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QUALITATIVE PROPERTIES OF CERTAIN C_0 SEMIGROUPS ARISING IN ELASTIC SYSTEMS WITH VARIOUS DAMPINGS

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Abstract. This paper studies various qualitative properties, such as exponential stability, spectrum determining growth property, differentiability, of Gevrey class and analyticity, for the semigroup $e^{\mathcal{A}t}$ generated by the operator of the form $\mathcal{A}=\begin{pmatrix}-A_0 & B\\ C & -A_1\end{pmatrix}$, where both $-A_0$ and $-A_1$ generate contraction semigroups on some Hilbert spaces, and B and C are certain closed densely defined linear operators.

1. Introduction. Let H_0 and H_1 be two Hilbert spaces, $-A_0:\mathcal{D}(A_0)\subseteq H_0\to H_0$ and $-A_1:\mathcal{D}(A_1)\subseteq H_1\to H_1$ be two densely defined closed linear operators generating C_0 contraction semigroups e^{-A_0t} and e^{-A_1t} on H_0 and H_1, respectively. Let $B:\mathcal{D}(B)\subseteq H_1\to H_0$ and $C:\mathcal{D}(C)\subseteq H_0\to H_1$ be other two densely defined and closed linear operators. Consider the following system of coupled linear evolution equations on $\mathcal{H}\stackrel{\Delta}{=}H_0\times H_1$:

$$\begin{cases}\dot{x}(t)=-A_0x(t)+By(t),\\ \dot{y}(t)=Cx(t)-A_1y(t),\end{cases}\qquad x(0)=x_0,\quad y(0)=y_0. \tag{1.1}$$

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Many elastic systems with various dampings can be put into such an abstract form (see [2, 11, 21, 23], for example, or 8 below). We keep in mind that an important special case of the above is that $C = -B^*$ (with identifications $H_0^* = H_0$ and $H_1^* = H_1$). Now, if we define

$$\begin{cases} \mathcal{A} = \begin{pmatrix} -A_0 & B \\ C & -A_1 \end{pmatrix} : \mathcal{D}(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}, \\ \mathcal{D}(\mathcal{A}) = [\mathcal{D}(A_0) \cap \mathcal{D}(C)] \times [\mathcal{D}(A_1) \cap \mathcal{D}(B)], \end{cases} \tag{1.2}$$

then (1.1) can be rewritten as follows:

$$\dot{z}(t) = \mathcal{A}z(t), \qquad t \in [0, \infty), \qquad z(0) = z_0, \tag{1.3}$$

with $z(\cdot) = (x(\cdot), y(\cdot))$ and $z_0 = (x_0, y_0)$. We raise the following natural questions:

(i) When does $\mathcal{A}$ generate a C_0 semigroup?
(ii) If $\mathcal{A}$ generates a C_0 semigroup $e^{\mathcal{A}t}$, when does it have the spectrum determining growth property (SDGP, for short)? And when is it strongly stable, exponentially stable, differentiable, of Gevrey class, or analytic? (See 2 for definitions.)

These two questions correspond to the well-posedness of (1.1), the asymptotic behavior and some regularity of the solutions to (1.1), respectively. In most applications, under very mild conditions, $\mathcal{A}$ does generate a C_0 semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$. Then the second question becomes very interesting. As is known, for example, the exponential stability is very crucial for further studying the optimal regulator problem associated with (1.1) in infinite time horizon; or, if we regard (1.1) as the closed-loop system of some control system, the exponential stability might be one of the goals that one wants to achieve (see [12, 13] and the references cited therein). On the other hand, differentiability, of Gevrey class, and analyticity of $e^{\mathcal{A}t}$ will give us some further regularity information about the solutions to (1.1), which will be very useful in analysis and/or design of the corresponding physical systems described by (1.1).

We point out that in the case

$$\|e^{\mathcal{A}t}\| \leq M, \quad \forall t \geq 0, \quad \rho(\mathcal{A}) \supset i\mathbb{R} \overset{\Delta}{=} \{i\beta : \beta \in \mathbb{R}\} \tag{1.4}$$

(according to [9], (1.4) implies the strong stability of $e^{\mathcal{A}t}$), one has the following chain of implications for $e^{\mathcal{A}t}$:

$$\begin{aligned} &\text{analytic} \Rightarrow \text{of Gevrey class } \delta > 1 \\ &\qquad \Rightarrow \text{differentiable} \Rightarrow \text{SDGP} \Rightarrow \text{exponentially stable.} \end{aligned} \tag{1.5}$$

Also, it is always true that the exponential stability implies the strong stability. There is a large number of papers on the study of various cases concerning (1.1). See for example [2–5, 8, 10, 11, 14, 21] and the references cited therein. The purpose of this paper is to give a systematic treatment of the above two questions. For the second question, our method is a combination of some direct estimation of the resolvent and the improved contradiction argument which was widely used recently in establishing the exponential stability of certain semigroups arising in elastic systems with dampings (see [1, 15–17]). Our results recover or extend relevant ones found in [1–4, 8, 10, 14–17, 21, 22].

This paper is organized as follows. In Section 2, after recalling some definitions, we state the main results and make some comments. In Section 3, some conditions are given to ensure that $\mathcal{A}$ generates a C_0 semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$. Section 4 is devoted to a presentation of some results related to our main theorems. In Sections 5 and 6, we prove our main results. In Section 7, multi-block operator cases are considered directly. Finally, in Section 8, a number of examples are given, which indicate the applicability of our results.

2. The main results. In this section, we state our main results of this paper. Also, we will point out some interesting cases covered by our results. To begin with, we recall some definitions for comparison purposes (see [13, 18, 24]).

Definition 2.1. Let e^{At} be a C_0 semigroup on some Hilbert space H.

(i) e^{At} is called a *contraction semigroup* if

$$\|e^{At}\| \leq 1, \qquad \forall t \geq 0. \tag{2.1}$$

(ii) e^{At} is said to be *strongly stable* if

$$\lim_{t\to\infty} \|e^{At}x\| = 0, \qquad \forall x \in H. \tag{2.2}$$

(iii) e^{At} is said to be *exponentially stable* if there exist constants $\omega > 0$ and $M \geq 1$, such that

$$\|e^{At}\| \leq Me^{-\omega t}, \qquad \forall t \geq 0. \tag{2.3}$$

(iv) e^{At} is said to have the *spectrum determining growth property* (SDGP, for short) if

$$\sigma_0(A) \overset{\Delta}{=} \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} = \lim_{t\to\infty} \frac{\log\|e^{At}\|}{t} \overset{\Delta}{=} \omega_0(A). \tag{2.4}$$

(v) e^{At} is said to be *differentiable* if for any $x \in H$, $t \mapsto e^{At}x$ is strongly differentiable on $(0,\infty)$; i.e., $Ae^{At} \in \mathcal{L}(H)$, for all $t > 0$, or equivalently,

$$\|Ae^{At}\| < \infty, \qquad \forall t > 0. \tag{2.5}$$

(vi) e^{At} is said to be of *Gevrey class* $\delta > 1$ if for any compact set $\mathcal{K} \subset (0,\infty)$ and any $\theta > 0$, there exists a constant $K = K(\theta, \mathcal{K})$, such that

$$\|A^n e^{At}\| \le K\theta^n (n!)^\delta, \qquad \forall t \in \mathcal{K},\ n \ge 0. \tag{2.6}$$

(vii) e^{At} is said to be *analytic* if e^{At} admits an extension $T(\lambda)$ for $\lambda \in \Delta_\theta \equiv \{\lambda \in \mathbb{C} : |\arg\lambda| < \theta\}$ for some $\theta > 0$, such that $\lambda \mapsto T(\lambda)$ is analytic and

$$\begin{cases} \lim_{\Delta_\theta \ni \lambda \to 0} \|T(\lambda)z - z\| = 0, & \forall z \in \mathcal{H}, \\ T(\lambda+\mu) = T(\lambda)T(\mu), & \forall \lambda, \mu \in \Delta_\theta, \end{cases} \tag{2.7}$$

or equivalently, there exists a constant $K > 0$, such that

$$\|Ae^{At}\| \le Kt^{-1}, \qquad \forall t > 0. \tag{2.8}$$

It is clear that (2.8) $\Rightarrow$ (2.6) $\Rightarrow$ (2.5). The number $\omega_0(A)$ defined in (2.4) is called the *growth abscissa* of A, which determines the growth (or decay) of the semigroup e^{At}. In general, we have (by the Hille–Yosida Theorem)

$$\sigma_0(A) \le \omega_0(A). \tag{2.9}$$

Hereafter, we keep the following basic assumptions:

$$\begin{aligned} & e^{-A_0 t} \text{ and } e^{-A_1 t} \text{ are contraction semigroups on } H_0 \text{ and } H_1, \text{ respectively,} \\ & B : \mathcal{D}(B) \subseteq H_1 \to H_0 \text{ and } C : \mathcal{D}(C) \subseteq H_0 \to H_1 \text{ are closed,} \\ & \overline{\mathcal{D}(A_0) \cap \mathcal{D}(C)} = H_0, \quad \overline{\mathcal{D}(A_1) \cap \mathcal{D}(B)} = H_1, \\ & \mathrm{Re}\,\{ \langle A_0 x, x \rangle + \langle A_1 y, y \rangle - \langle By, x \rangle - \langle Cx, y \rangle \} \ge 0, \\ & \forall x \in \mathcal{D}(A_0) \cap \mathcal{D}(C), y \in \mathcal{D}(A_1) \cap \mathcal{D}(B). \end{aligned} \tag{2.10}$$

By the Lumer–Phillips Theorem ([18]), we see that both $-A_0$ and $-A_1$ are *dissipative* and

$$(\lambda + A_0)^{-1} \in \mathcal{L}(H_0), \quad (\lambda + A_1)^{-1} \in \mathcal{L}(H_1), \quad \forall\, \mathrm{Re}\,\lambda > 0. \tag{2.11}$$

Furthermore, by (2.10), we have that the operator $\mathcal{A}$ defined by (1.2) is densely defined on $\mathcal{H}$ and it is dissipative:

$$\mathrm{Re}\,\langle\, \mathcal{A}\binom{x}{y}, \binom{x}{y}\,\rangle \le 0, \quad \forall \binom{x}{y} \in \mathcal{D}(\mathcal{A}). \tag{2.12}$$

Thus, if $\mathcal{A}$ generates a C_0 semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$, it must be a contraction semigroup. Note that in the case $C = -B^*$, the first condition in (2.10) implies the last in (2.10).

Now, suppose $\mathcal{A}$ generates a contraction semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$. We would like to look at some qualitative properties of this semigroup. Needless to say, the properties of $e^{\mathcal{A}t}$ not only depend on A_0 and A_1, but also depend on the *coupling operators* B and C. In what follows, we assume the following, which is true in many applications:

$$e^{-A_1 t} \text{ is analytic and exponentially stable.} \tag{2.13}$$

Thus, $e^{-A_1 t}$ is reasonably "good." One would like to know whether $e^{\mathcal{A}t}$ is also reasonably "good" if $e^{-A_0 t}$ happens not to be very "good." In this context, we will see that the coupling operators B and C play very interesting and subtle roles.

Note that under (2.13), the fractional power A_1^b (as well as $(A_1^*)^b = (A_1^b)^*$) is well-defined for all $b \in \mathbb{R}$; in particular, A_1^{-1} exists (see [18]). Concerning the coupling operators, we suppose that for some $b, c \ge 0$, it holds that

$$\mathcal{D}(A_1^b) \subseteq \mathcal{D}(B), \quad \mathcal{D}([A_1^c]^*) \subseteq \mathcal{D}(C^*). \tag{2.14}$$

We say that (1.1) is *weakly coupled* if $b, c \le 1$ and $b + c < 2$, and is *strongly coupled* if $b,\ c \ge 1$. If neither of the above happens, we say that (1.1) is *mixedly coupled*, which will not be considered in this paper. Note, however, that sometimes, by changing the underlying spaces properly, one can transform a mixedly coupled case to either a weakly or strongly coupled case. Roughly speaking, the weakly coupled case corresponds to the case that the "orders" of coupling operators B and C are lower than that of A_1. The strongly coupled case is the other way around.

The following two theorems are concerned with the properties of $e^{\mathcal{A}t}$ for the weakly coupled case. We first state some sufficient conditions for $e^{\mathcal{A}t}$'s having particular properties.

Theorem 2.2. *Let (2.10) and (2.13) hold. Let $0 \le b, c \le 1$ be such that (2.14) holds, and $e^{\mathcal{A}t}$ be a contraction semigroup.*

(i) *$e^{\mathcal{A}t}$ is exponentially stable if $i\mathbb{R} \subset \rho(\mathcal{A})$, $b + c < 1$, and one of the following holds:*

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta + A_0)^{-1}\| < \infty; \tag{2.15}$$

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta + A_0 - BA_1^{-1}C)^{-1}\| < \infty. \tag{2.16}$$

(ii) *$e^{\mathcal{A}t}$ is differentiable if $b + c \le 1$ and one of the following holds:*

$$\overline{\lim_{|\beta|\to\infty}} \log(1 + |\beta|)\|(i\beta + A_0)^{-1}\| = 0; \tag{2.17}$$

$$\overline{\lim_{|\beta|\to\infty}} \log(1 + |\beta|)\|(i\beta + A_0 - BA_1^{-1}C)^{-1}\| = 0. \tag{2.18}$$

(iii) *Let $\mu \in (0, 1)$. $e^{\mathcal{A}t}$ is of Gevrey class $\delta > 1/\mu$ if one of the following holds:*

$$b + c \le 1, \qquad \overline{\lim_{|\beta|\to\infty}} |\beta|^{\mu}\|(i\beta + A_0)^{-1}\| < \infty; \tag{2.19}$$

$$b + c < 1 + \mu, \qquad \overline{\lim_{|\beta|\to\infty}} |\beta|^{\mu}\|(i\beta + A_0 - BA_1^{-1}C)^{-1}\| < \infty. \tag{2.20}$$

(iv) *$e^{\mathcal{A}t}$ is analytic if one of the following holds:*

$$\begin{cases} b + c \le 1, \\ \overline{\lim}_{|\beta|\to\infty} |\beta|\|(i\beta + A_0)^{-1}\| < \infty; \ \textit{i.e.}, \ e^{-A_0 t} \ \textit{is analytic}; \end{cases} \tag{2.21}$$

$$\begin{cases} b + c < 2, \\ \overline{\lim}_{|\beta|\to\infty} |\beta|\|(i\beta + A_0 - BA_1^{-1}C)^{-1}\| < \infty; \\ \ \textit{i.e.}, \ e^{(-A_0 + BA_1^{-1}C)t} \ \textit{is analytic}. \end{cases} \tag{2.22}$$

In the above, when assuming (2.15), we have assumed that for $|\beta|$ large enough, $i\beta \in \rho(-A_0)$. It is not necessary to assume that $i\mathbb{R} \subset \rho(-A_0)$ (thus, it allows $A_0 = 0$). A similar remark applies to (2.16)–(2.22), as well as the following (2.24) and (2.25). Next, we look at some necessary conditions for $e^{\mathcal{A}t}$'s having some particular properties.

Theorem 2.3. *Let* (2.10) *and* (2.13) *hold. Let* $0 \le b, c \le 1$ *such that* (2.14) *holds, and* $e^{\mathcal{A}t}$ *and* $e^{(-A_0+BA_1^{-1}C)t}$ *be contraction semigroups.*

(i) *Let* $e^{\mathcal{A}t}$ *be exponentially stable and* $b+c<1$. *Then*

$$\begin{aligned} &i\mathbb{R} \subset \rho(-A_0) \Rightarrow e^{-A_0 t} \text{ is exponentially stable,}\\ &i\mathbb{R} \subset \rho(-A_0+BA_1^{-1}C) \Rightarrow e^{(-A_0+BA_1^{-1}C)t} \text{ is exponentially stable.}\end{aligned} \tag{2.23}$$

(ii) *Let* $e^{\mathcal{A}t}$ *be differentiable and* $b+c \le 1$.
Then both $e^{-A_0 t}$ *and* $e^{(-A_0+BA_1^{-1}C)t}$ *have the SDGP. Moreover, if*

$$\lim_{|\beta|\to\infty} \log(1+|\beta|)\|(i\beta-\mathcal{A})^{-1}\| = 0, \tag{2.24}$$

then both $e^{-A_0 t}$ *and* $e^{(-A_0+BA_1^{-1}C)t}$ *are differentiable.*

(iii) *Let* $\mu \in (0,1)$ *such that*

$$\overline{\lim_{|\beta|\to\infty}} |\beta|^{\mu}\|(i\beta-\mathcal{A})^{-1}\| < \infty. \tag{2.25}$$

Then, in the case $b+c \le 1$ *(respectively* $b+c<1+\mu$*),* $e^{-A_0 t}$ *(respectively* $e^{(-A_0+BA_1^{-1}C)t}$*) is of Gevrey class* $\delta > 1/\mu$.

(iv) *Let* $e^{\mathcal{A}t}$ *be analytic. Then, in the case* $b+c<2$ *(respectively* $b+c\le 1$*),* $e^{(-A_0+BA_1^{-1}C)t}$ *(respectively* $e^{-A_0 t}$*) is analytic.*

Let us make some comments on the above results. First of all, for the case $b+c \le 1$ (respectively $b+c<1$ and $i\mathbb{R} \subset \rho(\mathcal{A}) \cap \rho(-A_0)$), $e^{\mathcal{A}t}$ and $e^{-A_0 t}$ are both analytic (respectively exponentially stable) or neither. In particular, if $e^{-A_0 t}$ is not analytic (respectively exponentially stable), neither is $e^{\mathcal{A}t}$. This is the case if, say, A_0 is unbounded and skew-symmetric (see example in [8]). Thus, roughly speaking, if the coupling is too weak, the property of $e^{\mathcal{A}t}$ will only be as "good" as $e^{-A_0 t}$. On the other hand, if the coupling is reasonably strong (i.e., $b+c>1$), then due to the nicer property of $e^{-A_1 t}$, the property of $e^{\mathcal{A}t}$ could be better than $e^{-A_0 t}$, provided $e^{-[A_0-BA_1^{-1}C]t}$ is better. We note that the case $b+c=1$ for the exponential stability of $e^{\mathcal{A}t}$ and the case $b+c=2$ for the analyticity of $e^{\mathcal{A}t}$ are not included in the above. These cases turn out to be critical and will be included in the following results.

For the strongly coupled case, some additional (compatible) conditions are needed. We have the following results.

Theorem 2.4. *Let* (2.10) *and* (2.13) *hold. Let* $e^{\mathcal{A}t}$ *be a contraction semigroup. Assume further that there exist constants* $K, \delta > 0$ *and* $\mu \in (0,1]$, *and a self-adjoint positive definite operator* $Q : \mathcal{D}(Q) \subseteq H_1 \to H_1$, *such that*

$$\begin{aligned}\operatorname{Re}\Big\{ \langle A_0x, x \rangle + \langle A_1y, y \rangle - \langle By, x \rangle - \langle Cx, y \rangle \Big\} \\ \ge \delta \operatorname{Re}\big[\langle A_0x, x \rangle + \langle A_1y, y \rangle \big], \quad \forall (x, y) \in \mathcal{D}(\mathcal{A}),\end{aligned} \tag{2.26}$$

$$\big|\operatorname{Im}\langle A_0x, x \rangle\big|^{\mu} \le K\Big(\big|\operatorname{Re}\langle A_0x, x \rangle\big| + 1\Big), \quad \forall x \in \mathcal{D}(A_0), \ \|x\| \le 1, \tag{2.27}$$

$$\big|\operatorname{Im}\langle A_1y, y \rangle\big| \le K\Big(\big|\operatorname{Re}\langle A_1y, y \rangle\big| + \|y\|^2\Big), \quad \forall y \in \mathcal{D}(A_1), \tag{2.28}$$

$$|\langle By, x \rangle| \le K\Big(|\langle Cx, y \rangle| + \|x\|\|y\|\Big), \quad \forall (x, y) \in \mathcal{D}(\mathcal{A}), \tag{2.29}$$

$$\mathcal{D}(Q) \subseteq \mathcal{D}(A_1) \cap \mathcal{D}(A_1^*), \tag{2.30}$$

$$\langle Qy, y \rangle \le K \operatorname{Re}\langle A_1y, y \rangle, \qquad \forall y \in \mathcal{D}(A_1) \subseteq \mathcal{D}(Q), \tag{2.31}$$

$$\mathcal{D}(Q^{\frac{1}{\mu}}) \subseteq \mathcal{D}(B) \cap \mathcal{D}(C^*). \tag{2.32}$$

Then

$$\overline{\lim_{|\beta| \to \infty}} |\beta|^{\mu} \|(i\beta - \mathcal{A})^{-1}\| < \infty. \tag{2.33}$$

Thus, in the case $\mu = 1$, $e^{\mathcal{A}t}$ *is analytic and in the case* $\mu \in (0,1)$, $e^{\mathcal{A}t}$ *is of Gevrey class* $\delta > 1/\mu$.

Condition (2.30) *can be replaced by the following:*

$$\begin{cases} \|A_0x\| \le k_0\|Cx\| + K\|x\|, & \forall x \in \mathcal{D}(A_0) \cap \mathcal{D}(C), \\ \|A_1y\| \le k_1\|By\| + K\|y\|, & \forall y \in \mathcal{D}(A_1) \cap \mathcal{D}(B), \end{cases} \tag{2.34}$$

for some constants $k_0, k_1 > 0$ *with* $k_0k_1 < 1$. *In the case* $\mu = 1$, (2.27) *can be replaced by the analyticity of* $e^{-[A_0 - BA_1^{-1}C]t}$.

Let us make some comments on the conditions imposed in Theorem 2.4. Conditions (2.26) and (2.29) automatically hold if $C = -B^*$. Conditions (2.27) and (2.28) imply that e^{-A_0t} is of Gevrey class $\delta > 1/\mu$ and e^{-A_1t} is analytic (see Corollary 4.7). Condition (2.27) holds when A_0 is bounded, which is an interesting case in applications. Condition (2.34) (with $k_0k_1 < 1$) roughly means that we are in the strongly coupled case; and (2.31)–(2.32) mean that the coupling operators are "controlled" by $A_1^{1/\mu}$, through the self-adjoint positive definite operator Q. In the case that A_1 is self-adjoint, it is possible to take $Q = A_1$. Then (2.28), (2.30) and (2.31) hold. Finally, in the case that both A_0 and A_1 are self-adjoint and $C = -B^*$, (2.26)–(2.31) hold with $Q = A_1$. In such a case, the only real condition imposed is (2.32). This will lead to some known results (see below).

As we indicated in the theorem, when $\mu = 1$ and $e^{-[A_0 - BA_1^{-1}C]t}$ is analytic, (2.27) can be omitted. In this case, $e^{-A_0 t}$ can be very general. From this observation, we see that our result applies to several interesting coupled hyperbolic-parabolic systems such as linear thermoelastic system (see 8 for details).

A special case of our result is when $H_0 = H_1, A_0 = 0, A_1 = A^\alpha$, $B = -C = A^{\frac{1}{2}}$, with A being a self-adjoint positive definite operator. In this case, with $Q = A_1$, (2.26)–(2.31) are all satisfied and (2.32) is equivalent to $\frac{\alpha}{\mu} \geq \frac{1}{2}$. Thus, by taking $\mu = 1$, we see that when $\alpha \geq \frac{1}{2}$, $e^{\mathcal{A}t}$ is analytic; and for $0 < \alpha < \frac{1}{2}$, by taking $\mu \leq 2\alpha < 1$, we see that $e^{\mathcal{A}t}$ is of Gevrey class $\delta > \frac{1}{2\alpha}$. These results (with $0 < \alpha \leq 1$) had been obtained by Chen and Triggiani in [3–5]. They also gave an example to show that the semigroup is generally not analytic when $0 < \alpha < \frac{1}{2}$. Thus, we do not expect the analyticity of the semigroup under the conditions in Theorem 2.4 with $\mu < 1$, either.

To conclude this section, we would like to state the following result concerning the exponential stability of $e^{\mathcal{A}t}$.

Theorem 2.5. *Let $e^{\mathcal{A}t}$ be a contraction semigroup with $i\mathbb{R} \subseteq \rho(\mathcal{A})$. Assume that* (2.10), (2.26) *hold and $-A_1$ is strongly dissipative.*

(i) *Let*

$$\|By\| \leq K(\mathrm{Re}\,\langle A_1 y, y \rangle)^{\frac{1}{2}}, \qquad \forall\, y \in \mathcal{D}(B) \cap \mathcal{D}(A_1). \tag{2.35}$$

If either $e^{-A_0 t}$ is exponentially stable or $A_0 \in \mathcal{L}(H_0)$, then $e^{\mathcal{A}t}$ is exponentially stable.

(ii) *Let* (2.28) *and* (2.29) *hold. If $A_0 \in \mathcal{L}(H_0)$, then $e^{\mathcal{A}t}$ is exponentially stable.*

3. Conditions for $\mathcal{A}$'s generating a C_0 semigroup. The following result is concerned with the conditions under which $\mathcal{A}$ generates a contraction semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$.

Theorem 3.1. *Let (2.10) hold. Then the following are equivalent:*

(i) $\mathcal{A}$ *generates a contraction semigroup* $e^{\mathcal{A}t}$ *on* $\mathcal{H}$.

(ii) *For any* $\lambda > 0$,

$$\begin{cases} [\lambda + A_0 - B(\lambda + A_1)^{-1}C]^{-1} \in \mathcal{L}(H_0), \\ [\lambda + A_1 - C(\lambda + A_0)^{-1}B]^{-1} \in \mathcal{L}(H_1). \end{cases} \tag{3.1}$$

(iii) *For some* $\lambda > 0$, *(3.1) holds.*

(iv) *For any* $\lambda > 0$, $-[A_0 - B(\lambda + A_1)^{-1}C]$ *and* $-[A_1 - C(\lambda + A_0)^{-1}B]$ *generate contraction semigroups on* H_0 *and* H_1, *respectively.*

(v) *For some* $\lambda > 0$, $-[A_0 - B(\lambda + A_1)^{-1}C]$ *and* $-[A_1 - C(\lambda + A_0)^{-1}B]$ *generate contraction semigroups on* H_0 *and* H_1, *respectively.*

The above result amounts to saying that some compatibility conditions among A_0, A_1, B and C are sufficient for the operator $\mathcal{A}$ defined by (1.2)'s generating a contraction semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$. In many applications, A_0, A_1, B and C are differential operators. For these cases, conditions (ii)–(v) sometimes can be checked by looking at the orders (together with the boundary conditions) of the differential operators. We point out that conditions (iii) and (v) are very mild. We may take, say, $\lambda = 1$ to check one of these conditions.

In what follows, if S is a densely defined linear operator which admits a bounded extension, we simply denote its bounded extension by S itself.

Proof. (i) $\Rightarrow$ (ii). Suppose $\mathcal{A}$ generates a contraction semigroup $e^{\mathcal{A}t}$ over $\mathcal{H}$. Then, for any $\lambda > 0$,

$$(\lambda - \mathcal{A})^{-1} \in \mathcal{L}(\mathcal{H}). \tag{3.2}$$

Thus, for any $\xi \in H_0$, there exists

$$(x, y) \in \mathcal{D}(\mathcal{A}) \equiv [\mathcal{D}(A_0) \cap \mathcal{A}(C)] \times [\mathcal{D}(A_1) \cap \mathcal{D}(B)],$$

such that

$$\begin{pmatrix} \xi \\ 0 \end{pmatrix} = (\lambda - \mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (\lambda + A_0)x - By \\ -Cx + (\lambda + A_1)y \end{pmatrix}. \tag{3.3}$$

This leads to the following (note (2.11)):

$$\begin{cases} y = (\lambda + A_1)^{-1}Cx \in \mathcal{D}(B), \\ \xi = [\lambda + A_0 - B(\lambda + A_1)^{-1}C]x, \end{cases}$$

which implies $\mathcal{R}(\lambda + A_0 - B(\lambda + A_1)^{-1}C) = H_0$. Next, taking the inner product of (3.3) with (x, y), noting (2.12), we obtain

$$\begin{aligned}\lambda\|x\|^2 &\le \langle(\lambda - \mathcal{A})\tbinom{x}{y}, \tbinom{x}{y}\rangle \le \langle[\lambda + A_0 - B(\lambda + A_1)^{-1}C]x, x\rangle \\ &\le \|[\lambda + A_0 - B(\lambda + A_1)^{-1}C]x\|\,\|x\|, \quad \forall x \in \mathcal{D}\big(A_0 - B(\lambda + A_1)^{-1}C\big).\end{aligned}$$

Hence, the first relation in (3.1) follows. By considering $(0, \eta) \in \mathcal{H}$ and using (3.2), we can obtain the second relation in (3.1).

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i). A direct computation shows that in the case in which (3.1) holds for some $\lambda > 0$, we have

$$\begin{aligned}(\lambda - \mathcal{A})^{-1} &\equiv \begin{pmatrix} \lambda + A_0 & -B \\ -C & \lambda + A_1 \end{pmatrix}^{-1} \qquad (3.4) \\ &= \begin{pmatrix} [\lambda + A_0 - B(\lambda + A_1)^{-1}C]^{-1} & (\lambda + A_0)^{-1}B[\lambda + A_1 - C(\lambda + A_0)^{-1}B]^{-1} \\ (\lambda + A_1)^{-1}C[\lambda + A_0 - B(\lambda + A_1)^{-1}C]^{-1} & [\lambda + A_1 - C(\lambda + A_0)^{-1}B]^{-1} \end{pmatrix}.\end{aligned}$$

Thus, for such a $\lambda > 0$, (3.2) holds and the Lumer–Phillips theorem ([18]) applies.

(iii) $\Rightarrow$ (v). For $\lambda > 0$, by the last condition in (2.10), we can prove that both $-[A_0 - B(\lambda + A_1)^{-1}C]$ and $-[A_1 - C(\lambda + A_0)^{-1}B]$ are dissipative. Thus, by [18, p. 16, Theorem 4.6], we have that if (3.1) holds, then

$$\overline{\mathcal{D}(A_0 - B(\lambda + A_1)^{-1}C)} = H_0, \quad \overline{\mathcal{D}(A_1 - C(\lambda + A_0)^{-1}B)} = H_1.$$

Thus, the Lumer–Phillips theorem applies to obtain (v).

(v) $\Rightarrow$ (iii) is clear. Similarly, we can prove (ii) $\iff$ (iv). $\square$

In the case that A_0, A_1, B and C are differential operators (with proper boundary conditions), there are two interesting cases characterized by the comparison of the orders of the differential operators involved. We now look at these cases. First, we look at the case that the orders of coupling operators are "lower" than that of A_1 (or A_1^*); i.e., we are in the weakly coupled case.

Proposition 3.2. *Let* (2.10) *hold. Let*

$$\mathcal{D}(A_1) \subseteq \mathcal{D}(B). \qquad (3.5)$$

Then the first relation in (3.1) *implies the second. Further, if*

$$0 \in \rho(A_1), \qquad \mathcal{D}(A_1^*) \subseteq \mathcal{D}(C^*), \qquad (3.6)$$

then $\mathcal{A}$ generates a contraction semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$ if and only if $-[A_0 - BA_1^{-1}C]$ generates a contraction semigroup $e^{-[A_0-BA_1^{-1}C]t}$ on H_0.

Proof. For $\lambda > 0$, by (3.5), we have $B(\lambda + A_1)^{-1} \in \mathcal{L}(H_1, H_0)$, and

$$\begin{aligned}&[\lambda + A_1 - C(\lambda + A_0)^{-1}B]^{-1} \\ &= (\lambda + A_1)^{-1} + (\lambda + A_1)^{-1}C[\lambda + A_0 - B(\lambda + A_1)^{-1}C]^{-1}B(\lambda + A_1)^{-1}.\end{aligned} \tag{3.7}$$

This proves our first claim.

Next, let (3.6) hold. By Theorem 3.1, we see that it suffices to show that (3.1) holds if and only if $-[A_0 - BA_1^{-1}C]$ generates a contraction semigroup on H_0. We now prove this. Suppose (3.1) is true. By the proof of Theorem 3.1,we see that $-[A_0 - B(\lambda + A_1)^{-1}C]$ generates a contraction semigroup. On the other hand, by (3.5) and (3.6), we have

$$\begin{cases} \widetilde{A}_\lambda \triangleq -B(\lambda + A_1)^{-1}C + BA_1^{-1}C = \lambda B(\lambda + A_1)^{-1}A_1^{-1}C, \\ \mathcal{D}(\widetilde{A}_\lambda) = \mathcal{D}(C) \supseteq \mathcal{D}(A_0 - B(\lambda + A_1)^{-1}C), \end{cases} \tag{3.8}$$

and $\widetilde{A}_\lambda$ admits a bounded extension. Thus,

$$-[A_0 - BA_1^{-1}C] = -[A_0 - B(\lambda + A_1)^{-1}C - \widetilde{A}_\lambda] \tag{3.9}$$

generates a C_0 semigroup, which is necessarily a contraction semigroup.

Conversely, if $-[A_0 - BA_1^{-1}C]$ generates a contraction semigroup on H_0, then, by (3.8)–(3.9), we see that $-[A_0 - B(\lambda + A_1)^{-1}C]$ also generates a contraction semigroup. Hence, the first relation in (3.1) follows, and using what we just proved, one obtains the second relation in (3.1). $\square$

Next, we look at the case that the orders of coupling operators are "higher" than that of A_1 (or A_1^*); i.e., we are in the strongly coupled case. For such a case, the approach of Proposition 3.2 does not work any more since $\widetilde{A}_\lambda$ defined in (3.8) does not necessarily admit a bounded extension.

Recall that in a Hilbert space framework, $\mathcal{A}$ generates a contraction semigroup if and only if $\mathcal{A}$ is closed, densely defined, and both $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative (combining [18, p. 15, Cor. 4.4 and p. 41, Cor. 10.6]). Since we already have that $\mathcal{A}$ is densely defined and dissipative, it is enough for us to find conditions under which $\mathcal{A}$ is closed and $\mathcal{A}^*$ is also dissipative. Let us first look at the closedness of $\mathcal{A}$.

Proposition 3.3. *Let* (2.10) *and* (2.34) *hold for some constants* $k_0, k_1, K > 0$ *with* $k_0 k_1 < 1$. *Then* $\mathcal{A}$ *is closed.*

Proof. Let $(x_n, y_n) \in \mathcal{D}(\mathcal{A})$ such that

$$\binom{x_n}{y_n} \to \binom{x}{y}, \qquad \mathcal{A}\binom{x_n}{y_n} \to \binom{\xi}{\eta}. \tag{3.10}$$

This gives

$$\xi_n \stackrel{\Delta}{=} -A_0 x_n + B y_n \to \xi, \qquad \eta_n \stackrel{\Delta}{=} C x_n - A_1 y_n \to \eta. \tag{3.11}$$

Then $\|x_n\|$, $\|y_n\|$, $\|\xi_n\|$ and $\|\eta_n\|$ are all bounded. In what follows, $K > 0$ represents a generic constant which can be different at different places. By (2.34) and (3.11), we obtain the following estimates:

$$\begin{aligned} \|A_0 x_n\| &\le k_0 \|C x_n\| + K = k_0 \|\eta_n + A_1 y_n\| + K \le k_0 k_1 \|B y_n\| + K \\ &= k_0 k_1 \|\xi_n + A_0 x_n\| + K \le k_0 k_1 \|A_0 x_n\| + K. \end{aligned} \tag{3.12}$$

Thus, by $k_0 k_1 < 1$, we see that $\|A_0 x_n\|$ is bounded. Then it follows that $\|B y_n\|$, $\|A_1 y_n\|$ and $\|C x_n\|$ are all bounded. Now, using Mazur's theorem ([26]), we may let $(\widetilde{x}_n, \widetilde{y}_n)$ be some convex combination of (x_k, y_k), $k = 1, \dots, n$, such that

$$(\widetilde{x}_n, \widetilde{y}_n) \xrightarrow{s} (x, y), \quad A_0 \widetilde{x}_n,\ C\widetilde{x}_n,\ A_1 \widetilde{y}_n,\ B\widetilde{y}_n \ \text{ are all strongly convergent.}$$

By the closedness of A_0, A_1, B and C, we have $x \in \mathcal{D}(A_0) \cap \mathcal{D}(C)$ and $y \in \mathcal{D}(A_1) \cap \mathcal{D}(B)$, which means $(x, y) \in \mathcal{D}(\mathcal{A})$. We also easily see that

$$\mathcal{A}\binom{x}{y} = \binom{\xi}{\eta},$$

proving the closeness of $\mathcal{A}$. $\square$

Next, we assume (2.10) and we want to identify $\mathcal{A}^*$. To this end, let us define $\widehat{\mathcal{A}} : \mathcal{D}(\widehat{\mathcal{A}}) \subseteq \mathcal{H} \to \mathcal{H}$ as follows:

$$\mathcal{D}(\widehat{\mathcal{A}}) = [\mathcal{D}(A_0^*) \cap \mathcal{D}(B^*)] \times [\mathcal{D}(A_1^*) \cap \mathcal{D}(C^*)], \tag{3.13}$$

$$\widehat{\mathcal{A}} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \equiv \begin{pmatrix} -A_0^* & C^* \\ B^* & -A_1^* \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -A_0^* \xi + C^* \eta \\ B^* \xi - A_1^* \eta \end{pmatrix}, \ \forall \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathcal{D}(\widehat{\mathcal{A}}). \tag{3.14}$$

It is clear that $\mathcal{A}^*$ is an extension of $\widehat{\mathcal{A}}$: $\widehat{\mathcal{A}} \subseteq \mathcal{A}^*$. We now introduce the following (compare with the last condition in (2.10)):

$$\begin{aligned} \mathrm{Re}\Big\{ \langle A_0^*\xi, \xi \rangle + \langle A_1^*\eta, \eta \rangle - \langle B^*\xi, \eta \rangle - \langle C^*\eta, \xi \rangle \Big\} \geq 0, \\ \forall \xi \in \mathcal{D}(A_0^*) \cap \mathcal{D}(B^*),\ \eta \in \mathcal{D}(A_1^*) \cap \mathcal{D}(C^*). \end{aligned} \tag{3.15}$$

Note that in the case $C = -B^*$, (3.15) always holds. Under (3.15), we have

$$\mathrm{Re}\,\langle \widehat{\mathcal{A}}\tbinom{\xi}{r}\eta, \tbinom{\xi}{\eta} \rangle \leq 0, \qquad \forall \tbinom{\xi}{\eta} \in \mathcal{D}(\widehat{\mathcal{A}}). \tag{3.16}$$

Now, we would like to know when it holds that $\widehat{\mathcal{A}} = \mathcal{A}^*$.

Proposition 3.4. *Let* (2.10) *and* (3.15) *hold. Let* $0 \in \rho(A_1)$, *and*

$$\mathcal{D}(B) \subseteq \mathcal{D}(A_1),\ \mathcal{D}(C) \subseteq \mathcal{D}(A_0),\ \mathcal{D}(B^*) \subseteq \mathcal{D}(A_0^*),\ \mathcal{D}(C^*) \subseteq \mathcal{D}(A_1^*). \tag{3.17}$$

Let $-[A_0 - BA_1^{-1}C]$ *generate a contraction semigroup, and for some* $\lambda > 0$, *suppose* $[A_1 - C(\lambda + A_0)^{-1}B]^{-1} \in \mathcal{L}(H_1)$ *and*

$$\mathcal{D}\big((A_0 - BA_1^{-1}C)^*\big) \subseteq \mathcal{D}(B^*),\ \ \mathcal{D}\big([A_1 - C(\lambda + A_0)^{-1}B]^*\big) \subseteq \mathcal{D}(A_1^*). \tag{3.18}$$

Then $\widehat{\mathcal{A}} = \mathcal{A}^*$. *In addition, if* (2.34) *holds with* $k_0k_1 < 1$, *then* $\mathcal{A}$ *generates a contraction semigroup* $e^{\mathcal{A}t}$ *on* $\mathcal{H}$.

Proof. It suffices to prove that $\mathcal{D}(\mathcal{A}^*) \subseteq \mathcal{D}(\widehat{\mathcal{A}})$. For any fixed $(\xi, \eta) \in \mathcal{D}(\mathcal{A}^*)$, by definition, there exists a constant $K > 0$, such that

$$\begin{aligned} &|\,\langle -A_0x + By, \xi \rangle + \langle Cx - A_1y, \eta \rangle\,| \\ &\quad = \Big|\,\langle \mathcal{A}\tbinom{x}{y}, \tbinom{\xi}{\eta} \rangle\,\Big| \leq K\Big\|\tbinom{x}{y}\Big\|, \qquad \forall \tbinom{x}{y} \in \mathcal{D}(\mathcal{A}). \end{aligned} \tag{3.19}$$

Since $-A_0$ and $-[A_0 - BA_1^{-1}C]$ generate contraction semigroups, for any $\lambda > 0$, both $(\lambda + A_0)^{-1}$ and $(\lambda + A_0 - BA_1^{-1}C)^{-1}$ are bounded. Now, for any $z \in \mathcal{D}(B)$, we let

$$\begin{cases} x = -(\lambda + A_0 - BA_1^{-1}C)^{-1}Bz \in \mathcal{D}(A_0) \cap \mathcal{D}(C) = \mathcal{D}(C), \\ y = -A_1^{-1}C(\lambda + A_0 - BA_1^{-1}C)^{-1}Bz \in \mathcal{D}(A_1) \cap \mathcal{D}(B) = \mathcal{D}(B). \end{cases} \tag{3.20}$$

Then it follows that

$$\begin{cases} -A_0x + By = Bz + \lambda x, \\ Cx - A_1y = 0. \end{cases} \tag{3.21}$$

Note that for any $z \in \mathcal{D}(B) \subseteq \mathcal{D}(A_1)$, we have

$$\begin{aligned} & A_1^{-1}C(\lambda + A_0 - BA_1^{-1}C)^{-1}Bz \\ & = A_1^{-1}C[I - (\lambda + A_0)^{-1}BA_1^{-1}C]^{-1}(\lambda + A_0)^{-1}Bz \\ & = \big\{[I - A_1^{-1}C(\lambda + A_0)^{-1}B]^{-1} - I\big\}z = \big\{[A_1 - C(\lambda + A_0)^{-1}B]^{-1}A_1 - I\big\}z \\ & = \Big(A_1^*\{[A_1 - C(\lambda + A_0)^{-1}B]^*\}^{-1}\Big)^* z - z. \end{aligned}$$

Thus, for any $z \in \mathcal{D}(B)$, by taking $(x, y) \in \mathcal{D}(\mathcal{A})$ defined by (3.20) in (3.19), we obtain

$$\begin{aligned} |\langle Bz, \xi \rangle| \le K(\|x\| + \|y\|) \le K\big\{ & \|(\lambda + A_0 - BA_1C)^{-1}Bz\| \\ & + \|A_1^{-1}C(\lambda + A_0 - BA_1^{-1}C)^{-1}Bz\|\big\} \\ = K\Big\{ & \Big\|\big\{B^*[(\lambda + A_0 - BA_1^{-1}C)^{-1}]^*\big\}^* z\Big\| \\ & + \Big\|\Big(A_1^*\{[A_1 - C(\lambda + A_0)^{-1}B]^*\}^{-1}\Big)^* z - z\Big\|\Big\} \le K\|z\|. \end{aligned} \tag{3.22}$$

Hence, we have

$$\xi \in \mathcal{D}(B^*) = \mathcal{D}(A_0^*) \cap \mathcal{D}(B^*). \tag{3.23}$$

Combining (3.23) with (3.19), we see that

$$|\langle Cx - A_1y, \eta \rangle| \le K\big\{\|x\| + \|y\|\big\}, \qquad \forall (x, y) \in \mathcal{D}(\mathcal{A}). \tag{3.24}$$

Now, taking $y = 0$ and noting (3.17), we have

$$|\langle Cx, \eta \rangle| \le K\|x\|, \quad \forall x \in \mathcal{D}(A_0) \cap \mathcal{D}(C) = \mathcal{D}(C). \tag{3.25}$$

This gives

$$\eta \in \mathcal{D}(C^*) = \mathcal{D}(A_1^*) \cap \mathcal{D}(C^*). \tag{3.26}$$

Hence, $(\xi, \eta) \in \mathcal{D}(\widehat{\mathcal{A}})$, proving $\mathcal{D}(\mathcal{A}^*) \subseteq \mathcal{D}(\widehat{\mathcal{A}})$.

Finally, by Proposition 3.3, under (2.34), $\mathcal{A}$ is closed. Hence, the last conclusion follows.

4. Some relevant results. In this section, we present some results which are closely related to our main results.

Within this section, we let H be a Hilbert space and $A : \mathcal{D}(A) \subseteq H \to H$ generate a C_0 semigroup e^{At} on H. We define $\sigma_0(A)$ and $\omega_0(A)$ as in (2.4). It follows from the Hille–Yosida theorem ([18]) that for any $\varepsilon > 0$, there exists an $M_\varepsilon \geq 1$, such that

$$\|(\lambda - A)^{-1}\| \leq \frac{M_\varepsilon}{\operatorname{Re}\lambda - \omega_0(A) - \varepsilon}, \quad \forall \operatorname{Re}\lambda > \omega_0(A) + \varepsilon. \tag{4.1}$$

Our first result is concerned with $\sigma_0(A)$ and $\omega_0(A)$, which will be useful in studying the exponential stability of the C_0 semigroup e^{At}.

Proposition 4.1. *Let $\sigma_1 > \sigma_0(A)$. Then*

$$\omega_0(A) < \sigma_1, \tag{4.2}$$

if and only if

$$\sup_{\operatorname{Re}\lambda \geq \sigma_1} \|(\lambda - A)^{-1}\| \stackrel{\Delta}{=} K_1 < \infty. \tag{4.3}$$

Consequently,

$$\omega_0(A) = \inf\Big\{\sigma_1 > \sigma_0(A) : \sup_{\operatorname{Re}\lambda \geq \sigma_1} \|(\lambda - A)^{-1}\| < \infty\Big\}. \tag{4.4}$$

In the case (4.3) *holds, we further have the following estimate:*

$$\omega_0(A) \leq \sigma_1 - \frac{1}{K_1}. \tag{4.5}$$

The above result can be found in [7]. For the reader's convenience, we provide a detailed proof below. (A similar result can also be found in [19].)

Proof. Let (4.2) hold. By taking $\varepsilon = \frac{\sigma_1 - \omega_0(A)}{4}$ in (4.1), we see that (4.3) holds with $K_1 \leq \frac{4M_\varepsilon}{\sigma_1 - \omega_0(A)}$. Conversely, suppose (4.3) holds for $\sigma_1 > \sigma_0(A)$. Take $K > K_1$ and $n \geq 1$ such that

$$\omega_1 \stackrel{\Delta}{=} \sigma_1 + \frac{n}{K} > \omega_0(A). \tag{4.6}$$

As in [7], for any $x \in H$, one has

$$\|(\omega_1 + i \cdot - A)^{-1}x\|, \ \|(\omega_1 - i \cdot - A^*)^{-1}\| \in L^2(\mathbb{R}). \tag{4.7}$$

Now, for any $x \in H$ and $\alpha \in [\omega_1 - \frac{1}{K}, \omega_1]$, by (4.3), we have

$$\|(\alpha + i\beta - A)^{-1}x\| = \|[I - (\omega_1 - \alpha)(\omega_1 + i\beta - A)^{-1}]^{-1}(\omega_1 + i\beta - A)^{-1}x\|$$
$$\leq \frac{1}{1 - (\omega_1 - \alpha)K_1}\|(\omega_1 + i\beta - A)^{-1}x\| \leq \frac{K}{K - K_1}\|(\omega_1 + i\beta - A)^{-1}x\|.$$

Similarly, for any $\alpha \in [\omega_1 - \frac{2}{K}, \omega_1 - \frac{1}{K}]$, we have

$$\|(\alpha + i\beta - A)^{-1}x\| \leq \frac{K}{K - K_1}\|(\omega_1 - \frac{1}{K} + i\beta - A)^{-1}x\|$$
$$\leq \Big(\frac{K}{K - K_1}\Big)^2\|(\omega_1 + i\beta - A)^{-1}x\|.$$

By induction, we obtain

$$\|(\alpha + i\beta - A)^{-1}x\| \leq \Big(\frac{K}{K - K_1}\Big)^{n+1}\|(\omega_1 + i\beta - A)^{-1}x\|, \tag{4.8}$$

$\forall \alpha \in [\sigma_1 - \frac{1}{K}, \omega_1],\ \beta \in \mathbb{R},\ x \in H$. Note that (4.8) implies, in particular, that

$$\sigma_0(A) \leq \sigma_1 - \frac{1}{K}. \tag{4.9}$$

Combining (4.7) and (4.8), we see that

$$\|(\alpha + i \cdot - A)^{-1}x\|,\ \|(\alpha - i \cdot - A^*)^{-1}\| \in L^2(\mathbb{R}), \quad \forall \alpha \in [\sigma_1 - \frac{1}{K}, \omega_1]. \tag{4.10}$$

Also, by [7], we have

$$\lim_{|\beta|\to\infty}\|(\alpha + i\beta - A)^{-1}x\| = \lim_{|\beta|\to\infty}\|(\alpha - i\beta - A^*)^{-1}y\| = 0, \tag{4.11}$$

uniformly in $\alpha \in [\sigma_1 - \frac{1}{K}, \omega_1]$. Next, for any $0 < \varepsilon < \frac{1}{K}$, set $A_\varepsilon = A - (\sigma_1 + \varepsilon - \frac{1}{K})I$. We are going to prove that

$$\|e^{A_\varepsilon t}\| \leq M_\varepsilon, \qquad \forall t \geq 0, \tag{4.12}$$

for some $M_\varepsilon \geq 1$. This will imply

$$\omega_0(A) = \omega_0(A_\varepsilon) + \sigma_1 + \varepsilon - \frac{1}{K} \leq \sigma_1 + \varepsilon - \frac{1}{K}, \tag{4.13}$$

which yields (4.2) since $\varepsilon < \frac{1}{K}$. We now prove (4.12). From (4.9), we see that

$$\sigma_0(A_\varepsilon) = \sigma_0(A) - \sigma_1 - \varepsilon + \frac{1}{K} \leq -\varepsilon. \tag{4.14}$$

Also, by (4.6), we have (note $\varepsilon < \frac{1}{K}$)

$$\begin{aligned}\omega_0(A_\varepsilon) &= \omega_0(A) - \sigma_1 - \varepsilon + \frac{1}{K} < \omega_1 - \sigma_1 - \varepsilon + \frac{1}{K} \\ &= \frac{n+1}{K} - \varepsilon \stackrel{\Delta}{=} \mu > 0 \vee \omega_0(A_\varepsilon),\end{aligned} \tag{4.15}$$

and

$$\mu + i\beta - A_\varepsilon = \omega_1 + i\beta - A, \qquad \forall \beta \in \mathbb{R}. \tag{4.16}$$

Hence, for any $x, y \in \mathcal{D}(A^2)$, using (4.7) and (4.11), we have (see [18, p. 29, Corollary 7.5])

$$\begin{aligned}\langle e^{A_\varepsilon t} x, y \rangle &= \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\mu - i\beta}^{\mu + i\beta} \frac{e^{\lambda t}}{t} \langle (\lambda - A_\varepsilon)^{-2} x, y \rangle \, d\lambda \\ &= \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\mu - i\beta}^{\mu + i\beta} \frac{e^{\lambda t}}{t} \langle (\lambda - A_\varepsilon)^{-1} x, (\bar{\lambda} - A_\varepsilon^*)^{-1} y \rangle \, d\lambda \\ &= \frac{1}{2\pi t} \int_{-\infty}^{\infty} e^{(\mu + i\beta)t} \langle (\mu + i\beta - A_\varepsilon)^{-1} x, (\mu - i\beta - A_\varepsilon^*)^{-1} y \rangle \, d\beta.\end{aligned} \tag{4.17}$$

On the other hand, by (4.14), $\lambda \mapsto e^{\lambda t} \langle (\lambda - A_\varepsilon)^{-2} x, y \rangle$ is analytic in $\{\operatorname{Re} \lambda > -\varepsilon/2\}$. Thus, by (4.10)–(4.11), we may replace μ in (4.17) by 0. Then it follows that

$$\begin{aligned}|\langle e^{A_\varepsilon t} x, y \rangle| &\leq \frac{1}{2\pi t} \int_{-\infty}^{\infty} \|(i\beta - A_\varepsilon)^{-1} x\| \, \|(-i\beta - A_\varepsilon^*)^{-1} y\| d\beta \\ &= \frac{1}{2\pi t} \int_{-\infty}^{\infty} \|(\sigma_1 - \frac{1}{K} + \varepsilon + i\beta - A)^{-1} x\| \\ &\qquad \cdot \|(\sigma_1 - \frac{1}{K} + \varepsilon - i\beta - A^*)^{-1} y\| d\beta < \infty,\end{aligned}$$

for all $x, y \in H$ and $t \geq 1$. By the Principle of Uniform Boundedness (see [26]), we obtain (4.12).

Conclusion (4.4) is obvious. To obtain estimate (4.5), we send $\varepsilon \downarrow 0$ and $K \downarrow K_1$ in (4.13). This completes the proof.

Corollary 4.2. *Let $\sigma_0 \geq \sigma_0(A)$. Then the following four statements are equivalent:*

$$\omega_0(A) \leq \sigma_0; \tag{4.18}$$

$$\sup_{\operatorname{Re}\lambda\geq\alpha} \|(\lambda - A)^{-1}\| < \infty, \qquad \forall \alpha > \sigma_0; \tag{4.19}$$

$$K_\alpha \overset{\Delta}{=} \sup_{\beta\in\mathbb{R}} \|(\alpha + i\beta - A)^{-1}x\| < \infty, \qquad \forall \alpha > \sigma_0; \tag{4.20}$$

$$\overline{\lim_{|\beta|\to\infty}} \|(\alpha + i\beta - A)^{-1}\| < \infty, \qquad \forall \alpha > \sigma_0. \tag{4.21}$$

Proof. The implications (4.18) $\Longleftrightarrow$ (4.19) $\Rightarrow$ (4.20) $\Longleftrightarrow$ (4.21) are clear. We now prove (4.20) $\Rightarrow$ (4.19).

Let $\sigma_1 > \sigma_0 \geq \sigma_0(A)$ be arbitrary and fixed. For any $\alpha \in [\sigma_1, \omega_0(A) + 2]$, we have the following:

$$\begin{aligned} \|(\mu + i\beta - A)^{-1}\| &= \|(\alpha + i\beta - A)^{-1}[I + (\mu - \alpha)(\alpha + i\beta - A)^{-1}]^{-1}\| \\ &\leq \frac{K_\alpha}{1 - |\mu - \alpha| K_\alpha} \leq 2K_\alpha, \qquad \forall \mu \in [\alpha - \frac{1}{2K_\alpha}, \alpha + \frac{1}{2K_\alpha}]. \end{aligned} \tag{4.22}$$

Since $[\sigma_1, \omega_0(A) + 2]$ is compact, using the finite open cover theorem, we see that

$$\sup_{\operatorname{Re}\lambda\in[\sigma_1,\omega_0(A)+2]} \|(\lambda - A)^{-1}\| < \infty. \tag{4.23}$$

On the other hand, we have

$$\|(\lambda - A)^{-1}\| \leq \frac{M_1}{\operatorname{Re}\lambda - \omega_0(A) - 1} \leq M_1, \quad \forall \operatorname{Re}\lambda > \omega_0(A) + 2. \tag{4.24}$$

Combining (4.23) and (4.24), we obtain (4.19). □

By (2.9), we see that (4.18) with $\sigma_0 = \sigma_0(A)$ is equivalent to the SDGP of $e^{\mathcal{A}t}$ (see Definition 2.1). Thus, in this case, the above result gives necessary and sufficient conditions for e^{At} having the SDGP. This property has been used to verify the SDGP for C_0 semigroups associated with certain damped elastic systems (see [20]).

The following known result (also in [6]), which will be used later, is a direct consequence of the above.

Corollary 4.3. *Let* $\|e^{At}\| \le M$ *for all* $t \ge 0$*. Then* e^{At} *is exponentially stable if and only if*

$$\rho(A) \supset \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}, \tag{4.25}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta - A)^{-1}\| < \infty. \tag{4.26}$$

We also have the following interesting corollary.

Corollary 4.4. *Let* e^{At} *be a semigroup which is continuous in the uniform operator topology for* $t > 0$*. Then* e^{At} *has the SDGP.*

Proof. By Theorem 3.6 of [18, p. 50], there exists an increasing function $\psi : [0,\infty) \to [0,\infty)$, such that $\rho(A) \supset \{\alpha + i\beta : |\beta| \ge \psi(|\alpha|)\ \}$, and

$$\lim_{|\beta|\to\infty} \|(\alpha + i\beta - A)^{-1}\| = 0, \qquad \forall \alpha \in \mathbb{R}.$$

Then, for any $\alpha > \sigma_0(A)$, (4.21) holds and Corollary 4.2 applies. $\square$

We note that the condition of Corollary 4.4 holds if e^{At} is compact or differentiable (in particular, of Gevrey class or analytic). Thus, for this type of semigroups, SDGP holds (see (1.5) and [25]).

Next, we recall the following result (see [18] and [24]).

Proposition 4.5. *Let* $\|e^{At}\| \le M$ *for all* $t \ge 0$ *and* $i\beta \in \rho(A)$ *for* $|\beta|$ *large.*

(i) e^{At} *is analytic if and only if*

$$\overline{\lim_{|\beta|\to\infty}} \|\beta(i\beta - A)^{-1}\| < \infty. \tag{4.27}$$

(ii) e^{At} *is of Gevrey class* $\delta > 1/\mu$ $(0 < \mu < 1)$*, if*

$$\overline{\lim_{|\beta|\to\infty}} |\beta|^{\mu} \|(i\beta - A)^{-1}\| < \infty. \tag{4.28}$$

(iii) e^{At} *is differentiable if*

$$\overline{\lim_{|\beta|\to\infty}} \log(1 + |\beta|) \|(i\beta - A)^{-1}\| = 0. \tag{4.29}$$

We note that in the case that A is normal (i.e., A admits a resolution of identity), (4.28) and (4.29) are also necessary. Let us now present a result which will lead to some sufficient conditions for various properties of C_0 semigroups.

Proposition 4.6. *Let $f : [0,\infty) \to [0,\infty)$ be strictly increasing and concave. Let e^{At} be a contraction semigroup with (4.25) being true. Suppose for some $K \geq 0$,*

$$f(|\operatorname{Im}\langle Ax, x\rangle|) \leq |\operatorname{Re}\langle Ax, x\rangle| + K, \qquad \forall x \in \mathcal{D}(A),\ \|x\| = 1, \tag{4.30}$$

and

$$\lim_{\beta\to\infty} \frac{K}{f(\beta)} = 0. \tag{4.31}$$

Then

$$\overline{\lim_{|\beta|\to\infty}} f(|\beta|)\|(i\beta - A)^{-1}\| < \infty. \tag{4.32}$$

Proof. Suppose (4.32) is not true. Then we may assume that there exist $\beta_n \to \infty$ and $x_n \in \mathcal{D}(A)$ with $\|x_n\| = 1$, such that

$$\frac{1}{f(\beta_n)}(i\beta_n - A)x_n \to 0, \qquad n \to \infty. \tag{4.33}$$

We take the inner product of the above with x_n to get

$$\frac{i\beta_n}{f(\beta_n)} - \frac{1}{f(\beta_n)}\langle Ax_n, x_n\rangle \to 0, \qquad n \to \infty. \tag{4.34}$$

This implies (by taking real and imaginary parts, respectively)

$$\begin{cases} \frac{\operatorname{Re}\langle Ax_n, x_n\rangle}{f(\beta_n)} \to 0, & n \to \infty, \\ \frac{\beta_n}{f(\beta_n)}\Big(1 - \frac{\operatorname{Im}\langle Ax_n, x_n\rangle}{\beta_n}\Big) \to 0, & n \to \infty. \end{cases} \tag{4.35}$$

Since f is concave and $f(0) \geq 0$, for any $0 \leq \delta \leq 1$ and $\beta > 0$, one has

$$f(\delta\beta) \geq (1-\delta)f(0) + \delta f(\beta) \geq \delta f(\beta).$$

This implies (note f is strictly increasing)

$$f^{-1}(\delta f(\beta)) \leq \delta\beta, \qquad \forall 0 \leq \delta \leq 1,\ \beta > 0. \tag{4.36}$$

Then, for n large, we obtain (note (4.30)–(4.31) and (4.35)–(4.36))

$$\begin{aligned} \frac{|\operatorname{Im}\langle Ax_n, x_n\rangle|}{\beta_n} &\leq \frac{f^{-1}(|\operatorname{Re}\langle Ax_n, x_n\rangle| + K)}{\beta_n} = \frac{f^{-1}\Big(\frac{|\operatorname{Re}\langle Ax_n, x_n\rangle| + K}{f(\beta_n)} f(\beta_n)\Big)}{\beta_n} \\ &\leq \frac{|\operatorname{Re}\langle Ax_n, x_n\rangle| + K}{f(\beta_n)} \to 0, \qquad n \to \infty. \end{aligned} \tag{4.37}$$

On the other hand, since f is concave, it grows no more than a linear function. Thus,

$$\varliminf_{n\to\infty} \frac{\beta_n}{f(\beta_n)} > 0. \tag{4.38}$$

Then we see that (4.37)–(4.38) contradicts the second relation in (4.35).

Corollary 4.7. *Let the assumptions of Proposition* 4.6 *hold. Then*

(i) e^{At} *is analytic if*

$$\varlimsup_{\beta\to\infty} \frac{\beta}{f(\beta)} < \infty; \tag{4.39}$$

(ii) e^{At} *is of Gevrey class* $\delta > 1/\mu$ *with* $\mu \in (0,1)$ *if*

$$\varlimsup_{\beta\to\infty} \frac{\beta^\mu}{f(\beta)} < \infty; \tag{4.40}$$

(iii) e^{At} *is differentiable if*

$$\lim_{\beta\to\infty} \frac{\log\beta}{f(\beta)} = 0; \tag{4.41}$$

(iv) e^{At} *is exponential stable if* $K = 0$ *and*

$$\lim_{\beta\to\infty} f(\beta) < \infty. \tag{4.42}$$

The proof is clear.

We point out that conditions presented in Corollary 4.7 are only sufficient and, in general, they are not necessary. The advantage of these conditions is that they are easy to check. For example, (2.28) implies the analyticity of $e^{-A_1 t}$ and (2.27) implies that $e^{-A_0 t}$ is of Gevrey class $\delta > 1\mu$.

5. Properties of $e^{\mathcal{A}t}$. In this section, we are going to study various properties of the semigroup $e^{\mathcal{A}t}$. Among other things, we will prove Theorems 2.2 and 2.3. In what follows, we assume that $e^{\mathcal{A}t}$ is a contraction semigroup on $\mathcal{H}$. Thus, we have

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \subset \rho(\mathcal{A}). \tag{5.1}$$

Our first objective is to investigate when the following holds:

$$i\mathbb{R} \subset \rho(\mathcal{A}). \tag{5.2}$$

According to [9], this, together with the assumption that $e^{\mathcal{A}t}$ is a contraction semigroup, implies the *strong stability* of $e^{\mathcal{A}t}$ (see Definition 2.1).

Proposition 5.1. *Let $\mathcal{A}$ generate a contraction semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$. Then* (5.2) *holds provided any of the following* (i), (ii) *or* (iii) *holds.*

(i)
$$i\mathbb{R} \subset \rho(A_0) \cap \rho(A_1), \tag{5.3}$$

and

$$\begin{cases} [i\beta + A_0 - B(i\beta + A_1)^{-1}C]^{-1} \in \mathcal{L}(H_0), \\ [i\beta + A_1 - C(i\beta + A_0)^{-1}B]^{-1} \in \mathcal{L}(H_1). \end{cases} \qquad \forall \beta \in \mathbb{R}. \tag{5.4}$$

(ii)
$$\mathcal{D}(A_1) \subseteq \mathcal{D}(B), \qquad i\mathbb{R} \subset \rho(A_1), \tag{5.5}$$

and for any $\beta \in \mathbb{R}$, the first relation in (5.4) *holds.*

(iii) *There exists a $\delta > 0$, such that*

$$\mathrm{Re}\,\{\, \langle A_0x, x \rangle + \langle A_1y, y \rangle - \langle By, x \rangle - \langle Cx, y \rangle \,\} \geq \delta \|y\|^2, \tag{5.6}$$

$\forall (x, y) \in \mathcal{D}(\mathcal{A})$.

Further, it holds that

$$i\mathbb{R} \subset \rho(A_0 - BA_1^{-1}C), \quad \mathcal{D}\big([A_0 - BA_1^{-1}C]^*\big) \subseteq \mathcal{D}\big([BA_1^{-1}]^*\big). \tag{5.7}$$

In applications, condition (iii) seems the most useful one. We note the following: Condition (5.6) implies the exponential stability of e^{-A_1t} (since by taking $x = 0$ in (5.6), we have the strong dissipativeness of $-A_1$; consequently, A_1^{-1} exists). In the case where (5.5) holds, the second relation in (5.7) is automatically true since in this case $BA_1^{-1} \in \mathcal{L}(H_1, H_0)$; if $e^{-[A_0 - BA_1^{-1}C]t}$ is exponentially stable, then the first relation in (5.7) holds.

Proof. (i) In this case, we have (3.4) with $\lambda = i\beta$, for all $\beta \in \mathbb{R}$.

(ii) By (5.5), we know that for all $\beta \in \mathbb{R}$, $B(i\beta + A_1)^{-1} \in \mathcal{L}(H_1, H_0)$. A direct computation shows that (note the first relation in (5.4))

$$\begin{aligned}(i\beta - \mathcal{A})^{-1} &\equiv \begin{pmatrix} i\beta + A_0 & -B \\ -C & i\beta + A_1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} V(i\beta)^{-1} & V(i\beta)^{-1}B(i\beta + A_1)^{-1} \\ (i\beta + A_1)^{-1}CV(i\beta)^{-1} & (i\beta + A_1)^{-1} + (i\beta + A_1)^{-1}CV(i\beta)^{-1}B(\lambda + A_1)^{-1} \end{pmatrix},\end{aligned} \tag{5.8}$$

where $V(i\beta) \triangleq i\beta + A_0 - B(i\beta + A_1)^{-1}C$, $\beta \in \mathbb{R}$. This gives (5.2).

(iii) Since $e^{\mathcal{A}t}$ is a contraction semigroup, we have (5.1). Now, suppose (5.2) is not true. Then we may find $\beta \in \mathbb{R}$, $\alpha_n \downarrow 0$ and $(x_n, y_n) \in \mathcal{D}(\mathcal{A})$, with $\|x_n\|^2 + \|y_n\|^2 = 1$, such that

$$(\alpha_n + i\beta - \mathcal{A})\binom{x_n}{y_n} \to 0, \qquad n \to \infty. \tag{5.9}$$

This leads to the following (note $\alpha_n \to 0$):

$$\begin{cases} i\beta x_n + A_0 x_n - B y_n \to 0, \\ i\beta y_n + A_1 y_n - C x_n \to 0, \end{cases} \qquad n \to \infty. \tag{5.10}$$

By making inner products of (5.10) with x_n and y_n, respectively, we obtain

$$\begin{cases} i\beta\|x_n\|^2 + \langle A_0 x_n, x_n \rangle - \langle B y_n, x_n \rangle \to 0, \\ i\beta\|y_n\|^2 + \langle A_1 y_n, y_n \rangle - \langle C x_n, y_n \rangle \to 0, \end{cases} \qquad n \to \infty. \tag{5.11}$$

Adding the two limits in (5.11) together, one has

$$i\beta + \langle A_0 x_n, x_n \rangle + \langle A_1 y_n, y_n \rangle - \langle B y_n, x_n \rangle - \langle C x_n, y_n \rangle \to 0, \quad n \to \infty. \tag{5.12}$$

Taking the real part in the above, we have (by (5.6) and $\|x_n\|^2 + \|y_n\|^2 = 1$)

$$\|y_n\| \to 0, \ \|x_n\| \to 1, \qquad n \to \infty. \tag{5.13}$$

Consequently, from the second relation in (5.10), we have

$$\xi_n \triangleq A_1 y_n - C x_n \to 0, \qquad n \to \infty. \tag{5.14}$$

On the other hand, it follows from (5.6) (with $x = 0$) that $0 \in \rho(A_1)$. Thus, we can solve (5.14) for y_n:

$$y_n = A_1^{-1}\xi_n + A_1^{-1} C x_n. \tag{5.15}$$

Substituting (5.15) into the first relation in (5.10), we obtain

$$(i\beta + A_0 - BA_1^{-1}C)x_n - BA_1^{-1}\xi_n \to 0, \qquad n \to \infty. \tag{5.16}$$

Now, by the second relation in (5.7) and (5.14), we see that

$$\begin{aligned} &\lim_{n\to\infty} \|(i\beta + A_0 - BA_1^{-1}C)^{-1} BA_1^{-1}\xi_n\| \\ &\quad = \lim_{n\to\infty} \left\| \left\{ [BA_1^{-1}]^* [-i\beta + (A_0 - BA_1^{-1}C)^*]^{-1} \right\}^* \xi_n \right\| = 0. \end{aligned} \tag{5.17}$$

Hence, combining (5.16)–(5.17), we obtain

$$0 = \lim_{n\to\infty} \left\| x_n - (i\beta + A_0 - BA_1^{-1}C)^{-1} BA_1^{-1}\xi_n \right\| = \lim_{n\to\infty} \|x_n\| = 1,$$

which is a contradiction. $\square$

Now, we are going to give a proof of Theorems 2.2 and 2.3. To this end, we first prove several lemmas.

Lemma 5.2. *Let $e^{-A_1 t}$ be analytic and exponentially stable. Then, for any $a \le 1$, there exists a constant $K > 0$, such that*

$$\|A_1^a(\lambda + A_1)^{-1}\| \le \frac{K}{|\lambda|^{(1-a)\wedge 1}}, \qquad \forall \operatorname{Re}\lambda \ge 0,\ \lambda \ne 0, \tag{5.18}$$

where $(1-a)\wedge 1 = \min\{1-a, 1\}$.

Proof. For $a \le 0$, (5.18) is obvious. Let $0 \le a \le 1$. By [18, p.73, Theorem 6.10], we have

$$\|A_1^a y\| \le K\|y\|^{1-a}\|A_1 y\|^a, \qquad \forall y \in \mathcal{D}(A_1). \tag{5.19}$$

Now, for any $y_1 \in H_1$, taking $y = (\lambda + A_1)^{-1}y_1 \in \mathcal{D}(A_1)$ in the above, we obtain

$$\begin{aligned}\|A_1^a(\lambda + A_1)^{-1}y_1\| &\le K\|(\lambda + A_1)^{-1}y_1\|^{1-a}\|A_1(\lambda + A_1)^{-1}y_1\|^a \\ &\le \frac{K\|y_1\|^{1-a}}{|\lambda|^{1-a}}\|y_1 - \lambda(\lambda + A_1)^{-1}y_1\|^a \le \frac{K\|y_1\|}{|\lambda|^{1-a}}.\end{aligned} \tag{5.20}$$

This yields (5.18).

Lemma 5.3. *Let* (2.10), (2.13) *and* (2.14) *hold for some* $0 \le b, c \le 1$. *Then*

$$i\mathbb{R} \cap \rho(\mathcal{A}) = \{i\beta : [i\beta + A_0 - B(i\beta + A_1)^{-1}C]^{-1} \in \mathcal{L}(H_0)\}, \tag{5.21}$$

and for some constant $K > 0$,

$$\begin{aligned}\frac{1}{K}\|(i\beta - \mathcal{A})^{-1}\| &\le \|[i\beta + A_0 - B(i\beta + A_1)^{-1}C]^{-1}\| + \|(i\beta + A_1)^{-1}\| \\ &\le K\|(i\beta - \mathcal{A})^{-1}\|, \quad \forall i\beta \in \rho(\mathcal{A}).\end{aligned} \tag{5.22}$$

Proof. First of all, by (2.13) and (2.14) with $0 \le b, c \le 1$, we have

$$\begin{aligned}\|B(i\beta + A_1)^{-1}\| &\le \|BA_1^{-1}\|\,\|A_1(i\beta + A_1)^{-1}\| \\ &\le \|BA_1^{-1}\|\big\{1 + \|i\beta(i\beta + A_1)^{-1}\|\big\} \le K, \quad \forall \beta \in \mathbb{R}.\end{aligned} \tag{5.23}$$

Similarly (by our convention right before the proof of Theorem 3.1),

$$\|(i\beta + A_1)^{-1}C\| \le K, \qquad \forall \beta \in \mathbb{R}. \tag{5.24}$$

Now, suppose $i\beta$ is in the set on the right-hand side of (5.21). Then a direct computation shows that

$$(i\beta-\mathcal{A})^{-1}\equiv\begin{pmatrix} i\beta+A_0 & -B\\ -C & i\beta+A_1\end{pmatrix}^{-1}=\begin{pmatrix} I & 0\\ (i\beta+A_1)^{-1}C & I\end{pmatrix}$$
$$\times\begin{pmatrix} [i\beta+A_0-B(i\beta+A_1)^{-1}C]^{-1} & 0\\ 0 & (i\beta+A_1)^{-1}\end{pmatrix}\begin{pmatrix} I & B(i\beta+A_1)^{-1}\\ 0 & I\end{pmatrix}. \tag{5.25}$$

Using (5.23)–(5.24), we see that $i\beta\in\rho(\mathcal{A})$. Conversely, if $i\beta\in\rho(\mathcal{A})$, then we have

$$\begin{pmatrix} [i\beta+A_0-B(i\beta+A_1)^{-1}C]^{-1} & 0\\ 0 & (i\beta+A_1)^{-1}\end{pmatrix}$$
$$=\begin{pmatrix} I & 0\\ -(i\beta+A_1)^{-1}C & I\end{pmatrix}(i\beta-\mathcal{A})^{-1}\begin{pmatrix} I & -B(i\beta+A_1)^{-1}\\ 0 & I\end{pmatrix}. \tag{5.26}$$

Again, using (5.23)–(5.24), we obtain $i\beta$ is in the set on the right-hand side of (5.21). Finally, (5.22) follows immediately from (5.23)–(5.26).

Lemma 5.4. *Let* (2.10), (2.13) *and* (2.14) *hold with some* $0\le b,c\le 1$. *Let* $f:[0,\infty)\to[0,\infty)$ *be continuous and nondecreasing.*

(i) *Let* $b+c\le 1$ *and*

$$\lim_{\beta\to\infty} f(\beta)\beta^{1-b-c}=\infty. \tag{5.27}$$

Then

$$\overline{\lim_{|\beta|\to\infty}} f(|\beta|)\|[i\beta+A_0-B(i\beta+A_1)^{-1}C]^{-1}\|=\overline{\lim_{|\beta|\to\infty}} f(|\beta|)\|(i\beta+A_0)^{-1}\|. \tag{5.28}$$

(ii) *Let* $b+c<2$ *and*

$$\overline{\lim_{\beta\to\infty}} f(\beta)\beta^{(2-b-c)\wedge 1-1}=\infty. \tag{5.29}$$

Then

$$\overline{\lim_{|\beta|\to\infty}} f(|\beta|)\|[i\beta+A_0-B(i\beta+A_1)^{-1}C]^{-1}\|$$
$$=\overline{\lim_{|\beta|\to\infty}} f(|\beta|)\|(i\beta+A_0-BA_1^{-1}C)^{-1}\|. \tag{5.30}$$

Proof. We will prove only (ii). The proof of (i) is similar. By Lemma 5.2, we have (note (2.14) and $b+c\le 2$)

$$\begin{aligned}&\|BA_1^{-1}(i\beta+A_1)^{-1}C\|\\&\le \|BA_1^{-b}\|\|A_1^{b+c-1}(i\beta+A_1)^{-1}\|\,\|[C^*(A_1^*)^{-c}]^*\|\le \frac{K}{|\beta|^{(2-b-c)\wedge 1}}.\end{aligned}\tag{5.31}$$

Now, suppose the right-hand side of (5.30) is finite. Then we have

$$\begin{aligned}&\overline{\lim_{|\beta|\to\infty}} f(|\beta|)\|[i\beta+A_0-B(i\beta+A_1)^{-1}C]^{-1}\|\\&=\overline{\lim_{|\beta|\to\infty}} f(|\beta|)\|[i\beta+A_0-BA_1^{-1}C+i\beta BA_1^{-1}(i\beta+A_1)^{-1}C]^{-1}\|\\&\le \overline{\lim_{|\beta|\to\infty}} \frac{f(|\beta|)\|[i\beta+A_0-BA_1^{-1}C]^{-1}\|}{1-f(|\beta|)\|[i\beta+A_0-BA_1^{-1}C]^{-1}\|\frac{K}{f(|\beta|)|\beta|^{(2-b-c)\wedge 1-1}}}\\&\le \overline{\lim_{|\beta|\to\infty}} f(|\beta|)\|(i\beta+A_0-BA_1^{-1}C)^{-1}\|.\end{aligned}\tag{5.32}$$

We may prove the other direction of the inequality similarly. Thus, (5.30) follows.

Proof of Theorem 2.2. Combining Lemmas 5.3 and 5.4, we see that for a nondecreasing continuous function f satisfying (5.29) with $b+c<2$, we have that

$$\overline{\lim_{|\beta|\to\infty}} f(|\beta|)\|(i\beta-\mathcal{A})^{-1}\|<\infty \quad (\text{respectively } =0),\tag{5.33}$$

if and only if

$$\overline{\lim_{|\beta|\to\infty}} f(|\beta|)\|(i\beta+A_0-BA_1^{-1}C)^{-1}\|<\infty \quad (\text{respectively } =0).\tag{5.34}$$

In the case $b+c\le 1$, (5.33) holds if and only if

$$\overline{\lim_{|\beta|\to\infty}} f(|\beta|)\|(i\beta+A_0)^{-1}\|<\infty \quad (\text{respectively } =0).\tag{5.35}$$

Now, for (i)–(iv), we take $f(\beta)\equiv 1$, $f(\beta)=\log(1+\beta)$, $f(\beta)=\beta^\mu$ and $f(\beta)=\beta$, respectively, and use Corollary 4.3 and Proposition 4.5 to obtain the conclusions. $\square$

The proof of Theorem 2.3 is of similar nature and we omit the proof here.

6. Contradiction argument for resolvent estimate. In this section, we will prove Theorems 2.4 and 2.5.

Proof of Theorem 2.4. Let $\mu \in (0,1]$. Suppose

$$\overline{\lim_{|\beta|\to\infty}} |\beta|^\mu \|(i\beta - \mathcal{A})^{-1}\| = \infty. \tag{6.1}$$

Then we may assume that for some $\beta_n \to \infty$ and $z_n \equiv (x_n, y_n) \in \mathcal{D}(\mathcal{A}) \equiv [\mathcal{D}(A_0) \cap \mathcal{D}(C)] \times [\mathcal{D}(A_1) \cap \mathcal{D}(B)]$ with $\|z_n\| = 1$, one has

$$\beta_n^{-\mu} \|(i\beta - \mathcal{A}) z_n\| \to 0; \tag{6.2}$$

i.e.,

$$\begin{cases} i\beta_n^{1-\mu} x_n + \beta_n^{-\mu} A_0 x_n - \beta_n^{-\mu} B y_n \stackrel{\Delta}{=} f_n \to 0, & \text{in } H_0, \\ i\beta_n^{1-\mu} y_n + \beta_n^{-\mu} A_1 y_n - \beta_n^{-\mu} C x_n \stackrel{\Delta}{=} g_n \to 0, & \text{in } H_1. \end{cases} \tag{6.3}$$

Taking the inner product of the first equation in (6.3) with x_n and the second with y_n, we obtain

$$\begin{cases} i\beta_n^{1-\mu} \|x_n\|^2 + \beta_n^{-\mu} \langle A_0 x_n, x_n \rangle - \beta_n^{-\mu} \langle B y_n, x_n \rangle \to 0, \\ i\beta_n^{1-\mu} \|y_n\|^2 + \beta_n^{-\mu} \langle A_1 y_n, y_n \rangle - \beta_n^{-\mu} \langle C x_n, y_n \rangle \to 0. \end{cases} \tag{6.4}$$

Taking the real parts in the above and adding together, using (2.26), we obtain

$$\beta_n^{-\mu} \operatorname{Re} \langle A_1 y_n, y_n \rangle \to 0, \quad \beta_n^{-\mu} \operatorname{Re} \langle A_0 x_n, x_n \rangle \to 0. \tag{6.5}$$

Consequently, by (2.27), (2.28) and (2.31),

$$\begin{cases} \beta_n^{-\mu} \big| \operatorname{Im} \langle A_1 y_n, y_n \rangle \big| \le K \beta_n^{-\mu} \Big(\big| \operatorname{Re} \langle A_1 y_n, y_n \rangle \big| + 1 \Big) \to 0, \\ \beta_n^{-1} \big| \operatorname{Im} \langle A_0 x_n, x_n \rangle \big| \le K \beta_n^{-1} \big(| \operatorname{Re} \langle A_0 x_n, x_n \rangle |^{\frac{1}{\mu}} + 1 \big) \\ \qquad\qquad \le K \big\{ \big(\beta_n^{-\mu} \operatorname{Re} \langle A_0 x_n, x_n \rangle \big)^{\frac{1}{\mu}} + \beta_n^{-1} \big\} \to 0, \\ \|\beta_n^{-\frac{\mu}{2}} Q^{\frac{1}{2}} y_n\| \le K \beta_n^{-\mu} \operatorname{Re} \langle A_1 y_n, y_n \rangle \to 0. \end{cases} \tag{6.6}$$

Then (6.4)–(6.6) result in

$$i\|x_n\|^2 - \beta_n^{-1} \langle B y_n, x_n \rangle \to 0, \qquad i\|y_n\|^2 - \beta_n^{-1} \langle C x_n, y_n \rangle \to 0. \tag{6.7}$$

On the other hand, from (6.3), we see that

$$\begin{cases} ix_n + \beta_n^{-1}A_0x_n - \beta_n^{-1}By_n = \beta_n^{\mu-1}f_n \to 0, \\ iy_n + \beta_n^{-1}A_1y_n - \beta_n^{-1}Cx_n = \beta_n^{\mu-1}g_n \to 0. \end{cases} \tag{6.8}$$

We take the inner product of the second equation in (6.3) with

$$\beta_n^{-2+2\mu}Q^{-1}Cx_n$$

to get

$$\begin{aligned} \big\|\beta_n^{-(1-\frac{\mu}{2})}Q^{-\frac{1}{2}}Cx_n\big\|^2 &= \langle -\beta_n^{-1+\mu}g_n + iy_n, \beta_n^{-1+\mu}Q^{-1}Cx_n \rangle \\ &\quad + \langle \beta_n^{-(1-\frac{\mu}{2})}Q^{-\frac{1}{2}}A_1y_n, \beta_n^{-(1-\frac{\mu}{2})}Q^{-\frac{1}{2}}Cx_n \rangle . \end{aligned} \tag{6.9}$$

We note that (set $\alpha \stackrel{\Delta}{=} \frac{2(1-\mu)}{2-\mu} \in [0,1)$ since $\mu \in (0,1]$)

$$\begin{aligned} &\big\|\beta_n^{-1+\mu}Q^{-1}Cx_n\big\| = \beta_n^{-1+\mu}\big\|(Q^{\frac{1}{2(1-\alpha)}})^{\alpha}(Q^{-\frac{1}{\mu}}Cx_n)\big\| \\ &\quad \le K\beta_n^{-1+\mu}\big\|Q^{-\frac{1}{2}}Cx_n\big\|^{\alpha}\big\|Q^{-\frac{1}{\mu}}Cx_n\big\|^{1-\alpha} = K\Big(\big\|\beta_n^{-(1-\frac{\mu}{2})}Q^{-\frac{1}{2}}Cx_n\big\| + 1\Big), \end{aligned}$$

and (note $-(1-\frac{\mu}{2}) \le -\frac{\mu}{2}$, (2.30) and [8, Lemma 2.1])

$$\big\|\beta_n^{-(1-\frac{\mu}{2})}Q^{-\frac{1}{2}}A_1y_n\big\| \le \big\|Q^{-\frac{1}{2}}A_1Q^{-\frac{1}{2}}\big\|\big\|\beta_n^{\frac{-\mu}{2}}Q^{\frac{1}{2}}y_n\big\| \to 0.$$

Thus, (6.9) can be estimated as follows:

$$\big\|\beta_n^{-(1-\frac{\mu}{2})}Q^{-\frac{1}{2}}Cx_n\big\|^2 \le K\Big(\big\|\beta_n^{-(1-\frac{\mu}{2})}Q^{-\frac{1}{2}}Cx_n\big\| + 1\Big),$$

which further leads to

$$\big\|\beta_n^{-(1-\frac{\mu}{2})}Q^{-\frac{1}{2}}Cx_n\big\| \le K, \qquad \forall n \ge 1. \tag{6.10}$$

Hence,

$$\big|\beta_n^{-1}\langle Cx_n, y_n \rangle\big| \le \|\beta_n^{-(1-\frac{\mu}{2})}Q^{-\frac{1}{2}}Cx_n\|\,\|\beta_n^{-\frac{\mu}{2}}Q^{\frac{1}{2}}y_n\| \to 0. \tag{6.11}$$

Using (2.29), we obtain

$$\big|\beta_n^{-1}\langle By_n, x_n \rangle\big| \le \big|\beta_n^{-1}\langle Cx_n, y_n \rangle\big| + \beta_n^{-1}\|x_n\|\,\|y_n\| \to 0. \tag{6.12}$$

Now, combining (6.7), (6.11)–(6.12), we have $z_n \equiv (x_n, y_n) \to 0$, a contradiction. Hence, (2.33) holds.

Next, we assume (2.34) instead of (2.30). We still have everything up to (6.8). Using the same arguments as the proof of Proposition 3.3, we see from (6.8) that

$$\|\beta_n^{-1} A_0 x_n\|, \|\beta_n^{-1} A_1 y_n\|, \|\beta_n^{-1} B y_n\|, \|\beta_n^{-1} C x_n\| \le K, \quad \forall n \ge 1. \tag{6.13}$$

We note that (set $\alpha \stackrel{\Delta}{=} 1 - \mu/2 \in [1/2, 1)$)

$$\begin{aligned} &\big\|\beta_n^{-(1-\frac{\mu}{2})} Q^{-\frac{1}{2}} C x_n\big\| = \beta_n^{-\alpha} \big\|(Q^{\frac{1}{\mu}})^{\alpha} (Q^{-\frac{1}{\mu}} C x_n)\big\| \qquad (6.14)\\ &\le K \beta_n^{-\alpha} \big\|Q^{-\frac{1}{\mu}} C x_n\big\|^{1-\alpha} \big\|C x_n\big\|^{\alpha} \le K \big\|(C^* Q^{-\frac{1}{\mu}})^* x_n\big\|^{1-\alpha} \big\|\beta_n^{-1} C x_n\big\|^{\alpha} \le K. \end{aligned}$$

Hence, we have (6.11)–(6.12) and end up with a contradiction again.

Finally, we let $\mu = 1$ and $e^{-[A_0 - BA_1^{-1}C]t}$ be analytic. We do not assume (2.27). In this case, we still have (6.5), the first and the third convergence in (6.6), and the second convergence in (6.7). Then, in either case of (2.30) or (2.34), we have (6.11), which together with the second convergence in (6.7), gives $\|y_n\| \to 0$. Since $\mu = 1$, by (2.28), (2.31)–(2.32), we see that $BA_1^{-1} \in \mathcal{L}(H_1, H_0)$. Now, we apply BA_1^{-1} to the second equation in (6.8) and then add to the first equation in (6.8) to get

$$[i + \beta_n^{-1}(A_0 - BA_1^{-1}C)]x_n = \beta_n^{\mu-1} f_n + BA_1^{-1}(\beta_n^{\mu-1} g_n - i y_n) \stackrel{\Delta}{=} h_n \to 0. \tag{6.15}$$

Since $e^{-[A_0 - BA_1^{-1}C]t}$ is analytic, $[i - \beta_n^{-1}(A_0 - BA_1^{-1}C)]^{-1}$ is bounded. Thus,

$$\|x_n\| = \|[i - \beta_n^{-1}(A_0 - BA_1^{-1}C)]^{-1} h_n\| \le K \|h_n\| \to 0, \tag{6.16}$$

which leads to a contradiction again.

Proof of Theorem 2.5. It is enough to show that (2.33) holds with $\mu = 0$. Suppose it is not true. Then there exists a sequence $\beta_n \to \infty$, and a sequence $z_n = (x_n, y_n) \in \mathcal{D}(\mathcal{A})$ with $\|z_n\| = 1$, such that $\big(i\beta_n - \mathcal{A}\big) z_n \to 0$, in $\mathcal{H}$; i.e.,

$$\begin{cases} i\beta_n x_n + A_0 x_n - B y_n \to 0, & \text{in } H_0, \\ i\beta_n y_n - C x_n + A_1 y_n \to 0, & \text{in } H_1. \end{cases} \tag{6.17}$$

By (2.26) and the strong dissipativeness of $-A_1$, we have

$$\|y_n\|^2 \le K \operatorname{Re} \langle A_1 y_n, y_n \rangle \to 0. \tag{6.18}$$

(i) When (2.35) holds, (6.18) implies that $\|By_n\| \to 0$. Since $e^{-A_0 t}$ is exponentially stable or $A_0 \in \mathcal{L}(H_0)$, the first convergence in (6.17) implies the convergence of $\|x_n\|$ to zero, a contradiction.

(ii) From (2.28) and (6.18), we have $\langle A_1 y_n, y_n \rangle \to 0$. Then the second convergence in (6.17) leads to $\frac{1}{\beta_n} \langle Cx_n, y_n \rangle \to 0$. Applying this to the inner product of the first equation in (6.17) with x_n, we obtain (note (2.29))

$$i\|x_n\|^2 + \frac{1}{\beta_n} \langle A_0 x_n, x_n \rangle \to 0. \tag{6.19}$$

Since $A_0 \in \mathcal{L}(H_0)$, we again reach a contradiction: $\|x_n\|, \|y_n\| \to 0$.

7. Multi-block cases. In this section, we briefly consider the multi-block operator cases. Let $H_0, H_1, \dots, H_m$ be Hilbert spaces and let

$$\begin{cases} A_k : \mathcal{D}(A_k) \subseteq H_k \to H_k, & 0 \le k \le m, \\ B_k : \mathcal{D}(B_k) \subseteq H_k \to H_{k-1}, & 1 \le k \le m, \end{cases}$$

all be densely defined and closed operators. Consider the following equation:

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt} \begin{pmatrix} x^0 \\ x^1 \\ \vdots \\ x^{m-1} \\ x^m \end{pmatrix} \\ &= \begin{pmatrix} -A_0 & B_1 & \cdots & 0 & 0 \\ -B_1^* & -A_1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -A_{m-1} & B_m \\ 0 & 0 & \cdots & -B_m^* & -A_m \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ \vdots \\ x^{m-1} \\ x^m \end{pmatrix} = \mathcal{A}x \end{aligned} \tag{7.1}$$

on the Hilbert space $\mathcal{H} = H_0 \times H_1 \times \cdots \times H_m$. We may as well consider the case that all $-B_k^*$ are replaced by some $C_k : \mathcal{D}(C_k) \subseteq H_{k-1} \to H_k$. We prefer not to do that for the sake of simplicity.

Note that we may regard (7.1) as a special case of (1.1) by calling, say, the first k components of the state x and the last $m+1-k$ components y and redefining the operators A_0, A_1, B and C properly. Then the results of previous sections apply. In this section, however, we would like to briefly look at the multi-block operator defined in (7.1) directly so that we can take some advantage from its special structure.

Let us introduce the following:

$$\widetilde{A}_m = A_m, \quad \widetilde{A}_k = A_k + B_{k+1}\widetilde{A}_{k+1}^{-1}B_{k+1}^*, \qquad k = m-1, \dots, 1, \tag{7.2}$$

assuming all $\widetilde{A}_k^{-1}$ exist (for $k = 1, 2, \dots, m$). We have the following result.

Theorem 7.1. *Let $\mathcal{A}$ be defined in* (7.1) *which generates a contraction semigroup $e^{\mathcal{A}t}$ on the Hilbert space $\mathcal{H}$. Assume that A_k ($k = 0, \dots, m$) are self-adjoint, and $\widetilde{A}_k$ ($k = 1, \dots, m$) are all defined and invertible. Then $e^{\mathcal{A}t}$ is analytic if*

$$\mathcal{D}(\widetilde{A}_k) \subseteq \mathcal{D}(B_k), \qquad k = 1, \dots, m. \tag{7.3}$$

It seems possible to directly discuss some other properties of $e^{\mathcal{A}t}$, like differentiability, being of Gevrey class, etc. However, the situation seems not very clear at this moment. Thus, we would like to leave the detailed study of this aspect to our future publications.

Proof. Similar to the proof of Theorem 2.4, we will show that under our assumptions,

$$\lim_{|\beta|\to\infty} |\beta| \big\| (i\beta - \mathcal{A})^{-1} \big\| < \infty.$$

Suppose the above is false. Then we may assume that there exist a sequence $x_n = (x_n^1, \dots, x_n^m) \in \mathcal{D}(\mathcal{A})$ with $\|x_n\| = 1$, and a sequence $\beta_n > 0$ with $\beta_n \to \infty$ such that $\beta_n^{-1}\big\|(i\beta_n - \mathcal{A})x_n\big\| \to 0$; i.e.,

$$ix_n^k + \beta_n^{-1}B_k^*x_n^{k-1} + \beta_n^{-1}A_kx_n^k - \beta_n^{-1}B_{k+1}x_n^{k+1} \stackrel{\Delta}{=} f_n^k \to 0, \quad \text{in } H_k, \tag{7.4}$$

$k = 0, 1, \dots, m$, where $B_0^* \stackrel{\Delta}{=} 0$ and $B_{m+1} \stackrel{\Delta}{=} 0$. Taking the inner product of (7.4) with x_n^k, we have

$$\begin{aligned} i\|x_n^k\|^2 + \beta_n^{-1}\langle B_k^*x_n^{k-1}, x_n^k\rangle + \beta_n^{-1}\|A_k^{\frac{1}{2}}x_n^k\|^2 - \beta_n^{-1}\langle B_{k+1}x_n^{k+1}, x_n^k\rangle \\ = \langle f_n^k, x_n^k\rangle \to 0, \qquad k = 0, 1, \dots, m. \end{aligned} \tag{7.5}$$

Summing (7.5) for index k from 0 to m, then taking the real part yields

$$\|\beta_n^{-\frac{1}{2}}A_k^{\frac{1}{2}}x_n^k\|^2 \to 0, \qquad k = 0, 1, \dots, m. \tag{7.6}$$

Equation (7.5) together with (7.6) implies

$$i\|x_n^k\|^2 + \frac{1}{\beta_n}\langle B_k^* x_n^{k-1}, x_n^k \rangle - \frac{1}{\beta_n}\langle B_{k+1} x_n^{k+1}, x_n^k \rangle \to 0, \quad k = 0, 1, \dots, m. \tag{7.7}$$

We now show that

$$\|x_n^m\| \to 0, \qquad \text{as } n \to \infty. \tag{7.8}$$

To this end, we take the inner product of the m-th equation in (7.4) with $A_m^{-1} B_m^* x_n^{m-1}$ to get

$$\begin{aligned} &\big\|\beta_n^{-\frac{1}{2}} A_m^{-\frac{1}{2}} B_m^* x_n^{m-1}\big\|^2 \\ &= \langle f_n^m - i x_n^m, A_m^{-1} B_m^* x_n^{m-1} \rangle - \langle \beta_n^{-\frac{1}{2}} A_m^{\frac{1}{2}} x_n^m, \beta_n^{-\frac{1}{2}} A_m^{-\frac{1}{2}} B_m^* x_n^{m-1} \rangle \\ &\le (\|f_n^m\| + \|x_n^m\|)\|A_m^{-1} B_m^* x_n^{m-1}\| + \big\|\beta_n^{-\frac{1}{2}} A_m^{\frac{1}{2}} x_n^m\big\|\big\|\beta_n^{-\frac{1}{2}} A_m^{-\frac{1}{2}} B_m^* x_n^{m-1}\big\|. \end{aligned} \tag{7.9}$$

Note that $\|\widetilde{A}_m^{-1} B_m^* x_n^{m-1}\|$ is bounded due to condition (7.3). Thus,

$$\big\|\beta_n^{-\frac{1}{2}} A_m^{-\frac{1}{2}} B_m^* x_n^{m-1}\big\| \le K, \qquad \forall n \ge 1. \tag{7.10}$$

It then follows from (7.6) and (7.10) that

$$\begin{aligned} \frac{1}{\beta_n}|\langle B_m^* x_n^{m-1}, x_n^m \rangle| &= |\langle \beta_n^{-\frac{1}{2}} A_m^{-\frac{1}{2}} B_m^* x_n^{m-1}, \beta_n^{-\frac{1}{2}} A_m^{\frac{1}{2}} x_n^m \rangle| \\ &\le \big\|\beta_n^{-\frac{1}{2}} A_m^{-\frac{1}{2}} B_m^* x_n^{m-1}\big\|\big\|\beta_n^{-\frac{1}{2}} A_m^{\frac{1}{2}} x_n^m\big\| \to 0. \end{aligned} \tag{7.11}$$

Therefore, by the m-th relation in (7.7) and (7.11), we obtain (7.8).

Next, we prove

$$\|x_n^k\| \to 0, \qquad k = m-1, \dots, 0. \tag{7.12}$$

We use equation (7.4) for $k = m-1, m$; i.e.,

$$i x_n^{m-1} + \beta_n^{-1} B_{m-1}^* x_n^{m-2} + \beta_n^{-1} A_{m-1} x_n^{m-1} - \beta_n^{-1} B_m x_n^m \to 0, \qquad \text{in } H_{m-1}, \tag{7.13}$$

$$i x_n^m + \beta_n^{-1} B_m^* x_n^{m-1} + \beta_n^{-1} A_m x_n^m \to 0, \qquad \text{in } H_m. \tag{7.14}$$

By the condition (7.3), we can apply the bounded operator $B_m A_m^{-1}$ to (7.14), then add the result to (7.13) to get

$$i x_n^{m-1} + \beta_n^{-1} B_{m-1}^* x_n^{m-2} + \beta_n^{-1} \widetilde{A}_{m-1} x_n^{m-1} \to 0, \quad \text{in } H_{m-1}. \tag{7.15}$$

Combining (7.15) with the first $m-2$ equations in (7.4), we get back to the situation we begin with except that the system is now on $H_0 \times H_1 \times \cdots \times H_{m-1}$. Therefore, we can repeat the argument after (7.4) to prove (7.12), which contradicts $\|x_n\| = 1$. □

The self-adjointness of the operators $A_k, k = 0, \ldots, m$ is unnecessary. It can be replaced by conditions similar to those in Theorem 2.4. We prefer not to get into the details of this case.

Russell ([21]) initiated the study of system (7.1). He assumed that the A_k are nonnegative and self-adjoint, the B_k are normal, A_k, B_k, B_k^* are mutually commutative, and $H_k = H$ for all k. It was also assumed that

$$A_k = \int_{\mathbb{C}} a_k(\mu) E(d\mu), \qquad B_k = \int_{\mathbb{C}} b_k(\mu) E(d\mu),$$

where $a_k(\mu), b_k(\mu)$ satisfy certain comparison conditions, and $E(\cdot)$ is a resolution of identity. Under these conditions, the analyticity of $e^{\mathcal{A}t}$ was obtained. We know that a system with variable coefficients does not belong to such a "commuting case," in general. The result of Theorem 7.1 is much more general than the relevant results in [21].

To conclude this section, we consider a very special case of the exponential stability for the semigroup $e^{\mathcal{A}t}$, which will be useful in applications (see 8 for examples).

Theorem 7.2. *Let $\mathcal{A}$ be the operator defined in* (7.1) *with $m = 2$, $A_0 = A_1 = 0$, and $e^{\mathcal{A}t}$ be a contraction semigroup on the Hilbert space $\mathcal{H}$. Let $-A_2$ be strongly dissipative and there exist a bounded operator $F \in \mathcal{L}(H_0, H_2)$ such that*

$$|\langle B_1 x^1, x^0 \rangle| \le K |\langle B_2^* x^1, F x^0 \rangle|, \quad \forall x^1 \in \mathcal{D}(B_1) \cap \mathcal{D}(B_2^*). \tag{7.16}$$

If either

$$\begin{cases} |\langle A_2 x^2, F x^0 \rangle| \le K (\mathrm{Re}\,\langle A_2 x^2, x^2 \rangle)^{\frac{1}{2}} \|x^0\|, \\ |\langle B_2 x^2, x^1 \rangle| \le K (\mathrm{Re}\,\langle A_2 x^2, x^2 \rangle)^{\frac{1}{2}} \|B_1 x^1\|, \end{cases} \tag{7.17}$$

$\forall (x^0, x^1, x^2) \in \mathcal{D}(\mathcal{A})$, *or*

$$\begin{cases} |\langle A_2 x^2, F x^0 \rangle| \le K (\mathrm{Re}\,\langle A_2 x^2, x^2 \rangle)^{\frac{1}{2}} \|B_1^* x^0\|, \\ \|B_2 x^2\|^2 \le K\, \mathrm{Re}\,\langle A_2 x^2, x^2 \rangle, \end{cases} \tag{7.18}$$

$\forall (x^0, x^1, x^2) \in \mathcal{D}(\mathcal{A})$, *then* $e^{\mathcal{A}t}$ *is exponentially stable.*

Proof. Suppose the conclusion is not true. Then there exist a sequence $x_n = (x_n^0, x_n^1, x_n^2) \in \mathcal{D}(\mathcal{A})$ with $\|x_n\| = 1$, and a sequence $\beta_n > 0$ with $\beta_n \to \infty$, such that

$$\big\|(i\beta_n - \mathcal{A})x_n\big\| \to 0; \tag{7.19}$$

i.e.,

$$\begin{cases} i\beta_n x_n^0 - B_1 x_n^1 \to 0, & \text{in } H_0, \\ i\beta_n x_n^1 + B_1^* x_n^0 - B_2 x_n^2 \to 0, & \text{in } H_1, \\ i\beta_n x_n^2 + B_2^* x_n^1 + A_2 x_n^2 \to 0, & \text{in } H_2. \end{cases} \tag{7.20}$$

Since

$$\operatorname{Re}\langle A_2 x_n^2, x_n^2 \rangle = \operatorname{Re}\langle (i\beta_n - \mathcal{A})x_n, x_n \rangle \le \|(i\beta_n - \mathcal{A})x_n\| \to 0, \tag{7.21}$$

and $-A_2$ is strongly dissipative, we obtain

$$\|x_n^2\| \to 0. \tag{7.22}$$

From the first two equations in (7.20) we obtain

$$\begin{cases} i\|x_n^0\|^2 - \frac{1}{\beta_n}\langle B_1 x_n^1, x_n^0 \rangle \to 0, \\ i\|x_n^1\|^2 + \frac{1}{\beta_n}\langle B_1^* x_n^0, x_n^1 \rangle - \frac{1}{\beta_n}\langle B_2 x_n^2, x_n^1 \rangle \to 0. \end{cases} \tag{7.23}$$

When condition (7.17) or (7.18) is satisfied, it is easy to see that

$$\frac{1}{\beta_n}\langle B_2 x_n^2, x_n^1 \rangle$$

converges to zero, where we have used (7.21) and the boundedness of

$$\|\frac{1}{\beta_n} B_1 x_n^1\|$$

which follows from the first equation in (7.20). Thus we obtain from (7.23) that

$$\|x_n^0\|^2 - \|x_n^1\|^2 \to 0. \tag{7.24}$$

On the other hand, $\|x_n^0\|^2 + \|x_n^1\|^2 \to 1$ because of (7.22). Therefore,

$$\|x_n^0\|^2 \to \tfrac{1}{2}, \qquad \|x_n^1\|^2 \to \tfrac{1}{2}. \tag{7.25}$$

In what follows, we will show that $\|x_n^0\| \to 0$ to obtain a contradiction. Since $\|x_n^2\|$ converges to zero, the third equation in (7.20) implies

$$\frac{1}{\beta_n} B_2^* x_n^1 + \frac{1}{\beta_n} A_2 x_n^2 \to 0, \qquad \text{in } H_2. \tag{7.26}$$

We take the inner product of (7.26) with Fx_n^0 in H_2 to get

$$\frac{1}{\beta_n} \langle B_2^* x_n^1, Fx_n^0 \rangle + \frac{1}{\beta_n} \langle A_2 x_n^2, Fx_n^0 \rangle \to 0. \tag{7.27}$$

When the condition (7.17) holds, the second term in (7.27) converges to zero. Then the first term also converges to zero. By condition (7.17),

$$\frac{1}{\beta_n} \langle B_1 x_n^1, x_n^0 \rangle \to 0. \tag{7.28}$$

It follows from (7.28) and the first equation in (7.23) that $\|x_n^0\| \to 0$.

When the condition (7.18) holds, we see from the second equation in (7.20) that $\|\frac{1}{\beta_n} B_1^* x_n^0\|$ is bounded since $\|B_2 x_n^2\| \to 0$. Thus the second term in (7.27) again converges to zero which leads to the same contradiction as before.

8. Applications. In this section, we apply our results to various types of coupled PDEs.

Example 8.1. A thermoelastic plate (see [10, 14, 17]): Consider

$$\begin{aligned}
&w_{tt}(x,t) + \Delta^2 w(x,t) + \alpha \Delta\theta(x,t) = 0, \qquad (8.1)\\
&\theta_t(x,t) - \alpha\Delta w_t(x,t) - \beta\Delta\theta(x,t) + \gamma\theta(x,t) = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+,\\
&w|_{\partial\Omega} = \frac{\partial w}{\partial n}|_{\partial\Omega} = \theta|_{\partial\Omega} = 0, \quad w(x,0) = w_0(x),\ \theta(x,0) = \theta_0(x),
\end{aligned}$$

where $\Omega \subset \mathbb{R}^2$ is bounded with a smooth boundary, α, β, γ are positive constants.

Let the underlying Hilbert space be

$$\begin{cases} \mathcal{H} = H_0 \times H_1, \quad H_0 = H_0^2(\Omega) \times L^2(\Omega), \quad H_1 = L^2(\Omega), \\ \|(w,v)\|_{H_0}^2 = \|\Delta w\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2, \quad \forall (w,v) \in H_0, \\ \|\theta\|_{H_1} = \|\theta\|_{L^2(\Omega)}, \quad \forall \theta \in H_1. \end{cases}$$

The operator, associated with the evolution equation (8.1), is given by

$$\begin{cases} \mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ -\Delta^2 & 0 & -\alpha\Delta \\ 0 & \alpha\Delta & \beta\Delta - \gamma I \end{pmatrix}, \\ \mathcal{D}(\mathcal{A}) = \big\{(w, v, \theta) \in \mathcal{H} : w \in H^4(\Omega),\ v \in H_0^2(\Omega),\ \theta \in H^2(\Omega) \cap H_0^1(\Omega)\big\}, \end{cases}$$

and it generates a contraction semigroup on $\mathcal{H}$. We also define

$$\begin{cases} A_0 = \begin{pmatrix} 0 & -I \\ \Delta^2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ -\alpha\Delta \end{pmatrix}, \qquad A_1 = -\beta\Delta, \\ \mathcal{D}(A_0) = \Big\{ \begin{pmatrix} w \\ v \end{pmatrix} \in H_0 : w \in H^4(\Omega),\ v \in H_0^2(\Omega) \Big\}, \\ \mathcal{D}(B) = \mathcal{D}(A_1) = \{\theta \in H_1 : \theta \in H^2(\Omega) \cap H_0^1(\Omega)\}. \end{cases}$$

It is easy to see from Theorem 2.4 (with $\mu = 1$) that the operator

$$\begin{cases} \mathcal{A}_1 = -A_0 - BA_1^{-1}B^* = \begin{pmatrix} 0 & I \\ -\Delta^2 & \frac{\alpha^2}{\beta}\Delta \end{pmatrix}, \\ \mathcal{D}(\mathcal{A}_1) = \{(w, v) \in H_0 \mid w \in H^4(\Omega),\ v \in H_0^2(\Omega)\}, \end{cases}$$

generates an analytic semigroup on H_0. By Theorem 2.4 (with $\mu = 1$), $e^{\mathcal{A}t}$ is analytic.

When $\gamma > 0$, by the perturbation theory ([18, Theorem 2.1]), $e^{\mathcal{A}t}$ is also analytic.

Example 8.2. Euler-Bernoulli beam with shear diffusion (see [15, 22]): The problem is

$$\begin{aligned} &\rho w_{tt}(x,t) + EI[w_{xxxx}(x,t) + \beta_{xxx}(x,t)] = 0, \qquad (8.2)\\ &2\sigma\beta_t(x,t) + \tau\beta(x,t) - EI[w_{xxx}(x,t) + \beta_{xx}(x,t)] = 0,\ \ (x,t) \in (0,L) \times \mathbb{R}^+, \\ &w(x,t) = w_{xx}(x,t) + \beta_x(x,t) = 0, \quad x = 0, L, \\ &w(x,0) = w_0(x),\ w_t(x,0) = w_1(x),\ \beta(x,0) = \beta_0(x), \end{aligned}$$

where all coefficients are positive constants. Let $y = w_{xx} + \beta_x$, $v = w_t$, and take the underlying Hilbert space to be

$$\begin{cases} \mathcal{H} = H_0 \times H_1 = [L^2(0,L) \times L^2(0,L)] \times L^2(0,L), \\ \|(\beta, v, y)\|_{\mathcal{H}}^2 = \tau\|\beta\|^2 + \rho\|v\|^2 + EI\|y\|^2, \quad \forall (\beta, v, y) \in \mathcal{H}. \end{cases}$$

Hereafter, let $\|\cdot\|$ denote the L^2 norm, $D^i = \frac{\partial^i}{\partial x^i}$. The associated infinitesimal generator of the contraction semigroup on $\mathcal{H}$ for the system (8.2) is given by

$$\begin{cases} \mathcal{A} = \begin{pmatrix} -\frac{\tau}{2\sigma} I & 0 & \frac{EI}{2\sigma} D \\ 0 & 0 & -\frac{EI}{\rho} D^2 \\ -\frac{\tau}{2\sigma} D & D^2 & \frac{EI}{2\sigma} D^2 \end{pmatrix}, \\ \mathcal{D}(\mathcal{A}) = \big\{(\beta, v, y) \in \mathcal{H} : \beta \in H^1(0,L),\ v, y \in H^2(0,L) \cap H_0^1(0,L)\big\}. \end{cases}$$

Define

$$\begin{cases} A_0 = \begin{pmatrix} \frac{\tau}{2\sigma} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{EI}{2\sigma} D \\ -\frac{EI}{\rho} D^2 \end{pmatrix}, \quad A_1 = -\frac{EI}{2\sigma} D^2, \\ \mathcal{D}(A_0) = \big\{(\beta, v) \in H_0 : \beta \in H^1(0,L)\big\}, \\ \mathcal{D}(B) = \mathcal{D}(A_1) = \{y \in H_1 : y \in H^2(0,L) \cap H_0^1(0,L)\}. \end{cases}$$

Then

$$\begin{cases} B^* = \begin{pmatrix} \frac{\tau}{2\sigma} D \\ D^2 \end{pmatrix}, \\ \mathcal{D}(B^*) = \{(\beta, v) \in H_0 : \beta \in H^1(0,L),\ v \in H^2(0,L) \cap H_0^1(0,L)\}. \end{cases}$$

We see that A_1 is self-adjoint and positive definite. A straightforward computation gives

$$\begin{cases} \mathcal{A}_1 = -A_0 - BA_1^{-1}B^* = \begin{pmatrix} 0 & -D \\ -\frac{\tau}{\rho} D & \frac{2\sigma}{\rho} D^2 \end{pmatrix}, \\ \mathcal{D}(\mathcal{A}_1) = \{(\beta, v) \in H_0 : \beta \in H^1(0,L),\ v \in H^2(0,L) \cap H_0^1(0,L)\}. \end{cases}$$

Applying Theorem 2.4 (with $\mu = 1$) to $\mathcal{A}_1$, we see that $e^{\mathcal{A}_1 t}$ is analytic on H_0. Then applying Theorem 2.4 to $\mathcal{A}$, we obtain the analyticity of $e^{\mathcal{A}t}$.

When the thermal dynamics are taken into account for the system (8.2), we have the following equations (see [15, 22]):

$$\begin{cases} \rho w_{tt}(x,t) + EI[w_{xxxx}(x,t) + \beta_{xxx}(x,t)] + \kappa_1 T_{xx}(x,t) = 0, \\ 2\sigma\beta_t(x,t) + \tau\beta(x,t) - EI[w_{xxx}(x,t) + \beta_{xx}(x,t)] = 0, \\ T_t(x,t) + \kappa_2 T(x,t) - \kappa_1 w_{xxt}(x,t) = 0, \quad (x,t) \in (0,L) \times \mathbb{R}^+, \\ w(x,t) = w_{xx}(x,t) + \beta_x(x,t) = T(x,t) = 0, \quad x = 0, L. \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x),\ \beta(x,0) = \beta_0(x),\ T(x,0) = T_0(x), \end{cases}$$

where all coefficients are positive constants. By a similar analysis, we can also verify that the associated semigroup is analytic.

Example 8.3. A viscoelastic rod ([16]): Consider

$$\begin{aligned} &u_{tt}(x,t) = g(0)u_{xx}(x,t) + \int_0^\infty g'(s)u_{xx}(x,t-s)\,ds, \quad (x,t) \in [0,L]\times\mathbb{R}^+,\\ &u(0,t) = u(1,t) = 0, \qquad\qquad (8.3)\\ &u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ u(x,0) - u(x,-s) = w_0(x,s),\ s \geq 0, \end{aligned}$$

where the memory function $g(s)$ satisfies the following conditions:

(g1) $g \in C^2(0,\infty) \cap C[0,\infty), g' \in L^1(0,\infty)$;
(g2) $g(s) > 0, g'(s) < 0, g''(s) > 0$ for $s > 0$;
(g3) $g(\infty) > 0$;
(g4) $g''(s) + kg'(s) \geq 0$ for some $k > 0$ and all $s > 0$.

Let

$$\begin{cases} \mathcal{H} = H_0 \times H_1 \times H_2,\\ H_0 = \{f \in L^2(0,L) : \int_0^L f dx = 0\},\\ H_1 = L^2(0,L), \quad H_2 = L^2_{g'}(0,\infty;H_0),\\ \|(x_0,x_1,x_2)\|^2_{\mathcal{H}} = g(\infty)\|x_0\|^2 + \|x_1\|^2 + \int_0^\infty |g'(s)|\|x_2\|^2\,ds,\\ \qquad\qquad \forall (x_0,x_1,x_2) \in \mathcal{H}. \end{cases}$$

Introduce the variables $y = u_x(x,t)$, $v = u_t(x,t)$, $w(s) = u_x(x,t) - u_x(x,t-s)$. Define an operator $\mathcal{A}$ on $\mathcal{H}$ by the following:

$$\begin{cases} \mathcal{A} = \begin{pmatrix} 0 & B_1 & 0\\ -B_1^* & 0 & B_2\\ 0 & -B_2^* & -A_2 \end{pmatrix},\\ \mathcal{D}(\mathcal{A}) = \big\{(y,v,w) \in \mathcal{H} : \mathcal{A}(y,v,w) \in \mathcal{H}, v(0) = v(L) = 0, w(0) = 0\big\},\\ B_1 v = v_x, \quad B_1^* y = -g(\infty)y_x, \quad A_2 w = w_s,\\ B_2 w = -\int_0^\infty g'(s)w_x\,ds, \quad B_2^* v = -v_x. \end{cases}$$

Then $\mathcal{A}$ is the operator associated with the evolution equation (8.3), and it generates a contraction semigroup on $\mathcal{H}$ (see [16]).

By the conditions on the kernel $g(s)$, we have

$$\begin{aligned}\mathrm{Re}\,\langle A_2 w, w\rangle &= -\int_0^\infty g'(s)\,\langle w_s, w\rangle\, ds\\ &= \frac{1}{2}\int_0^\infty g''(s)\|w\|^2\, ds \geq -\frac{k}{2}\int_0^\infty g'(s)\|w\|^2\, ds = \frac{k}{2}\|w\|^2.\end{aligned}$$

Thus, $-A_2$ is strongly dissipative. Define $F \in \mathcal{L}(H_0, H_2)$, $Fy = sy$. Since

$$\langle B_2^* v, Fy\rangle = -\int_0^\infty g'(s)\,\langle v_x, sy\rangle = \delta\,\langle B_1 v, y\rangle,$$

where $\delta = -\int_0^\infty g'(s)s\, ds$, condition (7.16) holds. Condition (7.17) can be verified as follows.

$$\begin{aligned}|\langle B_2 w, v\rangle| &\leq \int_0^\infty |g'(s)|\|w\|\,\|v_x\|\, ds\\ &\leq \Big(\int_0^\infty |g'(s)|\, ds\Big)^{\frac{1}{2}}\|w\|_{H_2}\|B_1 v\| \leq K(\mathrm{Re}\,\langle A_2 w, w\rangle)^{\frac{1}{2}}\|B_1 v\|,\end{aligned}$$

and

$$\begin{aligned}&|\langle A_2 w, Fy\rangle| = \Big|\int_0^\infty s g'(s)\,\langle w_s, y\rangle\, ds\Big|\\ &= \Big|\int_0^\infty [g'(s) + s g''(s)]\,\langle w, y\rangle\, ds\Big|\\ &\leq \Big(\int_0^\infty |g'(s)|\, ds\Big)^{\frac{1}{2}}\|w\|_{H_2}\|y\| + \Big(\int_0^\infty s^2 g''(s)\, ds\Big)^{\frac{1}{2}}\Big(\int_0^\infty g''(s)\|w\|^2\, ds\Big)^{\frac{1}{2}}\|y\|\\ &\leq K(\mathrm{Re}\,\langle A_2 w, w\rangle)^{\frac{1}{2}}\|y\|.\end{aligned}$$

By Theorem 7.2, $e^{\mathcal{A}t}$ is exponentially stable.

A simple thermo-viscoelastic system is obtained by coupling equation (8.3) with a heat equation as follows.

$$\begin{cases} u_{tt}(x,t) = g(0)u_{xx}(x,t) + \int_0^\infty g'(s)u_{xx}(x,t-s)\, ds - \gamma\theta_x(x,t),\\ \theta_t(x,t) = -\gamma u_{xt}(x,t) + \theta_{xx}(x,t), \quad (x,t) \in [0,L]\times\mathbb{R}^+,\\ u(0,t) = u(L,t) = \theta(0,t) = \theta(L,t) = 0,\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x),\\ u(x,0) - u(x,-s) = w_0(x,s), \quad s \geq 0.\end{cases}$$

By a similar analysis, we can also verify that the associated semigroup is exponentially stable.

Example 8.4. A thermoelastic rod (see [1]): Consider

$$\begin{cases} u_{tt} = \alpha u_{xx} - c\gamma\theta_x, \\ \theta_t(x,t) = -\gamma u_{xt}(x,t) + k\theta_{xx}(x,t), \quad (x,t) \in (0,L) \times \mathbb{R}^+, \\ u(0,t) = u(L,t) = 0,\ \theta_x(0,t) = \lambda\theta(0,t),\ \theta_x(L,t) = -\lambda\theta(L,t), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{8.4}$$

Introduce the variables $y = u_x, v = u_t$. Let

$$\begin{cases} \mathcal{H} = H_0 \times H_1 \times H_2, \\ H_0 = \{f \in L^2(0,L) : \int_0^L f dx = 0\}, \quad H_1 = H_2 = L^2(0,L), \\ \|(y,v,\theta)\|^2 = \alpha\|y\|^2 + \|v\|^2 + c\|\theta\|^2, \quad \forall (y,v,\theta) \in \mathcal{H}. \end{cases}$$

We define an operator $\mathcal{A}$ on $\mathcal{H}$ by the following:

$$\begin{cases} \mathcal{A} = \begin{pmatrix} 0 & B_1 & 0 \\ -B_1^* & 0 & B_2 \\ 0 & -B_2^* & -A_2 \end{pmatrix}, \\ \mathcal{D}(\mathcal{A}) = \big\{z = (y,v,\theta) \in \mathcal{H} : \mathcal{A}z \in \mathcal{H}, v(0) = v(L) = 0, \\ \qquad\qquad\qquad \theta_x(0) = \lambda\theta(0), \theta_x(L) = -\lambda\theta(L)\big\}, \\ B_1 v = v_x, \quad B_1^* y = -\alpha y_x, \quad A_2\theta = -k\theta_{xx} \\ B_2\theta = -\gamma\theta_x, \quad B_2^* v = \gamma v_x. \end{cases}$$

Then $\mathcal{A}$ is the operator associated with the first-order evolution equation of (8.4), and it generates a contraction semigroup on $\mathcal{H}$. Since

$$\operatorname{Re}\langle A_2\theta, \theta\rangle = \lambda[\theta^2(L) + \theta^2(0)] + k\|\theta_x\|^2,$$

$-A_2$ is strongly dissipative. Define $F \in \mathcal{L}(H_0, H_2)$ by $Fy = y$. Then

$$\langle B_2^* v, Fy\rangle = \gamma\langle v_x, y\rangle = \frac{\gamma}{\alpha}\langle B_1 v, y\rangle.$$

Condition (7.16) holds. On the other hand,

$$\begin{aligned} |\langle A_2\theta, Fy\rangle| &= |\langle -k\theta_{xx}, y\rangle| \\ &\le |\lambda k(\theta(L)y(L) + \theta(0)y(0))| + |k\langle \theta_x, y_x\rangle| \le K(\operatorname{Re}\langle A_2\theta, \theta\rangle)^{\frac{1}{2}}\|B_1^* y\|, \end{aligned}$$

and $\|B_2\theta\|^2 = \|\gamma\theta_x\|^2 \le K \operatorname{Re}\langle A_2\theta, \theta\rangle$. Thus condition (7.18) also holds. By Theorem 7.2, $e^{\mathcal{A}t}$ is exponentially stable.

We can also verify that $e^{\mathcal{A}t}$ is not analytic. Let $-A_0$ be the upper left 2×2 block of the matrix operator $\mathcal{A}$ defined in this example; we see that e^{-A_0t} is not analytic. Since $b = c = \frac{1}{2}$ (see (2.14)), our conclusion follows from Theorem 2.3(iv).

Example 8.5. A thermoelastic rod with Kelvin-Voigt damping: Consider

$$\begin{cases} u_{tt}(x,t) = \alpha u_{xx}(x,t) + \beta u_{xxt}(x,t) - c\gamma\theta_x(x,t), \\ \theta_t(x,t) = -\gamma u_{xt}(x,t) + k\theta_{xx}(x,t), \quad (x,t)\in(0,L)\times\mathbb{R}^+, \\ u(0,t) = u(L,t) = 0, \quad \theta(0,t) = \theta(L,t) = 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{8.5}$$

Introduce the variables $y = u, v = u_t$. Let

$$\begin{cases} \mathcal{H} = H_0 \times H_1, \\ H_0 = H_0^1(0,L)\times L^2(0,L), \quad H_1 = L^2(0,L), \\ \|(y,v,\theta)\|_{\mathcal{H}}^2 = \alpha\|y_x\|^2 + \|v\|^2 + c\|\theta\|^2, \quad \forall (y,v,\theta)\in\mathcal{H}. \end{cases}$$

We define

$$\begin{cases} \mathcal{A} = \begin{pmatrix} -A_0 & B \\ C & -A_1 \end{pmatrix}, \\ \mathcal{D}(\mathcal{A}) = \big\{ z = (y,v,\theta)\in\mathcal{H} : y,v,\theta\in H^2(0,L)\cap H_0^1(0,L)\big\}, \\ -A_0 = \begin{pmatrix} 0 & I \\ \alpha D^2 & \beta D^2 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ -D \end{pmatrix},\ -A_1 = kD^2,\ C = (\,0 \quad -\gamma D\,), \\ \mathcal{D}(A_0) = [H^2(0,L)\cap H_0^1(0,L)]\times[H^2(0,L)\cap H_0^1(0,L)], \\ \mathcal{D}(B) = H_0^1(0,L), \\ \mathcal{D}(C) = H_0^1(0,L)\times H_0^1(0,L), \quad \mathcal{D}(A_1) = H^2(0,L)\cap H_0^1(0,L). \end{cases}$$

It is clear that $e^{\mathcal{A}t}$ is a C_0-semigroup of contractions on $\mathcal{H}$, and e^{-A_0t}, e^{-A_1t} are analytic semigroups on H_0, H_1, respectively. Since $b = c = \frac{1}{2}$, by Theorem 2.2(iv), $e^{\mathcal{A}t}$ is analytic.

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