{F(x,t_n v_n^+)}{t_n^{1+\beta^*}}dx \le o(1) \quad \text{as} \quad n \to \infty. \tag{2.7}$$

To study the third integral in (2.6), we first see from (1.2) and (2.5) that

$$\limsup_{n\to\infty} \frac{F(x,t_n v_n^+)}{t_n^{1+\beta^*}} \le \frac{1}{1+\beta^*} a_{+\infty}(x)(v^+)^{1+\beta^*}, \quad \forall x \in \{v(x) > 0\}. \tag{2.8}$$

Next we use (2.2), (2.5) and Fatou's Lemma to obtain

$$\limsup_{n\to\infty} \int_{\{v>0\}} \frac{F(x,t_n v_n^+)}{t_n^{1+\beta^*}} \le \int_{\{v>0\}} \frac{1}{1+\beta^*} a_{+\infty}(x)(v^+)^{1+\beta^*}. \tag{2.9}$$

A similar estimate can be obtained for the fourth integral in (2.6). Thus, from (2.6) and the estimates obtained above we find finally that

$$|Av|_{1+\beta^*}^{1+\beta^*} \le \int_\Omega a_{+\infty}(x)|v^+|^{1+\beta^*} + \int_\Omega a_{-\infty}(x)|v^-|^{1+\beta^*},$$

that contradicts the hypothesis $\lambda(A, a_\infty, a_{-\infty}) > 0$. □

The next lemma will allow us to show that the critical point we find is not trivial.

Lemma 2.2. *There is $v \in B$ such that $I(v) < 0$.*

Proof. We first see, from (1.2) and (f2), that

$$\liminf_{u \to 0^+} \frac{F(x,u)}{u^{1+\beta^*}} \geq \frac{a_{+0}(x)}{1+\beta^*}, \quad \forall x \in \Omega. \tag{2.10}$$

Next, since we are assuming $\lambda(A, a_{+0}, a_{-0}) < 0$, using (B2) we can find $w \in L^\infty(\Omega) \cap B$, $w \neq 0$ such that

$$||Aw||_{1+\beta^*}^{1+\beta^*} - \int_\Omega a_{+0}(x)|w^+|^{1+\beta^*} dx - \int_\Omega a_{-0}(x)|w^-|^{1+\beta^*} dx < 0. \tag{2.11}$$

Using (f2) and the fact that $w \in L^\infty(\Omega)$ we can find a constant $M \in \mathbb{R}$ so that

$$\frac{F(x, \epsilon w^+(x))}{\epsilon^{1+\beta^*}} \geq M \quad \text{and} \quad \frac{F(x, \epsilon w^-(x))}{\epsilon^{1+\beta^*}} \geq M \quad \forall x \in \Omega. \tag{2.12}$$

Then we can apply the Fatou's Lemma, and use (2.10) to conclude that

$$\liminf_{\epsilon \to 0} \int_\Omega \frac{F(x, \epsilon w)}{\epsilon^{1+\beta^*}} \geq \frac{1}{1+\beta^*} \int_\Omega a_{+0}(x)|w^+|^{1+\beta^*} + a_{-0}(x)|w^-|^{1+\beta^*}. \tag{2.13}$$

Consequently, from (2.11), we finally obtain that for a small ϵ

$$I(\epsilon w) = ||A(\epsilon w)||_{1+\beta^*}^{1+\beta^*} - \int_\Omega F(x, \epsilon w) dx < 0.$$

Proof of Theorem 2.1. Using standard arguments we can prove that the functional I is weakly lower semi continuous. Here it will be used, in particular, hypothesis (B1). Then, using Lemmas 2.1 and 2.2 we obtain that I attains its global minimum that is a nontrivial critical point. □

Remark 2.2. We would like to observe that hypothesis (B2) is not very restrictive, since whenever B is an appropriate Sobolev space then the C^∞ functions will be dense in B.

Remark 2.3. The existence theorem we just proved provides a nontrivial critical point for the functional I. This is interesting in the case $f(x,0) \equiv 0$, since then $u \equiv 0$ is always a solution.

With regard to the Remark 2.3, we have the following extension of Theorem 2.1.

Theorem 2.2. *Under the general hypotheses* **(B1)**, **(B2)**, **(A1)** *and assuming f satisfies* **(f1)**, *the condition*

$$\lambda(A, a_{+\infty}, a_{-\infty}) > 0$$

implies that the functional I has a critical point, characterized as a global minimum.

Remark 2.4. Theorem 2.2 permits to treat functions of the form $f(x,u) = g(u) + l(x)$.

We conclude this section considering the case of a pure power, that is the case when $f(x,u) = u^\alpha$ with $\alpha < \beta^*$. Here we do not need hypothesis (B2), and we can treat the case of a general measure space. Let (Ω, σ, μ) be a measure space, with μ a σ-finite measure and $(B, ||\cdot||)$ a real Banach space satisfying

(M1) *B is compactly contained in $L^{\alpha+1}(\Omega, \sigma, \mu)$.*

We also consider a continuous linear operator A satisfying

(M2) *$A : B \to L^{\beta^*+1}(\Omega, \sigma, \mu)$ is an isomorphism.*

For the functional $I : B \to \mathbb{R}$ defined as

$$I(u) = \frac{1}{1+\beta^*}\int_\Omega |Au|^{1+\beta^*} d\mu - \frac{1}{\alpha+1}\int_\Omega |u|^{\alpha+1} d\mu,$$

we have the following theorem

Theorem 2.3. *Under the hypotheses* **(M1)** *and* **(M2)**, *assuming $\alpha\beta < 1$, the functional I attains its global maximum and it is not trivial.*

Remark 2.5. The existence result for the superlinear case, that is when $\alpha\beta > 1$, was considered in the article [8], via the Mountain Pass Theorem.

3. Sublinear differential systems. Positive solutions. In this section we obtain, as a consequence of Theorem 2.1, 2.2 and 2.3, existence results for sublinear differential systems. By adding positivity hypotheses on the linear operators and nonlinearities, we give results on positive solutions for the corresponding systems. A brief discussion is finally given regarding systems (1.5) and (1.6).

We start introducing a second Banach space, $(B^*, ||\cdot||)$, and a second linear operator A^* satisfying

(B1)* *B^* is contained in $L^{\beta+1}(\Omega)$,*

(A1)* $A^* : B^* \to L^{\beta+1}(\Omega)$ *is an isomorphism.*

The spaces and operators are assumed to satisfy the following duality condition

(D) $\int_\Omega Auv = \int_\Omega uA^*v$ *for all* $u \in B$ *and* $v \in B^*$.

We want to study existence of solutions in $(u, v) \in B \times B^*$ for the following nonlinear system

$$\begin{aligned} A^*v &= f(x, u) \\ Au &= v^\beta. \end{aligned} \tag{3.1}$$

For this purpose we will find critical points of the functional I. Then, assuming the nonlinearity f satisfies a stronger growth condition, we can prove the critical point of I provides a solution for (3.1).

Specifically, we assume f satisfies

(f3) *There exists* $C > 0$ *such that* $|f(x, u)| \leq C(|u|^{\beta^*} + 1) \quad \forall x \in \Omega$ *a.e.*

Condition (f3) implies (f1) certainly.

Theorem 3.1. *Under the hypotheses* **(B1)**–**(B1)***, **(B2)**, **(A1)**–**(A1)***, **(D)**, *and assuming* f *satisfies* **(f2)** *and* **(f3)**, *the condition* **(E)** *implies that system (3.1) possesses a nontrivial solution* $(u, v) \in B \times B^*$.

Proof. From Theorem 2.1 the functional I possesses a nontrivial critical point $u \in B$, that satisfies

$$\int_\Omega (Au)^{\beta^*} Aw = \int_\Omega f(x, u)w \quad \forall w \in B. \tag{3.2}$$

Since $u \in B$, using (B1) and (f3) we find that $f(x, u) \in L^{\beta+1}(\Omega)$, so that (B1)* implies the existence of a unique $v \in B^*$ such that $A^*v = f(x, u)$. Then, from (3.2) and (D) we obtain

$$\int_\Omega (Au)^{\beta^*} Aw = \int_\Omega A^*vw = \int_\Omega vAw \quad \forall w \in B,$$

from where $(Au)^{\beta^*} = v$ or $Au = v^\beta$. $\square$

Next we give conditions for system (3.1) to have positive solutions. First we assume that the operators A and A^* are positive, that is

(P) *If* $w \in L^{\beta^*+1}(\Omega)$, $w \geq 0$ *then* $A^{-1}w \geq 0$, *and if* $z \in L^{\beta+1}(\Omega)$, $z \geq 0$ *then*

$(A^*)^{-1}z \geq 0$.

For a function $f : \Omega \times \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ we define $\bar{f}$ as $\bar{f}(x,u) = f(x,u)$ if $u \geq 0$ and $\bar{f}(x,u) = f(x,0)$ if $u < 0$. We observe that for $\bar{f}$ we have $a_{-\infty}(x) = 0$ and if $f(x,0) = 0$ then $a_{-0}(x) = 0$.

We have the following theorem

Theorem 3.2. *Under the general hypotheses of Theorem 3.1, we assume further that A and A^* satisfy* **(P)***, that $f(x,u) \geq 0$ and that $\bar{f}$ satisfy* **(f1)** *and* **(f3)** *then condition* **(E)** *implies that system* (3.1) *has a nontrivial positive solution.*

Proof. Direct from Theorem 3.1 and the positivity hypotheses of A, A^* and f. □

Remark 3.1. Using Theorems 2.2 and 2.3 we obtain existence theorems for system (3.1) like Theorems 3.1 and 3.2.

In the rest of the section we see how we apply the general existence theorems just discussed to study our two model systems (1.5) and (1.6).

We start with the elliptic system (1.5). We consider the spaces

$$B = W^{2,1+\beta^*}(\Omega) \cap W_0^{1,1+\beta^*}(\Omega) \quad \text{and} \quad B^* = W^{2,1+\beta}(\Omega) \cap W_0^{1,1+\beta}(\Omega),$$

and the operators $A = -\Delta : B \to L^{1+\beta^*}(\Omega)$ and $A^* = -\Delta : B^* \to L^{1+\beta}(\Omega)$.

From L^p theory for elliptic equations and Rellich embedding theorem we have that the spaces and operators satisfy the general hypotheses (B1)-(B1*), (A1)-(A1*). Further, Green Theorem implies (D), (B2) is clearly true, and the Maximum Principle implies (P). Thus if the function f satisfies the appropriate growth conditions we have strong solutions for system (1.5). Next, by the application of elliptic regularity theory we can prove that the solutions are actually classical solutions if f satisfies further

(H) *the function $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ belongs to $C^{0,\alpha}(\bar{\Omega} \times \mathbb{R})$.*

In particular we have the following

Theorem 3.3. *We assume the nonlinearity f satisfies $f(x,u) \geq 0$ and $\bar{f}$ satisfy* **(f1)**, **(f3)** *and* **(H)** *then condition* **(E)** *implies that system* (1.5) *has a nontrivial positive classical solution.*

Next we consider the infinite dimensional Hamiltonian system (1.6).

Let $\Omega_T = [-T,T] \times \Omega \subset \mathbb{R}^{N+1}$. For $1 < r < +\infty$ and for a Banach space E, we define $L_T^r(E)$ as $L^r([-T,T],E)$. We consider the Banach space

$$B_r = W_T^{1,r}(L^r(\Omega)) \cap L_T^r(W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)), \tag{3.4}$$

endowed with the norm

$$||u|| = \left[\int_{-T}^{T} \int_\Omega \{ |u(t,x)|^r + |\frac{\partial u}{\partial t}(t,x)|^r + \right. \\ \left. + \sum_{i=1}^{N} |\frac{\partial u}{\partial x_i}(t,x)|^r + \sum_{i,j=1}^{N} |\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x)|^r \} dxdt \right]^{\frac{1}{r}}, \tag{3.5}$$

where $W_T^{1,r}(L^r(\Omega))$ is the space of functions defined on $[-T,T]$ with values in $L^r(\Omega)$ with derivatives in $L^r(\Omega_T)$, and periodic in t. We define the Banach spaces $B = B_{1+\beta^*}$ and $B^* = B_{\beta+1}$. We identify $L_T^r(L^r(\Omega))$ with $L^r(\Omega_T)$ and as a consequence we have that $B_r \subset L^r(\Omega_T)$. If we further consider the Rellich theorem we find that B and B^* satisfies (B1)-(B1*). See Lemma A.1 in [8].

Next we introduce the operators A and A^*:

$$A = \frac{\partial u}{\partial t} - \Delta u : B_{1+\beta^*} \to L^{1+\beta^*}(\Omega_T) \quad \text{and}$$
$$A^* = -\frac{\partial u}{\partial t} - \Delta u : B_{1+\beta} \to L^{1+\beta}(\Omega_T).$$

This operators are isomorphism of Banach spaces. See Lemma 3.1 in [8]. Furthermore they satisfy

$$\int_{-T}^{T} \int_\Omega (\frac{\partial u}{\partial t} - \Delta u)(t,x) v(t,x) dxdt = \int_{-T}^{T} \int_\Omega u(t,x)(-\frac{\partial v}{\partial t} - \Delta v)(t,x) dxdt, \tag{3.6}$$

and so (A1)-(A1*) and (D) are satisfied. The operators A and A^* also satisfies the Maximum Principle, so that they are positive.

Thus we only need to assume appropriate hypotheses on the nonlinearity f in order to apply the general theorems to system (1.6). In addition to the growth conditions (f1) and (f3) we need to assume in this case that the function $f : \Omega_T \times \mathbb{R} \to \mathbb{R}$ is T-periodic in the temporal variable t.

In particular we will have the following theorem

Theorem 3.4. *We assume the nonlinearity f is nonnegative and T-periodic in the temporal variable. Further we assume $\bar{f}$ satisfy* **(f1)**, **(f3)** *and* **(H)**.

Then condition **(E)** *implies that system* (1.6) *has a nontrivial positive periodic classical solution.*

The regularity part in this theorem can be found in the Appendix of [8].

4. A general existence result: Necessary conditions. We devote this section to show that under certain conditions, hypothesis (E) on the generalized eigenvalues is also necessary for the existence of critical points of the functional I.

In order to do so we will need to make some additional assumptions on the function f. Instead of considering **(f1)** and **(f2)**, we will assume

(f0+) *f is a Caratheodory function, and the function $x \to f(x,u)$ is in $L^\infty(\Omega)$*

for all $u \in \mathbb{R}$.

(f4) *The function*

$$u \to \frac{f(x,u)}{|u|^{\beta^*}}$$

is strictly decreasing in $\mathbb{R} \setminus \{0\}$.

We note that if f satisfies **(f0+)** and **(f4)** then f satisfies **(f1)** and **(f2)**. Our first result deals with the eigenvalue associated to the behavior of f at zero.

Theorem 4.1. *Assume the hypothesis* **(B1)**, **(A1)**, **(f0+)** *and* **(f4)** *hold. Then, the existence of a nontrivial critical point of the functional I implies that*

$$\lambda(A, a_0, a_{-0}) < 0.$$

Proof. Let $u \in B$ be a nontrivial critical point of I, then u satisfies

$$\int_\Omega Au^{\beta^*} Av - \int_\Omega f(x,u)v = 0 \quad \forall v \in B. \tag{4.1}$$

From hypothesis **(f4)** and the fact that u is not trivial we see that

$$a_{+0}(x)|u^+(x)|^{1+\beta^*} + a_{-0}(x)|u^-(x)|^{1+\beta^*} > f(x,u(x))u(x),$$

on a set of positive measure. This inequality and the evaluation of (4.1) in $v = u$ give the result. □

Next we see that the condition on the eigenvalue at infinity is also necessary. For this we need to restrict our attention to positive operators, and assume that $\beta^* = 1$ and that f is nonnegative.

Theorem 4.2. *Under the hypothesis of Theorem 4.1. We further assume that $\beta^* = 1$, that A is positive and that $f(x,u) \geq 0$ for $u \geq 0$ and $f(x,u) = f(x,0)$ for $u \leq 0$. Then the existence of a nontrivial critical point u of I that is in $L^\infty(\Omega)$ and $u(x) > 0$ a.e. in Ω implies that*

$$\lambda(A, a_\infty, a_{-\infty}) > 0.$$

Proof. From our assumptions on f we have $a_{-\infty}(x) = 0$, so that if we define $\mu = \lambda(A, a_\infty, a_{-\infty})$, then

$$\mu = \inf_{||Au||_2=1} ||Au||_2^2 - \int_\Omega a_\infty(x)|u^+(x)|^2 dx. \tag{4.2}$$

Let us define

$$a(x) = \frac{f(x, \| u \|_\infty + 1)}{\| u \|_\infty + 1}.$$

From the positivity of f and (f4) we have

$$a(x) \geq 0 \quad \text{and} \quad a(x) > a_\infty(x) \quad \forall x \in \Omega, \tag{4.3}$$

and then, as $u > 0$ in Ω,

$$a(x)u < f(x,u) \quad \forall x \in \Omega. \tag{4.4}$$

We define an auxiliary generalized eigenvalue as

$$\lambda = \inf_{\|Au\|_2=1} ||Au||_{1+\beta^*}^{1+\beta^*} - \int_\Omega a(x)|u^+(x)|^2 dx. \tag{4.5}$$

We note that from (4.3) $\mu \geq \lambda$ and then we only need to prove that $\lambda > 0$. We do this assuming that $\lambda \leq 0$. We claim that, under this assumption, the infimum in (4.5) is achieved at a function ϕ.

Let us assume the claim is true for the moment. Let $\psi = A^{-1}|A\phi|$, then we have $||A\psi||_2 = ||A\phi||_2 = 1$ and $\psi \geq 0$ so that $\psi \geq \phi^+$. Thus the infimum is also achieved at ψ, and consequently

$$\lambda \int_\Omega A\psi Av = \int_\Omega A\psi Av - \int_\Omega a(x)\psi v \quad \forall v \in B. \tag{4.6}$$

On the other hand, since A and f are positive, for the critical point u of I, we have

$$\int_\Omega Auw = \int_\Omega f(x,u)A^{-1}w \geq 0, \tag{4.7}$$

for all $w \geq 0$ in $L^2(\Omega)$ and then $Au \geq 0$. Evaluating (4.6) in u and using again that u is a critical point of I we obtain

$$-\lambda \int_\Omega A\psi Au = \int_\Omega a(x)\psi u - \int_\Omega f(x,u)\psi. \tag{4.8}$$

From (4.4) we find $D \subset \Omega$ of positive measure, so that $a(x)\psi u - f(x,u)\psi < 0$ in D. Thus, from (4.8) and the facts that $Au \geq 0$ and $A\psi = |A\phi| \geq 0$, we conclude that $\lambda > 0$.

Now we prove the claim. Let $\{u_n\} \subset B$ be a minimizing sequence for (4.5). As $||Au_n||_2 = 1$, after a subsequence, we have $u_n \rightharpoonup w$ in B and $u_n \to w$ in $L^2(\Omega)$, as $n \to \infty$. Then we have $||Aw||_2^2 \leq 1$, and

$$||Aw||_2^2 - \int_\Omega a(x)|w^+(x)|^2 dx \leq \lambda. \tag{4.9}$$

If $\lambda < 0$, then from (4.9) $w \neq 0$, and we can define $v = w/||Aw||_2$, and we have $||Av||_2 = 1$. Then, as $||Aw||_2^2 \leq 1$ we conclude that infimum in (4.5) is achieved at v. Now we look at the case $\lambda = 0$. We have

$$1 = \lim_{n\to\infty} ||Au_n||_2^2 = \lim_{n\to\infty} \int_\Omega a(x)|u_n^+(x)|^2 dx, = \int_\Omega a(x)|w^+(x)|^2 dx, \tag{4.10}$$

so that $w \neq 0$. Now we proceed as in the case $\lambda < 0$ to obtain that the infimum is achieved at $v = w/||Aw||_2$. $\square$

Remark 4.1. It is interesting to note that if A satisfies the strong maximum principle, then the positivity hypothesis in Theorem 4.2 can be replaced by $u \neq 0$. The operators associated to equations (1.5) and (1.6) satisfy the strong maximum principle.

5. Uniqueness theorem. This section is devoted to study the uniqueness question for some of the systems considered in this article.

First we consider the case of system (1.5). Assuming that the nonlinearity f satisfies a monotonicity hypothesis, we are able to show that system (1.5) admits at most one positive solution. The proof of this result is based on a convexity property of the differential part of the operator together with the assumption on f.

A similar argument can be given to prove that system (1.6) also has the uniqueness property.

We will assume that the function f satisfies the condition (f4), but only on the positive axis.

(f5) *The function*

$$u \to \frac{f(x,u)}{u^{\beta^*}}$$

is strictly decreasing and nonegative in $\mathbb{R}^+$.

We observe that if f satisfies assumption (f5) then the function $u \to F(u^{1-\beta^*})$ is strictly concave.

We state our uniqueness theorem for system (1.5).

Theorem 5.1. *When the function f satisfies (f5) then system (1.5) possesses at most one positive solution.*

The following lemma is needed in our proof and it is a consequence of the maximum principle.

Lemma 5.1. *If $(u_1,v_1),(u_2,v_2) \in C^2(\bar\Omega)\times C^2(\bar\Omega)$ are solutions (1.5), then for all $r>0$ we have*

$$\frac{u_1^{r+1}}{u_2^r} \in C^1(\bar\Omega)\cap C^2(\Omega) \quad , \quad \frac{u_1^{r-1}}{u_2^r}v_1 \in C^0(\bar\Omega)\cap C^2(\Omega) \quad \textit{and}$$

$$\frac{u_1^{r+1}}{u_2^r} = 0 \quad \textit{on} \quad \partial\Omega.$$

Next we recall that if (u,v) is a classical positive solution of (1.5) then the function u is a classical solution of the fourth order equation

$$\begin{aligned} -\Delta(-\Delta u)^{1/\beta} &= f(x,u) \quad \text{in} \quad \Omega \\ u = -\Delta u &= 0 \quad \text{in} \quad \partial\Omega, \end{aligned} \tag{5.1}$$

and $u>0$, $v=-\Delta u>0$ in Ω. Then we see from the lemma that if $u_1,u_2\in C^2(\bar\Omega)$ are solutions of (5.1), then

$$\frac{u_1^{\beta^*+1}}{u_2^{\beta^*}}, \frac{u_2^{\beta^*+1}}{u_1^{\beta^*}} \in C^1(\bar\Omega)\cap C^2(\Omega), \quad \frac{u_1^{\beta^*+1}}{u_2^{\beta^*}} = \frac{u_2^{\beta^*+1}}{u_1^{\beta^*}} = 0 \quad \text{in} \quad \partial\Omega \tag{5.2}$$

$$\text{and} \quad \frac{u_2^{\beta^*-1}}{u_1^{\beta^*}}(-\Delta u_1)^{\beta^*}, \frac{u_1^{\beta^*-1}}{u_2^{\beta^*}}(-\Delta u_2)^{\beta^*} \in C^0(\bar\Omega)\cap C^2(\Omega). \tag{5.3}$$

Proof of Theorem 5.1. Let u_1 and u_2 be solutions of (5.1). Our goal is to show that the following inequality holds

$$\int_\Omega \left(\frac{-\Delta(-\Delta u_2)^{\beta^*}}{u_2^{\beta^*}} - \frac{-\Delta(-\Delta u_1)^{\beta^*}}{u_1^{\beta^*}} \right) \left(u_2^{\beta^*+1} - u_1^{\beta^*+1} \right) dx \geq 0. \tag{5.4}$$

This will finish the proof. In fact, if we call $I(u_1, u_2)$ the left hand side of (5.4), and we assume that u_1 and u_2 are distinct solutions of (5.1), then from (f5) we have

$$I(u_1, u_2) = \int_\Omega \left(\frac{f(x, u_2)}{u_2^{\beta^*}} - \frac{f(x, u_1)}{u_1^{\beta^*}} \right) \left(u_2^{\beta^*+1} - u_1^{\beta^*+1} \right) dx < 0, \tag{5.5}$$

that contradicts (5.4). In order to prove (5.4) the following identity is crucial

$$\begin{aligned} \Delta \left(\frac{u_2^{\beta^*+1}}{u_1^{\beta^*}} \right) &= (\beta^* + 1)\beta^* \frac{u_2^{\beta^*-1}}{u_1^{\beta^*}} |\nabla u_2 - \frac{u_2}{u_1} \nabla u_1|^2 \\ &\quad + (\beta^* + 1) \left(\frac{u_2}{u_1} \right)^{\beta^*} \Delta u_2 - \beta^* \left(\frac{u_2}{u_1} \right)^{\beta^*+1} \Delta u_1. \end{aligned} \tag{5.6}$$

A similar identity is obtained for $\Delta \left(u_1^{\beta^*+1} / u_2^{\beta^*} \right)$.

Using the Green Theorem for $I(u_1, u_2)$ and then the identities just mentioned we obtain

$$\begin{aligned} I(u_1, u_2) &= \int_\Omega (-\Delta u_2)^{\beta^*+1} dx + \int_\Omega (-\Delta u_1)^{\beta^*+1} dx \\ &\quad + \int_\Omega (-\Delta u_2)^{\beta^*} \left((\beta^* + 1) \left(\frac{u_1}{u_2} \right)^{\beta^*} \Delta u_1 - \beta^* \left(\frac{u_1}{u_2} \right)^{\beta^*+1} \Delta u_2 \right) dx \\ &\quad + \int_\Omega (-\Delta u_1)^{\beta^*} \left((\beta^* + 1) \left(\frac{u_2}{u_1} \right)^{\beta^*} \Delta u_2 - \beta^* \left(\frac{u_2}{u_1} \right)^{\beta^*+1} \Delta u_1 \right) dx \\ &\quad + \int_\Omega (-\Delta u_2)^{\beta^*} \left((\beta^* + 1)\, \beta^* \frac{u_1^{\beta^*-1}}{u_2^{\beta^*}} \left| \nabla u_1 - \frac{u_1}{u_2} \nabla u_2 \right|^2 \right) dx \\ &\quad + \int_{\Omega_n} (-\Delta u_1)^{\beta^*} \left((\beta^* + 1)\, \beta^* \frac{u_2^{\beta^*-1}}{u_1^{\beta^*}} \left| \nabla u_2 - \frac{u_2}{u_1} \nabla u_1 \right|^2 \right) dx. \end{aligned}$$

We observe that, since $u_1 \geq 0, u_2 \geq 0, -\Delta u_2 \geq 0$ and $-\Delta u_1 \geq 0$ the last two terms above are nonegative. Thus, if we define the auxiliary functional H as

$$H(u,v) = \int_\Omega (-\Delta v)^{\beta^*+1} - (\beta^*+1)\left(\frac{v}{u}\right)^{\beta^*} (-\Delta u)^{\beta^*}(-\Delta v) + \beta^* \left(\frac{v}{u}\right)^{\beta^*+1} (-\Delta u)^{\beta^*+1} dx, \tag{5.7}$$

we have that $I(u_1,u_2) \geq H(u_1,u_2) + H(u_2,u_1)$, and so we are only left to prove $H(u,v) \geq 0$ for all u, v. For this purpose we put $x = -\Delta v$ and $y = \frac{v}{u}(-\Delta u)$ and we note that the integrand in (5.7) is

$$\gamma(x,y) = x^{\beta^*+1} - (\beta^*+1)\, y^{\beta^*} x + \beta^* y^{\beta^*+1}.$$

The following calculus lemma finishes the proof.

Lemma 5.2. $\gamma(x,y) \geq 0$ *for all* $x, y \geq 0$.

Proof. Since the function γ is homogeneous we only need to prove that it is nonegative on $A = \{(x,y)/x+y=1, x,y>0\}$, that is equivalent to show that the function

$$g(x) = \gamma(x, 1-x) = x^{\beta^*+1} - (\beta^*+1)\,(1-x)^{\beta^*} x + \beta^*(1-x)^{\beta^*+1}$$

is nonnegative in $(0,1)$. Differentiating g we obtain that $g'(1/2) = g(1/2) = 0$ and that $g'(x) = 0$ only if $x\ x \in (0, \frac{\beta+1}{\beta+2})$. Next by differentiating we find that $g''(x) > 0$ if $0 < x < \frac{2\beta+1}{\beta+2}$. From here and the condition on the critical points of g we find that g has exactly one critical point in $(0,1)$, this is $x = 1/2$ and it is the global minimum, so that $g(x) \geq 0$ en $(0,1)$. $\square$

We finish with a uniqueness theorem for the infinite dimensional Hamiltonian system (1.6). We have

Theorem 5.2. *When the function f satisfies (f5) then system (1.6) possesses at most one positive solution.*

Proof. We prove this theorem following the lines of the proof of Theorem 5.1. At first we need to use the maximum Principle and the Hopf Lemma for parabolic operators to obtain a result like Lemma 5.1. Then we follow the steps in the proof of Theorem 5.1. In doing so we need the equivalent of identity (5.6), that here takes the surprisingly analogous form

$$A\left(\frac{v^{\beta^*+1}}{u^{\beta^*}}\right) = -(\beta^*+1)\beta^* \frac{u_2^{\beta^*-1}}{u_1^{\beta^*}} |\nabla u_2 - \frac{u_2}{u_1}\nabla u_1|^2 + (\beta^*+1)\left(\frac{v}{u}\right)^{\beta^*} Av - \beta^*\left(\frac{v}{u}\right)^{\beta^*+1} Au \tag{5.8}$$

where we use A for the operator

$$Au = \frac{\partial u}{\partial t} - \Delta u.$$

From here the proof follows exactly in the same way. □

Remark 5.1. As a consequence of this uniqueness theorem we see that, when system (1.6) is autonomous, that is f does not depend on t, then the solution does not depend on t. Moreover, in view of Theorem 5.1 the solution of (1.6) is the unique solution of (1.5).

We observe that for linear operators satisfying a maximum and Hopf principle, and also identity (5.8), a uniqueness theorem for the corresponding system will hold. However we do not have a characterization of such linear operators.

In any case, our theorems will apply to operators of the form given, but with variable coefficient.

REFERENCES

[1] V. Barbu, *Periodic solutions to unbounded hamiltonian systems*, Discrete and Continuous Dynamical Systems, 1 (1995), 277–283.

[2] R. Benguria, *The von Weizsäcker and exchange corrections in the Thomas-Fermi theory*, Dissertation, Princeton University, 1979.

[3] R. Benguria, H. Brezis, and E. Lieb, *The Thomas-Fermi-von Weizsäcker theory of atoms and molecules*, Communs. Math. Phys., 79 (1981), 167–180.

[4] H. Brezis and L. Oswald, *Remarks on Sublinear elliptic equations*, Nonlinear Analysis, TMA, Vol. 10 (1986), 55–64.

[5] Ph. Clément, D. de Figueiredo and E. Mitidieri. *Positive solutions of semilinear elliptic systems*, Comm. Partial Diff. Eq., 17, no 5-6, p 923–940.

[6] Ph. Clément, D. de Figueiredo and E. Mitidieri, *A priori estimates for positive solutions of semilinear elliptic systems via Hardy-Sobolev inequalities*, Pitman Res. Notes in Math. (1996), 73–91.

[7] Ph. Clément, P. Felmer and E. Mitidieri, *Solutions homoclines d'un système hamiltonien non-borné et superquadratique*, C.R. Acad. Sci. Paris, t.320, Série I, (1995), 1481–1484.

[8] Ph. Clément, P. Felmer and E. Mitidieri, *Homoclinic orbits for a class of infinite dimensional Hamiltonian systems*, Ann. Scuola Norm. Sup. Pisa., Serie IV, Vol. XXIV. Fasc. 2, (1997), 367–393.

[9] Ph. Clément and R.C.A.M. van der Vorst, *On a semilinear elliptic system*, Differential and Integral Eq., 8 (1995), 1317–1329.

[10] Ph. Clément, R. Manásevich, and E. Mitidieri, *Positive solutions for a quasilinear system via blow up*, Comm. in PDE., 18 (1993), 2071–2116.

[11] J.I. Diaz and J.E. Saa, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, C.R.A.S. Paris, 305, Série I (1987), 521–524.

[12] D. De Figueiredo and P. Felmer, *On Superquadratic elliptic systems*, Trans. Amer. Math. Soc., 343 (1994), 99–116.

[13] P. Felmer, R. Manásevich and F. De Thélin, *Existence and Uniqueness of positive solutions for certain quasilinear elliptic systems*, Comm. in PDE., 17 (1992), 2013–2029.

[14] J. Fleckinger, J. Hernández and F. De Thélin, *On maximum principles and existence of positive solutions for some cooperative elliptic systems*, Diff. and Int. Equations, 8 (1995), 69–85.

[15] J. Fleckinger and P. Takac, *Uniqueness of positive solutions for nonlinear cooperative systems with the p-Laplacian*, Indiana Univ. Math. Journal, 43 (1994), 1227–1253.

[16] J. Hulshof and R.A.C.M. van der Vorst, *Differential systems with strongly indefinite structure*, J. Funct. Anal., 114 (1993), 32–58.