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DERIVATION OF QUASI-GEOSTROPHIC POTENTIAL VORTICITY EQUATIONS

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Abstract. In this paper we derive the quasi-geostrophic potential vorticity equations, starting from primitive type equations, as the Rossby number goes to zero, and prove the convergence as long as the limit system has strong enough solutions. We in particular take account of the boundary layers and investigate the existence of global weak solutions of the limit system with physical boundary conditions.

1. Introduction. In this paper we address the derivation of the quasi-geostrophic potential vorticity equations. This classical equation in meteorology describes the motion of stratified, rotating flows. It can be derived from primitive-type equations, namely from incompressible Navier-Stokes equations where we make the Boussinesq approximation (we neglect the variation of ρ in the momentum equation, except in the gravity term), which filters out acoustic waves. After adimensionalization, two small parameters appear: the Rossby number, depending on the speed of rotation of the earth, and a stratification parameter, linked to the fact that the rest density of the ocean or of the atmosphere has a nonzero gradient. As the Rossby number goes to zero, the flow becomes two dimensional (Proudman theorem), a fact which is underlined by the stratification. By Archimedes force (buoyancy force) an element of fluid tends to move at the same height (if for instance it sinks, its density is less than the density of the neighboring fluid, and thus the Archimedes force pulls it back to its initial height, around which it oscillates). We refer to Chapter 6 of [24], [12], [13] for a physical discussion

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of this model, to the beginning of [3] for a more complete introduction, and to [20], [21] for a detailed derivation and study of the primitive equations.

Various mathematical works have been done during the last few years on the quasi-geostrophic limit; see in particular the papers of T. Beale and A. Bourgeois ([3]), J.-Y. Chemin ([5]), and D. Iftimie ([17]). However, [5] and [17] consider the whole space problem. Besides, [3] imposes simplified boundary conditions in order to reduce to a periodic problem.

In this paper, we study the quasi-geostrophic limit in $\Omega = \mathbb{T}^2 \times [0,1]$ or $\mathbb{R}^2 \times [0,1]$, with solid walls at $z = 0$ and $z = 1$. In the first part we fulfill a complete formal study including the analysis of the boundary layers and of the boundary conditions it induces on the quasi-geostrophic equations (following the analysis of [24] in the case of solid boundaries at $z = 0$ and $z = 1$). In the second part we prove the existence of global weak solutions of the quasi-geostrophic equations with the usual physical boundary conditions, and recall cases when the limit system has strong solutions. In the third part we prove convergence of weak solutions of the initial equations to solutions of the limit quasi-geostrophic equations as long as the latter solutions remain smooth. The last section is devoted to some possible extensions, in particular to the discussion of the boundary conditions for oceanic and atmospheric motions, which can be treated following the methods used in this paper.

This strategy follows the method used by one of the authors in [15] to handle boundary layers in rotating fluids (called Ekman layers) with the additional difficulties that the limit system is more complicated (since it is fully three dimensional and thus the existence of global strong solutions is open) and that there is no viscosity on the equation of the density (which, together with strong z dependence of the stratification, is an obstacle to a justification of approximate solutions "at any order"). We also refer to [6] and [16] for a treatment of other Ekman layers, or general layers appearing in the inviscid limit of parabolic equations.

2. Derivation of the model.

2.1. Boussinesq approximation in a rotating frame. We start from the Navier-Stokes equations on the velocity (u, v, w), density ρ and pressure p, where we have made the classical Boussinesq approximation

$$\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u - f_0 v + \frac{\partial_x p}{\rho_s} - \frac{\mathcal{F}_0 u}{\rho_s} = 0, \qquad (1)$$

$$\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v + f_0 u + \frac{\partial_y p}{\rho_s} - \frac{\mathcal{F}_0 v}{\rho_s} = 0, \quad (2)$$

$$\partial_t w + u\partial_x w + v\partial_y w + w\partial_z w + \frac{\partial_z p}{\rho_s} - \frac{\mathcal{F}_0 w}{\rho_s} = -g\frac{\rho}{\rho_s}, \quad (3)$$

$$\partial_t \rho + u\partial_x \rho + v\partial_y \rho + w\partial_z \rho = 0, \quad (4)$$

$$\partial_x u + \partial_y v + \partial_z w = 0, \quad (5)$$

where g is the acceleration by the gravity, f_0 is linked to the speed of rotation, ρ_s is a given density profile (rest profile) and where $\mathcal{F}_0 u$, $\mathcal{F}_0 v$ and $\mathcal{F}_0 w$ are viscosity terms which will be detailed later.

Let us recall that the Boussinesq approximation consists in taking the density to be constant in computing rates of change of momentum from accelerations, but taking full account of density variations when they give rise to buoyancy forces, i.e., when there is a multiplying factor g in the vertical component of the momentum equation (we refer to [12] page 130). We also impose the incompressibility of the fluid, the pressure p being the corresponding Lagrange multiplier.

2.2. Dimensionless equations. As usual in meteorology ([12], [13], [24]), we introduce typical quantities in order to adimensionalize the equations. Let L be a typical horizontal length of the domain of evolution, D a typical depth, U a typical horizontal speed and $\bar{\rho}$ a typical density. We rescale space, time, density and velocities according to

$$x = Lx', \quad y = Ly', \quad z = Dz', \quad t = \frac{L}{U}t',$$
$$u = Uu', \quad v = Uv', \quad w = \frac{D}{L}Uw', \quad \rho = \bar{\rho}\rho', \quad \rho_s = \bar{\rho}\rho_s', \quad p = U^2\bar{\rho}p'$$
$$\partial_t = \frac{U}{L}\partial_{t'}, \quad \partial_x = \frac{1}{L}\partial_{x'}, \quad \partial_y = \frac{1}{L}\partial_{y'}, \quad \partial_z = \frac{1}{D}\partial_{z'}.$$

Writing (1)–(5) in the new variables x', y', z', t', for the new functions u', v', w', ρ' and dropping the primes gives

$$\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u - \frac{L}{U}f_0 v + \frac{1}{\rho_s}\partial_x p - \frac{L}{U\bar{\rho}\rho_s}\mathcal{F}_0 u = 0 \quad (6)$$

$$\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v + \frac{L}{U}f_0 u + \frac{1}{\rho_s}\partial_y p - \frac{L}{U\bar{\rho}\rho_s}\mathcal{F}_0 u = 0, \quad (7)$$

$$\partial_t w + u\partial_x w + v\partial_y w + w\partial_z w + \frac{L^2}{D^2}\frac{1}{\rho_s}\partial_z p - \frac{L}{U\bar\rho\rho_s}\mathcal{F}_0 w = -\frac{L^2}{DU^2}g\frac{\rho}{\rho_s}, \quad (8)$$

$$\partial_t \rho + u\partial_x \rho + v\partial_y \rho + w\partial_z \rho = 0, \quad (9)$$

$$\partial_x u + \partial_y v + \partial_z w = 0. \quad (10)$$

Let us define three parameters: *the aspect ratio*

$$\delta = \frac{D}{L},$$

the Rossby number

$$\varepsilon = \frac{U}{Lf_0}$$

and *the Froude number*

$$Fr = \frac{U}{\sqrt{gD}}.$$

We assume that Fr is of order ε. Let

$$g_0 = \frac{\varepsilon^2}{Fr^2}.$$

In this paper the aspect ratio δ plays no role. Hence we will assume that it converges to a nonzero limit value, which is supposed to be one: $\delta = 1$ (or equivalently $L = D$). Equations (6), (7) and (8) then become

$$\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u - \frac{v}{\varepsilon} + \frac{1}{\rho_s}\partial_x p - \frac{L}{U\bar\rho}\frac{\mathcal{F}_0 u}{\rho_s} = 0, \quad (11)$$

$$\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v + \frac{u}{\varepsilon} + \frac{1}{\rho_s}\partial_y p - \frac{L}{U\bar\rho}\frac{\mathcal{F}_0 v}{\rho_s} = 0, \quad (12)$$

$$\partial_t w + u\partial_x w + v\partial_y w + w\partial_z w + \frac{g_0}{\varepsilon^2\delta^2}\frac{\rho}{\rho_s} + \frac{1}{\delta^2}\frac{1}{\rho_s}\partial_z p - \frac{L}{U\bar\rho}\frac{\mathcal{F}_0 w}{\rho_s} = 0, \quad (13)$$

since

$$\frac{gL^2}{DU^2} = \frac{L^2}{Fr^2D^2} = \frac{g_0}{\varepsilon^2\delta^2},$$

$$\partial_t \rho + u\partial_x \rho + v\partial_y \rho + w\partial_z \rho = 0, \quad (14)$$

$$\partial_x u + \partial_y v + \partial_z w = 0. \quad (15)$$

2.3. Deviation from the rest state. We consider motions where ρ is very close to its rest value ρ_s, $\rho_s(z)$ being a given decreasing function of the height only. Namely we set

$$\rho = \rho_s(1 + \varepsilon\tilde{\rho}),$$

$\varepsilon\rho_s\tilde{\rho}$ being the (small) deviation from the equilibrium. We define $\tilde{p}$ by

$$p = \frac{p_s}{\varepsilon^2} + \tilde{p},$$

where

$$\partial_z p_s = -g_0\rho_s.$$

We then get

$$\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u - \frac{v}{\varepsilon} - \frac{L}{U\bar{\rho}}\frac{\mathcal{F}_0 u}{\rho_s} = -\frac{1}{\rho_s}\partial_x\tilde{p} \tag{16}$$

$$\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v + \frac{u}{\varepsilon} - \frac{L}{U\bar{\rho}}\frac{\mathcal{F}_0 v}{\rho_s} = -\frac{1}{\rho_s}\partial_y\tilde{p} \tag{17}$$

$$\partial_t w + u\partial_x w + v\partial_y w + w\partial_z w + \frac{g_0}{\varepsilon\delta^2}\tilde{\rho} - \frac{L}{U\bar{\rho}}\frac{\mathcal{F}_0 w}{\rho_s} = -\frac{1}{\rho_s}\frac{1}{\delta^2}\partial_z\tilde{p} \tag{18}$$

$$\partial_t\tilde{\rho} + u\partial_x\tilde{\rho} + v\partial_y\tilde{\rho} + w\partial_z\tilde{\rho} + w\tilde{\rho}\frac{\partial_z\rho_s}{\rho_s} + \frac{w}{\varepsilon}\frac{\partial_z\rho_s}{\rho_s} = 0 \tag{19}$$

$$\partial_x u + \partial_y v + \partial_z w = 0. \tag{20}$$

2.4. Formal limit. Let us drop the viscosity terms and study the limit of (16, 17, 18, 19, 20) as ε goes to 0 in a formal way. We assume that u, v, w, $\tilde{\rho}$, $\varepsilon\tilde{p}$ converge to u^0, v^0, w^0, ρ^0, p^0. First using (16) and (17), we get

$$\rho_s v^0 = \partial_x p^0 \quad \text{and} \quad \rho_s u^0 = -\partial_y p^0, \tag{21}$$

whereas (19) gives

$$w^0 = 0.$$

Equation (18) gives

$$g_0\rho_s\rho^0 = -\partial_z p^0. \tag{22}$$

Notice that

$$\partial_x u^0 + \partial_y v^0 = 0. \tag{23}$$

Let us assume that u, v and w have expansions of the form $u = u^0 + \varepsilon u^1 + \cdots$. Taking the curl of (16) and (17) and using $w^0 = 0$, we get

$$\partial_t \zeta^0 + u^0 \partial_x \zeta^0 + v^0 \partial_y \zeta^0 = -\partial_x u^1 - \partial_y v^1,$$

where

$$\zeta^0 = \partial_x v^0 - \partial_y u^0$$

is the two-dimensional curl of $(u^0, v^0, w^0 = 0)$. Equation (20) gives

$$\partial_x u^1 + \partial_y v^1 + \partial_z w^1 = 0;$$

hence,

$$d_0 \zeta^0 = \partial_z w^1,$$

where

$$d_0 = \partial_t + u^0 \partial_x + v^0 \partial_y.$$

Now (19) gives, as $w^0 = 0$,

$$w^1 = \frac{g_0}{S} d_0 \rho^0,$$

where

$$S = -g_0 \rho_s^{-1} \partial_z \rho_s.$$

Therefore,

$$d_0 \zeta^0 = \partial_z \big(\frac{g_0}{S} d_0 \rho^0 \big) = d_0 \big[\partial_z \big(\frac{g_0}{S} \rho^0 \big) \big]$$

since ρ_s is independent on t, x, y. But $\rho_s \zeta^0 = \Delta_{x,y} p^0$ and $p^0 = \rho_s \psi$ where ψ is the stream function. Therefore the limit system is

$$d_0 \Big[\Delta_{x,y} \psi + \partial_z \Big(\frac{1}{S \rho_s} \partial_z (\rho_s \psi) \Big) \Big] = 0 \tag{24}$$

which can be rewritten

$$d_0 \Big[\Delta_{x,y} \psi + \frac{1}{\rho_s} \partial_z \Big(\frac{\rho_s}{S} \partial_z \psi \Big) \Big] = 0. \tag{25}$$

Equation (25) will have to be supplied with boundary conditions at $z = 0$ and $z = 1$. These boundary conditions come from a detailed analysis of the boundary layer near $z = 0$ and $z = 1$ and will be made precise later (see (34)). Notice that the formal limit is the same if we allow δ to go to 0 with ε.

2.5. Boundary conditions and viscosity. In order to take into account the viscosity of the fluid, and in particular the Ekman layers which appear near boundaries, we will study the quasi-geostrophic limit $\varepsilon \to 0$ with $\mathcal{F}_0 u, \mathcal{F}_0 v, \mathcal{F}_0 w \neq 0$. These viscosities come from a modelization of turbulence, which is of course far from being justified, and we refer to [24] for a discussion of turbulent viscosities in the framework of geophysical flows. Typically,

$$\frac{L}{U\bar{\rho}}\mathcal{F}_0 = \mathcal{F} = \nu_H \Delta_{x,y} + \nu_V \partial^2_{zz},$$

where ν_H and ν_V are dimensionless turbulent viscosity coefficients. Notice that in practice they are different, and $\nu_V \ll \nu_H$. Here we will assume, following [24],

$$\nu_V \sim C\varepsilon^\sigma, \quad \frac{\nu_H}{\varepsilon} \to +\infty, \quad \nu_H \to 0$$

for some $\sigma \geq 1$.

We will study the quasi-geostrophic limit in

$$\Omega = \mathbb{T}^2 \times [0,1],$$

where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the two-dimensional flat torus (square with periodic boundary conditions), or in

$$\Omega = \mathbb{R}^2 \times [0,1].$$

To fix the ideas and restrict ourselves to one case, we will consider the following boundary conditions:

$$u = v = w = 0 \quad \text{on} \ \ z = 0 \ \text{ and } \ z = 1, \tag{26}$$

which correspond to the motion of the fluid between two solid boundaries. Physical boundary conditions for the ocean and atmosphere and their mathematical treatment are discussed in Sections 5.2 and 5.3.

2.6. Boundary layers. The inviscid limit $\nu_H, \nu_V \to 0$ yields a modification of type of the equations, which go from a parabolic type (second order) to an hyperbolic one (first order). This leads to a change in boundary conditions we can enforce, and to boundary layers, that is, to changes of behavior of the solution when approaching the wall at very small scales. In the present case the boundary layers which arise are called Ekman layers ([10]). Ekman layers have been widely studied, and we refer in particular to [15] for the study of the small viscosity–high rotation limit of Navier-Stokes equations of rotating fluids, including the effect of Ekman layers near a flat boundary.

The boundary layers equations are obtained by looking for solutions $u^\varepsilon(t,x,y,z)$ of the form

$$u^\varepsilon(t,x,y,z) = u^0(t,x,y,z) + u_b(t,x,y,\zeta)$$

with

$$u_b(t,x,y,0) = -u^0(t,x,y,0),$$

where $\zeta = z/\theta$ (θ being the thickness of the expected boundary layer), and similarly for v^ε, w^ε and $\tilde\rho^\varepsilon$. Putting this Ansatz in (16, 17, 18, 19, 20) and keeping the predominant terms, we obtain, following [24],

$$-\frac{v_b}{\varepsilon} - \frac{\nu_V}{\theta^2}\frac{\partial^2_{\zeta\zeta}u_b}{\rho_s(0)} = -\frac{1}{\rho_s(0)\varepsilon}\partial_x p_b \tag{27}$$

$$\frac{u_b}{\varepsilon} - \frac{\nu_V}{\theta^2}\frac{\partial^2_{\zeta\zeta}v_b}{\rho_s(0)} = -\frac{1}{\rho_s(0)\varepsilon}\partial_y p_b \tag{28}$$

$$0 = \frac{1}{\rho_s(0)\delta^2\varepsilon\theta}\partial_\zeta p_b. \tag{29}$$

Hence, using (29), as usual in fluid boundary layers, the pressure p_b does not vary in the boundary layer and is given by the pressure inside the domain. Hence $p_b = 0$. Moreover, in order to have three terms of equal importance, we are led to choose $\theta = \sqrt{\varepsilon\nu_V/\rho_s(0)}$ as the size of the boundary layer. This leads to the study of

$$v_b = -\partial^2_{\zeta\zeta}u_b \quad \text{and} \quad u_b = \partial^2_{\zeta\zeta}v_b, \tag{30}$$

which are the equations of a boundary layer for incompressible rotating fluids (called an Ekman layer). We refer to [24] and [15] for a study of these Ekman layers. The solution of (30) such that $u^0 + u_b = v^0 + v_b = 0$ for $z = 0$ is given by

$$u_b(t,x,y,\zeta) = -\exp(-\tfrac{\zeta}{\sqrt{2}})\big(u^0(t,x,y,0)\cos(\tfrac{\zeta}{\sqrt{2}})+v^0(t,x,y,0)\sin(\tfrac{\zeta}{\sqrt{2}})\big) \tag{31}$$

$$v_b(t,x,y,\zeta) = \exp(-\tfrac{\zeta}{\sqrt{2}})\big(u^0(t,x,y,0)\sin(\tfrac{\zeta}{\sqrt{2}}) - v^0(t,x,y,0)\cos(\tfrac{\zeta}{\sqrt{2}})\big), \tag{32}$$

exactly as the Ekman layers for incompressible rotating fluids. Using the incompressibility condition, we in particular deduce

$$\begin{aligned} w_b(t,x,y,\zeta) = -\sqrt{\tfrac{\varepsilon\nu_V}{2\rho_s(0)}}\exp(-\tfrac{\zeta}{\sqrt{2}})(\partial_x v^0(t,x,y,0) \\ - \partial_y u^0(t,x,y,0)) \times \big(\sin(\tfrac{\zeta}{\sqrt{2}}) + \cos(\tfrac{\zeta}{\sqrt{2}})\big). \end{aligned} \tag{33}$$

We are therefore led to the following discussion.

- If $\nu_V \gg \varepsilon$, in order to require $w = 0$ on the boundary, we assume that

  $$w(t,x,y,z) = \sqrt{\varepsilon\nu_V}\,\tilde{w}^1(t,x,y,z) + w_b(t,x,y,\zeta)$$

  up to higher correctors, where $\tilde{w}^1$ describes the behavior of w in the interior of the domain. We recall that in the formal derivation, we have assumed $w = \varepsilon w^1$ inside the domain. These two Ansatz are therefore incompatible. In the framework of rotating flows, this corresponds to the case when the Ekman pumping is so strong that it immediately damps the flow, which decreases as $\exp(-\nu_V^{1/2} t/\varepsilon^{1/2})$. We will not detail this case, which is not the most physical one.
- If $\nu_V = r^2\varepsilon$, equation (33) in fact gives the boundary condition on w^1. Namely, on $z = 0$,

  $$\begin{aligned} \frac{g_0}{S(0)} d_0 \rho^0(t,x,y,0) &= w^1(t,x,y,0) \\ &= \frac{r}{\sqrt{2\rho_s(0)}}(\partial_x v^0(t,x,y,0) - \partial_y u^0(t,x,y,0)), \end{aligned}$$

  or

  $$\frac{1}{S(0)} d_0 \partial_z \psi = -\frac{r}{\sqrt{2\rho_s(0)}}\Delta_{x,y}\psi, \tag{34}$$

which is the boundary condition at $z=0$ which completes (25). At $z=1$, we have

$$S^{-1}(1)d_0\partial_z\psi = r\Delta_{x,y}\psi/\sqrt{2\rho_s(1)}.$$

Let

$$r^b = r\frac{S(0)}{\sqrt{2\rho_s(0)}} \tag{35}$$

and

$$r^t = r\frac{S(1)}{\sqrt{2\rho_s(1)}}. \tag{36}$$

- If $\nu_V \ll \varepsilon$, (34) simply becomes

$$d_0\partial_z\psi = 0 \tag{37}$$

at $z=0$ and $z=1$.

Notice that there is no boundary layer on the density at first order. However $\rho = \rho_s(1+\varepsilon\tilde{\rho})$, and ρ is transported by a flow which has a shear flow behavior near $z=0$ and $z=1$, shear of thickness $\sqrt{\varepsilon\nu_V}$. This leads to gradients of ρ of size $\sqrt{\varepsilon\nu_V}^{-1}$ and to gradients of ρ of order $\sqrt{\varepsilon/\nu_V}$ (see Section 4).

3. Some existence results for quasi-geostrophic equations.

3.1. Setup and main results. This section is devoted to the proof of a global existence theorem for solutions of the potential vorticity quasi-geostrophic system derived in the preceding sections. As before, we consider periodic boundary conditions in (x,y) variables, and rigid boundary conditions at top $z=1$ and bottom $z=0$ that take into account Ekman boundary layers

$$\begin{cases} d_0\omega = 0 \quad \text{in} \quad \mathcal{D}'((0,T)\times\Omega), \\ d_0 g^b = -r^b\Delta_0\Psi_{|z=0} \quad \text{in} \quad \mathcal{D}'((0,T)\times\mathbb{T}^2), \\ d_0 g^t = r^t\Big(\Delta_0\Psi_{|z=1} - \zeta^t\Big) \quad \text{in} \quad \mathcal{D}'((0,T)\times\mathbb{T}^2), \end{cases} \tag{38}$$

where r^b and r^t are defined by (35) and (36), $\Omega = \mathbb{T}^2\times[0,1]$, and ω, g^b, g^t denote

$$\begin{cases} \omega = \Delta_0\Psi + \rho_s^{-1}\partial_z\big(\frac{\rho_s}{S}\partial_z\Psi\big), \\ g^b = \partial_z\Psi_{|z=0} + \eta, \quad g^t = \partial_z\Psi_{|z=1}. \end{cases} \tag{39}$$

The horizontal transport operator $(\partial_t + u_0.\nabla_0)$ is denoted by d_0, where

$$\nabla_0 = \Big| \begin{matrix}\partial_x \\ \partial_y\end{matrix}, \quad \nabla_0^\perp = \Big| \begin{matrix}-\partial_y \\ \partial_x\end{matrix}, \quad \mathrm{div}_0 = \nabla_0. \quad , \quad \Delta_0 = \mathrm{div}_0 \nabla_0,$$

and u_0 is defined by $\nabla_0^\perp \Psi$. Although it has not been rigorously derived in the preceding sections, we can handle non-flat-bottom topographies (see [24], (6.6) and (4.3) for complete details), described by $z = \eta(x, y) \in C^\infty(\mathbb{T}^2)$. ζ^t is linked to the wind and given by the conditions we consider at the upper boundary $z = 1$ (see Section 5.2). The equilibrium density ρ_s and the stratification parameter S are C^∞ functions of z which are bounded and bounded from below by positive constants. Physically, $S = -g_0 \rho_s^{-1} \partial_z \rho_s$, but we will relax this constraint here. Finally, the initial conditions are taken such that

$$\begin{cases} \omega_{|t=0} = \omega_0 \ \in L^\infty \cap L^1(\Omega), \\ g^b_{|t=0} = g^b_0 \in L^2(\mathbb{T}^2), \quad g^t_{|t=0} = g^t_0 \in L^2(\mathbb{T}^2). \end{cases} \tag{40}$$

Without boundaries, the existence of global strong solutions has been proved in [4]. J.T. Beale and A.J. Bourgeois ([3]) proved a global existence theorem of smooth solutions for (38, 39, 40), in the case $g^b_0 \equiv 0$, $g^t_0 \equiv 0$ and $r^b = r^t = 0$ (homogeneous Neumann conditions). Here, we take into account the boundary layers leading to the Ekman pumping effect at top and bottom (see [24]). It turns out that (38, 39, 40) has global weak solutions. Let us emphasize that the Ekman pumping is essential to obtain global weak solutions. As a matter of fact, in the case $r = 0$, we do not know whether solutions persist for all times unless the initial data at the boundary identically vanish ([3]). More precisely, we establish the following theorem:

Theorem 3.1. *Let $r^t > 0$ and $r^b > 0$. There exists a global weak solution Ψ of the potential vorticity quasi-geostrophic system* (38, 39, 40) *such that for all $T > 0$, $\Psi \in L^2(0, T; H^2(\Omega)) \cap C([0, T]; H^s(\Omega))$ for all $s < 2$. Moreover, for almost every $t \geq 0$, we have the energy equality*

$$\begin{aligned} &\frac{1}{2}\int_\Omega \big(\rho_s |\nabla_0 \Psi(t, .)|^2 + \frac{\rho_s}{S} |\partial_z \Psi(t, .)|^2\big) \\ &\qquad + \int_0^t \int_{\mathbb{T}^2} \big(r^b |\nabla_0 \Psi|^2_{|z=0} + r^t |\nabla_0 \Psi|^2_{|z=1}\big)\, ds \\ &\quad = \frac{1}{2}\int_\Omega \big(\rho_s |\nabla_0 \Psi(0, .)|^2 + \frac{\rho_s}{S} |\partial_z \Psi(0, .)|^2\big) + r^t \int_0^t \int_{\mathbb{T}^2} \nabla_0 \zeta^t . \nabla_0 \Psi_{|z=1}\, ds, \end{aligned} \tag{41}$$

and for some constant C depending on the initial data

$$r\int_0^t |\Psi|^2_{H^2(\Omega)}\,ds + |g^b(t,.)|^2_{L^2(\mathbb{T}^2)} + |g^t(t,.)|^2_{L^2(\mathbb{T}^2)} \le C(1+tr). \tag{42}$$

3.2. Approximate solutions. In this section, we want to build approximate solutions to the potential vorticity quasi-geostrophic system using Leray-Schauder fixed-point theorem. Let $T>0$ and $n\in\mathbb{N}$ be given. Let $\mathcal{C}_T$ be the convex set defined by

$$\mathcal{C}_T = \big\{\Psi\in L^2(0,T;H^2(\Omega)) \text{ such that } |\Psi|_{L^2(0,T;H^2(\Omega))} \le R_0\big\}, \tag{43}$$

where R_0 is a constant to be determined. For any given $\overline{\Psi}\in\mathcal{C}_T$, we define $\Psi_n = F_n(\overline{\Psi})$ as the unique solution of the following linearized system in which we introduce additional diffusion terms at the boundary:

$$\begin{cases} \partial_t\omega_n + \operatorname{div}_0\,(\overline{u}_0^n\omega_n) = 0,\\ \partial_t g_n^b + \operatorname{div}_0\,(\overline{u}_0^n g_n^b) = -r^b\Delta_0\Psi_{n|z=0} + \frac{1}{n}\Delta_0 g_n^b,\\ \partial_t g_n^t + \operatorname{div}_0\,(\overline{u}_0^n g_n^t) = r^t\big(\Delta_0\Psi_{n|z=1} - \zeta^t\big) + \frac{1}{n}\Delta_0 g_n^t,\end{cases} \tag{44}$$

where ω_n, $\overline{u}_0^n$, g_n^b and g_n^t are defined by

$$\begin{cases} \omega_n = \Delta_0\Psi_n + \rho_s^{-1}\partial_z\big(\frac{\rho_s}{S}\partial_z\Psi_n\big),\\ \overline{u}_0^n = \nabla_0^\perp K_n(\overline{\Psi}), \quad g_n^b = \partial_z\Psi_{n|z=0} + \eta, \quad g_n^t = \partial_z\Psi_{n|z=1}.\end{cases} \tag{45}$$

The initial data satisfy

$$\omega_{n\ |t=0} = K_n(\omega_0), \quad g^b_{n\ |t=0} = \widetilde{K}_n(g_0^b), \quad g^t_{n\ |t=0} = \widetilde{K}_n(g_0^t), \tag{46}$$

where K_n and $\widetilde{K}_n$ are regularization operators defined respectively on $\mathcal{D}'(\Omega)$ and $\mathcal{D}'(\mathbb{T}^2)$ such that for all $s\ge 0$ and all $f\in H^s(\Omega)$, $K_n f$ converge to f in $H^s(\Omega)$, and for all $h\in H^s(\mathbb{T}^2)$, $\widetilde{K}_n h$ converge to h in $H^s(\mathbb{T}^2)$. Let us finally mention that the existence of global smooth solutions of (44, 45, 46) can be obtained using the Fourier transform in (x,y) variables.

3.2.1. First step: $L^2(H^2)$ bounds. In order to apply the Leray-Schauder theorem, we have to determine R_0 so that F_n maps $\mathcal{C}_T$ into itself.

The H^2 bounds that we derive come from the diffusion terms $\Delta_0\Psi$ at the boundaries and turn out to be uniform in n. Besides, this a priori bound formally holds for the original system. First of all, using the method of characteristics, we obtain for all $p \in [1,\infty]$

$$|\omega_n|_{L^\infty(0,T;L^p(\Omega))} \leq |K_n\omega_0|_{L^p(\Omega)}. \tag{47}$$

Next, integrating by parts, we observe that we have

$$\int_0^t\int_\Omega \big(\rho_s|\Delta_0\Psi_n|^2 + \frac{\rho_s}{S}|\nabla_0\partial_z\Psi_n|^2\big)ds = \int_0^t\int_\Omega \Delta_0\Psi_n\big(\rho_s\omega_n - \partial_z\big(\frac{\rho_s}{S}\partial_z\Psi_n\big)\big)ds$$
$$- \int_0^t\int_\Omega \big\{\partial_z\big(\frac{\rho_s}{S}\partial_z\Psi_n\Delta_0\Psi_n\big) - \Delta_0\Psi_n\partial_z\big(\frac{\rho_s}{S}\partial_z\Psi_n\big)\big\}\,ds, \tag{48}$$
$$= \int_0^t\int_\Omega \rho_s\omega_n\Delta_0\Psi_n\,ds - \int_0^t\int_\Omega \partial_z\big\{\frac{\rho_s}{S}\big(\partial_z\Psi_n + (1-z)\eta\big)\Delta_0\Psi_n\big\}\,ds$$
$$+ \int_0^t\int_\Omega \partial_z\big\{\frac{\rho_s}{S}\big((1-z)\eta\big)\Delta_0\Psi_n\big\}\,ds, \tag{49}$$
$$= \int_0^t\int_\Omega \rho_s\omega_n\Delta_0\Psi_n\,ds - \int_0^t\int_{\mathbb{T}^2} \frac{\rho_s}{S(1)}g_n^t\Delta_0\Psi_{n|z=1}\,ds$$
$$+ \int_0^t\int_{\mathbb{T}^2} \frac{\rho_s}{S(0)}g_n^b\Delta_0\Psi_{n|z=0}\,ds + \int_0^t\int_\Omega \partial_z\big\{\frac{\rho_s}{S}\big((1-z)\eta\big)\Delta_0\Psi_n\big\}\,ds. \tag{50}$$

Using the diffusive transport equations on $z=0$, we can write

$$\int_0^t\int_{\mathbb{T}^2} g_n^b\Delta_0\Psi_{n|z=0}\,ds = -\frac{1}{r^b}\int_0^t\int_{\mathbb{T}^2} g_n^b\big(\partial_t g_n^b + \operatorname{div}_0(\overline{u}_0^n g_n^b) - \frac{1}{n}\Delta_0 g_n^b\big)\,ds \tag{51}$$
$$= -\frac{1}{2r^b}\int_{\mathbb{T}^2}\big(g_n^b(t,.)\big)^2 + \frac{1}{2r^b}\int_{\mathbb{T}^2}\big(g_n^b(0,.)\big)^2 - \frac{1}{nr^b}\int_0^t\int_{\mathbb{T}^2}|\nabla_0 g_n^b|^2\,ds. \tag{52}$$

Similarly, on $z=1$, we have

$$\int_0^t\int_{\mathbb{T}^2} g_n^t\Delta_0\Psi_{n|z=1}\,ds$$
$$= \int_0^t\int_{\mathbb{T}^2} g_n^t\big\{\frac{1}{r^t}\big(\partial_t g_n^t + \operatorname{div}_0(\overline{u}_0^n g_n^t) - \frac{1}{n}\Delta_0 g_n^t\big) + \zeta^t\big\}\,ds, \tag{53}$$
$$= \frac{1}{2r^t}\int_{\mathbb{T}^2}\big(g_n^t(t,.)\big)^2 - \frac{1}{2r^t}\int_{\mathbb{T}^2}\big(g_n^t(0,.)\big)^2 + \int_0^t\int_{\mathbb{T}^2} g_n^t\zeta^t ds$$
$$+ \frac{1}{nr^t}\int_0^t\int_{\mathbb{T}^2}|\nabla_0 g_n^t|^2 ds. \tag{54}$$

Inserting (52) and (54) in (50), we obtain

$$\begin{aligned}
&\int_0^t\int_\Omega\big(\rho_s|\Delta_0\Psi_n|^2+\frac{\rho_s}{S}|\nabla_0\partial_z\Psi_n|^2\big)ds+\frac{\rho_s}{2r^bS(0)}\int_{\mathbb{T}^2}\big(g_n^b(t,.)\big)^2\\
&+\frac{\rho_s}{2r^tS(1)}\int_{\mathbb{T}^2}\big(g_n^t(t,.)\big)^2+\frac{1}{n}\int_0^t\int_{\mathbb{T}^2}\big(\frac{\rho_s}{r^tS(1)}|\nabla_0g_n^t|^2+\frac{\rho_s}{r^bS(0)}|\nabla_0g_n^b|^2\big)\,ds\\
&=\int_0^t\int_\Omega\rho_s\omega_n\Delta_0\Psi_n\,ds+\frac{\rho_s}{2r^bS(0)}\int_{\mathbb{T}^2}\big(g_n^b(0,.)\big)^2+\frac{\rho_s}{2r^tS(1)}\int_{\mathbb{T}^2}\big(g_n^t(0,.)\big)^2\\
&-\frac{\rho_s}{S(1)}\int_0^t\int_{\mathbb{T}^2}g_n^t\zeta^t\,ds \qquad (55)\\
&-\int_0^t\int_\Omega\nabla_0\big\{\frac{\rho_s}{S}\big((1-z)\eta\big)\big\}\nabla_0\partial_z\Psi_n\,ds+\int_0^t\int_\Omega\Delta_0\Psi_n\partial_z\big\{\frac{\rho_s}{S}\big((1-z)\eta\big)\big\}\,ds.
\end{aligned}$$

Therefore, there exists $C>0$ independent of n, R_0, r and $T>0$ such that for all $t\in(0,T)$

$$\begin{aligned}
&r\int_0^t\int_\Omega\big(|\Delta_0\Psi_n|^2+|\nabla_0\partial_z\Psi_n|^2\big)\,ds+\int_{\mathbb{T}^2}\big(g_n^b(t,.)\big)^2+\int_{\mathbb{T}^2}\big(g_n^t(t,.)\big)^2 \qquad (56)\\
&+\frac{1}{n}\int_0^t\int_{\mathbb{T}^2}\big(|\nabla_0g_n^t|^2+|\nabla_0g_n^b|^2\big)\,ds\\
&\le C\big(|\omega_n|^2_{L^\infty(0,T;L^2(\Omega))}+|\zeta^t|^2_{L^2(\mathbb{T}^2)}+\big|\frac{\rho_s}{S}\big((1-z)\eta\big)\big|^2_{H^1(\Omega)}\big)rt\\
&+C\int_{\mathbb{T}^2}\big(g_n^b(0,.)\big)^2+C\int_{\mathbb{T}^2}\big(g_n^t(0,.)\big)^2+Cr\int_0^t\int_{\mathbb{T}^2}|g_n^t|^2\,ds.
\end{aligned}$$

Hence using Gronwall's lemma on $\int_{\mathbb{T}^2}(g_n^t(t,.))^2$, R_0 can be determined in terms of norms of the initial data, r and T. Besides, it does not depend on n. Let us emphasize that the above estimate would be useless in the case $r=0$ since it provides only bounds on the normal derivatives of Ψ at the boundaries.

This leads to a priori $L^2(L^2)$ bounds on $\Delta_0\Psi_n$ and $\nabla_0\partial_z\Psi_n$. To get $L^2(H^2)$ bounds it remains to bound $\partial^2_{zz}\Psi_n$ in $L^2(L^2)$. For that we use

$$|\omega_n|_{L^\infty(L^2)}\le|K_n\omega_0|_{L^2}\quad\text{and}\quad\omega_n=\Delta_0\Psi+\rho_s^{-1}\big(\frac{\rho_s}{S}\partial_z\Psi_n\big).$$

3.2.2. Second step. We want now to derive further bounds for fixed n on the approximate solutions Ψ_n in order to prove that F_n is compact for any given n. Let us observe that the transport equation on ω_n yields for all $s \geq 0$

$$|w_n|^2_{L^\infty(0,T;H^s(\Omega))} + |\partial_t \omega_n|^2_{L^2(0,T;H^s(\Omega))} \leq C_{n,R_0,s}, \tag{57}$$

and specifically that $\partial_t \omega_n \in L^2(0,T;H^{-1/2}(\Omega))$ and $w_n \in L^\infty(0,T;H^{1/2}(\Omega))$. As a matter of fact, since $\overline{u}_0^n$ is $L^2(0,T;H^s(\Omega))$ for all $s \geq 0$, there exists a unique flow $X_n \in C([0,T];C^k(\Omega))$ for all k of $\overline{u}_0^n$ defined by

$$\partial_t X_n(t;s,x) = \overline{u}_0^n(t, X_n(t;s,x)), \quad X_n(t;s,x)_{|t=s} = x. \tag{58}$$

Thus ω_n has an explicit expression

$$\omega_n(t,x) = K_n(\omega_0)(X_n(0;t,x)), \tag{59}$$

so that $\overline{u}_0^n.\nabla_0 \omega_n$ has the claimed $L^2(0,T;H^s(\Omega))$ $(s \geq 0)$ regularity. Next, we prove the following proposition that we shall only use for the existence proof of approximate solutions:

Proposition 3.2. *There exists $C_0 > 0$ depending on T, n, r and R_0 such that, $\forall t \in [0,T]$,*

$$\begin{aligned} r|\Psi_n(t,.)|^2_{H^2(\Omega)} + \frac{1}{n}\int_0^t \big(|\nabla_0 g_n^b|^2_{L^2(\mathbb{T}^2)} + |\nabla_0 g_n^t|^2_{L^2(\mathbb{T}^2)}\big)\, ds \\ + \int_0^t \big(|\partial_t g_n^b|^2_{L^2(\Omega)} + |\partial_t g_n^t|^2_{L^2(\Omega)}\big)\, ds \leq C_0(T,r,n,R_0). \end{aligned} \tag{60}$$

Proof. We proceed as in the first step by writing

$$\begin{aligned} &\frac{1}{2}\int_\Omega \big(\rho_s|\Delta_0\Psi_n|^2 + \frac{\rho_s}{S}|\nabla_0\partial_z\Psi_n|^2\big)(t,.) \\ &= \frac{1}{2}\int_\Omega \big(\rho_s|\Delta_0\Psi_n|^2 + \frac{\rho_s}{S}|\nabla_0\partial_z\Psi_n|^2\big)(0,.) \\ &+ \int_0^t\int_\Omega \Delta_0\Psi_n\partial_t\big\{\rho_s\omega_n - \partial_z\big(\frac{\rho_s}{S}\partial_z\Psi_n\big)\big\}\, ds \\ &- \int_0^t\int_\Omega \big\{\partial_z\big(\Delta_0\Psi_n\partial_t\big(\frac{\rho_s}{S}\partial_z\Psi_n\big)\big) - \Delta_0\Psi_n\partial_z\partial_t\big(\frac{\rho_s}{S}\partial_z\Psi_n\big)\big\}\, ds, \end{aligned} \tag{61}$$

$$\begin{aligned} &= \frac{1}{2}\int_\Omega \big(\rho_s|\Delta_0\Psi_n|^2 + \frac{\rho_s}{S}|\nabla_0\partial_z\Psi_n|^2\big)(0,.) + \int_0^t\int_\Omega \rho_s\Delta_0\Psi_n\partial_t\omega_n\, ds \\ &- \int_0^t\int_\Omega \partial_z\big\{\Delta_0\Psi_n\partial_t\big(\frac{\rho_s}{S}\big(\partial_z\Psi_n + (1-z)\eta\big)\big)\big\}\, ds. \end{aligned} \tag{62}$$

Using once more the diffusive transport equations on $z = 0$ and $z = 1$, we rewrite the last term of the right-hand side as

$$
\begin{aligned}
&-\int_0^t \int_\Omega \partial_z \big\{ \Delta_0 \Psi_n \partial_t \big(\frac{\rho_s}{S} \big(\partial_z \Psi_n + (1-z)\eta \big) \big) \big\} \, ds && (63)\\
&= \int_0^t \int_{\mathbb{T}^2} \frac{\rho_s}{S(0)} \Delta_0 \Psi_{n|z=0} \partial_t g_n^b \, ds - \int_0^t \int_{\mathbb{T}^2} \frac{\rho_s}{S(1)} \Delta_0 \Psi_{n|z=1} \partial_t g_n^t \, ds \\
&= -\frac{\rho_s}{r^b S(0)} \int_0^t \int_{\mathbb{T}^2} \big(\partial_t g_n^b \big)^2 \, ds - \frac{\rho_s}{r^t S(1)} \int_0^t \int_{\mathbb{T}^2} \big(\partial_t g_n^t \big)^2 \, ds \\
&- \frac{1}{2nr^b} \int_{\mathbb{T}^2} |\nabla_0 g_n^b(t,.)|^2 - \frac{1}{2nr^t} \int_{\mathbb{T}^2} |\nabla_0 g_n^t(t,.)|^2 + \frac{1}{2nr^t} \int_{\mathbb{T}^2} |\nabla_0 g_n^t(0,.)|^2 \\
&+ \frac{1}{2nr^b} \int_{\mathbb{T}^2} |\nabla_0 g_n^b(0,.)|^2 - \frac{\rho_s}{r^b S(0)} \int_0^t \int_{\mathbb{T}^2} \overline{u}_0^n . \nabla_0 g_n^b \partial_t g_n^b \, ds \\
&- \frac{\rho_s}{r^t S(1)} \int_0^t \int_{\mathbb{T}^2} \big(\overline{u}_0^n . \nabla_0 g_n^t + r^t \zeta^t \big) \partial_t g_n^t \, ds. && (64)
\end{aligned}
$$

As a result, (62) combined with (64) yields

$$
\begin{aligned}
&r \int_\Omega \big(|\Delta_0 \Psi_n|^2 + |\nabla_0 \partial_z \Psi_n|^2 \big)(t,.) + \int_0^t \int_{\mathbb{T}^2} \big(\partial_t g_n^b \big)^2 \, ds + \int_0^t \int_{\mathbb{T}^2} \big(\partial_t g_n^t \big)^2 \, ds \\
&+ \frac{1}{n} \int_{\mathbb{T}^2} |\nabla_0 g_n^b(t,.)|^2 + \frac{1}{n} \int_{\mathbb{T}^2} |\nabla_0 g_n^t(t,.)|^2 \le C + Cr + Cr |\partial_t \omega_n|^2_{L^2((0,T)\times\Omega)} \\
&+ C \int_0^t \int_{\mathbb{T}^2} |\overline{u}_0^n . \nabla_0 g_n^b|^2 \, ds + C \int_0^t \int_{\mathbb{T}^2} |\overline{u}_0^n . \nabla_0 g_n^t|^2 \, ds + Ctr |\zeta^t|^2_{L^2(\Omega)} && (65)\\
&\quad \le C + Cr + C \int_0^t \int_{\mathbb{T}^2} |\nabla_0 g_n^b|^2 \, ds + C(1+r) \int_0^t \int_{\mathbb{T}^2} |\nabla_0 g_n^t|^2 \, ds, && (66)
\end{aligned}
$$

so that Proposition 3.2 is a straightforward consequence of (65).

3.2.3. Third step: existence of approximate solutions. Using (60) and (57), we first observe that $\partial_t \omega_n$ is bounded in $L^2(0,T;H^{-1/2}(\Omega))$, and $\partial_t \partial_z \Psi_{n|z=0}$ and $\partial_t \partial_z \Psi_{n|z=1}$ are bounded in $L^2(0,T;L^2(\mathbb{T}^2))$. Solving the corresponding Neumann problem in Ω, it follows that $\partial_t \Psi_n$ is bounded in $L^2(0,T;H^{3/2}(\Omega))$. Moreover, ω_n is bounded in $L^\infty(0,T;H^{1/2})$ and $\partial_z \psi_{n|z=0}$ and $\partial_z \psi_{n|z=1}$ are bounded in $L^2(0,T;H^1(\Omega))$; thus, ψ_n is bounded in $L^2(0, T;H^{5/2}(\Omega))$.

Hence $F_n(\mathcal{C}_T)$ is compact in $\mathcal{C}_T$ for fixed n. Consequently, using the fact that F_n is continuous, the Leray-Schauder fixed-point theorem applies and yields global existence of solutions for the approximate system with (46) as initial conditions

$$\begin{cases} d_0^n \omega_n = 0 \text{ in } \mathcal{D}'((0,T)\times\Omega), \\ d_0^n g_n^b = -r^b \Delta_0 \Psi_{n|z=0} + \frac{1}{n}\Delta_0 g_n^b \text{ in } \mathcal{D}'((0,T)\times\mathbb{T}^2), \\ d_0^n g_n^t = r^t\big(\Delta_0 \Psi_{n|z=1} - \zeta^t\big) + \frac{1}{n}\Delta_0 g_n^t \text{ in } \mathcal{D}'((0,T)\times\mathbb{T}^2), \end{cases} \tag{67}$$

where

$$d_0^n = \partial_t + K_n(u_0^n).\nabla_0, \tag{68}$$

$$u_0^n = \nabla_0^\perp \Psi_n, \tag{69}$$

and

$$\begin{cases} \omega_n = \Delta_0 \Psi_n + \rho_s^{-1}\partial_z\big(\frac{\rho_s}{S}\partial_z\Psi_n\big), \\ g_n^b = \partial_z \Psi_{n|z=0} + \eta, \quad g_n^t = \Psi_{n|z=1}. \end{cases} \tag{70}$$

3.3. Energy bounds. We want now to prove that the kinetic energy is conserved. This bound formally holds when $r = 0$. As in the first and second step, we use the transport equations on Ω, $\mathbb{T}^2\times\{0\}$, and $\mathbb{T}^2\times\{1\}$ to write

$$\begin{aligned} &\frac{1}{2}\int_\Omega \big(\rho_s|\nabla_0\Psi_n(t,.)|^2 + \frac{\rho_s}{S}|\partial_z\Psi_n(t,.)|^2\big) \\ &- \frac{1}{2}\int_\Omega \big(\rho_s|\nabla_0\Psi_n(0,.)|^2 + \frac{\rho_s}{S}|\partial_z\Psi_n(0,.)|^2\big) \\ &= -\int_0^t\int_\Omega \big\{\rho_s\Psi_n\partial_t\Delta_0\Psi_n + \Psi_n\partial_t\partial_z\big(\frac{\rho_s}{S}\partial_z\Psi_n\big)\big\}\,ds \\ &+ \int_0^t\int_\Omega \partial_z\big\{\Psi_n\frac{\rho_s}{S}\partial_t\big(\partial_z\Psi_n + \eta(1-z)\big)\big\}\,ds, \end{aligned} \tag{71}$$

$$\begin{aligned} &= -\int_0^t\int_\Omega \rho_s\omega_n u_0^n.\nabla_0\Psi_n\,ds + \frac{\rho_s}{S(1)}\int_0^t\int_{\mathbb{T}^2}\partial_t g_n^t\Psi_n\,ds \\ &- \frac{\rho_s}{S(0)}\int_0^t\int_{\mathbb{T}^2}\partial_t g_n^b\Psi_n\,ds, \end{aligned} \tag{72}$$

$$= \frac{\rho_s}{S(1)} \int_0^t \int_{\mathbb{T}^2} \big\{ g_n^t u_0^n . \nabla_0 \Psi_{n|z=1} - r^t \big(|\nabla_0 \Psi_n|^2_{|z=1} - \nabla_0 \zeta^t . \nabla_0 \Psi_{n|z=1} \big) \big\} \, ds$$
$$+ \frac{1}{n} \int_0^t \int_{\mathbb{T}^2} \big(\frac{\rho_s}{S(1)} \Psi_{n|z=1} \Delta_0 g_n^t - \frac{\rho_s}{S(0)} \Psi_{n|z=0} \Delta_0 g_n^b \big) \, ds$$
$$- \frac{\rho_s}{S(0)} \int_0^t \int_{\mathbb{T}^2} \big\{ g_n^b u_0^n . \nabla_0 \Psi_{n|z=0} + r^b |\nabla_0 \Psi_n|^2_{|z=0} \big\} \, ds. \tag{73}$$

Using the fact that $u_0^n . \nabla_0 \Psi_n = 0$, we deduce that

$$\frac{1}{2} \int_\Omega \big(\rho_s |\nabla_0 \Psi_n(t,.)|^2 + \frac{\rho_s}{S} |\partial_z \Psi_n(t,.)|^2 \big) \tag{74}$$
$$- \frac{1}{2} \int_\Omega \big(\rho_s |\nabla_0 \Psi_n(0,.)|^2 + \frac{\rho_s}{S} |\partial_z \Psi_n(0,.)|^2 \big)$$
$$+ \int_0^t \int_{\mathbb{T}^2} \big(\frac{r^b \rho_s}{S(0)} |\nabla_0 \Psi_n|^2_{|z=0} + \frac{r^t \rho_s}{S(1)} |\nabla_0 \Psi_n|^2_{|z=1} - \frac{r^t \rho_s}{S(1)} \nabla_0 \zeta^t . \nabla_0 \Psi_{n|z=1} \big) ds$$
$$= R_n,$$

where R_n is defined by

$$R_n = \frac{1}{n} \int_0^t \int_{\mathbb{T}^2} \big(\frac{\rho_s}{S(1)} \Psi_{n|z=1} \Delta_0 g_n^t - \frac{\rho_s}{S(0)} \Psi_{n|z=0} \Delta_0 g_n^b \big) \, ds. \tag{75}$$

One more time, we use an integration by parts in the z variable to express R_n as follows:

$$R_n = \frac{1}{n} \int_0^t \int_{\mathbb{T}^2} \partial_z \big\{ \frac{\rho_s}{S} \Psi_n \Delta_0 \big(\partial_z \Psi_n + \eta(1-z) \big) \big\} \, ds \tag{76}$$
$$= \frac{1}{n} \int_0^t \int_{\mathbb{T}^2} \Delta_0 \Psi_n \big\{ \rho_s \big(\omega_n - \Delta_0 \Psi_n \big) + \partial_z \big(\frac{\rho_s}{S} \eta(1-z) \big) \big\} \, ds$$
$$- \frac{1}{n} \int_0^t \int_{\mathbb{T}^2} \big\{ \frac{\rho_s}{S} |\nabla_0 \partial_z \Psi_n|^2 + \frac{\rho_s}{S} \nabla_0 \big(\eta(1-z) \big) . \nabla_0 \partial_z \Psi_n \big\} \, ds. \tag{77}$$

Hence, there exists $C_T > 0$ independent of n such that

$$|R_n| \leq \frac{C_T}{n}. \tag{78}$$

Let us observe that for the original system, we formally have $R = 0$.

3.4. Passage to the limit. Let us recall that the first step of the preceding section yields bounds (56) that do not depend on n. Namely, we have

$$\text{for all} \quad p \in [1, \infty], \quad |\omega_n|_{L^\infty(0,T;L^p(\Omega))} \le C_p, \tag{79}$$

$$r|\Psi_n|_{L^2(0,T;H^2(\Omega))} \le C, \tag{80}$$

$$|g_n^b|_{L^\infty(0,T;L^2(\Omega))} + |g_n^t|_{L^\infty(0,T;L^2(\Omega))} \le C. \tag{81}$$

As a matter of fact, the second derivatives $\partial_z^2 \Psi$ are recovered by using the $L^\infty(0, T;\ L^2(\Omega))$ bound on ω.

As a result, there exists $\Psi \in L^2(0, T; H^2(\Omega))$ and $\omega \in L^\infty(0, T; L^p(\Omega))$ for all $p \in [1, \infty]$ such that Ψ_n converges to Ψ in $\mathcal{D}'((0, T) \times \Omega)$ and ω_n converges to ω in $L^\infty((0, T) \times \Omega)$ weakly $*$. Moreover, ω and Ψ satisfy

$$\omega = \Delta_0 \Psi + \rho_s^{-1} \partial_z \big(\frac{\rho_s}{S} \partial_z \Psi \big). \tag{82}$$

Using the fact that $\partial_t \omega_n$ and u_0^n are respectively bounded in $L^2(0,T; H^{-1}(\Omega))$ and $L^2(0, T; H^1(\Omega))$, we deduce that ω is solution of the transport equation

$$\begin{cases} \partial_t \omega + \mathrm{div}_0 (u_0 \omega) = 0 \quad \text{in} \quad \mathcal{D}'((0, T) \times \Omega), \\ \omega_{|t=0} = \omega_0 \quad \text{in} \quad \mathcal{D}'(\Omega). \end{cases} \tag{83}$$

Furthermore, in view of the trace theorem, $\Psi_{n|z=0}$ is uniformly bounded in $L^2(0, T;\ H^{3/2}(\mathbb{T}^2))$; hence, $\partial_t g_n^b$ is uniformly bounded in $L^2(0, T; W^{-1,4/3}(\mathbb{T}^2)) + L^2(0, T; H^{-1/2}(\mathbb{T}^2))$. Therefore, $\Psi_{|z=0}$ and similarly $\Psi_{|z=1}$ are respectively solutions of

$$\begin{cases} \partial_t g^b + \mathrm{div}_0\ (u_0 g^b) = -r^b \Delta_0 \Psi_{|z=0} \quad \text{in} \quad \mathcal{D}'((0, T) \times \mathbb{T}^2), \\ g^b_{|t=0} = g_0^b \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^2) \end{cases} \tag{84}$$

and

$$\begin{cases} \partial_t g^t + \mathrm{div}_0\ (u_0 g^t) = r^t \big(\Delta_0 \Psi_{|z=1} - \zeta^t \big) \quad \text{in} \quad \mathcal{D}'((0, T) \times \mathbb{T}^2), \\ g^t_{|t=0} = g_0^t \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^2). \end{cases} \tag{85}$$

Next, using the fact that the limit stream function $\partial_t \partial_z \Psi_{|z=0}$ is bounded in $L^2(0, T; W^{-1,4/3}(\mathbb{T}^2)) + L^2(0, T; H^{-1/2}(\mathbb{T}^2))$ and that $\partial_z \Psi_{|z=0}$ is bounded in

$L^2(0,T;H^{1/2}(\mathbb{T}^2))$, we conclude that $\partial_z\Psi_{|z=0}$ belongs to $C([0,T];H^s(\mathbb{T}^2))$ for all $s<1/2$ and in view of the same arguments that $\partial_z\Psi_{|z=1}$ belongs to $C([0,T];H^s(\mathbb{T}^2))$ for all $s<1/2$. Solving the Neumann problem therefore yields the claimed $C([0,T];H^s(\Omega))$ bounds on Ψ for all $s<2$. Finally, the regularity (42) can easily be deduced from (56) so that the only thing that remains to prove in Theorem 3.1 is the energy equality. This will be achieved in the next section, that deals with weak stability issues. Let us notice that we can not reach the case $r=0$ in spite of the $L^\infty(0,T;W^{1+\frac{1}{p},p}(\Omega))$ a priori bounds on Ψ for all $p\in(1,\infty)$. The convergence in $(0,T)\times\Omega$ to the limit equation is straightforward. The major difficulty that prevents us from having global weak solutions is the passage to the limit in the nonlinear terms $u_0\partial_z\Psi_{|z=0}$ and $u_0\partial_z\Psi_{|z=1}$ at the boundary.

3.5. Weak stability.

Theorem 3.3. *Let Ψ_n be a sequence of approximate solutions of* (67) *such that ω_n, Ψ_n, $(\Psi_{n|z=0},\Psi_{n|z=1})$ are respectively bounded in $L^\infty(0,T;L^1\cap L^\infty(\Omega))$, $L^2(0,T;H^2(\Omega))$, $L^\infty(0,T;L^2(\mathbb{T}^2))^2$ uniformly in n. Then, denoting by Ψ and ω weak limits of Ψ_n and ω_n, Ψ is a global weak solution of* $(38,39,40)$. *Moreover, Ψ_n and ω_n respectively converge to Ψ in $C([0,T];H^s(\Omega))$ for all $s<2$ and to ω in $C([0,T];L^p(\Omega))$ for all $p<+\infty$.*

Let us remark that we could as well consider a sequence of weak solutions of the original system $(38,39,40)$ with initial data ω_0^n, $\partial_z\Psi^n_{|z=0,t=0}$ and $\partial_z\Psi^n_{|z=1,t=0}$ strongly converging to some ω_0, $\partial_z\Psi_{|z=0,t=0}$ and $\partial_z\Psi_{|z=1,t=0}$.

Proof. The above bounds lead to the existence of $\omega\in L^\infty((0,T)\times\Omega)$ such that ω_n converges to ω in $L^\infty((0,T)\times\Omega)$ for the weak $*$ topology. On the other hand, there exists $\Psi\in L^2(0,T;H^2(\Omega))$ such that Ψ_n converges weakly to Ψ in $\mathcal{D}'((0,T)\times\Omega)$. In view of the fact that u_0 is tangent to the boundaries $\mathbb{T}^2\times\{0\}$ and $\mathbb{T}^2\times\{1\}$, we deduce from the DiPerna-Lions method of renormalized solutions for linear transport equations that ω_n converges to ω in $C([0,T];L^p(\Omega))$ for all $p\in[1,\infty)$. We refer to [9] and [8] for a detailed proof.

Next, in order to prove the strong stability result of Ψ_n stated in Theorem 3.3, we observe that $\partial_t\partial_z\Psi_{n|z=0}$ is uniformly bounded in

$$L^2(0,T;W^{-1,4/3}(\mathbb{T}^2))+L^2(0,T;H^{-1/2}(\mathbb{T}^2)).$$

We deduce that $\partial_z \Psi_{n|z=0}$ is compact in $C([0,T]; H^{-s_0})$ for some positive, large-enough s_0. Since $\partial_z \Psi_{|z=0}$ is also bounded uniformly in n in $L^2(0,T; H^{1/2}(\mathbb{T}^2))$, $\partial_z \Psi_{n|z=0}$ converges to $\partial_z \Psi_{|z=0}$ in $C([0,T]; H^s(\mathbb{T}^2))$ for all $s < 1/2$. Finally, we can solve the Neumann problem in Ω to conclude that Ψ_n converges to Ψ in $C([0,T]; H^s(\Omega))$ for all $s < 2$.

As a corollary of the above stability result, we deduce the energy equality that only involves first derivatives of Ψ in Ω, $\mathbb{T}^2 \times \{0\}$ and $\mathbb{T}^2 \times \{1\}$.

Remark. We can also prove using similar arguments that there exist global weak solutions to (38, 39, 40) when $r = +\infty$. In that case, the conditions at the boundary $z = 0$ reduce to $\Delta \Psi_{|z=0} = 0$, and similarly $\Delta \Psi_{|z=1} = \zeta^t$ on $z = 1$. As a result, $\Psi_{|z=0}$ and $\Psi_{|z=1} - \Delta^{-1} \zeta^t$ are functions of t only, Δ^{-1} denoting the inverse Laplacian with zero mean value in the case when $\Omega = \mathbb{T}^2 \times [0,1]$. The main argument is based upon $L^2(0,T; H^2(\Omega))$ bounds on Ψ_r that are uniform on r, Ψ_r denoting a global weak solution for fixed r.

3.6. Strong solutions. We want now to discuss the existence of strong solutions in the case of smooth initial data. T. Beale and A. Bourgeois ([3]) proved global existence in $\Omega = \mathbb{T}^2 \times [0,1]$ of strong solutions for the following system:

$$\begin{cases} (\partial_t + u_0.\nabla_0)\omega = 0 \text{ in } \mathcal{D}'((0,T) \times \Omega), \\ \omega = \Delta_0 \Psi + \rho_s^{-1} \partial_z \left(\frac{\rho_s}{S} \partial_z \Psi \right) \text{ in } \mathcal{D}'((0,T) \times \Omega), \\ \partial_z \Psi_{|z=0} = \partial_z \Psi_{|z=1} \equiv 0 \text{ in } \mathcal{D}'((0,T) \times \mathbb{T}^2). \end{cases} \tag{86}$$

Let us observe that the above conditions at the boundary hold at $t = 0$ and thus limit the generality of initial configurations. A consequence of a result due to T. Colin ([7]) yields a similar theorem. He makes use of an argument based upon a model arising in numerical applications, namely the multilayer stratified quasi-geostrophic model, where the number of layers tends to infinity.

Let us mention that local existence of smooth solutions can easily be obtained by classical arguments. It is however not clear whether smooth solutions of the potential vorticity quasi-geostrophic system with physical boundary conditions (38, 39, 40) persist for all times. As a matter of fact, we can prove partial regularity properties inspired by Beale, Kato and Majda ([2]) in the context of the incompressible Euler equations.

We first introduce a few notations. Given $j \in \{1,2\}$, ∂_j^0 is defined by $\partial_1^0 = \partial_x$ when $j = 1$ and $\partial_2^0 = \partial_y$ when $j = 2$. Similarly, u_0^j denotes the j^{th} horizontal component of u_0. For all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ and $\beta \in \mathbb{N}^3$, D_0^α and

D^β are respectively defined as $(\partial_1^0)^{\alpha_1}(\partial_2^0)^{\alpha_2}$ and $(\partial_1^0)^{\beta_1}(\partial_2^0)^{\beta_2}\partial_z^{\beta_3}$. Finally, for any given $(s,k)\in\mathbb{N}^2$, we denote by $\widetilde{H}^{s,k}(\Omega)$ the Sobolev space

$$\begin{aligned}\widetilde{H}^{s,k}(\Omega) =&\{\phi\in L^2(\Omega) \text{ such that } D_0^\alpha\partial_z^j\phi\in L^2(\Omega)\\ &\text{for all } (\alpha,j)\in\mathbb{N}^2\times\mathbb{N} \text{ such that } j+|\alpha|\le s \text{ and } j\le k\},\end{aligned} \tag{87}$$

endowed with the norm

$$|\phi|^2_{\widetilde{H}^{s,k}(\Omega)} = \sum_{|\beta|+j\le s, j\le k} |D_0^\beta\partial_z^j\phi|^2_{L^2(\Omega)}. \tag{88}$$

Let $s\in\mathbb{N}$ such that $s\ge 3$. We assume that $\partial_z\Psi_{|z=0,t=0}$ and $\partial_z\Psi_{|z=1,t=0}$ belong to $H^s(\mathbb{T}^2)$, $\omega_{|t=0}\in\widetilde{H}^{s,0}(\Omega)\cap H^{s-1}(\Omega)$, and $\Psi\in\widetilde{H}^{s+2,1}(\Omega)$. Then, the following theorem holds for solutions of (38, 39, 40) whenever $r\ge 0$

Theorem 3.4. *As long as*

$$\int_0^T |\nabla_{x,y,z}u_0|_{L^\infty(\Omega)}\,ds < +\infty, \tag{89}$$

we have $\Psi\in L^\infty(0,T;\widetilde{H}^{s+2,2}(\Omega)\cap H^{s+1}(\Omega))\cap Lip([0,T];H^s(\Omega))$. *If* $r>0$ *and* $\omega_0\in H^s(\Omega)$, $\Psi\in L^2([0,T];H^{s+2}(\Omega))\cap C^{1/2}([0,T];H^{s+1/2}(\Omega))$.

For the sake of brevity, shall prove Theorem 3.4 in the case when $\eta\equiv 0$, $\zeta^t\equiv 0$, $r^t=r^b=r$, $\rho_s\equiv 1$ and $S\equiv 1$ (which is not physical since the first section leads to $S=-g_0\rho_s^{-1}\partial_z\rho_s$, but simplifies the already heavy bounds). In the following proof, the regularity in (x,y,z) is deduced from the tangential regularity in (x,y) variables. Let $T>0$ be given such that (89) holds. First of all, we obtain for all $\alpha\in\mathbb{N}^2$ such that $|\alpha|=s$

$$\partial_t D_0^\alpha\omega + u_0.\nabla_0 D_0^\alpha\omega = [u_0,\nabla_0 D_0^\alpha]\omega. \tag{90}$$

Let us recall a classical estimate on commutators in $\mathbb{T}^2$ where $\beta\in\mathbb{N}^2$ such that $|\beta|=\sigma$ for some $\sigma\ge 0$:

$$|[a,D_0^\beta]b|_{L^2(\mathbb{T}^2)} \le C\big(|D_0a|_{L^\infty(\mathbb{T}^2)}|b|_{H^{\sigma-1}(\mathbb{T}^2)} + |b|_{L^\infty(\mathbb{T}^2)}|a|_{H^\sigma(\mathbb{T}^2)}\big). \tag{91}$$

This bound relies upon Gagliardo-Nirenberg's inequality and can be found in the appendix of [18]. Hence, using (91) for fixed z and integrating over $(0,1)$ in the vertical variable z, we can write in view of the energy estimates

$$\begin{aligned}&\partial_t|\omega|^2_{\widetilde{H}^{s,0}(\Omega)}\\ &\le C\big(|\omega(t,.)|_{L^\infty(\Omega)}|u_0|_{\widetilde{H}^{s+1,0}(\Omega)} + |\nabla_0u_0|_{L^\infty(\Omega)}|\omega|_{\widetilde{H}^{s,0}(\Omega)}\big)|\omega|_{\widetilde{H}^{s,0}(\Omega)}.\end{aligned} \tag{92}$$

Multiplying (90) by $D_0^\alpha \Psi$ for $|\alpha| = s+1$ and integrating by parts, we obtain

$$\frac{1}{2}\partial_t \int_\Omega |\nabla_{x,y,z} D_0^\alpha \Psi|^2 - \int_\Omega \partial_z \big(D_0^\alpha \Psi \partial_t D_0^\alpha \partial_z \Psi \big) = \int_\Omega D_0^\alpha \Psi D_0^\alpha \mathrm{div}_0\ (u_0 \omega); \tag{93}$$

hence,

$$\begin{aligned} &\frac{1}{2}\partial_t \int_\Omega |\nabla_{x,y,z} D_0^\alpha \Psi|^2 + r \int_{\mathbb{T}^2} \big(|\nabla_0 D_0^\alpha \Psi_{|z=0}|^2 + |\nabla_0 D_0^\alpha \Psi_{|z=1}|^2 \big) \\ &= - \int_\Omega \partial_z \big(D_0^\alpha \Psi D_0^\alpha \mathrm{div}_0\ (u_0 \partial_z \Psi) \big) + \int_\Omega D_0^\alpha \Psi D_0^\alpha \mathrm{div}_0\ (u_0 \omega) \qquad (94) \\ &= \int_\Omega \big\{ D_0^\alpha \Psi D_0^\alpha \mathrm{div}_0\ (u_0 (\Delta_0 \Psi + \partial_z^2 \Psi)) - D_0^\alpha \Psi D_0^\alpha \mathrm{div}_0\ (u_0 \partial_z^2 \Psi) \\ &- D_0^\alpha \partial_z \Psi D_0^\alpha \mathrm{div}_0\ (u_0 \partial_z \Psi) \big\}, \qquad (95) \\ &= \int_\Omega \big\{ D_0^\alpha \partial_j^0 \Psi \big([D_0^\alpha, u_0^k] \partial_k^0 \partial_j^0 \Psi \big) - D_0^\alpha \partial_z \Psi \big([D_0^\alpha, u_0^k] \partial_z \partial_k^0 \Psi \big) \big\}. \qquad (96) \end{aligned}$$

Using (91), we can write

$$\begin{aligned} &\partial_t \int_\Omega |\nabla_{x,y,z} D_0^\alpha \Psi|^2 + r \int_{\mathbb{T}^2} \big(|\nabla_0 D_0^\alpha \Psi_{|z=0}|^2 + |\nabla_0 D_0^\alpha \Psi_{|z=1}|^2 \big) \\ &\qquad \le C |\nabla_{x,y,z} u_0|_{L^\infty(\Omega)} |\nabla_{x,y,z} \Psi|^2_{\widetilde{H}^{s+1,0}(\Omega)}. \end{aligned} \tag{97}$$

Next, deriving the transport equation on $z = 0$ and $z = 1$, and integrating by parts yields for any $\alpha \in \mathbb{N}^2$ such that $|\alpha| = s$

$$\begin{aligned} &\frac{1}{2}\partial_t \big(|D_0^\alpha \partial_z \Psi_{|z=0}|^2_{L^2(\mathbb{T}^2)} + |D_0^\alpha \partial_z \Psi_{|z=1}|^2_{L^2(\mathbb{T}^2)} \big) = r \int_\Omega \partial_z \big(\Delta_0 D_0^\alpha \Psi \partial_z D_0^\alpha \Psi \big) \\ &+ \int_\Omega \partial_z \big((1-2z) \partial_z D_0^\alpha \Psi D_0^\alpha \mathrm{div}_0\ (u_0 \partial_z \Psi) \big) \qquad (98) \\ &= -r \int_\Omega |D_0^\alpha \Delta_0 \Psi|^2 - r \int_\Omega |D_0^\alpha \nabla_0 \partial_z \Psi|^2 + r \int_\Omega D_0^\alpha \omega D_0^\alpha \Delta_0 \Psi \\ &- 2 \int_\Omega \partial_z D_0^\alpha \Psi D_0^\alpha \mathrm{div}_0\ (u_0 \partial_z \Psi) + \int_\Omega (1-2z) D_0^\alpha \partial_z^2 \Psi D_0^\alpha \mathrm{div}_0\ (u_0 \partial_z \Psi) \qquad (99) \\ &+ \int_\Omega (1-2z) \partial_z D_0^\alpha \Psi D_0^\alpha \mathrm{div}_0 (u_0 \partial_z^2 \Psi) + \int_\Omega (1-2z) \partial_z D_0^\alpha \Psi D_0^\alpha \big(\partial_z u_0 . \nabla_0 \partial_z \Psi \big). \end{aligned}$$

The last term of the above right-hand side vanishes since $\partial_z u_0.\nabla_0\partial_z\Psi \equiv 0$. Therefore, we have

$$
\begin{aligned}
&\tfrac{1}{2}\partial_t\big(|D_0^\alpha\partial_z\Psi_{|z=0}|^2_{L^2(\mathbb{T}^2)} + |D_0^\alpha\partial_z\Psi_{|z=1}|^2_{L^2(\mathbb{T}^2)}\big) + r\int_\Omega |D_0^\alpha\Delta\Psi|^2 \\
&\quad + r\int_\Omega |D_0^\alpha\nabla_0\partial_z\Psi|^2 = r\int_\Omega D_0^\alpha\omega D_0^\alpha\Delta_0\Psi - 2\int_\Omega \partial_z D_0^\alpha\Psi\big([D_0^\alpha, u_0^k]\partial_k^0\partial_z\Psi\big) \\
&\quad + \int_\Omega (1-2z)\Big\{D_0^\alpha(\omega-\Delta_0\Psi)\big([D_0^\alpha, u_0^k]\partial_k^0\partial_z\Psi\big) \\
&+ D_0^\alpha\partial_z\Psi\big([D_0^\alpha\partial_k^0, u_0^k]\omega\big) + D_0^\alpha\partial_z\partial_k^0\Psi\big([D_0^\alpha, u_0^k]\Delta_0\Psi\big)\big\}.
\end{aligned}
\tag{100}
$$

As a result, recovering estimates on $\partial_z^2\Psi$ from the identity $\partial_z^2\Psi = \omega - \Delta_0\Psi$, we deduce that (100) leads to

$$
\begin{aligned}
&\partial_t\big(|D_0^\alpha\partial_z\Psi_{|z=0}|^2_{L^2(\mathbb{T}^2)} + |D_0^\alpha\partial_z\Psi_{|z=1}|^2_{L^2(\mathbb{T}^2)}\big) + r|D_0^\alpha\Delta\Psi|^2_{L^2(\Omega)} \\
&+ r|D_0^\alpha\nabla_0\partial_z\Psi|^2_{L^2(\Omega)} + r|D_0^\alpha\partial_z^2\Psi|^2_{L^2(\Omega)} \le Cr|\omega|^2_{\widetilde{H}^{s,0}(\Omega)} \\
&+ Cr|\omega|_{\widetilde{H}^{s,0}(\Omega)}|\Delta_0\Psi|_{\widetilde{H}^{s,0}(\Omega)} + C|\nabla_{x,y,z}u_0|_{L^\infty(\Omega)}|\nabla_{x,y,z}\Psi|^2_{\widetilde{H}^{s,0}(\Omega)} \\
&+ C|\nabla_{x,y,z}u_0|_{L^\infty(\Omega)}|\nabla_{x,y,z}\Psi|_{\widetilde{H}^{s,0}(\Omega)}\big(|\omega|_{\widetilde{H}^{s,0}(\Omega)} + |\Delta_0\Psi|_{\widetilde{H}^{s,0}(\Omega)}\big) \\
&+ C|\nabla_{x,y,z}\Psi|_{\widetilde{H}^{s,0}(\Omega)}\big(|\omega|_{L^\infty(\Omega)}|u_0|_{\widetilde{H}^{s+1,0}(\Omega)} + |\omega|_{\widetilde{H}^{s,0}(\Omega)}|\nabla_{x,y,z}u_0|_{L^\infty(\Omega)}\big) \\
&+ |\nabla_{x,y,z}u_0|_{L^\infty(\Omega)}|\partial_z\nabla_0\Psi|_{\widetilde{H}^{s,0}(\Omega)}|\nabla_{x,y,z}\Psi|_{\widetilde{H}^{s,0}(\Omega)} \qquad (101) \\
&\le C(1+r)\big(1+|\nabla_{x,y,z}u_0|_{L^\infty(\Omega)}\big)\big(|\omega|^2_{\widetilde{H}^{s,0}(\Omega)} + |\nabla_{x,y,z}\Psi|^2_{\widetilde{H}^{s+1,0}(\Omega)}\big). \qquad (102)
\end{aligned}
$$

Using the energy bounds and summing (92), (97) and (101) finally yields

$$
\begin{aligned}
&\partial_t\big(|\omega|^2_{\widetilde{H}^{s,0}(\Omega)} + |\nabla_{x,y,z}\Psi|^2_{\widetilde{H}^{s+1,0}(\Omega)} + |\partial_z\Psi_{|z=0}|^2_{H^s(\mathbb{T}^2)} + |\partial_z\Psi_{|z=1}|^2_{H^s(\mathbb{T}^2)}\big) \\
&+ r\big\{|\Psi|^2_{\widetilde{H}^{s+2,2}(\Omega)} + |\Psi_{|z=0}|^2_{H^{s+2}(\mathbb{T}^2)} + |\Psi_{|z=1}|^2_{H^{s+2}(\mathbb{T}^2)}\big\} \\
&\le C(1+r)\big(1+|\nabla_{x,y,z}u_0|_{L^\infty(\Omega)}\big)\big(|\omega|^2_{\widetilde{H}^{s,0}(\Omega)} + |\nabla_{x,y,z}\Psi|^2_{\widetilde{H}^{s+1,0}(\Omega)}\big),
\end{aligned}
\tag{103}
$$

so that we deduce from (89) that

$$
\begin{aligned}
&\omega \in L^\infty(0,T;\widetilde{H}^{s,0}(\Omega)), \quad \nabla_{x,y,z}\Psi \in L^\infty(0,T;\widetilde{H}^{s+1,0}(\Omega)), \\
&\partial_z\Psi_{|z=0} \ \text{ and } \ \partial_z\Psi_{|z=1} \in L^\infty(0,T;H^s(\mathbb{T}^2)).
\end{aligned}
\tag{104}
$$

Moreover, since $\partial_z^2\Psi = \omega - \Delta_0\Psi$, we have $\Psi \in L^\infty(0,T;\widetilde{H}^{s+2,2}(\Omega))$ even if $r = 0$. Further regularity in z is obtained by applying D^β to the transport equation on the vorticity, where $\beta \in \mathbb{N}^3$ and $|\beta| = s-1$. Precisely, we can write

$$\partial_t D^\beta\omega + u_0.\nabla_0 D^\beta\omega = [u_0, \nabla_0 D^\beta]\omega; \tag{105}$$

hence, adapting (91) in Ω, we obtain

$$\partial_t|\omega|^2_{H^{s-1}(\Omega)} \le C|\omega|_{H^{s-1}(\Omega)}\big(|\nabla_{x,y,z}u_0|_{L^\infty(\Omega)}|\omega|_{H^{s-1}(\Omega)} + |\omega|_{L^\infty(\Omega)}|u_0|_{H^s(\Omega)}\big). \tag{106}$$

Next, solving the Neumann problem in Ω provides the following bound:

$$|u_0|_{H^s(\Omega)} \le C\big(|\partial_z\Psi_{|z=0}|_{H^{s-1/2}(\mathbb{T}^2)} + |\partial_z\Psi_{|z=1}|_{H^{s-1/2}(\mathbb{T}^2)} + |\omega|_{H^{s-1}(\Omega)}\big) \tag{107}$$

$$\le C + C|\omega|_{H^{s-1}(\Omega)}. \tag{108}$$

Consequently, we have

$$\partial_t|\omega|^2_{H^{s-1}(\Omega)} \le C\big(1 + |\nabla_{x,y,z}u_0|_{L^\infty(\Omega)}\big)|\omega|^2_{H^{s-1}(\Omega)} + C, \tag{109}$$

so that the proof is complete, solving once again the Neumann problem. The claimed regularity in time follows from the fact that

$$\begin{aligned}&\partial_t\omega \text{ is bounded in } L^\infty(0,T;H^{s-2}(\Omega)),\\&\partial_t\partial_z\Psi_{|z=0} \text{ and } \partial_t\partial_z\Psi_{|z=1} \text{ are bounded in } L^\infty(0,T;H^{s-3/2}(\Omega)).\end{aligned} \tag{110}$$

The additional bounds in the case $r > 0$ are straightforward consequences of the fact that $\Psi_{|z=0}$ and $\Psi_{|z=1}$ are bounded in $L^2(0,T;H^{s+2}(\mathbb{T}^2))$. Indeed, solving the Dirichlet problem in Ω, we obtain

$$|u_0|_{H^{s+1}(\Omega)} \le C\big(|\Psi_{|z=0}|_{H^{s+2}(\mathbb{T}^2)} + |\Psi_{|z=1}|_{H^{s+2}(\mathbb{T}^2)} + |\omega|_{H^s(\Omega)}\big), \tag{111}$$

so that we can estimate $|\omega|_{H^s(\Omega)}$ the same way as $|\omega|_{H^{s-1}(\Omega)}$in (109).

4. Convergence.

Theorem 4.1. *Let $\Psi \in L^\infty([0,T], H^s(\Omega))$ with s large enough be a smooth solution to the quasi-geostrophic equations* (38) *with $\zeta^t = 0$ and $\eta = 0$. Let*

$$U_l = \big(-\partial_y \Psi, \partial_x \Psi, 0, -\frac{\partial_z(\rho_s \Psi)}{g_0 \rho_s} \big).$$

Then there exists a sequence of global weak solutions $U^\varepsilon = (u^\varepsilon, v^\varepsilon, w^\varepsilon, \rho^\varepsilon)$ of $(16, 17, 18, 19, 20, 26)$ *such that*

$$\|U^\varepsilon - U_l\|_{L^\infty([0,T],L^2(\Omega))} \to 0 \quad as \quad \varepsilon \to 0.$$

Notice that we are vague on the initial conditions on U^ε in order to avoid technicalities in the boundary layer. This will be made precise in the next sections. The proof of this theorem involves two steps. First we build approximate solutions $U^{\varepsilon,app}$ for (16, 17, 18, 19, 20, 26) starting from U_l, then we make energy estimates on the difference $U^\varepsilon - U^{\varepsilon,app}$.

4.1. Approximate solutions.

Let us begin by a definition:

Definition 4.2. We say that $U^{\varepsilon,app} = (u^{\varepsilon,app}, v^{\varepsilon,app}, w^{\varepsilon,app}, \rho^{\varepsilon,app})$ is a sequence of approximate solutions of order N of (16, 17, 18, 19, 20, 26) on $[0,T]$ with initial data $(u^{ini}, v^{ini}, w^{ini}, \rho^{ini})$ if the following conditions are fulfilled:

(i) the first three components of U^ε are divergence free and satisfy (26),

(ii) the error terms obtained by putting $U^{\varepsilon,app}$ in (16, 17, 18, 19) are of the form $\varepsilon^N \mathcal{R}_u^\varepsilon$, $\varepsilon^N \mathcal{R}_v^\varepsilon$, $\varepsilon^N \mathcal{R}_w^\varepsilon$ and $\varepsilon^N \mathcal{R}_\rho^\varepsilon$, where $\mathcal{R}_u^\varepsilon$, $\mathcal{R}_v^\varepsilon$, $\mathcal{R}_w^\varepsilon$ are uniformly bounded in $L^2([0,T]; L^2(\Omega))$, and $\mathcal{R}_\rho^\varepsilon$ is uniformly bounded in $L^3([0,T]; L^3(\Omega))$,

(iii) $U^{\varepsilon,app}$ satisfies the following bounds:
- $u^{\varepsilon,app}, v^{\varepsilon,app}, w^{\varepsilon,app}$ are bounded in $L^\infty([0,T]; L^2(\Omega) \cap L^\infty(\Omega))$ and in $L^2([0,T]; H^1(\Omega))$, and $\rho^{\varepsilon,app}$ is bounded in $L^\infty([0,T] \times \Omega)$,
- $\partial_x U^{\varepsilon,app}$ and $\partial_y U^{\varepsilon,app}$ are bounded in $L^\infty([0,T] \times \Omega)$,
- $\rho^{\varepsilon,app} = \rho_i^{\varepsilon,app} + \rho_b^{\varepsilon,app}$, where $\partial_z \rho_i^{\varepsilon,app}$ is uniformly bounded in $L^\infty([0,T] \times \Omega)$, and

$$|\partial_z \rho_b^{\varepsilon,app}| \le \gamma(t) \sqrt{\frac{\varepsilon}{\nu_V}} \big[\exp\big(-\frac{z}{C\sqrt{\varepsilon \nu_V}} \big) + \exp\big(-\frac{(1-z)}{C\sqrt{\varepsilon \nu_V}} \big) \big], \quad (112)$$

where $\gamma(t) \in L^\infty([0,T])$,

- $u^{\varepsilon,app} = u_i^{\varepsilon,app} + u_b^{\varepsilon,app}$, where $\partial_z u_i^{\varepsilon,app}$ is uniformly bounded in $L^\infty([0,T] \times \Omega)$, and

$$|\partial_z u_b^{\varepsilon,app}| \leq \gamma(t) \frac{1}{\sqrt{\varepsilon \nu_V}} \big[\exp\big(- \frac{z}{C\sqrt{\varepsilon \nu_V}} \big) + \exp\big(- \frac{(1-z)}{C\sqrt{\varepsilon \nu_V}} \big) \big], \quad (113)$$

where $\gamma(t) \in L^\infty([0,T])$, and similarly for $v^{\varepsilon,app}$ and $w^{\varepsilon,app}$,

(iv) $U^{\varepsilon,app}(0)$ converges to $(u^{ini}, v^{ini}, w^{ini}, \rho^{ini})$ in $L^2(\Omega)$ as ε goes to 0.

The indices b and i respectively refer to boundary and interior behavior. Estimate (113) is classical in boundary-layer theory, and the ε factor gained in (112) with respect to (113) comes from $\rho = \rho_s(1 + \varepsilon\tilde{\rho})$. Notice also that no assumption is done on time derivatives.

Let us turn to the construction of approximate solutions:

Proposition 4.3. *Let $N \geq 0$, and let $(u^{ini}, v^{ini}, w^{ini}, \rho^{ini}) \in H^s(\Omega)$ with s large enough (depending on N), satisfying (21, 22, 23) for some $p^{ini} \in H^{s+1}$. Then there exists a sequence $U^{\varepsilon,app}$ of approximate solutions of order N of (16, 17, 18, 19, 20, 26), with initial data $(u^{ini}, v^{ini}, w^{ini}, \rho^{ini})$ on a time interval $[0,T]$, for all $T < T_0$, where T_0 is the existence time of strong solutions of the limit system (25) with boundary conditions (34).*

Remarks. The initial conditions are required to satisfy (21, 22, 23), which are called the "geostrophic constraints." In particular the data are "well prepared"; that is, the time derivatives of the solutions are uniformly bounded in ε and satisfy the "bounded derivatives principle" (H.-O. Kreiss). When these constraints are not satisfied, waves of amplitude $O(1)$ and of high speed propagate in the fluid. The energy method of the next section is still valid, but the construction of approximate solutions is more technical. Notice also that meteorologists usually want to filter out these waves in numerical computations, by "preparing" the initial data. We refer to [1], [11] and [25] for the study of "ill prepared data" for rotating fluids and general partial differential equations (without boundary).

Notice also that $(u^{ini}, v^{ini}, w^{ini}, \rho^{ini})$ does not satisfy in general (26) and hence is not the initial condition of (16, 17, 18, 19, 20, 26). We have in fact to add the boundary layers to match $u = v = 0$ on $\partial\Omega$. In particular $U^{\varepsilon,app}$ converges only in $L^2(\Omega)$ to $(u^{ini}, v^{ini}, w^{ini}, \rho^{ini})$ (in fact in $H^s(\Omega)$ for $s < 1/2$, but not in $L^\infty(\Omega)$ in general). This problem is classical in boundary-layer theory: we can not start from an arbitrary sequence of initial data since the boundary-layer behavior is prescribed (here by (31, 32)). Therefore we start from an initial data for the limit problem and build an initial data for the perturbed problem which has the right behavior near $\partial\Omega$.

Proof. We will only sketch the proof, since the complete construction is tedious and has been done thoroughly in nearby cases, for instance in [14] and [16] for the inviscid limit of quasi-linear parabolic systems.

For the sake of simplicity we will take $\nu_V = r^2\varepsilon$ and $\nu_H = s^2\varepsilon$, and will construct approximate solutions in the half space $z \geq 0$ (the case $0 \leq z \leq 1$ is similar). We look for approximate solutions of the form

$$u^{\varepsilon,app} = \sum_{j=0}^{M} \varepsilon^j u_i^j(t,x,y,z) + \varepsilon^j u_b^j(t,x,y,\varepsilon^{-1}z)$$

and proceed similarly for $v^{\varepsilon,app}$, $w^{\varepsilon,app}$ and $\rho^{\varepsilon,app}$.

As usual, the equations on u_i^j are given by putting $u^\varepsilon = \sum_{j=0}^M \varepsilon^j u_i^j$ in the equation and looking at coefficients of order ε^j, ignoring u_b^j. On the other hand, the boundary conditions on these equations (which are linearized versions of (25)) are given by the asymptotic behavior of u_b^j.

The equations on u_b^j are given by setting $\zeta = \varepsilon^{-1}z$ in the equation and identifying terms with the same power of ε. As boundary conditions we have $u_b^j \to 0$ as $\zeta \to +\infty$ and $u_b^j = -u_i^j$ for $\zeta = 0$.

Let us detail the first steps. First we solve (25) with boundary conditions (34) and initial data $(u^{ini}, v^{ini}, w^{ini}, \rho^{ini})$, and get a strong solution ψ^0 on $[0, T_0)$, with $T_0 \leq +\infty$. Notice that that $T_0 = +\infty$ if $\partial_z\psi^0 = 0$ on $z = 0$ and $z = 1$ and $r = 0$. This gives $u_i^0 = -\partial_y\psi^0$, $v_i^0 = \partial_x\psi^0$, $w_i^0 = 0$ and $\rho_i^0 = -g_0^{-1}\rho_s^{-1}\partial_z(\rho_s\psi^0)$. Then we construct the boundary layer u_b^0 and v_b^0 with (31) and (32). Using (20) at order ε^{-1}, we get $w_b^0 = 0$. Moreover (19) at order 0 gives w_i^1. Using again (20), at order ε^0, we get

$$\partial_x u_b^0 + \partial_y v_b^0 + \partial_\zeta w_b^1 = 0$$

which gives w_b^1, assuming $w_b^1 \to 0$ as $\zeta \to +\infty$. Notice that $w_b^1 + w_i^1 = 0$ for $z = 0$ is exactly the boundary condition (34) on (25), and hence holds automatically true.

Now using (19), we get ρ_b^0 simply by solving the transport equation with source term

$$\begin{aligned}\partial_t(\rho_i^0 + \rho_b^0) &+ (u_i^0 + u_b^0)\partial_x(\rho_i^0 + \rho_b^0) + (v_i^0 + v_b^0)\partial_y(\rho_i^0 + \rho_b^0)\\ &+ (w_i^1 + w_b^1)\partial_\zeta\rho_b^0 + (w_b^1 + w_i^1)\frac{\partial_z\rho_s}{\rho_s} = 0.\end{aligned}$$

At this point we completely get order 0 (u_i^0, v_i^0, w_i^0, ρ_i^0, u_b^0, v_b^0, w_b^0, ρ_b^0), w_i^1 and w_b^1. Next we turn to the equations on u_i^1, v_i^1, w_i^1 and ρ_i^1. These unknowns are solutions of

$$\partial_t u_i^1 + u_i^0 \partial_x u_i^1 + u_i^1 \partial_x u_i^0 + v_i^0 \partial_y u_i^1 + v_i^1 \partial_y u_i^0 + w_i^0 \partial_z u_i^1 \\ + w_i^1 \partial_z u_i^0 - v_i^2 - s^2 \Delta_{x,y} u_i^0 - r^2 \partial_{zz}^2 u_i^0 = -\frac{1}{\rho_s} \partial_i \tilde{p}^1$$

and proceed similarly for v_i^1, w_i^1 and ρ_i^1. From (16, 17, 18) at order 0, we already know

$$v_i^1 = \frac{1}{\rho_s} \partial_x \tilde{p}^0 + \text{ known}, \tag{114}$$

$$u_i^1 = -\frac{1}{\rho_s} \partial_y \tilde{p}^0 + \text{ known}, \tag{115}$$

$$\rho_i^1 = -\frac{1}{g_0 \rho_s} \partial_z \tilde{p}^0 + \text{known}, \tag{116}$$

(w_i^1 is already known), where the generic term *known* stands for terms which depend only on u_i^0, v_i^0, w_i^0 and ρ_i^0. Let

$$\zeta^1 = \partial_x v_i^1 - \partial_y u_i^1$$

and $\psi^1 = \tilde{p}^0$. We have

$$\zeta^1 = \Delta_{x,y} \psi^1 + \text{ known}.$$

Taking the curl of the equations on u_i^1 and v_i^1, we eliminate $\tilde{p}^1$, and using (19) to express w_2^i in terms of ρ_i^1, we get an equation on ψ^1, a linearized version of (25) (the boundary conditions will be detailed later). This gives u_i^1, v_i^1, ρ_i^1, using (114, 115, 116).

Next $\partial_\zeta \tilde{p}_b^0$ is given with (18) in terms of $\rho_i^0 + \rho_b^0$, which is known. Hence $\tilde{p}_b^0$ is known up to a function of z only, which disappears in what follows. The equations on v_b^1 and u_b^1 are then similar to (30) except that there is a source term $\partial_{x,y} \tilde{p}_b^0$. They can be solved as previously, which gives v_b^1 and u_b^1, and thus w_b^2, which is the boundary condition completing the equation on ψ^1. Moreover ρ_b^1 is computed using the transport equation, since we know w_i^2 and w_b^2.

We have thus gained one order in the expansion. Higher orders are similar and will not be detailed further.

4.2. Energy estimates. Adapting the proof of P.-L. Lions ([22]) in the case of the nonhomogeneous incompressible Navier-Stokes equations, we observe that (16, 17, 18, 19, 20, 26) has global weak solution $(u^\varepsilon, v^\varepsilon, w^\varepsilon, \rho^\varepsilon)$ such that for all $T > 0$,

$$u^\varepsilon, v^\varepsilon, w^\varepsilon \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$$
$$\rho^\varepsilon \in L^\infty((0,T)\times\Omega) \cap C([0,T]; L^p(\Omega))$$

for all $p \in [1, +\infty)$, as soon as the initial data satisfy $(u^{0,\varepsilon}, v^{0,\varepsilon}, w^{0,\varepsilon}) \in L^2(\Omega)^3$, $\partial_x u^{0,\varepsilon} + \partial_y v^{0,\varepsilon} + \partial_z w^{0,\varepsilon} = 0$ and $\rho^{0,\varepsilon} \in L^1 \cap L^\infty(\Omega)$.

Theorem 4.4. *Let $U^{\varepsilon,app} = (u^{\varepsilon,app}, v^{\varepsilon,app}, w^{\varepsilon,app}, \rho^{\varepsilon,app})$ be a sequence of approximate solutions of order N of* $(16, 17, 18, 19, 20, 26)$*, on a time interval* $[0,T]$*, in the sense of Definition* 4.2*, given by Proposition* 4.3*. Let us assume that $\nu_H \varepsilon^{-1} \to +\infty$ and $\nu_H \nu_V^{-1} \to +\infty$. Then, if N is large enough, there exists a global weak solution $U^\varepsilon = (u^\varepsilon, v^\varepsilon, w^\varepsilon, \rho^\varepsilon)$ of* $(16, 17, 18, 19, 20, 26)$ *such that*

$$\|U^\varepsilon - U^{\varepsilon,app}\|_{L^\infty([0,T]\times\Omega)} \to 0 \tag{117}$$

as $\varepsilon \to 0$.

Proof. Let $\bar U^\varepsilon = U^\varepsilon - U^{\varepsilon,app} = (\bar u, \bar v, \bar w, \bar\rho)$ which satisfy

$$\begin{aligned}\partial_t \bar u &+ (u^{app} + \bar u)\partial_x \bar u + \bar u \partial_x u^{app} + (v^{app} + \bar v)\partial_y \bar u + \bar v \partial_y u^{app}\\ &+ (w^{app} + \bar w)\partial_z \bar u + \bar w \partial_z u^{app} - \frac{\bar v}{\varepsilon} - \frac{1}{\rho_s}\mathcal{F}\bar u = -\frac{1}{\rho_s}\partial_x \bar p + \varepsilon^N \mathcal{R}_u\end{aligned} \tag{118}$$

$$\begin{aligned}\partial_t \bar v &+ (u^{app} + \bar u)\partial_x \bar v + \bar u \partial_x v^{app} + (v^{app} + \bar v)\partial_y \bar v + \bar v \partial_y v^{app}\\ &+ (w^{app} + \bar w)\partial_z \bar v + \bar w \partial_z v^{app} + \frac{\bar u}{\varepsilon} - \frac{1}{\rho_s}\mathcal{F}\bar v = -\frac{1}{\rho_s}\partial_y \bar p + \varepsilon^N \mathcal{R}_v\end{aligned} \tag{119}$$

$$\begin{aligned}\partial_t \bar w &+ (u^{app} + \bar u)\partial_x \bar w + \bar u \partial_x w^{app} + (v^{app} + \bar v)\partial_y \bar w + \bar v \partial_y w^{app}\\ &+ (w^{app} + \bar w)\partial_z \bar w + \bar w \partial_z w^{app} + \frac{g_0 \bar\rho}{\varepsilon} - \frac{1}{\rho_s}\mathcal{F}\bar w = -\frac{1}{\rho_s}\partial_z \bar p + \varepsilon^N \mathcal{R}_w\end{aligned} \tag{120}$$

$$\begin{aligned}\partial_t \bar\rho &+ (u^{app} + \bar u)\partial_x \bar\rho + \bar u \partial_x \rho^{app} + (v^{app} + \bar v)\partial_y \bar\rho + \bar v \partial_y \rho^{app} + (w^{app} + \bar w)\partial_z \bar\rho\\ &+ \bar w \partial_z \rho^{app} + \bar w \rho^{app}\frac{\partial_z \rho_s}{\rho_s} + w^{app}\bar\rho\frac{\partial_z \rho_s}{\rho_s} + \bar w \bar\rho \frac{\partial_z \rho_s}{\rho_s} + \frac{\bar w}{\varepsilon}\frac{\partial_z \rho_s}{\rho_s} = \varepsilon^N \mathcal{R}_\rho,\end{aligned} \tag{121}$$

with $\partial_x u^{app} + \partial_y v^{app} + \partial_z w^{app} = 0$ and $\partial_x \bar{u} + \partial_y \bar{v} + \partial_z \bar{w} = 0$. We will study the following norm, equivalent to the $L^2(\Omega)$ norm:

$$|||\bar{U}|||^2 = \int \rho_s(\bar{u}^2 + \bar{v}^2 + \bar{w}^2) + \int \phi\bar{\rho}^2,$$

where $\phi(z) = -\frac{\rho_s^2 g_0}{\partial_z \rho_s} > 0$. We will detail the estimates of the various terms occurring in $\partial_t |||\bar{U}|||^2/2$. First of all

$$\int \rho_s \bar{u}\big((u^{app}+\bar{u})\partial_x \bar{u} + (v^{app}+\bar{v})\partial_y \bar{u} + (w^{app}+\bar{w})\partial_z \bar{u}\big) = -\int \frac{\bar{u}^2}{2}\partial_z \rho_s (w^{app}+\bar{w}),$$

and similarly for the terms involving $\bar{v}$ and $\bar{w}$. Notice that $\int \bar{u}^2 w^{app}\partial_z\rho_s$ is bounded by $C|||\bar{U}|||^2$, whereas

$$\big|\int \bar{u}^2 \bar{w}\partial_z\rho_s\big| \le C\|\bar{w}\|_{L^2}\|\bar{u}\|_{L^2}^{1/2}\|\bar{u}\|_{H^1}^{3/2} \le \eta\nu_V\|\bar{u}\|_{H^1}^2 + \frac{C}{(\eta\nu_V)^3}\|\bar{w}\|_{L^2}^4\|\bar{u}\|_{L^2}^2.$$

Next

$$\big|\int \rho_s \bar{u}\big(\bar{u}\partial_x u^{app} + \bar{v}\partial_y u^{app}\big)\big| \le C|||\bar{U}|||^2$$

since $\partial_x U^{\varepsilon,app}$ and $\partial_y U^{\varepsilon,app}$ are bounded. Now $\int \rho_s \bar{u}\bar{w}\partial_z u^{app}$ can be treated as in [15]. The estimate on $\int \rho_s \bar{u}\bar{w}\partial_z u_i^{app}$ is straightforward. Let us bound

$$J = \int\int_0^{1/2} \rho_s \bar{u}\bar{w}\partial_z u_b^{app}\, dz\, dx\, dy,$$

the integral for z between $1/2$ and 1 being similar. We have

$$J = \int\int_0^{1/2}\rho_s\big(\int_0^z \partial_z\bar{u}d\tilde{z}\big)\big(\int_0^z \partial_z\bar{w}d\tilde{z}\big)\partial_z u_b^{app};$$

hence,

$$|J| \le C\int_{x,y}\big(\int_0^{1/2} z\partial_z u_b^{app}dz\big)\big(\int_0^1 |\partial_z\bar{u}|^2 d\tilde{z}\big)^{1/2}\big(\int_0^1 |\partial_z\bar{w}|^2 d\tilde{z}\big)^{1/2}.$$

But

$$\big|\partial_z u_b^{app}\big| \le C\sqrt{\varepsilon\nu_V}^{-1}\exp\big(-\frac{z}{C\sqrt{\varepsilon\nu_V}}\big);$$

therefore,

$$\Big|\int_0^{1/2} z\partial_z u_b^{app} dz\Big| \le C\sqrt{\varepsilon\nu_V},$$

which leads to

$$|J| \le C\sqrt{\varepsilon\nu_V}\|\partial_z\bar{u}\|_{L^2}\|\partial_x\bar{u}+\partial_y\bar{v}\|_{L^2},$$

where we used the incompressibility condition; thus,

$$|J| \le \eta\nu_V\|\partial_z\bar{u}\|_{L^2}^2 + \frac{C\varepsilon}{\eta}\|\partial_x\bar{u}\|_{L^2}^2 + \frac{C\varepsilon}{\eta}\|\partial_y\bar{v}\|_{L^2}^2$$

(η arbitrarily small). Since $\nu_H/\varepsilon \to +\infty$, we in fact get

$$|J| \le \eta\nu_V\|\partial_z\bar{u}\|_{L^2}^2 + \eta\nu_H\|\partial_x\bar{u}\|_{L^2}^2 + \eta\nu_H\|\partial_y\bar{v}\|_{L^2}^2.$$

The Coriolis force term yields a null contribution to the energy balance.

Moreover, the choice of the weight ρ_s leads to the estimate

$$\int \bar{u}\partial_x\bar{p} + \bar{v}\partial_y\bar{p} + \bar{w}\partial_z\bar{p}$$

which identically vanishes. The viscosity terms leads to

$$-\int \nu_H\bar{u}\Delta_{x,y}\bar{u} = \nu_H\int|\nabla_{x,y}u|^2$$

and

$$-\int \nu_V\bar{u}\partial_{zz}^2\bar{u} = \nu_V\int|\partial_z\bar{u}|^2.$$

Now

$$\int \rho_s g_0\bar{w}\bar{\rho} + \int \phi\bar{\rho}\bar{w}\frac{\partial_z\rho_s}{\rho_s} = 0$$

by definition of ϕ.

The transport term on $\bar{\rho}$ leads to $\int\bar{\rho}^2 w^{app}\partial_z\phi$, bounded by $C\int\bar{\rho}^2$, and to $\int\bar{\rho}^2\bar{w}\partial_z\phi$, which is bounded by

$$\Big|\int\bar{\rho}^2\bar{w}\partial_z\phi\Big| \le \int\bar{\rho}^3 + C\int\bar{w}^3,$$

and

$$\int \bar{w}^3 \le \|\bar{w}\|_{L^2}^{3/2}\|\bar{w}\|_{H^1}^{3/2} \le \frac{C}{(\eta\nu_V)^3}\|\bar{w}\|_{L^2}^6 + \eta\nu_V\|\bar{w}\|_{H^1}^2.$$

Moreover,

$$\Big|\int \phi\bar{\rho}\big(\bar{w}\rho^{app}\frac{\partial_z\rho_s}{\rho_s} + w^{app}\bar{\rho}\frac{\partial_z\rho_s}{\rho_s}\big)\Big| \le C\|\bar{w}\|_{L^2}^2 + C\|\bar{\rho}\|_{L^2}^2$$

and

$$\Big|\int \phi\bar{\rho}^2\bar{w}\frac{\partial_z\rho_s}{\rho_s}\Big| \le C\int \bar{\rho}^3 + C\int \bar{w}^3.$$

It remains to bound

$$\int \phi\bar{\rho}\big(\bar{u}\partial_x\rho^{app} + \bar{v}\partial_y\rho^{app} + \bar{w}\partial_z\rho^{app}\big)$$

which is bounded by $C|||\bar{U}|||^2$, except $\int \phi\bar{\rho}\bar{w}\partial_z\rho^{app}$ since $\partial_z\rho^{app}$ is bounded by $C + C\exp(-z/(C\sqrt{\varepsilon\nu_V}))\sqrt{\varepsilon/\nu_V}$ (for $z < 1/2$ and similarly for $z > 1/2$). Therefore we have to bound, using $\rho^{app} = \rho_i^{app} + \rho_b^{app}$,

$$J = \sqrt{\frac{\varepsilon}{\nu_V}}\int_{x,y}\int_0^{1/2}\phi\bar{\rho}\bar{w}\exp\big(-\frac{z}{C\sqrt{\varepsilon\nu_V}}\big).$$

As before we use the control by the viscosity in order to improve the behavior of w:

$$J = \sqrt{\frac{\varepsilon}{\nu_V}}\int_{x,y}\int_0^{1/2}\phi\bar{\rho}\big(\int_0^z \partial_z\bar{w}d\tilde{z}\big)\exp\big(-\frac{z}{C\sqrt{\varepsilon\nu_V}}\big);$$

hence,

$$\begin{aligned}
|J| &\le C\sqrt{\frac{\varepsilon}{\nu_V}}\int_{x,y}\big(\int_0^{1/2}\bar{\rho}\sqrt{z}\exp\big(-\frac{z}{C\sqrt{\varepsilon\nu_V}}\big)\big)\big(\int_0^1|\partial_z\bar{w}|^2d\tilde{z}\big)^{1/2}\\
&\le C\sqrt{\frac{\varepsilon}{\nu_V}}\int_{x,y}\big(\int_0^{1/2}\bar{\rho}^2\big)^{1/2}\big(\int_0^1 z\exp\big(-\frac{2z}{C\sqrt{\varepsilon\nu_V}}\big)\big)^{1/2}\big(\int_0^1|\partial_z\bar{w}|^2d\tilde{z}\big)^{1/2}\\
&\le C\varepsilon\|\bar{\rho}\|_{L^2}\|\partial_z\bar{w}\|_{L^2} \le \frac{C}{\eta}\|\bar{\rho}\|_{L^2}^2 + \eta\varepsilon^2\big(\|\partial_x\bar{u}\|_{L^2}^2 + \|\partial_y\bar{v}\|_{L^2}^2\big)\\
&\le \frac{C}{\eta}\|\bar{\rho}\|_{L^2}^2 + \eta\nu_H\big(\|\partial_x\bar{u}\|_{L^2}^2 + \|\partial_y\bar{v}\|_{L^2}^2\big)
\end{aligned}$$

since $\nu_H \gg \varepsilon$.

Summing the estimates, we get for η small enough

$$\partial_t |||\bar U|||^2 + \frac{\nu_H}{2}\int |\nabla_{x,y}\bar U_0|^2 + \frac{\nu_V}{2}\int |\partial_z \bar U_0|^2 \le C|||\bar U|||^2 + \frac{C}{(\eta\nu_V)^3}|||\bar U|||^6 + \int \bar\rho^3 + \varepsilon^{2N}\|\mathcal R\|_{L^2}^2, \tag{122}$$

where $\bar U_0 = (\bar u, \bar v, \bar w)$, or integrating in time

$$|||\bar U|||^2 + \frac{\nu_H}{2}\int_0^t\int |\nabla_{x,y}\bar U_0|^2 + \frac{\nu_V}{2}\int_0^t\int |\partial_z \bar U_0|^2 \le |||\bar U(0)|||^2 + C\int_0^t |||\bar U|||^2 + \frac{C}{(\eta\nu_V)^3}\int_0^t |||\bar U|||^6 + \int_0^t\int \bar\rho^3 + \varepsilon^{2N}\int_0^t \|\mathcal R\|_{L^2}^2. \tag{123}$$

Let us turn to the bound on $\int \bar\rho^3$. We have

$$\partial_t(\rho_s\bar\rho) + (u^{app}+\bar u)\partial_x(\rho_s\bar\rho) + (v^{app}+\bar v)\partial_y(\rho_s\bar\rho) + (w^{app}+\bar w)\partial_z(\rho_s\bar\rho) = \mathcal S^\varepsilon, \tag{124}$$

where

$$\mathcal S^\varepsilon = -\bar u\partial_x(\rho_s\rho^{app}) - \bar v\partial_y(\rho_s\rho^{app}) - \bar w\partial_z(\rho_s\rho^{app}) - \frac{\bar w}{\varepsilon}\partial_z\rho_s + \varepsilon^N\rho_s\mathcal R_\rho.$$

Using the divergence-free condition we get

$$\partial_t\int(\rho_s\bar\rho)^3 = \int 3\rho_s^2\bar\rho^2\mathcal S^\varepsilon \le C|\rho_s\bar\rho|_{L^3}^2|\mathcal S^\varepsilon|_{L^3} \le C|\mathcal S^\varepsilon|_{L^3}^3 + |\rho_s\bar\rho|_{L^3}^3;$$

hence,

$$|\bar\rho|_{L^3}^3 \le C\big(|\bar\rho(0)|_{L^3}^3 + \int_0^t |\mathcal S^\varepsilon|_{L^3}^3\big).$$

But

$$\int \bar u^3 \le C\|\bar u\|_{L^2}^{3/2}\|u\|_{H^1}^{3/2};$$

hence,

$$\int |\mathcal S^\varepsilon|^3 \le C\Big(\frac{1}{\varepsilon} + \sqrt{\frac{\varepsilon}{\nu_V}}\Big)^3\|\bar U_0\|_{L^2}^{3/2}\|\bar U_0\|_{H^1}^{3/2} + C\varepsilon^{3N}|\mathcal R_\rho|_{L^3}^3,$$

using the bounds on the derivatives of ρ^{app}, which leads to

$$\int_0^t \int \bar\rho^3 \le C\big(|\bar\rho(0)|_{L^3}^3 + \frac{C}{\eta^3\nu_V^3}\big(\frac{1}{\varepsilon} + \sqrt{\frac{\varepsilon}{\nu_V}}\big)^{12} \int_0^t \|\bar U_0\|_{L^2}^6 + \int_0^t \eta\nu_V \|\bar U_0\|_{H^1}^2 + \int_0^t \varepsilon^{3N} |\mathcal{R}_\rho|_{L^3}^3\big). \tag{125}$$

Summing this with (123) gives, for η small enough

$$\begin{aligned} |||\bar U|||^2 \le |||\bar U(0)|||^2 + C|\bar\rho(0)|_{L^3}^3 + C\int_0^t |||\bar U|||^2 \\ + \frac{C}{\eta^3\nu_V^3}\big(\frac{1}{\varepsilon} + \sqrt{\frac{\varepsilon}{\nu_V}}\big)^{12} \int_0^t \|\bar U\|_{L^2}^6 + C\varepsilon^{2N}\int_0^t \|\mathcal{R}\|_{L^2}^2 + C\varepsilon^{3N}\int_0^t \|\mathcal{R}_\rho\|_{L^3}^3. \end{aligned} \tag{126}$$

It remains to handle the term $|||\bar U|||^6$ in the right-hand side of (126). The main trick is to set

$$\hat U = \varepsilon^{-N'} \bar U,$$

where $N' < N$ to take advantage of the fact that $\bar U$ is small. We then get

$$\begin{aligned} |||\hat U|||^2 \le |||\hat U(0)|||^2 + C\varepsilon^{N'}|\hat\rho(0)|_{L^2}^3 + C\int_0^t |||\hat U|||^2 \\ + \frac{C\varepsilon^{4N'}}{\eta^3\nu_V^3}\big(\tfrac{1}{\varepsilon} + \sqrt{\tfrac{\varepsilon}{\nu_V}}\big)^{12} \int_0^t \|\hat U_0\|_{L^2}^6 + \varepsilon^{2N-2N'}\int_0^t \|\mathcal{R}\|_{L^2}^2 + \varepsilon^{3N-2N'}\int_0^t \|\mathcal{R}_\rho\|_{L^3}^3, \end{aligned} \tag{127}$$

where $\hat U_0 = (\hat u, \hat v, \hat w)$. We recall that η is small with respect to ν_H and ν_V. For N' large enough the end of the proof is straightforward, using a Gronwall lemma.

5. Remarks. The method used in this paper can be applied to numerous other nearby systems which arise in meteorology. This section is devoted to some hints on possible other tractable cases.

5.1. The limit $\delta \to 0$. The same energy method can be extended to the case $\delta \to 0$, with $\delta \sim \varepsilon^{\sigma'}$ for some σ', by taking

$$|||U|||^2 = \int \rho_s(\bar u^2 + \bar v^2 + \delta^2 \bar w^2) + \int \phi \bar\rho^2$$

as energy norm. We however emphasize that the limit flow satisfies $w^0 = 0$ not because $\delta \to 0$ (thin layer of fluid), but because the density is stratified (fluid particles can not change their heights, because of the buoyancy forces).

5.2. Boundary conditions for oceanic motions. The lower and upper boundary conditions in the case of oceanic motions are different, since the lower boundary is a solid (bottom of the ocean) whereas the upper boundary is a free surface (interface with the atmosphere).

Namely if we study oceanic motions between $z = 0$ and $z = 1$, it is usual to set

$$u = v = w = 0 \quad \text{at} \quad z = 0 \tag{128}$$

and

$$w = 0 \quad \text{and} \quad \partial_z(u, v) = \tau(x, y) \quad \text{at} \quad z = 1, \tag{129}$$

where τ is a given vector which describes the wind at the surface of the ocean (see [6] for a study of the Ekman layer in the case of homogeneous rotating fluids). The limit system is the same, except the boundary condition at $z = 1$ which becomes (see [24], Chapters 4 and 6)

$$d_0 \partial_z \psi = \operatorname{curl} \tau, \tag{130}$$

and the energy method works as well (Theorem 4.1 holds true). We do not know however whether we have global weak solutions or not since there is no Ekman pumping term on $z = 1$.

5.3. Boundary conditions for atmospheric motions. For the atmosphere, at $z = 0$, the boundary conditions are

$$w = 0 \quad \text{and} \quad \partial_z(u, v) = \tau(x, y) \tag{131}$$

if we consider atmospheric motions on the ocean, or

$$u = v = w = 0 \tag{132}$$

if we consider atmospheric motions on the earth.

Upper boundary conditions at $z = 1$ are more delicate, since there is no real upper boundary, and is an open question, even from a physical point of view (see [24]). One possibility is to enforce (132) at $z = 1$, which is the case treated in this paper (with (132) at $z = 0$). Another possibility is to consider the motion in the half space $z \geq 0$ (Theorem 4.4 remains valid).

5.4. Miscellaneous remarks. It is possible to handle the case when ν_H does not go to 0. We then get existence of global strong solutions for the limit system, can construct approximate solutions at any order for all times, and justify the convergence globally in time.

Notice that in [5] and [17] the authors call "quasi-geostrophic" a slightly different limit; namely, they replace the equation on the density, by an equation on the temperature where they put a Laplace operator.

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