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A STABILITY PROPERTY FOR THE GENERALIZED MEAN CURVATURE FLOW EQUATION

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Abstract. In this paper we will study stability properties for viscosity solutions of geometric equations. We will prove that, if the interface is regular (i.e., it is the boundary of an open set and it is not fat), the signed distance function from the front is stable for geometric perturbations of the equation. This result is based on representation formulas for viscosity solutions in terms of distance functions from the level sets. An application of the previous result to stability of approximation schemes is also presented.

1. Introduction. In this paper we will study stability properties for a class of degenerate parabolic equations, called *geometric*, whose typical example is the mean curvature flow equation. Geometric equations come from the so called *level set approach* to propagation of fronts. This approach is based on considering the evolving surface as the level set of the solution of an appropriate partial differential equation (see [22]).

In general geometric equations are highly degenerate and do not admit classical solutions. In [12] and [19] it was proved that a good notion of weak solution for this class of equations is the viscosity solution one introduced by Crandall-Lions [14]. It was shown that, given a function which represents the initial front, there exists a unique viscosity solution, defined for any time, of the geometric equation. Moreover the evolution of the level set at time t depends only on the level set at time 0 and not on the particular function chosen to represent it. In this way, the evolution of the front is well defined for any time.

A different way to describe the evolution of an interface is given by the study of the signed distance function from the front (see [24], [7]). This

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method is in some sense more intrinsic, but it has a drawback. The evolution of the front is uniquely defined until the front fattens; i.e., until the interior of the front becomes not empty. After this time the evolution of the front is not uniquely determined by its initial position (see [19], [21], [9] for some examples of fattening phenomena).

One of the main features of the viscosity solution theory introduced by Crandall and P.L. Lions is the stability under uniform convergence: the limit of an uniformly convergent sequence of viscosity solutions is still a viscosity solution. In [6] (see also [13]) it was shown that, if the limit equation satisfies a strong comparison principle, it is possible to pass to the limit with only a local uniform bound on the sequence. The limit of the sequence of viscosity solutions is uniquely determined by the equation and moreover the convergence is uniform on compact sets. This stability property holds also for geometric equations (see [12], [24]).

In this note, we are interested in studying stability properties for interfaces generated by viscosity solutions of geometric equations. Instead of considering just a single level set as in [24] and [7], we will study the whole family of distance functions from the α-level sets, for $\alpha \in \mathbb{R}$, of the viscosity solution. We will show that this family is uniquely determined by the viscosity solution and, vice versa, by a family of distance functions it is possible to reconstruct a viscosity solution of a geometric equation. Moreover the points of a level set which have, in some sense, a singular behavior can be identified looking at discontinuities in the α-variable of the distance functions.

As consequence of the previous correspondence, we will prove that families of distance functions are stable respect to the uniform convergence. Hence we will show that if the interface is regular (i.e., it is the boundary of an open set and doesn't fatten), then its signed distance function is the limit of the ones from the interfaces generated by perturbed propagation laws, for the perturbation going to zero. It is worth noting that convergence of the signed distance functions implies convergence of the interfaces in the sense of the Hausdorff distance.

In the last part of the paper we will show two applications of the previous results to level set convergence of approximation schemes: for a first order geometric Hamilton-Jacobi equation and for the Bence-Merriman-Osher scheme [10] for the mean curvature flow equation.

This paper is organized as follows. In Section 2 we recall some basic results about viscosity solutions and the level set approach. In Section 3 we study the properties of distance functions and we give the representation formula for viscosity solutions. Section 4 is devoted to the stability result. In Section 5 we study an application of the stability result to convergence of approximation schemes.

2. Preliminary results. This section is devoted to recall some results about the connection between viscosity solution theory and the level set flow for a geometric equations (we refer to [7], [12], [19] for a detailed discussion of this topic). We consider the equation

$$u_t - F(x,t,Du,D^2u) = 0 \qquad (x,t) \in \mathbb{R}^N \times (0,+\infty) \tag{2.1}$$

with the initial datum

$$u(x,0) = u_0(x) \qquad x \in \mathbb{R}^N. \tag{2.2}$$

We say that the equation (2.1) is geometric if the function $F : \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^N \times S^n \to \mathbb{R}$ satisfies the following condition

$$F(x,t,\lambda p, \lambda X + \mu(p \otimes p)) = \lambda F(x,t,p,X)$$

for any $\lambda > 0$, $\mu \in \mathbb{R}$ e $(x,t,p,X) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^N \times S^N$. Typical examples of geometric equations are the Hamilton-Jacobi equation (see [4], [18])

$$u_t - H(x,t,Du) = 0 \tag{2.3}$$

with $H(x,t,p)$ positive homogeneous of degree one in the variable p, and the mean curvature flow equation (see [12], [19])

$$u_t - \Delta u + \frac{\langle D^2uDu, Du\rangle}{|Du|^2} = 0. \tag{2.4}$$

We will always assume that F is (degenerate) elliptic, i.e.,

$$F(x,t,p,X) \le F(x,t,p,Y) \qquad \text{for any } X,Y \in S^n \text{ s.t. } X \le Y.$$

Let us define the weak limits of a sequence of functions (see [3], [15] for definitions and properties).

Definition 2.1.

(i) Given a sequence of functions $\phi_n : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R} \cup \{+\infty\}$ and $\phi : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R} \cup \{+\infty\}$, we say that $\phi = \limsup^*_n \phi_n$ in $\mathbb{R}^N \times [0,+\infty)$ if for any $(z,t) \in \mathbb{R}^N \times [0,+\infty)$ we have

$$\phi(z,t) = \sup\{\limsup_n \phi_n(z_n,t_n) : (z_n,t_n) \to (z,t)\}.$$

(ii) Given a sequence of functions $\phi_n : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R} \cup \{-\infty\}$ and $\phi : \mathbb{R}^N \times [0,\infty) \to \mathbb{R} \cup \{-\infty\}$, we say that $\phi = \liminf_{n\,*} \phi_n$ in $\mathbb{R}^N \times [0,+\infty)$ if for any $(z,t) \in \mathbb{R}^N \times [0,+\infty)$ we have

$$\phi(z,t) = \inf\{\liminf_n \phi_n(z_n,t_n) : (z_n,t_n) \to (z,t)\}.$$

Note that

$$\liminf_n {}_*\phi_n(x,t) = -\limsup_n {}_*\left(-\phi_n(x,t)\right).$$

Moreover if $\phi_n(x,t) = \phi(x,t)$ for every n, then the $\liminf_*$ (resp. $\limsup^*$) of ϕ_n, denoted by ϕ_* (resp. ϕ^*), is the l.s.c. envelope (resp. the u.s.c. envelope) of ϕ.

Definition 2.2. We say that

(i) A l.s.c. function $u : \mathbb{R}^N \times (0,+\infty) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a *viscosity supersolution* of (2.1) if for any $\phi \in C^{2,1}(\mathbb{R}^N \times (0,+\infty))$ such that $u - \phi$ has local minimum at (x_0,t_0) we have

$$\phi_t - F^*(x_0,t_0,D\phi,D^2\phi) \geq 0.$$

(ii) An u.s.c. function $u : \mathbb{R}^N \times (0,+\infty) \longrightarrow \mathbb{R} \cup \{-\infty\}$ is a *viscosity subsolution* of (2.1) if for any $\phi \in C^{2,1}(\mathbb{R}^N \times (0,+\infty))$ such that $u - \phi$ has a local maximum at (x_0,t_0) we have

$$\phi_t - F_*(x_0,t_0,D\phi,D^2\phi) \leq 0.$$

A function u is a *viscosity solution* of (2.1) if verifies (i) and (ii).

We will always intend that the initial datum is assumed in strong sense, i.e., if u is a subsolution (resp. a supersolution), then $u(x,0) \leq u_0(x)$ (resp. $u(x,0) \geq u_0(x)$) for any $x \in \mathbb{R}^N$.

For geometric equations we have an equivalent definition of viscosity solution (see [11] for the proof).

Proposition 2.1. *Let F be geometric. Then*

(i) *A l.s.c. function $u : \mathbb{R}^N \times (0,+\infty) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a viscosity supersolution of* (2.1) *if and only if for any $(x_0,t_0) \in \mathbb{R}^N \times (0,+\infty)$, $M \geq u(x_0,t_0)$ and for any $\psi \in C^{2,1}(\mathbb{R}^N \times (0,+\infty))$ which has a local maximum point on the set $\{(x,t) : u(x,t) \leq M\}$ at (x_0,t_0), we have*

$$\psi_t - F^*(x_0,t_0,D\psi,D^2\psi) \geq 0. \tag{2.5}$$

(ii) *An u.s.c. function $u : \mathbb{R}^N \times (0,+\infty) \longrightarrow \mathbb{R} \cup \{-\infty\}$ is a viscosity subsolution of* (2.1) *if and only if for any $(x_0,t_0) \in \mathbb{R}^N \times (0,+\infty)$, $M \leq u(x_0,t_0)$ and for any $\psi \in C^{2,1}(\mathbb{R}^N \times (0,+\infty))$ which has a local minimum point on the set $\{(x,t) : u(x,t) \geq M\}$ at (x_0,t_0), we have*

$$\psi_t - F_*(x_0,t_0,D\psi,D^2\psi) \leq 0. \tag{2.6}$$

The previous property relies on the following invariance property of geometric equations for a monotone transformation of the dependent variable (see [12], [19]).

Proposition 2.2. *If u is a viscosity subsolution* (*resp. supersolution*) *of* (2.1) *and $\Psi : \mathbb{R} \to \mathbb{R}$ is a non decreasing continuous function, then $\Psi(u)$ is a viscosity subsolution* (*resp. supersolution*) *of* (2.1).

Definition 2.3. We will say that (2.1) satisfies a *Comparison Principle for Discontinuous Solutions* if for any subsolution u and any supersolution v of (2.1) such that $u(x,0) \leq v(x,0)$, we have

$$u(x,t) \leq v(x,t) \qquad \text{in } \mathbb{R}^N \times [0,+\infty). \tag{2.7}$$

The following theorem summarizes the main results on the viscosity solution approach to geometric equations and corresponding motions of fronts. We refer to [7] and [20] for sets of assumptions on F which include equations (2.3), (2.4) and other relevant cases.

Theorem 2.1. *For any $u_0 \in UC(\mathbb{R}^N)$, there exists a unique solution $u \in UC(\mathbb{R}^N \times [0,+\infty))$ of* (2.1)–(2.2). *Moreover,* (2.1) *satisfies a Comparison Principle for discontinuous solutions* (*see Definition* 2.3). *Let u, $v \in UC(\mathbb{R}^N \times [0,+\infty))$ be two solutions of* (2.1) *such that*

$$\{x : u(x,0) > 0\} = \{x : v(x,0) > 0\},$$
$$\{x : u(x,0) < 0\} = \{x : v(x,0) < 0\},$$
$$\{x : u(x,0) = 0\} = \{x : v(x,0) = 0\}$$

and

$$\lim_{|x|\to+\infty} |u(x,0)|,\ \lim_{|x|\to+\infty} |v(x,0)| > 0.$$

Then, for any $t > 0$ we have

$$\begin{aligned}\{x : u(x,t) > 0\} &= \{x : v(x,t) > 0\},\\ \{x : u(x,t) < 0\} &= \{x : v(x,t) < 0\},\\ \{x : u(x,t) = 0\} &= \{x : v(x,t) = 0\}.\end{aligned}$$

The previous theorem asserts therefore that the evolution of level sets is defined for all time. Moreover the evolution of the level set is independent of the choice of function which represents the initial level set. Note the choice of the sign of the initial datum in the interior and the exterior regions determines the direction of the propagation. Note also that the level set 0 can be replaced by any level set α.

We conclude this section recalling the main stability result for viscosity solutions due to Barles-Perthame [6] (see also [3]).

Theorem 2.2. *Let u_ε be a locally uniformly bounded sequence of subsolutions* (*resp. supersolutions*) *of equation*

$$u_t - F_\varepsilon(x,t,Du,D^2u) = 0, \qquad (x,t) \in \mathbb{R}^N \times (0,+\infty) \tag{2.8}$$

with initial datum (2.2) *and let us suppose that F_ε is a locally uniformly bounded sequence satisfying*

$$\begin{gathered}\liminf_{\varepsilon\to0^+} {}_*F_{\varepsilon*}(x,t,p,X) \ge F_*(x,t,p,X)\\ \Big(resp.\quad \limsup_{\varepsilon\to0^+} {}^*F_\varepsilon{}^*(x,t,p,X) \le F^*(x,t,p,X)\Big).\end{gathered} \tag{2.9}$$

Then $\overline{u}(x,t) = \limsup_\varepsilon {}^*u_\varepsilon(x,t)$ (resp. $\underline{u}(x,t) = \liminf_\varepsilon {}_*u_\varepsilon(x,t)$) is a subsolution (*resp. a supersolution*) *of problem* (2.1)–(2.2).

Moreover, if (2.1) *satisfies a Comparison Principle for discontinuous solutions* (*cf. Definition* 2.3) *and u_ε is a locally uniformly bounded sequence of viscosity solutions of* (2.1)–(2.2)*, then u_ε converges to u, unique viscosity solution of* (2.1)–(2.2)*, locally uniformly in $\mathbb{R}^N \times [0,+\infty)$.*

We remark that the previous theorem holds independently of the fact that the equation is geometric. The key assumption to obtain convergence of the whole sequence u_ε to u is the Comparison Principle for discontinuous solutions which gives $\overline{u} \le \underline{u}$, while the reverse inequality holds for definition.

Remark 2.1. The definition of weak limits does not preserve the initial condition in strong sense. To obtain that the functions $\overline{u}$ and $\underline{u}$ in Theorem 2.2 satisfy the initial condition we can proceed in the following way. The initial condition is interpreted in weak (viscosity) sense, i.e. u is a supersolution (resp. a subsolution) of problem (2.1)–(2.2) if it satisfies Def. 2.2.(i) (resp. Def. 2.2.(ii)) in $\mathbb{R}^N \times (0, +\infty)$ and

$$\max\{\phi_t - F^*(x_0, 0, D\phi, D^2\phi), u(x,0) - u_0(x)\} \ge 0$$
$$\big(\text{resp. } \min\{\phi_t - F_*(x_0, 0, D\phi, D^2\phi), u(x,0) - u_0(x)\} \le 0\ \big)$$

for any function $\phi \in C^{2,1}(\mathbb{R}^N \times [0, +\infty))$ such that $u - \phi$ has a local minimum (resp. a local maximum) on $\mathbb{R}^N \times [0, +\infty)$ at $(x, 0)$. Since boundary conditions in viscosity sense are preserved by weak limits, $\overline{u}$ and $\underline{u}$ satisfy the initial condition in weak sense. Finally, it is possible to prove (see e.g. [3]) that if the initial conditions is satisfied in weak sense, then it is also satisfied in strong sense.

3. Representation formulas for viscosity solutions. Let us introduce some notations we will use in the following. Given a continuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$, we set for $\alpha \in \mathbb{R}$

$$\gamma_\alpha(t) = \big\{x : u(x,t) = \alpha\big\}, \tag{3.1}$$
$$\Delta_\alpha(t) = \big\{x : u(x,t) \le \alpha\big\}, \tag{3.2}$$
$$\Gamma_\alpha(t) = \big\{x : u(x,t) \ge \alpha\big\}. \tag{3.3}$$

For some $\alpha \in \mathbb{R}$, one among the previous sets may be empty. Observe that $\Delta_\alpha(t)$ is closed for a l.s.c. function u, $\Gamma_\alpha(t)$ is closed for an u.s.c. one. For $(x,t) \in \mathbb{R}^N \times [0, +\infty)$, we set

$$d_\alpha(x,t) = d(x, \gamma_\alpha(t)), \tag{3.4}$$
$$w_\alpha(x,t) = d(x, \Delta_\alpha(t)), \tag{3.5}$$
$$v_\alpha(x,t) = -d(x, \Gamma_\alpha(t)), \tag{3.6}$$

where in (3.4) d denotes the signed distance from γ_α, i.e. negative distance inside $\Gamma_a(t)$ and positive inside $\Delta_\alpha(t)$.

If, for some $\alpha \in \mathbb{R}$ and $t \in \mathbb{R}^+$, $\gamma_\alpha(t) = \emptyset$ and $u(x,t) < \alpha$ for all $x \in \mathbb{R}^N$, we set $d_\alpha(x,t) = -\infty$ for all x, while if $\gamma_\alpha(t) = \emptyset$ e $u(x,t) > \alpha$ for all x, we set $d_\alpha(x,t) = +\infty$ for all x. Similarly, if $\Delta_\alpha(t) = \emptyset$ we define $w_\alpha(x,t) = +\infty$ for all $x \in \mathbb{R}^N$, if $\Gamma_\alpha(t) = \emptyset$ we define $v_\alpha(x,t) = -\infty$ for all $x \in \mathbb{R}^N$. With the previous definitions we have

$$d_\alpha(x,t) = w_\alpha(x,t) + v_\alpha(x,t) \tag{3.7}$$

for all $\alpha \in \mathbb{R}$, $(x,t) \in \mathbb{R}^N \times [0,+\infty)$.

In the following proposition we will prove that there is a correspondence between subsolutions of the equation (2.1) and families of distance functions $v_\alpha(x,t)$ defined as in (3.6).

Proposition 3.1.

(i) *Let u be a subsolution of problem* (2.1)–(2.2). *If $v_\alpha(x,t)$, $\alpha \in \mathbb{R}$, is the family of distance functions defined as in* (3.6), *then the following properties are satisfied*

(V1) *For all $\alpha \in \mathbb{R}$, the function $v_\alpha : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R} \cup \{-\infty\}$ is u.s.c. and non positive.*

(V2) *Defined $\Gamma^0_\alpha = \{x : u_0(x) \geq \alpha\}$, then*

$$v_\alpha(x,0) \leq -d(x,\Gamma^0_\alpha)$$

for all $x \in \mathbb{R}^N$, $\alpha \in \mathbb{R}$.

(V3) *For all $(x,t) \in \mathbb{R}^N \times [0,+\infty)$, v_α is non increasing and u.s.c. in α. Moreover $\lim_{\alpha\to+\infty} v_\alpha(x,t) < 0$.*

(V4) *For all $\alpha \in \mathbb{R}$, v_α is a viscosity solution of*

$$\begin{cases} v_t(x,t) - F_*(z,t,Dv(x,t),D^2v(x,t)) \leq 0 \\ v(x,t) = -|x-z| \end{cases} \tag{3.8}$$

in $\mathbb{R}^n \times (0,+\infty)$.

(ii) *If v_α, $\alpha \in \mathbb{R}$, is a family of functions satisfying* (V1)–(V4), *let $u : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R} \cup \{-\infty\}$ defined by*

$$u(x,t) = \begin{cases} \max\{\alpha : v_\alpha(x,t) \geq 0\} \\ -\infty \ \textit{if}\ \{\alpha : v_\alpha(x,t) \geq 0\} = \emptyset. \end{cases} \tag{3.9}$$

Then $u(x,t)$ is a subsolution of problem (2.1)–(2.2).

Proof. Let us prove that for a given subsolution u of (2.1), the family v_a defined as in (3.6) satisfies (V1)–(V4).

Concerning (V1), $v_\alpha \le 0$ holds for definition. To prove the u.s.c. of v_α in (x,t) we distinguish two cases

(i) $v_\alpha(x_0,t_0) = -\infty$
(ii) $v_\alpha(x_0,t_0) > -\infty$.

In the first case, we have to prove that $v_\alpha(x_n,t_n) \to -\infty$ for any $(x_n,t_n) \to (x_0,t_0)$. Let us suppose that there exists a sequence (x_n,t_n) converging to (x_0,t_0) such that $v_\alpha(x_n,t_n) \to K$ for some $K \in \mathbb{R}$. It follows that $v_\alpha(x_n,t_n)$ is finite for n sufficiently large. Let $y_n \in \Gamma_\alpha(t_n)$ such that $v_\alpha(x_n,t_n) = -|x_n - y_n|$. Since $v_\alpha(x_n,t_n)$ is bounded and $x_n \to x_0$, then there exists $y_0 \in \mathbb{R}^N$ such that $y_n \to y_0$. By the u.s.c. of u and $\Gamma_\alpha(t_0) = \emptyset$ we get

$$\alpha \le \limsup_n u(y_n,t_n) \le u(y_0,t_0) < \alpha,$$

which yields a contradiction.

In the case (ii), let $(x_n,t_n) \to (x_0,t_0)$ such that the limit of $v_\alpha(x_n,t_n)$ exists and it is finite. Define $y_n \in \Gamma_\alpha(t_n)$ in such a way that $v_\alpha(x_n,t_n) = -|x_n - y_n|$. Since the sequence y_n is bounded, there exists y_0 such that, up to a subsequence, $y_n \to y_0$. Because of the u.s.c. of u we have

$$\limsup_n u(y_n,t_n) \le u(y_0,t_0).$$

Thus $\alpha \le u(y_0,t_0)$ which implies $y_0 \in \Gamma_\alpha(t_0)$. Therefore

$$\lim_n v_\alpha(x_n,t_n) = -|x_0 - y_0| \le -d(x_0,\Gamma_\alpha(t_0)) = v_\alpha(x_0,t_0).$$

Since $u(x,0) \le u_0(x)$ implies that $\{x : u(x,0) \ge \alpha\} \subset \{x : u_0(x) \ge \alpha\}$, then (V2) follows immediately. Let us consider (V3). The monotonicity in α is obvious, since $\Gamma_\alpha(t) \subset \Gamma_\beta(t)$ for $\alpha \le \beta$. Moreover, since $u(x,t) < +\infty$ for all $(x,t) \in \mathbb{R}^N \times [0,+\infty)$, it follows that $\lim_{\alpha\to+\infty} v_\alpha(x,t) < 0$. We have to prove that for any sequence α_n converging to α, then

$$\limsup_n v_{\alpha_n}(x,t) \le v_\alpha(x,t).$$

Because of the monotonicity in α, it is sufficient to consider a sequence α_n converging from left to α and to prove that

$$v_\alpha(x,t) \ge \lim_{n\to+\infty} v_{\alpha_n}(x,t). \tag{3.10}$$

If $v_\alpha(x,t) > -\infty$, then, by monotonicity, we have that v_{α_n} is finite for n large, therefore there exists $y_n \in \Gamma_{\alpha_n}(t)$ such that $v_{\alpha_n}(x,t) = -|x - y_n|$. Since the sequence y_n is bounded, passing to a subsequence, we get that $y_n \to \overline{y}$ for some $\overline{y} \in \mathbb{R}^N$. Moreover, since $u(y_n, t) \geq \alpha_n$ for all n, then

$$u(\overline{y}, t) \geq \limsup_n u(y_n, t) \geq \alpha,$$

and so $\overline{y} \in \Gamma_\alpha(t)$. Finally

$$v_\alpha(x,t) = -d(x, \Gamma_\alpha(t)) \geq -d(x, \overline{y}) = \lim_n -d(x, \Gamma_{\alpha_n}(t)) = \lim_n v_{\alpha_n}(x,t)$$

and therefore we have (3.10)

If $v_\alpha(x,t) = -\infty$, let us assume that $v_{\alpha_n}(x,t) > -\infty$ for all n, otherwise (3.10) is obvious. Let $y_n \in \Gamma_{\alpha_n}(t)$ such that $v_{\alpha_n}(x,t) = -|x - y_n|$. We are going to prove that $|y_n| \to +\infty$. If for a subsequence, still denoted with y_n, we have $y_n \to \overline{y}$, then it follows that

$$\alpha = \lim_n \alpha_n \leq \limsup_n u(y_n, t) \leq u(\overline{y}, t) < \alpha$$

which gives a contradiction. Therefore (V3) is completely proved.

We finally prove (V4). Fixed $\alpha \in \mathbb{R}$, let (x_0, t_0) be such that $v_\alpha(x_0, t_0) > -\infty$ and $\psi \in C^{2,1}(\mathbb{R}^N \times (0, +\infty))$ such that $v_\alpha(x_0, t_0) = \psi(x_0, t_0)$ and $v_\alpha(x,t) \leq \psi(x,t)$ for $(x,t) \neq (x_0, t_0)$. We have to show that

$$\psi_t(x_0, t_0) - F_*(z_0, t_0, D\psi(x_0, t_0), D^2\psi(x_0, t_0)) \leq 0, \tag{3.11}$$

where z_0 is any element of $\Gamma_\alpha(t_0)$ satisfying

$$|z_0 - x_0| = -v_\alpha(x_0, t_0). \tag{3.12}$$

Set $\beta = v_\alpha(x_0, t_0)$, $b = x_0 - z_0$, $\overline{\psi}(x,t) = \psi(x + b, t)$ and note that ψ attains a local minimum at (x_0, t_0) on the set $\{v_\alpha \geq \beta\}$. Denote by U a neighborhood of (x_0, t_0) such that the inequality $\psi(x,t) \geq \psi(x_0, t_0)$ holds for every $(x,t) \in U \cap \{v_\alpha \geq \beta\}$ and write $\widetilde{U} = \{(z,t) : (z + b, t) \in U\}$. If $(z,t) \in \widetilde{U} \cap \{u \geq \alpha\}$ then $(z + b, t) \in U \cap \{v \geq \beta\}$ since $|b| = -\beta$. So we obtain $\overline{\psi}(z,t) \geq \overline{\psi}(z_0, t_0)$ which shows that (z_0, t_0) is a local minimum point of $\overline{\psi}$ on $\{u \geq \alpha\}$. Therefore by Proposition 2.1 we find

$$\overline{\psi}_t(z_0, t_0) - F_*(z_0, t_0, D\overline{\psi}(z_0, t_0), D^2\overline{\psi}(z_0, t_0)) \leq 0$$

and so (3.11) follows.

Let us assume now that v_α, $\alpha \in \mathbb{R}$, is a family of functions which satisfies (V1)–(V4). Formula (3.9) is well defined since, because of (V3), if $\{\alpha : v_\alpha(x,t) \geq 0\}$ is not empty, it does not coincide with all $\mathbb{R}$. Therefore it is a closed bounded interval and there exists its maximum value.

Let us prove that $u(x,t)$ is u.s.c. in (x,t). We distinguish two cases

(i) $u(x_0,t_0) = -\infty$;
(ii) $u(x_0,t_0) = \alpha \in \mathbb{R}$.

In case (i), suppose that there exists a sequence $(x_n,t_n) \to (x_0,t_0)$ such that

$$\lim_n u(x_n,t_n) = K \in \mathbb{R}.$$

It follows that, for n large, $u(x_n,t_n) \geq K-1$ and $v_{K-1}(x_n,t_n) = 0$. For the u.s.c. of v_α in (x,t), we get $v_{K-1}(x_0,t_0) \geq 0$ and therefore a contradiction, since $\{\alpha : v_\alpha(x_0,t_0) \geq 0\} = \emptyset$.

If (ii) is verified, let us suppose that there exists $(x_n,t_n) \to (x_0,t_0)$ such that

$$\lim_n u(x_n,t_n) \geq u(x_0,t_0) + 2\varepsilon. \tag{3.13}$$

Set $\alpha_0 = u(x_0,t_0)$. The inequality (3.13) implies that, for n sufficiently large, $u(x_n,t_n) \geq \alpha_0 + \varepsilon$ and therefore $v_{\alpha_0+\varepsilon}(x_n,t_n) = 0$. The u.s.c. of v_α yields

$$v_{\alpha_0+\varepsilon}(x_0,t_0) \geq 0$$

and then $u(x_0,t_0) \geq \alpha_0 + \varepsilon$, which gives a contradiction.

To prove that $u(x,0) \leq u_0(x)$, observe that if $u(x,0) \geq \alpha$ then (3.9) implies that $v_\alpha(x,0) = 0$ and therefore by (V2) we get $u_0(x) \geq \alpha$. Similarly, if $u_0(x) < \beta$ then also $u(x,0) < \beta$ and therefore $u(x,0) \leq u_0(x)$.

Finally we prove that $u(x,t)$ is a subsolution of equation (2.1). By definition of u, for any $\alpha \in \mathbb{R}$, we have that

$$\big\{(x,t) : v_\alpha(x,t) \geq 0\big\} = \big\{(x,t) : u(x,t) \geq \alpha\big\}. \tag{3.14}$$

Thus (3.14) implies that, if ψ has a local minimum in (x_0,t_0) on $\{u \geq \alpha\}$, ψ has a local minimum in (x_0,t_0) on $\{v_\alpha \geq 0\}$. It follows that ψ is supertangent to v_α at (x_0,t_0), therefore

$$\begin{cases} \psi_t - F_*(z_0,t_0,D\psi,D^2\psi) \leq 0 \\ v_\alpha(x,t) = -|x_0 - z_0|. \end{cases}$$

Since $v_\alpha(x_0,t_0)=0$, we get $z_0=x_0$ and finally

$$\psi_t(x_0,t_0)-F_*(x_0,t_0,D\psi,D^2\psi)\le 0.$$

A correspondent result can be proved in similar way for supersolutions (note in this case the different representation formula).

Proposition 3.2.

(i) *Let u be a supersolution of* (2.1)–(2.2). *If $w_\alpha(x,t)$, $\alpha\in\mathbb{R}$, is the family of functions defined as in* (3.5)*, then the following properties are satisfied*

(W1) *For all $\alpha\in\mathbb{R}$, the function $w_\alpha:\mathbb{R}^N\times[0,+\infty)\to\mathbb{R}\cup\{+\infty\}$ is l.s.c. and non negative.*

(W2) *Defined $\Delta^0_\alpha=\{x:u_0(x)\le\alpha\}$, then*

$$w_\alpha(x,0)\ge d(x,\Delta^0_\alpha)$$

for all $x\in\mathbb{R}^N$, $\alpha\in\mathbb{R}$.

(W3) *For $(x,t)\in\mathbb{R}^N\times[0,+\infty)$, w_α is non decreasing and l.s.c. in α. Moreover $\lim_{\alpha\to-\infty}w_\alpha(x,t)>0$.*

(W4) *For all $\alpha\in\mathbb{R}$, w_α is a viscosity solution of*

$$\begin{cases} w_t(x,t)-F^*(z,t,Dw(x,t),D^2w(x,t))\ge 0\\ w(x,t)=|x-z| \end{cases}\tag{3.15}$$

in $\mathbb{R}^n\times(0,+\infty)$.

(ii) *If w_α, $\alpha\in\mathbb{R}$, is a family of functions satisfying* (W1)–(W4)*, let $u:\mathbb{R}^N\times[0,+\infty)\to\mathbb{R}\cup\{+\infty\}$ be defined by*

$$u(x,t)=\begin{cases}\min\{\alpha:w_\alpha(x,t)\le 0\}\\ +\infty \text{ if } \{\alpha:w_\alpha(x,t)\le 0\}=\emptyset.\end{cases}\tag{3.16}$$

Then $u(x,t)$ is a supersolution of problem (2.1)–(2.2).

We observe that, in general, there exists several families of functions which give a same subsolution or supersolution by means of formulas (3.9) or (3.16). We now prove that, if we assume that w_α or v_α are families of distance functions, then they are uniquely determined. We need the following lemma

Lemma 3.1. *Let u be an u.s.c. (respectively a l.s.c.) function in $\mathbb{R}^N \times [0,+\infty)$ and let $v_\alpha(x,t)$ be defined as in (3.6) (resp. $w_\alpha(x,t)$) as in (3.5)), then*

$$u(x,t) = \max\{\alpha \in \mathbb{R} : v_\alpha(x,t) \geq 0\}$$
$$\big(\textit{resp. } u(x,t) = \min\{\alpha \in \mathbb{R} : w_\alpha(x,t) \leq 0\}\big).$$

Proof. If we suppose that

$$u(x,t) < \max\{\alpha : -d(x, \Gamma_\alpha(t)) \geq 0\}$$

then, defined $\overline{\alpha} = u(x,t)$, by the previous inequality we deduce $d(x, \Gamma_{\overline{\alpha}}(t)) < 0$, and therefore a contradiction. The opposite inequality can be proved in the same way.

Theorem 3.1.

(i) *Let v_α be a family of functions which satisfies* (V1)–(V4) *and*

$$-|Dv| + 1 = 0 \qquad \textit{in } \{x : v_\alpha(x,t) < 0\} \tag{3.17}$$

for any $\alpha \in \mathbb{R}$ and $t \in \mathbb{R}^+$ such that $v_\alpha(\cdot,t)$ is not identically infinite. Then, for any $(x,t) \in \mathbb{R}^N \times [0,+\infty)$ and $\alpha \in \mathbb{R}$,

$$v_\alpha(x,t) = -d(x, \Gamma_\alpha(t)).$$

where u is defined as in (3.9) *and $\Gamma_\alpha(t)$ as in* (3.3).

(ii) *Let w_α be a family of functions which satisfies* (W1)–(W4) *and*

$$|Dw| - 1 = 0 \quad \textit{in } \{x : w_\alpha(x,t) > 0\} \tag{3.18}$$

for any $\alpha \in \mathbb{R}$ and $t \in \mathbb{R}^+$ such that $w_\alpha(\cdot,t)$ is not identically infinite. Then, for any $(x,t) \in \mathbb{R}^N \times [0,+\infty)$ and $\alpha \in \mathbb{R}$,

$$w_\alpha(x,t) = d(x, \Delta_\alpha(t)),$$

where u is defined as in (3.16) *and $\Delta_\alpha(t)$ as in* (3.2).

Proof. We will prove only (i), since (ii) is similar.

We observe that the set $\{x : v_\alpha(x,t) < 0\}$ which appears in (3.17) is open, since v_a is u.s.c. in x. Moreover $\{x : v_\alpha(x,t) < 0\} = \emptyset$ if and only if $\{x : v_\alpha(x,t) = 0\} = \mathbb{R}^N$, equivalently if and only if $\{x : u(x,t) \geq \alpha\} = \mathbb{R}^N$. In this case (3.17) does not give any condition on v_α. Let us prove now that

$$\{x : v_\alpha(x,t) < 0\} = \{x : -d(x,\Gamma_\alpha(t)) < 0\} \tag{3.19}$$

for any $t \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$. Let us suppose that for a fixed α and t, there exists x_0 such that

$$v_\alpha(x_0,t) \geq 0 > -d(x_0,\Gamma_\alpha(t)).$$

Then

$$\max\{\alpha : -d(x,\Gamma_\alpha(t)) \geq 0\} < \max\{\alpha : v_\alpha(x,t) \geq 0\}$$

and therefore from Lemma 3.1 and the definition of u (see (3.9)) we get a contradiction. The other inclusion can be proved similarly.

We have showed that $v_\alpha(\cdot,t)$ and $-d(\cdot,\Gamma_\alpha(t))$ are two positive solutions of problem

$$\begin{cases} -|Dz| + 1 = 0 & \text{in } \Omega \\ z = 0 & \text{in } \Omega^{\mathcal{C}}, \end{cases}$$

where $\Omega = \{x : v_\alpha(x,t) < 0\} = \{x : -d(x,\Gamma_\alpha(t)) < 0\}$. Therefore (see e.g. [2]), we get that

$$v_\alpha(x,t) = -d(x,\Gamma_\alpha(t)).$$

If for some $\alpha \in \mathbb{R}$ and $t \in \mathbb{R}^+$, $v_\alpha(x,t) = -\infty$ then it follows that $u(x,t) < \alpha$ for any $x \in \mathbb{R}^N$ and therefore $\Gamma_\alpha(t) = \emptyset$. By definition, we have $-d(x,\Gamma_\alpha(t)) = -\infty$ and therefore $v_\alpha(x,t) = -d(x,\Gamma_\alpha(t))$. ☐

We conclude this section proving an uniqueness result for families of distance functions which represent viscosity solutions of geometric equations.

Corollary 3.1. *Assume that the problem* (2.1), (2.2) *satisfies a Comparison Principle for discontinuous solutions* (*see Def.* 2.3). *Let* z_α, $\alpha \in \mathbb{R}$, *be a family of functions such that* $z_\alpha \wedge 0$ *satisfies properties* (V1)–(V4) *and* (3.17), $z_\alpha \vee 0$ *satisfies properties* (W1)–(W4) *and* (3.18) *and let* u *be the unique solution of* (2.1)–(2.2). *Then*

$$z_\alpha(x,t) = d(x,\gamma_\alpha(t)) \tag{3.20}$$

for all $(x,t) \in \mathbb{R}^N \times [0,+\infty)$, $\alpha \in \mathbb{R}$, *where* $d(x,\gamma_\alpha(t))$ *is defined as in* (3.1).

Proof. Let us define a function $\underline{u}$ by formula (3.9) with $v_\alpha = z_\alpha \wedge 0$ and a function $\overline{u}$ by (3.16) with $w_\alpha = z_\alpha \vee 0$. Because of the comparison principle, we have

$$\underline{u} \leq \overline{u}.$$

Moreover formula (3.9) and formula (3.16) imply that

$$\underline{u}(x,t) = \max\{\alpha : z_\alpha(x,t) \wedge 0 \geq 0\} \geq \min\{\alpha : z_\alpha(x,t) \vee 0 \leq 0\} = \overline{u}(x,t).$$

Thus $u = \overline{u} = \underline{u}$. Applying Theorem 3.1(i) and (ii) we deduce

$$\begin{aligned} z_\alpha(x,t) \wedge 0 &= -d(x,\Gamma_\alpha(t)) \\ z_\alpha(x,t) \vee 0 &= d(x,\Delta_\alpha(t)) \end{aligned}$$

and adding the two equality we get (3.20) (see (3.7)).

Remark 3.1. The previous corollary implies that a viscosity solution can be represented both by (3.9) and (3.16), i.e.

$$u(x,t) = \max\{\alpha : d(x,\gamma_\alpha(t)) \geq 0\} = \min\{\alpha : d(x,\gamma_\alpha(t)) \leq 0\}.$$

Moreover it is immediate to verify that there exists only one α such that $d(x,\gamma_\alpha(t)) = 0$ (otherwise $u(x,t)$ is discontinuous) and therefore, for any $(x,t) \in \mathbb{R}^N \times [0,+\infty)$, $u(x,t)$ is the global minimum point on $\mathbb{R}$ of the l.s.c. function $\alpha \to |d(x,\gamma_\alpha(t))|$, i.e.

$$u(x,t) = \{\alpha_0 : |d(x,\gamma_\alpha(t))| \text{ has a global minimun over } \mathbb{R} \text{ in } \alpha_0\}.$$

4. Stability of level sets. Along this section we will always assume that F, F_ε are geometric and satisfy (2.9). Moreover we will assume that F satisfies a set of hypothesis which guarantees a comparison principle for discontinuous solutions (see [12], [7]), u_ε is a sequence of solutions of problems (2.8)–(2.2) and u is the unique solution of (2.1)–(2.2).

Recall that, given two compact sets K_1 and K_2, the Hausdorff distance between K_1 and K_2 is defined as

$$d_{\mathcal{H}}(K_1,K_2) = \max\{\max_{x\in K_1}[\operatorname{dist}(x,K_2)], \max_{x\in K_2}[\operatorname{dist}(x,K_1)]\}.$$

Let $C^0_\gamma(\mathbb{R}^N)$, for $\gamma \in \mathbb{R}$, be the space of continuous functions $u : \mathbb{R}^N \to \mathbb{R}$ such that $u - \gamma$ has compact support in $\mathbb{R}^N$.

Proposition 4.1. *Let* $u_0 \in C^0_\gamma(\mathbb{R}^N)$ *and define*

$$\mathcal{F} = \cup_{\substack{t\in[0,T]\\ \alpha\in\mathbb{R}}}\{\alpha\}\times\{t\}\times\gamma_\alpha(t),$$
$$\mathcal{F}^\varepsilon = \cup_{\substack{t\in[0,T]\\ \alpha\in\mathbb{R}}}\{\alpha\}\times\{t\}\times\gamma_{\varepsilon\alpha}(t).$$

Then

$$d_{\mathcal{H}}(\mathcal{F}^\varepsilon,\mathcal{F}) \longrightarrow 0 \quad \textit{for } \varepsilon\to 0^+. \tag{4.1}$$

Proof. Under the hypothesis which guarantee the comparison principle for equation (2.1), it is possible to show by means of a barrier argument and employing (2.1) and the geometricity of F, F_ε, that if $u_0 \in C^0_\gamma(\mathbb{R}^N)$, then $u(\cdot,t)$, $u_\varepsilon(\cdot,t) \in C^0_\gamma(\mathbb{R}^N)$ for any $t>0$ (see [12]). Hence there exists a compact set $H\subset\mathbb{R}^N$ such that $u(x,t)=u_\varepsilon(x,t)$ for any $(x,t)\in(\mathbb{R}^N\setminus H)\times[0,T]$ and u_ε converges to u uniformly in $H\times[0,T]$. Moreover, as consequence of Proposition 2.2, we can always assume that u, u_ε are bounded. Therefore in (4.1), we can restrict to consider the distance between $\mathcal{F}^\varepsilon\cap K$ and $\mathcal{F}\cap K$ where K is a compact subset of $\mathbb{R}^{N+2}$.

We prove first that

$$\max_{\mathcal{F}^\varepsilon}\big\{\mathrm{dist}\big((x,t,\alpha),\mathcal{F}\big)\big\}\to 0 \qquad \text{for } \varepsilon\to 0^+. \tag{4.2}$$

If, by contradiction, (4.2) is not true, then there exists a sequence ε_n converging to 0 and $r\in\mathbb{R}^+$ such that

$$\max_{\mathcal{F}^{\varepsilon_n}}\big\{\mathrm{dist}\big((x,t,\alpha),\mathcal{F}\big)\big\}\ge r>0. \tag{4.3}$$

Let $(x_{\varepsilon_n},t_{\varepsilon_n},\alpha_{\varepsilon_n})\in\mathcal{F}^{\varepsilon_n}$ such that the maximum in (4.3) is achieved. Since K is compact, $(x_{\varepsilon_n},t_{\varepsilon_n},\alpha_{\varepsilon_n})\to(\overline{x},\overline{t},\overline{\alpha})\in K$. The convergence of u_ε to u and $u_{\varepsilon_n}(x_{\varepsilon_n},t_{\varepsilon_n})=\alpha_{\varepsilon_n}$ imply that $u(\overline{x},\overline{t})=\overline{\alpha}$. Therefore $(\overline{x},\overline{t},\overline{\alpha})\in\mathcal{F}$ and we get a contradiction to (4.3).

We prove now that

$$\max_{\mathcal{F}}\big\{\mathrm{dist}\big((x,t,\alpha),\mathcal{F}^\varepsilon\big)\big\}\to 0 \quad \text{if } \varepsilon\to 0^+.$$

By contradiction, let us assume that there exists a sequence $(x_{\varepsilon_n},t_{\varepsilon_n},\alpha_{\varepsilon_n})\in\mathcal{F}$ such that

$$\max_{\mathcal{F}}\big\{\mathrm{dist}\big((x,t,\alpha),\mathcal{F}^{\varepsilon_n}\big)\big\}=\mathrm{dist}\big((x_{\varepsilon_n},t_{\varepsilon_n},\alpha_{\varepsilon_n}),\mathcal{F}^{\varepsilon_n}\big)\ge r>0 \tag{4.4}$$

for ε_n converging to 0^+. Then, up to a subsequence, $(x_{\varepsilon_n}, t_{\varepsilon_n}, \alpha_{\varepsilon_n}) \to (\overline{x}, \overline{t}, \overline{\alpha}) \in \mathcal{F}$. From (4.4), it follows that

$$\mathrm{dist}\big((x, \alpha, t), \mathcal{F}^{\varepsilon_n}\big) \geq \frac{r}{2} \tag{4.5}$$

for any (x, α, t) such that $\mathrm{dist}((x_{\varepsilon_n}, t_{\varepsilon_n}, \alpha_{\varepsilon_n}), (x, \alpha, t)) \leq \frac{r}{2}$. Therefore, for n large enough, $\mathrm{dist}\big((\overline{x}, \overline{\alpha}, \overline{t}), \mathcal{F}^{\varepsilon_n}\big) \geq \frac{r}{2}$ and $|u_{\varepsilon_n}(\overline{x}, \overline{t}) - \overline{\alpha}| \geq \frac{r}{2}$. Inequality (4.5) is not possible since u_ε converges to u and $u(\overline{x}, \overline{t}) = \overline{\alpha}$. $\square$

We are now interested in studying convergence of a single level set, i.e. for α fixed. Consider the following example

$$u_t - \varepsilon|u_x| = 0, \quad (x,t) \in \mathbb{R} \times (0, +\infty),$$
$$u_0(x) = |x|, \quad x \in \mathbb{R}.$$

Then

$$u_\varepsilon(x,t) = |x| + \varepsilon t \quad (x,t) \in \mathbb{R} \times [0, +\infty)$$

and u_ε converges to u, where u is the unique viscosity solution of

$$u_t = 0, \quad (x,t) \in \mathbb{R} \times (0, +\infty),$$
$$u_0(x) = |x| \quad x \in \mathbb{R}.$$

For the 0-level sets, we have $\gamma_0(t) = \{0\}$ for any $t \geq 0$, while $\gamma_{\varepsilon 0}(t) = \emptyset$ if $t > 0$, for any $\varepsilon > 0$.

Therefore it is clear that we cannot hope to have convergence for each level set of the viscosity solution, but we need some additional assumptions. We look for conditions which give the convergence of $d(x, \gamma_{\varepsilon\alpha}(t))$ to $d(x, \gamma_\alpha(t))$.

Let us firstly recall a lemma due to Barles-Perthame on the convergence of extremal points under weak convergence (see Lemma 5.1 in [13]).

Lemma 4.1. *Let $u_n : \mathcal{O} \to \mathbb{R}$ be a sequence of l.s.c. functions (resp. u.c.s.) and let $\overline{U}(z) = \liminf_{*n} u_n(z)$ (resp. $\underline{U}(z) = \limsup^*_n u_n(z)$). If $\overline{U}(z_0) > -\infty$ (resp. $\underline{U}(z_0) < +\infty$) and z_0 is a local minimum point of $\overline{U}$ (resp. a local maximum point of $\underline{U}$), then there exists two subsequences $n_j \to +\infty$ and $x_j \to z_0$ such that $u_{n_j}(x_j) \to \overline{U}(z_0)$ (resp. $u_{n_j}(x_j) \to \underline{U}(z_0)$) and x_j is a local minimum point (resp. local maximum point) for u_{n_j} in $\mathcal{O}$.*

In the following we will denote with $d(x, \gamma_{\varepsilon\alpha}(t))$, $d(x, \Gamma_{\varepsilon\alpha}(t))$, and $d(x, \Delta_{\varepsilon\alpha}(t))$ the functions defined as in (3.4)–(3.6) relatively to u_ε.

Proposition 4.2. *Let us assume that F, F_ε are geometric and satisfy* (2.9). *For any $\varepsilon > 0$, let $v_{\alpha\varepsilon}$ be a family of functions verifying* (V1)–(V4) (*resp.* $w_{\alpha\varepsilon}$ *verifying* (W1)–(W4)) *relatively to the problem* (2.8)–(2.2). *Moreover assume that, for any compact set $K \subset \mathbb{R}^N \times [0,T]$, there exists $c \in \mathbb{R}^+$* (*depending on K*) *such that*

$$\lim_{\alpha\to+\infty} v_{\varepsilon\alpha}(x,t) \le -c \quad \big(\textit{resp.} \ \lim_{\alpha\to-\infty} w_{\varepsilon\alpha}(x,t) \ge c\big) \tag{4.6}$$

for any $(x,t) \in K$ and for any $\varepsilon > 0$. Then

$$\begin{aligned} \underline{v}_\alpha(x,t) &= \limsup_{\varepsilon\to0^+}{}^{*}\, v_{\varepsilon\alpha}(x,t) && (4.7)\\ &= \sup\{\limsup_{\varepsilon\to0^+} v_{\varepsilon\alpha_\varepsilon}(x_\varepsilon,t_\varepsilon) : (x_\varepsilon,t_\varepsilon,\alpha_\varepsilon) \to (x,t,\alpha)\} \end{aligned}$$

$$\begin{aligned} \Big(\textit{resp.}\ \overline{w}_\alpha(x,t) &= \liminf_{*\ \varepsilon\to0^+} w_{\varepsilon\alpha}(x,t) && (4.8)\\ &= \inf\{\liminf_{\varepsilon\to0^+} w_{\varepsilon\alpha_\varepsilon}(x_\varepsilon,t_\varepsilon) : (x_\varepsilon,t_\varepsilon,\alpha_\varepsilon) \to (x,t,\alpha)\}\Big) \end{aligned}$$

satisfies (V1)–(V4) (*resp.* (W1)–(W4)) *relatively to the problem* (2.1)–(2.2).

Proof. Let us prove that the family $\underline{v}_\alpha$, defined in (4.7), satisfies (V1)–(V4).

Property (V1) follows immediately from the upper semicontinuity of the $\limsup^*$ and from the non positiveness of $v_{\varepsilon\alpha}(x,t)$.

Property (V2) holds since $-d(x,\Gamma_\alpha^0)$ is u.s.c. in x and in α (see also Remark 2.1).

Property (V3) is satisfied because of the u.s.c. of $\limsup^*$ and because of hypothesis (4.6).

Let us prove (V4).

Fixed $\alpha_0 \in \mathbb{R}$, let $(x_0,t_0) \in \mathbb{R}^N \times (0,+\infty)$ be a local maximum point for $\underline{v}_{\alpha_0} - \psi$ where $\psi \in C^{2,1}(\mathbb{R}^N \times (0,+\infty))$ and $v_\alpha(x_0,t_0) = \psi(x_0,t_0)$. Set $\overline{\psi}(x,t\alpha) = \psi(x,t) + \omega(|\alpha-\alpha_0|)$ where ω is the upper semicontinuity modulus of the function $\underline{v}_\alpha(x,t)$ in a neighborhood of the point (x_0,t_0,α_0). Then (x_0,t_0,α_0) is a local maximum point for the function $\underline{v}_\alpha(x,t) - \overline{\psi}(x,t,\alpha)$. From Lemma 4.1, there exists a sequence $(x_\varepsilon,t_\varepsilon,\alpha_\varepsilon)$ of local maximum points for $v_{\varepsilon\alpha}(x,t) - \overline{\psi}(x,t,\alpha)$ converging to (x_0,t_0,α_0) and such that $v_{\varepsilon\alpha_\varepsilon}(x_\varepsilon,t_\varepsilon)$ converges to $\underline{v}_{\alpha_0}(x_0,t_0)$. Since $v_{\varepsilon\alpha}(x,t)$ satisfies (V4) relatively to F_ε, we have

$$\begin{cases} \psi_t - F_{\varepsilon*}(z_\varepsilon,t_\varepsilon,D\psi,D^2\psi) \le 0 \\ v_{\varepsilon\alpha_\varepsilon}(x_\varepsilon,t_\varepsilon) = -|x_\varepsilon - z_\varepsilon|. \end{cases} \tag{4.9}$$

Now $x_\varepsilon \to x_0$ and $v_{\varepsilon\alpha_\varepsilon}(x_0,t_0) \to \underline{v}_{\alpha_0}(x_0,t_0)$, so $|z_\varepsilon - x_\varepsilon|$ converges to $|z_0 - x_0|$, where z_0 is such that $\underline{v}_{\alpha_0}(x_0,t_0) = -|z_0 - x_0|$. For $\varepsilon \to 0^+$ in (4.9), we get

$$\{\ \psi_t - F_*(z_0,t_0,D\psi,D^2\psi) \le 0 \underline{v}_{\alpha_0}(x_0,t_0) = -|x_0 - z_0|.$$

Therefore also (V4) is proved.

In similar way it can be proved that $\overline{w}_\alpha(x,t)$ satisfies (W1)–(W4).

Remark 4.1. Hypothesis (4.6) in the previous proposition corresponds to the hypothesis of local uniform boundedness on the sequence u_ε in Theorem 2.2.

As consequence of the previous proposition, we have the following stability result for families of distance functions.

Theorem 4.1. *Under the same hypothesis of Proposition* 4.2*, let* u_ε *and* u *be a sequence of solutions of* (2.8)–(2.2) *and, respectively, the unique solution of* (2.1)–(2.2)*. Define, for any* $(x,t) \in \mathbb{R}^N \times [0,+\infty)$ *and* $\alpha \in \mathbb{R}$,

$$\begin{aligned}
\overline{d}_\alpha(x,t) &= \liminf_{*\ \varepsilon\to 0^+} d(x,\gamma_{\varepsilon\alpha}(t)) \\
&= \inf\{\liminf_{\varepsilon\to 0^+} d(x_\varepsilon,\gamma_{\varepsilon\alpha_\varepsilon}(t_\varepsilon)) : (x_\varepsilon,t_\varepsilon,\alpha_\varepsilon) \to (x,t,\alpha)\}, \\
\underline{d}_\alpha(x,t) &= \limsup_{\varepsilon\to 0^+}{}^{*}\, d(x,\gamma_{\varepsilon\alpha}(t)) \\
&= \sup\{\limsup_{\varepsilon\to 0^+} d(x_\varepsilon,\gamma_{\varepsilon\alpha_\varepsilon}(t_\varepsilon)) : (x_\varepsilon,t_\varepsilon,\alpha_\varepsilon) \to (x,t,\alpha)\}.
\end{aligned}$$

Then

(i) *Let* $\underline{u}$ *be given by formula* (3.9) *with* $v_\alpha(x,t) = \underline{d}_\alpha(x,t) \wedge 0$ *and* $\overline{u}$ *by formula* (3.16) *with* $w_\alpha(x,t) = \overline{d}_\alpha(x,t) \vee 0$*, then*

$$\underline{u} = \overline{u} = u \quad \text{for any } (x,t) \in \mathbb{R}^N \times [0,+\infty).$$

(ii) *For any* $\alpha \in \mathbb{R}$,

$$\begin{aligned}
\underline{d}_\alpha(x,t) &\le -d(x,\Gamma_\alpha(t)) \text{ in } \{\underline{d}_\alpha(x,t) < 0\} = \{-d(x,\Gamma_\alpha(t)) < 0\} \\
\overline{d}_\alpha(x,t) &\ge d(x,\Delta_\alpha(t)) \text{ in } \{\overline{d}_\alpha(x,t) > 0\} = \{d(x,\Delta_\alpha(t)) > 0\}.
\end{aligned} \tag{4.10}$$

(iii) *If, for* $(x,t) \in \mathbb{R}^N \times [0,+\infty)$ *fixed,* $d(x,\Gamma_\alpha(t))$ *(resp.* $d(x,\Delta_\alpha(t))$*) is continuous, as function of* α*, in* α_0 *then*

$$d(x,\Gamma_{\alpha_0}(t)) = \lim_{\varepsilon\to 0^+} d(x,\Gamma_{\varepsilon\alpha_0}(t))$$

$$\Big(\text{resp. } d(x,\Delta_{\alpha_0}(t)) = \lim_{\varepsilon\to 0^+} d(x,\Delta_{\varepsilon\alpha_0}(t))\Big).$$

Proof. By definition, it follows that

$$\underline{d}_\alpha(x,t) \geq \overline{d}_\alpha(x,t). \tag{4.11}$$

Observe that (see f.e. [15])

$$\begin{aligned}\underline{d}_\alpha(x,t)\wedge 0 &= \limsup_{\varepsilon\to 0^+}{}^{*}(d(x,\gamma_{\varepsilon\alpha}(t))\wedge 0) = \limsup_{\varepsilon\to 0^+}{}^{*}\, v_{\varepsilon\alpha}(x,t)\\ \overline{d}_\alpha(x,t)\vee 0 &= \liminf{}_{*\,\varepsilon\to 0^+}(d(x,\gamma_{\varepsilon\alpha}(t))\vee 0) = \liminf{}_{*\,\varepsilon\to 0^+}\, w_{\varepsilon\alpha}(x,t).\end{aligned}$$

Since we can always assume that u and u_ε are bounded, hypothesis (4.6) is satisfied. Proposition 4.2 implies that $\underline{d}_\alpha(x,t)\wedge 0$ satisfies (V1)–(V4), $\overline{d}_\alpha(x,t)\vee 0$ satisfies (W1)–(W4). Therefore we can define a function $\underline{u}$ by means of formula (3.9) with $v_\alpha(x,t) = \underline{d}_\alpha(x,t)\wedge 0$ and a function $\overline{u}$ by means of formula (3.16) with $w_\alpha(x,t) = \overline{d}_\alpha(x,t)\vee 0$. The comparison principle yields

$$\underline{u} \leq \overline{u} \qquad \text{in } \mathbb{R}^N\times[0,+\infty). \tag{4.12}$$

Moreover, because of (4.11), we have

$$\overline{u}(x,t) = \min\{\alpha : \overline{d}(x,t)\vee 0 \leq 0\} \leq \max\{\alpha : \underline{d}(x,t)\wedge 0 \geq 0\} = \underline{u}(x,t)$$

and therefore

$$\overline{u}(x,t) = \underline{u}(x,t) = u(x,t) \quad \text{for any } (x,t)\in\mathbb{R}^n\times[0,+\infty), \tag{4.13}$$

where u is the unique solution of (2.1)–(2.2).

Let us prove *(ii)*. Define

$$\begin{aligned}\gamma_\alpha^{out}(t) &= \{x\in\mathbb{R}^N : d(x,\Delta_\alpha(t)) > 0\}\\ \gamma_\alpha^{in}(t) &= \{x\in\mathbb{R}^N : -d(x,\Gamma_\alpha(t)) < 0\}.\end{aligned}$$

Arguing as in Theorem 3.1, we deduce from (4.13) that

$$\begin{aligned}\gamma_\alpha^{out}(t) &= \{x\in\mathbb{R}^N : \overline{d}_\alpha(x,t)\vee 0 > 0\}\\ \gamma_\alpha^{in}(t) &= \{x\in\mathbb{R}^N : \underline{d}_\alpha(x,t)\wedge 0 < 0\}\end{aligned} \tag{4.14}$$

for any $t > 0$ and $\alpha\in\mathbb{R}$.

Let us show that $\overline{d}_\alpha \vee 0$ is a supersolution of

$$\begin{cases} |Dz| - 1 = 0 & \text{in } \gamma_\alpha^{out}(t) \\ z = 0 & \text{in } \mathbb{R}^N \setminus \gamma_\alpha^{out}(t) \end{cases} \tag{4.15}$$

for any $\alpha \in \mathbb{R}$ and $t \in \mathbb{R}^+$ such that $\gamma_\alpha^{out}(t)$ is not empty, $\underline{d}_\alpha \wedge 0$ is a subsolution of

$$\begin{cases} -|Dz| + 1 = 0 & \text{in } \gamma_\alpha^{in}(t) \\ z = 0 & \text{in } \mathbb{R}^N \setminus \gamma_\alpha^{in}(t) \end{cases} \tag{4.16}$$

for any $\alpha \in \mathbb{R}$ and $t \in \mathbb{R}^+$ such that $\gamma_\alpha^{in}(t)$ is not empty.

We observe that for any $\varepsilon > 0$, $d(x, \gamma_{\varepsilon\alpha}(t))$ is a solution of

$$\begin{cases} |Dz| - 1 = 0 & \text{in } \{x : d(x, \gamma_{\varepsilon\alpha}(t)) > 0\} \\ -|Dz| + 1 = 0 & \text{in } \{x : d(x, \gamma_{\varepsilon\alpha}(t)) < 0\}. \end{cases}$$

Let x_0 be a local minimum point for $(\overline{d}_{\alpha_0}(\cdot, t_0) - \psi)(x)$, where $\psi \in C^2(\mathbb{R}^N)$. Setting $\overline{\psi}(x, t, \alpha) = \psi(x) + \omega(|\alpha - \alpha_0| + |t - t_0|)$, where ω is the upper semicontinuity modulus of $\underline{d}$, we get as in Proposition 4.2 that there exists a sequence $(x_\varepsilon, t_\varepsilon, \alpha_\varepsilon)$ of local minimum points for $d(x, \gamma_{\varepsilon\alpha}(t)) - \overline{\psi}(x, t, \alpha)$ converging to (x_0, t_0, α_0) and such that $d(x_\varepsilon, \gamma_{\varepsilon\alpha_\varepsilon}(t_\varepsilon))$ converges to $\overline{d}_{\alpha_0}(x_0, t_0)$. If $\overline{d}_{\alpha_0}(x_0, t_0) > 0$, then for n large enough $d(x_\varepsilon, \gamma_{\varepsilon\alpha_\varepsilon}(t_\varepsilon)) > 0$ and therefore

$$|D\psi(x_\varepsilon)| - 1 \geq 0.$$

Passing to the limit for $\varepsilon \to 0^+$ in the previous equation we have

$$|D\psi(x_0)| - 1 \geq 0.$$

We can argue similarly for (4.16).

Since $d(x, \Delta_\alpha(t))$ is a solution of (4.15) and $-d(x, \Gamma_\alpha(t))$ is a solution of (4.16), a comparison theorem for positive solutions of problem (4.15) and negative solutions of (4.16) (see [2]) implies that

$$\begin{aligned} d(x, \Delta_\alpha(t)) &\leq \overline{d}_\alpha(x, t) \vee 0 \\ -d(x, \Gamma_\alpha(t)) &\geq \underline{d}_\alpha(x, t) \wedge 0. \end{aligned}$$

and therefore (ii).

Let us prove (iii). Fixed (x,t), because of (ii), we have

$$-d(x,\Gamma_{\alpha_0}(t)) \geq \limsup_{\varepsilon\to 0^+}(-d(x,\Gamma_{\varepsilon\alpha_0}(t))).$$

For any fixed $\delta > 0$, let $\alpha_\delta > \alpha_0$ such that

$$-d(x,\Gamma_{\alpha_\delta}(t)) \geq -d(x,\Gamma_{\alpha_0}(t)) - \delta. \tag{4.17}$$

Let $\overline{x}$ be such that $-d(x,\Gamma_{\alpha_\delta}(t)) = -|x-\overline{x}|$. Since u_ε converges to u and $\alpha_\delta > \alpha_0$, there exists ε_δ such that for any $\varepsilon < \varepsilon_\delta$, $u_\varepsilon(\overline{x},t) \geq \alpha_0$. Therefore, for any $\varepsilon < \varepsilon_\delta$, $\overline{x} \in \Gamma_{\varepsilon\alpha_0}(t)$ and

$$-d(x,\Gamma_{\varepsilon\alpha_0}(t)) \geq -|x-\overline{x}| = -d(x,\Gamma_{\alpha_\delta}(t)) \geq -d(x,\Gamma_{\alpha_0}(t)) - \delta.$$

Then $-d(x,\Gamma_{\alpha_0}(t)) \leq \liminf_{\varepsilon\to 0^+}(-d(x,\Gamma_{\varepsilon\alpha_0}(t)))$ and *iii)* is proved. □

Recalling that $d(x,\gamma_\alpha(t)) = -d(x,\Gamma_\alpha(t)) + d(x,\Delta_\alpha(t))$, we easily get the following corollary.

Corollary 4.1. *If, for (x,t) fixed, $d(x,\gamma_\alpha(t))$ is continuous in α_0, then*

$$d(x,\gamma_{\varepsilon\alpha_0}(t)) \to d(x,\gamma_{\alpha_0}(t))$$

for $\varepsilon \to 0^+$.

There are essentially two reasons for which $d(x,\gamma_\alpha(t))$ can be discontinuous in α_0 ((x,t) fixed).

(i) $\gamma_\alpha(t) = \emptyset$ for any $\alpha < \alpha_0$ (or for any $\alpha > \alpha_0$) and $\gamma_{\alpha_0}(t) \neq \emptyset$;
(ii) $int\{\gamma_{\alpha_0}(t)\} \neq \emptyset$.

In the first case, $d(x,\gamma_\alpha(t))$ jumps from $\pm\infty$ to 0. In the second case, the distance between $x \in \gamma_{\alpha_0}(t)$ and any other α-level sets is positive, either for any $\alpha > \alpha_0$ or for any $\alpha < \alpha_0$.

Definition 4.1. We say that $x \in \gamma_\alpha(t)$ is a *regular point* of $\gamma_\alpha(t)$ if there exists a sequence $x_m \in \gamma_\alpha^{in}(t)$ and a sequence $y_m \in \gamma_\alpha^{out}(t)$ converging to x.

Lemma 4.2. *If x is a regular point of $\gamma_{\alpha_0}(t)$, then $d(x,\gamma_\alpha(t))$ is continuous, as function of α, in α_0.*

Proof. Since $d(x,\gamma_\alpha(t)) = -d(x,\Gamma_\alpha(t)) + d(x,\Delta_\alpha(t))$, it is sufficient to show that $-d(x,\Gamma_\alpha(t))$ and $d(x,\Delta_\alpha(t))$ are continuous in α_0.

Let us consider $-d(x, \Gamma_\alpha(t))$. Since this function is u.s.c. and non increasing in α, it is sufficient to show that for any sequence α_n converging to α_0 with $\alpha_n > \alpha_0$,

$$\lim_{n\to+\infty} -d(x, \Gamma_{\alpha_n}(t)) = -d(x, \Gamma_{\alpha_0}(t)). \tag{4.18}$$

Since x is a regular point, there exists $x_m \in \gamma^{in}_{\alpha_0}(t)$ converging to x with $u(x_m, t) = \beta_m > \alpha_0$. For the continuity of u, $\beta_m \to \alpha_0$, therefore (4.18) is verified for the sequence β_m. Then it is easy to see that (4.18) is verified for any other sequence $\alpha_n > \alpha_0$ converging to α_0.

Theorem 4.2. *If all the points of $\gamma_{\alpha_0}(t)$ are regular, then*

$$\lim_{\varepsilon\to 0} d(x, \gamma_{\varepsilon\alpha_0}(t)) = d(x, \gamma_{\alpha_0}(t)) \quad \textit{for any } x \in \mathbb{R}^N. \tag{4.19}$$

Proof. From Lemma 4.2 and Corollary 4.1, (4.19) is verified for $x \in \gamma_{\alpha_0}(t)$. Let $x \notin \gamma_{\alpha_0}(t)$ and, for example, $x \in \gamma^{in}_{\alpha_0}(t)$. Let $\overline{x} \in \gamma_{\alpha_0}(t)$ such that $d(x, \gamma_{\alpha_0}(t)) = -d(x, \Gamma_{\alpha_0}(t)) = -|x - \overline{x}|$. It is sufficient to show that $d(x, \Gamma_\alpha(t))$, as function of α, is continuous in α_0.

Let $x_m \in \gamma^{in}_{\alpha_0}(t)$ such that $u(x_m, t) = \alpha_m > \alpha$ and such that $x_m \to \overline{x}$. Since

$$-d(x, \Gamma_\alpha(t)) \geq -d(x, \Gamma_{\alpha_m}(t)) \geq -|x - x_m|$$

for any m and $|x - x_m|$ converges to $|x - \overline{x}|$, it follows that $-d(x, \Gamma_{\alpha_m}(t))$ converges to $-d(x, \Gamma_\alpha(t))$. Then it is easy to see that this is true for any other sequence α_n converging to α_0. $\square$

An interface is regular if, for example, it is the boundary of an open set and it is not fat. Moreover it is simple to see that the convergence of $d(x, \gamma_{\varepsilon\alpha}(t))$ to $d(x, \gamma_\alpha(t))$ for all $x \in \mathbb{R}^N$ implies the convergence of $\gamma_{\varepsilon\alpha}(t)$ to $\gamma_\alpha(t)$ in the sense of the Hausdorff distance, if $\gamma_\alpha(t)$ is compact. In general, we can conclude that only the regular part of $\gamma_\alpha(t)$, i.e. the subset of the regular points of $\gamma_\alpha(t)$, is approximated by $\gamma_{\varepsilon\alpha}(t)$.

5. Application to the convergence of approximation schemes. In this section we will show an application of the stability result to the convergence of level sets generated by approximation schemes both for a class of first order geometric equations and for the mean curvature flow equation.

Given a function $u_h\colon \mathbb{R}^N \times \mathbb{N} \to \mathbb{R}$, we set

$$\gamma^h_\alpha(n) = \{x : u_h(x,n) = \alpha\}, \tag{5.1}$$

$$\Delta^h_\alpha(n) = \{x : u_h(x,n) \le \alpha\}, \tag{5.2}$$

$$\Gamma^h_\alpha(n) = \{x : u_h(x,n) \ge \alpha\}, \tag{5.3}$$

and, for $(x,t) \in \mathbb{R}^N \times \mathbb{N}$,

$$d^h_\alpha(x,n) = d(x, \gamma^h_\alpha(n)), \tag{5.4}$$

$$w^h_\alpha(x,n) = d(x, \Delta^h_\alpha(n)), \tag{5.5}$$

$$v^h_\alpha(x,n) = -d(x, \Gamma^h_\alpha(n)), \tag{5.6}$$

with the same conventions of Section 3 if one of these sets is empty.

5.1. First order geometric equation. We consider the problem

$$u_t - H(x, Du) = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) \tag{5.7}$$

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{5.8}$$

where H is geometric and satisfies

$$|H(x,p) - H(z,q)| \le C\big((|p| \vee |q|)|x-z| + |p-q|\big). \tag{5.9}$$

Under the previous hypothesis, the problem (5.7)–(5.8) has been studied in [18], [25] by means of its controllistic interpretation. In fact, the Hamiltonian H admits the following representation formula

$$H(x,p) = \min_{|b|\le 1} \max_{|a|\le 1} \big\{ -f(x,a,b)\cdot p \big\}, \tag{5.10}$$

where $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function which satisfies

$$|f(x,a,b) - f(y,a,b)| \le L_f|x-y| \qquad x,y \in \mathbb{R}^N,\ a,b \in B(0,1).$$

By means of (5.10), the equation (5.7) can be rewrite as

$$u_t - \min_{|b|\le 1} \max_{|a|\le 1} \big\{ -f(x,a,b)\cdot Du \big\} = 0 \tag{5.11}$$

which is the dynamic programming equation for a finite horizon pursuit-evasion dynamic game. Under the previous hypothesis, problem (5.7)–(5.8) satisfies a comparison theorem and therefore it admits a unique viscosity solution u, given by the upper value function of the associated differential game.

The approximation scheme we consider is based on the controllistic form (5.11) of the equation and it has been studied in [1] for a pursuit-evasion differential game. For $h > 0$, discretization step, we set

$$u_h(x, n+1) = \min_{|b|\leq 1} \max_{|a|\leq 1} \{u_h(x - hf(x, a, b), n)\} \quad n \in \mathbb{N}, \tag{5.12}$$

$$u_h(x, 0) = u_0(x). \tag{5.13}$$

It is well known that the sequence u_h converges to u, unique solution of (5.7)–(5.8), locally uniformly in $\mathbb{R}^N \times [0, +\infty)$ (see f.e. [1], [8]). We will prove that the corresponding level sets satisfy a stability property.

It is simple to prove the following lemma

Lemma 5.1. *The function $u_h : \mathbb{R}^N \times \mathbb{N} \to \mathbb{R}$ is continuous in x and verifies*

$$|u_h(x, n) - u_h(y, n)| \leq \omega(|x - y|(1 + L_f h)^n)$$

where ω is the continuity modulus of u_0.

Since u_h is continuous, the sets $\Delta^h_\alpha(n)$, $\Gamma^h_\alpha(n)$ and $\gamma^h_a(n)$ are closed. The distance functions from the discrete fronts satisfy certain properties analogous to those which hold for the continuous problem (see Prop. 3.1).

Proposition 5.1. *The family v^h_α satisfies the following properties*

$(V1)_h$ *For every $\alpha \in \mathbb{R}$, $v^h_\alpha : \mathbb{R}^N \times \mathbb{N} \to \mathbb{R} \cup \{-\infty\}$ is Lipschitz continuous in x and non positive.*

$(V2)_h$ *Defined $\Gamma^0_\alpha = \{x : u_0(x) \geq \alpha\}$, we have*

$$v^h_\alpha(x, 0) \leq -d(x, \Gamma^0_\alpha)$$

for $x \in \mathbb{R}^N$, $\alpha \in \mathbb{R}$.

$(V3)_h$ *For every $(x, n) \in \mathbb{R}^N \times \mathbb{N}$, v^h_α is non increasing and u.s.c. in α. Moreover $\lim_{\alpha\to+\infty} v^h_\alpha(x, t) < 0$.*

$(V4)_h$ *For every $\alpha \in \mathbb{R}$, v^h_α is a solution of*

$$\begin{cases} v(x, n+1) \leq \min\limits_{|b|\leq 1} \max\limits_{|a|\leq 1} \{v(x - hf(z, a, b), n)\} \\ v(x, n+1) = -|x - z|. \end{cases} \tag{5.14}$$

Moreover, for a fixed $n \in \mathbb{N}$, v^h_α *is a viscosity solution of*

$$-|Dv| + 1 = 0 \qquad \text{in } \{x : v(x,n) < 0\} \tag{5.15}$$

for all α *and* n *such that* $v^h_\alpha(\cdot, n)$ *is not identically infinite.*

Proof. We will prove only property $(V4)_h$, since the other properties can be proved as in Prop. 3.1.

By definition of v^h_α, it is equivalent to show that

$$d(x, \Gamma^h_\alpha(n+1)) \geq \max_{|b|\leq 1} \min_{|a|\leq 1} \left\{ d(x - hf(z,a,b), \Gamma^h_\alpha(n)) \right\}.$$

Let $z \in \Gamma^h_\alpha(n+1)$ such that $|z - x| = d(x, \Gamma^h_\alpha(n+1))$. Since $z \in \Gamma^h_\alpha(n+1)$, it follows that for every $b \in B(0,1)$, there exists exists $a_b \in B(0,1)$ such that $z_b = z - hf(z, a_b, b) \in \Gamma^h_\alpha(n)$. Thus, for every $b \in B(0,1)$, we have

$$\begin{aligned} d(x,z) &= |x - z| = |x - z_b + z_b - z| = d(x - hf(z, a_b, b), z_0) \\ &\geq d(x - hf(z, a_b, b), \Gamma^h_\alpha(n)) \geq \min_{|a|\leq 1} \left\{ d(x - hf(z,a,b), \Gamma^h_\alpha(n)) \right\} \end{aligned}$$

and therefore

$$d(x, \Gamma^h_\alpha(n+1)) \geq \max_{|b|\leq 1} \min_{|a|\leq 1} \left\{ d(x - hf(z,a,b), \Gamma^h_\alpha(n)) \right\}.$$

Equation (5.15) follows from the fact that, for n and α fixed, v^h_α is a distance function on $\mathbb{R}^N$ and $\{x : v^h_\alpha(x,n) > 0\}$ is open.

For distance functions from sublevel sets we have

Proposition 5.2. *The family* w^h_α *satisfies the following properties*

$(W1)_h$ *For every* $\alpha \in \mathbb{R}$, *the function* $w^h_\alpha : \mathbb{R}^N \times \mathbb{N} \to \mathbb{R} \cup \{+\infty\}$ *is Lipschitz continuous in* x *and non negative.*

$(W2)_h$ *Defined* $\Delta^0_\alpha = \{x : u_0(x) \leq \alpha\}$, *we have*

$$w^h_\alpha(x, 0) \geq d(x, \Delta^0_\alpha)$$

for all $x \in \mathbb{R}^N$, $\alpha \in \mathbb{R}$.

$(W3)_h$ *For all* $(x,n) \in \mathbb{R}^N \times \mathbb{N}$, w^h_α *is non decreasing and l.s.c. in* α. *Moreover* $\lim_{\alpha \to -\infty} w^h_\alpha(x,t) > 0$.

$(W4)_h$ *For every fixed* $\alpha \in \mathbb{R}$, w^h_α *satisfies*

$$\begin{cases} w(x, n+1) \geq \min\limits_{|b|\leq 1} \max\limits_{|a|\leq 1} \left\{ w(x - hf(z,a,b), n) \right\} \\ w(x, n+1) = |x - z|. \end{cases} \tag{5.16}$$

Moreover w^h_α is a viscosity solution of the equation

$$|Dw| - 1 = 0 \qquad \text{in } \{x : w(x,n) > 0\} \tag{5.17}$$

for any n and α such that $w^h_\alpha(\cdot,n)$ is not identically infinite.

Proof. Also in this case we will only prove property $(W4)_h$. We have to show that

$$d(x, \Delta^h_\alpha(n+1)) \geq \min_{|b|\leq 1} \max_{|a|\leq 1} \left\{ d(x - hf(z,a,b), \Delta^h_\alpha(n)) \right\}$$

Let $z \in \Delta^h_\alpha(n+1)$ such that $d(x, \Delta^h_\alpha(n+1)) = |x - z|$. Since $z \in \Delta^h_\alpha(n+1)$, there exists $\bar{b} \in B(0,1)$ such that for every $a \in B(0,1)$ we have $z_{\bar{b}a} = z - hf(z,a,\bar{b}) \in \Delta^h_\alpha(n)$. Thus, for every $a \in B(0,1)$, we have

$$\begin{aligned} d(x,z) &= |x - z| = |x - z_{\bar{b}a} + z_{\bar{b}a} - z| = d(x - hf(z,a,\bar{b}), z_{\bar{b}a}) \\ &\geq d(x - hf(z,a,\bar{b}), \Delta^h_\alpha(n)). \end{aligned}$$

Thus, it follows that there exists $\bar{b} \in B(0,1)$ such that

$$d(x, \Delta^h_\alpha(n+1)) \geq \max_{a \in B(0,1)} \left\{ d(x - hf(z,a,\bar{b}), \Delta^h_\alpha(n)) \right\}$$

and therefore w^h_α satisfies (5.16).

We have the following stability result.

Theorem 5.1. *Let u_h be the sequence of the solution of* (5.12)–(5.13)*, u the unique viscosity solution of* (5.7)–(5.8) *and let $\gamma_\alpha(t)$ and $\gamma^h_\alpha(n)$ be defined as in* (3.4) *and respectively in* (5.4)*. Assume that, for any compact set $K \subset \mathbb{R}^N$ and $N \in \mathbb{N}$, there exists $c \in \mathbb{R}^+$ (depending on K and N) such that*

$$\lim_{\alpha \to +\infty} v_{h\alpha}(x,n) \leq -c \qquad \Big(\textit{resp.} \quad \lim_{\alpha \to -\infty} w_{h\alpha}(x,n) \geq c\Big)$$

for any $x \in K$ and $n \leq N$ and for any $h > 0$. Set

$$\begin{aligned} \overline{d}_\alpha(x,t) &= \liminf_{*\; h \to 0^+} d(x, \gamma^h_\alpha(n)) \\ \underline{d}_\alpha(x,t) &= \limsup^{*}_{h \to 0^+} d(x, \gamma^h_\alpha(n)). \end{aligned}$$

Then

(i) *Let* $\underline{u}$ *be given by formula* (3.9) *with* $v_\alpha(x,t) = \underline{d}_\alpha(x,t) \wedge 0$ *and* $\overline{u}$ *by formula* (3.16) *with* $w_\alpha(x,t) = \overline{d}_\alpha(x,t) \vee 0$, *then*

$$\underline{u} = \overline{u} = u \qquad \text{for any } (x,t) \in \mathbb{R}^N \times [0,+\infty)$$

where u *is the unique solution of* (5.7)–(5.8).

(ii)

$$\underline{d}_\alpha(x,t) \le -d(x,\Gamma_\alpha(t)) \quad \text{in } \{\underline{d}_\alpha(x,t) < 0\} = \{-d(x,\Gamma_\alpha(t)) < 0\}$$
$$\overline{d}_\alpha(x,t) \ge d(x,\Delta_\alpha(t)) \quad \text{in } \{\overline{d}_\alpha(x,t) > 0\} = \{d(x,\Delta_\alpha(t)) > 0\}$$

(iii) *If, for* $(x,t) \in \mathbb{R}^N \times [0,+\infty)$ *fixed,* $d(x,\Gamma_\alpha(t))$ *(resp.* $d(x,\Delta_\alpha(t))$*) is continuous in* α_0, *then, for* $nh \to t$, *we have*

$$d(x,\Gamma_{\alpha_0}(t)) = \lim_{h\to 0^+} d(x,\Gamma^h_{\alpha_0}(nh))$$
$$\Big(\textit{resp.} \qquad d(x,\Delta_{\alpha_0}(t)) = \lim_{h\to 0^+} d(x,\Delta^h_{\alpha_0}(t))\Big).$$

Proof. The proof of the result follows exactly the same line of Theorem 4.1. In fact, observe that (5.14) is *consistent* in the sense of Barles-Souganidis (see [8]) with (3.8), i.e., for any sequence $(x_h, n_h h) \to (x,t)$ and for any sequence $z_h \to z$ for $h \to 0^+$, we have that

$$\lim_{h\to 0} \frac{\phi(x_h,(n_h+1)h) - \phi(x_h,n_h h)}{h} - \min_{|b|\le 1}\max_{|a|\le 1}\Big\{\frac{1}{h}\big[\phi(x_h - hf(z_h,a,b), n_h h) - \phi(x_h,n_h h)\big]\Big\} = \frac{\partial\phi}{\partial t}(x,t) - H(z, D\phi(x,t))$$

for any $\phi \in C^1(\mathbb{R}^N \times [0,+\infty))$. Similarly for (5.16) and (3.15). □

If the interface is regular, as in the continuous case, we have convergence of distance functions and convergence in the sense of Hausdorff distance if the interface $\gamma_{\alpha_0}(t)$ is compact.

Theorem 5.2. *If all the points of* $\gamma_{\alpha_0}(t)$ *are regular, then for* $nh \to t$,

$$\lim_{h\to 0} d(x,\gamma^h_{\alpha_0}(nh)) = d(x,\gamma_{\alpha_0}(t)) \qquad \text{for any } x \in \mathbb{R}^N.$$

5.2. Mean curvature flow equation. In this part we will study convergence properties of an approximation scheme proposed by Bence, Merriman and Osher [10] for the mean curvature flow equation (2.4). Convergence of approximate solutions to the viscosity solution of equation (2.4) has been proved in [16], [5].

Let us describe the scheme. First define an operator $\mathcal{H}$ on the class of closed subsets of $\mathbb{R}^N$ by

$$\mathcal{H}(t)C = \left\{x \in \mathbb{R}^N : u(x,t) \geq \tfrac{1}{2}\right\}$$

where u is the solution of the problem

$$\begin{cases} u_t - \Delta u = 0 & (x,t) \in \mathbb{R}^N \times (0,+\infty) \\ u(x,0) = \chi_C(x) & x \in \mathbb{R}^N \end{cases} \tag{5.18}$$

(χ denotes the indicator function of the set C). Note that u is given explicitly by

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} \chi_C(y) dy \tag{5.19}$$

For a discretization step $h > 0$, the approximation scheme is given by

$$\begin{cases} u_h(x, n+1) = \big[H(h)u_h(\cdot, n)\big](x) & n > 0 \\ u_h(x,0) = u_0(x) \end{cases} \tag{5.20}$$

where $H(t)$ is the operator on $BUC(\mathbb{R}^N)$ defined by

$$\big[H(t)f\big](x) = \sup\{\lambda \in \mathbb{R} : x \in \mathcal{H}(t)[f \geq \lambda]\} \tag{5.21}$$

or, equivalently, by

$$\big[H(t)f\big](x) = \inf\{\lambda \in \mathbb{R} : x \in \mathcal{H}(t)[f \leq \lambda]\} \tag{5.22}$$

We refer to [16] for properties of the operators $\mathcal{H}$ and H.

Remark 5.1. We observe that formula (5.21) can be rewritten as

$$\big[H(t)f\big](x) = \sup\{\lambda \in \mathbb{R} : v_\lambda^h(x,n) \geq 0\}.$$

where v_λ^h defined as in (5.6) and similarly for (5.22). Therefore for the discrete problem we have a representation formula similar to that which holds for the continuous problem (see (3.9) and (3.16)) with the heat flow $\mathcal{H}(t)$ replacing the mean curvature flow.

Let us first prove a lemma, which essentially tells that also the discrete scheme has a geometric property.

Lemma 5.2. *Let u_h be the sequence given by* (5.20) *with $u_h(x,0) = u_0(x)$ and let $\Psi : \mathbb{R} \to \mathbb{R}$ a strictly increasing continuous function. Then $\hat{u}_h(x,n) = \Psi(u_h(x,n))$ is the sequence given by* (5.20) *with initial datum $\hat{u}_h(x,0) = \Psi(u_0(x))$.*

Proof. By induction, for $n = 0$ the assertion is true. Let us suppose that it is true for n. Then

$$\begin{aligned}&\hat{u}_h(x,n+1) = \sup\{\lambda \in \mathbb{R} : x \in \mathcal{H}(h)[\Psi(u_h(x,n)) \geq \lambda]\}\\ &= \sup\{\lambda \in \mathbb{R} : x \in \mathcal{H}(h)[u_h(x,n) \geq \Psi^{-1}(\lambda)]\}.\end{aligned}$$

Similarly

$$\hat{u}_h(x,n+1) = \inf\{\lambda \in \mathbb{R} : x \in \mathcal{H}(h)[u_h(x,n) \leq \Psi^{-1}(\lambda)]\}$$

and therefore

$$\hat{u}_h(x,n+1) = \Psi(u_h(x,n+1)).$$

We define the distance functions from the discrete level sets as in (5.4)–(5.6). We have

Proposition 5.3. *The family v^h_α satisfies the following properties*

$(V1)_h$ *For every $\alpha \in \mathbb{R}$, $v^h_\alpha : \mathbb{R}^N \times \mathbb{N} \to \mathbb{R} \cup \{-\infty\}$ is Lipschitz continuous in x and non positive.*

$(V2)_h$ *Defined $\Gamma^0_\alpha = \{x : u_0(x) \geq \alpha\}$, we have*

$$v^h_\alpha(x,0) \leq -d(x,\Gamma^0_\alpha)$$

for $x \in \mathbb{R}^N$, $\alpha \in \mathbb{R}$.

$(V3)_h$ *For every $(x,n) \in \mathbb{R}^N \times \mathbb{N}$, v^h_α is non increasing and u.s.c. in α. Moreover $\lim_{\alpha\to+\infty} v_\alpha(x,t) < 0$.*

$(V4)_h$ *For every $\alpha \in \mathbb{R}$, v^h_α satisfies*

$$\begin{cases} \text{for any } \phi \in C^0(\mathbb{R}^N \times \mathbb{N}) \text{ s.t. } v^h_\alpha - \phi \text{ has a global} \\ \text{maximum point at } (\hat{x},\hat{n}), \text{ then} \\ \frac{1}{2} - \frac{1}{(4\pi h)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|\hat{x}-y|^2}{4h}} \chi_{\{\phi(\cdot,\hat{n}-1)-\phi(\hat{x},\hat{n})\geq 0\}}(y)dy \leq 0 \end{cases} \tag{5.23}$$

Moreover, for a fixed $n \in \mathbb{N}$, v^h_α is a viscosity solution of

$$-|Dv| + 1 = 0 \qquad in\ \{x : v(x,n) < 0\}$$

for all α and $n \in \mathbb{N}$ such that v^h_α is not identically infinite.

Proof. We will prove only $(V4)_h$. Let $\phi \in C^0(\mathbb{R}^N \times \mathbb{N})$ and let $(\hat{x}, \hat{n})$ be a global maximum point for $v^h_\alpha - \phi$. Let $\hat{z} \in \Gamma^h_\alpha(\hat{n})$ such that $v^h_\alpha(\hat{x}, \hat{n}) = -|\hat{x} - \hat{z}|$. Set $\overline{\phi}(z,n) = \phi(\hat{x} - \hat{z} + z, n)$. Then there exists an increasing continuous function Ψ (see [17], Theorem 2.2 for the explicit construction of Ψ) such that $\Psi(u_h) - \overline{\phi}$ has a global maximum point at $(\hat{z}, \hat{n})$. Thus

$$u_h(z, \hat{n} - 1) - u_h(\hat{z}, \hat{n}) \le \overline{\phi}(z, \hat{n} - 1) - \overline{\phi}(\hat{z}, \hat{n})$$

for any $z \in \mathbb{R}^N$. By monotonicity of $H(t)$, we get

$$0 = \big[H(h)(u_h(\cdot, \hat{n} - 1) - u_h(\hat{z}, \hat{n}))\big](\hat{z}) \le \big[H(h)(\overline{\phi}(\cdot, \hat{n} - 1) - \overline{\phi}(\hat{z}, \hat{n}))\big](\hat{z}).$$

It follows that $\hat{z} \in \mathcal{H}(h)[\overline{\phi}(\cdot, \hat{n} - 1) - \overline{\phi}(\hat{z}, \hat{n}) \ge 0]$, i.e., $\hat{x} \in \mathcal{H}(h)[\phi(\cdot, \hat{n} - 1) - \phi(\hat{x}, \hat{n}) \ge 0]$. From the definition of $\mathcal{H}(t)$ and (5.19), we get property $(V4)_h$.

Correspondent properties hold also for the distance function form the sublevel sets. In [5] it has been proved that (5.23) is equivalent to say that the discrete scheme is *consistent* in the sense of Barles-Souganidis with the mean curvature flow equation. Therefore a result completely analogous to Theorem 5.1 and the convergence of distance functions in the regular case hold also for the Bence-Merriman-Osher scheme.

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