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FUNCTIONAL CALCULI FOR LINEAR OPERATORS IN VECTOR-VALUED L^p-SPACES VIA THE TRANSFERENCE PRINCIPLE

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Abstract. Let $-A$ be the generator of a bounded C_0-group or of a positive contraction semigroup, respectively, on $L^p(\Omega,\mu,Y)$, where (Ω,μ) is measure space, Y is a Banach space of class $\mathcal{HT}$ and $1<p<\infty$. If $Y=\mathbb{C}$, it is shown by means of the transference principle due to Coifman and Weiss that A admits an H^∞-calculus on each double cone $C_\theta=\{\lambda\in\mathbb{C}\backslash\{0\}:|\arg\lambda\pm\pi/2|<\theta\}$, where $\theta>0$ and on each sector $\Sigma_\theta=\{\lambda\in\mathbb{C}\backslash\{0\}:|\arg\lambda|<\theta\}$ with $\theta>\pi/2$, respectively. Several extensions of these results to the vector-valued case $L^p(\Omega,\mu,Y)$ are presented. In particular, let $-A$ be the generator of a bounded group on a Banach spaces of class $\mathcal{HT}$. Then it is shown that A admits an H^∞-calculus on each double cone C_θ, $\theta>0$, and that $-A^2$ admits an H^∞-calculus on each sector Σ_θ, where $\theta>0$. Applications of these results deal with elliptic boundary value problems on cylindrical domains and on domains with non smooth boundary.

1. Introduction. Let X be a complex Banach space and let A be a closed, linear, densely defined operator in X. If $G\subset\mathbb{C}$ is open, then $H^\infty(G)$ denotes the space of all functions $f:G\to\mathbb{C}$ which are holomorphic and bounded. Equipped with the sup-norm $|\cdot|_\infty$, $H^\infty(G)$ is a Banach algebra. Given $G\subset\mathbb{C}$, A is said to *admit an H^∞-calculus* for G, if the spectrum $\sigma(A)\subset\overline{G}$ and there is a continuous algebra homomorphism $\Phi:H^\infty(G)\to\mathcal{B}(X)$ such that the functions $f_\mu(\lambda):=(\mu-\lambda)^{-1}$, $\mu\notin\overline{G}$, are mapped to $\Phi f_\mu=(\mu-A)^{-1}$. This definition was introduced by McIntosh [23] in 1986. We refer to [9] for detailed information on this topic.

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Obviously, the functional calculus for normal operators A in a Hilbert space X implies that such operators admit an H^∞-calculus for G, whenever $\overline{G} \supset \sigma(A)$. Based on the theory of unitary dilations, Foias and Nagy [18] developed an H^∞-calculus for $G = \mathbb{D}$, the unit disk, whenever $|A| \leq 1$, and via the Cayley transform for $G = \mathbb{C}_+$, the positive halfplane, whenever A is m-accretive.

On the other hand, if the Banach space X is arbitrary and A is bounded, then the Dunford calculus implies that A admits an H^∞-calculus for G, whenever $G \supset \sigma(A)$. However, in the case of unbounded operators in non-Hilbert spaces such results are rare and, generally speaking, difficult to prove. Still, affirmative answers are available for important classes of operators in certain Banach spaces.

To mention one important example, consider elliptic partial differential operators on $L^p(\Omega, \mathbb{R}^n)$ of Agmon-Douglis-Nirenberg type. If the boundary of $\Omega \subset \mathbb{R}^n$ is C^∞ and if the coefficients of the differential operator as well as of the boundary operators are C^∞ too, Duong [15] proved that such operators admit an H^∞-calculus, hereby extending a famous result of Seeley [26], [27] on the boundedness of the imaginary powers of the operators under consideration. For the special case $\Omega = \mathbb{R}^n$, Amann, Hieber and Simonett [2] were able to relax the assumptions on the coefficients of the principle part of the operator to bounded and uniformly Hölder continuous; see also a related result of Prüss and Sohr [25] on bounded imaginary powers of second order operators with Dirichlet boundary conditions. Recently, Duong and Simonett [16] proved that, again for the case $\Omega = \mathbb{R}^n$, the result remains true if the coefficients of the principal part are only bounded and uniformly continuous.

In this paper we deal with functional calculi in a more abstract framework. More precisely, we consider two special classes of operators $-A$ in spaces $X = L^p(\Omega, \mu, Y)$, where (Ω, μ) is a measure space, Y is a Banach space and $1 < p < \infty$, namely

(a) generators $-A$ of bounded C_0-groups T on X and
(b) generators $-A$ of positive contraction C_0-semigroups T on X.

We note that results on H^∞-calculus in vector-valued L^p spaces are important in connection with arguments of Dore-Venni type (see [13], [6], [24]). In fact, in sections 5 and 6 below, we study regularity properties for solutions of elliptic boundary value problems on non smooth domains by means of functional calculi techniques in vector-valued spaces.

For the case $Y = \mathbb{C}$, the transference principle of Coifman and Weiss

[7] is available for the class of operators A described under a) and b). It was used by them in particular to prove the boundedness of the imaginary powers of A. In this connection the theory of diffusion semigroups on L^p-spaces due to Stein [28] has to be mentioned. For the generators $-A$ of such semigroups, Stein developed a certain functional calculus which allowed for the first time to prove the boundedness of the imaginary powers of elliptic differential operators in L^p-spaces by means of abstract arguments. For the same class of operators, Cowling [8] later extended Stein's result by means of the transference principle and interpolation theory and obtained an H^∞-calculus on $L^p(\Omega,\mu)$. Duong [14] showed that Cowling's result remains true for arbitrary negative generators of positive contraction semigroups on $L^p(\Omega,\mu)$. In particular, he obtained an estimate of the form

$$|f(A)|_{\mathcal{B}(L^p(\Omega,\mu))} \le C(\theta)|f|_\infty, \qquad f \in H^\infty(\Sigma_\theta),$$

where $C(\theta)$ is a constant depending on θ and $\Sigma_\theta = \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg\lambda| < \theta\}$ with $\theta > \pi/2$.

It is the main purpose of this paper to prove generalizations and extensions of these results to the case of

i) generators of bounded groups on $L^p(\Omega,\mu)$ and to

ii) vector-valued L^p-spaces.

In order to do so, we apply the transference principle of Coifman and Weiss and obtain in case a) an $H_1^\infty(\mathbb{R})$-calculus, which in particular implies an H^∞-calculus for each double cone

$$G = C_\theta, \quad \theta > 0, \quad \text{where } C_\theta = \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg\lambda \pm \pi/2| < \theta\}.$$

For the case (b) we obtain an $H_1^\infty(\mathbb{C}_+)$-calculus, which also implies an H^∞-calculus for each sector

$$G = \Sigma_\theta, \quad \theta > \pi/2, \quad \text{where } \Sigma_\theta = \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg\lambda| < \theta\}.$$

Observe that related functional calculi as the Hörmander and Davies-Helffer-Sjöstrand calculi are only defined for generators of holomorphic semigroups of angle $\pi/2$. Hence our approach allows to investigate different classes of functions and operators as compared to ones examined in [20], [11] and [12]. Moreover, even in case b) the approach presented allows a larger class of admissable functions and leads to stronger estimates as compared to the results in [14]. This program is carried out in Sections 2 and 3.

Extensions of these results to the vector-valued case are closely related to the UMD-property of Y, or equivalently to the boundedness of the Hilbert transform in $L^2(\mathbb{R}, Y)$; i.e., to Banach spaces of class $\mathcal{HT}$. In fact, it is known that $A = -\frac{d}{dt}$ admits an H^∞-calculus on $L^p(\mathbb{R}, Y)$ for $G = \Sigma_\theta$, $\theta > \pi/2$, if and only if Y belongs to the class $\mathcal{HT}$ (see [24]).

Combining the vector-valued Coifman-Weiss inequality for groups and the vector-valued Mikhlin theorem we obtain in Theorem 5 an $H_1^\infty(\mathbb{R})$-calculus for generators of bounded groups on $L^p(\Omega, \mu, Y)$, provided the Banach space Y is of class $\mathcal{HT}$. As a consequence we obtain H^∞-calculi for generators $-A$ of bounded groups acting on Banach spaces of class $\mathcal{HT}$ for each double cone C_θ, $\theta > 0$ and for $-A^2$ on each sector Σ_θ, where $\theta > 0$. For the semigroup case, the situation is quite different. Until now, the validity of the transference principle in $L^p(\Omega, \mu, Y)$ is only known for semigroups of the form $\mathcal{T} = TId_Y$, where T is a positive contraction semigroup on $L^p(\Omega, \mu)$ (see [6]). Hence, we have to restrict ourselves here also to this situation.

The last two sections of this paper are devoted to applications of our results to elliptic boundary value problems on domains which are unbounded and do not necessarily possess a smooth boundary. Via the H^∞-calculus method, we first establish an existence and uniqueness result as well as an explicit representation formula of the solution. Second, combining these results with Dore-Venni type arguments, we deduce maximal regularity properties of the solution in L^p-spaces.

Throughout this paper we denote by $D(A)$, $N(A)$, $R(A)$, $\sigma(A)$, $\rho(A)$ the domain, kernel, range, spectrum and resolvent set of A and $\mathcal{B}(X)$ denotes the space of bounded, linear operators on X.

2. A functional calculus for generators of positive contraction semigroups on L^p-spaces. Let $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a bounded C_0-semigroup on a Banach space X with $M := \sup\{|T(t)| : t \geq 0\}$ and let $-A$ denote its infinitesimal generator. We first recall the well known Phillips calculus for such operators. For this purpose let $BV(\mathbb{R}_+)$ be the space of all functions $b : \mathbb{R}_+ \to \mathbb{C}$ of bounded variation, which are normalized by left-continuity and by $b(0) = 0$. The space $BV(\mathbb{R}_+)$ becomes a Banach space when normed by the total variation $Var\,|_0^\infty$ and even a Banach algebra with unit, where the multiplication on $BV(\mathbb{R}_+)$ is defined as the convolution

$$(a \cdot b)(t) := \int_0^t a(t-s)db(s) = \int_0^t b(t-s)da(s), \quad t \geq 0. \tag{2.1}$$

The unit is the Heaviside function $e(t)$ defined by $e(t) = 1$ for $t > 0$ and $e(0) = 0$. The Laplace transform

$$f(\lambda) = \int_0^\infty e^{-\lambda t} db(t), \quad Re\lambda > 0, \tag{2.2}$$

induces an algebra homomorphism $b \mapsto f = \widehat{db}$ from $BV(\mathbb{R}_+)$ to $H^\infty(\mathbb{C}_+)$, the Banach algebra of all functions $f : \mathbb{C}_+ \to \mathbb{C}$ which are holomorphic on $\mathbb{C}_+$ and uniformly bounded; the multiplication on $H^\infty(\mathbb{C}_+)$ being pointwise multiplication. The estimate

$$||f||_\infty := \sup_{Re\lambda > 0} |f(\lambda)| \leq Var\, b|_0^\infty, \quad b \in BV(\mathbb{R}_+), \tag{2.3}$$

shows that $\widehat{\ }$ is continuous. Moreover, it is injective by the uniqueness theorem of Laplace transforms. The range of $: \widehat{\ }$ will be denoted by $\widehat{BV}(\mathbb{C}_+)$. It is well known that $\widehat{BV}(\mathbb{C}_+)$ is dense in $H^\infty(\mathbb{C}_+)$ with respect to uniform convergence on compact subsets; i.e., local uniform convergence, but that $\widehat{BV}(\mathbb{C}_+) \neq H^\infty(\mathbb{C}_+)$.

Formula (2.2) is the basis for the Phillips calculus which is defined by the formula

$$f(A) = \int_0^\infty T(t) db(t), \quad b \in BV(\mathbb{R}_+). \tag{2.4}$$

Equation (2.4) yields the algebra homomorphism $\Phi : \widehat{BV}(\mathbb{C}_+) \to \mathcal{B}(X)$ or equivalently $\hat{\Phi} : BV(\mathbb{R}_+) \to \mathcal{B}(X)$ via

$$\hat{\Phi}(b) = \Phi(\widehat{db}) = \int_0^\infty T(t) db(t), \tag{2.5}$$

which is continuous by the estimate

$$|f(A)|_{\mathcal{B}(X)} \leq M \cdot Var\, b|_0^\infty, \quad b \in BV(\mathbb{R}_+). \tag{2.6}$$

Obviously, the functions $e_\mu(t) = \int_0^t e^{-\mu s} ds$, $Re\mu > 0$, belong to $BV(\mathbb{R}_+)$, and

$$f_\mu(\lambda) := \widehat{de_\mu}(\lambda) = (\lambda + \mu)^{-1}, \quad Re\lambda > 0,$$

yields

$$f_\mu(A) = (\mu + A)^{-1}, \quad Re\mu > 0. \tag{2.7}$$

Therefore, the Phillips calculus is consistent with the Dunford calculus.

Example 1. For the special case of $X = L^1(\mathbb{R})$ and $(T_0(t)u)(\xi) = u(\xi - t)$ being the translation group on $L^1(\mathbb{R})$, we have $A_0 = d/d\xi$ and

$$(f(A_0)u)(\xi) = \int_0^\infty (T_0(t)u)(\xi)db(t) = \int_0^\infty u(\xi - t)db(t), \tag{2.8}$$

is the convolution operator with kernel $db(t)$. Since its norm on $L^1(\mathbb{R})$ equals $Var\, b|_0^\infty$, this example shows that (2.6) cannot be improved, in general.

The situation for the translation group on $X = L^p(\mathbb{R})$, $1 < p < \infty$, is different. In fact, taking Fourier transforms in (2.8) we have

$$(f(A_0)u)^\sim(\rho) = f(i\rho)\tilde{u}(\rho), \quad \rho \in \mathbb{R}, \tag{2.9}$$

i.e., the operator $f(A_0)$ is represented by the Fourier multiplier $f(i\rho)$. Therefore, for $p = 2$ estimate (2.6) can be improved to

$$|f(A_0)|_{\mathcal{B}(L^2(\mathbb{R}))} \leq ||f||_\infty, \tag{2.10}$$

while for $p \in (1, \infty)$ we obtain by Mikhlin's theorem

$$|f(A_0)|_{\mathcal{B}(L^p(\mathbb{R}))} \leq C_p ||f||_{1,\infty}, \tag{2.11}$$

where C_p denotes a constant only depending on p and $||\cdot||_{1,\infty}$ is defined by

$$||f||_{1,\infty} := \sup_{Re\lambda > 0} \{|f(\lambda)| + |\lambda f'(\lambda)|\}. \tag{2.12}$$

Observe that (2.10) holds for all $f \in \widehat{BV}(\mathbb{C}_+)$, while (2.11) makes sense only for $f \in \widehat{BV}(\mathbb{C}_+)$ satisfying $||f||_{1,\infty} < \infty$.

Next we recall the transference principle for positive contraction C_0-semigroups on L^p-spaces and the Coifman-Weiss inequality.

Theorem 1. (Coifman and Weiss [7]). *Let (Ω, μ) be a measure space and let T be a positive contraction C_0-semigroup on $L^p(\Omega, \mu)$, where $1 < p < \infty$. Let $-A$ be the generator of T. Then for each $f \in \widehat{BV}(\mathbb{C}_+)$,*

$$|f(A)|_{\mathcal{B}(L^p(\Omega,\mu))} \leq |f(A_0)|_{\mathcal{B}(L^p(\mathbb{R}))}, \tag{2.13}$$

where $A_0 = d/d\xi$ denotes the generator of the translation group on $L^p(\mathbb{R})$.

Combining the Coifman-Weiss inequality (2.13) with estimates (2.10) or (2.11) which result from Fourier multiplier theory we obtain an extension of the Phillips calculus for generates of positive contraction semigroups on L^p-spaces. For this purpose we define the Banach algebra $H_1^\infty(\mathbb{C}_+)$ by

$$H_1^\infty(\mathbb{C}_+) = \{f \in H^\infty(\mathbb{C}_+) : ||f||_{1,\infty} < \infty\},$$

where $||f||_{1,\infty}$ is given by (2.12). Observe that neither $H_1^\infty(\mathbb{C}_+) \subset \widehat{BV}(\mathbb{C}_+)$ nor $\widehat{BV}(\mathbb{C}_+) \subset H_1^\infty(\mathbb{C}_+)$ hold. However, $\widehat{BV}(\mathbb{C}_+) \cap H_1^\infty(\mathbb{C}_+)$ is dense in $H_1^\infty(\mathbb{C}_+)$ with respect to local uniform convergence as part (i) of the following lemma shows.

Lemma 1. *Let $\varphi \in H_1^\infty(\mathbb{C}_+)$ be defined by $\varphi(\lambda) = \lambda(1+\lambda)^{-2}$. Fix $f_0 \in H_1^\infty(\mathbb{C}_+)$ and for $\varepsilon \geq 0$ let $f_\varepsilon(\lambda) = f_0(\lambda)\varphi(\lambda)^\varepsilon$. Then the following holds.*

(i) *For each $\varepsilon > 0$ there exists a function $a_\varepsilon \in L^1(\mathbb{R}_+)$ such that*

$$\hat{a}_\varepsilon(\lambda) = f_\varepsilon(\lambda), \quad Re\lambda > 0.$$

(ii) *For each $\varepsilon \geq 0$ there exists $c_\varepsilon \in L^2(\mathbb{R}_+)$, absolutely continuous on $(0,\infty)$ with $t\dot{c}_\varepsilon \in L^2(\mathbb{R}_+)$ such that*

$$\hat{c}_\varepsilon(\lambda) = g_\varepsilon(\lambda) := f_\varepsilon(\lambda)(1+\lambda)^{-1}, \quad Re\lambda > 0, \quad and \quad |c_\varepsilon|_2 + |t\dot{c}_\varepsilon|_2 \leq C||f_0||_{1,\infty}, \tag{2.14}$$

where the constant $C > 0$ is independent of $\varepsilon \in [0,1]$.

(iii) *As $\varepsilon \to 0+$ we have $c_\varepsilon \to c_0$ and $t\dot{c}_\varepsilon \to t\dot{c}_0$ in $L^2(\mathbb{R}_+)$.*

Proof. (i) The assumption $f_0 \in H_1^\infty(\mathbb{C}_+)$ and the definition of $\varphi(\lambda)$ imply that $f_\varepsilon \in H^\infty(\mathbb{C}_+)$, $||f_\varepsilon||_\infty \leq ||f_0||_\infty$ and that $||f_\varepsilon||_{1,\infty} \leq (1+\varepsilon)||f_0||_{1,\infty}$. Moreover, $f'_\varepsilon \in H^1(\mathbb{C}_+)$, where $H^p(\mathbb{C}_+)$, $p > 0$, denotes the Hardy space of all functions $g : \mathbb{C}_+ \to \mathbb{C}$ such that

$$\sup_{\sigma>0} \int_{-\infty}^{\infty} |g(\sigma + i\rho)|^p d\rho < \infty.$$

Therefore, by Hardy's inequality (cf. Duren [17]), there exist functions $a_\varepsilon \in L^1(\mathbb{R}_+)$ such that $\widehat{a_\varepsilon}(\lambda) = f_\varepsilon(\lambda)$, $Re\lambda > 0$.

(ii) Note that $g_\varepsilon(\lambda) = f_\varepsilon(\lambda)(1+\lambda)^{-1}$ belongs to the Hardy space $H^2(\mathbb{C}_+)$ and that $||g_\varepsilon||_2 \leq ||f_\varepsilon||_\infty \sqrt{\pi} \leq ||f_0||_\infty \sqrt{\pi}$. The Paley-Wiener theorem implies that there is $c_\varepsilon \in L^2(\mathbb{R}_+)$ such that $\hat{c}_\varepsilon(\lambda) = g_\varepsilon(\lambda)$, $Re\lambda > 0$, and that $|c_\varepsilon|_2 \leq ||g_\varepsilon||_2/\pi \leq ||f_0||_\infty/\sqrt{\pi}$. Moreover, by means of the identity

$$\lambda g_\varepsilon'(\lambda) = \lambda f_\varepsilon'(\lambda)(1+\lambda)^{-1} - g_\varepsilon(\lambda)\lambda(1+\lambda)^{-1} \tag{2.15}$$

we also have $\lambda g_\varepsilon' \in H^2(\mathbb{C}_+)$ and $||\lambda g_\varepsilon'||_2 \leq ||f_\varepsilon||_{1,\infty}\sqrt{\pi}$. The Paley-Wiener theorem implies that there is a function $d_\varepsilon \in L^2(\mathbb{R}_+)$ such that $\hat{d}_\varepsilon(\lambda) = \lambda g_\varepsilon'(\lambda)$ and $|d_\varepsilon|_2 \leq ||\lambda g_\varepsilon'(\lambda)||_2/\pi \leq ||f_\varepsilon||_{1,\infty}/\sqrt{\pi} \leq (1+\varepsilon)||f_0||_{1,\infty}/\sqrt{\pi}$. The uniqueness theorem of Laplace transforms and the equation

$$\widehat{tc_\varepsilon}(\lambda) = -g_\varepsilon'(\lambda) = \frac{-1}{\lambda}\hat{d}_\varepsilon(\lambda), \quad Re\lambda > 0,$$

yield

$$tc_\varepsilon(t) = -\int_0^t d_\varepsilon(s)ds \quad \text{ for a.a. } s > 0.$$

Hence $c_\varepsilon(t)$ is absolutely continuous on $(0,\infty)$, differentiable a.e. and

$$t\dot{c}_\varepsilon(t) = -c_\varepsilon(t) - d_\varepsilon(t) \text{ for a.a. } t > 0.$$

Thus $t\dot{c}_\varepsilon \in L^2(\mathbb{R}_+)$ and we have

$$|c_\varepsilon|_2 + |t\dot{c}_\varepsilon|_2 \leq 2|c_\varepsilon|_2 + |d_\varepsilon|_2 \leq 3(1+\varepsilon)||f_0||_{1,\infty}/\sqrt{\pi}.$$

This implies (2.14).

(iii) Since $\varphi(\lambda)^\varepsilon \to 1$ locally uniformly on $\mathbb{C}_+$ as $\varepsilon \to 0$, we obtain $f_\varepsilon(\lambda) \to f_0(\lambda)$, $g_\varepsilon(\lambda) \to g_0(\lambda)$ locally uniformly on $\mathbb{C}_+$. Moreover, (2.15) and the fact that

$$\lambda f_\varepsilon'(\lambda) = \lambda f_0'(\lambda)\varphi(\lambda)^\varepsilon + \varepsilon f_\varepsilon(\lambda) \cdot \frac{1-\lambda}{1+\lambda} \to \lambda f_0'(\lambda), \quad Re\lambda > 0,$$

imply also that $\lambda g_\varepsilon'(\lambda) \to \lambda g_0'(\lambda)$ locally uniformly on $\mathbb{C}_+$. Therefore the bounds

$$|g_\varepsilon(\lambda)| \leq ||f_0||_\infty |1+\lambda|^{-1}, \ |\lambda g_\varepsilon'(\lambda)| \leq (1+\varepsilon)||f_0||_{1,\infty}|1+\lambda|^{-1}, \quad Re\lambda > 0$$

imply via Lebesgue's dominated convergence theorem that

$$g_\varepsilon \to g_0 \quad \text{and} \quad \lambda g_\varepsilon' \to \lambda g_0' \quad \text{in} \quad H^2(\mathbb{C}_+).$$

Hence

$$c_\varepsilon \to c_0 \quad \text{and} \quad t\dot{c}_\varepsilon \to t\dot{c}_0 \quad \text{in} \quad L^2(\mathbb{R}_+),$$

by the Paley-Wiener theorem. □

Fix now a function $f_0 \in H_1^\infty(\mathbb{C}_+)$ and let f_ε, g_ε, a_ε, c_ε be defined as in Lemma 1. Then $\hat{c}_\varepsilon(\lambda) = \hat{a}_\varepsilon(\lambda)(1+\lambda)^{-1}$. Hence by the Phillips calculus

$$g_\varepsilon(A) = f_\varepsilon(A)(I+A)^{-1},$$

which implies for $x \in D(A) \cap R(A)$

$$\begin{aligned} f_\varepsilon(A)x = g_\varepsilon(A)(x+Ax) &= \int_0^\infty T(t)(x+Ax)c_\varepsilon(t)dt \\ &= \int_0^1 T(t)(x+Ax)c_\varepsilon(t)dt - \int_1^\infty \dot{T}(t)(x+A^{-1}x)c_\varepsilon(t)dt \\ &= \int_0^1 T(t)(x+Ax)c_\varepsilon(t)dt + \int_1^\infty T(t)(x+A^{-1}x)\dot{c}_\varepsilon(t)dt \\ &\quad + T(1)(x+A^{-1}x)c_\varepsilon(1). \end{aligned}$$

Since $c_\varepsilon \to c_0$ in $L^2(\mathbb{R}_+)$ and $t\dot{c}_\varepsilon \to t\dot{c}_0$ in $L^2(\mathbb{R}_+)$ by Lemma 1, we obtain $\dot{c}_\varepsilon \to \dot{c}_0$ in $L^1(1,\infty)$ as well as $c_\varepsilon(t) \to c_0(t)$ for each $t > 0$. Therefore, the right hand side of the last identity converges as $\varepsilon \to 0$ with limit

$$\begin{aligned} f_0(A)x := &\int_0^1 T(t)(x+Ax)c_0(t)dt + \int_1^\infty T(t)(x+A^{-1}x)\dot{c}_0(t)dt + \\ &+ T(1)(x+A^{-1}x)c_0(1), \quad x \in D(A)\cap R(A). \end{aligned} \tag{2.16}$$

This equation serves as a definition of $f_0(A)$ on the subset $D(A) \cap R(A)$ of X, where $f_0 \in H_1^\infty(\mathbb{C}_+)$ is arbitrary.

Theorem 2. *Let (Ω, μ) be a measure space and let T be a positive contraction C_0-semigroup on $X = L^p(\Omega,\mu)$, where $1 < p < \infty$. Denote its generator by $-A$ and assume that $N(A) = \{0\}$. Then the functional calculus $\Phi : \widehat{BV}(\mathbb{C}_+) \to \mathcal{B}(X)$ extends uniquely to $H_1^\infty(\mathbb{C}_+)$ by (2.16). Moreover,*

$\Phi : H_1^\infty(\mathbb{C}_+) \to \mathcal{B}(X)$ is continuous, i.e., there is a constant C_p depending only on p, such that

$$|f(A)|_{\mathcal{B}(L^p(\Omega,\mu))} \le C_p ||f||_{1,\infty} \quad \text{for all } f \in H_1^\infty(\mathbb{C}_+). \tag{2.17}$$

Furthermore, $(f_n) \subset H_1^\infty(\mathbb{C}_+)$, $\sup_{n\in\mathbb{N}} ||f_n||_{1,\infty} < \infty$, $f_n \to f$ locally uniformly, imply $f_n(A) \to f(A)$ strongly.

Proof. Let $f_0 \in H_1^\infty(\mathbb{C}_+)$ be given. Choose $f_\varepsilon(\lambda) = f_0(\lambda)\varphi(\lambda)^\varepsilon$ as in Lemma 1. Then for each $\varepsilon > 0$, $f_\varepsilon \in H_1^\infty(\mathbb{C}_+) \cap \widehat{BV}(\mathbb{C}_+)$. Let $a_\varepsilon \in L^1(\mathbb{R}_+)$ be such that $\hat{a}_\varepsilon(\lambda) = f_\varepsilon(\lambda)$, $Re\lambda > 0$, according to Lemma 1. Then the operators $f_\varepsilon(A)$ are well defined by the Phillips calculus and by Theorem 1 we obtain

$$|f_\varepsilon(A)|_{\mathcal{B}(L^p(\Omega,\mu))} \le |f_\varepsilon(A_0)|_{\mathcal{B}(L^p(\mathbb{R}))},$$

where $A_0 = d/d\xi$ denotes the negative generator of the translation group $T_0(t)$ on $L^p(\mathbb{R})$. By Example 1 we have

$$|f_\varepsilon(A_0)|_{\mathcal{B}(L^p(\mathbb{R}))} \le C_p||f_\varepsilon||_{1,\infty} \le C_p(1+\varepsilon)||f_0||_{1,\infty}.$$

Hence

$$|f_\varepsilon(A)|_{\mathcal{B}(L^p(\Omega,\mu))} \le C_p||f_0||_{1,\infty}, \quad \varepsilon > 0, \tag{2.18}$$

with a possibly different constant $C_p > 0$.

Next, observe that $X = L^p(\Omega,\mu) = \overline{R(A)}$ (since X is reflexive), $|\lambda(\lambda + A)^{-1}| \le 1$, $\lambda > 0$ and that $N(A) = \{0\}$ by the well known ergodic theorem of C_0-semigroups (cf. [21]). Therefore, $R(A) \cap D(A)$ is dense in X. Let $c_\varepsilon(t)$ be defined according to Lemma 1. Since $c_\varepsilon \to c_0$ in $L^1(0,1)$, $\dot{c}_\varepsilon \to \dot{c}_0$ in $L^1(1,\infty)$ and $c_\varepsilon \to c_0$ in $C[\eta, 1/\eta]$, for each $\eta > 0$ by (iii) of Lemma 1, the identity

$$f_\varepsilon(A)x = \int_0^1 T(t)(x + Ax)c_\varepsilon(t)dt + \int_1^\infty T(t)(x + A^{-1}x)\dot{c}_\varepsilon(t)dt \\ + T(1)(x + A^{-1}x)c_\varepsilon(1)$$

shows that $(f_\varepsilon(A)x)$ is Cauchy for each $x \in R(A) \cap D(A)$. Thus the Banach-Steinhaus theorem implies that the family $(f_\varepsilon(A))_{\varepsilon>0}$ has a strong limit as $\varepsilon \to 0$, which will be denoted by $f_0(A)$. Then $f_0(A) \in \mathcal{B}(L^p(\Omega,\mu))$ and by (2.18) we have

$$|f_0(A)|_{\mathcal{B}(L^p(\Omega,\mu))} \le \lim_{\varepsilon\to 0} |f_\varepsilon(A)|_{\mathcal{B}(L^p(\Omega,\mu))} \le C_p||f_0||_{1,\infty}.$$

This completes the construction of Φ in $H_1^\infty(\mathbb{C}_+)$ and proves the estimate (2.17).

Evidently, Φ is an algebra homomorphism on $H_1^\infty(\mathbb{C}_+)$ and it is continuous by (2.17). It is uniquely determined by (2.16), since $R(A) \cap D(A)$ is dense in X, and the function $c_0(t)$ from Lemma 1 is uniquely determined by f_0. To prove the last statement, let $(f_n) \subset H_1^\infty(\mathbb{C}_+)$ be such that $||f_n||_{1,\infty} \leq m < 0$ and $f_n \to f$ locally uniformly on $\mathbb{C}_+$. Then $|f_n(A)|_{\mathcal{B}(X)} \leq mC_p < \infty$ by (2.17). Furthermore, $f_n(A)x \to f(A)x$ for each $x \in D(A) \cap R(A)$ by (2.16), since as in the proof of (iii) of Lemma 1 we obtain $c_n \to c$ and $t\dot{c}_n \to t\dot{c}$ in $L^2(\mathbb{R}_+)$. Therefore the Banach-Steinhaus theorem implies $f_n(A) \to f(A)$ strongly. The proof is complete. □

At this point a remark concerning the assumption $N(A) = \{0\}$ in Theorem 2 seems to be necessary. If $N(A) \neq \{0\}$, then by the ergodic theorem we have the decomposition $X = N(A) \oplus \overline{R(A)}$, which is respected by the semigroup T. Therefore, on $\overline{R(A)}$, Theorem 2 applies and hence the functional calculus can be constructed on $\overline{R(A)}$. On the other hand, on $N(A)$ we have $A = 0$ or equivalently $T(t) \equiv I$, so that here the functional calculus is trivial. Therefore, there is no loss of generality in assuming $N(A) = \{0\}$.

3. A functional calculus for bounded C_0-groups on L^p-spaces. Let $\{T(t)\}_{t\in\mathbb{R}} \subset \mathcal{B}(X)$ be a bounded C_0-group on a Banach space X, set $M := \sup\{|T(t)| : t > \mathbb{R}\}$ and denote by $-A$ the generator of T. The Banach algebra $BV(\mathbb{R})$ is defined as the class of functions of bounded variation on $\mathbb{R}$ normalized by left-continuity and by the condition $b(-\infty) = 0$. The Fourier transform

$$h(\rho) = \int_{-\infty}^{\infty} e^{-i\rho t} db(t), \quad \rho \in \mathbb{R}, \tag{3.1}$$

where $b \in BV(\mathbb{R})$, gives rise to the Phillips functional calculus for bounded C_0-groups by means of the formula

$$h(A) = \int_{-\infty}^{\infty} T(t) db(t), \quad b \in BV(\mathbb{R}), \tag{3.2}$$

which has to be understood in the strong sense. The estimate

$$|h|_\infty \leq Var\, b|_{-\infty}^{\infty}$$

shows that the Fourier transform induces a continuous algebra homomorphism $\tilde{}$ of $BV(\mathbb{R})$ into $L^\infty(\mathbb{R})$, which is also injective by uniqueness of the Fourier transform. Its range will be denoted by $\widetilde{BV}(\mathbb{R})$. The Phillips calculus $\Psi : \widetilde{BV}(\mathbb{R}) \to \mathcal{B}(X)$ or equivalently $\tilde{\Psi} : BV(\mathbb{R}) \to \mathcal{B}(X)$ is given by

$$\Psi(\widetilde{db}) := \tilde{\Psi}(db) := \int_{-\infty}^{\infty} T(t)db(t). \tag{3.3}$$

It is a continuous algebra homomorphism and the estimate

$$|h(A)|_{\mathcal{B}(X)} \leq M \cdot Var\, b|_{-\infty}^{\infty}, \quad b \in BV(\mathbb{R}) \tag{3.4}$$

is easily derived. The connection with the Phillips calculus for semigroups is evident by extending $b \in BV(\mathbb{R}_+)$ by 0 to a function of bounded variation on $\mathbb{R}$ and by observing that then $h(\rho)$ defined by (3.1) is the boundary value $f(i\rho)$ of $f(\lambda)$ given by (2.2).

The transference principle for bounded C_0-groups on L^p-spaces reads as follows.

Theorem 3. (Coifman and Weiss [7]). *Let (Ω, μ) be a measure space and let T be a bounded C_0-group on $L^p(\Omega, \mu)$, where $1 < p < \infty$. Let $-A$ be the generator of T. Then for each $h \in \widetilde{BV}(\mathbb{R})$,*

$$|h(A)|_{\mathcal{B}(L^p(\Omega,\mu))} \leq M^2 |h(A_0)|_{\mathcal{B}(L^p(\mathbb{R}))}, \tag{3.5}$$

where $A_0 = d/d\xi$ denotes the generator of the translation group $T_0(t)$ on $L^p(\mathbb{R})$ and M is given by $M = \sup\{|T(t)|_{\mathcal{B}(L^p(\Omega,\mu))} : t \in \mathbb{R}\}$.

If $h \in \widetilde{BV}(\mathbb{R})$, say $h(\rho) = \widetilde{db}(\rho)$, $b \in BV(\mathbb{R})$, then as in Example 1

$$(h(A_0)u)(\xi) = \int_{-\infty}^{\infty} (T_0(t)u)(\xi)db(t) = \int_{-\infty}^{\infty} u(\xi - t)db(t), \quad \xi \in \mathbb{R}.$$

This means that $h(A_0)$ is the convolution operator on $L^p(\mathbb{R})$ with kernel $db(t)$. Therefore, the Fourier multiplier theorem of Mikhlin implies that $h(A_0) \in \mathcal{B}(L^p(\mathbb{R}))$ and that

$$|h(A_0)|_{\mathcal{B}(L^p(\mathbb{R})} \leq C_p ||h||_{1,\infty}, \quad h \in \widetilde{BV}(\mathbb{R}), \tag{3.6}$$

where $C_p > 0$ is a constant only depending on p. The norm $||\cdot||_{1,\infty}$ is defined by

$$||h||_{1,\infty} := \operatorname{ess\,sup}_{\rho \in \mathbb{R}} \{|h(\rho)| + |\rho h'(\rho)|\}. \tag{3.7}$$

Of course, (3.6) is only meaningful in case $h \in L^\infty(\mathbb{R})$, $h \in H^{1,\infty}_{loc}(\mathbb{R}\backslash\{0\})$, and $||h||_{1,\infty} < \infty$.

Similarly to the case of semigroups in Section 2, Theorem 3 combined with (3.6), (3.7) yield an extension of the Phillips calculus for groups to the Banach algebra

$$H_1^\infty(\mathbb{R}) = \{h \in L^\infty(\mathbb{R}) \cap H^{1,\infty}_{loc}(\mathbb{R}\backslash\{0\}) : ||h||_{1,\infty} < \infty\} \tag{3.8}$$

equipped with the norm $||\cdot||_{1,\infty}$. We first prove an analog of Lemma 1 for the case of the line.

Lemma 2. *Let $\chi \in H_1^\infty(\mathbb{R})$ be defined by*

$$\chi(\rho) = \begin{cases} 1 & |\rho| \le 1 \\ 2 - |\rho| & 1 < |\rho| < 2, \\ 0 & |\rho| \ge 2. \end{cases}$$

Fix $h_0 \in H_1^\infty(\mathbb{R})$ and for $\varepsilon, \rho > 0$ set $h_\varepsilon(\rho) = h_0(\rho)\chi(\varepsilon\rho)(1 - \chi(\rho/\varepsilon))$. Then the following holds.

(i) *For each $\varepsilon > 0$ there exists a function $a_\varepsilon \in L^1(\mathbb{R})$ such that $\tilde{a}_\varepsilon(\rho) = h_\varepsilon(\rho)$, $\rho \in \mathbb{R}$.*

(ii) *For each $\varepsilon \ge 0$ there exists $c_\varepsilon \in L^2(\mathbb{R})$, absolutely continuous on $\mathbb{R}\backslash\{0\}$, with $t\dot{c}_\varepsilon \in L^2(\mathbb{R})$ such that*

$$\tilde{c}_\varepsilon(\rho) = g_\varepsilon(\rho) := h_\varepsilon(\rho)(1 + i\rho)^{-1}, \quad \rho \in \mathbb{R}, \quad \textit{and} \tag{3.9}$$

$$|c_\varepsilon|_2 + |t\dot{c}_\varepsilon|_2 \le C||h_0||_{1,\infty}, \tag{3.10}$$

where the constant $C > 0$ is independent of $\varepsilon \in [0, 1]$.

(iii) *As $\varepsilon \to 0+$, we have $c_\varepsilon \to c_0$ and $t\dot{c}_\varepsilon \to t\dot{c}_0$ in $L^2(\mathbb{R})$.*

Proof. (i) Fix $\varepsilon > 0$. Since $\operatorname{supp} h_\varepsilon \subset \{\rho \in \mathbb{R} : 2\varepsilon \le |\rho| \le 2/\varepsilon\}$ is compact, the assumption $h_0 \in H_1^\infty(\mathbb{R})$ implies that h_ε, $h'_\varepsilon \in L^2(\mathbb{R})$. Hence, thanks to Plancherel's theorem, there is a function $a_\varepsilon \in L^2(\mathbb{R})$ with $ta_\varepsilon \in L^2(\mathbb{R})$ such that $\tilde{a}_\varepsilon(\rho) = h_\varepsilon(\rho)$, $\rho \in \mathbb{R}$. Consequently, we have

$$|a_\varepsilon|_1 = \int_{-1}^1 |a_\varepsilon(t)|dt + \int_{|t|\ge 1} |ta_\varepsilon(t)|\frac{dt}{|t|} \le (|a_\varepsilon|_2 + |ta_\varepsilon|_2)\sqrt{2},$$

which shows that $a_\varepsilon \in L^1(\mathbb{R})$.

(ii) Since $h_0 \in L^\infty$ and $0 \le \chi(\rho) \le 1$ for all $\rho \in \mathbb{R}$, we have

$$|g_\varepsilon(\rho)| \le |h_0(\rho)|(1+\rho^2)^{-1/2} \le ||h_0||_{1,\infty}(1+\rho^2)^{-1/2}, \quad \rho \in \mathbb{R},$$

i.e., $g_\varepsilon \in L^2(\mathbb{R})$. Hence, by Plancherel's theorem, there exist functions $c_\varepsilon \in L^2(\mathbb{R})$ with $\tilde{c}_\varepsilon(\rho) = g_\varepsilon(\rho)$, $\rho \in \mathbb{R}$, satisfying

$$|c_\varepsilon|_2 = \sqrt{2\pi}|h_\varepsilon|_2 \le C||h_0||_{1,\infty}\,, \quad \varepsilon \ge 0. \tag{3.11}$$

On the other hand, the identity

$$\begin{aligned}\rho g_\varepsilon'(\rho) &= \rho h_\varepsilon'(\rho)(1+i\rho)^{-1} - i\rho h_\varepsilon(\rho)(1+i\rho)^{-2} \\ &= (\rho h_0'(\rho) + h_0(\rho)(\varepsilon\rho\chi'(\varepsilon\rho) + (\rho/\varepsilon)\chi'(\rho/\varepsilon) - \frac{i\rho}{1+i\rho})(1+i\rho)^{-1}\end{aligned}$$

implies that $\rho g_\varepsilon'(\rho)$ in $L^2(\mathbb{R})$ as well and that

$$|\rho g_\varepsilon'|_2 \le ||h_0||_{1,\infty}, \quad \varepsilon \ge 0.$$

Applying Plancherel's theorem once more we obtain functions $d_\varepsilon \in L^2(\mathbb{R})$ satisfying

$$\tilde{d}_\varepsilon(\rho) = \rho g_\varepsilon'(\rho), \quad \varepsilon \ge 0, \tag{3.12}$$

and

$$|d_\varepsilon|_2 \le C||h_0||_{1,\infty},\ \varepsilon \ge 0, \tag{3.13}$$

where the constant $C > 0$ is independent of ε. For $\varepsilon > 0$, the support of $h_\varepsilon(\rho)$ is compact and bounded away from zero. Therefore, a_ε, c_ε, d_ε are of class $H^{\infty,2}(\mathbb{R}) = \cap_{m\ge 0} H^{m,2}(\mathbb{R})$. Hence the inversion formula of the Fourier transform implies

$$d_\varepsilon(t) = -(tc_\varepsilon(t))^\cdot = -c_\varepsilon(t) - t\dot{c}_\varepsilon(t), \quad t \in \mathbb{R}. \tag{3.14}$$

Hence $t\dot{c}_\varepsilon \in L^2(\mathbb{R})$ and by (3.11), (3.13) we obtain

$$|c_\varepsilon|_2 + |t\dot{c}_\varepsilon|_2 \le C||h_0||_{1,\infty}, \quad \varepsilon > 0,$$

where the constant $C > 0$ is independent of $\varepsilon > 0$. This proves (ii) for $\varepsilon > 0$.

(iii) As $\varepsilon \to 0+$ the definition of χ implies

$$h_\varepsilon(\rho) \to h_0(\rho),\ g_\varepsilon(\rho) \to g_0(\rho),\ \rho g_\varepsilon'(\rho) \to \rho g_0'(\rho) \ \text{ for a.a. } \rho \in \mathbb{R}.$$

Since $g_\varepsilon(\rho)$ as well as $\rho g_\varepsilon'(\rho)$ are bounded by the L^2-function $(1+\rho^2)^{-1}$, Lebesgue's theorem implies that

$$g_\varepsilon \to g_0 \quad \text{and that} \quad \rho g_\varepsilon' \to \rho g_0' \quad \text{in} \quad L^2(\mathbb{R}) \quad \text{as} \quad \varepsilon \to 0.$$

Plancherel's theorem yields

$$c_\varepsilon \to c_0 \quad \text{and} \quad d_\varepsilon \to d_0 \text{ in } L^2(\mathbb{R}) \text{ as } \varepsilon \to 0.$$

Hence identity (3.14) implies that $c_0 \in L^2(\mathbb{R})$ is absolutely continuous on $\mathbb{R}\backslash\{0\}$ and that

$$d_0(t) = -c_0(t) = t\dot{c}_0(t) \quad \text{a. a. } t \in \mathbb{R}.$$

In particular $t\dot{c}_0 \in L^2(\mathbb{R})$ and (3.10) holds also for $\varepsilon = 0$. The proof is complete. □

Fix now a function $h_0 \in H_1^\infty(\mathbb{R})$ and let a_ε, c_ε, h_ε, g_ε be defined as in Lemma 2. Then $\tilde{c}_\varepsilon(\rho) = \tilde{a}_\varepsilon(\rho)/(1+i\rho)$. Hence the Phillips calculus for bounded C_0-groups yields

$$g_\varepsilon(A) = h_\varepsilon(A)(I+A)^{-1}, \quad \varepsilon > 0,$$

and for $x \in D(A) \cap R(A)$ we obtain

$$\begin{aligned} h_\varepsilon(A)x = g_\varepsilon(A)(x+Ax) &= \int_{-\infty}^{\infty} T(t)(x+Ax)c_\varepsilon(t)dt \\ &= \int_{-1}^{1} T(t)(x+Ax)c_\varepsilon(t)dt - \int_{|t|\geq 1} \dot{T}(t)(x+A^{-1}x)c_\varepsilon(t)dt \\ &= \int_{-1}^{1} T(t)(x+Ax)c_\varepsilon(t)dt + \int_{|t|\geq 1} T(t)(x+A^{-1}x)\dot{c}_\varepsilon(t)dt \\ &\quad + T(1)(x+A^{-1}x)c_\varepsilon(1) - T(-1)(x+A^{-1}x)c_\varepsilon(-1). \end{aligned}$$

Since $c_\varepsilon \to c_0$ and $t\dot{c}_\varepsilon \to t\dot{c}_0$ in $L^2(\mathbb{R})$ as $\varepsilon \to 0$ by Lemma 2, we obtain $c_\varepsilon \to c_0$ in $L^1(-1,1)$, $\dot{c}_\varepsilon \to \dot{c}_0$ in $L^1(\mathbb{R}\backslash[-1,1])$ and $c_\varepsilon(t) \to c_0(t)$ for each $t \neq 0$. Therefore the right hand side of the last identity converges as $\varepsilon \to 0$ with limit

$$\begin{aligned} h_0(A)x := \int_{-1}^{1} T(t)(x+Ax)c_0(t)dt &+ \int_{|t|\geq 1} T(t)(x+A^{-1}x)\dot{c}_0(t)dt \\ + c_0(1)T(1)(x+A^{-1}x) &- c_0(-1)T(-1)(x+A^{-1}x), \\ x \in D(A) \cap R(A). & \end{aligned} \tag{3.15}$$

This equation serves as the definition of $h_0(A)$ on the dense subset $D(A) \cap R(A)$ of X for arbitrary $h_0 \in H_1^\infty(\mathbb{R})$. Combining Theorem 3 with estimate (3.6), which results from the Mikhlin Multiplier theorem, we obtain in the case $X = L^p(\Omega, \mu)$, $1 < p < \infty$, the estimate

$$\begin{aligned} |h_\varepsilon(A)|_{\mathcal{B}(L^p(\Omega,\mu))} &\leq M^2 |h_\varepsilon(A_0)|_{\mathcal{B}(L^p(\mathbb{R}))} \\ &\leq M^2 C_p ||h_\varepsilon||_{1,\infty} \leq ||h_0||_{1,\infty}, \end{aligned} \tag{3.16}$$

which as in proof of Theorem 2 leads to the following result.

Theorem 4. *Let (Ω, μ) be a measure space and let T be a bounded C_0-group on $X = L^p(\Omega, \mu)$, where $1 < p < \infty$. Denote by $-A$ its generator and assume that $N(A) = \{0\}$. Then the functional calculus $\Psi : \widetilde{BV}(\mathbb{R}) \to \mathcal{B}(X)$ defined by (3.3) extends uniquely to $H_1^\infty(\mathbb{R})$ by (3.15). Moreover, $\Psi : H_1^\infty(\mathbb{R}) \to \mathcal{B}(X)$ is continuous, i.e., there is a constant C_p only depending on p such that*

$$|h(A)|_{\mathcal{B}(L^p(\Omega,\mu))} \leq C_p ||h||_{1,\infty} \quad \text{for all } h \in H_1^\infty(\mathbb{R}). \tag{3.17}$$

Furthermore, $(h_n) \subset H_1^\infty(\mathbb{R})$, $||h_n||_{1,\infty} \leq m < \infty$, $h_n(\rho) \to h(\rho)$ for a.a. $\rho \in \mathbb{R}$ imply $h_n(A) \to h(A)$ strongly.

4. H^∞-calculi and vector-valued extensions. We begin this section by showing that the functional calculi established in Theorems 2 and 4 imply in particular the existence of an H^∞-calculus for the operators under consideration.

Consider first the case of negative generators of positive semigroups. To this end, set $\Sigma_\theta = \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg \lambda| < \theta\}$, where $\theta > \pi/2$. Let

$$H^\infty(\Sigma_\theta) = \{f : \Sigma_\theta \to \mathbb{C} \quad \text{bounded and holomorphic}\},$$

equipped with the norm

$$||f||_\infty := \sup\{|f(\lambda)| : \lambda \in \Sigma_\theta\}.$$

Then $H^\infty(\Sigma_\theta)$ is a Banach algebra which is continuously embedded into $H_1^\infty(\mathbb{C}_+)$. In fact, by Cauchy's theorem we obtain

$$f'(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_\lambda} \frac{f(z)}{(z-\lambda)^2} dz, \quad \lambda \in \mathbb{C}_+, \tag{4.1}$$

where Γ_λ denotes the circle around λ with radius $r_\lambda = |\lambda||\cos\theta|$. Formula (4.1) implies that for $f \in H^\infty(\Sigma_\theta)$ we have

$$|f'(\lambda)| \leq ||f||_\infty \cdot \frac{1}{2\pi r_\lambda^2} \cdot 2\pi r_\lambda = ||f||_\infty \cdot r_\lambda^{-1}.$$

Hence

$$|\lambda f'(\lambda)| \leq ||f||_\infty / |\cos\theta|, \quad \lambda \in \mathbb{C}_+. \tag{4.2}$$

Thus $H^\infty(\Sigma_\theta) \hookrightarrow H_1^\infty(\mathbb{C}_+)$ and $||f||_{1,\infty} \leq C(\theta)||f||_\infty$, where the constant $C(\theta)$ depends only on $\theta > \pi/2$. Hence, as a corollary of Theorem 2, we obtain the following result which was proved first by Duong by different techniques.

Corollary 1. (Duong [14]). *Let (Ω, μ) be a measure space and let $-A$ be the generator of a positive contraction C_0-semigroup on $L^p(\Omega, \mu)$, where $1 < p < \infty$. Assume that $N(A) = \{0\}$. Then A admits an H^∞-calculus for each sector Σ_θ, $\theta > \pi/2$.*

We now consider the case of generators of bounded groups. Denote by $C_\theta = \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg\lambda \pm \pi/2| < \theta\}$ the vertical double cone with vertex in 0 and opening angle 2θ. Then $H^\infty(C_\theta)$ is a Banach algebra which is continuously embedded into $H_1^\infty(\mathbb{R})$. Apply (4.1) for $\lambda = i\rho$, $\rho \in \mathbb{R}\backslash\{0\}$, to see this. Therefore, as a corollary to Theorem 4 we obtain the following result.

Corollary 2. *Let (Ω, μ) be a measure space and let $-A$ be the generator of a bounded C_0-group on $L^p(\Omega, \mu)$, where $1 < p < \infty$. Assume that $N(A) = \{0\}$. Then A admits an H^∞-calculus for each cone C_θ, $\theta > 0$.*

We now discuss extensions of Theorems 2 and 4 and of Corollaries 1 and 2 to the vector-valued situation. For this purpose we recall the two main ingredients of the proofs of Theorems 2 and 4, namely

(a) the Coifman-Weiss inequalities, i.e., (2.13) and (3.5), respectively, and

(b) the Mikhlin multiplier theorem for $L^p(\mathbb{R})$, $1 < p < \infty$.

Beginning with the group case, it is known that there is a Coifman-Weiss inequality in the vector-valued case, too. More precisely, let (Ω, μ) be a measure space, Y be a Banach space and $\mathcal{T}$ be a bounded C_0-group on $L^p(\Omega, \mu, Y)$. Denote by $-\mathcal{A}$ the generator of $\mathcal{T}$. Then for each $h \in \widetilde{BV}(\mathbb{R})$,

$$|h(\mathcal{A})|_{\mathcal{B}(L^p(\Omega,\mu,Y))} \leq M^2 |h(\mathcal{A}_0)|_{\mathcal{B}(L^p(\mathbb{R},Y))}, \tag{4.3}$$

where M is defined by $M = \sup\{|\mathcal{T}(t)|_{\mathcal{B}(L^p(\Omega,\mu,Y))} : t \in \mathbb{R}\}$. Here $\mathcal{A}_0$ denotes the negative generator of the C_0-group of translations $\mathcal{T}_0$ on $L^p(\mathbb{R}, Y)$, $1 \leq p < \infty$. The proof of (4.3) is a direct extension of that of Coifman and Weiss [7] in the scalar case. For the discrete case this has been carried through in [6].

Next we have to estimate the convolution operator $h(\mathcal{A}_0)$ in $L^p(\mathbb{R}, Y)$, i.e., in the vector-valued case, like in (3.6) for the scalar case. Observe that this is not possible for an arbitrary Banach space Y, but it is known that the Mikhlin multiplier theorem remains valid for $L^p(\mathbb{R}, Y)$ if and only if Y belongs to the class $\mathcal{H}T$. We recall that a Banach space Y belongs to the class $\mathcal{H}T$ if the Hilbert transform H defined by

$$(Hf)(t) = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \frac{1}{\pi} \int_{\varepsilon \leq |s| \leq R} f(t-s) \frac{ds}{s}, \quad t \in \mathbb{R}, \tag{4.4}$$

where $f \in C_0^\infty(\mathbb{R}, Y)$, is bounded in $L^2(\mathbb{R}, Y)$. We refer to the papers of Bourgain [3], Mc Connell [22], Zimmermann [29] and to the monograph of Prüss [24] for proofs of the latter statement and for further results. Combining the vector-valued Coifman-Weiss inequality (4.3) for groups and the vector-valued Mikhlin theorem we obtain the following extension of Theorem 4 to the vector-valued situation.

Theorem 5. *Let (Ω, μ) be a measure space, Y be a Banach space of class $\mathcal{H}T$ and let $\mathcal{T}$ be a bounded C_0-group on $X = L^p(\Omega, \mu, Y)$, where $1 < p < \infty$. Denote the generator of $\mathcal{T}$ by $-\mathcal{A}$ and assume that $N(\mathcal{A}) = \{0\}$. Then the functional calculus $\Psi : \widetilde{BV}(\mathbb{R}) \to \mathcal{B}(X)$ defined by*

$$\Psi(\widetilde{db}) = \int_{-\infty}^{\infty} \mathcal{T}(t)\, db(t) = \widetilde{db}(\mathcal{A}) \tag{4.5}$$

extends uniquely to $H_1^\infty(\mathbb{R})$. Moreover, $\Psi : H_1^\infty(\mathbb{R}) \to \mathcal{B}(X)$ is continuous and $(h_n) \subset H_1^\infty(\mathbb{R})$, $||h_n||_{1,\infty} \leq m < \infty$, $h_n(\rho) \to h(\rho)$ for a.a. $\rho \in \mathbb{R}$ imply $h_n(\mathcal{A}) \to h(\mathcal{A})$ strongly. In particular, $\mathcal{A}$ admits an H^∞-calculus for each double cone C_θ, $\theta > 0$.

Specializing to (Ω, μ) a one-point space, we obtain the following corollary.

Corollary 3. *Let X be a Banach space of class $\mathcal{H}T$ and let $\mathcal{T}$ be a bounded C_0-group on X with generator $-A$. Assume that $N(\mathcal{A}) = \{0\}$. Then the functional calculus $\Psi : \widetilde{BV}(\mathbb{R}) \to \mathcal{B}(X)$ defined by*

$$\Psi(\widetilde{db}) = \int_{-\infty}^{\infty} T(t)\, db(t) = \widetilde{db}(A) \tag{4.6}$$

extends uniquely to $H_1^\infty(\mathbb{R})$*. Moreover,* $\Psi : H_1^\infty(\mathbb{R}) \to \mathcal{B}(X)$ *is continuous and* $(h_n) \subset H_1^\infty(\mathbb{R})$*,* $||h_n||_{1,\infty} \le m < \infty$*,* $h_n(\rho) \to h(\rho)$ *for a.a.* $\rho \in \mathbb{R}$ *imply* $h_n(A) \to h(A)$ *strongly. In particular,* A *admits an* H^∞*-calculus for each double cone* C_θ*,* $\theta > 0$*.*

A particularly interesting situation arises if $\mathcal{A} = -\mathcal{A}_0^2$, where $\mathcal{A}_0$ generates a bounded C_0-group on a Banach space X belonging to the class $\mathcal{H}T$. The transformation $h(\rho) = f(-\rho^2)$ induces a bounded linear map from $H_1^\infty(i\mathbb{R})$ to $H_1^\infty(\mathbb{R}_+)$ and from $H^\infty(C_{\theta/2})$ to $H^\infty(\Sigma_\theta)$, $\theta > 0$. Therefore Corollary 3 yields an $H_1^\infty(\mathbb{R}_+)$-calculus as well as an $H^\infty(\Sigma_\theta)$-calculus for $\mathcal{A}$.

Corollary 4. *Let* X *be a Banach space of class* $\mathcal{H}T$ *and let* $\mathcal{T}$ *be a bounded* C_0*-group on* X *with generator* $-\mathcal{A}_0$*. Assume that* $N(\mathcal{A}_0) = \{0\}$ *and set* $\mathcal{A} = -\mathcal{A}_0^2$*. Then* $\mathcal{A}$ *admits an* H^∞*-calculus for each sector* Σ_θ*,* $\theta > 0$ *and an* H_1^∞*-calculus for* $\mathbb{R}_+$*.*

We now turn to the case of generators of positive contraction semigroups. At present, it seems to be unknown whether Theorem 1 extends to the vector-valued case. There is, however, an important special case for which Theorem 1 may be extended to the vector-valued situation (see [6]). To describe the situation, let (Ω, μ) be a measure space, T be a positive C_0-semigroup of contractions on $L^p(\Omega, \mu)$ with generator $-A$. If Y is a Banach space, define $\mathcal{T}(t)$ by means of

$$\mathcal{T}(t)(\varphi y) = (T(t)\varphi)y, \quad t > 0,\ \varphi \in L^p(\Omega, \mu),\ y \in Y. \tag{4.7}$$

The family $\mathcal{T}(t)$ defined in this way extends uniquely to a C_0-semigroup of contractions on $L^p(\Omega, \mu, Y)$, which is also positive in the sense that, given any cone $K \subset Y$, the induced cone $\mathcal{K} = L^p(\Omega, \mu; K) \subset L^p(\Omega, \mu; Y)$ is invariant under the semigroup $\mathcal{T}(t)$. For details and proofs we refer to [4]. We call $\mathcal{T}$ *the tensor product extension* of T. Denote by $-\mathcal{A}$ the generator of $\mathcal{T}$. Then it has been shown in [6] that

$$|f(\mathcal{A})|_{\mathcal{B}(L^p(\Omega,\mu,Y))} \le |f(\mathcal{A}_0)|_{\mathcal{B}(L^p(\mathbb{R},Y))} \tag{4.8}$$

for each $f \in \widehat{BV}(\mathbb{C}_+)$. Here $\mathcal{A}_0$ denotes as before the negative generator of the C_0-group of translations $\mathcal{T}_0$ on $L^p(\mathbb{R}, Y)$, $1 \le p < \infty$. Combining the latter result with the vector-valued Mikhlin multiplier theorem, we obtain the following result.

Theorem 6. *Let (Ω, μ) be a measure space, Y be a Banach space of class $\mathcal{H}T$ and let T be a positive C_0-semigroup of contractions on $L^p(\Omega, \mu)$, where $1 < p < \infty$. Let $\mathcal{T}$ be its tensor product extension to $X = L^p(\Omega, \mu, Y)$ and denote by $-\mathcal{A}$ its generator. Assume that $N(\mathcal{A}) = \{0\}$. Then the functional calculus $\phi : \widehat{BV}(\mathbb{C}_+) \to \mathcal{B}(X)$ defined by*

$$\phi(\widehat{db}) = \int_0^\infty \mathcal{T}(t)db(t) = \widehat{db}(\mathcal{A}) \tag{4.9}$$

extends uniquely to $H_1^\infty(\mathbb{C}_+)$. Moreover, $\phi : H_1^\infty(\mathbb{C}_+) \to \mathcal{B}(X)$ is continuous and $(f_n) \subset H_1^\infty(\mathbb{C}_+)$, $||f_n||_{1,\infty} \leq m < \infty$, $f_n \to f$ locally uniformly imply $f_n(\mathcal{A}) \to f(\mathcal{A})$ strongly. In particular, $\mathcal{A}$ admits an H^∞-calculus for each sector Σ_θ, $\theta > \pi/2$.

5. Elliptic boundary value problems on cylindrical domains. Let (Ω, μ) be a measure space. For $p \in (1, \infty)$ we put $X := L^p(\Omega, \mu, Y)$ where Y is assumed to belong to the class $\mathcal{H}T$. Assume that $-A$ is a linear operator in X satisfying the assumptions of Theorem 5 or 6 and for $f \in L^1([0,1], X)$ consider the following boundary value problem

$$\begin{aligned} u'' - Au &= f, \qquad t \in [0,1] \\ u(0) &= u(1) = 0. \end{aligned} \tag{5.1}$$

In this section we show that the results of the proceeding sections imply not only the existence of a solution u of (5.1) but also an explicit representation of u. In fact, for $t, s \in [0,1]$ let $G_0(t, s, z)$ denote Green's function for the scalar boundary value problem

$$w'' - z^2 w = g, \quad t \in [0,1], \quad w(0) = w(1) = 0,$$

which is given by

$$G_0(t, s, z) := \frac{1}{zshz} \begin{cases} sh(tz)sh(z(1-s)), & t < s \\ sh(sz)sh(z(1-t)), & t > s. \end{cases}$$

Then $G_0(t, s, \cdot)$ is an analytic function on $\mathbb{C} \backslash \{i\pi\mathbb{Z}\}$. Moreover, since

$$G_0(t, s, z) = \frac{e^{-|t-s|z}}{2z(1-e^{-2z})} \begin{cases} (1-e^{-2tz})(1-e^{-2z(1-s)}), & t < s \\ (1-e^{-2sz})(1-e^{-2z(1-t)}), & t > s \end{cases}$$

it follows that $|G_0(t,s,z)| \leq C(\phi,\eta)/|z|$ provided $|\arg z| \leq \phi < \pi/2$ and $|z| \geq \eta > 0$. Moreover, writing

$$G_0(t,s,z) = \frac{e^{-|t-s|z}2z}{(1-e^{-2z})} \begin{cases} (\frac{1-e^{-2tz}}{2tz})(\frac{1-e^{-2z(1-s)}}{2z(1-s)})t(1-s), & t < s \\ (\frac{1-e^{-2sz}}{2sz})(\frac{1-e^{-2z(1-t)}}{2z(1-t)})s(1-t), & t > s \end{cases}$$

we conclude that $|G_0(t,s,z)| \leq C(\phi)$ for all $z \in \mathbb{C}$ satisfying $|argz| \leq \phi < \pi/2$. Put

$$H_0(t,s,\lambda) := G_0(t,s,\lambda^{1/2}).$$

Then $H_0(t,s,\cdot) \in H^\infty(\Sigma_\Theta)$ for every $\Theta \in (0,\pi)$. Hence, by Theorem 5 or Theorem 6, $H_0(t,s,A) \in \mathcal{B}(\mathrm{X})$ for all $t,s \in [0,1]$ and there exists a constant $C = C(p,M,Y)$ such that

$$\|H_0(t,s,A)\| \leq C$$

for all $s,t \in [0,1]$. In a similar way one may prove that $\partial_t H_0(s,t,\cdot)$ is a bounded holomorphic functions in Σ_Θ. Hence, for each $f \in L^1([0,1],X)$, the function $u : [0,1] \to X$ given by

$$u(t,A) := \int_0^1 H_0(t,s,A)f(s)ds$$

belongs to $C^1([0,1],X)$. Moreover, defining $G_1(t,s,\cdot)$ and $H_1(t,s,\cdot)$ by

$$\begin{aligned} G_1(t,s,z) &= zG_0(t,s,z) \qquad |\arg z| \leq \phi < \pi/2 \\ H_1(t,s,\lambda) &= G_1(t,s,\lambda^{1/2}) \qquad |\arg \lambda| \leq \Theta < \pi, \end{aligned}$$

by the above estimates $|G_1(t,s,z)| \leq C(\phi)$ for $|\arg z| \leq \phi$ as well, hence $H_1(t,s,\cdot) \in H^\infty(\Sigma_\Theta)$. Hence, for each $f \in L^1([0,1],X)$, the function $v : [0,1] \to X$ given by

$$v(t,A) := \int_0^1 H_1(t,s,A)f(s)ds, \quad t \in [0,1],$$

belongs to $C([0,1],X)$, which by the relation

$$v(t,A) = A^{1/2}u(t,A), \quad t \in [0,1],$$

implies

$$u(t) \in D(A^{1/2}) \quad \text{for all} \quad t > 0 \quad \text{and} A^{1/2}u(\cdot, A) \quad \text{is continuous on } [0,1].$$

Similarly, since A commutes with $H_j(t,s,A)$ we see for functions

$$f \in L^1([0,1], X)$$

such that $f(t) \in D(A^{1/2})$ a.e. and $A^{1/2}f \in L^1([0,1],X)$, that u has the regularity properties

$$u \in C^2([0,1],X)) \cap C^1([0,1], D(A^{1/2})) \cap C_0([0,1], D(A)),$$

and is a solution of (5.1).

Finally, recall from Theorem 5 or 6 that A admits a bounded $H^\infty(\Sigma_\Theta)$ calculus provided $\Theta > \pi/2$. Since the operator $B := \partial_t^2$ in X with domain $W^{2,p}((0,1),X) \cap W_0^{1,p}((0,1),X)$ admits an H^∞ calculus on each sector $\Sigma_\Theta, \Theta > 0$, the Dore-Venni theorem (cf. [13] or [24]) implies the following maximal regularity for the solution u of (5.1):
Let $1 < q < \infty$ and $f \in L^q((0,1),X)$. Then the solution u of (5.1) satisfies

$$u \in W^{2,q}([0,1],X)) \cap W_0^{1,q}([0,1],X) \cap L^q([0,1], D(A)).$$

6. The Dirichlet problem in a cone. Let $\Phi \in (0,\pi)$ and consider the subset Ω of $\mathbb{R}^2$ given by $\Omega := \{r\sigma; r > 0, \sigma \in S\}$ where $S := \{e^{i\phi}; |\phi| \le \Phi\}$. Notice that the boundary of Ω has a corner at the point 0. In the following we consider the Dirichlet problem in Ω, i.e., we consider the equation

$$\begin{aligned} \Delta u &= f, \qquad \text{in} \quad \Omega \\ u &= 0, \qquad \text{on} \quad \partial\Omega \end{aligned} \tag{6.1}$$

where $f \in L^p(\Omega)$. If Ω were bounded and its boundary smooth, then a famous result of Agmon, Douglis and Nirenberg [1] implies the existence of a function $u \in W^{2,p}(\Omega)$ satisfying (6.1). Problem 6.1 has been studied in detail in [5] and [19]. It is the aim of this section to apply the results of the proceeding sections to our situation where Ω is the cone described above, i.e., is unbounded and has a nonsmooth boundary. In doing this we obtain more direct proofs of the regularity results contained in [5] and [19].

To this end, we rewrite equation (6.1) in polar coordinates and obtain

$$\begin{aligned} r(ru_r)_r + u_{\phi\phi} &= r^2 f, \qquad r > 0,\ \phi \in (-\Phi, \Phi) \\ u(r, \pm\Phi) &= 0, \qquad r > 0. \end{aligned} \tag{6.2}$$

We now sketch our approach for two possible choices of the underlying Banach space.

We first choose $X := L^p(\mathbb{R}_+, \frac{dr}{r}, Y)$, where $Y := L^p(-\Phi, \Phi)$ and $1 < p < \infty$. Define the operator B_0 in X by

$$B_0 := r\partial_r \qquad D(B_0) := \{u \in X :\ r\partial_r u \in X\}.$$

We observe that $-B_0$ generates an isometric group $(T(t))_{t\geq 0}$ on X given by

$$(T(t)u)(r, \phi) := u(e^{-t}r, \phi).$$

Put $B := -B_0^2 = -r\partial_r(r\partial_r)$. It then follows from Corollary 4 that B admits a bounded $H^\infty(\Sigma_\Theta)$-calculus on X for each $\Theta > 0$. Moreover, define the operator A by $Au := -u_{\phi\phi}$ for $u \in D(A) := L^p(\mathbb{R}_+, r\frac{d}{dr}, D(\partial_\phi^2))$, where

$$D(\partial_\phi^2) := W^{2,p}(-\Phi, \Phi) \cap W_0^{1,p}(-\Phi, \Phi).$$

It is well known that A admits a bounded $H^\infty(\Sigma_\Theta)$-calculus on X for each $\Theta > 0$ (cf. [25]). Furthermore assume that $f \in L^p(\Omega, (x^2 + y^2)^{p-1} dxdy)$. Then g given by $g(r, \phi) := r^2 f(r, \phi)$ belongs to X.

Applying the Dore-Venni theorem to our situation we obtain the following result.

Proposition 1. *Let $g \in X$. Then there exists a unique solution of* (6.2) *satisfying*

$$u \in L^p(\mathbb{R}_+, \frac{dr}{r}, W^{2,p}(-\Phi, \Phi) \cap W_0^{1,p}(-\Phi, \Phi)) \cap D(B).$$

Next, we describe a second approach to problem (6.2) by choosing as underlying Banach space the space $X := L^p(\mathbb{R}, Y)$ where $Y := L^p(-\Phi, \Phi)$.

The change of variable $r = e^t$ which maps Ω into $\mathbb{R} \times (-\Phi, \Phi)$ leads us to the equation

$$u_{tt} + u_{\phi\phi} = g$$

in $\mathbb{R} \times (-\Phi, \Phi)$, where $g = e^{2t} f$. Assuming $f \in L^p(\Omega)$ it follows that $e^{(2/p-2)t} g \in L^p(\mathbb{R} \times (-\Phi, \Phi))$. Introducing the functions $v := e^{\alpha t} u$ and $h := e^{\alpha t} g$ with $\alpha = 2/p - 2$, we obtain the equation

$$\begin{aligned} v_{tt} - 2\alpha v_t + \alpha^2 v + v_{\phi\phi} &= h, \qquad \text{in} \quad \mathbb{R} \times (-\Phi, \Phi) \\ v(t, \pm\Phi) &= 0, \qquad \text{in} \quad \mathbb{R}. \end{aligned} \tag{6.3}$$

Define the operator A in X by

$$Av := -v_{\phi\phi} \qquad \text{with} \quad D(A) := L^p(\mathbb{R}, W^{2,p}(-\Phi, \Phi) \cap W_0^{1,p}(-\Phi, \Phi)).$$

Then A admits an H^∞-calculus on X for each sector $\Sigma_\Theta, \Theta > 0$. Moreover, define the operator B in X by

$$Bv := -\partial_t^2 v + 2\alpha \partial_t v - \alpha^2 v = -(\partial_t - \alpha)^2 v \quad \text{with} \quad D(B) := W^{2,p}(\mathbb{R}, Y).$$

Observe that $\sigma(A)$ consists of a sequence $(\lambda_j)_{j\in\mathbb{N}}$ of strictly positive eigenvalues and that $\sigma(B)$ is given by the set

$$\{-\xi^2 - 2\alpha i\xi + \alpha^2); \xi \in \mathbb{R}\}.$$

By the Mikhlin multiplier theorem, the operator $B + \alpha^2$ in X admits an H^∞-calculus on each halfplane Σ_θ, for each $\theta > \pi/2$. Hence, by $h \in X$ and assuming that $\lambda_j \neq \alpha^2$ for all $j \in \mathbb{N}$, it follows from Theorem 8.5 in [24] that A+B is closed and there exists a unique solution v of (6.3) satisfying

$$v \in W^{2,p}(\mathbb{R}, Y) \cap L^p(\mathbb{R}, W^{2,p}(-\Phi, \Phi) \cap W_0^{1,p}(-\Phi, \Phi)).$$

In particular we conclude that $v \in W^{2,p}(\mathbb{R} \times (-\Phi, \Phi))$. Summarizing, we proved the following result.

Proposition 2. *Let $h \in L^p(\mathbb{R} \times (-\Phi, \Phi))$ and assume that $\lambda_j \neq (2/p-2)^2$ for all $j \in \mathbb{N}$. Then equation* (6.3) *has a unique solution $v \in W^{2,p}(\mathbb{R} \times (-\Phi, \Phi)) \cap W_0^{1,p}(\mathbb{R} \times (-\Phi, \Phi))$.*

Returning to the original variables it follows that for given $f \in L^p(\Omega)$ there exists a solution u of (6.1) such that

$$\frac{u}{r^2}, \frac{\partial_i u}{r}, \partial_i \partial_j u \in L^p(\Omega) \qquad (i, j = 1, 2).$$

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