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CRYSTALLINE VERSION OF THE STEFAN PROBLEM WITH GIBBS–THOMPSON LAW AND KINETIC UNDERCOOLING

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Abstract. The author studies the modified Stefan problem in the plane with surface tension and kinetic undercooling when the interfacial curve is a polygon. Existence of local-in-time solutions is shown. Geometric properties of the flow are studied if the Wulff shape is a regular N-sided polygon. The author shows that an initial interface being a scaled Wulff shape with sufficiently small perimeter shrinks to a point. Moreover, at each time the interface remains a scaled Wulff shape.

1. Introduction. We study a version of the modified Stefan problem in the plane. The special feature of our approach is that we assume that the interfacial curve is a polygon. We stress that admitting nonsmooth interfaces is natural from the viewpoint of modeling crystal evolution. We shall pursue this direction.

Describing the process of melting or growing a crystal requires setting the problem in the framework of two-phase thermodynamics. This is done in the book of Gurtin, [10]. The author of the book pays special attention to the evolution of nonsmooth interfaces (see Section 12 in [10]). Developing this theory Gurtin and Matias proposed in their paper [11] the particular problem we study here. The setting is the following: a crystal $\Omega_1(t)$ is in a container Ω filled with melt $\Omega_2(t)$; i.e., $\Omega_2(t) = \Omega \setminus \overline{\Omega_1}(t)$ (the notation shall be explained in detail in the next section). The heat transport is described by the equation

$$e_i u_t = -\text{div}\, \mathbf{q} \quad \text{in} \quad \bigcup_{0<t<T} \Omega_i(t), \quad i = 1, 2, \tag{1.1}$$

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which is complemented by the Fourier law

$$\mathbf{q} = -k_i \nabla u, \quad i = 1, 2. \tag{1.2}$$

The temperature u is continuous across the interface $s = \partial\Omega_1 \cap \partial\Omega_2$ being a polygon with facets s_i, $s = \cup_{i=1}^N s_i$. We assume the number of facets is constant. We are fully aware that this assumption can be the subject of discussion. We claim that in certain circumstances this hypothesis is valid. We will address this issue in Section 4.

Continuing the description of our problem we denote by V_i the velocity of s_i in the direction of the outer normal ν_i. This velocity satisfies the equation

$$[\![\mathbf{q}]\!]\nu_j = V_j, \quad j = 1, \dots, N. \tag{1.3}$$

Finally, we need a condition that the temperature u must satisfy on the interface. The approximation to the balance of capillary forces yields

$$\int_{s_j(t)} u = \Gamma_j - \beta_j L_j(t) V_j(t), \quad j = 1, \dots, N. \tag{1.4}$$

We remark here that the above problem was formulated by Herring in the metallurgical literature in the fifties; see [14]. Later, it was independently rediscovered by Ben Amar–Pomeau ([2]) and Gurtin–Matias ([12]).

One might expect here a version of Gibbs–Thompson law (with or without kinetic undercooling). This law states that the temperature on the interface is proportional to the curvature of the interface. In our consideration u is a normalized temperature, so u is zero at melting temperature on a flat interface. Indeed, (1.4) is a version of the Gibbs–Thompson law, where because of lack of smoothness the definition of curvature is adjusted so that it is well defined also for polygons. As we shall see later Γ_i/L_i plays the role of curvature of the edge s_i, $i = 1, \dots, N$ (Γ_i is an appropriate constant).

We note here that the modified Stefan problem for smooth interfaces (without kinetic undercooling) has been already studied. The first paper is by Luckhaus ([15]) who considered weak solutions and C^1-interfaces. Almgren and Wang ([1]) studied this problem in a more general setting allowing for anisotropic surface energy densities.

The presence of nonzero kinetic undercooling makes the analysis somewhat simpler. The modified Stefan problem with undercooling was studied in particular by Chen and Reitich (see [3]). They showed local-in-time existence of smooth temperature u (away from the interface) as well as smooth interface. Independently, Radkevich ([17]) studied the same problem. The advantage of [17] is that the author allows a slightly more general form of the heat equation.

We restrict our attention to polygonal interfaces, so our treatment is different from the one already mentioned above. We hope that working with polygons allows us to use a relatively simple technique as well as to obtain more precise information on behavior of solutions. In order to start our work we formulate a weak form of (1.1)–(1.4) augmented with initial and boundary conditions. This is done in Section 2. For the sake of simplicity we work assuming that the bulk specific heats, e_i, are equal, $e_1 = e_2 = \epsilon > 0$, and similarly we set the coefficients of conductivity k_i, $i = 1, 2$, to be 1. In Section 3 we show local-in-time existence of weak solutions. We cannot expect global existence since we anticipate that geometry may change during the evolution. Some of the properties of solutions will be exposed in Section 4.

Investigation of "crystalline motion"(i.e., evolution of fully faceted curves) is natural also from the view-point of numerical simulations. But we shall not dwell on numerics; we would rather refer the interested reader to the literature, e.g., [4, 7, 8, 9, 18] and references therein. We concentrate on some of the theoretical aspects of the problem. In particular the work on viscosity-type-solution framework ([5]) and a variational formulation of the crystalline motion in the space ([6]) are beyond the scope of this paper. However, of special interest to us are the papers [21], [20]. J. Taylor ([21]) considers the system where the i-th facet is governed by an ODE

$$\Gamma_i = \beta_i L_i V_i. \tag{1.5}$$

System (1.5) seems to be the "zero-temperature limit" of (1.1)–(1.4). Interestingly, both flows behave similarly if the initial interface is a scaled Wulff shape and its perimeter is small. Briefly speaking, it is known that solutions of (1.5) approach the underlying Wulff shape (see [20]). We show for weak solutions of (1.1)–(1.4) a similar but weaker statement in Theorem 4.1 of Section 4. It states that $s(t)$ remains a scaled Wulff shape and $s(t)$ shrinks to a point in finite time. Such a result is also true for a so-called quasi-steady approximation of (1.1)–(1.4), i.e., for $\epsilon = 0$ (cf. Proposition 13 in [19]).

2. Weak formulation. Before stating the problem (1.1)–(1.4) in the weak form let us explain the setting and our basic assumptions. We shall consider only admissible polygonal interfaces, where the edges s_i are numbered counterclockwise. Admissibility means here that the outer normals

ν_i to the facets s_i belong to the set of normals of a given Wulff shape W (cf. Sections 7 and 12 in [10]). Moreover, we require that normals to successive facets in s must be neighboring normals to W. For the sake of present analysis we may think of W as being a given, convex polygon with M edges numbered counterclockwise. Let us note that $N \geq M$ and the equality holds if s is convex.

The length of facet s_i determined by its vertices v_i, v_{i+1} is denoted by L_i, $L_i = |v_i - v_{i+1}|$. The perimeter L of s is equal to $\sum_{j=1}^{N} L_i$. The velocity of V_i of the edge s_i in the direction of the outer normal ν_i is defined by

$$V_i(t) = \frac{d}{dt} z_i(t),$$

where

$$z_i(t) = \begin{cases} \mathrm{dist}\,(l_i(t), l_i(0)) & \text{if } (v_i(t) - v_i(0)) \cdot \nu_i > 0, \\ -\mathrm{dist}\,(l_i(t), l_i(0)) & \text{if } (v_i(t) - v_i(0)) \cdot \nu_i < 0, \end{cases} \tag{2.1}$$

and $l_i(t)$ is the line containing $s_i(t)$. The definition of V_i in (1.3) involves the jump $[\![\cdot]\!]$ across $s(t)$. This quantity is given by

$$[\![\phi]\!](x_0) = \lim_{\Omega_2(t) \ni x \to x_0} \phi(x) - \lim_{\Omega_1(t) \ni x \to x_0} \phi(x), \quad x_0 \in \partial\Omega_1(t) \cap \partial\Omega_2(t),$$

where $x_0 \in s(t)$. We assume that the sets Ω, $\Omega_1(t)$, $\Omega_2(t)$ are regions, where $\Omega_1(t) \subset\subset \Omega$ and $\Omega = \Omega_1(t) \cup s(t) \cup \Omega_2(t)$. At last we assume that the boundary $\partial\Omega$ of Ω is smooth.

The kinetic coefficients $\beta_j > 0$ are constants; so are Γ_j, $j = 1, \ldots, N$, and they are defined depending on s as follows (see Section 12.5 in [10]):

$$\Gamma_j = \begin{cases} -\ell_j & \text{if } s \text{ is locally convex near both vertices } v_i,\ v_{i+1}, \\ \ell_j & \text{if } s \text{ is locally concave near both vertices } v_i,\ v_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

where ℓ_j is the length of the edge of the Wulff shape with normal ν_j.

Interestingly, Γ_j / L_j may be related to the curvature of s_j. We make use of the definition of curvature which does not take into account the differential structure of s. The appropriate notion is crystalline curvature; see [21, p. 423]. If the z_i are as defined above and $\mathbf{z} = (z_1, \ldots, z_N)$, i.e., $s(\mathbf{z})$ is a polygon resulting from moving the entire facet s_i by z_i in the direction of the normal ν_i, $A(\mathbf{z})$ is the area surrounded by $s(\mathbf{z})$ and $L(\mathbf{z})$ is the perimeter of $s(\mathbf{z})$, then the *crystalline curvature* $\mathcal{K}_i$ *of* s_i is

$$\mathcal{K}_i = - \lim_{\Delta z_i \to 0} \frac{L(\mathbf{z} + \mathbf{e}_i \Delta z_i) - L(\mathbf{z})}{A(\mathbf{z} + \mathbf{e}_i \Delta z_i) - A(\mathbf{z})}$$

where $\mathbf{e}_i$, $i = 1, \dots, N$, are the standard unit vectors of the coordinate axis in $\mathbb{R}^N$. This limit may be evaluated with the aid of the lemma below, whose proof we leave to the reader (cf. also [19]).

Lemma 2.1. *Let us suppose we are given a polygon* s *with its edges* s_i *numbered counterclockwise,* $i = 1, \dots, N$. *If* L_i *is the length of edge* s_i *and* θ_i *is the (oriented) angle between normals* ν_{i-1} *and* ν_i *to* s_{i-1} *and* s_i*, respectively, then*

$$\begin{aligned} \Delta L_i :&= L_i(\mathbf{z} + \Delta \mathbf{z}) - L_i(\mathbf{z}) \\ &= -\Delta z_i (\operatorname{ctan} \theta_i + \operatorname{ctan} \theta_{i+1}) + \frac{\Delta z_{i-1}}{\sin \theta_i} + \frac{\Delta z_{i+1}}{\sin \theta_{i+1}}. \end{aligned} \tag{2.2}$$

If we further assume that s *is a convex polygon, the origin belongs to the region bounded by* s *and* d_i *is the distance from the origin to* s_i*, then*

$$L_i = -d_i (\operatorname{ctan} \theta_i + \operatorname{ctan} \theta_{i+1}) + \frac{d_{i-1}}{\sin \theta_i} + \frac{d_{i+1}}{\sin \theta_{i+1}}.$$

Here, by convention $s_{N+1} = s_1$*, etc.*

We note that Lemma 2.1 implies that $\mathcal{K}_i = \frac{\kappa_i}{L_i} < 0$, where

$$\kappa_i = -(\operatorname{ctan}\theta_i + \operatorname{ctan}\theta_{i+1}) + \frac{1}{\sin \theta_i} + \frac{1}{\sin \theta_{i+1}} < 0. \tag{2.3}$$

In the special case of the Wulff shape W being a regular N-gon formula (2.3) reduces to

$$\kappa_i = \kappa = 2\operatorname{ctan}\theta - \frac{2}{\sin \theta} = -2 \tan \frac{\pi}{N}, \tag{2.4}$$

because $\theta_i = \theta = \frac{2\pi}{N}$. Moreover, if $\Gamma_i \neq 0$ we have

$$|\Gamma_i| = \ell_i = |\kappa| d, \tag{2.5}$$

where d is the distance from the center of symmetry (or the center of the circumscribed circle) to the i-th edge.

In order to obtain a closed system we augment equations (1.1)–(1.4) with initial and boundary data. We consider here only homogeneous Dirichlet boundary data.

$$u(0,x) = u_0(x), \quad s(0) = s_0, \qquad u|_{\partial\Omega} = 0 \text{ for } t \geq 0. \tag{2.6}$$

This choice gives us some technical advantages. We shall not consider the Neumann condition, which is physically relevant, because our tools do not apply directly to it.

The process of multiplying (1.1) by a test function, then integrating by parts using (1.3) leads to the following definition: a pair $(\mathbf{z}, u)$ where $\mathbf{z}$ is as in (2.1) is called a *weak solution to* (1.1)–(1.4) *and* (2.6) *on* $[0,T)$, if $\mathbf{z} \in C^1([0,T);\mathbb{R}^N)$, $\mathbf{z}(0) = 0$, $u \in C^\alpha([0,T), H^1_0(\Omega))$ with $u(0) = u_0$, $u_t \in L^\infty_{loc}([0,T), H^{-1}(\Omega))$ (where $H^{-1}(\Omega)$ is the dual of $H^1_0(\Omega)$), and the identities

$$\epsilon\langle u_t, h\rangle = -\int_\Omega \nabla u(x)\cdot\nabla h(x)\,dx + \sum_{j=1}^N \int_{s_j(t)} V_j(t)h(x)\,dl,\ \forall h \in H^1_0(\Omega), \tag{2.7a}$$

$$\int_{s_j(t)} u\,dl = \Gamma_j - \beta_j L_j(t) V_j(t), \quad j = 1,\dots,N \tag{2.7b}$$

hold, where $\langle\cdot,\cdot\rangle$ is the pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

Existence of a weak solution $(\mathbf{z}, u)$ on a maximal interval of existence $[0, T_{max})$ is shown in the next section. A global existence result cannot be expected, especially if we fix the number of edges, because topological catastrophes like self-intersection, collapsing of a facet to a point, or bumping into the boundary are imminent. On the other hand, keeping the number of edges fixed may lead to uniqueness. This view is supported by the uniqueness result for the quasi-steady approximation of (2.7), i.e., for $\epsilon = 0$; see Theorem 4 in [19]. In the last section we study some qualitative properties of solutions in the case where the Wulff shape is a regular polygon. Throughout the paper vector quantities are set in bold, e.g., $\mathbf{z} = (z_1,\dots,z_N)$, the inner product in $\mathbb{R}^k$ is denoted by dot: $\mathbf{a}\cdot\mathbf{b} = \sum_{i=1}^k a_i b_i$, $|\mathbf{a}|$ is the Euclidean norm $|\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{a}$, and finally (f,g) is the inner product in $H^1_0(\Omega)$; i.e., $(f,g) = \int_\Omega \nabla f(x)\cdot\nabla g(x)\,dx$ and $\|f\|^2 = (f,f)$.

3. Existence. In this section we show existence of solutions to (2.7). Our main tool is semigroup theory as in [13]. In order to make this theory work we transform (2.7), and we exploit fully smoothness of the source terms in (2.7a). On the way we use some ideas which proved useful in the study of the quasi-steady approximation. We first rewrite equation (2.7a) in a unified form. Let us recall that if $g \in H^{-1}(\Omega)$, then there exist $g_i \in L^2(\Omega)$, $i = 1, 2$, such that $g = \sum_{i=1}^{2} \frac{\partial g_i}{\partial x_i}$; i.e., the pairing $\langle g, h \rangle$ is given by

$$\langle g, h \rangle = -\int_\Omega \sum_{i=1}^{2} g_i \frac{\partial h}{\partial x_i}. \tag{3.1}$$

Let us also recall that the mapping

$$H_0^1(\Omega) \ni f \mapsto -\Delta f \in H^{-1}(\Omega)$$

is an isomorphism of Hilbert spaces. If we now set for any $g \in H^{-1}(\Omega)$

$$G = -\Delta^{-1} g,$$

then we can rewrite (3.1) as

$$\langle g, h \rangle = \langle -\Delta G, h \rangle = \langle -\mathrm{div} \nabla G, h \rangle = \int_\Omega \nabla G(x) \cdot \nabla h(x)\, dx = (G, h).$$

Let us now look at the right-hand side of (2.7 a). We note that the mapping

$$H_0^1(\Omega) \ni h \mapsto \int_{s_i} h\, dl \in \mathbb{R}$$

is a continuous functional over $H_0^1(\Omega)$; we will denote it by δ_{s_i}. We define now elements f_i, $i = 1, \dots, N$, as follows:

$$f_i = -\Delta^{-1} \delta_{s_i}. \tag{3.2}$$

Thus, by (3.2) we obtain

$$\int_{s_i} h\, dl = -\langle \Delta f_i, h \rangle = (f_i, h).$$

In the sequel, since s is defined by $\mathbf{z}$ we will write $f(\mathbf{z})$ in order to stress this dependence. So, after taking into account the above remarks and after setting $U_t = -\Delta^{-1} u_t$ we can rewrite (2.7 a) as

$$\epsilon(U_t, h) = -(u, h) + \sum_{j=1}^{N} V_j\,(f_j(\mathbf{z}), h), \quad \forall h \in H_0^1(\Omega).$$

For the sake of consistency we had better write

$$\epsilon(U_t, h) = (\Delta U, h) + \sum_{j=1}^{N} V_j\,(f_j(\mathbf{z}), h), \quad \forall h \in H_0^1(\Omega),$$

where $-\Delta^{-1} u = U \in H_0^1(\Omega) \cap H^3(\Omega)$, because $u \in H_0^1(\Omega)$. Therefore, (2.7a) is equivalent to

$$\epsilon U_t = \Delta U + \sum_{j=1}^{N} V_j f_j(\mathbf{z}), \quad U(0) = U_0, \tag{3.3a}$$

and (2.7b) becomes

$$\frac{dz_i}{dt} = \frac{\Gamma_i + (\Delta U, f_i(\mathbf{z}))}{\beta_i L_i}, \quad z_i(0) = 0, \qquad i = 1, \ldots, N. \tag{3.3b}$$

Theorem 3.1. *Let us suppose that $\beta_i > 0$, Γ_i, $i = 1, \ldots, N$, are as in Section 2, the $f_i(\mathbf{z})$ are defined by (3.2) and $\epsilon > 0$. We also assume that $U_0 = -\Delta^{-1} u_0$ is compatible with the problem (3.3); i.e., $u_0 \in H_0^1(\Omega)$ and*

$$u_0 - \sum_{j=1}^{N} f_j(0) V_j(0) \in H^2(\Omega) \cap H_0^1(\Omega),$$

where the $V_i(0)$ are given by (3.3b),

$$V_i(0) = \frac{\Gamma_i - \int_{s_i(0)} u_0\, dl}{\beta_i L_i(0)}.$$

Then, there exists $T_{max} > 0$ and $\mathbf{z}$, U solutions of (3.3) such that

$$U \in C^\alpha([0, T_{max}), H^3(\Omega)), \quad U_t \in C^\alpha([0, T_{max}), H_0^1(\Omega)),$$
$$z_i \in C^{1,\alpha}([0, T_{max})), \quad i = 1, \ldots, N,$$

for any $\frac{1}{2} > \alpha > 0$.

This theorem immediately yields

Proposition 3.2. *If Γ_i, β_i, ϵ, u_0 are as in the previous theorem, then there exist $T_{max} > 0$ and a pair $(u, \mathbf{z})$ such that $(u, \mathbf{z})$ is a weak solution to* (2.7) *on the interval $[0, T_{max})$ and*

$$u \in C^\alpha([0, T_{max}), H_0^1(\Omega)), \qquad z_i \in C^{1,\alpha}([0, T_{max})), \quad i = 1, \dots, N.$$

Remark. The method of proof does not guarantee uniqueness. The work on the smooth counterpart of the problem ([3], [17]), as well as on the quasi-steady approximation of (2.7) ([19]), suggest that uniqueness might be expected. We will address this question elsewhere.

It is apparent that the transformed system (3.3) is a parabolic equation coupled to an ODE. In the proof we shall use the semigroup theory of [13]. But experience with this theory suggests that we need to find some additional smoothness in the problem. A closer scrutiny reveals that the f_i are smoother than just $H_0^1(\Omega)$. A related fact for distributions on $\mathbb{R}^k$ is well known to experts (e.g., see [16] and references therein). What we actually need is expressed in the two following lemmas.

Lemma 3.3. *Let us suppose that $\partial\Omega$ is smooth, a segment $s \subset \Omega$ is such that for all $x \in s$ we have $\operatorname{dist}(x, \partial\Omega) > \eta > 0$ for a fixed number η. We also assume that $f \in H_0^1(\Omega)$ is given by* (3.2)*, and $\varphi \in C_0^\infty(\Omega)$ is such that $0 \le \varphi \le 1$ and*

$$\varphi(x) = \begin{cases} 1 & \text{if } \operatorname{dist}(x, \partial\Omega) \ge \eta, \\ 0 & \text{if } \operatorname{dist}(x, \partial\Omega) \le \eta/2; \end{cases}$$

then, for all σ, $1 \le \sigma < 3/2$, there exists a $g \in H^\sigma(\mathbb{R}^2)$ and a unique $r \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$f = \varphi g + r,$$

and for any α, $0 < \alpha < \frac{1}{2}$

$$\|g\|_{H^\sigma(\mathbb{R}^2)} \le L^\alpha c_1(0, \sigma) c_2(L, \alpha), \qquad \|r\|_{H^2(\Omega)} \le 2\pi c_e c_3(\varphi) c_2(L, \alpha) L^\alpha, \quad (3.4)$$

where

$$c_1^2(\alpha, \sigma) = \frac{1}{2\pi} \int_{\mathbb{R}} |t|^{2\alpha} (1 + t^2)^{2-\sigma} \, dt, \quad c_2^2(L, \alpha) = 2L^{2(1-\alpha)} + 2\frac{(2\pi)^{2(1-\alpha)}}{1 - 2\alpha},$$
$$c_3(\varphi) = \max\{\max_{x \in \Omega} |\varphi + \Delta\varphi|, 2 \max_{x \in \Omega} |\nabla\varphi|\}.$$

Proof. Let us consider a unique solution $g \in H^1(\mathbb{R}^2)$ of the equation

$$g - \Delta g = \delta_s. \tag{3.5}$$

After an orthogonal change of coordinate system we may assume that $s = [0, L] \times \{0\}$. We recall that any orthogonal transformation of variables leaves equation (3.5) intact. The Fourier transform applied to (3.5) yields

$$(1 + |\xi|^2)\hat{g} = (2\pi)^{-1/2} \frac{e^{iL\xi_1} - 1}{i\xi_1}.$$

It is now obvious that

$$\|g\|^2_{H^\sigma(\mathbb{R}^2)} = \int_{\mathbb{R}^2} (1 + |\xi|^2)^\sigma |\hat{g}|^2 \, d\xi = \frac{1}{2\pi} \int_{\mathbb{R}^2} (1 + |\xi|^2)^{\sigma - 2} |\xi_1^{-2}| |e^{i\xi_1 L} - 1|^2 \, d\xi.$$

It is easy to note, taking the periodicity of e^{ix} into account, that

$$|e^{ix} - 1| \leq (2\pi)^{1-\alpha} |x|^\alpha, \tag{3.6}$$

for all $0 \leq \alpha \leq 1$.

It now follows that $\|g\|_{H^\sigma(\mathbb{R}^2)}$ is finite for $\sigma < 3/2$ and any $0 < \alpha < \frac{1}{2}$; precisely, (3.6) yields

$$\begin{aligned}
\|g\|^2_{H^\sigma(\mathbb{R}^2)} &\leq \frac{1}{2\pi} \int_{\mathbb{R}} (1 + |\xi_2|^2)^{\sigma - 2} \, d\xi_2 \int_{|\xi_1| < 1} L^2 \, d\xi_1 \\
&\quad + \frac{1}{2\pi} \int_{\mathbb{R}} (1 + |\xi_2|^2)^{\sigma - 2} \, d\xi_2 \int_{|\xi_1| \geq 1} (2\pi)^{2(1-\alpha)} L^{2\alpha} |\xi_1|^{-2(1-\alpha)} \, d\xi_1 \\
&= L^{2\alpha} c_1^2(0, \sigma) c_2^2(L, \alpha).
\end{aligned}$$

We define $r \in H^1_0(\Omega)$ by

$$r = f - \varphi g,$$

and it is unique after fixing φ since f and g are unique. We check that

$$\Delta r = -g(\varphi + \Delta\varphi) + 2\nabla\varphi\nabla g;$$

i.e., the right-hand side is in $L^2(\Omega)$; hence, by regularity theory for the Laplacian we obtain that $r \in H^2(\Omega) \cap H^1_0(\Omega)$ and

$$\|r\|_{H^2(\Omega)} \leq c_e \|g(\varphi + \Delta\varphi) + 2\nabla\varphi\nabla g\|_{L^2(\Omega)}.$$

Using (3.4) for $\sigma = 1$ and the definition of $c_3(\varphi)$ we arrive at

$$\|r\|_{H^2(\Omega)} \leq 2\pi c_e c_3(\varphi) c_2(L, \alpha) L^\alpha.$$

Lemma 3.4. *Let us suppose that $\partial\Omega$ is smooth, s_1, s_2 are like s in Lemma 3.3 and φ is also as in the previous lemma. Moreover, we assume that s_1, s_2 are parallel and they form a trapezium T, which is such that:*

- *the distance between the lines containing s_1 and s_2 is less than h;*
- *the length of the two remaining sides of T is less than $c(\mathbf{k})h$, where $c(\mathbf{k})$ is a universal constant depending only on the vector $\mathbf{k} = (\kappa_1, \dots, \kappa_N)$, where the κ_i, $i = 1, \dots, N$, are given by (2.3). Hence $|L_1 - L_2| \leq 2c(\mathbf{k})h$.*

The elements f_1 and f_2 are given by (3.2) and g_i, r_i, $i = 1, 2$, are defined in the previous lemma. Then, for any positive α, σ such that $\alpha + \sigma < 3/2$ we have

$$\|g_1 - g_2\|_{H^\sigma(\mathbb{R}^2)} \leq h^\alpha c_4(\sigma, \alpha, L_2), \qquad \|r_1 - r_2\|_{H^2(\Omega)} \leq c_e c_3(\varphi) c_4(\sigma, \alpha, L_2) h^\alpha,$$

where

$$c_4^2(\sigma, \alpha, L) = 8c_1^2(0, \sigma) 4^\alpha (c(\mathbf{k}) + 1)^{2\alpha} \big(4^{1-\alpha}(c(\mathbf{k}) + 1)^{2(1-\alpha)} + \frac{(2\pi)^{2(1-\alpha)}}{1 - 2\alpha} + \frac{L^2 + 4}{1 - 2\alpha} c_1(\alpha, \sigma)\big).$$

Proof. We proceed as in the proof of Lemma 3.3. After an orthogonal change of variables and taking the Fourier transform of equation (3.5) we obtain

$$(1 + |\xi|^2)\hat{g}_k = (2\pi)^{-1/2} \frac{e^{i(L_k + a_k)\xi_1} - e^{ia_k\xi_1}}{i\xi_1} e^{ih_k\xi_2}$$

where $a_1 = 0 = h_1$, $|h_2| = h$, and $|a_2| \leq c(\mathbf{k})h$. After taking the difference of the two above equations and after some calculations we obtain

$$\begin{aligned} &\|g_1 - g_2\|^2_{H^\sigma(\mathbb{R}^2)} \\ &\leq \tfrac{2}{\pi} \int_{\mathbb{R}^2} (1 + |\xi|^2)^{\sigma - 2} \Big| \frac{e^{i(L_1 + a_1)\xi_1} - e^{(L_2 + a_2)\xi_1}}{i\xi_1} + \frac{e^{ia_1\xi_1} - e^{ia_2\xi_1}}{i\xi_1} \Big|^2 d\xi \\ &+ \tfrac{2}{\pi} \int_{\mathbb{R}^2} (1 + |\xi|^2)^{\sigma - 2} \Big| \frac{e^{i(L_2 + a_2)\xi_1} - e^{ia_2\xi_1}}{i\xi_1} \Big|^2 |e^{ih_1\xi_2} - e^{h_2\xi_2}|^2 \, d\xi. \end{aligned}$$

We can estimate the right-hand side of the above equation using the same methods which led to (3.4); i.e., we obtain

$$\|g_1 - g_2\|_{H^\sigma(\mathbb{R}^2)} \leq h^\alpha c_4(\sigma, \alpha, L_2).$$

We now estimate the difference $r_1 - r_2$. We note that

$$\Delta(r_1 - r_2) = (g_2 - g_1)(\varphi + \Delta\varphi) + 2\nabla\varphi\nabla(g_1 - g_2);$$

hence,

$$\|r_1 - r_2\|_{H^2(\Omega)} \leq c_e c_3(\varphi)\|g_1 - g_2\|_{H^1(\mathbb{R}^2)} \leq c_e c_3(\varphi) c_4(\sigma, \alpha, L_2) h^\alpha.$$

Proof of the theorem. System (3.3) does not seem to fit directly the framework of semigroup theory as presented in [13]. That is why we will apply the method of successive approximation. We set

$$U^0(t) \equiv U_0, \quad t \in [0, \infty),$$

and let $\mathbf{z}^0$ be a unique solution of

$$\frac{d}{dt} z_i^0 = \frac{\Gamma_i + (\Delta U^0, f_i(\mathbf{z}^0))}{\beta_i L_i^0}, \quad z_i^0(0) = 0.$$

Its maximal interval of existence is $[0, T_0)$. Subsequently, we define U^{n+1} as a unique solution (on $[0, T_n)$) of the linear parabolic equation

$$\epsilon U_t^{n+1} = \Delta U^{n+1} + \sum_{j=1}^N V_j^n f_j(\mathbf{z}^n), \qquad U^{n+1}(0) = U^0, \tag{3.7}$$

and we define $\mathbf{z}_i^{n+1}$ as a unique solution to

$$\frac{d}{dt} z_i^{n+1} = \frac{\Gamma_i + (\Delta U^{n+1}, f_i(\mathbf{z}^{n+1}))}{\beta_i L_i^{n+1}}, \quad i = 1, \ldots, N; \tag{3.8}$$

its maximal interval of existence is $[0, T_{n+1})$; hence, $T_{n+1} \leq T_n$.

We shall show uniform bounds for U^n and V_i^n permitting us to choose a convergent subsequence.

As is well known, $-\Delta$ with the domain $D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega)$ is sectorial (see Section 1.6 in [13]). We can use the variation-of-constants formula for solution of (3.7) (Lemma 3.3.2 in [13]); thus, we have

$$U^{n+1}(t) = e^{\Delta t/\epsilon} U_0 + \frac{1}{\epsilon}\int_0^t e^{\Delta(t-\tau)/\epsilon} \sum_{j=1}^N V_j^n(\tau) f_j(\mathbf{z}^n(\tau))\, d\tau. \tag{3.9}$$

Lemma 3.3 suggests that the source terms in (3.7) are quite regular. The natural way of measuring the smoothness associated with a parabolic equation (3.7) is given by the scale of domains of fractional powers of $(-\Delta)^\alpha$ (or equivalently $(Id - \Delta)^\alpha$; see [13]). A universal measure is given by the scale of Sobolev spaces $H^\sigma(\mathbb{R}^2)$. We must check that the two scales of spaces $H^\sigma(\mathbb{R}^2)$ and $D((-\Delta)^\alpha)$ agree in the cases of our interest. We actually have

Lemma 3.5. *If* $0 \le \sigma$*, then* $H_0^{2\sigma}(\Omega) \subset D((Id - \Delta)^\sigma)$ *and*

$$\|u\|_{H_0^{2\sigma}(\Omega)} = \|u\|_{D((Id-\Delta)^\sigma)}.$$

We postpone the proof until the end of the proof of the theorem. But we note here a well-known fact that if $D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, then $D((-\Delta)^{1/2}) = H_0^1(\Omega)$ (see Theorems 1.15.3 and 4.3.3 in [22]).

In order to simplify the notation we shall introduce the shorthand

$$\|u\|_\sigma = \|u\|_{D((-\Delta)^\sigma)}$$

and

$$[u]_{\alpha,\sigma,t} = \sup_{0\le t_1<t_2\le t} \frac{\|u(t_1) - u(t_2)\|_\sigma}{|t_1 - t_2|^\alpha}, \quad [V]_{\alpha,t} = \sup_{0\le t_1<t_2\le t} \frac{|V(t_1) - V(t_2)|}{|t_1 - t_2|^\alpha}.$$

Let us fix φ and $\eta > 0$ satisfying the assumptions of Lemma 3.3 for all s_i, $i = 1, \dots, N$. Let us take $\delta > 0$, $\alpha > 0$ such that $\alpha + \delta < 1/2$ but otherwise arbitrary. We set $\sigma = 3 + \delta$. We need estimates for $[U^{n+1}]_{\alpha,\sigma,\rho}$ for $\rho < T_n$. We infer from (3.9) for $n > 0$ and $h > 0$, $t + h \le \rho < T_n$

$$\begin{aligned}
&U^{n+1}(t+h) - U^{n+1}(t)\\
&= e^{\Delta t/\epsilon}(e^{\Delta h/\epsilon} - I)U_0 + \int_0^h e^{\Delta(t+h-\tau)/\epsilon} \sum_{j=1}^N f_j(\mathbf{z}(\tau))V_j(\tau)\, d\tau\\
&+ \int_0^t e^{\Delta(t-\tau)/\epsilon} \sum_{j=1}^N (f_j(\mathbf{z}(\tau+h))V_j(\tau+h) - f_i(\mathbf{z}(\tau))V_i(\tau))\, d\tau.
\end{aligned}$$

It is well known that (Theorem 1.4.3 in [13])

$$\|(e^{\Delta t} - Id)x\|_0 \le c_5(\alpha) t^\alpha \|x\|_\alpha. \tag{3.10}$$

Hence, by (3.10) we obtain

$$\begin{aligned}
\|U^{n+1}(t+h) - U^{n+1}(t)\|_{\sigma/2} &\le \frac{c_5(\alpha)}{\epsilon^\alpha} e^{-\lambda t/\epsilon} h^\alpha \|U_0\|_{\sigma/2+\alpha}\\
&+ \int_0^h \|(-\Delta)^{1-\alpha} e^{\Delta(t+h-\tau)/\epsilon} \sum_{j=1}^N (-\Delta)^{\sigma/2-1+\alpha} f_j(\mathbf{z}^n(\tau))V_j(\tau)\|_0\, d\tau\\
&+ \int_0^t \|(-\Delta)^{1-\alpha/2} e^{\Delta(t-\tau)/\epsilon}\| \sum_{j=1}^N \|f_j(\mathbf{z}^n(\tau+h))V_j(\tau+h)\\
&\qquad - f_j(\mathbf{z}^n(\tau))V_j(\tau)\|_{(1+\delta+\alpha)/2}\, d\tau,
\end{aligned}$$

where λ is the smallest eigenvalue of $-\Delta$ in Ω.

It is also well known (see Theorem 1.4.4 and the remark after it in [13]) that

$$\|(-\Delta)^\alpha e^{\Delta t}\| \le c_6(\alpha) t^{-\alpha} e^{-\lambda t}. \tag{3.11}$$

Now, our Lemma 3.4 and Lemma 3.5 above combined with (3.11) yield

$$\begin{aligned}
&\|U^{n+1}(t+h) - U^{n+1}(t)\|_{\sigma/2} \le \frac{c_5(\alpha)}{\epsilon^\alpha} e^{-\lambda t/\epsilon} h^\alpha \|u_0\|_{(1+\delta)/2+\alpha} \\
&+ c_6(1-\alpha)\epsilon^{1-\alpha} \int_0^h (t+h-\sigma)^{-1+\alpha} \sum_{j=1}^N \sup_{t\le\tau} |V_j^n(t)|\, \|f_j(\mathbf{z}^n(t))\|_{(1+\delta)/2+\alpha} \\
&+ c_6(1-\frac{\alpha}{2})\epsilon^{1-\alpha/2} \int_0^t \frac{e^{-\lambda(t-\tau)}}{(t-\tau)^{1-\alpha/2}} \sum_{j=1}^N [f_j(\mathbf{z})V_j]_{\alpha,(1+\delta+\alpha)/2,\tau} h^\alpha \, d\tau.
\end{aligned}$$

After some simple calculations we arrive at

$$\begin{aligned}
[U^{n+1}]_{\alpha,\sigma,\rho} &\le \frac{c_5(\alpha)}{\epsilon^\alpha} e^{-\lambda t/\epsilon} \|u_0\|_{\sigma/2-1+\alpha} \\
&+ \frac{\epsilon^{1-\alpha}}{\alpha} C_h c_6(1-\alpha) \sum_{j=1}^N \max_{0\le t\le\rho} |V_j^n(t)| \|f_j(\mathbf{z}^n(t)\|_{\sigma/2+\alpha-1} \\
&+ c_6(1-\alpha/2) \frac{2\epsilon^{1-\alpha/2}}{\alpha} \rho^{\alpha/2} \sum_{j=1}^N [f_j(\mathbf{z}^n)]_{\alpha,(1+\delta)/2+\alpha,\rho} \max_{0\le t\le\rho} |V_j(t)| \\
&+ c_6(1-\alpha/2) \frac{2\epsilon^{1-\alpha/2}}{\alpha} \rho^{\alpha/2} \sum_{j=1}^N [V_j^n]_{\alpha,\rho} \max_{0\le t\le\rho} \|f_j(\mathbf{z}^n(t))\|_{(1+\delta)/2+\alpha}.
\end{aligned}$$

We want to express $[U^{n+1}]_{\alpha,\sigma,\rho}$ only in terms of data and $\max_i [V_i^n]_{\alpha,\rho}$. To this end let us note that Lemma 3.4 implies that

$$[f_i]_{\alpha,(\sigma+2\alpha)/2-1,\rho} \le C_U \sum_{j=1}^N \max_{0\le t\le\rho} |V_j^n(t)|^\alpha, \tag{3.12}$$

where $C_U = C_U(\mathbf{k}, \sigma, \alpha, \operatorname{diam}\Omega)$ is a universal constant depending only on $\mathbf{k}$, α, σ and the diameter of Ω. On the other hand

$$\max_{0\le t\le\rho} |V_i^n(t)| \le |V_i(0)| + \rho^\alpha [V_i^n]_{\alpha,\rho}. \tag{3.13}$$

By the same token using (3.12) and (3.13) we see

$$\max_{0\le t\le\rho} \|f_i(\mathbf{z}^n)\|_{(1+\delta+2\alpha)/2} \le \|f_i(0)\|_{(1+\delta+2\alpha)/2} + C_U\rho^\alpha \sum_{j=1}^{N}(|V_j(0)| + \rho^\alpha [V_j^n]_{\alpha,\rho})^\alpha. \tag{3.14}$$

Finally, taking into account estimates (3.12)–(3.14) we obtain

$$[U^{n+1}]_{\alpha,\sigma,\rho} \le \frac{c_5(\alpha)}{\epsilon^\alpha}\|u_0\|_{\sigma/2-1+\alpha} + F_U(\max_i |V_i(0)|, \max_i \|f_i(0)\|_{(1+\delta+2\alpha)/2}, \rho, \max_i [V_i^n]_{\alpha,\rho}), \tag{3.15}$$

where

$$F_U(d_V, d_f, t, w) = \frac{\max\{\epsilon^{\alpha/2}, \epsilon^\alpha\}}{\alpha}\big[(d_f + C_U t^\alpha N(d_V + t^\alpha w)^\alpha) C_h(d_V + t^\alpha w) + 2t^{\alpha/2}N^2 C_U (d_V + t^\alpha w)^{1+\alpha}\big].$$

We turn our attention to estimating $[V_i^{n+1}]_{\alpha,\rho}$ for $\rho < T_{n+1}$. Calculations based on the definition of the V_i^{n+1}'s and using the ideas presented above lead to

$$[V_i^{n+1}]_{\alpha,\rho} \le F_V\big(\max_i |V_i(0)|, \max_i \|f_i(0)\|_{(1+\delta+2\alpha)/2}, \min_i L_i(0), \rho, \max_i [V_i^n]_{\alpha,\rho}, \max_i [V_i^{n+1}]_{\alpha,\rho}\big). \tag{3.16}$$

We will spare the reader the exact form of $F_V(d_V, d_f, d_L, t, w_1, w_2)$. We had better state the important properties of F_V. They are the following:

(a) F_V is defined for all positive d_V, d_f, d_L, w_1 and positive t, w_2 such that
$$d_L > C_U t^\alpha N(d_V + t^\alpha w_2)^\alpha,$$
and
$$\lim_{t\to t_c} F_V(d_V, d_f, d_L, t, w_1, w_2) = +\infty,$$
where t_c is a unique solution of $d_L = C_U t^\alpha N(d_V + t^\alpha w_2)^\alpha$;

(b) F_V is continuous;

(c) for the arguments from the domain
$$F_V(d_V, d_f, d_L, 0, w_1, w_2) = F_V(d_V, d_f, d_L, 0, 0, 0) > 0;$$

(d) for d_V, d_f, d_L fixed and two out of t, w_1, w_2 fixed too, F_V is strictly increasing in the remaining variable.

We shall now show that $\inf T_n > 0$ and $[V_i^n]_{\alpha,T}$ are uniformly bounded for some $T > 0$, $T \leq \inf T_n$. We start with some comments on restrictions on T_n coming from the ODE (3.8). In particular we shall not consider any possible topological catastrophes. If a change of topology occurs at T_n^{top} then $T_n^{top} \leq T_n$. We cannot extend a solution of (3.8) beyond T_n if V_i^n blows up or if the solution reaches the boundary of the domain of definition of the right-hand side of (3.8). The latter occurs if

(i) $\min_i L_i(t_k) \to 0$ if $t_k \to T_n$; or
(ii) there exists $x(t_k) \in s_i(t_k)$ for some i and such that $\operatorname{dist}(x(t_k), \partial\Omega) \to \eta > 0$.

We recall here that fixing a φ in Lemma 3.3 prevents in our analysis $s(t)$ from approaching $\partial\Omega$ closer than $\eta > 0$.

If we are able to bound $[V_i^n]_{\alpha,\rho}$ by a constant $K > 0$ then we are able to estimate when each of the events (i) or (ii) could happen at the earliest. The event (i) could happen no earlier then at $T_z > 0$ being a unique solution of

$$\min_i L_i(0) = C_U T_z^\alpha N(d_V + T_z^\alpha K)^\alpha.$$

It is easy to check that the time of event (ii) is estimated below by a unique solution of

$$d_p - \eta = C_U T_b^\alpha N(d_V + T_b^\alpha K)^\alpha,$$

where d_p is the sup of distances by which all the sides of s_0 can be moved simultaneously in the direction of the outer normals to the edges without intersecting $\partial\Omega$. Since both equations differ just by their left-hand sides we can assume that $\min_i L_i(0) \leq d - \eta$ and F_V is such that $F_V \to \infty$ if t_k, w_2^k are such that

$$\min_i L_i(0) - C_U t_k^\alpha N(d_V + t_k^\alpha w_2^k)^\alpha \to 0.$$

Let us now observe that properties (a)–(d) of F_V lead to the following fact, which is obvious due to the Darboux principle.

Fact. If d_V, d_f, d_L are arbitrary positive numbers and $K > 0$ is such that $K > F_V(d_V, d_f, d_L, 0, 0, 0)$, then there exists a unique $T > 0$ solution of

$$F_V(d_V, d_f, d_L, T, K, K) = K. \tag{3.17}$$

We now prove uniform estimates for $[V_i^n]_{\alpha,\rho}$ and T_n. Let us first choose $K > 0$ such that

$$\begin{aligned} &K > F_V(d_V, d_f, d_L, 0, 0, 0),\\ &\min_i L_i(0) \leq C_U T_0^\alpha N(\max_i |V_i(0)| + T_0^\alpha K)^\alpha. \end{aligned} \tag{3.18}$$

It follows that if T is a solution of (3.17) for K, then $T_0 > T > 0$. We also assume that

$$\max_i [V_i^0]_{\alpha,T} \leq K. \tag{3.19}$$

We can always assume (3.17) because increasing K leads to diminishing T. That is why we have

$$\max_i [V_i^0]_{\alpha,T} \leq K, \qquad T < T_0.$$

We will prove that if $\max_i [V_i^n]_{\alpha,T} \leq K$, $T \leq T_n$, then also $\max_i [V_i^{n+1}]_{\alpha,T} \leq K$ and $T \leq T_{n+1}$, because the events (i) nor (ii) cannot occur on $[0, T]$.

Inequality (3.16) yields

$$\max_i [V_i^{n+1}]_{\alpha,t} \leq F_V(d_V, d_f, d_L, t, K, \max_i [V_i^{n+1}]_{\alpha,t}),$$

for $t < T_{n+1}$. The function $t \to \max_i [V_i^{n+1}]_{\alpha,t}$ is in principle defined only for $t > 0$. Let us set then

$$\max_i [V_i^{n+1}]_{\alpha,0} := \limsup_{t \to 0^+} \max_i [V_i^{n+1}]_{\alpha,t}$$

and $E = \{t \in [0, T_{n+1}) : \max_i [V_i^{n+1}]_{\alpha,t} < K\}$. By (3.18) $0 \in E$ and $E \neq \emptyset$. Let us set $c = \sup E$. We will see that $c \geq T$. We take any sequence $\{t_k\} \subset E$ converging to c. We have

$$\begin{aligned}\max_i [V_i^{n+1}]_{\alpha,t_k} &\leq F_V(d_V, d_f, d_L, t_k, K, \max_i [V_i^{n+1}]_{\alpha,t_k}) \\ &< F_V(d_V, d_f, d_L, t_k, K, K).\end{aligned}$$

After taking the sup we obtain

$$\sup_k \max_i [V_i^{n+1}]_{\alpha,t_k} \leq F_V(d_V, d_f, d_L, c, K, K).$$

If it happened that $c < T$, then

$$K = \sup_{t<c} \max_i [V_i^{n+1}]_{\alpha,t} < F_V(d_V, d_f, d_L, T, K, K) = K,$$

which is impossible. Thus, we are able to conclude that $\max_i [V_i^n]_{\alpha,T} \leq K$ and $0 < T \leq T_n$ for all natural numbers n.

Therefore, the $\mathbf{V}^n$ are equicontinuous on $[0,T]$; they are also equibounded on that interval, because $V_j^n(0) = V_j(0)$, $j = 1,\dots,N$, and the $V_j(0)$'s are defined in the statement of Theorem 3.1. Thus, by Ascoli's Theorem we can extract a subsequence $\mathbf{V}^{n_k}$ converging to $\mathbf{V}^\infty$ in $C^{\alpha'}([0,T])$, $\alpha' < \alpha$. By the uniform estimate (3.15) and the fact that the embedding $D((-\Delta)^\beta) \subset D((-\Delta)^{\beta'})$ is compact if $\beta > \beta'$ we can extract a subsequence $U^{n_{k_j}}$ converging to U^∞ in $C^{\alpha'}([0,T], D((-\Delta)^{(3+\delta')/2})$, $\delta' < \delta$. Hence, we may pass to the limit in (3.9), and we obtain

$$\begin{aligned} U^\infty(t) &= e^{\Delta t/\epsilon} U_0 + \int_0^t e^{\Delta(t-\tau)/\epsilon} \sum_{j=1}^N f_j(\mathbf{z}(\tau)) V_j^\infty(\tau)\, d\tau \\ \frac{dz_i^\infty}{dt} &= \frac{1}{\beta_j L_j}(\Gamma_j + (\Delta U^\infty, f_j(\mathbf{z}))). \end{aligned} \tag{3.20}$$

Existence of time derivatives now follows from Lemma 3.2.1 in [13]. □

We now prove Lemma 3.5. It seems obvious, but some care is needed. By the very definition of $H_0^t(\Omega)$ it is the closure of $C_0^\infty(\Omega)$ in the topology of $H^t(\mathbb{R}^2)$. In particular, elements of $H_0^t(\Omega)$ belong to $H^t(\mathbb{R}^2)$, and if $u \in H_0^t(\Omega)$, then $(Id - \Delta)u \in H_0^{t-2}(\Omega)$.

We can use the Fourier transform (it is denoted by $\hat{}$ or $\mathcal{F}$). So, if $u \in H_0^2(\Omega)$, then

$$((Id - \Delta)u)\hat{} = (1 + |\xi|^2)\hat{u}. \tag{3.21}$$

It is clear that $H_0^2(\Omega) \subset D(-\Delta) \equiv D(Id - \Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. We can define an operator $A : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by the formula

$$(Af)(\xi) = (1 + |\xi|^2) f(\xi).$$

So, $D(A) = \{f \in L^2(\mathbb{R}^2) \colon (1 + |\xi|^2) f(\xi) \in L^2(\mathbb{R}^2)\}$, and it is clear that A is closed and self-adjoint.

It follows from (3.21) that for $Id - \Delta$ restricted to $H_0^2(\Omega)$ and A restricted to $\mathcal{F}(H_0^2(\Omega))$ the Fourier transform $\mathcal{F}$ is a unitary equivalence; i.e., $(Id - \Delta)u = \mathcal{F}^{-1}(A\mathcal{F}u)$. It is rather easy to check that $(A^\alpha f)(\xi) = (1 + |\xi|^2)^\alpha f(\xi)$ and also $D(A^\alpha) = \{f \in L^2(\mathbb{R}^2) \colon (1 + |\xi|^2)^\alpha f(\xi) \in L^2(\mathbb{R}^2)\}$. Thus, $\mathcal{F}$ is again a unitary equivalence of $(Id - \Delta)^\alpha$ restricted to $H_0^2(\Omega)$ and A^α restricted to $\mathcal{F}H_0^2(\Omega)$; then, we have

$$\|(Id - \Delta)^\alpha u\|_{L^2(\Omega)} = \|(1 + |\xi|^2)^\alpha \hat{u}\|_{L^2(\mathbb{R}^2)},$$

for $u \in H_0^2(\Omega)$. But the norm on the right-hand side of the above formula is finite also for u belonging to $H_0^{2\alpha}(\Omega)$, because

$$\|(1+|\xi|^2)^\alpha \hat{u}\|_{L^2(\mathbb{R}^2)} = \|u\|_{H_0^{2\alpha}(\Omega)}.$$

Thus we showed that $H_0^{2\alpha}(\Omega) \subset D((Id-\Delta)^\alpha)$, as desired. □

4. The case of small perimeter. We would like to exhibit in this section some geometric properties of weak solutions, as well as to justify the fixing of the number N of edges. We consider a special case of interface with small perimeter. Of course $L(0)$ must be in some balance with the size of initial distribution of temperature u_0. The small size of $s(0)$ means that the surface tension prevails over the destabilizing bulk forces. Thus, $s(t)$ should shrink to a point. The results for the curvature flow suggest that $s(t)$ approaches the Wulff shape while shrinking. Thus, it seems that there is no need for creating new facets during the evolution provided $L(0)$ is already small.

In our analysis of this section we make the following simplifying assumptions:

- the Wulff shape W is a regular polygon with N sides;
- all β_i are equal to $\beta > 0$.

We consider here a special case of s_0 being a scaled Wulff shape. We show in Theorem 4.1 below that $s(t)$ shrinks to a point while it remains a scaled Wulff shape. Such results are known for the flow of (1.5) (see [20] and [19]) and for quasi-steady approximation of (2.7), i.e., for $\epsilon = 0$ (see [19]). The method we use here is based on showing that the time derivative of the isoperimetric quotient L^2/A is nonpositive. This suggests that the inequality $\frac{d}{dt}(L^2/A) \le 0$ is true in a more general situation, but we are not able to show it.

In the course of our analysis we compare $s(t)$ to the flow of (1.5) for the same initial data, establishing some properties of (1.5) on the way. That is another restriction in our analysis, because this approach requires that both flows were 'close' to each other, meaning in particular that $L(0)$ must be small. Let us recall that the flow of (1.5) always shrinks $s(0)$ to a point if $s(0) = tW$, $t > 0$. To the contrary, we expect that a solution of (2.7) will expand $s(0)$ if $L(0)$ is sufficiently large.

Before stating our theorem let us remark that the assumptions of Section 3 imply that if $s_0 = tW$, $t > 0$, then $\Gamma_i = \Gamma < 0$, $i = 1, \dots, N$.

Theorem 4.1. *Let us assume that* s_0 *is a regular polygon* (*i.e., a scaled Wulff shape*)*; we fix arbitrary numbers* $0 < \delta < 1$, $\frac{1}{2} < \sigma < 1$, $0 < \alpha < 1/2$, $1/2 < r < 3/4$, *satisfying*

$$r + \alpha/2 > \sigma/2 + 1/2; \tag{4.1}$$

moreover, $L(0) < 1$, *and* $L(0)$ *and* u_0 *satisfy the relations*

$$u_0(x) \leq 0 \quad \text{for all } x \in \Omega, \tag{4.2}$$

$$\gamma L(0)^{1/2} F(\|u_0\|_{\sigma/2}, T^U) < \delta|\Gamma|, \tag{4.3}$$

where

- $\gamma > 0$ *is a constant in the inequality* $\|u\|_{L^2(line\cap\Omega)} \leq \gamma\|u\|_{\sigma/2}$ (*see the remark below*)*;*
- F *is a continuous function which is strictly increasing in the second variable for the first variable fixed* (F *is precisely described below*)*;*
- $T^U = \frac{L^2(0)\beta}{d\kappa^2(1-\delta)}$.

Then, under the conditions of Theorem 3.1, and provided that T_{max} *is the maximal time of existence of a weak solution* $(\mathbf{z}, u)$, *we have*

a) $s(t)$ *is regular polygon and* $V_i(t) < 0$ *for* $t \in [0, T_{max})$*; moreover,*

$$\lim_{t\to T_{max}} L(t) = 0;$$

b) $u(t,x) < 0$ *for all* $t > 0$, $x \in \Omega$;

c) $\|u(t)\|_{\sigma/2} \leq F(\|u_0\|_{\sigma/2}, t)$ *for* $t \in [0, T_{max})$;

d) $T_D < T_{max} < T^U$, *where* $T_D = \frac{L^2(0)\beta}{d\kappa^2(1+\delta)}$.

Remark. It is well known that if $u \in H^\sigma(\mathbb{R}^2)$, $\sigma > \frac{1}{2}$, then the map $R : H^\sigma(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ given by

$$u \mapsto Ru = u|_{line}$$

is well defined and $\|Ru\|_{L^2(line)} \leq \gamma_0\|u\|_{H^\sigma(\mathbb{R}^2)}$. So, for u a weak solution we have that $u(t) \in H^1_0(\Omega)$; hence, $u \in H^\sigma_0(\Omega)$, $\sigma \leq 1$. By Lemma 3.5

$$\begin{aligned}|\int_{s_i} u\,dl| &\leq L_i^{1/2}\|Ru\|_{L^2(line)} \leq \gamma_0 L_i^{1/2}\|u\|_{H^\sigma_0(\Omega)}\\ &= \gamma_0 L_i^{1/2}\|u\|_{D((Id-\Delta)^{\sigma/2})} \leq \gamma L_i^{1/2}\|u\|_{\sigma/2}.\end{aligned}$$

The definition of $F(a,t)$ takes few more lines. We begin with a simple

Lemma 4.2. *Let us suppose* $c_d > c_u > 0$, $L_0 > 0$, $T^U = L_0^2/c_u^2$. *If we define* $t_1 = \frac{L_0^2}{c_d^2}$, $t_{k+1} = \frac{c_u^2}{c_d^2}(T^U - t_k)$, *then*

$$0 < t_k < T^U, \quad \textit{and} \quad \sum_{k=1}^{\infty} t_k = \infty.$$

Proof. Let us note first that

$$t_1 = \frac{L_0^2}{c_d^2} = T^U \frac{c_u^2}{c_d^2} < T^U.$$

So, if $t_k < T^U$ then

$$\frac{c_u^2}{c_d^2}(T - t_k) < \frac{c_u^2}{c_d^2} T^U < T^U,$$

and the first part follows by induction.

In order to show the second part let us sum up t_k from 2 to $r+1$ for an arbitrary natural number r; we obtain

$$\sum_{k=1}^{r} t_{k+1} = \sum_{k=1}^{r} \frac{c_u^2}{c_d^2}(T^U - t_k);$$

then,

$$\frac{c_u^2}{c_d^2} r T^U = \sum_{k=1}^{r} t_{k+1} + \sum_{k=1}^{r} \frac{c_u^2}{c_d^2} t_k < (1 + \frac{c_u^2}{c_d^2}) \sum_{k=1}^{r+1} t_k.$$

The lemma follows. □

The above Lemma implies that the number

$$\tau(c_d, c_u, L_0) = \min\{r : \sum_{k=1}^{r} t_k > T^U\}$$

is well defined. Only now we are ready for the definition of $F(a,t)$. We set $I_k = [t_{k-1}, t_k)$ for $k = 1, \dots, \tau(c_d, c_u, L_0)$ (assuming that $t_0 = 0$); hence, the family of intervals $\{I_k\}_{k=1}^{\tau(c_d,c_u,L_0)}$ covers $[0, T^U)$. For $t \in I_1$ we set

$$F(a,t) := (a + K) E_{\bar{\beta}}(\theta t);$$

on the other hand, if $t \in I_k$, $k > 1$, then

$$F(a,t) := (F(a,t_{k-1}) + K)E_{\bar\beta}(\theta(t-t_{k-1})),$$

where

$$\begin{aligned}
&\theta = (b\Gamma(\bar\beta))^{1/\bar\beta}, \qquad \bar\beta = r + \alpha/2 - \sigma/2 - 1/4 > 0,\\
&b = N\frac{\gamma}{\beta}c_w^{-1/2}c_1(0,2r)c_2(\operatorname{diam}\Omega,\alpha)c_6(3/4-\bar\beta)\epsilon^{3/4-\bar\beta},\\
&E_{\bar\beta}(z) = \sum_{n=0}^{\infty}\frac{z^{n/\bar\beta}}{\Gamma(n(\bar\beta+1))},\\
&K = \frac{|\Gamma|c_1(0,2r)c_2(\operatorname{diam}\Omega,\alpha)\Gamma(\bar\beta-1/4)c_6(1+\sigma/2-r)}{\beta c_w^{1-\alpha}\lambda^{r+\alpha/2-\sigma/2-5/2}\epsilon^{r-1-\sigma/2}},\\
&a = \|u_0\|_{\sigma/2}, \qquad c_u = N\Big(\frac{2\kappa\Gamma(1-\delta)}{\beta}\Big)^{1/2},\\
&c_d = N\Big(\frac{2\kappa\Gamma(1+\delta)}{\beta}\Big)^{1/2}, \qquad c_w = N\Big(\frac{2\kappa\Gamma}{\beta}\Big)^{1/2}
\end{aligned} \tag{4.4}$$

where $\Gamma(z)$ is Euler's Gamma function and λ is the smallest eigenvalue of $-\Delta$ in Ω.

It follows that F is specified solely by the initial data and some free parameters specified in the theorem.

We start the proof of the theorem; it relies on two lemmas, the first of which gives an estimate for the extinction time for the flow (1.5).

Lemma 4.3. *If s_0 is a regular polygon, $\mathbf{z}^T$ is a unique solution to*

$$\frac{dz_i^T}{dt} = \frac{\Gamma}{\beta L_i^T}, \quad \mathbf{z}^T(0) = 0,$$

defined on the maximal interval of existence $[0,t_0)$, then $s^T(t)$ is a regular polygon and $L_i^T(t) = c_w(t_0-t)^{1/2}$, where $c_w = (2\kappa\Gamma/\beta)^{1/2}$.

Proof. For the sake of notational convenience let us drop the superscript T. It is known that $s(t)$ is a regular polygon shrinking to a point (e.g., Corollary 14 in [19]; see also [20] for a more general independent statement). It thus follows that at each instant t, $NL_i(t) = L(t)$. We note that by Lemma 2.1

$$L(t_0) - L(t) = \int_t^{t_0}\frac{d}{d\tau}L(\tau)\,d\tau = -\kappa\int_t^{t_0}\sum_{j=1}^{N}V_j(\tau)\,d\tau,$$

where κ is given by (2.4). Hence, by definition of V_i and by the fact $L(t_0) = 0$ we infer

$$L(t) = \frac{1}{\beta}\kappa\Gamma N^2 \int_t^{t_0} L^{-1}(\tau)\, d\tau.$$

It is now easy to see that $L(t) = c_w(t_0 - t)^{1/2}$.

Lemma 4.4. *If $L_i \geq c_w(t-t_0)^{1/2}$, where c_w is given in the previous lemma, r, α and σ satisfy* (4.1)*, and the number K is defined by* (4.4$_4$)*, then for all $t \in [0, t_0)$ we have*

$$\| \int_0^t e^{\Delta(t-\tau)/\epsilon} \frac{\Gamma}{\beta L_i} f_i(\mathbf{z}(\tau))\, d\tau \|_{\sigma/2+1} \leq K < \infty.$$

Proof. We recall that for the above range of r and α Lemma 3.3 is applicable, and it yields

$$\|f_i\|_r \leq L_i^\alpha c_1(0, 2r) c_2(L_i, \alpha).$$

In this way we obtain using (4.1)

$$\begin{aligned}
&\| \int_0^t e^{\Delta(t-\tau)/\epsilon} \frac{\Gamma}{\beta L_i} f_i(\mathbf{z}(\tau))\, d\tau \|_{\sigma/2+1} \\
&\leq \frac{|\Gamma|}{\beta} \int_0^t \|(-\Delta)^{1+\sigma/2-r} e^{\Delta(t-\tau)/\epsilon} (-\Delta)^r f_i(\mathbf{z}(\tau)) L_i^{-1}(\tau)\|_0\, d\tau \\
&\leq \frac{|\Gamma| c_1(0,2r)}{\beta \epsilon^{r-1-\sigma/2}} \max_{0 \leq \tau \leq t} c_2(L_i(\tau), \alpha) c_6(1 + \frac{\sigma}{2} - r) \int_0^t \frac{e^{-\lambda(t-\tau)/\epsilon} L_i(\tau)^{\alpha-1}}{(t-\tau)^{1+\sigma/2-r}}\, d\tau \\
&\leq \frac{|\Gamma| c_1(0,2r) c_2(\operatorname{diam}\Omega, \alpha)}{\beta c_w^{1-\alpha} \epsilon^{r-1-\sigma/2}} c_6(1+\sigma/2-r) \int_0^{t_0} \frac{e^{-\lambda(t_0-\tau)/\epsilon}}{(t_0-\tau)^{1+\sigma/2-r-(\alpha-1)/2}}\, d\tau \\
&\leq K.
\end{aligned}$$

The proof of part (b) of our theorem requires another lemma.

Lemma 4.5. *If f_i is given by* (3.2)*, $t > 0$, then*

$$-\Delta e^{\Delta t} f_i \geq 0.$$

Remark. Since $e^{\Delta t}$ is a smoothing operator the inequality is pointwise.

Proof. It is well known that if A is a positive self-adjoint operator on a Hilbert space $\mathcal{H}$, then so is e^{-At}; e.g., one can see this from the representation of e^{-At} given in Theorem 1.3.4 of [13]. It is also well known that $-\Delta : D(\Delta) \subset L^2(\Omega) \to L^2(\Omega)$, with $D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ self-adjoint. It follows that $e^{\Delta t}$ is self-adjoint too.

Suppose now to the contrary of our claim that there exists x_0 such that

$$-\Delta e^{\Delta t} f_i(x_0) < 0.$$

Then, by continuity of $\Delta e^{\Delta t} f_i$ there exists a neighborhood U of x_0 such that

$$-\Delta e^{\Delta t} f_i(x) < 0 \quad \text{for all } x \in U.$$

Thus, we may take a function $\phi \in C_0^\infty(U)$ and such that $\phi \geq 0$. Then,

$$\begin{aligned} 0 > -\int_\Omega \phi(x) \Delta e^{\Delta t} f_i(x)\, dx &= -\int_\Omega \Delta e^{\Delta t} \phi(x) f_i(x)\, dx \\ = \int_\Omega \nabla(e^{\Delta t}\phi(x)) \nabla f_i(x)\, dx &= \int_{s_i} e^{\Delta t}\phi(x)\, dl(x) > 0, \end{aligned}$$

a contradiction. We used in the last line the Maximum Principle for the heat equation. The lemma is proved. $\square$

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. Let us set

$$E = \left\{ a \in [0, T_{max}) : \begin{array}{c} L(t) \leq L(0),\ \|u(t)\|_{\sigma/2} \leq F(\|u_0\|_{\sigma/2}, t) \text{ and} \\ s(t) \text{ is a regular polygon for all } 0 \leq t \leq a \end{array} \right\}.$$

Of course $E \neq \emptyset$ because our assumptions imply that $0 \in E$. Let us set $\omega = \sup E$. We shall show that $\omega = T_{max}$. Let us observe that if $\omega < T_{max}$ then by the remark following the statement of the theorem and by (4.3)

$$\left| \int_{s_i(\omega)} u(\omega)\, dl \right| \leq \gamma L^{1/2}(\omega) \|u(\omega)\|_{\sigma/2} < \delta |\Gamma|;$$

thus, there exists $\eta > 0$ such that

$$\left| \int_{s_i} u\, dl \right| < \delta |\Gamma| \quad \text{on } [\omega, \omega + \eta). \tag{4.5}$$

The above inequality immediately implies that $V_i(t) < 0$ and $L(t) \leq L(0)$ on $[\omega, \omega + \eta)$. Moreover, it implies that $u(t) \leq 0$, too. In order to see this we apply $-\Delta$ to both sides of (3.20_1). We obtain

$$u(t) = -\Delta U(t) = e^{\Delta t/\epsilon} u_0 + \int_0^t (-\Delta) e^{\Delta(t-\tau)/\epsilon} \sum_{j=1}^N f_j(\mathbf{z}(\tau)) V_j(\mathbf{z}(\tau))\, d\tau.$$

The right-hand side is well defined because of the regularity of f_j's. By Lemma 4.5 and $V_j < 0$, $j = 1, \dots, N$, it follows that the integrand above is nonpositive for $\tau < t$. So, the integral is nonpositive. Since $u_0 \leq 0$, the Maximum Principle implies that $e^{\Delta t/\epsilon} u_0 < 0$ for $t > 0$, inside Ω. Part (b) follows on $[0, \omega + \eta)$.

We may now calculate the derivative of the isoperimetric quotient. Since $u(t) < 0$ for $(0, \omega + \eta)$ we apply the reasoning as in the proof of Theorem 10 in [19], and we come to

$$\frac{d}{dt}\Big(\frac{L^2}{A}\Big) \leq -\frac{L}{A\beta}\Big(\sum_{j=1}^N \frac{\kappa^2}{L_j} + \frac{1}{2}\frac{L}{A} N\kappa\Big)\Big(d - \frac{1}{\kappa N}\sum_{j=1}^N \int_{s_j} u\, dl\Big).$$

By Lemmas 11 and 12 of [19] it follows that

$$\sum_{j=1}^N \frac{\kappa^2}{L_j} + \frac{1}{2}\frac{L}{A} N\kappa \geq 0.$$

This combined with (4.5) and (2.5) yields

$$\frac{d}{dt}\Big(\frac{L^2}{A}\Big) \leq 0 \quad \text{on } [\omega, \omega + \eta).$$

It follows that $s(t)$ is a regular polygon on that interval. We compare $s(t)$ to the curves given by solving (1.5). If we set $V_i^D = \frac{\Gamma(1+\delta)}{\beta L_i}$ then

$$V_i^D(t) \leq \frac{\Gamma - \int_{s_i} u\, dl}{\beta L_i(t)} = V_i(t),$$

and it follows that $L_i^D(t) \leq L_i(t)$ and we can apply the two preceding lemmas with c_d, c_u given by (4.4).

Since $\omega \in I_k$ for some $k \leq \tau(c_d, c_u, L(0))$ we may assume that η is such that $\omega + \eta \in I_k$. We now estimate $\|u(t)\|_{\sigma/2}$ on $[\omega, \omega+\eta)$ using the integral representation of solution (3.5) and Lemmas 4.3 and 4.4; namely, we have

$$\|u(t)\|_{\sigma/2} \leq \|u(t_{k-1})\|_{\sigma/2} + \| \int_{t_{k-1}}^{t} e^{\Delta(t-\tau)/\epsilon} \sum_{j=1}^{N} \frac{\Gamma_j f_j(\mathbf{z}(\tau))}{\beta_j L_j} \, d\tau \|_{1+\sigma/2}$$

$$+ \| \int_{t_{k-1}}^{t} e^{\Delta(t-\tau)/\epsilon} \sum_{j=1}^{N} \int_{s_j} u \, dl \frac{f_j}{\beta L_j} \, d\tau \|_{1+\sigma/2}$$

$$\leq \|u(t_{k-1})\|_{\sigma/2} + K$$

$$+ \frac{\gamma}{\beta} \sum_{j=1}^{N} \int_{t_{k-1}}^{t} \|(-\Delta)^{1+\sigma/2-r-\alpha/2} e^{\Delta(t-\tau)/\epsilon}\| \, \|u(\tau)\|_{\sigma/2} \frac{\|f_j(\mathbf{z}(\tau))\|_r}{L_j^{1/2}(\tau)} \, d\tau$$

$$\leq \|u(t_{k-1})\|_{\sigma/2} + K + b \int_{t_{k-1}}^{t} (t-\tau)^{-5/4-\sigma/2+r+\alpha/2} \|u(\tau)\|_{\sigma/2} \, d\tau.$$

We apply now a variant of Gronwall's inequality, see Lemma 7.1.1 in [13], which yields

$$\|u(t)\|_{\sigma/2} \leq (\|u(t_{k-1})\|_{\sigma/2} + K) E_{\bar{\beta}}(\theta(t - t_{k-1})) \leq F(\|u_0\|_{\sigma/2}, t).$$

This contradicts the maximality of ω. It follows from the presented argument that $L(t) \to 0$ as $t \to T_{max}$.

Because of the estimate for $\|u(t)\|_{\sigma/2}$ we can compare s to the curve s^U obtained by solving (1.5) for $V_i^U = \Gamma(1-\delta)/(\beta L_i)$. Since $V_i \leq V_i^U$ we have $L(t) \leq L^U(t)$, and Lemma 4.3 yields the extinction time for the flows (1.5) V_i^U and V_i^D.

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