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BLOW UP OF CRITICAL AND SUBCRITICAL NORMS IN SEMILINEAR HEAT EQUATIONS

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1. Introduction. We consider the semilinear heat equation

$$\begin{aligned} &u_t - \Delta u = |u|^{p-1}u, \quad x \in \Omega, \ \ t \in [0,T], \\ &u(t,x) = 0, \quad x \in \partial\Omega, \ \ t \in [0,T], \\ &u(0,x) = u_0(x), \quad x \in \Omega, \end{aligned} \tag{1}$$

with $p > 1$ and $u_0 \in L^\infty(\Omega)$, where Ω is a smooth, bounded domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$.

Let $u \in C((0,T_m) \times \overline{\Omega})$ be a solution of (1) which blows up in finite time, i.e., u is defined on the maximal interval of time $[0,T_m)$, with $T_m < +\infty$. By the blow up alternative, we know that u blows up in $L^\infty(\Omega)$, i.e., $\lim_{t\uparrow T_m} \|u(t)\|_{L^\infty(\Omega)} = +\infty$. A classical problem is to determine whether $\|u(t)\|_{L^q(\Omega)}$ also blows up as $t \uparrow T_m$, for some $q < +\infty$, and if so, at what rate.

We first note that the value $q = \frac{N(p-1)}{2}$ plays a critical role as in the problem of local existence for (1) with $u_0 \in L^q(\Omega)$. The case where q is supercritical, i.e., $q > \frac{N(p-1)}{2}$, is essentially known. Indeed,

$$\lim_{t\uparrow T_m} \|u(t)\|_{L^q(\Omega)} = +\infty,$$

for all $\frac{N(p-1)}{2} < q \leq +\infty$, $q \geq 1$. Moreover,

$$\liminf_{t\uparrow T_m} (T_m - t)^\delta \|u(t)\|_{L^q(\Omega)} > 0, \tag{2}$$

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with $\delta = \frac{1}{p-1} - \frac{N}{2q} > 0$. These results can be deduced from the local in time existence theory (cf. Brezis and Cazenave [3], Weissler [32]). We refer to Friedman and McLeod [8] and Giga and Kohn [11] for different arguments.

Here, we are interested in the cases where q is *subcritical* or *critical*, i.e., $q < \frac{N(p-1)}{2}$ or $q = \frac{N(p-1)}{2}$. In the case of q critical, we may still deduce the blow up of the critical norm from a general local existence result for $u_0 \in L^q(\Omega)$ (which has been recently proved by Brezis and Cazenave [4]), but under the hypothesis $u_t \geq 0$ (see Section 5). Without this assumption, the question of the blow up of the critical norm remains mostly open (for some ranges of p) and seems delicate. Whereas for q subcritical, the main point is that no local existence and uniqueness theory with L^q initial data holds (cf. Weissler [31], [34], Haraux and Weissler [14]). In fact, concerning the subcritical norms, the only known result is obtained in [8], for radial solutions in a ball under some assumptions on the initial data.

In this paper, we give several blow up results concerning critical and subcritical norms. The first one deals with the blow up of the critical norm for radial solutions, when $N \geq 3$ and $p \geq \frac{N+2}{N-2}$, and under a weaker form of a classical growth assumption on the blow up rate. We state our first main theorem.

Theorem 1. *Let Ω be a ball in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. Assume that $N \geq 3$ and $p \geq \frac{N+2}{N-2}$. Let $u_0 \in L^\infty(\Omega) \cap H^1(\Omega)$ be nonnegative, radially symmetric and nonincreasing as a function of $r = |x|$, such that the solution u of* (1) *blows up in finite time. If*

$$\liminf_{t \uparrow T_m} (T_m - t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty(\Omega)} < +\infty, \tag{3}$$

then

$$\limsup_{t \uparrow T_m} \|u(t)\|_{L^q(\Omega)} = +\infty, \quad \text{for} \;\; q = \frac{N(p-1)}{2}. \tag{4}$$

Remark 1. The growth condition (3) implies that there exist a sequence $t_n \uparrow T_m$ and $M > 0$ such that

$$\lim_n \|u(t_n)\|_{L^\infty} = +\infty \quad \text{and} \quad (T_m - t_n)^{\frac{1}{p-1}} \|u(t_n)\|_{L^\infty} \leq M. \tag{5}$$

Recall that Giga and Kohn (see [10], [11], [12]) characterized the asymptotic behavior of the solutions $u(t,x)$ of (1) near a blow up singularity, provided $N = 1, 2$ or $N \geq 3$ and $p < \frac{N+2}{N-2}$, and Ω is convex. More precisely, they proved under the above conditions that

$$\lim_{t \uparrow T_m} (T_m - t)^{\frac{1}{p-1}} u(t, a + (T_m - t)^{1/2} y) = \pm k, \tag{6}$$

uniformly on each compact set $|y| \leq C$ ($C > 0$), with $k = (p-1)^{-\frac{1}{p-1}}$, for every blow up point (T_m, a) of u. This implies that the $L^{\frac{N(p-1)}{2}}(\Omega)$ norm of u blows up, and for any larger q, the norm $L^q(\Omega)$ blows up faster then the self-similar rate (by (2)). Moreover, they proved that u satisfies the following upper bound on the blow up rate

$$\|u(t)\|_{L^\infty(\Omega)} \leq M(T_m - t)^{-\frac{1}{p-1}}, \quad t \in [0, T_m), \tag{7}$$

with $M > 0$, provided that either $1 < p < \frac{3N+8}{3N-4}$ or $N = 1$, or

$$N = 1, 2 \ \text{ or } \ N \geq 3 \ \text{ and } \ 1 < p < \frac{N+2}{N-2}, \ \text{ and } \ u \geq 0. \tag{8}$$

The first result is obtained using interpolation inequalities and parabolic regularity theory and the second one uses a blow up argument by contradiction.

The upper bound (7) was first proved by Weissler [33], for radial solutions in a ball with special initial data. In [8], it was proved for solutions on a bounded convex domain, with initial data $u_0 \in L^\infty(\Omega) \cap H^1_0(\Omega) \cap H^2(\Omega)$ satisfying

$$u_0 \geq 0, \qquad \Delta u + {u_0}^p \geq 0,$$

which assures that $u \geq 0$ and $u_t \geq 0$ for all $0 < t < T_m$, eliminating any possibility of oscillation in time.

The real remaining difficulty in understanding how the single point blow up occurs for (1) (in the radial case) rests on determining the global bounded solutions of the associated steady state equation

$$w_{rr} + \big(\frac{N-1}{r} - \frac{r}{2}\big) w_r + |w|^{p-1} w - \frac{w}{p-1} = 0 \ \text{ in } \ (0, +\infty). \tag{9}$$

In the cases $N = 1, 2$ or $N \geq 3$ and $p \leq \frac{N+2}{N-2}$, the only such positive solution of (9) is $w \equiv k$. On the other hand, Budd and Qi proved in [5] that (9) has infinitely many global bounded solutions on $[0,\infty)$, for $2 < N \leq 10$ and $p > \frac{N+2}{N-2}$, or $N \geq 11$ and $\frac{N+2}{N-2} < p < p^*$, with

$$p^* = \frac{N - 2(N-1)^{1/2}}{N - 4 - 2(N-1)^{1/2}}. \tag{10}$$

This result was proved by Troy in [27] for the case $N = 3$, $6 \leq p \leq 12$. The existence of at least a finite number of such solutions of (9) was obtained for some values of p when $p > p^*$, by Lepin [22].

This multiple existence of solutions complicates the stability analysis required to describe precisely the evolution of the time-dependent radial solutions u of (1) near a blow up singularity.

In [2], Bebernes and Eberly extended the above results to dimensions $N \geq 3$ by proving that, in spite of the multiple existence of solutions of (9), the asymptotic formula (6) remains the same as in dimensions 1 and 2, provided that Ω is a ball in $\mathbb{R}^N$ and the initial data u_0 is nonnegative, radially symmetric, nonincreasing as a function of $r = |x|$ and

$$\Delta u_0 + {u_0}^p \geq 0 \text{ in } \Omega. \tag{11}$$

Although no restriction on p is made, these hypotheses on u_0 imply (7), as was mentioned above (see [8]).

In Theorem 1, we prove the blow up of the critical norm (at least along a sequence $t_n \uparrow T_m$), eliminating the condition (11), at the expense of a condition upon the blow up rate, although weaker than (7). The idea is to use the method developed in [10], [11], [12] based on similarity variables (see Section 3), and the results of Budd and Qi [5], about the asymptotic behavior of the positive bounded solutions of (9).

Next, we prove that the growth assumption (3) is not a necessary condition for the blow up of the critical norm. Indeed, as an application of the results of Herrero and Velázquez in [18], for $N \geq 11$ and $p > p^*$, in the case $\Omega = \mathbb{R}^N$, we exhibit some solutions which blow up in finite time faster than the self-similar rate, i.e., (3) does not hold, and also blow up for the critical norm. Our second main result is the following.

Theorem 2. *Let $\Omega = \mathbb{R}^N$. Assume that $N \geq 11$ and $p > p^*$ (as in (10)). Then for any $T_m > 0$ there exists a solution u of (1) which blows up at $x = 0$, $t = T_m$ in such a way that*

$$\lim_{t\uparrow T_m} (T_m - t)^{\frac{1}{p-1}} u(t,0) = +\infty,$$

and

$$\lim_{t\uparrow T_m} \|u(t)\|_{L^q(P_R(t))} = +\infty,$$

for $q = \frac{N(p-1)}{2}$ *and all* $R > 0$, *where* $P_R(t)$ *denotes the space-time parabola prior to* $(T_m, 0) : P_R(t) = \{x \in \Omega : |x| \le R(T_m - t)^{1/2}\}$. *Moreover,*

$$\limsup_{t\uparrow T_m} \|u(t)\|_{L^q(B_R)} < +\infty,$$

for all $1 \le q < \frac{N(p-1)}{2}$ *and* $R > 0$.

In view of these results, we conjecture that the critical norm always blows up for nonnegative blow up solutions of (1).

On the other hand, for $\Omega \subset \mathbb{R}^N$, we show that the behavior of u in a suitable exterior domain does not influence the behavior of $\|u(t)\|_{L^q(\Omega)}$ as $t \uparrow T_m$, for any $q \ge 1$. We establish the following proposition.

Proposition 3. *Let* $\Omega = \mathbb{R}^N$. *Let* $u_0 \in L^\infty(\Omega) \cap H^1(\Omega)$ *be such that the solution* u *of* (1) *blows up in finite time* $t = T_m$. *Assume in addition that* (7) *holds for some constant* $M > 0$. *Then there exist a compact set* $K \subset \Omega$ *and* $M_1 > 0$ *such that*

$$\sup_{0\le t<T_m} \|u(t)\|_{L^q(\Omega\setminus K)} \le (\|u_0\|_{L^q(\Omega)} + \Gamma)e^{CT_m},$$

provided $u_0 \in L^q(\Omega)$, *for any* $q \ge 1$, *where* $C = {M_1}^{p-1}$ *and* $\Gamma > 0$ *is a constant which depends only on* N, T_m, K *and* M_1.

Finally, as an application of the results of Herrero and Velázquez in [15], [16] and of Velázquez in [28] concerning the possible blow up behaviors of (1) in the one-dimensional case, we present some blow up results concerning the subcritical norms. Whereas it is known that, for suitable initial data, the subcritical norms remain bounded at the blow up time (see [8, Theorem 2.4]), we prove that for blow up solutions of (1) exhibiting a certain asymptotic behavior, some subcritical norms actually blow up. More precisely, we prove the following result.

Theorem 4. *Assume* $N = 1$. *Let* $u_0 \in L^\infty(\Omega)$ *be continuous nonnegative and let* u *be the corresponding solution of* (1). *Assume that* u *blows up in finite time* T_m *at* $x = a$. *Then one of the following possibilities occurs*

(a) $\lim_{t\uparrow T_m} \|u(t)\|_{L^q(V)} = +\infty$, *for all* $\frac{p-1}{2} \le q \le +\infty$, $q \ge 1$. *Moreover,*

$$\liminf_{t\uparrow T_m} (T_m - t)^\delta |\log(T_m - t)|^{-\mu} \|u(t)\|_{L^q(V)} > 0,$$

with $\delta = \frac{1}{p-1} - \frac{1}{2q} \geq 0$ *and* $\mu = \frac{1}{2q}$,

(b) $\lim_{t \uparrow T_m} \|u(t)\|_{L^q(V)} = +\infty$, *for all* $\frac{p-1}{m} \leq q \leq +\infty$, $q \geq 1$, *and some even integer* $m \geq 4$. *Moreover,*

$$\liminf_{t \uparrow T_m} (T_m - t)^{\gamma} \|u(t)\|_{L^q(V)} > 0, \quad \text{with } \gamma = \frac{1}{p-1} - \frac{1}{mq} \geq 0,$$

where $V \subset \Omega$ *is any neighborhood of* $x = a$.

Note that in this last theorem condition (b) implies (a), however these conditions are consequence of different blow up behaviors of u around its blow up singularity $x = a$. For a more precise statement of this result, see Theorem 15 in Section 7.

In particular, we present an example of *blow up of subcritical norms*, in any subinterval of $\mathbb{R}$.

Theorem 5. *For any* $0 < R \leq +\infty$, *there exists an initial value* $u_0 \in L^{\infty}(\Omega) \cap H^1(\Omega)$, $u_0 \geq 0$, *such that the corresponding solution of* (1), *with* $\Omega = (-R, R)$, *blows up in finite time* T_m *at* $x = 0$ *and satisfies*

$$\lim_{t \uparrow T_m} \|u(t)\|_{L^q(\Omega)} = +\infty,$$

for all $\frac{p-1}{4} \leq q \leq +\infty$, $q \geq 1$.

In the case $\Omega = \mathbb{R}^N$, we are able to extend Theorem 4, under the general assumptions on the blow up rate (6) and (7) (see Section 7).

For the sake of completeness and clarity about the question of blow up of the critical norm, we next present a summary of the cases where

$$\lim_{t \uparrow T_m} \|u(t)\|_{L^{\frac{N(p-1)}{2}}(\Omega)} = +\infty, \tag{12}$$

or (4) holds. Note that the results which are deduced from the local existence theory (cf. [3], [4]) do not require the convexity of the domain Ω.

Let Ω be a smooth, bounded domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$.

• Assuming Ω bounded, then (12) holds if one of the following conditions is satisfied (cf. [3], [4], [34]).

- ∘ $\frac{N(p-1)}{2} = 2$, i.e., $p = 1 + \frac{4}{N}$.
- ∘ $\frac{N(p-1)}{2} > 1$ and $u_t \geq 0$.
- ∘ $N \geq 3$, $\frac{N(p-1)}{2} = 1$ and $u_t \geq 0$.

• Assuming Ω convex, $N = 1, 2$ or $N \geq 3$ and $p < \frac{N+2}{N-2}$, then (12) holds (cf. [10], [11], [12]).

Let now Ω be a ball in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. Assume that $u_0 \in L^\infty(\Omega)$ is radially symmetric.

• Assuming $u \geq 0$, $u_r \leq 0$ and (3), then (4) holds (Theorem 1).

• Assuming (7) and $x = 0$ is a blow up point of u, then (12) holds (Corollary 7, Section 3).

The plan of this paper is the following. In Section 2, we obtain some results concerning the parabolic problem obtained from (1) by the change of similarity variables, applying semigroup theory. In Section 3, we prove Theorem 1 and derive a corollary. Next, in Section 4, we present the proof of Theorem 2. In Section 5, we give an additional critical norm blow up result for equation (1) with a more general nonlinearity, whereas in Section 6 we prove Proposition 3. Finally, in Section 7, we obtain estimates on the L^q-rate of blow up and in particular we prove Theorems 4 and 5.

2. Preliminary results. In this section, we shall first perform a change of variables in equation (1) following [10]. Let u be a classical solution of (1), defined on a maximal time interval $[0, T_m)$. Then, $u \in C((0, T_m), L^\infty(\Omega)) \cap C^{1,2}((0, T_m) \times \overline{\Omega})$. Moreover, since $u_0 \in H^1(\Omega)$, $u \in C([0, T_m), H^1(\Omega))$. Assume that $T_m < +\infty$. Then, we say that (T_m, a) is a blow up point of u if there exist $t_n \uparrow T_m$ and $x_n \to a$, with $x_n \in \Omega$ for each $n \in \mathbb{N}$, such that $u(t_n, x_n) \to +\infty$ as $n \to \infty$.

Rescaling equation (1) by similarity variables around a blow up point (T_m, a),

$$s = \log\Big(\frac{T_m}{T_m - t}\Big),\ y = (T_m - t)^{-1/2}(x - a), \tag{13}$$

we obtain the associated problem

$$\begin{cases} w_s - \Delta w + \frac{1}{2} y \cdot \nabla w = |w|^{p-1} w - \frac{1}{p-1} w,\ (s, y) \in \mathcal{W}_a, \\ w(0, y) = w_0(y),\ a + T_m^{1/2} y \in \Omega, \end{cases} \tag{14}$$

with $w(s, y) = w_a(s, y)$ defined by

$$w_a(s, y) = (T_m - t)^{\frac{1}{p-1}} u(t, x), \tag{15}$$

on the space-time domain

$$\mathcal{W}_a = \{(s, y) : s > 0,\ a + T_m^{1/2} e^{-s/2} y \in \Omega\}, \tag{16}$$

and $w_0(y) = T_m{}^{\frac{1}{p-1}} u_0(a + T_m{}^{1/2} y)$. Note that, if $\Omega = \mathbb{R}^N$ then $\mathcal{W}_a = (0, +\infty) \times \mathbb{R}^N$.

If Ω is bounded, for each $s_1 > 0$, we denote by $\Omega_a(s_1)$ the set

$$\Omega_a(s_1) = \mathcal{W}_a \cap \{s = s_1\} = T_m^{-1/2} e^{s_1/2}(\Omega - a).$$

To simplify the notation, we shall suppress the subscript a. Then, the rescaled solution $w(s,y)$ satisfies the Dirichlet boundary condition

$$w = 0 \ \text{ on } \bigcup_{s>0} \partial\Omega(s) \times \{s\}. \tag{27}$$

Since the blow up time $t = T_m$ corresponds to $s = +\infty$, studying solutions of (1) near blow up is therefore equivalent to analyzing the large time asymptotic behavior of solutions of (14).

Next, we apply semigroup theory to derive some properties of the problem (14). For $1 \le q < +\infty$ and any positive integer $k \ge 1$, we define the weighted spaces

$$L^q_\rho(\mathbb{R}^N) = \{f \in L^q_{loc}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |f(x)|^q \rho(x)\, dx < +\infty\},$$
$$H^k_\rho(\mathbb{R}^N) = \{f \in L^2_w(\mathbb{R}^N) : D^\alpha f \in L^2_\rho(\mathbb{R}^N), \alpha = (\alpha_1, \dots, \alpha_N),\ |\alpha| \le k\},$$

with $\rho(x) = \exp(-\frac{|x|^2}{4})$. We use the standard notation

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_N} x_N}, \text{ where } \alpha = (\alpha_1, \dots, \alpha_N)$$

and $|\alpha| = \sum_{i=1}^N \alpha_i$. Norms in $L^q_\rho(\mathbb{R}^N)$ and $H^k_\rho(\mathbb{R}^N)$ are defined in a natural way. In particular,

$$\|f\|^q_{q,\rho} = \int_{\mathbb{R}^N} |f(x)|^q \rho(x)\, dx,\ f \in L^q_\rho(\mathbb{R}^N).$$

Clearly, for $k \ge 1$, $H^k_\rho(\mathbb{R}^N)$ can be given a structure of Hilbert space in a straightforward way. For example in $L^2_\rho(\mathbb{R}^N)$, we can define the inner product by

$$< f, g >_{2,\rho} = \int_{\mathbb{R}^N} f(x) g(x) \rho(x)\, dx,\ f, g \in L^2_\rho(\mathbb{R}^N).$$

Now, we define the operator A in the Hilbert space $L^2_\rho(\mathbb{R}^N)$ by

$$\begin{cases} D(A) = H^2_\rho(\mathbb{R}^N), \\ Au = \Delta u - \frac{1}{2}x \cdot \nabla u, \quad u \in D(A). \end{cases}$$

The operator $-A$ is accretive and self-adjoint with dense domain in $L^2_\rho(\mathbb{R}^N)$. Therefore, A is the generator of a semigroup of contractions on $L^2_\rho(\mathbb{R}^N)$ which we denote by $(S(s))_{s\geq 0}$.

Note that, for $u \in D(A)$,

$$Au = \frac{1}{\rho}\nabla \cdot (\rho \nabla u).$$

Let now w be a solution of the problem

$$\begin{cases} w_s - Aw \equiv w_s - \Delta w + \frac{1}{2}y \cdot \nabla w = 0 \text{ in } (0,+\infty) \times \mathbb{R}^N, \\ w(0,y) = w_0(y) \text{ in } \mathbb{R}^N, \end{cases} \tag{18}$$

for $w_0 \in L^2_\rho(\mathbb{R}^N)$. The rescaled function $v(t,x)$, defined by

$$v(t,x) = w(-\log(1-t), (1-t)^{-1/2}x),$$

solves

$$\begin{cases} v_t - \Delta v = 0, \quad (t,x) \in (0,1) \times \mathbb{R}^N, \\ v(0,x) = w_0(x), \quad x \in \mathbb{R}^N. \end{cases} \tag{19}$$

If $w_0 \in L^q(\mathbb{R}^N)$, for a certain $q \geq 1$, then the maximal interval of definition of v is in fact $(0,+\infty)$. It is well known that $v(t) = K(t) \star w_0$, where $K(t)$ is the heat equation kernel in $\mathbb{R}^N$, i.e., $K(t,x) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$. It follows that

$$w(s,y) = S(s)w_0(y) = \frac{1}{\big(4\pi(1-e^{-s})\big)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|e^{-\frac{s}{2}}y - x|^2}{4(1-e^{-s})}} w_0(x)\,dx.$$

Furthermore, we have the following properties

- $\|w(s)\|_{L^\infty(\mathbb{R}^N)} \leq \|w_0\|_{L^\infty(\mathbb{R}^N)}$, if $w_0 \in L^\infty(\mathbb{R}^N) \subset L^2_\rho(\mathbb{R}^N)$.
- $\|w(s)\|_{L^\infty(\mathbb{R}^N)} \leq (4\pi(1-e^{-s}))^{-\frac{N}{2}} \|w_0\|_{L^1(\mathbb{R}^N)}$, if $w_0 \in L^1(\mathbb{R}^N)$.
- $\|w(s)\|_{L^q(\mathbb{R}^N)} \leq e^{\frac{Ns}{2q}} \|w_0\|_{L^q(\mathbb{R}^N)}$, if $w_0 \in L^q(\mathbb{R}^N)$, for $q \geq 1$.

We shall need the following, to prove Theorem 1.

Lemma 6. *Assume that* $N \geq 3$ *and* $p > 1$. *Let* $M > 0$. *There exists* $\tau > 0$ *such that, given* $w_0 \in L^\infty(\mathbb{R}^N)$, *if*

$$\|w_0\|_{L^\infty(\mathbb{R}^N)} \leq M \tag{20}$$

and if w *is a nonnegative subsolution of the problem*

$$\begin{cases} w_s - Aw = |w|^{p-1}w \ \text{in } (0,+\infty)\times\mathbb{R}^N, \\ w(0,y) = w_0(y) \ \text{in } \mathbb{R}^N, \end{cases} \tag{21}$$

then

$$\|w(s)\|_{L^\infty(\mathbb{R}^N)} \leq 2M, \tag{22}$$

for every $0 \leq s \leq \tau$.

Proof. Let $s > 0$. Since w is a nonnegative subsolution of (21), we have

$$0 \leq w(s) \leq S(s)w_0 + \int_0^s S(s-\sigma)|w(\sigma)|^{p-1}w(\sigma)\,d\sigma.$$

Then, it follows from (20) that

$$\begin{aligned} \|w(s)\|_{L^\infty(\mathbb{R}^N)} &\leq \|w_0\|_{L^\infty(\mathbb{R}^N)} + \int_0^s \|w(\sigma)\|^p{}_{L^\infty(\mathbb{R}^N)}\,d\sigma \\ &\leq M + \big(\sup_{0\leq\sigma\leq s}\|w(\sigma)\|_{L^\infty(\mathbb{R}^N)}\big)^p s. \end{aligned}$$

Given $s > 0$, we set

$$M(s) = \sup_{0\leq\sigma\leq s}\|w(\sigma)\|_{L^\infty(\mathbb{R}^N)}.$$

Hence,

$$M(s) \leq M + M(s)^p s, \tag{23}$$

where M is a constant independent of $s \geq 0$.

Note that the function $s \mapsto M(s)$ is continuous and nondecreasing, and

$$M(0) = \|w_0\|_{L^\infty(\mathbb{R}^N)} \leq M.$$

If $\tau > 0$ is such that $M(\tau) = 2M$, it follows from (23) that $\tau \geq 2^{-p}M^{-(p-1)}$. Therefore, taking $\tau = 2^{-(p+1)}M^{-(p-1)}$, the result follows.

3. Proof of Theorem 1. We are now in a position to prove our first main theorem.

Proof of Theorem 1. Note that u is nonnegative, radially symmetric and nonincreasing as a function of $r = |x|$. Therefore, it blows up at the single point $x = 0$ (see [8] for the case of $\Omega = B_R(0)$ a ball in $\mathbb{R}^N$, Mueller and Weissler [25] for the case $\Omega = \mathbb{R}^N$). Then, we rescale u by similarity variables around $(T_m, 0)$ as in Section 2, and we denote by w the rescaled solution. Recall that w satisfies (14). We now proceed in four steps.
Step 1. We show that there exist $\tau > 0$, $s_n \uparrow +\infty$ and $C > 0$ such that

$$\|w(s_n + s)\|_{L^\infty(\Omega(s_n+s))} \le C,$$

for every $s \in [0, \tau]$ and $n \in \mathbb{N}$.

By hypothesis (3), there exist $t_n \uparrow T_m$ and $M > 0$ such that (5) holds. Defining

$$s_n = \log\big(\frac{T_m}{T_m - t_n}\big),$$

we have $s_n \uparrow +\infty$ and

$$\|w(s_n)\|_{L^\infty(\Omega(s_n))} \le M.$$

For each $s > 0$, the Dirichlet boundary condition (17) implies that $w(s)$ can be continuously extended by zero to $\mathbb{R}^N$. We still denote this extension by $w(s)$.

For each $n \in \mathbb{N}$, let W_n be the solution of the problem

$$\begin{cases} \frac{\partial W_n}{\partial s} - \Delta W_n + \frac{1}{2} y \cdot \nabla W_n = {w_n}^p \text{ in } (0, +\infty) \times \mathbb{R}^N, \\ W_n(0, y) = w_n(0, y) \text{ in } \mathbb{R}^N, \end{cases} \tag{24}$$

where $w_n(s, y) = w(s_n + s, y)$. The comparison principle holds in $(0, +\infty) \times \mathbb{R}^N$, then $w_n(s) \le W_n(s)$, for every $s > 0$ and $n \in \mathbb{N}$. Hence, $\frac{\partial W_n}{\partial s} - AW_n \le {W_n}^p$ in $(0, +\infty) \times \mathbb{R}^N$ (i.e., W_n is a subsolution of (21)) and $\|W_n(0)\|_{L^\infty(\mathbb{R}^N)} \le M$, for every $n \in \mathbb{N}$.

It follows from Lemma 6 that there exists $\tau > 0$ (independent of n) such that

$$\|w_n(s)\|_{L^\infty(\Omega_n(s))} \le \|W_n(s)\|_{L^\infty(\mathbb{R}^N)} \le 2M,$$

with $\Omega_n(s) = \Omega(s_n + s)$, for every $0 \le s \le \tau$ and $n \in \mathbb{N}$.

Step 2. We show that $w_n \to \overline{w}$ uniformly on each compact set of $(0,\tau]\times\mathbb{R}^N$, for some $\overline{w} \in L^\infty(\mathbb{R}^N)$ which solves

$$-\Delta\overline{w} + \frac{1}{2}y\cdot\nabla\overline{w} + \frac{1}{p-1}\overline{w} - \overline{w}^p = 0, \quad y \in \mathbb{R}^N. \tag{25}$$

We can assume without loss of generality that s_n is increasing and such that $s_{n+1} - s_n \to +\infty$. For each $R > \frac{1}{\tau}$, let $\mathcal{Q}(R)$ be the cylinder defined by

$$\mathcal{Q}(R) = \{(s,y):\ \frac{1}{R} < s \le \tau,\ |y| \le R\}.$$

By Step 1, there exists $M > 0$ such that $w_n(s,y) \le M$ in $\Omega_n(s)$, for any $s > 0$ and $n \in \mathbb{N}$. It follows from L^q regularity theory for parabolic equations (see Ladyzenskaja, Solonnikov and Uralceva [21], Chapter 4) that (∇w_n), $(\nabla^2 w_n)$ and $(\partial w_n/\partial s)$ are bounded sequences in $L^q(\mathcal{Q}(R))$, for any $1 \le q < +\infty$ and $R > \frac{1}{\tau}$, uniformly with respect to $n \in \mathbb{N}$. Therefore, by Sobolev inequalities, w_n is Hölder continuous for each $n \in \mathbb{N}$. Then, applying Schauder estimates (see Friedman [7], Chapter 3), we conclude that $(\nabla^2 w_n)$ and $(\partial w_n/\partial s)$ are Hölder continuous on each $\mathcal{Q}(R)$, uniformly with respect to n.

Finally, by Arzela-Ascoli theorem and applying a diagonal argument, there exists a subsequence, still denoted by (w_n), and a function $\overline{w}$, such that w_n converges to $\overline{w}$ uniformly on each $\mathcal{Q}(R)$. This function $\overline{w}$ is defined and uniformly bounded (by M) in $(0,\tau]\times\mathbb{R}^N$, and it satisfies

$$\overline{w}_s - \Delta\overline{w} + \frac{1}{2}y\cdot\nabla\overline{w} + \frac{1}{p-1}\overline{w} - |\overline{w}|^{p-1}\overline{w} = 0,\ y \in \mathbb{R}^N,\ 0 < s < \tau.$$

We may assume, taking yet another subsequence if necessary, that $(\partial w_n/\partial s)$ converges weakly to $\overline{w}_s$ in $L^2(\mathcal{Q}(R))$, for each $R > \frac{1}{\tau}$.

We denote by $E[w](s)$ the energy associated to problem (14) at a time s,

$$E[w](s) = \int_{\Omega(s)} \big(\frac{|\nabla w(s)|^2}{2} + \frac{|w(s)|^2}{2(p-1)} - \frac{|w(s)|^{p+1}}{p+1}\big)\rho\,dy, \tag{26}$$

where $\rho(y) = \exp(-\frac{|y|^2}{4})$. The rescaled function $w(s,y)$ satisfies, by Proposition 2.1 of Giga and Kohn [11],

$$\int_{\Omega(s)} |w_s|^2\rho\,dy = -\frac{d}{ds}E[w](s) - \frac{1}{4}\int_{\partial\Omega(s)} (y\cdot\nu)\big|\frac{\partial w}{\partial\nu}\big|^2\rho\,d\sigma, \tag{27}$$

where ν is the exterior unit normal vector to $\partial\Omega(s)$ and $d\sigma$ is the surface area element. Then, the energy $E[w]$ is nonincreasing. Moreover, the global existence of w keeps $E[w](s) \geq 0$, for every $s \geq 0$. Therefore, there exists $E_\infty \geq 0$ such that $E[w](s) \downarrow E_\infty$ when $s \to +\infty$.

From (27), with respect to w_n, we have

$$\int_0^\tau \int_{\Omega_n(s)} |w_{ns}|^2 \rho \, dy \, ds \leq E[w_n](0) - E[w_n](\tau) = E[w](s_n) - E[w](s_n + \tau).$$

It follows that

$$\lim_n \int_0^\tau \int_{\Omega_n(s)} |w_{ns}|^2 \rho \, dy \, ds = 0. \tag{28}$$

Since ρ decreases exponentially as $|y| \to \infty$, the integral in (28) is lower semi-continuous, and since $(\partial w_n / \partial s)$ converges weakly to $\overline{w}_s$, we conclude that $\overline{w}$ is independent of s. This implies that $\overline{w}$ is a stationary solution of (14). Hence $\overline{w} \in L^\infty(\mathbb{R}^N)$ is a solution of (25).

Step 3. We show that $\overline{w}$ is not identically zero in $\mathbb{R}^N$.

Since u is a nonnegative, radially symmetric and nonincreasing as a function of $r = |x|$, there exists $c > 0$ such that

$$u(t, 0) \geq c(T_m - t)^{-\frac{1}{p-1}}, \tag{29}$$

for all $t \in [0, T_m)$ (cf. [8]). Therefore,

$$w_n(\tau, 0) = w(s_n + \tau, 0) = (T_m - \tilde{t}_n)^{\frac{1}{p-1}} u(\tilde{t}_n, 0) \geq c,$$

for each $n \in \mathbb{N}$, where $\tilde{t}_n = e^{-\tau} t_n + T_m(1 - e^{-\tau})$ (see (15)). Hence, by uniform convergence, $\overline{w}(0) \geq c > 0$. It is easy to show that we can take $c = k \equiv (p-1)^{-\frac{1}{p-1}}$.

Step 4. We show that there is $t_{n'} \uparrow T_m$ (as in Step 1) such that

$$\lim_{n'} \|u(t_{n'})\|_{L^q(\Omega)} = +\infty.$$

Let $\overline{w}$ be given by Step 2. Then, $\overline{w}$ is a positive bounded solution of (25). Since it is a radial solution, i.e., $\overline{w}(y) = \overline{w}(r)$ for $r = |y|$, it solves

$$\begin{cases} \overline{w}_{rr} + \left(\frac{N-1}{r} - \frac{r}{2}\right)\overline{w}_r + \overline{w}^p - \frac{1}{p-1}\overline{w} = 0, \text{ in } (0, +\infty), \\ \overline{w}(0) = \alpha, \ \ \overline{w}_r(0) = 0, \end{cases} \tag{30}$$

with $\alpha \geq k$ (by Step 3).

If $p = \frac{N+2}{N-2}$ then $\overline{w} \equiv k$ (cf. [10]).

If $p > \frac{N+2}{N-2}$, it follows from a result of Budd and Qi [5, Theorem 3], that there exists $C > 0$ such that for all $r \geq 1$,

$$\overline{w}(r) \geq Cr^{-2(p-1)}. \tag{31}$$

Therefore, in both cases, $\overline{w}$ does not belong to $L^q(\mathbb{R}^N)$, for $q = \frac{N(p-1)}{2}$. For this value of q, we have

$$\|u(t)\|_{L^q(\Omega)} = \|w(s)\|_{L^q(\Omega(s))}, \tag{32}$$

where s is defined in (13) and $t \in [0, T_m)$.

Suppose that $p > \frac{N+2}{N-2}$. Let $R > 1$ and $0 < \lambda < C$. Take $0 < \varepsilon \leq (C-\lambda)R^{-\frac{2}{p-1}}$. Let $s_n \uparrow +\infty$ and $\tau > 0$ be given by Steps 1 and 2. Since $\lim_n w(s_n + \tau, y) = \overline{w}(y)$ uniformly on the compact sets of $\mathbb{R}^N$, there is $n_{R,\varepsilon} \in \mathbb{N}$ such that

$$w(s_n + \tau, y) \geq \overline{w}(y) - \varepsilon \geq \lambda |y|^{-\frac{2}{p-1}},$$

for every $n \geq n_{R,\varepsilon}$ and $1 \leq |y| \leq R$. For each $n \in \mathbb{N}$, denote $\tilde{s}_n = s_n + \tau$ and let $\tilde{t}_n \in (0, T_m)$ being given by $\tilde{t}_n = T_m(1 - e^{-\tilde{s}_n}) = e^{-\tau} t_n + T_m(1 - e^{-\tau})$ (see (13)). Without loss of generality, we can assume that $B_R(0) \subset \Omega(\tilde{s}_n)$, for every $n \geq n_{R,\varepsilon}$. It follows from (32) that

$$\begin{aligned} \|u(\tilde{t}_n)\|^q{}_{L^q(\Omega)} &\geq \int_{1 \leq |y| \leq R} |w(\tilde{s}_n)|^q \, dy \geq \lambda^q \int_{1 \leq |y| \leq R} |y|^{-\frac{2q}{p-1}} \, dy \\ &= \lambda^q \int_{1 \leq |y| \leq R} |y|^{-N} dy, \end{aligned}$$

for every $n \geq n_{R,\epsilon}$. Then,

$$\liminf_n \|u(\tilde{t}_n)\|^q{}_{L^q(\Omega)} \geq \lambda^q \int_{1 \leq |y| \leq R} |y|^{-N} \, dy.$$

Since $R > 1$ is arbitrary and λ is independent of R, we conclude that $\|u(\tilde{t}_n)\|_{L^q(\Omega)} \to +\infty$ when $n \to +\infty$. The argument is even simpler if $p = \frac{N+2}{N-2}$.

Remark 2. The hypotheses on u_0 are to ensure that $x = 0$ is a blow up point in a way that when we rescale u by similarity variables (around $(T_m, 0)$) we preserve the radially symmetry of the solutions.

Remark 3. The hypothesis of radial symmetry seems to be essential in our argument, since only in the radial case we can describe the asymptotic behavior (in y) of any bounded solution of (25).

Now recall the following criterion for excluding blow up which was obtained by Giga and Kohn in [12], [13].

- Let u be a solution of (1) which blows up in finite time T_m. Assume that (7) holds. Then,

$$\lim_{t\uparrow T_m} (T_m - t)^{\frac{1}{p-1}} u(t, (T_m - t)^{1/2} y) = 0$$

uniformly on the compact sets $|y| \le C$ $(C > 0)$, if and only if $x = 0$ is not a blow up point of u.

Putting together this result and the argument of Theorem 1, we obtain the following corollary.

Corollary 7. *Let Ω be a ball in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. Assume that $N \ge 3$ and $p \ge \frac{N+2}{N-2}$. Let $u_0 \in L^\infty(\Omega) \cap H^1(\Omega)$ be radially symmetric such that the solution u of* (1) *blows up in finite time $t = T_m$ at $x = 0$. Assume that* (7) *holds. Then, for $q = \frac{N(p-1)}{2}$,*

$$\lim_{t\uparrow T_m} \|u(t)\|_{L^q(\Omega)} = +\infty.$$

Proof. Let $s_n \to +\infty$. Since (7) holds, we prove arguing as in the proof of Theorem 1 that there exists a subsequence, which we still denote by (s_n), such that $w_n \to \overline{w}$ uniformly on each compact set of $\mathbb{R} \times \mathbb{R}^N$, for some $\overline{w} \in L^\infty(\mathbb{R}^N)$ which solves (25), where $w_n(s, y) = w(s_n + s, y)$ (see [10]). Moreover, we may assume that, for each integer $m \in \mathbb{N}$, $\nabla w_n(m, y) \to \nabla \overline{w}(y)$ a.e. in $y \in \mathbb{R}^N$.

We claim that $E[w_n](m) \to E[\overline{w}]$, as $m \to +\infty$, where

$$E[\overline{w}] = \int_{\mathbb{R}^N} \Big(\frac{|\nabla \overline{w}|^2}{2} + \frac{|\overline{w}|^2}{2(p-1)} - \frac{|\overline{w}|^{p+1}}{p+1}\Big) \rho \, dy,$$

and ρ is defined as in (26). Since

$$\|w_n(s)\|_{W^{1,\infty}(\Omega_n(s))} \le C_1,$$

for a certain $C_1 > 0$ (see Proposition 1' in [10]), and ρ decreases exponentially as $|y| \to \infty$, there exists $C > 0$ such that

$$\int_{\Omega_n(s)\cap\{|y|>C\}} \Big(\frac{|\nabla w_n(s)|^2}{2} + \frac{|w_n(s)|^2}{2(p-1)} - \frac{|w_n(s)|^{p+1}}{p+1}\Big)\rho\, dy \le \frac{\varepsilon}{3},$$

for every $s > 0$ and $n \in \mathbb{N}$. We may assume without loss of generality that $B_C(0) \subset \Omega_n(s)$, for every $n \in \mathbb{N}$ and $s > 0$. On the other hand, since $E[\overline{w}] < +\infty$ (this follows from the fact that $\overline{w} \in W^{1,\infty}(\mathbb{R}^N)$), we can also assume $C > 0$ being such that

$$\Big|\int_{\{|y|>C\}} \Big(\frac{|\nabla \overline{w}|^2}{2} + \frac{|\overline{w}|^2}{2(p-1)} - \frac{|\overline{w}|^{p+1}}{p+1}\Big)\rho\, dy\Big| \le \frac{\varepsilon}{3}.$$

Finally, using the fact that $w_n(m) \to \overline{w}$ uniformly on $B_C(0)$ and $\nabla w_n(m) \to \nabla\overline{w}(m)$ a.e. in B_C and by the dominated convergence theorem, there exists $n_0 \in N$ such that $0 \le E[w_n](m) - E[\overline{w}] \le \varepsilon$, for all $n \ge n_0$.

Using the notation of Step 2 of the proof of Theorem 1, $E[w](s) \downarrow E_\infty$ when $s \uparrow +\infty$, for some $E_\infty \ge 0$. Moreover, we have $E_\infty = E[\overline{w}]$. Therefore the limit of $E[w_n](s)$ as $n \to +\infty$ is independent of the sequence $s_n \to +\infty$. Since $\overline{w}$ is a solution of (25), it is easily seen that $E[\overline{w}] = 0$ if and only if $\overline{w} \equiv 0$. It follows from the criterion for excluding blow up mentioned above (cf. [12], [13]) that there exists $s_n \to +\infty$ such that $\overline{w} \not\equiv 0$, and then $E_\infty > 0$.

Note that, if $\overline{w}$ is a nontrivial bounded solution of (25) then $\overline{w}$ is either positive or negative for every $r > 0$ sufficiently large. This follows from the argument of Theorem 3 in [5]. Moreover, there exists $C > 0$ such that

$$|\overline{w}(r)| \ge C r^{-\frac{2}{p-1}},$$

for all $r \ge 1$. Finally, it follows that $\lim_{t\uparrow T_m} \|u(t)\|_{L^{\frac{N(p-1)}{2}}(\Omega)} = +\infty$ as in Step 4 of Theorem 1.

Remark 4. For $p > \frac{N+2}{N-2}$, the multiple existence of solutions of (9) complicates the analysis required to precisely describe the blow up behavior of solutions of (1) around its blow up singularities. However, a result in this direction was proved recently by the author in [23]. More precisely, it was proved that there exists a class of positive radially symmetric solutions of (1) which blow up in finite time $t = T_m$ at $x = 0$, such that (7) holds, and such that the limit in (6) exists and is a function φ, where either $\varphi \equiv k$ or φ admits the following asymptotic expansion

$$\varphi(y) = C|y|^{-\frac{2}{p-1}}\big\{1 - C_1\frac{1}{|y|^2} - C_2\frac{1}{|y|^4} + o\big(\frac{1}{|y|^4}\big)\big\}, \text{ as } |y| \to +\infty, \quad (33)$$

with $C > 0$, and C_1, C_2 constants which depend only on C, N and p (see [23, Theorem 1]).

4. Blow up of the critical norm without growth assumptions. In this section we shall consider problem (1) with $\Omega = \mathbb{R}^N$.

Assume that

$$N \geq 11 \text{ and } p > p^* = \frac{N - 2(N-1)^{1/2}}{N - 4 - 2(N-1)^{1/2}}. \quad (34)$$

In this case, Herrero and Velázquez in [18] (see [19] for more details) produced a blow up mechanism where the a priori bound (7) is not satisfied. More precisely, they proved that there exist positive and radial solutions of (1) which blow up at $t = T_m$ and $x = 0$, and are such that

$$\lim_{t \uparrow T_m} (T_m - t)^{\frac{1}{p-1}} u(t, 0) = +\infty. \quad (35)$$

Note then that the growth assumption (3) does not hold. We shall prove that the $L^{\frac{N(p-1)}{2}}(\mathbb{R}^N)$ norm of these solutions blows up as $t \uparrow T_m$.

To this end, we first rescale equation (1) by similarity variables around $(T_m, 0)$ as in (13) and (15). Recall that

$$\phi(y) = c_{p,N}|y|^{-\frac{2}{p-1}}; \;\; c_{p,N}^{p-1} = \frac{2}{p-1}\big((N-2) - \frac{2}{p-1}\big), \quad (36)$$

is a singular solution of (14) if $p > \frac{N}{N-2}$; it actually belongs to H^1_{loc} near $y = 0$ when $p > \frac{N+2}{N-2}$. To obtain (35), we first linearize around ϕ in (36) by setting in (14)

$$w(s, y) = c_{p,N}|y|^{-\frac{2}{p-1}} + \psi(s, y).$$

Then $\psi(s, y)$ satisfies formally the following equation

$$\begin{aligned}\psi_s =& \Delta\psi - \frac{1}{2}y \cdot \nabla\psi + p\frac{c_{p,N}^{p-1}}{|y|^2}\psi - \frac{\psi}{p-1} + \\ &+ \big(\big(c_{p,N}|y|^{-\frac{2}{p-1}} + \psi\big)^p - c_{p,N}^p|y|^{-\frac{2}{p-1}} - p\frac{c_{p,N}^{p-1}}{|y|^2}\psi\big) \equiv A\psi + f(\psi).\end{aligned}$$

In the radial case, it can be shown that the corresponding Friedrichs extension of the linear operator

$$-A\psi = \Delta\psi - \frac{1}{2}y\cdot\nabla\psi + p\frac{c_{p,N}^{p-1}}{|y|^2}\psi - \frac{\psi}{p-1},$$

can be represented by the differential operator

$$-\tilde{A}\psi = \psi'' + \big(\frac{N-1}{r} - \frac{r}{2}\big)\psi' + \big(p\frac{c_{p,N}^{p-1}}{r^2} - \frac{1}{p-1}\big)\psi$$

in a suitable functional space (cf. [19]). Recall that the eigenvalues (λ_j) and the eigenfunctions (φ_j) of $\tilde{A}$ are given by

$$\lambda_j = \frac{1}{p-1} + \frac{\alpha_+}{2} + j; \quad j = 0, 1, 2, \dots , \tag{37}$$

where

$$\alpha_+ = \frac{1}{2}\big(- (N-2) + \big(\beta_{p,N}^2 - 4(p-1)c_{p,N}^{p-1}\big)^{1/2}\big) \tag{38}$$

with $\beta_{p,N} = (N-2) - \frac{4}{p-1}$,

$$\varphi_j(r) = c_j r^{\alpha_+} M\big(- \lambda_j + \frac{1}{p-1} + \frac{\alpha_+}{2}, \alpha_+ + \frac{N}{2}; \frac{r^2}{4}\big), \tag{39}$$

where $M(a,b;x)$ is the standard Kummer function (cf. for instance the book of Galindo and Pascual [9]), and c_j is a suitable normalization constant. The proof of (35) then proceeds by showing that one can match near $x = 0$ scaled stationary solutions of (1) with suitable eigenfunctions (39). The behavior obtained is characterized by the appearance of a stationary solution of (1) in a small boundary layer near the origin.

Let l be an integer for which λ_l in (37) is positive, and define positive numbers η and λ as follows

$$\eta = \lambda_l\big(|\alpha_+| - \frac{2}{p-1}\big)^{-1}, \quad \lambda = \frac{2\eta}{p-1}. \tag{40}$$

The following theorem was proved in [19].

Theorem 8. ([19]) *Let* $\Omega = \mathbb{R}^N$. *Assume that* (34) *holds and let* v *be any radial, nonnegative and bounded solution of* $\Delta v + v^p = 0$ *in* $\mathbb{R}^N$. *Then for any* $T_m > 0$ *there exist* $C > 0$ *and a solution* u *of* (1) *which blows up at* $x = 0$, $t = T_m$ *in such a way that* (35) *holds and*

$$|w(s,y) - e^{\lambda s} v(e^{\eta s}|y|)| = o(e^{\lambda s}) \text{ as } s \to +\infty, \tag{41}$$

uniformly on bounded sets of $e^{\eta s}|y|$,

$$|w(s,|y|) - c_{p,N}|y|^{-\frac{2}{p-1}} + Ce^{-\lambda_l s}\varphi_l(|y|)| = o(e^{-\lambda_l s}) \text{ as } s \to +\infty, \tag{42}$$

uniformly on sets $\varepsilon \leq |y| \leq 1/\varepsilon$, *with* $\varepsilon \in (0,1)$.

Instead of Theorem 2 we prove a more precise result.

Proposition 9. *Let* $\Omega = \mathbb{R}^N$. *Assume that* (34) *holds and let* v *be any radial, nonnegative and bounded solution of* $\Delta v + v^p = 0$ *in* $\mathbb{R}^N$. *Let* $T_m > 0$ *and let* u *be a solution of* (1) *which blows up at* $x = 0$, $t = T_m$ *in such a way that* (35), (41) *and* (42) *hold. Then,*

$$\lim_{t \uparrow T_m} \|u(t)\|_{L^q(P_R(t))} = +\infty,$$

for $q = \frac{N(p-1)}{2}$ *and all* $R > 0$, *where* $P_R(t)$ *denotes the space-time parabola prior to* $(T_m, 0)$ *defined in Theorem 2, and*

$$\limsup_{t \uparrow T_m} \|u(t)\|_{L^q(B_R)} < +\infty,$$

for all $1 \leq q < \frac{N(p-1)}{2}$ *and all* $R > 0$.

Proof. Let $R > 0$ and let u be a solution given by Theorem 8 with v as the associated radial, nonnegative and bounded solution of $\Delta v + v^p = 0$ in $\mathbb{R}^N$.

Let $q = \frac{N(p-1)}{2}$. We have,

$$\|u(t)\|^q_{L^q(P_R(t))} = \|w(s)\|^q_{L^q(B_R)}. \tag{43}$$

It follows from (41) that

$$w(s,y) \geq e^{\lambda s}(v(e^{\eta s}|y|) - o(1)),$$

where $o(1) = o(1)(s,y) \to 0$ as $s \to +\infty$, uniformly on bounded sets of $e^{\eta s}|y|$. Therefore, using the rescaled variable $z = e^{\eta s} y$,

$$\int_{B_R} |w(s,y)|^q \, dy \geq \int_{B_R} e^{\lambda q s} |(v(e^{\eta s}|y|) - o(1))^+|^q \, dy$$
$$= \int_{\{|z| \leq e^{\eta s} R\}} |(v(|z|) - \tilde{o}(1))^+|^q \, dz,$$

where $\tilde{o}(1) = \tilde{o}(1)(s,z) = o(1)(s,y) \to 0$ as $s \to +\infty$, uniformly on $|z| \leq K$, with $K > 0$. Note that $\lambda q - \eta N = 0$ by (40).

Since p is supercritical, it follows from classical results (cf. Joseph and Lundgren [20]) that there exists $C_1 > 0$ such that

$$v(|z|) \geq C_1 |z|^{-\frac{2}{p-1}}, \text{ for } |z| \geq 1.$$

Let $\Gamma > 1$. Take $C_2 \in (0, C_1)$ and $0 < \varepsilon \leq \frac{C_1 - C_2}{\Gamma^{\frac{2}{p-1}}}$ (then $\Gamma \leq \left(\frac{C_1 - C_2}{\varepsilon}\right)^{\frac{p-1}{2}}$). Note that $\varepsilon = \varepsilon(\Gamma) \to 0$ as $\Gamma \to +\infty$. Let $s_\varepsilon > 0$ be such that $\Gamma \leq e^{\eta s} R$ and $|\tilde{o}(1)| \leq \varepsilon$, for every $s \geq s_\varepsilon$ and $|z| \leq \Gamma$. Therefore,

$$\int_{\{|z| \leq e^{\eta s} R\}} |(v(|z|) - \tilde{o}(1))^+|^q \, dz \geq \int_{\{1 \leq |z| \leq \Gamma\}} |(v(|z|) - \tilde{o}(1))^+|^q \, dz$$
$$\geq C_2^q \int_{\{1 \leq |z| \leq \Gamma\}} |z|^{-\frac{2q}{p-1}} \, dz = C_2^q \int_{\{1 \leq |z| \leq \Gamma\}} |z|^{-N} \, dz,$$

for every $s \geq s_\varepsilon$. Finally, using (43), we have

$$\liminf_{t \uparrow T_m} \|u(t)\|^q_{L^q(P_R(t))} = \liminf_{s \uparrow +\infty} \|w(s)\|^q_{L^q(B_R)} \geq C_2^q \int_{\{1 \leq |z| \leq \Gamma\}} |z|^{-N} \, dz,$$

for every $\Gamma > 1$. Thus, $\|u(t)\|_{L^q(P_R(t))} \to +\infty$ when $t \uparrow T_m$.

Next, let $1 \leq q < \frac{N(p-1)}{2}$. Note that in this case, we shall need more information about the asymptotic behavior of $w(s,y)$ than the one given by (41) and (42). Recalling the functional framework where the topological argument was applied in [19] (see Section 3 in that paper), we conclude that there exists $s_0 > 0$ such that w satisfies the following estimates

(a) $w(s,y) \leq c_{p,N} |y|^{-\frac{2}{p-1}}$ for $|y| \leq 1$ and $s \geq s_0$.

(b) $w(s,y) \leq c_{p,N}|y|^{-\frac{2}{p-1}} + |y|^{2\lambda_l} e^{-\lambda_l s}$ for $1 \leq |y| \leq e^{\sigma s}$ and $s \geq s_0$, where

$$\lambda_l\big(2\lambda_l + \frac{2p}{p-1}\big)^{-1} < \sigma < \lambda_l\big(2\lambda_l + \frac{2}{p-1}\big)^{-1}.$$

(c) $w(s,y) \leq (1+c_{p,N})|y|^{-\frac{2}{p-1}}$ for $|y| \geq e^{\sigma s}$ and $s \geq s_0$.

It follows from the choice of σ (cf. [19] Sections 4–5]) that there exists a constant $K > 0$ such that

$$|w(s,y)| \leq K|y|^{-\frac{2}{p-1}},$$

for every $y \in \mathbb{R}^N$ and $s \geq s_0$. Therefore, for every $t \geq t_0 = T_m(1-e^{-s_0})$,

$$\begin{aligned}\|u(t)\|^q_{L^q(B_R)} &= (T_m - t)^{\frac{N}{2}-\frac{q}{p-1}} \int_{\{|y| \leq T_m{}^{-1/2} e^{s/2} R\}} |w(s,y)|^q \, dy \\ &\leq C_3 e^{-(\frac{N}{2}-\frac{q}{p-1})s} \int_{\{|y| \leq T_m{}^{-1/2} e^{s/2} R\}} |y|^{-\frac{2q}{p-1}} \, dy \\ &\leq C_3 \omega_N T_m{}^{-(\frac{N}{2}-\frac{q}{p-1})} R^{N-\frac{2q}{p-1}} \int_0^1 r^{-\frac{2q}{p-1}+N-1} \, dr = C_4,\end{aligned}$$

for some constants $C_3, C_4 > 0$ (since q is subcritical), and where ω_N is the volume of the unit ball in $\mathbb{R}^N$. Hence the result follows.

Remark 5. In particular, we conclude that for such solutions of (1) the critical norm blows up and that the subcritical norms remain bounded at the blow up time, in every bounded neighborhood of $x = 0$. Moreover, the estimates obtained in [19] allow us to conclude that there exists a constant $\kappa > 0$ such that

$$|w(s,y)| \geq \kappa |y|^{-\frac{2}{p-1}},$$

provided that $s \geq s_0 \gg 1$ and $|y| \geq e^{\sigma s}$, where σ is chosen as in (b). Thus, $u(t) \notin L^q(\mathbb{R}^N)$ for every $t \geq 0$ and every $1 \leq q < \frac{N(p-1)}{2}$ (see Proposition 3).

5. Blow up of the critical norm: further results. We shall now consider problem (1) with a more general nonlinearity. More precisely, we consider the nonlinear heat equation

$$\begin{cases} u_t - \Delta u = g(u), \quad x \in \Omega, \quad t \in [0,T], \\ u(t,x) = 0, \quad x \in \partial\Omega, \quad t \in [0,T], \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{44}$$

where $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and Ω is a smooth, bounded domain in $\mathbb{R}^N$.

We obtain the following critical norm blow up result, using regularity properties of the linear heat semigroup.

Theorem 10. *Assume that $N \geq 1$ and that*

$$|g(u)| \leq C(1 + |u|^p), \tag{45}$$

with $p > 1 + \frac{2}{N}$. Let $u_0 \in L^\infty(\Omega) \cap H^2(\Omega) \cap H_0^1(\Omega)$ be such that the solution u of (44) blows up in finite time. If $\Delta u_0 + g(u_0) \geq 0$ in Ω, then

$$\lim_{t \uparrow T_m} \|u(t)\|_{L^q(\Omega)} = +\infty,$$

for $q = \frac{N(p-1)}{2}$.

Remark 6. This result was already known under more restrictive assumptions on g (for example $g(u) = |u|^{p-1}u$, cf. [3, Chapter 3]).

Remark 7. In the "doubly critical" case $q = \frac{N(p-1)}{2}$ and $q = 1$, i.e., $p = 1 + \frac{2}{N}$, this result still holds when $N \geq 3$ (see [3, Theorem 3.8.3]). Therefore the remaining two open cases are $N = 1$ and $p = 3$, $N = 2$ and $p = 2$.

Proof. Let $u_0 \in L^\infty(\Omega) \cap H^2(\Omega) \cap H_0^1(\Omega)$ and let u be the solution of (40) defined on the maximal interval of time $[0, T_m)$. It follows from the assumptions on u_0 that $u_t(t, x) \geq 0$, for all $t \in [0, T_m)$ and all $x \in \Omega$. We now proceed in two steps.

Step 1. We show that $\limsup_{t \uparrow T_m} \|u(t)\|_{L^q} = +\infty$. Assume by contradiction that $\limsup_{t \uparrow T_m} \|u(t)\|_{L^q} < +\infty$. Since $u(t, x)$ is a nondecreasing function on t for every $x \in \Omega$, it follows that $(u(t))_{t \in [0, T_m)}$ is contained in a compact set $K \subset L^q(\Omega)$.

Given $0 < t < T_m$, we have

$$u(t + s) = T(s)u(t) + \int_0^s T(s - \sigma) g(u(t + \sigma))\, d\sigma,$$

for $0 \leq s < T_m - t$, where $(T(t))_{t \geq 0}$ denotes the linear heat semigroup with Dirichlet boundary condition on Ω.

Fix any $r \in (q, pq)$, $r \geq p$. We have, using the smoothing effect of $T(t) : L^{\frac{r}{p}}(\Omega) \to L^r(\Omega)$ and (45),

$$\begin{aligned}
\|u(t+s)\|_{L^r} &\leq \|T(s)u(t)\|_{L^r} + \int_0^s \|T(s-\sigma)g(u(t+\sigma))\|_{L^r}\, d\sigma \\
&\leq \|T(s)u(t)\|_{L^r} + \int_0^s (s-\sigma)^{-\frac{N(p-1)}{2r}} \|g(u(t+\sigma))\|_{L^{r/p}}\, d\sigma \\
&\leq \|T(s)u(t)\|_{L^r} + C_1 \int_0^s (s-\sigma)^{-\frac{N(p-1)}{2r}} (1 + \|u(t+\sigma)\|_{L^r}^p)\, d\sigma \\
&\leq \|T(s)u(t)\|_{L^r} + C_2 s^{1-\frac{N(p-1)}{2r}} + C_1 \int_0^s (s-\sigma)^{-\frac{N(p-1)}{2r}} \|u(t+\sigma)\|_{L^r}^p\, d\sigma,
\end{aligned}$$

for some constants $C_1, C_2 > 0$ which depend only on r, $|\Omega|$ and on the constant in (45), and with $\frac{N(p-1)}{2r} < 1$.

With $\alpha = \frac{N}{2}\left(\frac{1}{q} - \frac{1}{r}\right) < \frac{1}{p} < 1$, we then have

$$s^\alpha \|u(t+s)\|_{L^r} \leq s^\alpha \|T(s)u(t)\|_{L^r} + C_2 s^{p\alpha} + C_3 \big(\sup_{0 \leq \sigma \leq s} \sigma^\alpha \|u(t+\sigma)\|_{L^r} \big)^p,$$

with $C_3 = C_1 s^\alpha \int_0^s (s-\sigma)^{-\frac{N(p-1)}{2r}} \sigma^{-p\alpha}\, d\sigma < +\infty$, since $p\alpha + \frac{N(p-1)}{2r} = \alpha + 1$.

Given $0 < t < T_m$, we set

$$M(s) = \sup_{0 \leq \sigma \leq s} \sigma^\alpha \|u(t+\sigma)\|_{L^r},$$

for $0 \leq s < T_m - t$. Hence,

$$M(s) \leq s^\alpha \|T(s)u(t)\|_{L^r} + C_2 s^{p\alpha} + C_3 M(s)^p, \tag{46}$$

for all $0 \leq s < T_m - t$.

Let f be the function defined in $[0, +\infty)$ by $f(x) = x - C_3 x^p$. The unique maximum of f is achieved for $x_0 = \left(\frac{1}{pC_3}\right)^{\frac{1}{p-1}}$ and is positive ($f(x_0) = \frac{p-1}{p} x_0$). We now take $0 < \varepsilon < f(x_0)$. Let $0 < \alpha < \beta$ be the two solutions of $f(x) = \varepsilon$.

Since $K \subset L^q(\Omega)$ is compact, it follows from Lemma 8 in [4] that there exists $s_0 > 0$ such that

$$s^\alpha \|T(s)u(t)\|_{L^r} + C_2 s^{p\alpha} \leq \varepsilon,$$

for any $0 \leq s \leq s_0$ and $t + s < T_m$. Therefore, (46) yields

$$M(s) \leq \varepsilon + C_3 M(s)^p,$$

for all $0 \leq s \leq s_0$ and $t+s < T_m$. Then, we have for every $0 \leq s \leq s_0$ either $M(s) \leq \alpha$ or $M(s) \geq \beta$. Since the function $s \mapsto M(s)$ is continuous and $M(0) = 0$, we conclude that $M(s) \leq \alpha$ in $[0, s_0]$. Consequently, there exists $C > 0$ such that

$$s^\alpha \|u(t+s)\|_{L^r} \leq C,$$

for every $0 \leq s \leq s_0$ and $t+s < T_m$. It follows that $u \in L^\infty((0,T_m), L^r(\Omega))$, this yields a contradiction with $r > q = \frac{N(p-1)}{2}$.

Step 2. Conclusion. Since $u(t,x)$ is a nondecreasing function of t for every $x \in \Omega$, we have clearly

$$\limsup_{t \uparrow T_m} \|u(t)\|_{L^q(\Omega)} = \lim_{t \uparrow T_m} \|u(t)\|_{L^q(\Omega)},$$

and the result follows from Step 1.

6. Proof of Proposition 3. We begin by proving the following lemma.

Lemma 11. *Let $\Omega = \mathbb{R}^N$. Let $u_0 \in L^\infty(\Omega) \cap H^1(\Omega)$ be such that the solution u of (1) blows up in finite time $t = T_m$. Assume that there exist a compact set $K \subset \Omega$ and $M_1 > 0$ such that*

$$|u(t,x)| \leq M_1 \text{ whenever } x \in \Omega \setminus K, \tag{47}$$

for all $t \in (0, T_m)$. Then

$$\sup_{0 \leq t < T_m} \|u(t)\|_{L^q(\Omega \setminus K)} \leq (\|u_0\|_{L^q(\Omega)} + \Gamma) e^{CT_m}, \tag{48}$$

provided $u_0 \in L^q(\Omega)$, for any $q \geq 1$, where $C = {M_1}^{p-1}$ and $\Gamma > 0$ is a constant which depends only on N, T_m, K and M_1.

Proof. Let $q \geq 1$ be such that $u_0 \in L^q(\Omega)$. Set $\Omega_e = \Omega \setminus K$, where K is the compact set given by (47). Let $\varphi \in C_c^\infty(\Omega)$ be such that $\varphi(x) = M_1$ on ∂K, and let $w(t,x)$ be the solution of the parabolic problem

$$\begin{cases} w_t - \Delta w = Cw + h(x), \quad x \in \Omega_e, \quad t > 0, \\ w(t,x) = 0, \quad x \in \partial\Omega_e, \quad t > 0, \\ w(0,x) = u_0(x) - \varphi(x), \quad x \in \Omega_e, \end{cases}$$

where $C = {M_1}^{p-1}$ and $h(x) = C\varphi(x) + \Delta\varphi(x)$, $h \in C_c^\infty(\Omega)$. It follows from classical results of semigroup theory that such solution w always exists and is global. Then, $z = w + \varphi$ satisfies the following problem

$$\begin{cases} z_t - \Delta z = Cz, \quad x \in \Omega_e, \quad t > 0, \\ z(t,x) = M_1, \quad x \in \partial\Omega_e, \quad t > 0, \\ z(0,x) = u_0(x), \quad x \in \Omega_e. \end{cases} \tag{49}$$

Denote by $(T(t))_{t\geq 0}$ the linear heat semigroup with Dirichlet boundary condition on the exterior domain Ω_e. Given $t > 0$, we have

$$w(t,x) = T(t)(u_0 - \varphi) + \int_0^t T(t-s)(Cw(s) + h(x))\, ds.$$

Then

$$\|w(t)\|_{L^q(\Omega_e)} \leq \|u_0 - \varphi\|_{L^q(\Omega_e)} + C\int_0^t \|w(s)\|_{L^q(\Omega_e)}\, ds + t\|h\|_{L^q(\Omega_e)},$$

for every $t > 0$. Thus, by Gronwall's inequality,

$$\|z(t)\|_{L^q(\Omega_e)} \leq \big(\|u_0\|_{L^q(\Omega_e)} + (T_mC + 2)\|\varphi\|_{L^q(\Omega_e)} + T_m\|\Delta\varphi\|_{L^q(\Omega_e)}\big)e^{CT_m},$$

for every $0 < t < T_m$. Since u is a subsolution of (49) (by (47)) and φ was chosen arbitrarily, applying the comparison principle and setting $\Gamma = \max\{T_mC + 2, T_m\}\Gamma_1$ where

$$\Gamma_1 = \inf\{\|\varphi\|_{L^q(\Omega)} + \|\Delta\varphi\|_{L^q(\Omega)} : \varphi \in C_c^\infty(\Omega),\ \varphi_{|\partial K} \equiv M_1\},$$

we obtain (48). □

Next, we assume that Ω is convex. Then, using the argument of Merle and Zaag [24, Proposition 2.1], we show that, for every solution u of (1) such that the upper bound (7) holds, blow up must occur inside a compact set $K \subset \Omega$ and u, ∇u and Δu are bounded in $\Omega \setminus K$.

Proposition 12. *Let Ω be a smooth, bounded convex domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. Let $u_0 \in L^\infty(\Omega) \cap H^1(\Omega)$ be such that the solution u of* (1) *blows up in finite time T_m. Assume in addition that*

$$\|u(t)\|_{L^\infty(\Omega)} \leq M(T_m - t)^{-\frac{1}{p-1}}, \ \ t \in [0, T_m), \tag{50}$$

for some constant $M > 0$. Then there exists a compact set $K \subset \Omega$, $M_1 > 0$ and $0 < \bar{t} < T_m$ such that (47) *holds for all $t \in [\bar{t}, T_m)$.*

Proof. We proceed as in the proof of Proposition 2.1 in [24]. We rescale u by similarity variables around a point (T_m, a) as in (13), (15). Let us recall the expression of the energy associated to $w_a(s,y)$ which was introduced in (26),

$$E[w_a](s) = \int_{\Omega(s)} \Big(\frac{|\nabla w_a(s)|^2}{2} + \frac{|w_a(s)|^2}{2(p-1)} - \frac{|w_a(s)|^{p+1}}{p+1}\Big)\rho\, dy,$$

where $\rho(y) = \exp(-\frac{|y|^2}{4})$. When expressed in the original variables this yields a local energy for equation (1)

$$\begin{aligned}\mathcal{E}_{a,t}[u] = t^{\frac{2}{p-1}-N+1} &\int_\Omega \Big(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\Big)e^{-\frac{|x-a|^2}{4t}}\, dx \\ &+ \frac{t^{\frac{2}{p-1}-\frac{N}{2}}}{2(p-1)}\int_\Omega |u|^2 e^{-\frac{|x-a|^2}{4t}}\, dx.\end{aligned}$$

Since $u_0 \in H^1(\Omega)$, we have $u(t) \in H^1(\Omega)$ for all $t \in [0, T_m)$. Then, we conclude as in [24], using the fact that Giga and Kohn proved in [12], [13], that uniform estimates on $\mathcal{E}_{a,t}$ give uniform estimates in $L^\infty_{loc}(\Omega)$ on the solutions of (1). For the sake of completeness we state below this result.

Proposition 13. ([12], [13]) *Let u be a solution of* (1) *which blows up in finite time T_m. Assume that $(N-2)p < N+2$ or that the upper bound* (50) *holds. Then*

(i) *If for all $x \in B(x_0, \delta)$, $\mathcal{E}_{x,T_m-t_0}[u(t_0)] \le \sigma$, then*

$$|u(t,x)| \le \eta(\sigma)(T_m - t)^{-\frac{1}{p-1}}$$

for all $x \in B(x_0, \frac{\delta}{2})$, $t \in (\frac{t_0+T_m}{2}, T_m)$ where $\eta(\sigma) \le c\sigma^\theta$, $\theta > 0$, and $c > 0$ and θ depend only on p.

(ii) *Assume in addition that for all $x \in B(x_0, \delta)$, $|u(\frac{t_0+T_m}{2}, x)| \le M$. There exists $\eta_0 = \eta_0(p) > 0$ such that if $\eta(\sigma) \le \eta_0$, then $|u(t,x)| \le M^*$ for all $x \in B(x_0, \frac{\delta}{4})$, $t \in (\frac{t_0+T_m}{2}, T_m)$ where M^* depends only on M, T and t_0.*

Remark 8. Note that in Proposition 12 we replaced the assumption $(N-2)p < N+2$ of Proposition 2.1 in [24] by the growth rate estimate (50).

Proof of Proposition 3. The result follows from Lemma 11 and Proposition 12.

7. Blow up of subcritical norms. We shall first consider problem (1) in the one-dimensional case. More precisely, we consider the semilinear heat equation

$$\begin{cases} u_t - u_{xx} = u^p, \quad x \in I, \quad t \in [0,T], \\ u(t,x) = 0, \quad x \in \partial I, \quad t \in [0,T], \\ u(0,x) = u_0(x), \quad x \in I, \end{cases} \tag{51}$$

with $p > 1$, u_0 is continuous, nonnegative and bounded, and where $I = (-R, R)$ with $0 < R \le +\infty$.

The following result about the local behavior of solutions of (51) when the blow up time is approached was proved by Herrero and Velázquez in [15], [16] for the case $I = \mathbb{R}$, and by Velázquez in [28] for the case I bounded interval in $\mathbb{R}$.

Theorem 14. ([15], [16], [28]) *Let u be a solution of* (51)*, and assume that u blows up in finite time T_m at $x = a$. Then one of the following three possibilities occurs*

(i) $u(t,x) = ((p-1)(T_m - t))^{-\frac{1}{p-1}}$.

(ii) $\lim_{t \uparrow T_m} (T_m - t)^{\frac{1}{p-1}} u(t, a + (T_m - t)^{1/2} |\log(T_m - t)|^{1/2} y) =$ $(p-1)^{-\frac{1}{p-1}} \big(1 + \frac{(p-1)}{4p}|y|^2\big)^{-\frac{1}{p-1}}$ *uniformly on sets $|y| \le R$ with $R > 0$.*

(iii) $\lim\limits_{t \uparrow T_m} (T_m - t)^{\frac{1}{p-1}} u(t, a + (T_m - t)^{1/m} y) = (p-1)^{-\frac{1}{p-1}} \big(1 + C|y|^m\big)^{-\frac{1}{p-1}}$ *uniformly on sets $|y| \le R$ with $R > 0$, for some constant $C > 0$ and some even integer $m \ge 4$.*

Indeed, in [28] it was studied the asymptotic profile of singularities appearing at blow up time for solutions of (51), where I is a bounded interval of $\mathbb{R}$ and without assuming boundary conditions. Theorem 14 was proved for solutions whose blow up takes place at the interior of I. Now recall that it follows from the classical result of Chen and Matano [6, Theorem E′] that the blow up set of a solution u of (51) consists of finitely many points. Moreover, it is a compact subset of I (cf. [8, Theorem 3.3]). Therefore, the blow up set of u is finite and is confined to the interior of the interval $(-R, R)$.

Thus, the result of Vélazquez still holds for the Dirichlet boundary value problem.

Recall that, for $I = \mathbb{R}$, (ii) holds true whenever u_0 has a single maximum, which will be the case when u_0 is radially symmetric and nonincreasing as a function of $r = |x|$ (cf. [15]).

The existence of flat blow up behaviors as those given by (iii) with $m = 4$ is proved in [16], for $I = \mathbb{R}$. The solution exhibiting such behavior is constructed in such a way that two maxima are made to collapse exactly at $x = a$ and $t = T_m$. At the end of this section, we shall show that a similar result holds for any I bounded interval in $\mathbb{R}$ (see the proof of Theorem 5). More recently, Amadori [1] developed a general method for constructing solutions that behave as described in (iii), for any even integer $m \geq 4$ (and any dimension N), with a prescribed asymptotic behavior near the blow up time.

For $I = \mathbb{R}$, Herrero and Velázquez showed in [17] that (ii) and (iii) correspond, respectively, to stable and unstable behaviors. Moreover, they proved that solutions satisfying (iii) actually possess $\left(\frac{m}{2}\right)$ maxima for t close to T_m. Therefore, the blow up profile of u will depend on the number of maxima which reach $x = a$ at $t = T_m$.

Applying the above theorem, we derive some conclusions about the $L^q(I)$ norms of $u(t)$ as $t \uparrow T_m$, where u is a blow up solution of (51). We improve (2) and also estimate the growth rate of the $L^{\frac{N(p-1)}{2}}(I)$-norm blow up. More precisely, we have the following L^q-norm blow up result which is a restatement of Theorem 4.

Theorem 15. *Let u be a solution of* (51)*, and assume that u blows up in finite time $t = T_m$ at $x = a$. Then*

- *If* (ii) *holds, condition* (*a*) *of Theorem* 4 *occurs.*
- *If* (iii) *holds, condition* (b) *of Theorem* 4 *occurs.*

Proof. To simplify the proof, we shall assume that u blows up at $x = 0$. Let $J = (-R_0, R_0) \subset I$, with $0 < R_0 \leq +\infty$.

Case 1. Condition (ii) holds.

We have, for every $q \geq 1$,

$$\|u(t)\|^q_{L^q(J)} = (T_m - t)^{\frac{1}{2} - \frac{q}{p-1}} |\log(T_m - t)|^{\frac{1}{2}} \times \int_{B_{R_0(s)}} |(T_m - t)^{\frac{1}{p-1}} u(t, (T_m - t)^{\frac{1}{2}} |\log(T_m - t)|^{\frac{1}{2}} y)|^q \, dy, \tag{52}$$

where

$$R_0(s) = R_0 T_m^{-1/2} e^{s/2} |s - \log T_m|^{-1/2}, \tag{53}$$

and $B_{R_0(s)} = (-R_0(s), R_0(s))$, for each $s > 0$ given by (13). Let $\frac{p-1}{2} \le q \le +\infty$ and $q \ge 1$. Then

$$\begin{aligned} &\big((T_m - t)^{\delta} |\log(T_m - t)|^{-\frac{1}{2q}} \|u(t)\|_{L^q(J)}\big)^q \\ &\quad = \int_{B_{R_0(s)}} |(T_m - t)^{\frac{1}{p-1}} u(t, (T_m - t)^{1/2} |\log(T_m - t)|^{1/2} y)|^q \, dy, \end{aligned} \tag{54}$$

with $\delta = \frac{1}{p-1} - \frac{1}{2q} \ge 0$.

Let $R > 0$ be such that $\varphi(y) \ge K|y|^{-\frac{2}{p-1}}$, for every $|y| \ge R$, where

$$\varphi(y) = (p-1)^{-\frac{1}{p-1}} \big(1 + \frac{(p-1)}{4p} |y|^2\big)^{-\frac{1}{p-1}}$$

and $K = \frac{1}{2}\big(\frac{4p}{(p-1)^2}\big)^{\frac{1}{p-1}}$. Take $0 < \lambda < K$ and let $\varepsilon_0 > 0$ be such that $R < R_\varepsilon \equiv \big(\frac{K-\lambda}{\varepsilon}\big)^{\frac{p-1}{2}}$ for all $0 < \varepsilon \le \varepsilon_0$. Fix $0 < \varepsilon \le \varepsilon_0$. It follows from (ii) that there exists $\gamma = \gamma(\varepsilon) > 0$ such that

$$(T_m - t)^{\frac{1}{p-1}} u(t, (T_m - t)^{1/2} |\log(T_m - t)|^{1/2} y) \ge \varphi(y) - \varepsilon$$

for $|y| \le R_\varepsilon$ and $t \in (T_m - \gamma, T_m)$. We may assume without loss of generality that $\gamma > 0$ is also such that $R_\varepsilon < R_0(s)$ for any $s > 0$ given by (13) with $t \in (T_m - \gamma, T_m)$.

Using (54), we conclude that

$$\begin{aligned} \big((T_m - t)^{\delta} |\log(T_m - t)|^{-\frac{1}{2q}} \|u(t)\|_{L^q(J)}\big)^q &\ge \int_{B_{R_\varepsilon}} |(\varphi(y) - \varepsilon)^+|^q \, dy \\ &\ge \lambda^q \int_{B_{R_\varepsilon} \setminus B_R} |y|^{-\frac{2q}{p-1}} \, dy, \end{aligned} \tag{55}$$

for every $t \in (T_m - \gamma, T_m)$. Note that $R_\varepsilon \to +\infty$ when $\varepsilon \to 0$. Therefore, for every $0 < \varepsilon \le \varepsilon_0$,

$$\liminf_{t \uparrow T_m} (T_m - t)^{\delta} |\log(T_m - t)|^{-\frac{1}{2q}} \|u(t)\|_{L^q(J)} \ge \lambda \Big(\int_{B_{R_\varepsilon} \setminus B_R} |y|^{-\frac{2q}{p-1}} \, dy\Big)^{1/q}.$$

Recall that $|y|^{-\frac{2q}{p-1}} \in L^1(\mathbb{R} \setminus B_R)$ if and only if $q > \frac{p-1}{2}$. Now (a) follows easily. Moreover, for $q = \frac{p-1}{2}$,

$$|\log(T_m - t)|^{-\frac{1}{p-1}} \|u(t)\|_{L^q(J)} \to +\infty \text{ when } t \uparrow T_m.$$

Case 2. Condition (iii) holds.

We have, for every $q \geq 1$,

$$\|u(t)\|^q_{L^q(J)} = (T_m - t)^{\frac{1}{m} - \frac{q}{p-1}} \int_{B_{R_0(s)}} |(T_m - t)^{\frac{1}{p-1}} u(t, (T_m - t)^{1/m} y)|^q \, dy, \tag{56}$$

where $R_0(s)$ is given by (53). Let $\frac{p-1}{m} \leq q \leq +\infty$ and $q \geq 1$. Let $R > 0$ be such that $\psi(y) \geq K|y|^{-\frac{m}{p-1}}$, for every $|y| \geq R$, where

$$\psi(y) = (p-1)^{-\frac{1}{p-1}} \big(1 + C|y|^m\big)^{-\frac{1}{p-1}}$$

and $K = \frac{1}{2}((p-1)C)^{-\frac{1}{p-1}}$. Take $0 < \lambda < K$ and let $\varepsilon_0 > 0$ be such that $R < R_\varepsilon \equiv \big(\frac{K-\lambda}{\varepsilon}\big)^{\frac{p-1}{m}}$ for all $0 < \varepsilon \leq \varepsilon_0$. Fix $0 < \varepsilon \leq \varepsilon_0$. It follows from (iii) that there exists $\gamma = \gamma(\varepsilon) > 0$ such that

$$(T_m - t)^{\frac{1}{p-1}} u(t, (T_m - t)^{1/m} y) \geq \psi(y) - \varepsilon$$

for $|y| \leq R_\varepsilon$ and $t \in (T_m - \gamma, T_m)$. We also assume $\gamma > 0$ being such that $R_\varepsilon < R_0(s)$ for any $s = \log\big(\frac{T_m}{T_m - t}\big)$, with $t \in (T_m - \gamma, T_m)$.

Finally, by (56), we conclude that, for every $t \in (T_m - \gamma, T_m)$,

$$\begin{aligned} \big((T_m - t)^{\frac{1}{p-1} - \frac{1}{mq}} \|u(t)\|_{L^q(J)}\big)^q &\geq \int_{B_{R_\varepsilon}} |(\varphi(y) - \varepsilon)^+|^q \, dy \\ &\geq \lambda^q \int_{B_{R_\varepsilon} \setminus B_R} |y|^{-\frac{mq}{p-1}} \, dy. \end{aligned}$$

If $q > \frac{p-1}{m}$, which is equivalent to say that $\frac{1}{mq} - \frac{1}{p-1} < 0$, we conclude as in the Case 1. that

$$\liminf_{t \uparrow T_m} (T_m - t)^{\frac{1}{p-1} - \frac{1}{mq}} \|u(t)\|_{L^q(J)} > 0, \tag{57}$$

and (b) follows. If $q = \frac{p-1}{m}$, we conclude that

$$\|u(t)\|_{L^q(J)} \to +\infty \text{ when } t \uparrow T_m,$$

as stated in (b).

Remark 9. In particular, we conclude that whenever u is a blow up solution of (1) such that (iii) holds then certain subcritical norms of u blow up as $t \uparrow T_m$. More precisely,

$$\lim_{t \uparrow T_m} \|u(t)\|_{L^q(I)} = +\infty,$$

for *all* $\frac{p-1}{m} \le q < \frac{p-1}{2}$, where m is given in (iii). Recall that it was proved in [8, Theorem 2.4] that if $u_0 \in L^\infty(I)$ is nonnegative, radially symmetric and nonincreasing as a function of $r = |x|$, then the solution of (51), with $R < +\infty$, blows up at the single point $x = 0$, and

$$\limsup_{t \uparrow T_m} \|u(t)\|_{L^q(I)} < +\infty \tag{58}$$

for all $1 \le q < \frac{p-1}{2}$. (This results also holds for the N-dimensional problem (1), with Ω a ball in $\mathbb{R}^N$). Note that, under these assumptions on the initial data, (ii) holds true.

Next, we shall prove Theorem 5.

Proof of Theorem 5. We shall consider separately two cases.
Case 1. $I = \mathbb{R}$.

The result follows from Theorem 2 in [16] and Theorem 15.

Case 2. I is a bounded interval in $\mathbb{R}$.

Take $R > 1$. Let $\xi_1 \in C^2(\mathbb{R})$ be a nonnegative function such that $\xi_1(s) = 0$ if $s \le 0$ and assume that ξ_1 has a single maximum at $x = 1$ such that ${\xi'}_1(s) > 0$ for every $0 < s < 1$. Consider the function $\xi(x) = k\xi_1(1 - |x|)$. Then, ξ is radially symmetric and decreasing on $r = |x|$, and is supported on $(-1, 1)$.

We may assume $k > 0$ sufficiently large such that the corresponding solution of (51), where $I = (-R, R)$, with initial value $\xi(x)$ blows up in finite time. Note that we can choose $k > 0$ independently of $R > 1$. Indeed, it is enough to have

$$E(\xi) = \frac{1}{2}\int_{-1}^{1} |\nabla \xi|^2 - \frac{1}{p+1}\int_{-1}^{1} |\xi|^{p+1} < 0.$$

Define

$$u_0^R(x) = \xi(x - \frac{R}{2}) + \xi(x + \frac{R}{2})$$

and denote by $u^R(t,x)$ the corresponding solution of (51) with initial value $u_0^R(x)$. Note that, $\operatorname{supp} u_0^R \subset (-1 - \frac{R}{2}, 1 - \frac{R}{2}) \cup (-1 + \frac{R}{2}, 1 + \frac{R}{2})$, for any $R > 2$. We now proceed in three steps.

Step 1. We show that u^R blows up at exactly two points, for R sufficiently large.

Clearly, $u^R(t,x)$ blows up in finite time for any $R > 2$. The initial condition is radially symmetric and has at most two local maxima therefore, it follows from a classical result (see [6]) that the blow up set of u^R consists of one or two points. By symmetry, blow up may only occur at $x = 0$ (single point case) or at two points $x = \pm x_0$ for some $x_0 > 0$.

Next, we shall use Corollary 3.6 of Giga and Kohn [12], which eliminates the possibility of blow up at $x = 0$ if the weighted energy $\mathcal{E}[u_0^R]$ is sufficiently small, where

$$\mathcal{E}[u_0^R] = T(R)^{\frac{2}{p-1}+\frac{1}{2}} \int_{-R}^{R} (\frac{1}{2}|\nabla u_0^R|^2 - \frac{1}{p+1}|u_0^R|^{p+1}) e^{-\frac{|x|^2}{4T(R)}}\, dx + \frac{T(R)^{\frac{2}{p-1}-\frac{1}{2}}}{2(p-1)} \int_{-R}^{R} |u_0^R|^2 e^{-\frac{|x|^2}{4T(R)}}\, dx,$$

and $T(R)$ denotes the blow up time of $u^R(t,x)$ (see Proposition 13 in Section 6). We show that

$$\limsup_{R\to+\infty} \mathcal{E}[u_0^R] \le 0,$$

using the argument of Proposition 5.6 in [12]. Then the result follows.

Step 2. Let $R_0 > 0$ be such that the solution of (51), with $I = (-R_0, R_0)$ and initial data $u_0^{R_0}$, blows up at exactly two points. Next, for each $0 \le R \le R_0$, we consider the problem

$$\begin{cases} u_t - u_{xx} = u^p, \quad x \in (-R_0, R_0), \quad t \in [0,T], \\ u(t, \pm R_0) = 0, \quad t \in [0,T], \\ u(0,x) = u_0^R(x), \quad x \in (-R_0, R_0), \end{cases} \tag{59}$$

and denote by $v^R(t,x)$ its solution.

Clearly, $v^R(t,x)$ blows up in finite time for any $0 \le R \le R_0$. We denote by $T'(R)$ the blow up time of $v^R(t,x)$.

Step 3. Define

$$R^* = \sup\{R \in (0, R_0) : \text{ blow up happens only at } x = 0\}.$$

Following the argument of the proof of Theorem 2 in [16], we conclude that $v^{R^*}(t, x)$ satisfies (iii) of Theorem 14, with $m = 4$. We first note that Lemmas 4.1 and 4.2 in [16] still hold for problem (59). Therefore, concerning $v^{R^*}(t, x)$, two maxima exist for any $t < T_m$ which collapse at $x = 0$, $t = T_m$. To prove the analogue of Lemma 4.1, it is enough to do slight modifications in the proof given in [16]. Next, note that Lemma 4.2 (for $\Omega = \mathbb{R}$) relies mainly on Proposition 3.1 and Lemma 2.4 in [16]. Even if this last result concerns the problem posed in $\mathbb{R}$, it still holds for the Dirichlet boundary value problem as in the proof of Lemma 3.9 in [28]. On the other hand, the proposition can be applied directly to our problem. Finally, we conclude using the extension of the results of Theorems A and 1 ([16]) to the case of the Dirichlet boundary value problem (see Theorem 1 and Proposition 3.6 in [28]).

It follows from Theorem 15 that this gives the desired example on the interval $(-R_0, R_0)$ and scaling leads easily to an example on any interval. Namely, for $I = (-R, R)$ it is enough to take $u(t, x) = \lambda^{\frac{2}{p-1}} v^{R^*}(\lambda^2 t, \lambda x)$, where $\lambda = R_0/R$. □

In what follows, we shall assume that $\Omega = \mathbb{R}^N$ and u is a nonnegative solution of (1) which blows up in finite time $t = T_m$ at $x = 0$. Moreover, we shall assume that the conditions on the blow up rate (6) and (7) hold, which is the case under current assumptions on the exponent p and the initial value u_0, as was mentioned in the introduction. Also recall that in this case the critical norm of u always blows up when $t \uparrow T_m$.

In [29], Velázquez extended Theorem 14 to the higher dimensional case. More precisely, he proved the following theorem.

Theorem 16. ([29]) *Let $\Omega = \mathbb{R}^N$ and let u be a nonnegative solution of* (1) *which blows up in finite time T_m at $x = a$, and such that* (6) *and* (7) *hold. Then the following alternative arises. Either there exists an orthogonal transformation of the coordinate axes such that, still denoting by y the new coordinates,*

(I) $\lim_{t\uparrow T_m}(T_m - t)^{\frac{1}{p-1}} u(t, a + (T_m - t)^{1/2}|\log(T_m - t)|^{1/2} y) = (p - 1)^{-\frac{1}{p-1}}\big(1 + \frac{(p-1)}{4p}\sum_{k=1}^{l} y_k^2\big)^{-\frac{1}{p-1}}$ *uniformly on sets $|y| \le R$ with $R > 0$, where $1 \le l \le N$ and $y = (y_1, \dots, y_N)$.*

or there exists an even integer $m \geq 4$, and constants C_α not all zero, such that

(II) $\lim_{t\uparrow T_m}(T_m-t)^{\frac{1}{p-1}}u(t,a+(T_m-t)^{1/m}y) = (p-1)^{-\frac{1}{p-1}}\big(1+(p-1)\sum_{|\alpha|=m}\overline{c}_\alpha C_\alpha y^\alpha\big)^{-\frac{1}{p-1}}$ *uniformly on sets $|y| \leq R$ with $R>0$, where the homogeneous multilinear form $B(x) = \sum_{|\alpha|=m}\overline{c}_\alpha C_\alpha x^\alpha$ is nonnegative. Here $\alpha = (\alpha_1,\dots,\alpha_N)$, α_i is a nonnegative integer for $1 \leq i \leq N$, $x^\alpha = x_1^{\alpha_1}\dots x_N^{\alpha_N}$, and $\overline{c}_\alpha = c_{\alpha_1}\dots c_{\alpha_N}$, with $c_m = (2^{m/2}(4\pi)^{1/4}(m!)^{1/2})^{-1}$ for any m nonnegative integer.*

Now we can state the N-dimensional extension of Theorem 15.

Theorem 17. *Let $\Omega = \mathbb{R}^N$ and let u be a nonnegative solution of* (1) *which blows up in finite time $t = T_m$ at $x = a$, and such that* (6) *and* (7) *hold. Then*

(a) *If* (I) *holds,*

$$\lim_{t\uparrow T_m}\|u(t)\|_{L^q(B_R(a))} = +\infty,$$

for all $\frac{N(p-1)}{2} \leq q \leq +\infty$, $q \geq 1$ and $R > 0$. Moreover,

$$\liminf_{t\uparrow T_m}(T_m-t)^\delta|\log(T_m-t)|^{-\mu}\|u(t)\|_{L^q(B_R(a))} > 0,$$

with $\delta = \frac{1}{p-1} - \frac{N}{2q} \geq 0$ and $\mu = \frac{N}{2q}$.

(b) *If* (II) *holds,*

$$\lim_{t\uparrow T_m}\|u(t)\|_{L^q(B_R(a))} = +\infty,$$

for all $\frac{N(p-1)}{m} \leq q \leq +\infty$, $q \geq 1$ and $R > 0$. Moreover,

$$\liminf_{t\uparrow T_m}(T_m-t)^\gamma\|u(t)\|_{L^q(B_R(a))} > 0,$$

with $\gamma = \frac{1}{p-1} - \frac{N}{mq} \geq 0$.

Remark 10. Putting together this result, Theorem 1 in [1] and Theorem 1 in [30], we conclude that for every even integer $m \geq 4$ and every $a \in \mathbb{R}^N$, there exists a nonnegative initial condition $u_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that the corresponding solution of (1) blows up in finite time $t = T_m$ at the single point $x = a$, and such that (b) holds. In particular, $\|u(t)\|_{L^q(B_R(a))} \to +\infty$ as $t \uparrow T_m$ for all $q \geq 1$ and $R > 0$, provided that m is sufficiently large.

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