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THE THIN FILM EQUATION WITH $2 \leq n < 3$: FINITE SPEED OF PROPAGATION IN TERMS OF THE L^1-NORM*

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Abstract. We consider the equation $u_t + (u^n u_{xxx})_x = 0$ with $2 \leq n < 3$ and establish an estimate for the finite speed of propagation of the support of compactly supported nonnegative solutions. The estimate depends only on the L^1-norm and is valid *a posteriori* for strong solutions obtained through a Bernis-Friedman regularization.

1. Introduction. In this paper we consider compactly supported solutions of the Thin Film Equation

$$u_t + (u^n u_{xxx})_x = 0, \tag{1.1}$$

on $Q_T = \{(x,t) : -R < x < R,\ 0 < t \leq T\}$ with nonnegative initial data

$$u(x,0) = u_0(x), \quad 0 \leq u_0 \in H^1(-R,R), \tag{1.2}$$

and lateral boundary conditions

$$u_x(\pm R, t) = u_{xxx}(\pm R, t) = 0. \tag{1.3}$$

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Here R is for the moment a finite positive number, but for compactly supported solutions we may of course consider $R = \infty$ (i.e., the Cauchy problem).

Equation (1.1) is a nonlinear degenerate fourth order diffusion equation which, for different values of the positive exponent n, arises in a number of applications, see e.g. [2] for references. We mention in particular the case $n = 3$, describing the height $u(x,t)$ of a thin film of slowly flowing viscous fluid on a horizontal plate when the dominating driving force is the surface tension.

One of the main difficulties of equation (1.1) is that a weak formulation using smooth test functions and integration by parts leads to considerable technical difficulties if one tries to put the space derivatives on the test functions. In a first paper on the rigorous mathematical treatment of equation (1.1) with $n \geq 1$, posed with boundary conditions (1.3) and initial conditions (1.2), Bernis and Friedman [5], use a parabolic regularization to establish the existence of a continuous weak nonnegative solution which is smooth away from $u = 0$ and $t = 0$, satisfying the initial condition (1.2) strongly in H^1. The solution satisfies (1.1) only in the sense that

$$\int\int_{Q_T} u\phi_t + \int\int_{\mathcal{P}} u^n u_{xxx}\phi_x = 0, \tag{1.4}$$

for all smooth test functions which vanish near $t = 0$ and $t = T$. Here $\mathcal{P}$ is the set in Q_T where $u \neq 0$. In particular $u^{\frac{n}{2}} u_{xxx} \in L^2(\mathcal{P})$. The lateral boundary conditions are satisfied only in points where $u \neq 0$. Thus this concept of a solution is too weak, which is illustrated by the fact that every function of the form $(x-b)^+(c-x)^+$ is a weak stationary solution in this sense. The second space derivative u_{xx} of such a function is not even a measurable function. Under additional positivity assumptions on the initial data, which are certainly satisfied if $u_0 > 0$ on $[-R, R]$, Bernis and Friedman show that the constructed solution does have $u_{xx} \in L^2(Q_T)$ and satisfies

$$\int\int_{Q_T} u\phi_t = \int\int_{Q_T} u^n u_{xx}\phi_{xx} + \int\int_{Q_T} nu^{n-1}u_x u_{xx}\phi_x, \tag{1.5}$$

for smooth test functions satisfying the additional condition that $\phi_x = 0$ on the lateral boundary. Note that $u_{xx} \in L^2(Q_T)$ implies that this weak solution has $u \in C^1([-R, R])$ for almost every $t \in (0, T]$. Such weak solutions are called strong solutions [2]. They are conjectured to be unique. The standard parabolic regularization though in general does not produce a strong solution for all nonnegative $u_0 \in H^1$. We note that most of the qualitative properties (in particular the properties described below) for strong solutions apply *only* to strong solutions obtained by some limit procedure.

For $n \geq 4$ Bernis and Friedman show that the weak solution is strictly positive and unique if $u_0 > 0$ on $[-R, R]$, motivating a regularization of u^n with

$$f_\epsilon(u) = \frac{u^{n+4}}{\epsilon u^n + u^4} \to u^n, \tag{1.6}$$

rather than with $u^n + \epsilon$. The corresponding regularized problem (which includes an approximation of the initial data by $u_0 + \epsilon^\theta$ with an appropriate choice of θ) has a unique positive smooth solution u_ϵ. In [1] Beretta, Bertsch and Dal Passo show for all $0 < n < 3$, choosing $0 < \theta < \frac{2}{5}$, that the solutions u_ϵ converge (along a subsequence) to a strong solution, see also [7]. Subsequently they establish various interesting properties of the behaviour of the solution near the boundary of its support. In particular they recover the exponents found by Bernis, Peletier and Williams [6], who establish the existence of compactly supported self-similar source-type solutions for $0 < n < 3$: at the moving free boundary these profiles have $u^{\frac{1}{\beta}} \in C^1$, where $\beta < 2$ for $0 < n \leq \frac{3}{2}$ and $\beta < \frac{3}{n}$ for $\frac{3}{2} < n < 3$. Another important result in [1] is that for $n \geq 2$ the constructed strong solution is strictly positive for almost all times if it is strictly positive initially. This will be used in our approach for $2 \leq n < 3$.

The compactly supported self-similar solutions strongly suggest that there is a finite (positive) speed of propagation property for strong solutions of (1.1) when $0 < n < 3$.

For $0 < n < 2$ this was proved independently by Bernis [2] *a posteriori* for (strong) solutions obtained through (1.6) and *a priori* by Kersner and Shishkov [8] for "energy" solutions.

In [2] the continuity of the free boundary of the support, as well as its large time behaviour (choosing sufficiently large domains and solving in fact the Cauchy Problem) are also discussed. In [4] the results are extended to the range $2 \leq n < 3$. The methods in [4] involve new integral inequalities for positive functions on an interval satisfying certain homogeneous boundary conditions [3]. We will refer to these integral inequalities as the Bernis inequalities.

In the proof of the results on strong solutions described above, the regularity of the initial data, i.e., the assumption that u_0 belongs to H^1, is important. In fact the H^1-norm of the initial data appears explicitly in the estimate for the finite speed of propagation in [4] (formula's (5.1) and (5.2)) in the range $2 \le n < 3$. In [2] (for the range $0 < n < 2$), the estimate involves only the L^1-norm of the initial data and is invariant under the scaling which leaves the source-type solutions invariant. This will be useful if one tries to describe large time behaviour of compactly supported solutions in terms of the source type solutions using a scaling limit instead of the large time limit. It is the purpose of this work to provide such an estimate for $2 \le n < 3$.

The energy method we use is based on integral estimates reminiscent of Saint-Venant's principle. For linear parabolic equations such an approach was suggested and used by Oleinik, see for example [10]. These techniques were adapted to higher order quasilinear (degenerate and nondegenerate) equations in [11]. We note that we use the Bernis inequalities from [3] and not their weighted versions as used in [4]. A further difference is that we use functional inequalities rather than differential inequalities.

The paper is divided as follows. In Section 2 we give the precise formulation of the main results. An outline of the method of proof for $2 \le n < 3$ is given in Section 3. The details of the proofs are given in Section 4.

2. The main result. The result in this paper concerns strong nonnegative solutions obtained through (1.6).

Theorem 1. *Let $2 \le n < 3$, $R > 0$, $T > 0$, and let $0 \le u_0 \in H^1(-R,R)$. The strong solution u of* (1.1), (1.2), (1.3) *can be obtained as a limit of smooth positive solutions u_ϵ using the regularization* (1.6) *with initial data $u(x,0) \ge u_0 + \epsilon^\theta$ with $0 < \theta < \frac{2}{5}$, in such a way that, if, for some $R_0 \in (-R,R)$,*

$$\text{supp } u_0 \subset [-R_0, R_0], \tag{2.1}$$

then

$$f(t) = \sup\{x \in (-R,R) : \ u(x,t) > 0\} < R_0 + Ct^{\frac{1}{n+4}}, \tag{2.2}$$

where C is a constant depending only on n and $\int_{-R}^{R} u_0$.

The test functions used in our proofs are basically variants of u_{xx}. The first one is $(u\psi)_{xx}\psi g$, where $\psi = \psi(x)$ is a smooth nonnegative function with a compactly supported derivative ψ_x, and where $g = g(t)$ is a smooth nonnegative function, e.g. $g(t) = \exp(-\frac{t}{T})$. Since this test function does not belong to the test function class of any of the weak formulations, we have to justify its use through the approximating problems. It is instructive to discuss this in some detail. We consider strictly positive unique smooth solutions u_ϵ with initial data

$$u_\epsilon(x,0) = u_0(x) + \epsilon^\theta, \quad 0 < \theta < \tfrac{2}{5} \tag{2.3}$$

of the regularized equation (cf. (1.6))

$$u_t + (f_\epsilon(u)u_{xxx})_x = 0, \tag{2.4}$$

with lateral boundary conditions (1.3). For these smooth solutions we have, testing with $(u_\epsilon\psi)_{xx}\psi g$ and omitting the subscript ϵ, that

$$\int\int_{Q_T} gf(u)u_{xxx}((u\psi)_{xx}\psi)_x - \frac{1}{2}\int\int_{Q_T} g'((u\psi)_x)^2 + \frac{1}{2}g(T)\int_{\Omega_T}((u\psi)_x)^2 = \frac{1}{2}g(0)\int_\Omega ((u_0\psi)')^2. \tag{2.5}$$

Here $\Omega_T = \Omega \times \{T\}$. Since we only have

$$(f_\epsilon(u_\epsilon))^{\frac{1}{2}}u_{\epsilon xxx} \to u^{\frac{n}{2}}u_{xxx}\chi_{\{u>0\}} \text{ weakly in } L^2(Q_T), \tag{2.6}$$

and

$$(u_\epsilon\psi)_x \to (u\psi)_x \text{ weakly in } L^2(\Omega_T), \tag{2.7}$$

along a subsequence as $\epsilon \to 0$, we cannot expect to have this identity for the strong limit solution. Instead we have for the limit solution

$$\begin{aligned}\int\int_{Q_T\cap\{u>0\}} gu^n u_{xxx}((u\psi)_{xx}\psi)_x - \frac{1}{2}\int\int_{Q_T} g'((u\psi)_x)^2 \\ + \frac{1}{2}g(T)\int_{\Omega_T}((u\psi)_x)^2 \le \frac{1}{2}g(0)\int_\Omega((u_0\psi)')^2.\end{aligned} \tag{2.8}$$

The terms containing the L^2-integrals of (2.6) and (2.7) in (2.5) are the only terms which may cause the equality in (2.5) to change in an inequality in the limit. All the other (lower order) terms converge. This follows from the strong convergence results in [2] (Lemma 3.3).

The relation (2.8) with equality would simplify the proofs. It is not completely clear yet whether our result can be proved *a priori* for all strong solutions satisfying (2.8). For the moment we will prove Theorem 1 through the approximation with (1.6). At this point we remark that the notation in (2.6), although used in some of the papers on (1.1), is not completely correct because u_{xxx} is only defined in the positivity set of u. It would be better to say that the left hand side of (2.6) converges weakly to a function which equals $u^{\frac{n}{2}}u_{xxx}$ in $\mathcal{P}$ and zero outside.

3. An outline of the proof. The proof will be based on the following lemma, which may be seen as a variant of Stampacchia's Lemma, see [12].

Lemma 1. *Suppose a nonnegative nonincreasing function $J(\tau)$ satisfies, for some τ_0 and for some $0 < \theta < 1$, the relation*

$$J(\tau + J(\tau)) < \theta J(\tau) \text{ for all } \tau > \tau_0. \tag{3.1}$$

Then, for any $\tau_1 \geq \tau_0$,

$$J(\tau) = 0 \quad \text{for all } \tau > \tau_1 + \frac{J(\tau_1)}{1-\theta}. \tag{3.2}$$

The proof of this lemma is elementary. Define $\tau_{k+1} = \tau_k + J(\tau_k)$, $(k = 1, 2, \dots)$. Then $J(\tau_{k+1}) \leq \theta J(\tau_k)$, whence $\tau_{k+1} = \tau_1 + J(\tau_1) + J(\tau_2) + \cdots + J(\tau_k) \leq \tau_1 + J(\tau_1)(1+\theta+\cdots+\theta^{k-1}) \leq \tau_1 + J(\tau_1)/(1-\theta)$. Since $J(\tau_{k+1}) \to 0$ as $k \to \infty$ and J is nonnegative and nonincreasing, it follows that $J(\tau) = 0$ for all $\tau \geq \tau_1 + J(\tau_1)/(1-\theta)$ and Lemma 1 is proved.

In our context $J(\tau) = 0$ will correspond to the support being contained in a ball with radius τ. Before explaining how Lemma 1 can be applied we need some notation. We shall write $Q = \Omega \times (0, \infty)$, $Q_T = \Omega \times (0, T)$, $\Omega(\tau) = \Omega \cap \{|x| > \tau\}$, $Q_T(\tau) = \Omega(\tau) \times (0, T)$, $\Omega(\tau, \delta) = \Omega(\tau) \backslash \Omega(\tau + \delta)$, $Q_T(\tau, \delta) = Q_T(\tau) \backslash Q_T(\tau+\delta)$, $\Omega_T = \Omega \times \{T\}$, $\Omega_T(\tau) = \Omega(\tau) \times \{T\}$, $\Omega_T(\tau, \delta) = \Omega_T(\tau) \backslash \Omega_T(\tau + \delta)$. In addition we use, given a function u, for each of these sets the notation $S^+ = S \cap \{u > 0\}$.

Now let

$$I_T(\tau) = \int\int_{Q_T(\tau)} u^{n+2}. \tag{3.3}$$

Suppose that we have an estimate for $I_T(\tau)$ of the form

$$I_T(\tau + \delta) \leq \frac{C(I_T(\tau) - I_T(\tau + \delta))^{1+\beta} A(T)}{\delta^m}, \tag{3.4}$$

with m, β and $A(T)$ positive. Then the function $J_T(\tau)$ defined by

$$J_T(\tau)^m = A(T) I_T(\tau)^\beta, \tag{3.5}$$

taking $\delta = J_T(\tau)$ in (3.4), satisfies the assumption of Lemma 1 with $\theta = C/(1 + C)$.

The first part of the proof will consist of proving an estimate of the form (3.4). For fixed $\tau > 0$ and $\delta > 0$ we introduce a smooth cut-off function $\eta(x)$ with

$$0 \leq \eta \leq 1, \quad \eta(x) = 0 \text{ if } |x| < \tau, \quad \eta(x) = 1 \text{ if } |x| > \tau + \delta, \quad |\frac{d^i \eta}{dx^i}| \leq C\delta^{-i}, \tag{3.6}$$

for $i = 1, 2, 3$, where C is a constant independent of δ and τ.

Lemma 2. *The strong solution in Theorem* 1 *may be constructed in such a way that, for all $\tau > R_0$, $\delta > 0$ and $\epsilon > 0$,*

$$\begin{aligned}\frac{1}{e}\int_{\Omega_T}(u\psi)_x^2 + \frac{1}{T}\int\int_{Q_T}(u\psi)_x^2\exp(-\frac{t}{T}) + d(n)\int\int_{Q_T^+}U\exp(-\frac{t}{T})\\ \leq \epsilon\int\int_{Q_T^+(\tau,\delta)}U_0\exp(-\frac{t}{T}) + \frac{C(\epsilon)}{\delta^6}\int\int_{Q_T(\tau,\delta)}u^{n+2}\exp(-\frac{t}{T}),\end{aligned} \tag{3.7}$$

where $\psi = \eta^{\frac{n+2}{2}}$, $U = (u\eta)^{n-4}|(u\eta)_x|^6 + (u\eta)^{n-1}|(u\eta)_{xx}|^3 + (u\eta)^n|(u\eta)_{xxx}|^2$, $U_0 = u^{n-4}|u_x|^6 + u^{n-1}|u_{xx}|^3 + u^n|u_{xxx}|^2$, and where $d(n)$ and $C(\epsilon)$ are positive constants independent of u, T, τ and δ.

Lemma 2 is proved using the function $g(t) = \exp(-t/T)$ in the inequality (2.8). In the proof the homogeneous Bernis estimates [3] for positive functions are used. Thus we first use (2.8) for solutions which are positive for almost every t. Our strong solution is the limit of strong solutions of (1.1) with this property obtained by Beretta, Bertsch and Dal Passo in [1]. Let us remark here that it is conjectured (though certainly not straightforward) that the Bernis estimates also hold for nonnegative C^1-functions with compact support. If this conjecture is true, an appeal to the positivity property in [1] is not necessary and the restriction $n \geq 2$ can be omitted.

In the second term on the right hand side of (3.7) we already recognize the right hand side of (3.4). If we restrict the integrals on the left hand side of (3.7) to $|x| \geq \tau + \delta$, the integrals become smaller and the factors ψ disappear. In the resulting estimate we can, using an iteration argument, control the constant $C(\epsilon)$ as $\epsilon \to 0$. This yields the next lemma.

Lemma 3. *Under the same conditions as in Lemma* 2 *we have, with new constants,*

$$\begin{aligned}\int_{\Omega_T(\tau+\delta)}|u_x|^2 + \frac{1}{T}\int\int_{Q_T(\tau+\delta)}|u_x|^2 + d(n)\int\int_{Q_T^+(\tau+\delta)}U_0\\ \leq \frac{C(n)}{\delta^6}\int\int_{Q_T^+(\tau,\delta)}u^{n+2},\end{aligned} \tag{3.8}$$

where $C(n)$ and $d(n)$ are positive constants independent of u, T, τ and δ.

If the support of u is compact, the first two terms on the left hand side of (3.8) may be bounded from below using Poincaré's inequality.

Corollary 1. *Suppose*

$$\mathrm{supp}\, u(x,t) \subset [-f(T), f(T)] \text{ for all } 0 < t \leq T \text{ and } f(T) < R.$$

Then, with new constants,

$$\int_{\Omega_T(\tau+\delta)} u^2 + \frac{1}{T}\int\int_{Q_T(\tau+\delta)} u^2 + d(n)(f(T)-\tau-\delta)_+^2 \int\int_{Q_T^+(\tau+\delta)} U_0$$
$$\leq \frac{C}{\delta^6}(f(T)-\tau-\delta)_+^2 \int\int_{Q_T(\tau,\delta)} u^{n+2} \text{ for all } \tau > R_0, \delta > 0. \tag{3.9}$$

Next we consider the left hand side of the desired estimate (3.4) and observe that, assuming $f(T) < R$, by the homogeneous Gagliardo-Nirenberg inequalities,

$$I_T(\tau+\delta) \leq C(n)\Big(\int\int_{Q_T^+(\tau+\delta)} (u^{\frac{n+2}{2}})_{xxx}^2\Big)^{\frac{n}{n+12}} \big(\Psi_{\frac{n+2}{2}}(\tau+\delta, T)\big)^{\frac{12}{n+12}}, \tag{3.10}$$

where

$$\Psi_\lambda(\tau, T) = \int_0^T \Big(\int_{\Omega_t(\tau)} u^2\Big)^\lambda. \tag{3.11}$$

The first factor on the right hand side of (3.10) is controlled by U_0 in (3.8). Estimating also the second factor on the right hand side of (3.10) we shall arrive at an estimate of the form (3.4). Here the estimates (3.8) and (3.9) are not yet good enough because they only control (3.11) with $\lambda = 1$. Therefore we need a version for $\lambda > 1$. To this purpose we define the function

$$\zeta_\lambda(t) = \int_0^t \Big(\int_{\Omega_s} (u\psi)_x^2\Big)^\lambda ds, \quad \lambda > 0, \tag{3.12}$$

and employ as test function $(u\psi)_{xx}\psi\zeta_\lambda$. Note that we omit the t-dependence in the notation. Here again we have to go through the regularization. We obtain the recurrence estimate

$$\zeta_{\lambda+1}(T) \leq C(n)\zeta_\lambda(T)\Big(\frac{I_T(\tau-\delta) - I_T(\tau+\delta)}{\delta^6}\Big), \tag{3.13}$$

which leads to

$$\zeta_{\frac{n+2}{2}}(T) \leq C(n)^{\frac{n}{2}}\zeta_1(T)\Big(\frac{I_T(\tau-\delta) - I_T(\tau+\delta)}{\delta^6}\Big)^{\frac{n}{2}}. \tag{3.14}$$

Combining (3.14), (3.10) with (Poincaré)

$$\Psi_{\frac{n+2}{2}}(\tau+\delta, T) \leq C(f(T)-\tau-\delta)_+^{n+2}\zeta_{\frac{n+2}{2}}(T), \tag{3.15}$$

and controlling $\zeta_1(T)$ by (3.7), we shall arrive at an estimate of the form (3.4).

Lemma 4. *Let u be as in Lemma 2. Then, for all $\tau > R_0$ and $\delta > 0$,*

$$I_T(\tau+\delta) \le C(n)(f(T)-\tau-\delta)_+^{\frac{12(n+2)}{n+12}} T^{\frac{12}{n+12}} \Big(\frac{I_T(\tau)-I_T(\tau+\delta)}{\delta^6}\Big)^{\frac{7n+12}{n+12}}. \tag{3.16}$$

The constant does not depend on u, τ, δ and T.

Fixing $\tau_1 > R_0$, whence for $\tau \ge \tau_1$, $f(T)-\tau-\delta < f(T)-\tau_1$, and using Lemma 1 as explained at the beginning of this section we now obtain:

Lemma 5. *For all $\tau_1 > R_0$ we have for the support of the solution u from Lemma 2 that*

$$f(T)-\tau_1 \le C(n) T^{\frac{2}{5n+8}} I_T(\tau_1)^{\frac{n}{5n+8}}. \tag{3.17}$$

This completes the first part of the proof. To prove the main theorem, it remains to estimate $I_T(\tau)$ and to choose an optimal value of τ_1. Before doing that we apply the inhomogeneous Gagliardo-Nirenberg estimates to the right hand side of (3.8). Using mass conservation plus an integration argument we obtain an L^1-version of Lemma 3.

Lemma 6. *The solution u considered in the previous lemma's satisfies the estimate*

$$\int_{\Omega_T(\tau+\delta)} u_x^2 + \frac{1}{T}\int\int_{Q_T(\tau+\delta)} u_x^2 + d(n)\int\int_{Q_T(\tau+\delta)} U_0 \le \frac{CK^{n+2}T}{\delta^{n+7}}, \tag{3.18}$$

for all $\tau > R_0$, where $K = \int u_0$.

Using the homogeneous Gagliardo-Nirenberg inequalities again (in view of the compactness of the support), it follows from this last estimate that

$$I_T(\tau+\delta) \le \frac{C(n)K^{n+2}T}{\delta^{n+1}}. \tag{3.19}$$

Now replace $\tau+\delta$ by τ and then put (for the new τ) $\delta = \tau - R_0$. The estimate above becomes

$$I_T(\tau) \le \frac{C(n)K^{n+2}T}{(\tau-R_0)^{n+1}}.$$

Combining with Lemma 5 we have

$$f(T) \le \tau + CT^{\frac{2}{5n+8}} \Big(\frac{CK^{n+2}T}{(\tau-R_0)^{n+1}}\Big)^{\frac{n}{5n+8}} = \tau + \frac{C(n)T^{\frac{2+n}{5n+8}} K^{\frac{n(n+2)}{5n+8}}}{(\tau-R_0)^{\frac{n(n+1)}{5n+8}}}. \tag{3.20}$$

We minimize the right hand side by setting

$$\tau = \tau_{min} = (\frac{n(n+1)}{5n+8})^{\frac{5n+8}{(n+2)(n+4)}} C(n)^{\frac{5n+8}{(n+2)(n+4)}} K^{\frac{n}{n+4}} T^{\frac{1}{n+4}}$$
$$= C(n) K^{\frac{n}{n+4}} T^{\frac{1}{n+4}},$$

thus obtaining the final estimate

$$f(T) \leq C(n) T^{\frac{1}{n+4}} K^{\frac{n}{n+4}}. \tag{3.21}$$

with $C(n)$ independent of T and u.

4. Details. In this section we prove the estimates stated in Section 3. Let us first recall from [3] that for any $\frac{1}{2} < n < 3$, for any interval $[a, b]$ and for any strictly positive $v \in C^1([a, b])$ with $v'(a) = v'(b) = 0$, the following estimates hold:

$$\int_a^b v^{n-4}|v'|^6 \leq C(n) \int_a^b v^{n-1}|v''|^3; \; \int_a^b v^{n-1}|v''|^3 \leq C(n) \int_a^b v^n|v'''|^2; \tag{4.1}$$

$$\int_a^b |(v^{\frac{n+2}{2}})'''|^2 \leq C(n) \int_a^b v^n|v'''|^2. \tag{4.2}$$

The constants depend only on n and can be computed explicitly. A relaxation of the strict positivity assumption would be most welcome.

These estimates will be used in combination with Young's inequality applied to the "lower order terms"

$$\frac{u^n u_j u_k}{\delta^i} \text{ with } i \geq 1, j, k \geq 0, \; i + j + k = 6, \quad u_j = |\frac{\partial^j u}{\partial x^j}|. \tag{4.3}$$

Young's inequality may be written as

$$\frac{abc}{\delta^i} \leq \epsilon(\frac{a^p}{p} + \frac{b^q}{q}) + \frac{c^r}{r\epsilon^{r-1}\delta^{ir}}, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 (q = \infty \text{ if } b = 1). \tag{4.4}$$

Proof of Lemma 2. We substitute $\psi(x) = \eta(x)^{\frac{n+2}{2}}$ and $g(t) = \exp(-\frac{t}{T})$ in (2.8). Let us first evaluate the integrand in the first term. We have

$$(u^n u_{xxx})((u\psi)_{xx}\psi)_x = (u^n u_{xxx})(u\psi)_{xxx}\psi + (u^n u_{xxx})(u\psi)_{xx}\psi_x$$
$$= B_1 + B_2. \tag{4.5}$$

Then B_1 equals

$$B_{1,1}+B_{1,2}=u^n u_{xxx}^2\psi^2 \\ +(u^n u_{xxx}u\psi_{xxx}\psi+3u^n u_{xxx}u_{xx}\psi_x\psi+3u^n u_{xxx}u_x\psi_{xx}\psi). \tag{4.6}$$

For the first term we have

$$\begin{aligned} B_{1,1}=&(u\eta)^n((u\eta)_{xxx}-3u_{xx}\eta_x-3u_x\eta_{xx}-u\eta_{xxx})^2\\ =&(u\eta)^n(u\eta)_{xxx}^2-\Big(2(u\eta)^n(u\eta)_{xxx}(u\eta_{xxx}+3u_x\eta_{xx}+3u_{xx}\eta_x)\\ &-(u\eta)^n(u\eta_{xxx}+3u_x\eta_{xx}+3u_{xx}\eta_x)^2\Big)=B_{1,1}^1-B_{1,1}^2. \end{aligned} \tag{4.7}$$

We want to apply the Bernis estimates to $u\psi$ and conclude that

$$\int_{\Omega^+}B_{1,1}^1=\int_{\Omega^+}(u\eta)^n(u\eta)_{xxx}^2\geq d(n)\int_{\Omega^+}U, \tag{4.8}$$

where $U=(u\eta)^{n-4}(u\eta)_x^6+(u\eta)^{n-1}(u\eta)_{xx}^3+(u\eta)^n(u\eta)_{xxx}^2$, $d(n)>0$. Then we can evaluate (2.8) as

$$\begin{aligned} &\frac{1}{2e}\int_{\Omega_T^+}(u\psi)_x^2+\frac{1}{T}\int\int_{Q_T^+}(u\psi)_x^2\exp(-\frac{t}{T})+d(n)\int\int_{Q_T^+}U\exp(-\frac{t}{T})\\ &\quad\leq\int_0^T\exp(-\frac{t}{T})\int_{\Omega_t^+(\tau,\delta)}(B_{1,1}^2-B_{1,2}-B_2)+\frac{1}{2}\int_{\Omega_0}(u\psi)_x^2, \end{aligned} \tag{4.9}$$

and continue with the right hand side.

It is in (4.8) that we need the positivity of u which holds for almost every t if u is the strong solution in [1] with strictly positive initial data (the cut-off function ψ is harmless here). For the moment we assume that $u(x,0)>0$.

Using the bounds on η and its derivatives the terms in $B_{1,1}^2-B_{1,2}-B_2$ can be estimated by the terms in (4.3). All these lower order terms are supported between $x=\tau$ and $x=\tau+\delta$ and bounded by, using Young's inequality,

$$\epsilon U_0+\frac{C(\epsilon)}{\delta^6}u^{n+2}. \tag{4.10}$$

Thus from (4.9), apart from the last term $\int(u\psi)_x^2$, we arrive at (3.7).

Finally we observe that we can construct a strong solution with nonnegative initial data by first lifting the data, taking the strong solution from (4.8), and then taking the limit. It is not clear however whether any strong solution constructed directly can also be obtained in this way. This completes the proof of Lemma 2.

Proof of Lemma 3. From (3.7) we also have, writing $\tau' + \delta' = (\tau + \frac{\delta}{2}) + \frac{\delta}{2}$, absorbing the terms with e in the constants, and taking the first integral on the right hand side over the whole of $Q_T^+(\tau') = Q_T^+(\tau + \frac{\delta}{2})$, that

$$\begin{aligned}\int_{\Omega_T(\tau+\delta)} |u_x^2| + \frac{1}{T}\int\int_{Q_T(\tau+\delta)} |u_x|^2 + d(n)\int\int_{Q_T^+(\tau+\delta)} U_0 \\ \leq \epsilon \int\int_{Q_T^+(\tau+\frac{\delta}{2})} U_0 + \frac{C(\epsilon)}{(\frac{\delta}{2})^6}\int\int_{Q_T(\tau+\frac{\delta}{2},\frac{\delta}{2})} u^{n+2}.\end{aligned} \tag{4.11}$$

Introducing the δ-dependent quantities

$$\begin{aligned}A(\delta) &= \delta^6\Big(\int_{\Omega_T(\tau+\delta)} |u_x|^2 + \frac{1}{T}\int_{Q_T(\tau+\delta)} |u_x|^2\Big),\\ H(\delta) &= \delta^6 \int_{Q_T^+(\tau+\delta)} U_0, \quad F(\delta) = \int_{Q_T(\tau+\delta)} u^{n+2},\end{aligned}$$

multiplying by δ^6 and fixing $\epsilon = d(n)2^{-7}$, this last estimate (4.11) can be rewritten and iterated as

$$\begin{aligned}A(\delta) + d(n)H(\delta) &\leq \frac{d(n)}{2}H(\frac{\delta}{2}) + C(n)(F(\frac{\delta}{2}) - F(\delta))\\ &\leq d(n)\frac{1}{2^2}H(\frac{\delta}{2^2}) + C(n)(1+\frac{1}{2})(F(\frac{\delta}{2^2}) - F(\delta))\\ &\leq \frac{1}{2^2}\Big(\frac{1}{2}d(n)H(\frac{\delta}{2^3}) + C(n)(F(\frac{\delta}{2^3}) - F(\frac{\delta}{2^2}))\Big)\\ &\quad + C(n)(1+\frac{1}{2})(F(\frac{\delta}{2^2}) - F(\delta))\\ &\leq d(n)\frac{1}{2^3}H(\frac{\delta}{2^3}) + C(n)(1+\frac{1}{2}+\frac{1}{2^2})(F(\frac{\delta}{2^3}) - F(\delta))\\ &\leq \cdots \leq d(n)\frac{1}{2^j}H(\frac{\delta}{2^j}) + C(n)\sum_{i=0}^{j}\frac{1}{2^i}(F(\frac{\delta}{2^j}) - F(\delta)).\end{aligned} \tag{4.12}$$

The dependence on n of $C(n)$ is through the choice of ϵ above. Letting $j \to \infty$ we deduce that

$$A(\delta) + d(n)H(\delta) \leq 2C(n)(F(0) - F(\delta)), \tag{4.13}$$

which completes the proof of Lemma 3.

In the proofs of Lemma 4, Lemma 6 and finally Theorem 1 we use the Gagliardo-Nirenberg interpolation inequalities (see e.g. the book of Maz'ja [9]) in the form (with the same subscript notation as in (4.3))

$$\Big(\int_a^b |v_i|^\alpha\Big)^{\frac{1}{\alpha}} \le C_1\Big(\int_a^b |v_m|^\beta\Big)^{\frac{\theta}{\beta}}\Big(\int_a^b |v|^\gamma\Big)^{\frac{1-\theta}{\gamma}} + \frac{C_2}{(b-a)^{i+\frac{\alpha-\gamma}{\alpha\gamma}}}\Big(\int_a^b |v|^\gamma\Big)^{\frac{1}{\gamma}}, \tag{4.14}$$

for $\alpha > 0$, $\beta \ge 1$, $\gamma > 0$, $\theta \in [\frac{i}{m}, 1)$, $\frac{1}{\alpha} - i = (\frac{1}{\beta} - m)\theta + \frac{1-\theta}{\gamma}$, with constants only depending on the parameters. If v has support smaller than $[a, b]$, the constant C_2 may be taken equal to zero. The resulting estimate is then homogeneous. We shall apply these estimates to $v = u^{\frac{n+2}{2}}$ to control the L^{n+2}-norm of u by the L^2-norm of v_{xxx} and the L^2- or L^1-norm of u.

Proof of Lemma 4. We first derive (3.10). ¿From (4.14) above with $C_2 = 0$, $\alpha = \beta = 2$, $\gamma = \frac{4}{n+2}$, $i = 0$, $m = 3$ and $\theta = \frac{n}{n+12}$ it follows that

$$\int_{\Omega_t(\tau+\delta)} u^{n+2} \le C(n)(\int_{\Omega(\tau+\delta)} (u^{\frac{n+2}{2}})^2_{xxx})^{\frac{n}{n+12}}(\int_{\Omega(\tau+\delta)} u^2)^{\frac{6(n+2)}{n+12}},$$

whence, using Hölder's inequality, (3.10). The first factor on the right of (3.10) is controlled by (3.8), since $(u^{\frac{n+2}{2}})^2_{xxx} \le C(n)U_0$. To control the second factor on the right of (3.10), we would like to use (2.8) with g replaced by (3.12). Unfortunately the recurrence inequality (3.13) can only be derived with an equality in (2.8). Thus we have to go back to the regularized problem again and to (2.5) with g replaced by

$$\zeta^\epsilon_\lambda(t) = \int_0^t (\int_{\Omega_s} (u_\epsilon\psi)^2_x)^\lambda ds, \quad \lambda > 0. \tag{4.15}$$

We have from (2.5) that

$$\frac{1}{2}\zeta^\epsilon_\lambda(T)\int_{\Omega_T}(u_\epsilon\psi)^2_x - \frac{1}{2}\zeta^\epsilon_{\lambda+1}(T) + \int\int_{Q_T}(f_\epsilon(u_\epsilon)u_{\epsilon xxx})((u_\epsilon\psi)_{xx}\psi)_x\zeta^\epsilon_\lambda = 0,$$

whence

$$\begin{aligned}\zeta^\epsilon_{\lambda+1}(T) &= \zeta^\epsilon_\lambda(T)\int_{\Omega_T}(u_\epsilon\psi)^2_x + 2\int\int_{Q_T}(f_\epsilon(u_\epsilon)u_{\epsilon xxx})((u_\epsilon\psi)_{xx}\psi)_x\zeta^\epsilon_\lambda \\ &= \zeta^\epsilon_\lambda(T)\int_{\Omega_T}(u_\epsilon\psi)^2_x + 2\int\int_{Q_T}\zeta^\epsilon_\lambda f_\epsilon(u_\epsilon)u^2_{\epsilon xxx}\psi^2 + 2\int\int_{Q_T}\zeta^\epsilon_\lambda(\dots) \\ &\le \zeta^\epsilon_\lambda(T)\Big(\int_{\Omega_T}(u_\epsilon\psi)^2_x + 2\int\int_{Q_T}f_\epsilon(u_\epsilon)u^2_{\epsilon xxx}\psi^2 + 2\int\int_{Q_T}|\dots|\Big)\end{aligned} \tag{4.16}$$

The dots contain terms of the form, omitting the epsilons from the notation,

$$f(u)u_3u_i\psi_j\psi_k, \quad i+j+k=3,\ i\le 2, \tag{4.17}$$

where subscripts denote again numbers of x-derivatives. All these terms are supported between $x=\tau$ and $x=\tau+\delta$, and can be estimated by Young's inequality:

$$|f(u)u_3u_i\psi_j\psi_k| \le \delta f(u)u_3^2\psi^2 + C(\delta)f(u)u_i^2\frac{\psi_j^2\psi_k^2}{\psi^2}. \tag{4.18}$$

Here we note that we may adjust the choice of the cut-off function in such a way that, in addition to (3.6), also

$$\frac{\psi_j^2\psi_k^2}{\psi^2} \le \frac{C}{\delta^{j+k}}. \tag{4.19}$$

The estimate (4.16) may be rewritten as

$$\begin{aligned}\zeta^\epsilon_{\lambda+1}(T) \le & \zeta^\epsilon_\lambda(T)\Big(\int_{\Omega_T}(u_\epsilon\psi)_x^2 + (2+\delta)\int\int_{Q_T} f_\epsilon(u_\epsilon)u_{\epsilon xxx}^2\psi^2 \\ & + C(\delta)\int\int_{Q_T(\tau,\delta)}(\dots)\Big),\end{aligned} \tag{4.20}$$

the dots containing the lower order terms on the right hand side of (4.18).

In (4.20) we cannot take the limit $\epsilon\to 0$, because the first two terms on the right hand side are not under control, cf. (2.6). Therefore we rewrite the highest order term using (2.5) with $g=1$,

$$\int\int_{Q_T} f_\epsilon(u_\epsilon)u_{\epsilon xxx}^2\psi^2 = -\frac{1}{2}\int_{\Omega_T}(u_\epsilon\psi)_x^2 + \frac{1}{2}\int_{\Omega_0}(u_\epsilon\psi)_x^2 - 2\int\int_{Q_T}(\dots),$$

where the dots contain again terms of the form (4.17). These terms are controlled as before by (4.18), so we derive that

$$\begin{aligned}(1-\delta)\int\int_{Q_T} f_\epsilon(u_\epsilon)u_{\epsilon xxx}^2\psi^2 = -\frac{1}{2}\int_{\Omega_T}(u_\epsilon\psi)_x^2 + \frac{1}{2}\int_{\Omega_0}(u_\epsilon\psi)_x^2 \\ C(\delta)\int\int_{Q_T(\tau,\delta)}(\dots),\end{aligned}$$

but now the dots contain the same lower order terms as in (4.20). With this estimate it follows from (4.20) that, with a new constant,

$$\zeta^\epsilon_{\lambda+1}(T) \le \zeta^\epsilon_\lambda(T)\Big(\frac{2+\delta}{1-\delta}\int_{\Omega_0}(u_\epsilon\psi)^2_x + C(\delta)\int\int_{Q_T(\tau,\delta)}(\dots)\Big), \tag{4.21}$$

in which the dots are the same as in (4.20) and contain only terms which converge as $\epsilon \to 0$. Note that also the first term on the right hand side of (4.20) has dropped out with a negative sign. Thus we may now take the limit and drop the epsilons in (4.16). By assumption also the first term on the right hand side disappears. Fixing e.g. $\delta = \frac{1}{2}$ we obtain

$$\zeta_{\lambda+1}(T) \le C\zeta_\lambda(T)\int\int_{Q_T(\tau,\delta)}(\dots),$$

the dots containing terms of the form

$$u^n u_i^2 \frac{\psi_j^2\psi_k^2}{\psi^2}, \quad i+j+k=3,\ i\le 2. \tag{4.22}$$

As in the proof of Lemma 2 these terms may be estimated by (4.10). Combining with Lemma 3 it follows that

$$\zeta^\epsilon_{\lambda+1}(T) \le C(n)\Big(\frac{1}{\delta^6}\int\int_{Q_T(\tau-\delta,2\delta)} u^{n+2}\Big)\zeta^\epsilon_\lambda(T), \tag{4.23}$$

i.e., (3.13). Now observe that, writing

$$A(t) = \int_{\Omega_t}(u\psi)^2_x, \quad ||A||_\lambda = \Big(\int_0^T A^\lambda\Big)^{\frac{1}{\lambda}},$$

this estimate is of the form

$$||A||^{\lambda+1}_{\lambda+1} \le C||A||^\lambda_\lambda,$$

whence

$$||A||^{m+p}_{m+p} \le C^m||A||^p_p, \text{ for } m = 0,1,2,3.\dots.$$

Since, for $1 \le p \le 2$,

$$||A||_p \le ||A||_1^{\frac{2-p}{p}}\,||A||_2^{\frac{2p-2}{p}} \le ||A||_1^{\frac{2-p}{p}}\Big(C||A||_1\Big)^{\frac{p-1}{p}} = C^{\frac{p-1}{p}}||A||_1^{\frac{1}{p}},$$

it follows that

$$||A||_{m+p}^{m+p} \le C^{m+p-1}||A||_1.$$

Thus (3.14) follows from (3.13).

The second factor on the right of (3.10) is now controlled by

$$\Psi_{\frac{n+2}{2}}(\tau+\delta),T) \le C(f(T)-\tau-\delta)_+^{n+2}\zeta_{\frac{n+2}{2}}(T)$$
$$\le (f(T)-\tau-\delta)_+^{n+2}\zeta_1(T)\Big(\frac{C(n)}{\delta^6}\int\int_{Q_T(\tau-\delta,2\delta)} u^{n+2}\Big)^{\frac{n}{2}},$$

in which, in view of (3.7) and (3.8), also

$$\zeta_1(T) \le \frac{C(n)}{\delta^6}T\int\int_{Q_T(\tau-\delta,2\delta)} u^{n+2}.$$

Combining the control on both factors on the right hand side of (3.10), and taking all integrals over the larger set $Q_T(\tau-\delta,2\delta)$, we arrive at (3.16) with $I_T(\tau-\delta)$ instead of $I_T(\tau)$. Changing by $\tau'=\tau-\delta$, $\delta'=2\delta$, the proof of Lemma 4 is complete.

Proof of Lemma 5. As explained in the beginning of Section 3, we apply Lemma 1 to

$$J_T(\tau) = T^{\frac{2}{7n+12}}(f(T)-\tau_1)^{\frac{2(n+2)}{7n+12}}I_T(\tau)^{\frac{n}{7n+12}},$$

which satisfies the assumption of Lemma 1 with $\theta = (\frac{C(n)}{1+C(n)})^{\frac{n}{7n+12}}$. It follows that

$$f(T) \le \tau_1 + \frac{J_T(\tau_1)}{1-\theta} = \tau_1 + C(n)T^{\frac{2}{7n+12}}(f(T)-\tau_1)^{\frac{2(n+2)}{7n+12}}I_T(\tau)^{\frac{n}{7n+12}},$$

with a new constant $C(n)$, which implies (3.17).

Proof of Lemma 6. We go back to (3.8). Using the inhomogeneous (4.14) with $C_2>0$, $i=0$, $m=3$, $\alpha=\beta=2$, $\gamma=\frac{2}{n+2}$ and $\theta=\frac{n+1}{n+7}$, concerning the second term on the right hand of (3.8), we have

$$\int_{\Omega_t(\tau,\delta)} u^{n+2} = \int_{\Omega_t(\tau,\delta)} (u^{\frac{n+2}{2}})^2$$
$$\le C(n)\Big(\int_{\Omega_t^+(\tau,\delta)} (u^{\frac{n+2}{2}})_{xxx}^2\Big)^{\frac{n+1}{n+7}}\Big(\int_{\Omega_t(\tau,\delta)} u\Big)^{\frac{6(n+2)}{n+7}} + \frac{C(n)}{\delta^{n+1}}\Big(\int_{\Omega_t(\tau,\delta)} u\Big)^{n+2}.$$

This implies that

$$\int\int_{Q_T(\tau,\delta)} u^{n+2} \leq C(n)\Big(\int\int_{Q_T^+(\tau,\delta)} (u^{\frac{n+2}{2}})^2_{xxx}\Big)^{\frac{n+1}{n+7}} \Big(\int_0^T (\int_{\Omega_t(\tau,\delta)} u)^{n+2}\Big)^{\frac{6}{n+7}}$$
$$+ \frac{C(n)}{\delta^{n+1}} \int_0^T \Big(\int_{\Omega_t(\tau,\delta)} u\Big)^{n+2} \leq (\text{since } \int_{\Omega_t} u = \int_\Omega u_0 = K),$$
$$\leq C(n)\Big(\int\int_{Q_T^+(\tau,\delta)} (u^{\frac{n+2}{2}})^2_{xxx}\Big)^{\frac{n+1}{n+7}} (K^{n+2}T)^{\frac{6}{n+7}} + \frac{C(n)}{\delta^{n+1}} K^{n+2}T.$$

We may now estimate the right hand side of (3.8) by

$$\frac{C(n)}{\delta^6}\Big(\int\int_{Q_T^+(\tau,\delta)} U_0\Big)^{\frac{n+1}{n+7}} (K^{n+2}T)^{\frac{6}{n+7}} + \frac{C(n)}{\delta^{n+7}} K^{n+2}T$$
$$\leq \epsilon \int\int_{Q_T^+(\tau,\delta)} U_0 + (C(n) + C(n)^{\frac{n+7}{6}} C(\epsilon)) \frac{1}{\delta^{n+7}} K^{n+2}T.$$

Using the same iteration procedure as in the proof of Lemma 3 we can take $\epsilon = 0$. This completes the proof of Lemma 6.

Proof of Theorem 1. We prove (3.19), from which Theorem 1 follows as explained at the end of Section 3. We apply again (4.14) with $C_2 = 0$, $i = 0$, $m = 3$, $\alpha = \beta = 2$, $\gamma = \frac{2}{n+2}$ and $\theta = \frac{n+1}{n+7}$ to obtain that

$$\int_{\Omega_t(\tau+\delta)} u^{n+2} \leq C(n)(\int_{\Omega(\tau+\delta)} (u^{\frac{n+2}{2}})^2_{xxx})^{\frac{n+1}{n+7}} (\int_{\Omega(\tau+\delta)} u)^{\frac{6(n+2)}{n+7}},$$

whence, using Hölder's inequality,

$$I_T(\tau+\delta) \leq C(n)\Big(\int\int_{Q_T^+(\tau+\delta)} (u^{\frac{n+2}{2}})^2_{xxx}\Big)^{\frac{n+1}{n+7}} \Big(\int_0^T \Big(\int_{\Omega_t(\tau+\delta)} u\Big)^{n+2}\Big)^{\frac{6}{n+7}}$$
$$\leq C(n) K^{\frac{6(n+2)}{n+7}} T^{\frac{6}{n+7}} \Big(\int\int_{Q_T(\tau+\delta)} U_0\Big)^{\frac{n+1}{n+7}}$$

(in view of (3.18))

$$\leq C(n) K^{\frac{6(n+2)}{n+7}} T^{\frac{6}{n+7}} (K^{n+2} \frac{1}{\delta^{n+7}} T)^{\frac{n+1}{n+7}} = C(n) K^{n+2} T \frac{1}{\delta^{n+1}}.$$

This completes the proof of Theorem 1.

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