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ON SOME NONLINEAR EQUATIONS INVOLVING THE p-LAPLACIAN WITH CRITICAL SOBOLEV GROWTH

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I. Introduction. Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Let also $p \in (1, n)$ real, and a, f two smooth functions on $\overline{\Omega}$. We are concerned here with the existence of solutions $u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$, positive or not, to the problem

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + a(x)|u|^{p-2}u = f(x)|u|^{p^\star-2}u \ \text{ in } \ \Omega \\ u \not\equiv 0\ ,\ u = 0 \ \text{ on } \ \partial\Omega, \end{cases} \qquad (\star)$$

where $p^\star = np/(n-p)$. The first term in the LHS of $(\star)$ is called the p-laplacian of u, while changing sign solutions to $(\star)$ are called nodal solutions. When $p = 2$, the p-Laplacian is just the usual laplacian (with the minus sign convention). Equation $(\star)$ is then said to be of scalar curvature type, in reference with the equation relating the scalar curvatures of two conformal Riemannian metrics. Such an equation has been studied by many authors. A remark one has to do here is that since $q = p^\star$ is critical for the embedding of $W^{1,p}$ in L^q, it is not possible to obtain solutions of $(\star)$ via simple variational arguments. Problem $(\star)$ has been studied by Guedda and Veron [8]. Among other results, these authors proved that for $a \equiv 0$, the equation

$$-\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = |u|^{p^\star-1}$$

admits no nonzero solution in $W_0^{1,p}(\Omega)$ if Ω is starshaped with respect to some point, while for a constant negative, greater than minus the best Poincaré constant of $W_0^{1,p}(\Omega)$, and $p^2 \le n$, the problem

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + a(x)u^{p-1} = u^{p^\star-1} \ \text{ in } \ \Omega \\ u > 0 \ \text{ in } \ \Omega\ ,\ u = 0 \ \text{ on } \ \partial\Omega \end{cases}$$

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admits a solution in $W_0^{1,p}(\Omega)$. We assume here that the operator

$$L(u) = -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + a(x)|u|^{p-2}u$$

is coercive in the sense that there exists $\lambda > 0$ such that for any $u \in W_0^{1,p}(\Omega)$,

$$\int_\Omega \big(|\nabla u|^p dx + a(x)|u|^p\big)dx \geq \lambda \int_\Omega |u|^p dx.$$

This is automatically satisfied when a is nonnegative, and more generally when a is greater than minus the best Poincaré constant of $W_0^{1,p}(\Omega)$. First, we present general existence results in the presence of symmetries. Then, we derive some specific examples of applications.

II. A general theorem concerning the existence of positive solutions. In what follows, let Ω be a smooth bounded domain in $\mathbb{R}^n$. Let also G be some subgroup of $O(n)$. We assume that Ω is stable under the action of G, in the sense that $\tau(\Omega) = \Omega$ for all $\tau \in G$. For $x \in \Omega$, we denote by $O_G(x)$ the G-orbit of x. By definition $O_G(x) = \big\{\tau(x),\ \tau \in G\big\}$. Let a and f be two smooth G-invariant functions on $\overline{\Omega}$, and let $1 < p < n$ be some given real number. We are concerned here with the existence of G-invariant solutions $u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ to the problem

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + a(x)u^{p-1} = f(x)u^{p^\star-1} \ \text{ in } \ \Omega \\ u > 0 \ \text{ in } \ \Omega,\ u = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{I}$$

where $p^\star = np/(n-p)$ is the critical exponent for the Sobolev embedding of $W^{1,p}$ in L^q. Without loss of generality we can assume that G is compact. If not, one can replace G by its closure $\overline{G}$ in $O(n)$. (Note that any G-invariant function is $\overline{G}$-invariant). As already mentioned, we assume in what follows that the operator

$$L(u) = -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + a(x)|u|^{p-2}u$$

is coercive in the sense that there exists $\lambda > 0$ such that for any $u \in W_0^{1,p}(\Omega)$,

$$\int_\Omega \big(|\nabla u|^p dx + a(x)|u|^p\big)dx \geq \lambda \int_\Omega |u|^p dx.$$

It is then clear that a necessary condition for the problem to have a solution is that $\sup_{x\in\Omega} f(x) > 0$. Now, we define the functional I_p on $W_0^{1,p}(\Omega)$ by

$$I_p(u) = \int_\Omega |\nabla u|^p dx + \int_\Omega a(x)|u|^p dx$$

and the subset of $W_0^{1,p}(\Omega)$,

$$\mathcal{W}_{p^\star}(G) = \big\{u \in W_0^{1,p}(\Omega) : \forall\tau \in G,\ u\circ\tau = u \text{ a.e, and } \int_\Omega f(x)|u|^{p^\star} dx = +1\big\}.$$

Let also $\mu(G)$ be defined by

$$\mu(G) = \inf_{u\in\mathcal{W}_{p^\star}(G)} I_p(u).$$

Let us note here that the fact that $\mu(G)$ is achieved, implies that there exists a solution to problem (I). First, if $\mu(G)$ is achieved by some u, one can suppose that u is nonnegative since $|u|$ also realizes $\mu(G)$. The Euler equation associated to $\mu(G)$ then provides a nonnegative solution of

$$-\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + a(x)u^{p-1} = \mu(G)f(x)u^{p^\star-1}.$$

By regularity results, as developed by Guedda-Veron [8] (see also Lewis [12], Tolksdorf [15]), and by Vazquez strict maximum principle [16], one gets that $u \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, and that $u > 0$ in Ω. Finally, by multiplying u by a convenient constant, one obtains a solution to (I).

Independently, one should also note that $\mathcal{W}_{p^\star}(G)$ is never empty (of course under the assumption that f is positive somewhere). The point here is just to get the existence of some G-invariant function $u \in W_0^{1,p}(\Omega)$ such that $\int_\Omega f|u|^{p^\star} dx > 0$. Let q be some real number such that $q > p^\star$. If G is a finite group, and if $u \in \mathcal{D}(\Omega)$ is nonnegative and such that $\int_\Omega f u^q dx > 0$, one easily checks that

$$u_G = \big(\sum_{\tau\in G} \big(u\circ\tau\big)^q\big)^{1/p^\star}$$

is G-invariant, belongs to $W_0^{1,p}(\Omega)$, and satisfies

$$\int_\Omega f u_G^{p^\star} dx = \big(Card G\big) \int_\Omega f u^q dx > 0$$

(since f is G-invariant). This proves the claim in this special case. In the general case, if $d\mu$ denotes the Haar measure on G, and taking u as previously, one easily gets that the function

$$u_G(x) = \Big(\int_G u\big(\tau(x)\big)^q d\mu(\tau)\Big)^{1/p^\star}$$

is G-invariant, belongs to $W_0^{1,p}(\Omega)$, and is such that

$$\int_\Omega f u_G^{p^\star} dx = Vol_{d\mu}\big(G\big)\int_\Omega f u^q dx > 0.$$

This proves the claim in the general case.

The purpose of this section is to prove the following theorem. Here, $K(n,p)$ denotes the best constant for the embedding of $W^{1,p}(\mathbb{R}^n)$ in $L^{p^\star}(\mathbb{R}^n)$. In other words,

$$K(n,p) \;=\; \sup_{u\in\mathcal{D}(R^n)} \frac{\big(\int_{R^n}|u|^{p^\star}dx\big)^{1/p^\star}}{\big(\int_{R^n}|\nabla u|^p dx\big)^{1/p}}.$$

One has that

$$K(n,p) = \frac{p-1}{n-p}\Big(\frac{n-p}{n(p-1)}\Big)^{1/p}\Big(\frac{\Gamma\big(n+1\big)}{\Gamma\big(n/p\big)\Gamma\big(n+1-n/p\big)\omega_{n-1}}\Big)^{1/n},$$

where ω_{n-1} is the volume of the standard unit sphere of $\mathbb{R}^n$. See for instance Aubin [1] and Talenti [13].

Theorem 1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, G be a compact subgroup of $O(n)$, and $p\in(1,n)$ be some real number. We assume that Ω is stable under the action of G. Let also a and f be two smooth G-invariant functions on $\overline{\Omega}$, a being such that L is coercive (where L is as above), and f being positive somewhere. If for all $x\in\overline{\Omega}$ such that $f(x)>0$,*

$$K(n,p)^p\mu(G)f(x)^{1-p/n} < \big(CardO_G(x)\big)^{p/n},$$

then there exists $u\in W_0^{1,p}(\Omega)\cap C^{1,\alpha}(\overline{\Omega})$, $\alpha\in(0,1)$, u G-invariant, which is a solution of (I).

The Proof of Theorem 1 proceeds in several steps. The first step deals with the subcritical case: we replace the second member of (I) by $f(x)u^{q-1}$ where $p<q<p^\star$. We are lead to introduce

$$\mathcal{W}_q(G) = \big\{u\in W_0^{1,p}(\Omega)\ /\ \forall\tau\in G,\ u\circ\tau = u\text{a.e, and } \int_\Omega f(x)|u|^q dx = +1\big\}.$$

Here again, it is clear that $\mathcal{W}_q(G)$ is not empty. We let

$$\mu_q(G) = \inf_{u \in \mathcal{W}_q(G)} I_p(u)$$

and we have the following existence result.

Proposition 1. *Let Ω, G, p, a, and f, be as in Theorem* 1. *Let also $q \in (p, p^\star)$ be some real number. There exists $u_q \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, u_q G-invariant, which is a solution of*

$$\begin{cases} -\mathrm{div}\big(|\nabla u_q|^{p-2}\nabla u_q\big) + a(x)u_q^{p-1} = \mu_q(G)f(x)u_q^{q-1} \text{ in } \Omega \\ u_q > 0 \quad in \quad \Omega, \; u_q = 0 \quad on \quad \partial\Omega \\ \int_\Omega f(x)u_q^q dx = 1. \end{cases} \tag{II}$$

In particular, $\mu_q(G)$ is achieved.

Proof. We use standard variational technics. Let (v_i) be a minimizing sequence for $\mu_q(G)$. Without loss of generality, up to replacing v_i by $|v_i|$, one can assume that v_i is nonnegative. Clearly, since L is coercive, (v_i) is bounded in $W_0^{1,p}(\Omega)$. As a consequence, we may extract from it a subsequence, still denoted (v_i), which converges weakly in $W_0^{1,p}(\Omega)$ to some function $u_q \in W_0^{1,p}(\Omega)$. Furthermore, by the Rellich-Kondrakov theorem, we can assume that (v_i) converges to u_q in $L^q(\Omega)$. One can also assume that (v_i) converges to u_q a.e. One then gets that u_q is G-invariant and nonnegative. One also gets that

$$\int_\Omega f u_q^q \, dx = 1.$$

Hence, $u_q \in \mathcal{W}_q(G)$. Finally, and by property of the weak convergence, one has that u_q realizes $\mu_q(G)$. As a consequence, we obtain that for every G-invariant function $h \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u_q|^{p-2}\big(\nabla u_q.\nabla h\big)dx + \int_\Omega a(x)u_q^{p-1}hdx = \mu_q(G)\int_\Omega f(x)u_q^{q-1}hdx \tag{1}$$

Now, we claim that (1) holds for every $h \in W_0^{1,p}(\Omega)$, not necessarily G-invariant. This will imply that u_q is a solution of (II). In order to prove the claim, let us consider $\phi \in W_0^{1,p}(\Omega)$, and let ϕ_G be the function defined by

$$\phi_G(x) = \int_G \phi\big(\tau(x)\big)d\mu(\tau),$$

where $d\mu$ denotes the Haar measure on G. One easily gets that $\phi_G \in W_0^{1,p}(\Omega)$, while by construction, ϕ_G is G-invariant. Writing (1) for $h = \phi_G$, and since u_q, a, and f are G-invariant, we get that (1) is true for $h = \phi$. This proves the claim. Regularity results, as developed in Guedda-Veron [8], and Vazquez strict maximum principle [16], end the proof of the proposition. □

The general idea in what follows, is to get the solution u of Theorem 1 as the limit of (a subsequence of) (u_q), $q \to p^\star$. The first result we need here, is the following.

Lemma 1. *For $\mu_q(G)$ and $\mu(G)$ as above,* $\limsup_{q\to p^\star} \mu_q(G) \le \mu(G)$.

Proof. Let $\epsilon > 0$ given, and v be a G-invariant and nonnegative function of $W_0^{1,p}(\Omega)$ which satisfies $I_p(v) \le \mu(G) + \epsilon$ and $\int_\Omega f(x)|v|^{p^\star} dx = 1$. For q close to $p^\star$,

$$v_q = \Big(\int_\Omega f(x)|v|^q dx\Big)^{-1/q} v$$

makes sense and belongs to $\mathcal{W}_q(G)$. Hence, $\mu_q(G) \le I_p(v_q)$. Moreover, $\lim_{q\to p^\star} I_p(v_q) = I_p(v)$. As a consequence, one has that for any $\epsilon > 0$,

$$\limsup_{q\to p^\star} \mu_q(G) \le \mu(G) + \epsilon.$$

Clearly, this proves Lemma 1. □

In what follows, up to a subsequence, we assume that $\lim_{q\to p^\star} \mu_q(G)$ exists. We let $\mu = \lim_{q\to p^\star} \mu_q(G)$. One then has the following.

Proposition 2. *Let (u_q) be as in Proposition 1, with the additional property that $\lim_{q\to p^\star} \mu_q(G) = \mu$ exists. Suppose that a subsequence of (u_q) converges in some $L^k(\Omega)$, $k > 1$, to a function $u \not\equiv 0$. Then $u \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, u is G-invariant, and u satisfies*

$$\begin{cases} -div\big(|\nabla u|^{p-2}\nabla u\big) + a(x)u^{p-1} = \mu f(x) u^{p^\star-1} \quad in \quad \Omega \\ u > 0 \quad in \quad \Omega, \; u = 0 \quad on \quad \partial\Omega. \end{cases}$$

In particular, problem (I) *has a solution.*

Proof. It is clear that (u_q) is bounded in $W_0^{1,p}(\Omega)$. By standard arguments, one can then assume that when q goes to $p^\star$,

(i) (u_q) converges weakly to u in $W_0^{1,p}(\Omega)$
(ii) (u_q) converges to u in $L^{p^\star-1}(\Omega)$
(iii) (u_q) converges to u a.e.

In addition, and since $|\nabla u_q|$ is bounded in $L^p(\Omega)$, one can assume that

(iv) $\big(|\nabla u_q|^{p-2}\nabla u_q\big)$ converges weakly to some Σ in $L^{\tilde{p}}(\Omega)$,

where $\tilde{p} = p/(p-1)$ is the conjugate of p. By passing to the limit $q \to p^\star$ in (II), one then gets that

$$-\mathrm{div}(\Sigma) + a(x)u^{p-1} = \mu f(x)u^{p^\star-1}.$$

Of course, by (iii), u is nonnegative and G-invariant. Independently, it is clear that $\big(\mu_q(G)f(x)u_q^{q-1} - a(x)u_q^{p-1}\big)$ is bounded in $L^1(\Omega)$. By Lemma 2 below, this implies that $\Sigma = |\nabla u|^{p-2}\nabla u$. As a consequence, u is a solution of

$$-\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + a(x)u^{p-1} = \mu f(x)u^{p^\star-1}. \tag{2}$$

By regularity arguments, as developed in Guedda-Veron [8], $u \in C^{1,\alpha}(\overline{\Omega})$, while by Vazquez strict maximum principle [16], u is positive in Ω. Multiplying (2) by u and integrating the result over Ω, one clearly sees that μ and $\int_\Omega f(x)u^{p^\star}dx$ are necessarily positive (since $u \not\equiv 0$).

Remark. As a matter of fact, $\mu = \mu(G)$, $u \in \mathcal{W}_{p^\star}(G)$, and u realizes $\mu(G)$. Indeed, by multiplying (2) by u and integrating the result over Ω, we get that

$$\begin{aligned}\mu\int_\Omega f(x)u^{p^\star}dx &= \int_\Omega |\nabla u|^p dx + \int_\Omega a(x)u^p dx \\ &\le \liminf_{q\to p^\star}\Big(\int_\Omega |\nabla u_q|^p dx + \int_\Omega a(x)u_q^p dx\Big) = \liminf_{q\to p^\star}\mu_q(G).\end{aligned}$$

As a consequence,

$$\int_\Omega f(x)u^{p^\star}dx \le 1.$$

From now on, let

$$v = \Big(\int_\Omega f(x)u^{p^\star}dx\Big)^{-1/p^\star} u.$$

Then $v \in \mathcal{W}_{p^\star}(G)$, and

$$\mu \le \mu(G) \le I_p(v) = \mu\Big(\int_\Omega f(x)u^{p^\star}dx\Big)^{1-p/p^\star}.$$

This in turn implies that

$$\int_\Omega f(x)u^{p^\star}dx \ge 1,$$

and one gets that $\int_\Omega f(x)u^{p^\star}dx = 1$ and $\mu = \mu(G)$. This proves the claim.

Lemma 2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, let $p > 1$ be a real number, and let (u_q) be a sequence in $W_0^{1,p}(\Omega)$. Assume that*

(i) *(u_q) is bounded in $W_0^{1,p}(\Omega)$*
(ii) *$\mathrm{div}\big(|\nabla u_q|^{p-2}\nabla u_q\big)$ is bounded in $L^1(\Omega)$.*

Then, there exists $u \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, (u_q) converges to u a.e, (∇u_q) converges to ∇u a.e, and $\big(|\nabla u_q|^{p-2}\nabla u_q\big)$ converges a.e and weakly in $L^{\tilde{p}}(\Omega)$ to $|\nabla u|^{p-2}\nabla u$, where $\tilde{p} = p/(p-1)$.

Proof. We borrow ideas from Evans [5]. For simplicity, we denote by Σ_q the function $|\nabla u_q|^{p-2}\nabla u_q$. As a consequence of (i), (Σ_q) is bounded in $L^{\tilde{p}}(\Omega)$. Up to a subsequence, we can assume that

(a) (u_q) converges to u a.e
(b) (Σ_q) converges weakly in $L^{\tilde{p}}(\Omega)$ to some function $\Sigma \in L^{\tilde{p}}(\Omega)$.

Let $\delta > 0$ be given. By Egoroff's theorem, there exists $E_\delta \subset\subset \Omega$ such that $meas\big(\Omega \backslash E_\delta\big) < \delta$, and (u_q) converges uniformly to u on E_δ. As a consequence, $\epsilon > 0$ being given, one can take q close enough to $p^\star$ in order to have $|u_q(x) - u(x)| < \epsilon/2$, $\forall x \in E_\delta$. We define a truncation function β_ϵ by

$$\beta_\epsilon(x) = \begin{cases} x & \text{if } |x| < \epsilon \\ \frac{\epsilon x}{|x|} & \text{if } |x| \geq \epsilon. \end{cases}$$

Let us now denote by Θ the function $\Theta = |\nabla u|^{p-2}\nabla u$. One clearly has that

$$\big(\Sigma_q - \Theta\big).\nabla\big(\beta_\epsilon \circ (u_q - u)\big) \geq 0$$

almost everywhere in Ω. Since for q close to $p^\star$,

$$\nabla\big(\beta_\epsilon \circ (u_q - u)\big) = \nabla(u_q - u)$$

in E_δ, one has that

$$\int_{E_\delta}\big(\big(\Sigma_q - \Theta\big).\nabla(u_q - u)\big)(x)dx \leq \int_\Omega\big(\big(\Sigma_q - \Theta\big).\nabla\big(\beta_\epsilon \circ (u_q - u)\big)\big)(x)dx$$
$$= -\int_\Omega\big(\big(\mathrm{div}(\Sigma_q)\big)\big(\beta_\epsilon \circ (u_q - u)\big)\big)(x)dx - \int_\Omega\big(\Theta.\nabla\big(\beta_\epsilon \circ (u_q - u)\big)\big)(x)dx.$$

Now, note that $\beta_\epsilon \circ (u_q - u)$ converges weakly to 0 in $W_0^{1,p}(\Omega)$, so that

$$\lim_{q \to p^\star}\int_\Omega\big(\Theta.\nabla\big(\beta_\epsilon \circ (u_q - u)\big)\big)(x)dx = 0.$$

On the other hand, one has by (ii) that there exists $C > 0$ such that for any q,

$$\int_\Omega \Big(\big(\mathrm{div}(\Sigma_q)\big)\big(\beta_\epsilon \circ (u_q - u)\big)\Big)(x)dx \leq \epsilon \int_\Omega |div(\Sigma_q)|(x)dx \leq C\epsilon.$$

As a consequence,

$$\limsup_{q \to p^\star} \int_{E_\delta} \big((\Sigma_q - \Theta).\nabla(u_q - u)\big)(x)dx \leq C\epsilon.$$

Since $\epsilon > 0$ is arbitrary, this proves in particular that $\big(\Sigma_q - \Theta\big).\nabla(u_q - u)$ converges a.e to 0 on E_δ. By Lemma 3 below, we obtain that ∇u_q converges to ∇u a.e on E_δ. Since $\delta > 0$ is arbitrary, this implies that ∇u_q converges to ∇u a.e on Ω. Clearly, this in turn implies that $\big(|\nabla u_q|^{p-2}\nabla u_q\big)$ converges to Θ a.e on Ω. As a consequence, and since $\big(|\nabla u_q|^{p-2}\nabla u_q\big)$ is bounded in $L^{\tilde{p}}(\Omega)$, $\Sigma = \Theta$. (Recall that a bounded sequence in L^α, $\alpha > 1$, which converges a.e to some h, converges weakly to h in L^α.) This proves the lemma.

Lemma 3. *Let (X_k) be a sequence in $\mathbb{R}^n$ such that*

$$\big(|X_k|^{p-2}X_k - |X|^{p-2}X\big).\big(X_k - X\big)$$

converges to 0. Then (X_k) converges to X.

Proof. First, we claim that (X_k) is bounded. Indeed, if not, there would exist a subsequence, still denoted (X_k), such that $|X_k| \to +\infty$. But,

$$\big(|X_k|^{p-2}X_k - |X|^{p-2}X\big).\big(X_k - X\big) \sim |X_k|^p,$$

which is absurd. This proves the claim, and, up to a subsequence, $X_k \to \alpha$ for some $\alpha \in \mathbb{R}^n$. One then gets that

$$\big(|\alpha|^{p-2}\alpha - |X|^{p-2}X\big).\big(\alpha - X\big) = 0$$

which implies that $\alpha = X$. Noting that this is true for any subsequence, one obtains that (X_k) converges to X. □

From now on, we assume that every subsequence of (u_q) which converges in a $L^k(\Omega)$, $k > 1$, converges to zero. In other words, we assume that the situation of Proposition 2 does not occur. We denote by $(\mathcal{R}\mathcal{A})$ this assumption.

Let $P \in \overline{\Omega}$, and $\eta \in C^\infty(\mathbb{R}^n)$ be such that $0 \le \eta \le 1$, $\eta = 1$ in $B_p(\delta/2)$, and $\eta = 0$ in $\mathbb{R}^n \backslash B_P(\delta)$. ($B_P(r)$ denotes the euclidean ball of center P and radius r). We multiply equation (II) by $\eta^p u_q^k$ where $1 < k < p^\star/p$. This leads to

$$-\eta^p u_q^k div\big(|\nabla u_q|^{p-2}\nabla u_q\big) + a(x)\eta^p u_q^{p+k-1} = \mu_q(G)f(x)\eta^p u_q^{k+q-1} \qquad (3).$$

Our aim in what follows is to prove that there exists $C > 0$, independent of q, such that

$$\int_\Omega |\nabla\big(\eta u_q^{(k+p-1)/p}\big)|^p(x)dx \le C.$$

For that purpose, we integrate by part the first term of the LHS of (3). We get that

$$\begin{aligned} &-\int_\Omega \eta^p u_q^k div\big(|\nabla u_q|^{p-2}\nabla u_q\big)dx = \int_\Omega |\nabla u_q|^{p-2}\nabla u_q.\nabla\big(\eta^p u_q^k\big)dx \\ &= p\int_\Omega \eta^{p-1}u_q^k|\nabla u_q|^{p-2}\nabla u_q.\nabla\eta dx + k\int_\Omega \eta^p u_q^{k-1}|\nabla u_q|^p dx. \end{aligned}$$

The first of the two last integrals can be bounded as follows

$$\begin{aligned} \int_\Omega \eta^{p-1}u_q^k|\nabla u_q|^{p-2}\nabla u_q.\nabla\eta dx &\le \big(\max\eta\big)^{p-1}\big(\max|\nabla\eta|\big)\int_\Omega u_q^k|\nabla u_q|^{p-1}dx \\ &\le C_1\Big(\int_\Omega u_q^{kp}dx\Big)^{1/p}\Big(\int_\Omega |\nabla u_q|^p dx\Big)^{(p-1)/p} \end{aligned}$$

by Hölder and where $C_1 > 0$ does not depend on q. This implies that for $kp < p^\star$,

$$\int_\Omega \eta^{p-1}u_q^k|\nabla u_q|^{p-2}\nabla u_q.\nabla\eta dx \le C_2,$$

where $C_2 > 0$ is independent of q. Now, we want to compare $\int\eta^p|\nabla u_q|^p u_q^{k-1}dx$ with $\frac{p^p}{(p+k-1)^p}\int|\nabla\big(\eta u_q^{(k+p-1)/p}\big)|^p dx$. For that aim, we use the inequality

$$\big||X+Y|^p - |Y|^p\big| \le C\big(|X|^{p-1} + |Y|^{p-1}\big)|X|$$

that we apply here with

$$X = \frac{p}{p+k-1} u_q^{(k+p-1)/p} \nabla \eta, \quad Y = \eta u_q^{(k-1)/p} \nabla u_q.$$

As a consequence, we are led to bound from above

$$\int_\Omega |X|^p dx \text{ and } \int_\Omega |Y|^{p-1} |X| dx.$$

Clearly, one has that

$$\int_\Omega |X|^p dx \le C_3 \int_\Omega u_q^{k+p-1} dx \le C_4,$$

where $C_3 > 0$ and $C_4 > 0$ are independent of q, and since $kp < p^\star$ implies that $k + p - 1 \le p^\star$. On the other hand,

$$\begin{aligned}\int_\Omega |Y|^{p-1} |X| dx &\le C_5 \int_\Omega |\nabla u_q|^{p-1} u_q^k dx \\ &\le C_5 \big(\int_\Omega |\nabla u_q|^p dx \big)^{(p-1)/p} \big(\int_\Omega u_q^{kp} dx \big)^{1/p} \le C_6\end{aligned}$$

by Hölder, where $C_5 > 0$ and $C_6 > 0$ are independent of q, and since $kp < p^\star$. To summarize, one gets by integrating (3) that

$$\frac{kp^p}{(k+p-1)^p} \int_\Omega |\nabla \big(\eta u_q^{(k+p-1)/p} \big)|^p dx \le C_7 + \mu_q(G) \int_\Omega \eta^p f(x) u_q^{k+q-1} dx,$$

where $C_7 > 0$ is independent of q.

Assume first that $f(P) < 0$. Then, by choosing δ small enough, we get that

$$\int_\Omega |\nabla \big(\eta u_q^{(k+p-1)/p} \big)|^p dx \le C_8 = \frac{(k+p-1)^p}{kp^p} C_7.$$

Assume now that $f(P) \ge 0$. We write that

$$\begin{aligned}&\frac{kp^p}{(k+p-1)^p} \int_\Omega |\nabla \big(\eta u_q^{(k+p-1)/p} \big)|^p dx \\ \le& C_7 + \big(\sup_{B_P(\delta)} |f| \big)^{p/q} \mu_q(G) \int_\Omega \big(u_q^{(k+p-1)/p} \eta \big)^p u_q^{q-p} |f|^{1-p/q} dx \\ \le& C_7 + \big(\sup_{B_P(\delta)} |f| \big)^{p/q} \mu_q(G) \big(\int_\Omega \big(\eta u_q^{(k+p-1)/p} \big)^q dx \big)^{p/q} \big(\int_{B_P(\delta)} |f| u_q^q dx \big)^{1-p/q}.\end{aligned}$$

By the Sobolev embedding theorem, one then gets that for $\epsilon > 0$ arbitrary small, and q close to $p^\star$,

$$\Big(\int_\Omega \big(\eta u_q^{(k+p-1)/p}\big)^q dx\Big)^{p/q} \le \big(K(n,p)^p + \epsilon\big) \int_\Omega |\nabla\big(\eta u_q^{(k+p-1)/p}\big)|^p dx.$$

If $f(P) = 0$, by choosing $\delta > 0$ small, there exists $C_9 > 0$ and $C_{10} \in (0,1)$ independent of q such that

$$\int_\Omega |\nabla\big(\eta u_q^{(k+p-1)/p}\big)|^p dx \le C_9 + C_{10} \int_\Omega |\nabla\big(\eta u_q^{(k+p-1)/p}\big)|^p dx.$$

In the same manner, if $f(P) > 0$, we choose $\delta > 0$ small enough such that $f > 0$ in $B_P(\delta)$. If

$$K(n,p)^p \mu(G) f(P)^{p/p^\star} \limsup_{q \to p^\star} \Big(\int_{B_P(\delta)} |f| u_q^q dx\Big)^{1-p/p^\star} < 1$$

one then gets that for $k > 1$ close to one, there exist $C_{11} > 0$ and $C_{12} \in (0,1)$ independent of q, such that

$$\int_\Omega |\nabla\big(\eta u_q^{(k+p-1)/p}\big)|^p dx \le C_{10} + C_{11} \int_\Omega |\nabla\big(\eta u_q^{(k+p-1)/p}\big)|^p dx.$$

In both cases, we have obtained that there exists $C_{13} > 0$, independent of q, such that

$$\int_\Omega |\nabla\big(\eta u_q^{(k+p-1)/p}\big)|^p dx \le C_{13}.$$

Now, we prove the following.

Lemma 4. *Assume that* $(\mathcal{RA})$ *holds. Assume in addition that for some* $P \in \overline{\Omega}$ *there exists* η *as above, and there exists* $C > 0$ *and* $k > 1$, *independent of* q, *such that for all* q

$$\int_\Omega |\nabla\big(\eta u_q^{(k+p-1)/p}\big)|^p dx \le C.$$

Then $\limsup_{q \to p^\star} \int_{B_P(\delta/2)} u_q^q dx = 0$.

Proof. Assume by contradiction that

$$\limsup_{q \to p^\star} \int_{B_P(\delta/2)} |u_q|^q dx > 0.$$

We have that

$$\int_{B_P(\delta/2)} u_q^q dx \le C_{14}\Big(\int_{B_P(\delta/2)} u_q^{p^\star} dx\Big)^{q/p^\star},$$

where $C_{14} > 0$ is independent of q, and we write that

$$\begin{aligned} p^\star &= \big(p^\star - 1\big) + 1 \\ &= \Big(\frac{(k+p-1)p^\star}{p}\Big)\Big(\frac{p^\star - 1}{p^\star}\Big)\Big(\frac{p}{k+p-1}\Big) + \Big(\frac{n(k+p-1)}{nk-p}\Big)\Big(\frac{nk-p}{n(k+p-1)}\Big). \end{aligned}$$

By Hölder's inequality, and by the Sobolev embedding theorem,

$$\begin{aligned} &\Big(\int_{B_P(\delta/2)} u_q^{p^\star} dx\Big)^{q/p^\star} \\ \le&\Big(\int_{B_P(\delta/2)} u_q^{(k+p-1)p^\star/p} dx\Big)^{q(p^\star-1)p/(p^\star)^2(k+p-1)} \\ &\times \Big(\int_{B_P(\delta/2)} u_q^{n(k+p-1)/(nk-p)} dx\Big)^{(nk-p)q/n(k+1-p)p^\star} \\ \le& C_{15}\Big(\int_{B_P(\delta/2)} u_q^{n(k+p-1)/(nk-p)} dx\Big)^{(nk-p)q/n(k+p-1)p^\star}, \end{aligned}$$

where $k > 1$ is as in the lemma. Let

$$k_1 = \frac{n(k+p-1)}{nk-p}.$$

Then $1 < k_1 < p^\star$, and we get that

$$\limsup_{q\to p^\star} \int_{B_P(\delta/2)} u_q^{k_1} dx > 0.$$

Clearly, this is in contradiction with $(\mathcal{RA})$. Lemma 4 is proved. □

In order to make things clear, and as a consequence of the previous discussion, we have proved the following: Under the assumption $(\mathcal{RA})$, and if for all $P \in \overline{\Omega}$, where $f(P) > 0$, there exists $\delta_P > 0$ such that

$$K(n,p)^p \mu(G) f(P)^{p/p^\star} \limsup_{q\to p^\star}\Big(\int_{B_P(\delta)} |f| u_q^q dx\Big)^{1-p/p^\star} < 1$$

then, and for all $x \in \overline{\Omega}$, there exists $\delta(x) > 0$ such that

$$\limsup_{q \to p^\star} \int_{B_x\left(\delta(x)\right)} u_q^q dx = 0.$$

We are now in position to prove Theorem 1.

Proof of Theorem 1. Suppose that for all $P \in \overline{\Omega}$, where $f(P) > 0$, there exists $\delta_P > 0$ such that

$$K(n,p)^p \mu(G) f(P)^{p/p^\star} \limsup_{q \to p^\star} \Big(\int_{B_P(\delta)} |f| u_q^q dx \Big)^{1-p/p^\star} < 1.$$

Since $\overline{\Omega}$ is compact, it can be covered by a finite number of $B_x\big(\delta(x)\big)$, where $B_x\big(\delta(x)\big)$ is as above. The fact that

$$\int_\Omega f(x) u_q^q dx = 1$$

then clearly leads to a contradiction. As a consequence, there exists some $P \in \overline{\Omega}$ such that $f(P) > 0$, and such that for all $\delta > 0$,

$$K(n,p)^p \mu(G) f(P)^{p/p^\star} \limsup_{q \to p^\star} \Big(\int_{B_P(\delta)} |f| u_q^q \, dx \Big)^{1-p/p^\star} \geq 1. \tag{4}$$

Suppose now that $CardO_G(P) < +\infty$. Then, we can choose $\delta > 0$ small enough in order to get that

$$\limsup_{q \to p^\star} \int_{B_P(\delta)} f u_q^q dx \leq \frac{1}{CardO_G(P)}.$$

The point here is that for $\tau \in G$, and $\delta > 0$ small enough,

$$\int_{B_P(\delta)} f u_q^q dx = \int_{B_{\tau(P)}(\delta)} f u_q^q dx$$

and that according to the previous discussion, (u_q^q) converges to 0 in L^1 in a neighbourhood of any x such that $f(x) \leq 0$. One then gets that (4) is in contradiction with the assumption of the theorem. In the same order of ideas, assume now that $CardO_G(P) = +\infty$. Then, and with the same kind of arguments, one has that for any $\epsilon > 0$, there exists $\delta > 0$ small, such that

$$\limsup_{q \to p^\star} \int_{B_P(\delta)} f u_q^q dx \le \epsilon.$$

Here, one gets the contradiction with (4). As a conclusion, we have proved that under the assumption of the theorem, $(\mathcal{RA})$ does not hold. By Proposition 2, this proves Theorem 1.

III. A general theorem concerning the existence of nodal solutions. Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Let G be a subgroup of $O(n)$. As in Section II, we assume that Ω is stable under the action of G, and that G is compact. Let also σ be an involution of $O(n)$, and assume here again that Ω is stable under the action of σ. We denote by $H = [G, \sigma]$ the group generated by σ and G. We denote by $O_H(x)$ the H-orbit of x. Let a and f be two smooth H-invariant functions on $\overline{\Omega}$, a being such that L is coercive (where L is as in the introduction), and let $1 < p < n$ be some given real number. We are concerned here with the existence of G-invariant and σ-antisymmetrical functions $u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ to the problem

$$\begin{cases} -\text{div}\big(|\nabla u|^{p-2} \nabla u\big) + a(x)|u|^{p-2}u = f(x)|u|^{p^\star - 2}u \text{ in } \Omega \\ u \not\equiv 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega, \end{cases} \tag{III}$$

where $p^\star = np/(n-p)$ is the critical exponent for the Sobolev embedding of $W^{1,p}$ in L^q. Here again, and since L is assumed to be coercive, it is clear that a necessary condition for (III) to have a solution is that $\sup_{x \in \Omega} f(x) > 0$. Independently, and by σ-antisymmetrical, we mean that $u \circ \sigma = -u$, so that u has to change sign. In what follows, we will say that σ cuts Ω in two domains Ω_1 and Ω_2 if $\sigma(\Omega_1) = \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, and $\Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{F}_\sigma$ with the additional property that $meas(\mathcal{F}_\sigma) = 0$, where $\mathcal{F}_\sigma = \{x \in \Omega : \sigma(x) = x\}$. One will easily check that this occurs if and only if σ is an orthogonal symmetry with respect to some hyperplane. As in Section II, we define the functional I_p on $W_0^{1,p}(\Omega)$ by

$$I_p(u) = \int_\Omega |\nabla u|^p dx + \int_\Omega a(x)|u|^p dx.$$

We also define

$$\mathcal{W}_{p^\star}^\sigma(G) = \big\{u \in W_0^{1,p}(\Omega) \ / \ \forall \tau \in G,$$
$$u \circ \tau = u \text{ a.e }, u \circ \sigma = -u \text{ a.e, and } \int_\Omega f(x)|u|^{p^\star} dx = +1\big\}$$

and let

$$\mu_\sigma(G) = \inf_{u \in \mathcal{W}^\sigma_{p^\star}(G)} I_p(u)$$

with the convention that $\mu_\sigma(G) = +\infty$ if $\mathcal{W}^\sigma_{p^\star}(G) = \emptyset$. For sake of simplicity, we assume in what follows that G and σ commute weakly in the sense that for any x, $\sigma(O_G(x)) = O_G(\sigma(x))$. We claim here that under this assumption, $\mathcal{W}^\sigma_{p^\star}(G) \neq \emptyset$ if and only if

$$\exists x \in \Omega \;/\; f(x) > 0 \text{ and } \sigma\big(O_G(x)\big) \cap O_G(x) = \emptyset. \tag{5}$$

Indeed, (5) is a necessary condition for $\mathcal{W}^\sigma_{p^\star}(G)$ not to be empty, since if u is G-invariant and σ-antisymmetrical, and if for some $x \in \Omega$, $\sigma\big(O_G(x)\big) \cap O_G(x) \neq \emptyset$, then $u(x) = 0$. Conversely, we claim that (5) is a sufficient condition for $\mathcal{W}^\sigma_{p^\star}(G)$ not to be empty. For that purpose, let x be as in (5), let $\delta > 0$ real, and let $\phi \in C^\infty(\mathbb{R})$ be a nonnegative function such that $\phi(t) = 1$ if $t \leq \delta/2$, and $\phi(t) = 0$ if $t \geq \delta$. Set

$$u_G(y) = \phi\big(d\big(y, O_G(x)\big)\big),$$

where d is the euclidean distance. Then u_G is G-invariant, while for $\delta > 0$ small enough, u_G is smooth, with compact support in Ω, and such that $\int_\Omega f u_G^{p^\star}\, dx > 0$. (Recall that $O_G(x)$ is a compact submanifold of $\mathbb{R}^n$. Recall also that f is G-invariant). For $\delta > 0$ small enough, let $u_G^\sigma = u_G - u_G \circ \sigma$. Clearly, u_G^σ is σ-antisymmetrical, while by the weak commutation of G and σ at x, one also gets that u_G^σ is G-invariant. In addition, and for $\delta > 0$ small enough,

$$\int_\Omega f|u_G^\sigma|^{p^\star}\, dx = 2\int_\Omega f u_G^{p^\star}\, dx > 0.$$

Hence, $\mathcal{W}^\sigma_{p^\star}(G) \neq \emptyset$, since

$$\Big(\int_\Omega f|u_G^\sigma|^{p^\star}\, dx\Big)^{-1} u_G^\sigma \;\in\; \mathcal{W}^\sigma_{p^\star}(G).$$

This proves the claim. As a remark, note that if G is finite, or countable, (5) is equivalent to the condition $\sigma \notin G$. Indeed,

$$\big\{x \in \mathbb{R}^n \;/\; \sigma\big(O_G(x)\big) \cap O_G(x) \neq \emptyset\big\} = \bigcup_{\tau_1, \tau_2 \in G} \mathrm{Ker}\big(\tau_2 \circ \sigma \circ \tau_1 - Id\big)$$

and since $\mathrm{meas}\big(\big\{x \in \Omega \;/\; f(x) > 0\big\}\big) > 0$, one clearly gets the result.

The purpose of this section is to prove the following theorem. Here again, $K(n,p)$ denotes the best constant for the embedding of $W^{1,p}(\mathbb{R}^n)$ in $L^{p^\star}(\mathbb{R}^n)$.

Theorem 2. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, G be a compact subgroup of $O(n)$, σ be an involution of $O(n)$, and $p \in (1,n)$ be some real number. Let $H = [G,\sigma]$, and let also a and f be two smooth H-invariant functions on $\overline{\Omega}$, a being such that L is coercive (where L is as in the introduction), and f being positive somewhere. We assume that Ω is stable under the action of H, that G and σ commute weakly, and that relation* (5) *above holds. If for all $x \in \overline{\Omega}$ such that $f(x) > 0$,*

$$K(n,p)^p \mu_\sigma(G) f(x)^{1-p/n} < \big(CardO_H(x)\big)^{p/n}$$

then there exists $u \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, u G-invariant and σ-antisymmetrical, which is a solution of (III)*. If in addition σ cuts Ω in two domains stables under the action of G, one can prescribe the zero set of u, in the sense that $x \in \Omega$ is a point where u vanishes if and only if $x \in \mathcal{F}_\sigma$, where $\mathcal{F}_\sigma$, as above, is the set of the fixed points of σ in Ω.*

The Proof of Theorem 2 proceeds in several steps. Since almost each of them presents only slight changes when compared to their analogous of Section II, we shall not give all the details, and will only enumerate the different steps and changes to bring. The first step consists in considering the subcritical case. We replace the second member of (III) by $f(x)|u|^{q-2}u$ where $p < q < p^\star$. We are then led to introduce

$$\begin{aligned} \mathcal{W}_q^\sigma(G) = \big\{ & u \in W_0^{1,p}(\Omega) \ / \ \forall \tau \in G, \\ & u \circ \tau = u \text{ a.e}, u \circ \sigma = -u \text{ a.e, and } \int_\Omega f(x)|u|^q dx = +1 \big\}. \end{aligned}$$

As above, it is clear that under our assumptions, $\mathcal{W}_q^\sigma(G) \neq \emptyset$. We define

$$\mu_q^\sigma(G) = \inf_{u \in \mathcal{W}_q^\sigma(G)} I_p(u)$$

and we have the following existence result.

Proposition 3. *Let Ω, G, σ, p, a, and f be as in Theorem 2. Let also $q \in (p,p^\star)$ be some real number. There exists $u_q \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, u_q G-invariant and σ-antisymmetrical, which is a solution of*

$$\begin{cases} -\text{div}\big(|\nabla u_q|^{p-2}\nabla u_q\big) + a(x)|u_q|^{p-2}u_q = \mu_q^\sigma(G) f(x)|u_q|^{q-2}u_q \ \textit{in } \Omega \\ u_q \not\equiv 0 \ \textit{in } \Omega, \ u_q = 0 \ \textit{on } \partial\Omega \\ \int_\Omega f(x)|u_q|^q dx = 1. \end{cases} \tag{IV}$$

In particular, $\mu_q^\sigma(G)$ is achieved.

Proof. By standard variational technics, as in the proof of Proposition 1, we obtain the existence of $u_q \in \mathcal{W}_q^\sigma(G)$ which realizes $\mu_q^\sigma(G)$. As a consequence, one has that for any G-invariant and σ-antisymmetrical function $h \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}\int_\Omega |\nabla u_q|^{p-2}\big(\nabla u_q, \nabla h\big)dx + \int_\Omega a(x)|u_q|^{p-2}u_q h\, dx \\ = \mu_q^\sigma(G)\int_\Omega f(x)|u_q|^{q-2}u_q h\, dx.\end{aligned} \tag{6}$$

By similar arguments to those used in the Proof of Proposition 1, (6) is in fact true for every $h \in W_0^{1,p}(\Omega)$. Indeed, consider $\phi \in W_0^{1,p}(\Omega)$, and let ϕ_G be the function defined by

$$\phi_G(x) = \int_G \phi\big(\tau(x)\big)d\mu(\tau),$$

where $d\mu$ denotes the Haar measure on G. Set $\phi_G^\sigma = \phi_G \circ \sigma - \phi_G$. Clearly, $\phi_G^\sigma \in W_0^{1,p}(G)$, ϕ_G^σ is G-invariant, and ϕ_G^σ is σ-antisymmetrical. (Recall that G and σ commute weakly). Writing (6) for $h = \phi_G^\sigma$, and according to the invariances of u_q, a, and f, we get that (6) is true for $h = \phi$. Regularity results, as developed in Guedda-Veron [8], end the proof of the proposition. □

As in Section II, the general idea is to get the solution u of Theorem 2 as the limit of (a subsequence of) (u_q), $q \to p^\star$. The following result follows from a mere adaptation of the argument employed in the proof of Lemma 1.

Lemma 5. *For $\mu_q^\sigma(G)$ and $\mu_\sigma(G)$ as above,* $\limsup_{q\to p^\star} \mu_q^\sigma(G) = \mu_\sigma(G)$.

We now assume that $\lim_{q\to p^\star} \mu_q^\sigma(G)$ exists. Following the arguments employed in the Proof of Proposition 2, one obtains the following proposition. (See also the remark following the Proof of Proposition 2).

Proposition 4. *Let (u_q) be as in Proposition 3. Suppose that a subsequence of (u_q) converges in some $L^k(\Omega)$, $k > 1$, to a function $u \not\equiv 0$. Then $u \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, u is G-invariant and σ-antisymmetrical, and u satisfies*

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + a(x)|u|^{p-2}u = \mu_\sigma(G)f(x)|u|^{q-2}u \ \text{in } \Omega \\ u \not\equiv 0 \ \text{in } \Omega, u = 0 \ \text{on } \partial\Omega \\ \int_\Omega f(x)|u|^{p^\star}dx = 1. \end{cases}$$

In particular, u realizes $\mu_\sigma(G)$, *and problem* (III) *has a solution.*

As a remark, suppose that σ cuts Ω in two domains Ω_1 and Ω_2 stables under the action of G. Set $v = |u|$ in Ω_1 and $v = -|u| \circ \sigma$ in Ω_2, where u is as in Proposition 4. Clearly, v is G-invariant and σ-antisymmetrical, and v also realizes $\mu_\sigma(G)$. In particular, v is a solution of problem (III). By regularity results as developed in Guedda-Veron [8], one gets that $v \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$. By Vazquez strict maximum principle [16], one then gets that the zero set of v is exactly $\mathcal{F}_\sigma$, the set of the fixed points of σ in Ω. In other words, if σ cuts Ω in two domains stables under the action of G, and under the assumptions of Proposition 4, problem (III) possesses a solution $v \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, G-invariant and σ-antisymmetrical, whose zero set is exactly the set of the fixed points of σ in Ω.

As in Section II, we now assume that every subsequence of (u_q) which converges in a $L^k(\Omega)$, $k > 1$, converges to zero. In other words, we assume that the situation of Proposition 4 does not occur. We denote by $(\mathcal{RA})$ this assumption.

Let $P \in \overline{\Omega}$, and $\eta \in C^\infty(\mathbb{R}^n)$ be such that $0 \le \eta \le 1$, $\eta = 1$ in $B_p(\delta/2)$, and $\eta = 0$ in $\mathbb{R}^n \backslash B_P(\delta)$. (As in Section II, $B_P(r)$ denotes the euclidean ball of center P and radius r). We multiply equation (IV) by $\eta^p |u_q|^{k-1} u_q$ where $1 < k < p^\star/p$. This leads to

$$-\eta^p |u_q|^{k-1} u_q div\big(|\nabla u_q|^{p-2} \nabla u_q\big) + a(x)\eta^p |u_q|^{p+k-1} = \mu_q^\sigma(G) f(x) \eta^p |u_q|^{k+q-1}.$$

Following the calculations developed in Section II, we obtain that if $f(P) \le 0$, or $f(P) > 0$ and

$$K(n,p)^p \mu_\sigma(G) f(P)^{p/p^\star} \limsup_{q \to p^\star} \Big(\int_{B_P(\delta)} |f|.|u_q|^q dx \Big)^{1-p/p^\star} < 1,$$

then there exists $C > 0$, independent of q, such that

$$\int_\Omega |\nabla\big(\eta |u_q|^{(k+p-1)/p}\big)|^p dx \le C. \tag{7}$$

As in Section II, (7) implies that

$$\limsup_{q \to p^\star} \int_{B_P(\delta/2)} |u_q|^q dx = 0.$$

(See the Proof of Lemma 4). The Proof of Theorem 2 then proceeds as follows.

Proof of Theorem 2. Suppose that for all $P \in \overline{\Omega}$, where $f(P) > 0$, there exists $\delta_P > 0$ such that

$$K(n,p)^p \mu_\sigma(G) f(P)^{p/p^\star} \limsup_{q \to p^\star} \Big(\int_{B_P(\delta)} |f|.|u_q|^q dx \Big)^{1-p/p^\star} < 1.$$

Since $\overline{\Omega}$ is compact, it can be covered by a finite number of $B_x(\delta_x)$, where $\delta_x > 0$ is such that

$$\limsup_{q \to p^\star} \int_{B_x(\delta_x)} |u_q|^q dx = 0.$$

The fact that

$$\int_\Omega f(x)|u_q|^q dx = 1$$

then clearly leads to a contradiction. As a consequence, there exists some $P \in \overline{\Omega}$ such that $f(P) > 0$, and such that for all $\delta > 0$,

$$K(n,p)^p \mu_\sigma(G) f(P)^{p/p^\star} \limsup_{q \to p^\star} \Big(\int_{B_P(\delta)} |f|.|u_q|^q dx \Big)^{1-p/p^\star} \geq 1. \qquad (8)$$

Suppose now that $CardO_H(P) < +\infty$. Then, we can choose $\delta > 0$ small enough in order to get that

$$\limsup_{q \to p^\star} \int_{B_P(\delta)} f|u_q|^q dx \leq \frac{1}{CardO_H(P)}.$$

The point here is that for $\tau \in H$, and $\delta > 0$ small enough,

$$\int_{B_P(\delta)} f|u_q|^q dx = \int_{B_{\tau(P)}(\delta)} f|u_q|^q dx$$

and that according to the previous discussion, $(|u_q|^q)$ converges to 0 in L^1 in a neighbourhood of any x such that $f(x) \leq 0$. One then gets that (8) is in contradiction with the assumption of the theorem. In the same order of ideas, assume now that $CardO_H(P) = +\infty$. Then, and with the same kind of arguments, one has that for any $\epsilon > 0$, there exists $\delta > 0$ small, such that

$$\limsup_{q \to p^\star} \int_{B_P(\delta)} f|u_q|^q dx \leq \epsilon.$$

Here, one gets the contradiction with (8). As a conclusion, we have proved that under the assumption of the theorem, $(\mathcal{RA})$ does not hold. By Proposition 4, and what has been said just after Proposition 4, this proves Theorem 2.

IV. Estimates and test functions. Let $x_0 \in \mathbb{R}^n$ and $r = |x - x_0|$ the euclidean distance from x_0 to x. For $p > 1$ given, p real such that $p < n$, we define the function u_ϵ by $u_\epsilon(x) = (\epsilon + r^{p/p-1})^{1-n/p}$ and the function v_ϵ by $v_\epsilon(x) = (\epsilon + r^{p/p-1})^{1-n/p}\phi(r)$, where $\phi : \mathbb{R} \to \mathbb{R}$, nonnegative and smooth, is such that $\phi(r) = 1$ for $r \leq \delta/4$ and $\phi(r) = 0$ for $r \geq \delta$, $\delta > 0$ small. Recall here that $u_1(x) = (1 + r^{p/p-1})^{1-n/p}$ realizes the best constant for the embedding of $W^{1,p}(\mathbb{R}^n)$ in $L^{p^\star}(\mathbb{R}^n)$. Let a and f be two smooth functions defined in a neighbourhood Ω of x_0, f being such that $f(x_0) > 0$. We assume in what follows that $f > 0$ in Ω, and that $B_{x_0}(\delta) \subset \Omega$. For $u \in W_0^{1,p}(\Omega)$ a function such that $\int_\Omega f(x)|u|^{p^\star} dx > 0$, we define

$$J_p(u) = \frac{\int_\Omega |\nabla u|^p dx + \int_\Omega a(x)|u|^p dx}{\left(\int_\Omega f(x)|u|^{p^\star} dx\right)^{p/p^\star}}.$$

We also introduce

$$k_a = \inf\left\{j \in \mathbf{N} \ / \ \Delta^j a(x_0) \neq 0\right\}, \quad k_f = \inf\left\{j \in \mathbf{N}^\star \ / \ \Delta^j f(x_0) \neq 0\right\}$$

with the convention that $k_a = +\infty$ (resp. $k_f = +\infty$) if the corresponding set above is empty. Here, $\Delta^j = \Delta^{j-1} \circ \Delta$, $j \geq 1$, where Δ is the usual laplacian so that $\Delta = \sum_i \partial_i^2$, and $\Delta^0 \psi = \psi$. When $n > p^2$, we define

$$k = \sup\{m \in \mathbf{N} \ / \ n > p^2 + 2m(p-1)\}$$

and for j integer, we set

$$\alpha_{n,j} = \frac{\Gamma\left(j + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)^{n-1}\left(2j + n\right)}{\Gamma\left(j + \frac{n}{2} + 1\right)}$$

and

$$\tilde{\alpha}_j^{p,n} = \frac{\alpha_{n,j}}{(2j)!}\int_0^{+\infty} \frac{r^{n+2j-1}}{\left(1 + r^{p/(p-1)}\right)^{n-p}} dr$$

$$\tilde{\beta}_j^{p,n} = \frac{\alpha_{n,j}}{(2j)!}\frac{(n-p)^p}{(p-1)^{p-1}}\int_0^{+\infty} \frac{r^{n+2j-1}}{\left(1 + r^{p/(p-1)}\right)^{n}} dr.$$

Note here that $\tilde{\alpha}_j^{p,n}$ exists as soon as $n > p^2 + 2j(p-1)$, that $\tilde{\beta}_j^{p,n}$ exists as soon as $n > 2j(p-1)$, and that the precise values of $\tilde{\alpha}_j^{p,n}$ and $\tilde{\beta}_j^{p,n}$ are given by Lemma 7 below. First we prove the following.

Proposition 5. *Suppose that $1 < p^2 < n$. For $\epsilon > 0$ small, one has that*

$$J_p(v_\epsilon) \; < \; \frac{1}{K(n,p)^p f(x_0)^{1-p/n}}$$

in each of the following cases

(i) $k \geq k_a$, $k_f > k_a + \frac{p}{2}$, *and* $\Delta^{k_a} a(x_0) < 0$
(ii) $k \geq k_a$, $k_f < k_a + \frac{p}{2}$, *and* $\Delta^{k_f} f(x_0) > 0$
(iii) $k \geq k_a$, $k_f = k_a + \frac{p}{2}$, *and* $\tilde{\alpha}^{p,n}_{k_a}\big(\Delta^{k_a} a(x_0)\big) f(x_0) - \tilde{\beta}^{p,n}_{k_f} \Delta^{k_f} f(x_0) < 0$
(iv) $k < k_a$, $k_f \leq k + \frac{p}{2}$, *and* $\Delta^{k_f} f(x_0) > 0$,

where $K(n,p)$ is the best constant for the embedding of $W^{1,p}(\mathbb{R}^n)$ in $L^{p^\star}(\mathbb{R}^n)$, and v_ϵ, k_a, k_f, k, $\tilde{\alpha}^{p,n}_j$, $\tilde{\beta}^{p,n}_j$, and J_p are as above.

In order to illustrate this result by simple examples, note that the following corollaries are just particular situations that occur in Proposition 5.

Corollary 1. *Suppose that $1 < p^2 < n$. For $\epsilon > 0$ small, one has that*

$$J_p(v_\epsilon) \; < \; \frac{1}{K(n,p)^p f(x_0)^{1-p/n}}$$

in each of the following cases

(i) $1 < p < 2$ *and* $a(x_0) < 0$
(ii) $p = 2$ *and* $\frac{8(n-1)}{(n-2)(n-4)} a(x_0) f(x_0) - \Delta f(x_0) < 0$
(iii) $p > 2$ *and* $\Delta f(x_0) > 0$,

where $K(n,p)$ is the best constant for the embedding of $W^{1,p}(\mathbb{R}^n)$ in $L^{p^\star}(\mathbb{R}^n)$, and v_ϵ and J_p are as above.

Corollary 2. *Suppose that $n > 4 + 2k_a$. For $\epsilon > 0$ small, one has that*

$$J_2(v_\epsilon) \; < \; \frac{1}{K(n,2)^2 f(x_0)^{1-2/n}}$$

in each of the following cases

(i) $k_f > k_a + 1$ *and* $\Delta^{k_a} a(x_0) < 0$
(ii) $k_f = k_a + 1$ *and* $\tilde{\alpha}^{2,n}_{k_a}\big(\Delta^{k_a} a(x_0)\big) f(x_0) - \tilde{\beta}^{2,n}_{k_f} \Delta^{k_f} f(x_0) < 0$
(iii) $k_f < k_a + 1$ *and* $\Delta^{k_f} f(x_0) > 0$,

where $K(n,2)$ is the best constant for the embedding of $W^{1,2}(\mathbb{R}^n)$ in $L^{\frac{2n}{n-2}}$ $(\mathbb{R}^n)$, and v_ϵ, k_a, k_f, $\tilde{\alpha}^{2,n}_j$, $\tilde{\beta}^{2,n}_j$, and J_2 are as above.

Corollary 3. *Suppose that $1 < p^2 < n$ and $n > p + 2k_f(p-1)$. Assume that a is flat at x_0 in the sense that $k_a = +\infty$. Then, for $\epsilon > 0$ small,*

$$J_p(v_\epsilon) \; < \; \frac{1}{K(n,p)^p f(x_0)^{1-p/n}}$$

as soon as $\Delta^{k_f} f(x_0) > 0$, where $K(n,p)$ is the best constant for the embedding of $W^{1,p}(\mathbb{R}^n)$ in $L^{p^\star}(\mathbb{R}^n)$, and v_ϵ, k_a, k_f, and J_p are as above.

If $a \equiv 0$, we refer to Proposition 6 below for a better result. Anyway, the idea of the proof of Proposition 5 consists in developing the integrals

$$\int |\nabla v_\epsilon|^p dx, \; \int a(x) u_\epsilon^p dx, \; \int f(x) u_\epsilon^{p^\star} dx$$

in powers of ϵ, using the development of a and f around x_0 for the last two integrals. Roughly speaking, we will obtain that

$$\frac{\int |\nabla v_\epsilon|^p dx + \int a(x) v_\epsilon^p dx}{\left(\int f(x) v_\epsilon^{p^\star} dx\right)^{p/p^\star}} \leq \frac{1}{K(n,p)^p f(x_0)^{p/p^\star}} \Big\{ \frac{1 + O(\epsilon^{\tilde{k}_a})}{\left(1 + O(\epsilon^{\tilde{k}_f})\right)^{p/p^\star}} \Big\},$$

where $\tilde{k}_a$ and $\tilde{k}_f$ are positive real numbers which depends on k_a, k_f, and p. As a consequence, the quantity above will be less than $\frac{1}{K(n,p)^p f(x_0)^{p/p^\star}}$, when $O(\epsilon^{\tilde{k}_a}) - \frac{p}{p^\star} O(\epsilon^{\tilde{k}_f})$ is negative. In order to prove Proposition 5, we need the two following technical lemmas. For sake of clearness, their proofs are postponed after the proof of Proposition 6.

Lemma 6. *Let ψ be a smooth function defined around some point $x_0 \in \mathbb{R}^n$. Then, for any integer j,*

$$\int_{S^{n-1}} \left(D^{(2j+1)} \psi(x_0).x^{2j+1}\right) d\sigma(x) = 0$$
$$\int_{S^{n-1}} \left(D^{(2j)} \psi(x_0).x^{2j}\right) d\sigma(x) = \alpha_{n,j} \Delta^j \psi(x_0),$$

where $\alpha_{n,j} = \frac{\Gamma(j+\frac{1}{2})\Gamma(\frac{1}{2})^{n-1}(2j+n)}{\Gamma(j+\frac{n}{2}+1)}$ is as above, and $d\sigma$ denotes the volume element on S^{n-1}.

Lemma 7. *For j integer,*

$$\int_0^{+\infty}\frac{r^{n+2j-1}}{\left(1+r^{p/(p-1)}\right)^{n-p}}dr=\frac{p-1}{p}\;\frac{\Gamma\left(\frac{(n+2j)(p-1)}{p}\right)\Gamma\left(\frac{n-p^2-2j(p-1)}{p}\right)}{\Gamma\left(n-p\right)}$$

$$\int_0^{+\infty}\frac{r^{n+2j-1}}{\left(1+r^{p/(p-1)}\right)^{n}}dr\quad=\frac{p-1}{p}\;\frac{\Gamma\left(\frac{(n+2j)(p-1)}{p}\right)\Gamma\left(\frac{n-2j(p-1)}{p}\right)}{\Gamma\left(n\right)}$$

$$\int_0^{+\infty}\frac{r^{p/(p-1)+n-1}}{\left(1+r^{p/(p-1)}\right)^{n}}dr\quad=\frac{p-1}{p}\;\frac{\Gamma\left(n-\frac{n}{p}+1\right)\Gamma\left(\frac{n}{p}-1\right)}{\Gamma\left(n\right)}$$

as soon as $n>p^2+2j(p-1)$ for the first integral, $n>2j(p-1)$ for the second integral, and $n>p$ for the third integral.

Note here that for u_1 as above,

$$\int_{R^n}u_1^{p^\star}dx=\omega_{n-1}\int_0^{+\infty}\frac{r^{n-1}}{\left(1+r^{p/(p-1)}\right)^{n}}dr$$

$$\int_{R^n}u_1^{p}dx=\omega_{n-1}\int_0^{+\infty}\frac{r^{n-1}}{\left(1+r^{p/(p-1)}\right)^{n-p}}dr$$

$$\int_{R^n}|\nabla u_1|^{p}dx=\frac{(n-p)^p}{(p-1)^p}\omega_{n-1}\int_0^{+\infty}\frac{r^{p/(p-1)+n-1}}{\left(1+r^{p/(p-1)}\right)^{n-p}}dr,$$

where ω_{n-1} is the volume of the standard unit sphere in $\mathbb{R}^n$

Proof of Proposition 5. Let j_1 be some integer such that $2j_1<\frac{n-p^2}{p-1}$ and $\Delta^j a(x_0)=0$ for all integers $0\le j<j_1$, with the convention that $j_1=0$ if $a(x_0)\neq 0$. Since $\phi\equiv 1$ near x_0, one has that

$$\left(a\phi^p\right)(x)=\sum_{j=0}^{2j_1-1}\frac{1}{j!}D^j a(x_0).(x-x_0)^j$$
$$+\frac{1}{(2j_1)!}D^{(2j_1)}a(x_0).(x-x_0)^{(2j_1)}+r^{2j_1+1}O(1).$$

Let $\eta>0$ be such that $2j_1+\eta<(n-p^2)/(p-1)$. We write that $r^{2j_1+1}=$

$r^{2j_1+\eta}r^{1-\eta}$. According to Lemma 6,

$$\int_\Omega a(x)v_\epsilon^p dx = \frac{\alpha_{n,j_1}}{(2j_1)!}\big(\Delta^{j_1}a(x_0)\big)\int_0^\delta \frac{r^{n+2j_1-1}}{\big(\epsilon + r^{p/(p-1)}\big)^{n-p}}dr$$
$$+\int_0^\delta\int_{S^{n-1}}\frac{r^{n+2j_1+\eta-1}O(1)}{\big(\epsilon + r^{p/(p-1)}\big)^{n-p}}drd\sigma.$$

The change of variables $r = \epsilon^{(p-1)/p}s$ then leads to

$$\int_\Omega a(x)v_\epsilon^p dx = \frac{\alpha_{n,j_1}}{(2j_1)!}\big(\Delta^{j_1}a(x_0)\big)\epsilon^{[p^2-n+2j_1(p-1)]/p}$$
$$\times\int_0^{\delta/\epsilon^{(p-1)/p}}\frac{s^{n+2j_1-1}}{\big(1+s^{p/(p-1)}\big)^{n-p}}ds + o\Big(\epsilon^{[p^2-n+2j_1(p-1)]/p}\Big)$$
$$= \frac{\alpha_{n,j_1}}{(2j_1)!}\big(\Delta^{j_1}a(x_0)\big)\epsilon^{[p^2-n+2j_1(p-1)]/p}$$
$$\times\int_0^{+\infty}\frac{s^{n+2j_1-1}}{\big(1+s^{p/(p-1)}\big)^{n-p}}ds + o\Big(\epsilon^{[p^2-n+2j_1(p-1)]/p}\Big).$$

The point here is that since $2j_1 < (n-p^2)/(p-1)$, the integrals

$$\int_0^{+\infty}\frac{s^{n+2j-1}}{\big(1+s^{p/(p-1)}\big)^{n-p}}ds$$

do exist for $j \le j_1$, and that according to the choice of $\eta > 0$, the integral

$$\int_0^{+\infty}\frac{s^{n+2j_1+\eta-1}}{\big(1+s^{p/(p-1)}\big)^{n-p}}ds$$

also exists.

In the same order of ideas, let j_2 be an integer such that $2j_2 < \frac{n}{p-1}$ and $\Delta^j f(x_0) = 0$ for all integers $1 \le j < j_2$, with the convention that $j_2 = 1$ if $\Delta f(x_0) \ne 0$. Here again, and since $\phi \equiv 1$ near x_0,

$$\big(f\phi^{p^\star}\big)(x) = f(x_0) + \sum_{j=1}^{2j_2-1}\frac{1}{j!}D^j f(x_0).(x-x_0)^j$$
$$+\frac{1}{(2j_2)!}D^{(2j_2)}f(x_0).(x-x_0)^{2j_2} + r^{2j_2+1}O(1).$$

Let $\eta > 0$ be such that $2j_2 + \eta < n/(p-1)$. We write that $r^{2j_2+1} = r^{2j_2+\eta}r^{1-\eta}$. According to Lemma 6,

$$\int_\Omega f(x)v_\epsilon^{p^\star}dx = f(x_0)\omega_{n-1}\int_0^\delta \frac{r^{n-1}}{\left(\epsilon + r^{p/(p-1)}\right)^n}dr$$
$$+ \frac{\alpha_{n,j_2}}{(2j_2)!}\left(\Delta^{j_2}f(x_0)\right)\int_0^\delta \frac{r^{n+2j_2-1}}{\left(\epsilon + r^{p/(p-1)}\right)^n}\,dr + \int_0^\delta\int_{S^{n-1}}\frac{r^{n+2j_2+\eta-1}O(1)}{\left(\epsilon + r^{p/(p-1)}\right)^n}dr,$$

where ω_{n-1} denotes the volume of the standard unit sphere S^{n-1}. The change of variables $r = \epsilon^{(p-1)/p}s$ then leads to

$$\int_\Omega f(x)v_\epsilon^{p^\star}dx = f(x_0)\omega_{n-1}\epsilon^{-n/p}\int_0^{\delta/\epsilon^{(p-1)/p}}\frac{s^{n-1}}{\left(1+s^{p/(p-1)}\right)^n}ds$$
$$+ \frac{\alpha_{n,j_2}}{(2j_2)!}\left(\Delta^{j_2}f(x_0)\right)\epsilon^{[-n+2j_2(p-1)]/p}\int_0^{\delta/\epsilon^{(p-1)/p}}\frac{s^{n+2j_2-1}}{\left(1+s^{p/(p-1)}\right)^n}ds$$
$$+ o\left(\epsilon^{[-n+2j_2(p-1)]/p}\right)$$
$$= f(x_0)\epsilon^{-n/p}\omega_{n-1}\Big\{\int_0^{\delta/\epsilon^{(p-1)/p}}\frac{s^{n-1}}{\left(1+s^{p/(p-1)}\right)^n}ds$$
$$+ \frac{\alpha_{n,j_2}}{(2j_2)!\omega_{n-1}}\frac{\Delta^{j_2}f(x_0)}{f(x_0)}\epsilon^{2j_2(p-1)/p}\int_0^{+\infty}\frac{s^{n+2j_2-1}}{\left(1+s^{p/(p-1)}\right)^n}ds + o\left(\epsilon^{2j_2(p-1)/p}\right)\Big\}.$$

One can then write that

$$\int_0^{\delta/\epsilon^{(p-1)/p}}\frac{s^{n-1}}{\left(1+s^{p/(p-1)}\right)^n}ds$$
$$= \int_0^{+\infty}\frac{s^{n-1}}{\left(1+s^{p/(p-1)}\right)^n}ds - \int_{\delta/\epsilon^{(p-1)/p}}^{+\infty}\frac{s^{n-1}}{\left(1+s^{p/(p-1)}\right)^n}ds$$

and that

$$\int_{\delta/\epsilon^{(p-1)/p}}^{+\infty}\frac{s^{n-1}}{\left(1+s^{p/(p-1)}\right)^n}ds = \int_{\delta/\epsilon^{(p-1)/p}}^{+\infty}\frac{s^{n+2j_2-1}}{\left(1+s^{p/(p-1)}\right)^n s^{2j_2}}ds$$
$$\leq \frac{1}{\delta^{2j_2}}\;\epsilon^{2j_2(p-1)/p}\int_{\delta/\epsilon^{(p-1)/p}}^{+\infty}\frac{s^{n+2j_2-1}}{\left(1+s^{p/(p-1)}\right)^n}ds.$$

As a consequence, one has that

$$\int_0^{\delta/\epsilon^{(p-1)/p}} \frac{s^{n-1}}{\left(1+s^{p/(p-1)}\right)^n} ds = \int_0^{+\infty} \frac{s^{n-1}}{\left(1+s^{p/(p-1)}\right)^n} ds + o\left(\epsilon^{2j_2(p-1)/p}\right)$$

and one can write that

$$\begin{aligned}
&\int_\Omega f(x) v_\epsilon^{p^\star} dx = f(x_0)\epsilon^{-n/p}\omega_{n-1}\Big\{ \int_0^{+\infty} \frac{s^{n-1}}{\left(1+s^{p/(p-1)}\right)^n} ds \\
&+ \frac{\alpha_{n,j_2}}{(2j_2)!\omega_{n-1}} \frac{\Delta^{j_2} f(x_0)}{f(x_0)} \epsilon^{2j_2(p-1)/p} \int_0^{+\infty} \frac{s^{n+2j_2-1}}{\left(1+s^{p/(p-1)}\right)^n} ds + o\left(\epsilon^{2j_2(p-1)/p}\right)\Big\} \\
&= f(x_0)\epsilon^{-n/p}\Big(\omega_{n-1}\int_0^{+\infty} \frac{s^{n-1}}{\left(1+s^{p/(p-1)}\right)^n} ds\Big)\Big\{1 + \epsilon^{2j_2(p-1)/p} \\
&\times \frac{\alpha_{n,j_2}}{(2j_2)!\omega_{n-1}} \frac{\Delta^{j_2} f(x_0)}{f(x_0)} \Big(\int_0^{+\infty} \frac{s^{n+2j_2-1}}{\left(1+s^{p/(p-1)}\right)^n} ds\Big)\Big(\int_0^{+\infty} \frac{s^{n-1}}{\left(1+s^{p/(p-1)}\right)^n} ds\Big)^{-1} \\
&\qquad + o\left(\epsilon^{2j_2(p-1)/p}\right)\Big\}.
\end{aligned}$$

Here again, the point is that since $2j_2 < n/(p-1)$, the integrals

$$\int_0^{+\infty} \frac{s^{n+2j-1}}{\left(1+s^{p/(p-1)}\right)^n} ds$$

do exist for $j \leq j_2$, and that according to the choice of $\eta > 0$, the integral

$$\int_0^{+\infty} \frac{s^{n+2j_2+\eta-1}}{\left(1+s^{p/(p-1)}\right)^n} ds$$

also exists. Finally, one can write that

$$\int_\Omega |\nabla v_\epsilon|^p dx \leq \int_{R^n} |\nabla u_\epsilon|^p dx + \int_{r\geq\delta} |\nabla v_\epsilon|^p dx \leq \int_{R^n} |\nabla u_\epsilon|^p dx + C_0,$$

where $C_0 > 0$ is a positive constant independent of ϵ. Independently, one easily checks that

$$\begin{aligned}
\int_{R^n} |\nabla u_\epsilon|^p dx &= \frac{(n-p)^p}{(p-1)^p}\omega_{n-1}\int_0^{+\infty} \frac{r^{p/(p-1)+n-1}}{\left(\epsilon + r^{p/(p-1)}\right)^n} dr \\
&= \frac{(n-p)^p}{(p-1)^p}\omega_{n-1}\epsilon^{1-n/p}\int_0^{+\infty} \frac{r^{p/(p-1)+n-1}}{\left(1 + r^{p/(p-1)}\right)^n} dr.
\end{aligned}$$

Note here that this integral exists as soon as $n > p$. Recall now that $u_1(x) = (1+r^{p/(p-1)})^{1-n/p}$ realizes the best constant for the embedding of $W^{1,p}(\mathbb{R}^n)$ in $L^{p^\star}(\mathbb{R}^n)$. As a consequence,

$$\big(\omega_{n-1}\int_0^{+\infty}\frac{s^{n-1}}{\big(1+s^{p/(p-1)}\big)^n}ds\big)^{p/p^\star} = K(n,p)^p\frac{(n-p)^p}{(p-1)^p}\omega_{n-1}\int_0^{+\infty}\frac{s^{p/(p-1)+n-1}}{\big(1+s^{p/(p-1)}\big)^n}ds$$

and one has that

$$\begin{aligned}&\int_\Omega|\nabla v_\epsilon|^p dx+\int_\Omega a(x)v_\epsilon^{p^\star}dx\\ &\leq\frac{1}{K(n,p)^p}\big(\omega_{n-1}\int_0^{+\infty}\frac{s^{n-1}}{\big(1+s^{p/(p-1)}\big)^n}ds\big)^{p/p^\star}\epsilon^{1-n/p}\\ &\times\big\{1+C_0\epsilon^{n/p-1}+\epsilon^{n/p-1}\epsilon^{[p^2-n+2j_1(p-1)]/p}K(n,p)^p\frac{\alpha_{n,j_1}}{(2j_1)!}\big(\Delta^{j_1}a(x_0)\big)\\ &\times\big(\omega_{n-1}\int_0^{+\infty}\frac{s^{n-1}}{\big(1+s^{p/(p-1)}\big)^n}ds\big)^{-p/p^\star}\int_0^{+\infty}\frac{s^{n+2j_1-1}}{\big(1+s^{p/(p-1)}\big)^{n-p}}ds\\ &+o\big(\epsilon^{(n/p-1)+[p^2-n+2j_1(p-1)]/p}\big)\big\}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\big(\int_\Omega f(x)v_\epsilon^{p^\star}dx\big)^{p/p^\star}=f(x_0)^{p/p^\star}\epsilon^{1-n/p}\big(\omega_{n-1}\int_0^{+\infty}\frac{s^{n-1}}{\big(1+s^{p/(p-1)}\big)^n}ds\big)^{p/p^\star}\\ &\times\big\{1+\epsilon^{2j_2(p-1)/p}\frac{\alpha_{n,j_2}}{(2j_2)!\omega_{n-1}}\frac{\Delta^{j_2}f(x_0)}{f(x_0)}\big(\int_0^{+\infty}\frac{s^{n-1}}{\big(1+s^{p/(p-1)}\big)^n}ds\big)^{-1}\\ &\times\int_0^{+\infty}\frac{s^{n+2j_2-1}}{\big(1+s^{p/(p-1)}\big)^n}ds+o\big(\epsilon^{2j_2(p-1)/p}\big)\big\}^{p/p^\star}\end{aligned}$$

and, as a conclusion, one gets that

$$\begin{aligned}J_p(v_\epsilon)\leq\ &\frac{1}{K(n,p)^pf(x_0)^{p/p^\star}}\ \big\{1+C_0\epsilon^{n/p-1}\\ &+\epsilon^{n/p-1}\epsilon^{[p^2-n+2j_1(p-1)]/p}K(n,p)^p\frac{\alpha_{n,j_1}}{(2j_1)!}\big(\Delta^{j_1}a(x_0)\big)\\ &\times\big(\omega_{n-1}\int_0^{+\infty}\frac{s^{n-1}}{\big(1+s^{p/(p-1)}\big)^n}ds\big)^{-p/p^\star}\int_0^{+\infty}\frac{s^{n+2j_1-1}}{\big(1+s^{p/(p-1)}\big)^{n-p}}ds\end{aligned}$$

$$- \epsilon^{n/p-1}\epsilon^{[p-n+2j_2(p-1)]/p}\frac{p}{p^\star}\frac{\alpha_{n,j_2}}{(2j_2)!\omega_{n-1}}\frac{\Delta^{j_2}f(x_0)}{f(x_0)}$$
$$\times\Big(\int_0^{+\infty}\frac{s^{n-1}}{\big(1+s^{p/(p-1)}\big)^n}ds\Big)^{-1}\int_0^{+\infty}\frac{s^{n+2j_2-1}}{\big(1+s^{p/(p-1)}\big)^n}ds$$
$$+ o\Big(\epsilon^{(n/p-1)+[p^2-n+2j_1(p-1)]/p}\Big) + o\Big(\epsilon^{(n/p-1)+[p-n+2j_2(p-1)]/p}\Big)\Big\}.$$

From now on, and for j real, set

$$A_{n,p} = \frac{n}{p} - 1, \quad B_{n,p}(j) = \frac{n}{p} - 1 + \frac{p^2 - n + 2j(p-1)}{p}$$
$$C_{n,p}(j) = \frac{n}{p} - 1 + \frac{p - n + 2j(p-1)}{p}.$$

Clearly, one has that $B_{n,p}(j) = C_{n,p}(j + \frac{p}{2})$. First, we prove points (ii), (iii), and (iv) of Proposition 5. Then, we prove point (i).

(ii) Suppose that $k \geq k_a$ and $k_f < k_a + p/2$. According to the definition of k, we can take here $j_1 = k_a$ and $j_2 = k_f$. It is then easy to check that $A_{n,p} > B_{n,p}(j_1) > C_{n,p}(j_2)$. As a consequence of the development above, one then gets that for $\epsilon > 0$ small,

$$J_p(v_\epsilon) \; < \; \frac{1}{K(n,p)^p f(x_0)^{1-p/n}}$$

as soon as $\Delta^{k_f} f(x_0) > 0$. This proves point (ii) of Proposition 5.

(iii) Suppose that $k \geq k_a$ and $k_f = k_a + p/2$. (This implies that $p/2$ is an integer). Here again, and according to the definition of k, one can take $j_1 = k_a$ and $j_2 = k_f$. One then has that

$$A_{n,p} > B_{n,p}(j_1) = C_{n,p}(j_2).$$

As a consequence of the development above, and by Lemma 6 and Lemma 7, one then gets that for $\epsilon > 0$ small,

$$J_p(v_\epsilon) \; < \; \frac{1}{K(n,p)^p f(x_0)^{1-p/n}}$$

as soon as

$$\tilde{\alpha}_{k_a}^{p,n}\big(\Delta^{k_a} a(x_0)\big) f(x_0) - \tilde{\beta}_{k_f}^{p,n} \Delta^{k_f} f(x_0) < 0.$$

This proves point (iii) of Proposition 5.

(iv) Suppose that $k < k_a$ and $k_f \le k + p/2$. We take $j_1 = k$ and $j_2 = k_f$. Then,

$$A_{n,p} > B_{n,p}(j_1) \ge C_{n,p}(j_2).$$

Since $\Delta^k a(x_0) = 0$, and according to the development above, one gets that for $\epsilon > 0$ small,

$$J_p(v_\epsilon) \;<\; \frac{1}{K(n,p)^p f(x_0)^{1-p/n}}$$

as soon as $\Delta^{k_f} f(x_0) > 0$. This proves point (iv) of Proposition 5.

(i) Let us now prove point (i) of Proposition 5. We assume that $k \ge k_a$ and that $k_f > k_a + p/2$. By the definition of k, we can take $j_1 = k_a$. Now we go back to the development of $\int_\Omega f(x) v_\epsilon^{p^\star} dx$ and take $j_2 = k_a + [p/2]$, where $[p/2]$ denotes the greatest integer not exceeding $p/2$. Since $\Delta^{j_2} f(x_0) = 0$, and for $\eta > 0$ such that $2j_2 + \eta < n/(p-1)$, one has that

$$\int_\Omega f(x) v_\epsilon^{p^\star} dx = f(x_0)\omega_{n-1} \int_0^\delta \frac{r^{n-1}}{(\epsilon + r^{p/(p-1)})^n} dr + \int_0^\delta \int_{S^{n-1}} \frac{r^{n+2j_2+\eta_1} O(1)}{(\epsilon + r^{p/(p-1)})^n} dr.$$

The change of variables $r = \epsilon^{(p-1)/p} s$ then leads to

$$\begin{aligned}\int_\Omega f(x) v_\epsilon^{p^\star} dx =& f(x_0)\omega_{n-1}\epsilon^{-n/p} \int_0^{\delta/\epsilon^{(p-1)/p}} \frac{s^{n-1}}{\big(1 + s^{p/(p-1)}\big)^n} ds \\ &+ O\big(\epsilon^{[-n+(2j_2+\eta)(p-1)]/p}\big)\end{aligned}$$

and, here again, one can write that

$$\begin{aligned}\int_{\delta/\epsilon^{(p-1)/p}}^{+\infty} \frac{s^{n-1}}{\big(1 + s^{p/(p-1)}\big)^n} ds &= \int_{\delta/\epsilon^{(p-1)/p}}^{+\infty} \frac{s^{n+2j_2+\eta-1}}{\big(1 + s^{p/(p-1)}\big)^n s^{2j_2+\eta}} ds \\ &\le \frac{1}{\delta^{2j_2+\eta}}\ \epsilon^{(2j_2+\eta)(p-1)/p} \int_{\delta/\epsilon^{(p-1)/p}}^{+\infty} \frac{s^{n+2j_2+\eta-1}}{\big(1 + s^{p/(p-1)}\big)^n} ds.\end{aligned}$$

As a consequence, one has that for any $\eta > 0$ such that $2j_2 + \eta < n/(p-1)$,

$$\begin{aligned}&\int_\Omega f(x) v_\epsilon^{p^\star} dx \\ =& f(x_0)\epsilon^{-n/p}\omega_{n-1} \int_0^{+\infty} \frac{s^{n-1}}{\big(1 + s^{p/(p-1)}\big)^n} ds\ \Big\{1 + O\big(\epsilon^{(2j_2+\eta)(p-1)/p}\big)\Big\}.\end{aligned}$$

This leads to the fact that for any $\eta > 0$ such that $2j_2 + \eta < n/(p-1)$,

$$J_p(v_\epsilon) \le \frac{1}{K(n,p)^p f(x_0)^{p/p^\star}} \big\{ 1 + C_0 \epsilon^{n/p-1} + \epsilon^{n/p-1} \epsilon^{[p^2-n+2j_1(p-1)]/p} K(n,p)^p \frac{\alpha_{n,j_1}}{(2j_1)!} \big(\Delta^{j_1} a(x_0) \big) \times \Big(\omega_{n-1} \int_0^{+\infty} \frac{s^{n-1}}{\big(1+s^{p/(p-1)}\big)^n} ds \Big)^{-p/p^\star} \int_0^{+\infty} \frac{s^{n+2j_1-1}}{\big(1+s^{p/(p-1)}\big)^{n-p}} ds + o\Big(\epsilon^{(n/p-1)+[p^2-n+2j_1(p-1)]/p}\Big) + o\Big(\epsilon^{(n/p-1)+[p-n+(2j_2+\eta)(p-1)]/p}\Big) \big\}.$$

Now we choose $\eta > 0$ such that $2(k_a + [p/2]) + \eta < \frac{n}{p-1}$ and $2[p/2] + \eta > p$. This can be done since $n > 2k_a(p-1) + p(p-1)$. With this choice of η, one has that

$$p - n + (2j_2 + \eta)(p-1) > p^2 - n + 2j_1(p-1).$$

Since one also has that $p^2 - n + 2j_1(p-1) < 0$, one gets by the development above that for $\epsilon > 0$ small,

$$J_p(v_\epsilon) \; < \; \frac{1}{K(n,p)^p f(x_0)^{1-p/n}}$$

as soon as $\Delta^{k_a} a(x_0) < 0$. This proves point (i) of Proposition 5. □

As one easily checks from the proof of Proposition 5, we get a better result than the one of Corollary 3 if instead of assuming that a is flat at x_0, one assumes that $a \equiv 0$. The point here is that when $a \equiv 0$, we do not need anymore to assume that $n > p^2$. Note that when $p = 2$, the fact that $k_f \ge 1$ and the inequality $n > p + 2k_f(p-1)$, imply that $n > p^2$. More precisely, the following proposition holds.

Proposition 6. *Suppose that $n > p + 2k_f(p-1)$. For $\epsilon > 0$ small, one has that*

$$\frac{\int |\nabla v_\epsilon|^p dx}{\big(\int f(x) v_\epsilon^{p^\star} dx\big)^{p/p^\star}} < \frac{1}{K(n,p)^p f(x_0)^{p/p^\star}}$$

as soon as $\Delta^{k_f} f(x_0) > 0$, where $K(n,p)$ is the best constant for the embedding of $W^{1,p}(\mathbb{R}^n)$ in $L^{p^\star}(\mathbb{R}^n)$, and v_ϵ and k_f are as above.

Proof of Proposition 6. The point here is that we do not need anymore to develop the term $\int_\Omega a(x) v_\epsilon^p dx$. As a consequence, according to the developments made in the proof of Proposition 5, and taking $j_2 = k_f$, one has that for $n > p + 2k_f(p-1)$,

$$\begin{aligned} J_p(v_\epsilon) \;\le\; & \frac{1}{K(n,p)^p f(x_0)^{p/p^\star}} \; \Big\{1 + C_0 \epsilon^{n/p-1} \\ & - \epsilon^{n/p-1} \epsilon^{[p-n+2k_f(p-1)]/p} \frac{p}{p^\star} \frac{\alpha_{n,k_f}}{(2k_f)!\omega_{n-1}} \frac{\Delta^{k_f} f(x_0)}{f(x_0)} \\ & \times \Big(\int_0^{+\infty} \frac{s^{n-1}}{\big(1+s^{p/(p-1)}\big)^n} ds \Big)^{-1} \int_0^{+\infty} \frac{s^{n+2k_f-1}}{\big(1+s^{p/(p-1)}\big)^n} ds \\ & + o\Big(\epsilon^{(n/p-1)+[p-n+2k_f(p-1)]/p}\Big)\Big\} \end{aligned}$$

with the property that

$$\frac{n}{p} - 1 > \frac{n}{p} - 1 + \frac{p-n+2k_f(p-1)}{p}.$$

Clearly, this proves Proposition 6.

Proof of Lemma 6. We begin to write that

$$\begin{aligned} & \int_{S^{n-1}} \big(D^{(j)}\psi(x_0).x^j\big) d\sigma(x) \\ & = \sum_{I_j} C_j^{j_1 \dots j_n} \big(\partial_1^{j_1} \dots \partial_n^{j_n} \psi\big)(x_0) \int_{S^{n-1}} x_1^{j_1} \dots x_n^{j_n} d\sigma(x), \end{aligned}$$

where

$$I_j = \Big\{(j_1, \dots, j_n) \;/\; \sum_{k=1}^n j_k = j\Big\} \text{ and } C_j^{j_1 \dots j_n} = \frac{j!}{j_1! \dots j_n!}.$$

Suppose now that j is some odd number. Then, in every n-uple $(j_1, \dots, j_n)$ there exists at least one odd number. By the symmetries of S^{n-1}, one immediately sees that

$$\int_{S^{n-1}} x_1^{j_1} \dots x_n^{j_n} d\sigma(x) = 0.$$

This proves the first part of Lemma 6. □

Suppose now that j is even. Due to the previous remark, we are led to consider the n-uple which are only composed of even numbers. In other words, we are concerned with the following expression

$$\sum_{I_j} C_{2j}^{2j_1\ldots 2j_n}\big(\partial_1^{2j_1}\ldots\partial_n^{2j_n}\psi\big)(x_0)\int_{S^{n-1}} x_1^{2j_1}\ldots x_n^{2j_n}\,d\sigma(x).$$

Independently, we note here that

$$\Delta^j\psi(x_0)=\sum_{I_j} C_j^{j_1\ldots j_n}\big(\partial_1^{2j_1}\ldots\partial_n^{2j_n}\psi\big)(x_0).$$

As a consequence, Lemma 6 will be proved if we show that for any $(j_1,\ldots,j_n)$ $\in I_j$,

$$C_j^{j_1\ldots j_n}\int_{S^{n-1}} x_1^{2j}\,d\sigma(x)=C_{2j}^{2j_1\ldots 2j_n}\int_{S^{n-1}} x_1^{2j_1}\ldots x_n^{2j_n}\,d\sigma(x) \tag{9}$$

and that

$$\int_{S^{n-1}} x_1^{2j}\,d\sigma(x)=\frac{\Gamma\big(j+\frac12\big)\Gamma\big(\frac12\big)^{n-1}\big(2j+n\big)}{\Gamma\big(j+\frac n2+1\big)}. \tag{10}$$

We claim here that for any $(j_1,\ldots,j_n)\in I_j$,

$$\int_{S^{n-1}} x_1^{2j_1}\ldots x_n^{2j_n}\,d\sigma(x)=\frac{\big(2j+n\big)\prod_{k=1}^n\Gamma\big(j_k+\frac12\big)}{\Gamma\big(j+\frac n2+1\big)}. \tag{11}$$

The proof of this identity will be given later. As a consequence of (11), taking $j_1=j$, and $j_2=\cdots=j_n=0$, one has that

$$\int_{S^{n-1}} x_1^{2j}\,d\sigma(x)=\frac{\Gamma\big(j+\frac12\big)\Gamma\big(\frac12\big)^{n-1}\big(2j+n\big)}{\Gamma\big(j+\frac n2+1\big)}.$$

In other words, (10) is a consequence of (11). Independently, by standard properties of Γ, one has that for any $k=1,\ldots,n$,

$$\Gamma\Big(j_k+\frac12\Big)=\frac{(2j_k-1)!\,\Gamma\big(\frac12\big)}{2^{j_k}\big(2(j_k-1)\big)!}.$$

As a consequence,

$$\prod_{k=1}^{n} \Gamma\big(j_k + \frac{1}{2}\big) = \frac{\prod_{k=1}^{n}(2j_k)!\Gamma\big(\frac{1}{2}\big)^n}{\prod_{k=1}^{n}\big(2^{2j_k} j_k!\big)}.$$

Here again, taking $j_1 = j$, and $j_2 = \cdots = j_n = 0$, one gets that

$$\Gamma\big(j + \frac{1}{2}\big) = \frac{(2j)!\Gamma\big(\frac{1}{2}\big)}{2^{2j} j!}$$

and we have obtained that

$$\int_{S^{n-1}} x_1^{2j_1} \ldots x_n^{2j_n} d\sigma(x) = \frac{\big(\prod_{k=1}^{n}(2j_k)!\big) j!}{\big(\prod_{k=1}^{n} j_k!\big)(2j)!} \int_{S^{n-1}} x_1^{2j} d\sigma(x).$$

This proves (9), and we are led to prove (11). Let $B_n(\rho)$ be the ball of $\mathbb{R}^n$ of center 0 and radius ρ. Clearly, (11) is equivalent to the fact that for any $(j_1, \ldots, j_n) \in I_j$, and any $\rho > 0$,

$$\int_{B_n(\rho)} x_1^{2j_1} \ldots x_n^{2j_n} dx_1 \ldots dx_n = \rho^{2j+n} \frac{\prod_{k=1}^{n} \Gamma\big(j_k + \frac{1}{2}\big)}{\Gamma\big(j + \frac{n}{2} + 1\big)}. \tag{12}$$

First, we prove (12) for $n = 2$. Here,

$$\begin{aligned}
&\int_{B_2(\rho)} x_1^{2j_1} x_2^{2j_2} dx_1 dx_2 = 4\rho^{2(j+1)} \int_0^1 x_1^{2j_1} \Big(\int_0^{\sqrt{1-x_1^2}} x_2^{2j_2} dx_2\Big) dx_1 \\
&= \frac{4\rho^{2(j+1)}}{2j_2 + 1} \int_0^1 x_1^{2j_1} \big(1 - x_1^2\big)^{(2j_2+1)/2} dx_1 = \frac{\rho^{2(j+1)}}{j_2 + \frac{1}{2}} \int_0^1 u^{j_1 - 1/2} \big(1 - u\big)^{j_2 + 1/2} du \\
&= \rho^{2(j+1)} \ \frac{\Gamma\big(j_1 + \frac{1}{2}\big)\Gamma\big(j_2 + \frac{3}{2}\big)}{\big(j_2 + \frac{1}{2}\big)\Gamma\big(j_1 + j_2 + 1\big)} = \rho^{2(j+1)} \ \frac{\Gamma\big(j_1 + \frac{1}{2}\big)\Gamma\big(j_2 + \frac{1}{2}\big)}{\Gamma\big(j_1 + j_2 + 1\big)}
\end{aligned}$$

so that (12) holds for $n = 2$. Note that we use here the fact that for any $\alpha, \beta > 0$,

$$\int_0^1 u^{\alpha - 1} \big(1 - u\big)^{\beta - 1} du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

Suppose now that (12) is true at the order n. Then

$$\begin{aligned}
&\int_{B_{n+1}(\rho)} x_1^{2j_1}\dots x_{n+1}^{2j_{n+1}}dx_1\dots dx_{n+1}\\
&= 2\rho^{2j+n+1}\int_0^1 x_1^{2j_1}\Big(\int_{B_n(\sqrt{1-x_1^2})} x_2^{2j_2}\dots x_{n+1}^{2j_{n+1}}dx_2\dots dx_{n+1}\Big)dx_1\\
&= 2\rho^{2j+n+1}\frac{\prod_{k=2}^{n+1}\Gamma\big(j_k+\frac{1}{2}\big)}{\Gamma\big(j-j_1+\frac{n}{2}+1\big)}\int_0^1 x_1^{2j_1}\big(1-x_1^2\big)^{(2(j-j_1)+n)/2}\\
&= \rho^{2j+n+1}\frac{\prod_{k=2}^{n+1}\Gamma\big(j_k+\frac{1}{2}\big)}{\Gamma\big(j-j_1+\frac{n}{2}+1\big)}\ \int_0^1 u^{j_1-1/2}\big(1-u\big)^{(j-j_1)+n/2}du\\
&= \rho^{2j+n+1}\frac{\Gamma\big(j_1+\frac{1}{2}\big)\Gamma\big(j-j_1+\frac{n}{2}+1\big)\prod_{k=2}^{n+1}\Gamma\big(j_k+\frac{1}{2}\big)}{\Gamma\big(j+\frac{n}{2}+\frac{3}{2}\big)\Gamma\big(j-j_1+\frac{n}{2}+1\big)}\\
&= \rho^{2j+n+1}\frac{\prod_{k=1}^{n+1}\Gamma\big(j_k+\frac{1}{2}\big)}{\Gamma\big(j+\frac{n+1}{2}+1\big)}
\end{aligned}$$

and (12) is proved. As already mentioned, this ends the proof of the lemma.

Proof of Lemma 7. The three integrals of the lemma are easily computable with the change of variable $u = r^{p/(p-1)}$. As an example, and if $n > p^2 + 2j(p-1)$, we obtain

$$\begin{aligned}
\int_0^{+\infty}\frac{r^{n+2j-1}}{\big(1+r^{p/(p-1)}\big)^{n-p}}dr &= \frac{p-1}{p}\int_0^{+\infty}\frac{u^{(n+2j)(p-1)/p-1}}{\big(1+u\big)^{n-p}}du\\
&= \frac{p-1}{p}\frac{\Gamma\big(\frac{(n+2j)(p-1)}{p}\big)\Gamma\big(\frac{n-p^2-2j(p-1)}{p}\big)}{\Gamma\big(n-p\big)}
\end{aligned}$$

which proves the first part of the lemma. Here, we use the fact that for any $\alpha, \beta > 0$,

$$\int_0^{+\infty}\frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

The computations for the second integral and the third integral are left to the reader. □

As an easy consequence of Propositions 5 and 6, one has the following propositions. For a and f two smooth functions defined in a neighbourhood of some point x_0, we denote by $k_a(x_0)$ and $k_f(x_0)$ the integers k_a and k_f of Proposition 5. In other words

$$k_a(x_0) = \inf\big\{j \in \mathbf{N} \ / \ \Delta^j a(x_0) \neq 0\big\}$$
$$k_f(x_0) = \inf\big\{j \in \mathbf{N}^\star \ / \ \Delta^j f(x_0) \neq 0\big\}$$

with the convention that $k_a(x_0) = +\infty$ (resp. $k_f(x_0) = +\infty$) if the corresponding set above is empty. As above, we also let

$$k = \sup\big\{m \in \mathbf{N} \ / \ n > p^2 + 2m(p-1)\big\}.$$

The first proposition we state is the following. Here, $\mu(G)$ is as in Section II.

Proposition 7. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, G be a compact subgroup of $O(n)$, and $p \in (1,n)$ be some real number. We assume that $n > p^2$ and that Ω is stable under the action of G. Let also a and f be two smooth G-invariant functions on $\overline{\Omega}$, and let $x_0 \in \Omega$ be some point (of finite G-orbit) where $f(x_0) > 0$. Then*

$$K(n,p)^p \mu(G) f(x_0)^{1-p/n} < \big(Card O_G(x_0)\big)^{p/n} \tag{13}$$

in each of the following cases

(i) $k \geq k_a$, $k_f > k_a + \frac{p}{2}$, *and* $\Delta^{k_a} a(x_0) < 0$

(ii) $k \geq k_a$, $k_f < k_a + \frac{p}{2}$, *and* $\Delta^{k_f} f(x_0) > 0$

(iii) $k \geq k_a$, $k_f = k_a + \frac{p}{2}$, *and* $\tilde{\alpha}^{p,n}_{k_a}\big(\Delta^{k_a} a(x_0)\big) f(x_0) - \tilde{\beta}^{p,n}_{k_f} \Delta^{k_f} f(x_0) < 0$

(iv) $k < k_a$, $k_f \leq k + \frac{p}{2}$, *and* $\Delta^{k_f} f(x_0) > 0$

where $K(n,p)$ is the best constant for the embedding of $W^{1,p}(\mathbb{R}^n)$ in $L^{p^\star}(\mathbb{R}^n)$, $k_a = k_a(x_0)$, $k_f = k_f(x_0)$, and $\tilde{\alpha}^{p,n}_j$, $\tilde{\beta}^{p,n}_j$ are as above. In particular, (13) *is true in each of the different situations described by Corollaries* 1, 2, *and* 3. *Moreover, and if $a \equiv 0$ on Ω, one do not need anymore to assume that $n > p^2$, in the sense that* (13) *holds under the assumptions that $n > p + 2k_f(p-1)$ and $\Delta^{k_f} f(x_0) > 0$.*

Proof. Let $\delta > 0$ be such that $|y - x| > \delta$ for any $x, y \in O_G(x_0)$ such that $x \neq y$. Let also v_ϵ be the function defined at the beginning of this section. For $x \in O_G(x_0)$, and if $\sigma \in G$ is such that $\sigma(x) = x_0$, we let $v_{\epsilon,x} = v_\epsilon \circ \sigma$. The function

$$w_\epsilon = \sum_{x \in O_G(x_0)} v_{\epsilon,x}$$

is then G-invariant. Furthermore, one clearly sees that

$$I_p(w_\epsilon) = \big(CardO_G(x_0)\big)I_p(v_\epsilon)$$

and

$$\int_\Omega f(x)w_\epsilon^{p^\star}dx = \big(CardO_G(x_0)\big)\int_\Omega f(x)v_\epsilon^{p^\star}dx$$

Proposition 7 is then a straightforward consequence of Propositions 5 and 6. □

The second proposition we state is the following. Here, $\mu_\sigma(G)$ is as in Section III.

Proposition 8. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, G be a compact subgroup of $O(n)$, σ be an involution of $O(n)$, and $p \in (1,n)$ be some real number. We assume that $n > p^2$, that Ω is stable under the action of $H = [G,\sigma]$, and that G and σ commute weakly. Let also a and f be two smooth H-invariant functions on $\overline{\Omega}$, and let $x_0 \in \Omega$ be some point (of finite G-orbit) where relation (5) above holds. Then*

$$K(n,p)^p\mu_\sigma(G)f(x_0)^{1-p/n} < \big(CardO_H(x_0)\big)^{p/n} \tag{14}$$

in each of the following cases

(i) $k \geq k_a$, $k_f > k_a + \frac{p}{2}$, *and* $\Delta^{k_a}a(x_0) < 0$
(ii) $k \geq k_a$, $k_f < k_a + \frac{p}{2}$, *and* $\Delta^{k_f}f(x_0) > 0$
(iii) $k \geq k_a$, $k_f = k_a + \frac{p}{2}$, *and* $\tilde{\alpha}_{k_a}^{p,n}\big(\Delta^{k_a}a(x_0)\big)f(x_0) - \tilde{\beta}_{k_f}^{p,n}\Delta^{k_f}f(x_0) < 0$
(iv) $k < k_a$, $k_f \leq k + \frac{p}{2}$, *and* $\Delta^{k_f}f(x_0) > 0$

where $K(n,p)$ is the best constant for the embedding of $W^{1,p}(\mathbb{R}^n)$ in $L^{p^\star}(\mathbb{R}^n)$, $k_a = k_a(x_0)$, $k_f = k_f(x_0)$, and $\tilde{\alpha}_j^{p,n}$, $\tilde{\beta}_j^{p,n}$ are as above. In particular, (14) is true in each of the different situations described by Corollaries 1, 2, and 3. Moreover, and if $a \equiv 0$ on Ω, one do not need anymore to assume that $n > p^2$, in the sense that (14) holds under the assumptions that $n > p+2k_f(p-1)$ and $\Delta^{k_f}f(x_0) > 0$.

Proof. Let $\delta > 0$ be such that $|y - x| > \delta$ for any $x, y \in O_H(x_0)$ such that $x \neq y$. Let also v_ϵ be the function defined at the beginning of this section. For $x \in O_G(x_0)$, and if $\tau \in G$ is such that $\tau(x) = x_0$, we let $v_{\epsilon,x} = v_\epsilon \circ \tau$. The function

$$v_G^\epsilon = \sum_{x \in O_G(x_0)} v_{\epsilon,x}$$

is then G-invariant, so that by the weak commutation of G and σ, the function $w_\epsilon = v_G^\epsilon \circ \sigma - v_G^\epsilon$ is G-invariant and σ-antisymmetrical. Here again, one clearly sees that

$$I_p(w_\epsilon) = \big(CardO_H(x_0)\big) I_p(v_\epsilon)$$

and

$$\int_\Omega f(x) w_\epsilon^{p^\star} dx = \big(CardO_H(x_0)\big) \int_\Omega f(x) v_\epsilon^{p^\star} dx.$$

Proposition 8 is then a straightforward consequence of Propositions 5 and 6.

V. Specific examples. We present in this section some concrete situations where the results of the preceding sections can be applied. The following list is of course far to be exhaustive. By Theorems 1 and 2, and what has been said in Section IV, one will easily construct many other examples where solutions to problems (I) and (II) exist. The first result we mention here is the following generalization of the existence result of Guedda and Veron [8].

Proposition 9. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, and $p > 1$ be some real number such that $p^2 < n$. Let also a and f be two smooth functions on $\overline{\Omega}$, a being such that L is coercive (where L is as in the introduction), and f being positive somewhere. Assume that there exists $x_0 \in \Omega$ such that*

$$f(x_0) = \max_{x \in \overline{\Omega}} f(x) \textit{ and } a(x_0) < 0.$$

If $p \geq 2$, assume in addition that $\Delta^j f(x_0) = 0$ for any $1 \leq j \leq [p/2]$, where $[p/2]$ denotes the greatest integer not exceeding $p/2$. Then, and under these assumptions, problem (I) *has a solution $u \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$.*

Proof. In this result, $G = \{Id\}$ and we use Theorem 1. By Proposition 5, we get that

$$K(n,p)^p \mu(G) f(x_0)^{1-1/p} < 1,$$

where $\mu(G)$ is as in Section II. Since x_0 is a point where f is maximum, this inequality implies that for any $x \in \overline{\Omega}$ such that $f(x) > 0$,

$$K(n,p)^p \mu(G) f(x)^{1-1/p} < 1.$$

The inequality of theorem 1 then holds, and Proposition 9 is proved. □

In Proposition 9, a is assumed to be negative somewhere. When this is not the case, a general idea to get simple results from Theorems 1 and 2 is to deal with groups having the property that all the points in $\overline{\Omega}$ have an infinite orbit. In this situation, one does not need the results of Section IV. Basic examples here are given by annulii or solid torii. As an example, the following proposition holds.

Proposition 10. *Let Ω be a solid torus of $\mathbb{R}^3$, obtained by rotation around the z axis of a ball centered on the y-axis. Let G be the group of the rotations around the z-axis, let σ be the orthogonal symmetry with respect to the (x, y)-plane, and let $H = [G, \sigma]$. Let also a and f be two smooth H-invariant functions on $\overline{\Omega}$, a being such that L is coercive (where L is as in the introduction), and f being positive somewhere. For any $1 < p < 3$, the problem*

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + a(x)|u|^{p-2}u = f(x)|u|^{p^\star-2}u \ \textit{in}\ \Omega \\ u \not\equiv 0,\ u = 0 \ \textit{on}\ \partial\Omega \end{cases}$$

possesses a positive G-invariant solution $u \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$, and a nodal G-invariant and σ-antisymmetrical solution $u \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$, whose zero set is exactly the intersection of Ω with the (x, y)-plane.

More generally, by Theorems 1 and 2, and Propositions 7 and 8, one easily deals with situations where almost all the points in Ω have an infinite orbit under the action of the group considered. Basic examples here are given by the two following propositions.

Proposition 11. *Let Ω be an annulus of $\mathbb{R}^n$ centered at 0, and $a < 0$ a real number such that L is coercive (where L is as in the introduction). For any $p > 1$ such that $n > p^2$, the problem*

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + a|u|^{p-2}u = |u|^{p^\star-2}u \ \textit{in}\ \Omega \\ u \not\equiv 0,\ u = 0 \ \textit{on}\ \partial\Omega \end{cases}$$

possesses a positive radial solution $u_0 \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$, and an infinity of nodal solutions $u_i \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$.

Proof. Let $G = O(n)$. By theorem 1, and since for any $x \in \Omega$, $CardO_G(x) = +\infty$, one gets that the problem possesses a positive radial solution. (Here, one does not need to assume that $n > p^2$, but just $n > p$). This proves the first part of the proposition. On what concerns the second part, let P be some hyperplane of $\mathbb{R}^n$ such that $0 \in P$, and let D_P be the orthogonal complement of P, $0 \in D_P$. We denote by G_P the group of rotations around D_P, and by σ_P the orthogonal symmetry with respect to P. Since a and f are constants, with a negative, one gets by Proposition 8 (with $x_0 \in D_P \cap \Omega$) that for any $p > 1$ such that $n > p^2$,

$$\mu_\sigma(G) < \frac{2^{p/n}}{K(n,p)^p},$$

where $\mu_\sigma(G)$ is as in Section III. Since for any $x \in \Omega \backslash D_P$, $CardO_{G_P}(x) = +\infty$, by Theorem 2 this leads to the existence of a nodal solution $u_P \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, whose zero set is exactly $P \cap \Omega$. In particular, if $P_1 \neq P_2$, then $u_{P_1} \neq u_{P_2}$. Since P is arbitrary, this proves the second part of the proposition.

Proposition 12. *Let B be a ball of $\mathbb{R}^n$ centered at 0, and let a and f be two smooth radial functions on $\overline{B}$, a being such that L is coercive (where L is as in the introduction), and f being positive somewhere. If $f(0) \leq 0$, then for any $1 < p < n$, problem* (I) *has a radial solution $u \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$. If $f(0) > 0$, and if $p > 1$ is such that $p^2 < n$, assume that one of the following is true*

(i) $1 < p < 2$ *and* $a(0) < 0$
(ii) $p = 2$ *and* $\frac{8(n-1)}{(n-2)(n-4)} a(0) f(0) - \Delta f(0) < 0$
(iii) $p > 2$ *and* $\Delta f(0) > 0$.

Then problem (I) *has a radial solution $u \in W_0^{1,p}(B) \cap C^{1,\alpha}(\overline{B})$, $\alpha \in (0,1)$.*

Proof. We apply Theorem 1 with $G = O(n)$. If $f(0) \leq 0$, this immediately gives the result since for $x \neq 0$, $CardO_G(x) = +\infty$. If $f(0) > 0$, we have to prove that

$$K(n,p)^p \mu(G) f(0)^{1-p/n} < 1,$$

where $\mu(G)$ is as in Section II. By Proposition 5 (see also Corollary 1), such an inequality is true. The proposition is proved. □

In the situation of Proposition 12, and if $a \equiv 0$, one can use Proposition 6 instead of Proposition 5. This leads for instance to the fact that for any $p > 1$ such that $n > 3p - 2$, the problem

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2} \nabla u\big) = f(x) u^{p^\star - 1} \text{ in } B \\ u > 0 \text{ in } B,\ u = 0 \text{ on } \partial B \end{cases}$$

possesses a radial solution $u \in W_0^{1,p}(B) \cap C^{1,\alpha}(\overline{B})$, $\alpha \in (0,1)$, if $\Delta f(0) > 0$. Recall that for $f \equiv 1$, such a problem does not have any solution. In the same order of ideas, and when dealing with nodal solutions, let B a ball of $\mathbb{R}^n$ centered at 0, and let a and f two smooth σ-invariant functions on $\overline{B}$, where $\sigma(x) = -x$. Let also $1 < p \leq 2$ be such that $n > p^2$. Assume that a is such that L is coercive (where L is as in the introduction), that $f(0) \leq 0$, and that f achieves its positive maximum at some point $x_0 \in B$ where one of the following is true

(i) $1 < p < 2$ and $a(x_0) < 0$

(ii) $p = 2$ and $\frac{8(n-1)}{(n-2)(n-4)} a(x_0) f(x_0) - \Delta f(x_0) < 0$.

By Theorem 2 and Proposition 8 with $G = \{Id\}$, one gets that problem (II) possesses a nodal σ-antisymmetrical solution $u \in W_0^{1,p}(B) \cap C^{1,\alpha}(\overline{B})$, $\alpha \in (0,1)$.

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