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ELLIPTIC BOUNDARY VALUE PROBLEMS INVOLVING MEASURES: EXISTENCE, REGULARITY, AND MULTIPLICITY

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Abstract. In this paper we study linear and semilinear second order elliptic boundary value problems with measures as data. We are particularly interested in the noncoercive case where we establish existence and multiplicity results, given suitable growth and monotonicity restrictions for the nonlinearities.

1. Introduction. This paper is concerned with semilinear elliptic boundary value problems (BVPs) of the types

$$\left.\begin{aligned} -\Delta u &= f(x,u,\nabla u)+\mu && \text{in } \Omega\ , \\ u &= 0 && \text{on } \Gamma_0\ , \\ \partial_\nu u &= g(x,u)+\sigma_1 && \text{on } \Gamma_1\ , \end{aligned}\right\} \tag{1.1}$$

and

$$\left.\begin{aligned} -\Delta u &= h(x,u)+\mu && \text{in } \Omega\ , \\ u &= \sigma_0 && \text{on } \Gamma_0\ , \\ \partial_\nu u &= \sigma_1 && \text{on } \Gamma_1\ . \end{aligned}\right\} \tag{1.2}$$

Here Ω is a bounded smooth domain in $\mathbb{R}^n$, where $n \geq 2$, and $\Gamma := \Gamma_0 \cup \Gamma_1$ denotes its boundary, the 'Dirichlet boundary' Γ_0 and the 'Neumann boundary' Γ_1 being open in Γ and disjoint. Moreover, μ, σ_0, and σ_1 are bounded

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Radon measures, and f, g, and h are given continuous functions satisfying the following growth restrictions:

$$\left.\begin{aligned} |f(x,\xi,\eta)| &\le c(1+|\xi|^{r_1}+|\eta|^{r_0})\ , \\ |g(y,\xi)| &\le c(1+|\xi|^{r_2})\ , \\ |h(x,\xi)| &\le c(1+|\xi|^{r_0}) \end{aligned}\right\} \tag{1.3}$$

for $(\xi,\eta)\in\mathbb{R}\times\mathbb{R}^n$ and $(x,y)\in\Omega\times\Gamma$, where

$$1<r_0<\frac{n}{n-1}\ ,\quad 1<r_1<\frac{n}{n-2}\ ,\quad 1<r_2<\frac{n-1}{n-2}\ . \tag{1.4}$$

We show that these problems — in fact, their generalizations which are obtained by replacing $-\Delta$ by a general second order strongly uniformly elliptic operator in divergence form, and ∂_ν by a corresponding boundary operator — can be put in a functional-analytical framework. Using this fact we will be able to derive existence, regularity, and multiplicity results by invoking methods from nonlinear functional analysis.

Elliptic problems involving measures have been studied by many authors. Much of the published work deals with the problem of obtaining precise information on the singularities of (possible) solutions. For this aspect, which we do not touch here, we refer to [31], [25], and the references therein.

As for existence results: in almost all papers known to us existence is proven by approximating the given problems by smooth ones and by subsequent limiting procedures. This requires a priori estimates, of course. Thus this approach is essentially restricted to coercive problems, that is, to the case where f is independent of η, and f, g, and h are decreasing in ξ (at least asymptotically). In this situation it is, however, possible to replace $-\Delta$ by a second order quasilinear operator satisfying suitable monotonicity conditions (see the survey [11] and the references therein, as well as [21], [26], [15], [16], for example; in this connection we also refer to [12] and [9]).

In our paper we are primarily concerned with the noncoercive case. In this situation only very few existence results seem to be known. In fact, Baras and Pierre [8] study the existence of positive solutions for

$$-\Delta u = u^{r_1}+\lambda\mu \text{ in } \Omega\ ,\quad u=0 \text{ on } \Gamma\ , \tag{$1.5)_\lambda$}$$

where μ is a positive bounded Radon measure and $\lambda\in\mathbb{R}^+$. They show that $(1.5)_\lambda$ has for each sufficiently small $\lambda>0$ at least one positive solution (in a suitably generalized sense) belonging to $L_{1,\mathrm{loc}}$. They also show that the restriction for r_1 specified in (1.4) is optimal: if $r_1 \geq n/(n-2)$, then there exists a positive $\mu \in L_1(\Omega)$ such that $(1.5)_\lambda$ has no solution for $\lambda > 0$.

The methods of [8] rely heavily on the fact that $(1.5)_\lambda$ involves the Laplace operator. Indeed, it is one of the main points of [1] that the results of [8] remain valid if $-\Delta$ is replaced by $-\nabla \cdot (\boldsymbol{a}\nabla \cdot) + a_0$, where $\boldsymbol{a}(x)$ is a positive definite $(n \times n)$-matrix depending continuously differentiably on $x \in \overline{\Omega}$, and $a_0 \in L_\infty(\Omega)$ is nonnegative.

In [31, Theorems 3.9 and 3.10] it is shown that there exists $\lambda^* > 0$ such that $(1.5)_\lambda$ has for each $\lambda \in (0, \lambda^*)$ at least one positive solution and no such solution for $\lambda > \lambda^*$, provided $0 \in \Omega$ and $\mu = \delta_0$, the Dirac mass at zero. The proof uses in an essential way the facts that the differential operator is the Laplacian and that μ is Dirac's measure.

As a very special case of our multiplicity result it follows that there exists $\lambda^* := \lambda^*(\mu) > 0$ such that $(1.5)_\lambda$ has at least two positive solutions for $0 < \lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$, provided $\mu \neq 0$.

Another multiplicity result for radial solutions of the problem

$$-\Delta u = \lambda_1 e^u - \lambda_2 \delta_0 \text{ for } |x| < 1 \;, \quad u = 0 \text{ for } |x| = 1 \;,$$

where λ_1, λ_2 are suitable positive numbers, was recently obtained by Spielmann [28]. His result is based on a phase plane analysis and it cannot be extended to non-radial situations.

In [2] the authors consider problem (1.1) with $\Gamma_1 = \emptyset$ and assume that f is nonnegative and convex with respect to η. Then they show that (1.1) does not possess a positive solution if μ is large enough, provided f is independent of ξ and superlinear in η (see [2, Theorem 2.1]). In particular, it follows that there exists $\lambda^* > 0$ such that

$$-\Delta u = |\nabla u|^{r_0} + \lambda\mu \text{ in } \Omega \;, \quad u = 0 \text{ on } \Gamma \;, \tag{1.6$_\lambda$}$$

does not have a positive (weak) solution for $\lambda > \lambda^*$. But in [2] there is no existence result for $(1.6)_\lambda$. The conditional existence assertion of [2, Theorem 5.1] guarantees that $(1.6)_\lambda$ possesses a positive solution if there exists a positive supersolution (in the distributional sense). However, the existence of such a supersolution has not been established.

It is an easy consequence of our Theorem 9.4 and of the positivity assertion of Theorem 5.1 that there exists $\lambda_* > 0$ such that $(1.6)_\lambda$ has for each

$\lambda \in (0, \lambda_*)$ a positive solution u_λ. In fact, it is the only small positive solution and the map $\lambda \mapsto u_\lambda$ is continuously differentiable from $(0, \lambda_*)$ into $W_p^1(\Omega)$ with $1 \le p < n/(n-1)$.

Now we describe some of our main results for problems (1.1) and (1.2). For easy statements we impose more stringent conditions than are needed in some cases. More precise and more general theorems are found in the main body of this paper. We also restrict ourselves to the case where f is independent of η. Thus we can handle problems (1.1) and (1.2) simultaneously. However, there is an important difference: if the Dirichlet boundary is charged by a nontrivial measure, we cannot allow a nonlinearity on the Neumann boundary and we have to impose a more severe growth restriction.

To be more precise, we consider the problem

$$\left.\begin{aligned} -\Delta u &= f(x,u) + \lambda\mu && \text{in } \Omega \ , \\ u &= \lambda\sigma && \text{on } \Gamma_0 \ , \\ \partial_\nu u &= g(x,u) + \lambda\sigma && \text{on } \Gamma_1 \ , \end{aligned}\right\} \tag{1.7$_\lambda$}$$

where μ and σ are bounded positive Radon measures on Ω and Γ, respectively, such that $(\mu,\sigma) \ne (0,0)$, and $\lambda \in \mathbb{R}^+$. We also suppose that f and g are continuously differentiable and that

$$|\partial_2 f(\cdot,\xi)| \le c(1 + |\xi|^{r-1}) \ , \quad |\partial_2 g(\cdot,\xi)| \le c(1 + |\xi|^{\rho-1}) \ , \qquad \xi \in \mathbb{R} \ ,$$

where either

$$\left.\begin{aligned} &\sigma\,|\,\Gamma_0 = 0 \ , \quad 1 < r < n/(n-2) \ , \quad 1 < \rho < (n-1)/(n-2) \ , \\ &\text{or} \\ &\sigma\,|\,\Gamma_0 > 0 \ , \quad 1 < r < n/(n-1) \ , \quad g = 0 \ . \end{aligned}\right\} \tag{1.8}$$

If $\sigma\,|\,\Gamma_0 = 0$, we can assume that

$$n/(n-1) < r < n/(n-2) \quad \text{and} \quad \rho \le (1 - 1/n)r \tag{1.9}$$

by increasing r, if necessary. Then we put $p_\bullet := nr/(n+r)$.

By a solution u of $(1.7)_\lambda$ we mean a weak solution in the following sense:

- if $\sigma\,|\,\Gamma_0 = 0$, then $u \in W_{p_\bullet}^1(\Omega)$ with $u\,|\,\Gamma_0 = 0$ and

$$\int_\Omega \nabla v \cdot \nabla u \, dx = \int_\Omega v f(\cdot,u)\, dx + \lambda \int_\Omega v\, d\mu + \int_{\Gamma_1} v g(\cdot,u)\, d\Gamma + \lambda \int_{\Gamma_1} v\, d\sigma$$

for all $v \in C^1(\overline{\Omega})$ satisfying $v = 0$ on Γ_0;

- if $\sigma \,|\, \Gamma_0 > 0$, then $u \in L_r(\Omega)$ and

$$-\int_\Omega u\Delta v\,dx = \int_\Omega v f(\cdot,u)\,dx + \lambda\int_\Omega v\,d\mu - \lambda\int_{\Gamma_0} \partial_\nu v\,d\sigma + \lambda\int_{\Gamma_1} v\,d\sigma$$

for all $v \in C^2(\overline{\Omega})$ satisfying $v = 0$ on Γ_0 and $\partial_\nu v = 0$ on Γ_1.

Of course, it is understood that all conditions on Γ hold in the sense of traces. Note that $p_\bullet$, resp. r, is the smallest number such that $f(\cdot,u) \in L_1(\Omega)$ and $g(\cdot,u) \in L_1(\Gamma_1)$ for all $u \in W^1_{p_\bullet}$ if $\sigma \,|\, \Gamma_0 = 0$, and $f(\cdot,u) \in L_1(\Omega)$ for all $u \in L_r(\Omega)$ if $\sigma \,|\, \Gamma_0 > 0$, respectively.

It should be observed that, thanks to the results of Baras and Pierre [8], the growth restriction on f is optimal if $\sigma \,|\, \Gamma_0 = 0$. On the other hand, if $\sigma \,|\, \Gamma_0 > 0$, then the optimal condition should perhaps be $r < (n+1)/(n-1)$ as is suggested by [31, Section 4.1]. Note, however, that in this reference another concept of solution is employed that is weaker than ours.

The following result shows that solutions to $(1.7)_\lambda$ possess much better regularity properties than required for the well-posedness of the problem.

Theorem 1.1. *If u is a solution of $(1.7)_\lambda$, then*

$$u \in \begin{cases} W_1^{2-\varepsilon}(\Omega) & \text{if } \sigma \,|\, \Gamma_0 = 0\ , \\ W_1^{1-\varepsilon}(\Omega) & \text{if } \sigma \,|\, \Gamma_0 > 0\ , \end{cases}$$

for each $\varepsilon \in (0,1)$. Thus, by Sobolev's embedding theorem,

$$u \in \begin{cases} W_p^1(\Omega) & \text{if } \sigma \,|\, \Gamma_0 = 0\ , \\ L_p(\Omega) & \text{if } \sigma \,|\, \Gamma_0 > 0\ , \end{cases}$$

for $1 \le p < n/(n-1)$.

The first part of this regularity theorem, which is a special case of Theorem 9.3, is new even in the simplest case. All the other works dealing with nonlinear elliptic problems involving measures guarantee only that $u \in W_p^1(\Omega)$ for $1 \le p < n/(n-1)$ if $\sigma \,|\, \Gamma_0 = 0$.

Next we describe some of our existence results, where we restrict ourselves to the case of positive solutions. For this we assume, in addition to the above growth restrictions, of course, that

$$\left.\begin{aligned}&\big(f(\cdot,0),g(\cdot,0)\big)=(0,0),\\&\text{there exist } \omega\geq 0 \text{ and } \omega_\Gamma\geq 0 \text{ such that}\\&\partial_2 f(\cdot,\xi)\geq -\omega\ ,\quad \partial_2 g(\cdot,\xi)\geq -\omega_\Gamma\ ,\qquad \xi\in\mathbb{R}^+\ ,\\&\text{and}\\&\partial_2 f(\cdot,0)<\alpha_0\ ,\quad \partial_2 g(\cdot,0)\leq 0\ ,\end{aligned}\right\}\tag{1.10}$$

where α_0 is the least eigenvalue of the elliptic eigenvalue problem

$$-\Delta u=\alpha u \text{ in } \Omega\ ,\quad u=0 \text{ on } \Gamma_0\ ,\quad \partial_\nu u=0 \text{ on } \Gamma_1\ .$$

The following theorem is a vast generalization of all the known existence results for the noncoercive case. Its proof is given in Section 14.

Theorem 1.2. *Let* (1.10) *be satisfied. Then there exists* $\lambda^*\in(0,\infty]$ *such that problem* $(1.7)_\lambda$ *has for each* $\lambda\in(0,\lambda^*)$ *a positive solution* $\overline{u}_\lambda$ *and no solution at all for* $\lambda>\lambda^*$. *If* $\lambda\in[0,\lambda^*)$, *then every solution* u_λ *of* $(1.7)_\lambda$ *satisfies* $u_\lambda\geq\overline{u}_\lambda$, *where* $\overline{u}_0:=0$. *The map* $\lambda\mapsto\overline{u}_\lambda$ *is left continuous and strictly increasing from* $[0,\lambda^*)$ *into* $W^1_{p_\bullet}(\Omega)$ *if* $\sigma\,|\,\Gamma_0=0$, *and into* $L_r(\Omega)$ *if* $\sigma\,|\,\Gamma_0>0$. *If the maps* $\xi\mapsto f(x,\xi)$ *and* $\xi\mapsto g(y,\xi)$ *are either both convex or both concave on* $\mathbb{R}^+$ *for* $(x,y)\in\overline{\Omega}\times\Gamma$, *then* $\lambda\mapsto\overline{u}_\lambda$ *is continuously differentiable, and* $\overline{u}_\lambda$ *is an isolated solution of* $(1.7)_\lambda$ *in* $W^1_{p_\bullet}(\Omega)$ *or* $L_r(\Omega)$, *respectively. If there exists* $\beta>0$ *such that* $f(\cdot,\xi)\geq\alpha_0\xi-\beta$ *for* $\xi\in\mathbb{R}^+$ *then* $\lambda^*<\infty$.

Finally, we describe our main multiplicity result, a special instant of Theorem 14.1, in the present situation. For this we impose a superlinearity condition on f with a precise asymptotic behavior. Namely, we assume that

$$\left.\begin{aligned}&\text{there exist } t\in(1,r] \text{ and a bounded and everywhere}\\&\text{strictly positive function } \ell \text{ such that}\\&|\partial_2 f(\cdot,\xi)|\leq c\big(1+|\xi|^{t-1}\big)\\&\text{for } \xi\in\mathbb{R},\ \text{and}\\&\xi^{-t}f(x,\xi)\to\ell(x)\ \ (\xi\to\infty)\ ,\\&\text{uniformly with respect to } x\in\overline{\Omega}\ .\\&\text{If } \Gamma_0\neq\Gamma \text{ then } t<(n+1)/(n-1) \text{ and } \rho<(t+1)/2\ .\end{aligned}\right\}\tag{1.11}$$

It should be noted that t is restricted to be smaller than $(n+1)/(n-1)$ if $\Gamma_1 \neq \emptyset$, even if $g = 0$. (Of course, the boundedness of ℓ is a consequence of the uniform convergence in (1.11), which implies $\ell \in C(\overline{\Omega})$. However, in the main body of this paper we drop the continuity assumption on f with respect to x. Then ℓ is no longer automatically bounded. Moreover, it would seem natural to put $t = r$. But since we assume (1.9) if $\sigma \,|\, \Gamma_0 = 0$, for convenience, this would impose an unnecessary restriction.)

Theorem 1.3. *Let assumptions* (1.10) *and* (1.11) *be satisfied. Then there exists* $\lambda_* \in (0, \lambda^*]$ *such that problem* $(1.7)_\lambda$ *has for each* $\lambda \in [0, \lambda_*)$ *at least two solutions. If the maps* $\xi \mapsto f(x,\xi)$ *and* $\xi \mapsto g(y,\xi)$ *are both convex on* $\mathbb{R}^+$ *then* $\lambda_* = \lambda^*$*.*

For example, there exists $\lambda^* \in (0, \infty)$ such that

$$-\Delta u = u^t \text{ in } \Omega \;, \quad \partial_\nu u = u^\rho + \lambda\sigma \text{ on } \Gamma \;, \quad u \geq 0 \;,$$

has at least two solutions for $0 \leq \lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$, provided $t < (n+1)/(n-1)$ and $\rho < (t+1)/2$. Or there exists $\lambda^* \in (0, \infty)$ such that

$$-\Delta u = u^r + \lambda\mu \text{ in } \Omega \;, \quad u = 0 \text{ on } \Gamma \;, \quad u \geq 0 \;, \tag{1.12}$$

has at least two solutions for $0 \leq \lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$, provided $1 < r < n/(n-2)$.

There remains, of course, the question whether there exists a solution if $\lambda = \lambda^*$. For this we give a positive answer in Section 15 if we restrain the growth restriction even more. For simplicity, we confine ourselves to problem (1.12).

Theorem 1.4. *Suppose that* $1 < r < 2/(n-2)$*. Then problem* (1.12) *possesses exactly one solution for* $\lambda = \lambda^*$*.*

It should be observed that Theorem 1.4 restricts n to be 2 or 3.

Now we briefly describe the contents of the following sections. In Section 2 we collect the basic results from [5] on linear elliptic boundary value problems in interpolation-extrapolation spaces, and we discuss the relations between these spaces and bounded Radon measures. In Section 3 we establish local and global a priori estimates in these weak settings. The results of those two sections are basic for the whole paper.

In Section 4 we introduce the concepts of weak and very weak solutions for linear elliptic boundary value problems involving measures. As an easy application of the results of Section 2 we prove a regularity theorem which is optimal in the framework of Sobolev-Slobodeckii spaces. We also formulate global a priori estimates.

It is worthwhile to point out that everything proven in Sections 2–4 holds for (normally) elliptic systems as well.

Section 5 discusses positivity properties. In particular, we establish the inverse positivity of elliptic boundary value problems in the weak settings as well as existence and monotonicity properties of a principal eigenvalue.

It should be mentioned that the results of Sections 2–5 are also of importance for the study of semilinear parabolic equations involving measures. This will be done in a forthcoming paper.

In Section 6 we start with the discussion of semilinear problems. In particular, we establish our general regularity result and a local existence and uniqueness theorem. In the following section we generalize the well-known sub- and supersolution theorem to semilinear problems involving measures.

In Section 8 we turn our interest to parameter-dependent problems and positive solutions. By specializing the general results of the preceding section we show that the set of parameters for which there exists a positive solution is an interval and that there exists a least positive solution $\overline{u}_\lambda$ such that the map $\lambda \mapsto \overline{u}_\lambda$ is increasing and left continuous. We also derive a sufficient condition for this interval to be bounded. In addition, we establish an abstract condition — in terms of fixed point index properties of $\overline{u}_\lambda$ and a priori bounds for families of solutions — guaranteeing the existence of at least two solutions.

It should be observed that Sections 6–8 deal with general, possibly nonlocal nonlinearities depending nonlinearly on measures as well.

Sections 9–11 are devoted to problems of the types (1.1) and (1.2) involving local nonlinearities and general elliptic differential operators (in Section 9 even systems are admissible). Besides of establishing some technical lemmas we adapt the results of Sections 6–8 to that setting.

It should be mentioned that the existence and structure theorems proven in Sections 6–11 are more or less straightforward applications of some of the results on fixed point equations in ordered Banach spaces contained in [3]. This is due to the fact that — on the basis of our results for linear elliptic boundary value problems in spaces of measures, developed in the earlier sections — we can reformulate the boundary value problems as equivalent fixed point equations in suitable ordered Banach spaces, where the nonlinear maps are order-preserving. However, in the present case we have to work in spaces whose positive cones have empy interiors. This is due to the fact that the solutions of our problem are unbounded, in general, and so do not belong to $C(\overline{\Omega})$. This is in contrast to the situation for classical boundary value problems. Thus we cannot rely on those results of [3] which use properties of positive cones with nonempty interiors. This is the case for the multiplicity results proven in [3] and the computations of local fixed point indices, in particular. Hence, in order to determine fixed point indices we have to work much harder. This is done in Section 12 where we also derive a condition guaranteeing that the λ-interval, for which $\overline{u}_\lambda$ exists, is not reduced to $\{0\}$.

In order to use the information on the fixed point index established in Section 12 to prove multiplicity theorems, we have to establish a priori bounds for all solutions. This is done in Section 13 by building on ideas of Gidas and Spruck [19]. In Section 14 we combine the results of the preceding sections to prove the main result of this paper as far as the nonlinear case is being concerned. In the last section we briefly discuss the question of the existence of a solution for $\lambda = \lambda^*$.

2. Linear Elliptic Boundary Value Problems. We write $\mathbb{E} := \mathbb{E}(\Omega)$ for the Banach space

$$C^2(\overline{\Omega}, \mathbb{R}^{n\times n}_{\mathrm{sym}}) \times C^1(\overline{\Omega}, \mathbb{R}^n) \times C^1(\overline{\Omega}, \mathbb{R}^n) \times C(\overline{\Omega}, \mathbb{R}) \times C^1(\Gamma, \mathbb{R}) ,$$

with $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ being the space of symmetric $(n \times n)$-matrices. We define for each $(\boldsymbol{a}, \vec{b}, \vec{c}, a_0, d) \in \mathbb{E}$ a linear differential operator $\mathcal{A}$ on Ω by

$$\mathcal{A}u := -\nabla \cdot (\boldsymbol{a}\nabla u + \vec{b}u) + \vec{c} \cdot \nabla u + a_0 u$$

and a boundary operator $\mathcal{B}$ on Γ by

$$\mathcal{B}u := \delta\big\{\partial_{\nu_a} u + (\gamma\vec{b} \cdot \vec{\nu} + d)\gamma u\big\} + (1 - \delta)\gamma u \ .$$

Here $\delta : \Gamma \to \{0, 1\}$ is defined by $\delta^{-1}(j) := \Gamma_j$ for $j \in \{0, 1\}$, and γ denotes the trace operator. Moreover, $\partial_{\nu_a} u := \vec{\nu} \cdot \gamma(\boldsymbol{a}\nabla u)$ is the conormal derivative with respect to $\mathcal{A}$, where $\vec{\nu}$ is the outer unit-normal on Γ.

Then $(\mathcal{A}, \mathcal{B})$ is said to be a linear BVP on Ω (of order at most two in divergence form). We topologize the set of these BVPs by identifying them with $\mathbb{E}$ by means of $(\mathcal{A}, \mathcal{B}) \leftrightarrow (\boldsymbol{a}, \vec{b}, \vec{c}, a_0, d)$. Then $\mathcal{E} := \mathcal{E}(\Omega)$ is defined to be the subset of all strongly uniformly elliptic BVPs, that is, $(\mathcal{A},\mathcal{B})\in\mathbb{E}$ belongs to $\mathcal{E}$ iff $\boldsymbol{a}(x)$ is positive definite for each $x\in\overline{\Omega}$. Note that $\mathcal{E}$ is open in $\mathbb{E}$.

We associate with each $(\mathcal{A},\mathcal{B})\in\mathcal{E}$ its formally adjoint BVP, defined by

$$\mathcal{A}^\sharp v := -\nabla\cdot(\boldsymbol{a}\nabla v+\vec{c}v)+\vec{b}\cdot\nabla v+a_0 v$$

and

$$\mathcal{B}^\sharp v := \delta\big\{\partial_{\nu_a}v+(\gamma\vec{c}\cdot\vec{\nu}+d)\gamma v\big\}+(1-\delta)\gamma v\ ,$$

as well as its Dirichlet form

$$\mathfrak{a}(v,u) := \langle\nabla v,\boldsymbol{a}\nabla u+\vec{b}u\rangle+\langle v,\vec{c}\cdot\nabla u+a_0u\rangle+\langle\gamma v,d\gamma u\rangle_\partial\ .$$

Here $\langle u,v\rangle := \int_\Omega uv\,dx$ is the $L_p(\Omega)$-duality pairing and $\langle u,v\rangle_\partial := \int_\Gamma uv\,d\Gamma$ stands for the $L_p(\Gamma)$-pairing.

We denote by $W_p^s := W_p^s(\Omega)$ the Sobolev-Slobodeckii spaces for $s\in\mathbb{R}$ and $1\le p<\infty$, and write $\|\cdot\|_{s,p}$ for their norms, where $\|\cdot\|_{0,p}=\|\cdot\|_p$. We also put

$$\partial W_p^s := W_p^{s-1/p}(\Gamma_0)\times W_p^{s-1-1/p}(\Gamma_1)\ ,\qquad s\in\mathbb{R}\ ,\quad 1<p<\infty\ .$$

Here and in the following it is understood that obvious modifications are employed if Γ_0 or Γ_1 is empty.

If E and F are locally convex spaces, we write $\mathcal{L}(E,F)$ for the space of all continuous linear operators from E into F, endowed with the topology of uniform convergence on bounded sets, and $\mathcal{L}\mathrm{is}(E,F)$ is the subset of all isomorphisms in $\mathcal{L}(E,F)$. Recall that $\mathcal{L}\mathrm{is}(E,F)$ is open in the Banach space $\mathcal{L}(E,F)$, if E and F are Banach spaces.

It follows from the trace theorem that $(\mathcal{A},\mathcal{B})\in\mathcal{L}(W_p^2,L_p\times\partial W_p^2)$ for $1<p<\infty$ and $(\mathcal{A},\mathcal{B})\in\mathbb{E}$. This fact can be expressed more precisely by

$$\mathbb{E}\hookrightarrow\mathcal{L}(W_p^2,L_p\times\partial W_p^2)\ ,\qquad 1<p<\infty\ , \tag{2.1}$$

where $\hookrightarrow$ denotes continuous injection.

It is the purpose of the following considerations to associate with an elliptic BVP $(\mathcal{A},\mathcal{B})\in\mathcal{E}$ suitable weak formulations. For this we introduce the Banach spaces $W_{p,\mathcal{B}}^s$ for $s\in[-2,2]\backslash(\mathbb{Z}+1/p)$ and $1<p<\infty$ by

$$W_{p,\mathcal{B}}^s := \begin{cases}\{\,u\in W_p^s\ ;\ \mathcal{B}u=0\,\}\ , & 1+1/p<s\le 2\ ,\\ \{\,u\in W_p^s\ ;\ (1-\delta)\gamma u=0\,\}\ , & 1/p<s<1+1/p\ ,\\ W_p^s\ , & 0\le s<1/p\ ,\\ (W_{p',\mathcal{B}^\sharp}^{-s})'\ , & s\in[-2,0)\backslash(\mathbb{Z}+1/p)\ ,\end{cases}$$

where $p+p'=pp'$ and the dual space is formed with respect to the duality pairing $\langle\cdot,\cdot\rangle$ induced by the L_p-pairing. We denote the norm in $W^s_{p,\mathcal{B}}$ by $\|\cdot\|_{s,p}$ as well. This will not lead to any confusion since the spaces W^s_p will only be used for $s>-1+1/p$ in which case $W^s_{p,\mathcal{B}}$ is a closed linear subspace of W^s_p (see Remark 2.4(b)).

First we show that the 'negative' spaces $W^{-s}_{p,\mathcal{B}}$ and ∂W^{-s}_p contain bounded Radon measures if s and p satisfy suitable restrictions.

Let X be a σ-compact metric space and denote by $C_0(X)$ the Banach space of all continuous real-valued functions on X vanishing at infinity, endowed with the maximum norm. Then $\mathcal{M}(X)$, the Banach space of bounded Radon mesures on X, is the dual of $C_0(X)$, that is,

$$\mathcal{M}(X)=\big[C_0(X)\big]'\ , \tag{2.2}$$

with the usual identification of continuous linear functionals on $C_0(X)$ with bounded Radon measures.

Lemma 2.1. *Fix $(\mathcal{A},\mathcal{B})\in\mathcal{E}$ and $p\in(1,1^*)$, where $1^*:=n/(n-1)$.*

(i) *If $1/p<s<2-n/p'$ then*

$$\mathcal{M}(\Omega\cup\Gamma_1)\times\mathcal{M}(\Gamma_1)\hookrightarrow W^{s-2}_{p,\mathcal{B}}\times W^{s-1-1/p}_p(\Gamma_1)\ .$$

(ii) *If $0\le s<1-n/p'$ then*

$$\mathcal{M}(\Omega\cup\Gamma_1)\times\mathcal{M}(\Gamma)\hookrightarrow W^{s-2}_{p,\mathcal{B}}\times\partial W^s_p\ .$$

Proof. First note that $k<k+1-n/p'<k+1/p$ for $k\in\mathbb{N}$. Hence the spaces $W^s_{p,\mathcal{B}}$ are well-defined if the above restrictions for s are met. Also note that $2-s>n/p'>1/p'$ implies $(1-\delta)\gamma v=0$ for $v\in W^{2-s}_{p',\mathcal{B}^\sharp}$ and $s\ne 1/p$. Hence Sobolev's embedding theorem entails

$$W^{2-s}_{p',\mathcal{B}^\sharp}\overset{d}{\hookrightarrow}C_0(\Omega\cup\Gamma_1)\ ,\qquad 0\le s<2-n/p'\ ,\quad s\ne 1/p\ ,$$

as well as

$$W^{k+1/p-s}_{p'}(\Gamma_k)\overset{d}{\hookrightarrow}C(\Gamma_k)\ ,\qquad 0\le s<k+1-n/p'\ ,\quad k=0,1\ , \tag{2.3}$$

where $\overset{d}{\hookrightarrow}$ denotes dense injection. Thus (2.2) implies

$$\mathcal{M}(\Omega \cup \Gamma_1) \hookrightarrow W_{p,\mathcal{B}}^{s-2} , \qquad 0 \le s < 2 - n/p' , \quad s \ne 1/p ,$$

and $\left[W_{p'}^t(\Gamma)\right]' = W_p^{-t}(\Gamma)$ for $0 \le t \le 2$ guarantees, together with (2.2) and (2.3), that

$$\mathcal{M}(\Gamma_k) \hookrightarrow W_p^{s-k-1/p}(\Gamma_k) , \qquad k = 0, 1 .$$

This proves the assertions. $\square$

Remark 2.2. Suppose that $1 < p < 1^*$ and $0 \le s < 1 - n/p'$. Then $W_{p,\mathcal{B}}^{s-2}$ and $W_p^{s-1-1/p}(\Gamma_1)$ contain singular distributions which are not measures.

Proof. Suppose, for example, that $\mu_j \in \mathcal{M}(\Omega)$ has compact support in Ω for $1 \le j \le n$. Put $\vec{\mu} := (\mu_1, \ldots, \mu_n)$ and observe that $\nabla \cdot \vec{\mu}$ is a distribution of order 1 with compact support in Ω, which is not a measure, in general. Fix $\psi \in \mathcal{D}$, where $\mathcal{D} := \mathcal{D}(\Omega)$ is the space of test functions on Ω, such that ψ equals 1 on the support of $\vec{\mu}$, and set

$$\langle \nabla \cdot \vec{\mu}, v \rangle := \langle \nabla \cdot \vec{\mu}, \psi v \rangle = -\sum_{j=1}^{n} \int_\Omega \partial_j(\psi v)\, d\mu_j$$

for $v \in W_{p',\mathcal{B}^\sharp}^{2-s} \hookrightarrow C^1(\overline{\Omega})$. Then $\nabla \cdot \vec{\mu}$ belongs to $(W_{p',\mathcal{B}^\sharp}^{2-s})' = W_{p,\mathcal{B}}^{s-2}$, and the assertion is proven for $W_{p,\mathcal{B}}^{s-2}$.

Note that

$$W_{p'}^{1+1/p-s}(\Gamma_1) \overset{d}{\hookrightarrow} C^1(\Gamma_1) \overset{d}{\hookrightarrow} C(\Gamma_1)$$

so that $C^1(\Gamma_1)' \hookrightarrow W_p^{s-1-1/p}(\Gamma_1)$, which entails the assertion for the boundary space. $\square$

Suppose that $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}$ and that $s \in [0,2] \backslash (\mathbb{N} + 1/p)$. Consider the elliptic BVP

$$\mathcal{A}u = f \text{ in } \Omega , \quad \mathcal{B}u = g \text{ on } \Gamma , \tag{2.4}$$

with $(f,g) \in W_{p,\mathcal{B}}^{s-2} \times \partial W_p^s$. Then u is said to be a W_p^s**-solution** of (2.4) if the following is true: $u \in W_p^s$ and

- if $1 + 1/p < s \le 2$ then $\mathcal{A}u = f$ holds in Ω in the sense of distributions and $\mathcal{B}u = g$ on Γ in the sense of traces;

- If $1/p < s < 1 + 1/p$ then $(1-\delta)\gamma u = (1-\delta)g$ and
$$\mathfrak{a}(v,u) = \langle v, f\rangle + \langle \gamma v, \delta g\rangle_\partial \ , \qquad v \in W^{2-s}_{p',\mathcal{B}^\sharp} \ ;$$
- if $0 \le s < 1/p$ then
$$\langle \mathcal{A}^\sharp v, u\rangle = \langle v, f\rangle + \big\langle (\delta-1)\partial_{\nu_a} v + \delta\gamma v, g\big\rangle_\partial \ , \qquad v \in W^{2-s}_{p',\mathcal{B}^\sharp} \ .$$

More precisely: a W_p^s-solution is said to be

- strong if $1 + 1/p < s \le 2$;
- weak if $1/p < s < 1 + 1/p$;
- very weak if $0 \le s < 1/p$.

Note that a weak W_2^1-solution is a weak solution in the variational sense.

It is well-known that for each $(\mathcal{A},\mathcal{B}) \in \mathcal{E}$ there exists $\omega \in \mathbb{R}$ such that $(\lambda + \mathcal{A}, \mathcal{B})$ belongs to $\mathcal{L}\mathrm{is}(W_p^2, L_p \times \partial W_p^2)$ for $\operatorname{Re}\lambda > \omega$ and $1 < p < \infty$. This and (2.1) imply that
$$\mathcal{E}_0 := \mathcal{E}_0(\Omega) := \big\{\, (\mathcal{A},\mathcal{B}) \in \mathcal{E} \ ; \ (\mathcal{A},\mathcal{B}) \in \mathcal{L}\mathrm{is}(W_p^2, L_p \times \partial W_p^2), \ 1 < p < \infty \,\big\}$$
is a nonempty open subset of $\mathcal{E}$.

The following extension theorem is the basic solvability and continuity result for problem (2.4).

Theorem 2.3. *Suppose that $(\mathcal{A},\mathcal{B}) \in \mathcal{E}_0$. There exists, for each $p \in (1,\infty)$ and each $s \in [0,2]\backslash(\mathbb{N} + 1/p)$, a unique extension*
$$\mathfrak{A}_{s,p} \in \mathcal{L}\mathrm{is}(W_p^s, W^{s-2}_{p,\mathcal{B}} \times \partial W_p^s)$$
of $(\mathcal{A},\mathcal{B})$. If $1 < q \le p < \infty$ and $0 \le t \le s \le 2$ with $t \notin \mathbb{N} + 1/q$ then
$$\mathfrak{A}_{t,q} \supset \mathfrak{A}_{s,p} \ . \tag{2.5}$$
The map
$$\mathcal{E}_0 \to \mathcal{L}\mathrm{is}(W_p^s, W^{s-2}_{p,\mathcal{B}} \times \partial W_p^s) \ , \quad (\mathcal{A},\mathcal{B}) \mapsto \mathfrak{A}_{s,p} \tag{2.6}$$
is analytic.

Lastly, $u = \mathfrak{A}^{-1}_{s,p}(f,g)$ for $(f,g) \in W^{s-2}_{p,\mathcal{B}} \times \partial W_p^s$ iff u is a W_p^s-solution of (2.4).

Proof. This is a consequence of the results in [5, Sections 4–9] (also see [6]), where the analyticity of (2.6) follows from the fact that this map is the restriction to $\mathcal{E}_0$ of a continuous linear one. ☐

Since $\mathfrak{A}_{s,p}$ is uniquely determined by $(\mathcal{A},\mathcal{B})$ we denote it again by $(\mathcal{A},\mathcal{B})$ without fearing confusion.

Remarks 2.4. **(a)** Suppose that $\mathcal{K}$ is a compact subset of $\mathcal{E}$. Then there exists $\omega \in \mathbb{R}$ such that

$$(\lambda+\mathcal{A},\mathcal{B}) \in \mathcal{L}\mathrm{is}(W_p^s, W_{p,\mathcal{B}}^{s-2} \times \partial W_p^s) , \qquad 1 < p < \infty , \quad s \in [0,2]\backslash(\mathbb{N}+1/p) ,$$

for $\operatorname{Re}\lambda > \omega$ and $(\mathcal{A},\mathcal{B}) \in \mathcal{K}$.

Proof. This follows from Theorem 2.3 and the fact that the assertion is true for $s = 2$, as is well-known from the standard L_p-theory of elliptic BVPs (e.g., [6]). □

(b) Observe that $1/p < s < 1 + 1/p$ iff $1/p' < 2 - s < 1 + 1/p'$. Thus

$$W_{p',\mathcal{B}^\sharp}^{2-s} = W_{p',\mathcal{B}}^{2-s} = \left\{ v \in W_{p'}^{2-s} \ ; \ (1-\delta)\gamma v = 0 \right\} , \qquad 1/p < s < 1 + 1/p .$$

Also note that

$$W_{p,\mathcal{B}}^{-s} = W_p^{-s} = (\mathring{W}_{p'}^s)' , \qquad -1 + 1/p < -s \le 0 . \tag{2.7}$$

Proof. Equality (2.7) follows from the density of $\mathcal{D}$ in $W_{p'}^s$, that is, from $\mathring{W}_{p'}^s = W_{p'}^s = W_{p',\mathcal{B}^\sharp}^s$ for $0 \le s < 1/p'$. □

It should be mentioned that the above theory of generalized linear elliptic BVPs is an extension and simplification of the Lions-Magenes theory of generalized L_p-solutions (see [22]). However, there is an important difference: in [22] the right-hand side of $\mathcal{A}u = f$ is restricted to lie in a space of distributions on Ω, which, in general, cannot be explicitly described. In our case f has to belong to $W_{p,\mathcal{B}}^{s-2}$, which is a space of distributions iff the space $\mathcal{D}$ of test functions is dense in $W_{p',\mathcal{B}^\sharp}^{2-s}$. However, $W_{p,\mathcal{B}}^{s-2}$ has a natural description as the dual of $W_{p',\mathcal{B}^\sharp}^{2-s}$, which is most useful — as we shall see below — and makes it a natural and optimal choice.

3. A Priori Estimates. It is not difficult to derive a priori estimates from Theorem 2.3. For later use we include local estimates. Here and below $\mathbf{1}$ denotes the function which is equal to 1 everywhere.

Theorem 3.1. *Suppose that $p \in (1,\infty)$ and $s \in [0,2]\backslash(\mathbb{N}+1/p)$. Also suppose that $\varphi, \psi \in C^3(\overline{\Omega})$ with $\psi \,|\, \mathrm{supp}(\varphi) = \mathbf{1}$, and $\gamma(\boldsymbol{a}\nabla\varphi) = 0$ if $s < 1/p$. Then*

$$\|\varphi u\|_{s,p} \le c\big(\|\varphi\mathcal{A}u\|_{s-2,p} + \|\gamma\varphi\mathcal{B}u\|_{\partial W_p^s} + \|\psi u\|_{s-1,p}\big) , \qquad u \in W_p^s ,$$

uniformly with respect to $(\mathcal{A}, \mathcal{B})$ *in compact subsets of* $\mathcal{E}$.

Proof. First we note that

$$\partial_j \in \mathcal{L}(W^s_{p,\mathcal{B}}, W^{s-1}_{p,\mathcal{B}}) , \qquad 1/p < s < 1 + 1/p , \quad 1 \le j \le n . \tag{3.1}$$

Indeed, this is clear if $s \ge 1$. Thus assume $1/p < s < 1$. Then $W^{s-1}_{p,\mathcal{B}} = W^{s-1}_p$ by Remark 2.4(b). If $\Gamma = \Gamma_0$ then $W^s_{p,\mathcal{B}} = \mathring{W}^s_p$ and (3.1) follows from the well-known fact that

$$\partial_j \in \mathcal{L}(\mathring{W}^k_p, \mathring{W}^{k-1}_p) , \qquad k \in \mathbb{Z} , \quad 1 \le j \le n ,$$

and by interpolation, thanks to

$$\mathring{W}^s_p \doteq (\mathring{W}^k_p, \mathring{W}^{k+1}_p)_{s-k,p} , \qquad k < s < k+1 , \quad k \in \mathbb{Z} ,$$

where $(\cdot)_{\theta,p}$ is the real interpolation functor and $\doteq$ means 'equivalent norms' (e.g., [30] or [6]). If $\Gamma = \Gamma_1$ then $W^s_{p,\mathcal{B}} = W^s_p$ and (3.1) is obtained by the usual restriction procedure from $\partial_j \in \mathcal{L}\big(W^s_p(\mathbb{R}^n), W^{s-1}_p(\mathbb{R}^n)\big)$ (cf. [30, Theorem 4.2.2] and recall that $W^s_p = B^s_{p,p}$). Finally, in the intermediate case where $\Gamma \ne \Gamma_j$ for $j = 0, 1$, assertion (3.1) follows from the two cases just discussed and the fact that ∂_j is a local operator (by reducing the question by means of local coordinates to full-space and half-space problems, respectively).

Denote by M_a point-wise multiplication $u \mapsto au$. Then it is obvious that

$$(a \mapsto M_a) \in \mathcal{L}\big(C^1(\overline{\Omega}), \mathcal{L}(W^k_{p,\mathcal{B}})\big) , \qquad k \in \{0,1\} .$$

Since

$$W^s_{p,\mathcal{B}} \doteq (L_p, W^1_{p,\mathcal{B}})_{s,p} , \qquad s \in (0,1) \backslash \{1/p\} ,$$

(e.g., [5]), it follows, by employing duality and $(M_a)' \supset M_a$ as well, that

$$(a \mapsto M_a) \in \mathcal{L}\big(C^1(\overline{\Omega}), \mathcal{L}(W^s_{p,\mathcal{B}})\big) , \qquad s \in [-1,1] \backslash (\mathbb{Z} + 1/p) . \tag{3.2}$$

Now suppose that $1 + 1/p < s \le 2$. Then

$$\mathcal{A}(\varphi u) = \varphi \mathcal{A} u + [\mathcal{A}, \varphi](\psi u) , \quad \mathcal{B}(\varphi u) = \gamma \varphi \mathcal{B} u + [\mathcal{B}, \varphi](\psi u) , \tag{3.3}$$

with

$$[\mathcal{A},\varphi]u := -2\nabla\varphi\cdot\boldsymbol{a}\nabla u + \big((\vec{c}-\vec{b})\cdot\nabla\varphi - \nabla\cdot(\boldsymbol{a}\nabla\varphi)\big)u$$

and $[\mathcal{B},\varphi]u := \delta(\partial_{\nu_a}\varphi)\gamma u$. It is an easy consequence of (3.1) and (3.2) that

$$\big((\mathcal{A},\mathcal{B})\mapsto\big([\mathcal{A},\varphi],[\mathcal{B},\varphi]\big)\big)\in\mathcal{L}\big(\mathbb{E},\mathcal{L}(W_p^{s-1},W_{p,\mathcal{B}}^{s-2}\times\partial W_p^s)\big)\ . \tag{3.4}$$

Let $\mathcal{K}$ be a compact subset of $\mathcal{E}$. By Remark 2.4(a) we can fix $\omega>0$ such that $(\lambda+\mathcal{A},\mathcal{B})\in\mathcal{L}\mathrm{is}(W_{p,\mathcal{B}}^s,W_{p,\mathcal{B}}^{s-2})$. Then we obtain from (3.3) that

$$\varphi u = (\omega+\mathcal{A},\mathcal{B})^{-1}\big(\varphi\mathcal{A}u+\omega\varphi u+[\mathcal{A},\varphi](\psi u),\gamma\varphi\mathcal{B}u+[\mathcal{B},\varphi](\psi u)\big)\ . \tag{3.5}$$

Since the map $B\mapsto B^{-1}$ is smooth from $\mathcal{L}\mathrm{is}(E,F)$ into $\mathcal{L}(F,E)$, with E and F being Banach spaces, it follows that

$$\begin{aligned}\|\varphi u\|_{s,p}\le c\Big(&\|\varphi\mathcal{A}u\|_{s-2,p}+\omega\,\|\varphi u\|_{s-2,p}+\|[\mathcal{A},\varphi](\psi u)\|_{s-2,p}\\&+\|\gamma\varphi\mathcal{B}u\|_{\partial W_p^s}+\|[\mathcal{B},\varphi](\psi u)\|_{\partial W_p^s}\Big)\end{aligned}$$

for $u\in W_p^s$ and $(\mathcal{A},\mathcal{B})\in\mathcal{K}$. Now (3.4) implies the assertion.

Next suppose that $1/p<s<1+1/p$. Then

$$\mathfrak{a}(v,\varphi u)=\mathfrak{a}(\varphi v,u)+\big\langle v,C_\varphi(\psi u)\big\rangle \tag{3.6}$$

with

$$\langle v,C_\varphi u\rangle := \big\langle\nabla\varphi\cdot\boldsymbol{a}\nabla v+(\vec{c}\cdot\nabla\varphi)v,u\big\rangle=\big\langle\nabla v,(\boldsymbol{a}\nabla\varphi)u\big\rangle+\big\langle v,(\vec{c}\cdot\nabla\varphi)u\big\rangle\ .$$

Thus, using (3.1) and (3.2) together with the definition of the spaces $W_{p,\mathcal{B}}^s$, it is not difficult to verify that

$$C_\varphi\in\mathcal{L}\big(\mathbb{E},\mathcal{L}(W_{p,\mathcal{B}}^{s-1},W_{p,\mathcal{B}}^{s-2})\big)\ . \tag{3.7}$$

Since $v\in W_{p',\mathcal{B}^\sharp}^{2-s}$ implies $\varphi v\in W_{p',\mathcal{B}^\sharp}^{2-s}$ by Remark 2.4(b), it follows from (3.6) and the weak formulation of problem (2.4) that u being a W_p^s-solution of (2.4) implies $(1-\delta)\gamma(\varphi u)=(1-\delta)(\gamma\varphi)g$ and

$$\mathfrak{a}_\omega(v,\varphi u)=\big\langle v,\varphi f+\omega\varphi u+C_\varphi(\psi u)\big\rangle+\big\langle\gamma v,\delta(\gamma\varphi)g\big\rangle_\partial\ ,\qquad v\in W_{p',\mathcal{B}^\sharp}^{2-s}\ ,$$

where $\mathfrak{a}_\omega := \mathfrak{a} + \omega\langle\cdot,\cdot\rangle$. Consequently, by Theorem 2.3,

$$\varphi u = (\omega + \mathcal{A}, \mathcal{B})^{-1}\big(\varphi\mathcal{A}u + \omega\varphi u + C_\varphi(\psi u), \gamma\varphi\mathcal{B}u\big)\ , \qquad u \in W_p^s\ . \tag{3.8}$$

Now the assertion is implied by (3.7) and the arguments used in the preceding case.

Lastly, suppose that $0 \le s < 1/p$. Then

$$\langle \mathcal{A}^\sharp v, \varphi u\rangle = \big\langle \mathcal{A}^\sharp(\varphi v), u\big\rangle + \big\langle v, C_\varphi(\psi u)\big\rangle \tag{3.9}$$

with $\langle v, C_\varphi u\rangle := -\big\langle [\mathcal{A}^\sharp, \varphi]v, u\big\rangle$. Since

$$-[\mathcal{A}^\sharp, \varphi]v = 2\nabla\varphi\cdot\boldsymbol{a}\nabla v + av = 2\nabla v\cdot\boldsymbol{a}\nabla\varphi + av$$

where $a := (\vec{c} - \vec{b})\cdot\nabla\varphi + \nabla\cdot(\boldsymbol{a}\nabla\varphi) \in C^1(\overline{\Omega})$, it follows that $\gamma(\boldsymbol{a}\nabla\varphi) = 0$ implies $(1-\delta)\gamma\big([\mathcal{A}^\sharp, \varphi]v\big) = 0$ for $v \in W^{2-s}_{p',\mathcal{B}^\sharp}$. Hence

$$\big((\mathcal{A}, \mathcal{B}) \mapsto [\mathcal{A}^\sharp, \varphi]\big) \in \mathcal{L}\big(\mathbb{E}, \mathcal{L}(W^{2-s}_{p',\mathcal{B}^\sharp}, W^{1-s}_{p',\mathcal{B}^\sharp})\big)\ ,$$

thanks to (3.2). Consequently,

$$|\langle v, C_\varphi u\rangle| \le \|[\mathcal{A}^\sharp, \varphi]\|_{\mathcal{L}(W^{2-s}_{p',\mathcal{B}^\sharp}, W^{1-s}_{p',\mathcal{B}^\sharp})}\, \|v\|_{2-s,p'}\, \|u\|_{s-1,p}$$

for $(v, u) \in W^{2-s}_{p',\mathcal{B}^\sharp} \times W^{s-1}_{p,\mathcal{B}}$, which shows that

$$C_\varphi \in \mathcal{L}\big(\mathbb{E}, \mathcal{L}(W^{s-1}_{p,\mathcal{B}}, W^{s-2}_{p,\mathcal{B}})\big)\ . \tag{3.10}$$

Also note that

$$\mathcal{B}^\sharp(\varphi v) = \gamma\varphi\mathcal{B}^\sharp v + \delta(\partial_{\nu_a}\varphi)\gamma v = 0\ , \qquad v \in W^{2-s}_{p',\mathcal{B}^\sharp}\ , \tag{3.11}$$

thanks to $\gamma(\boldsymbol{a}\nabla\varphi) = 0$. Thus $\varphi v \in W^{2-s}_{p',\mathcal{B}^\sharp}$ whenever $v \in W^{2-s}_{p',\mathcal{B}^\sharp}$. Thus we infer from (3.9) that

$$\varphi u = (\omega + \mathcal{A}, \mathcal{B})^{-1}\big(\varphi\mathcal{A}u + \omega\varphi u + C_\varphi(\psi u), (\gamma\varphi)\mathcal{B}u\big) \tag{3.12}$$

and the assertion follows from (3.10) and the arguments of the first case. $\square$

Corollary 3.2. *Suppose that* $1 < p < \infty$ *and* $s \in [0,2]\backslash(\mathbb{N}+1/p)$. *Then*

$$\|u\|_{s,p} \le c\big(\|\mathcal{A}u\|_{s-2,p} + \|\mathcal{B}u\|_{\partial W_p^s} + \|u\|_{s-2,p}\big) , \qquad u \in W_p^s ,$$

uniformly with respect to $(\mathcal{A},\mathcal{B})$ *in compact subsets of* $\mathcal{E}$.

Proof. This follows from (3.5), (3.8), and (3.12), respectively, by setting $\varphi = \psi = \mathbf{1}$, and by observing that $[\mathcal{A},\mathbf{1}] = 0$, $[\mathcal{B},\mathbf{1}] = 0$, and $C_{\mathbf{1}} = 0$. □

Remarks 3.3.

(a) Suppose that $1 < p < \infty$ and $0 \le s < 1/p$. Also assume $\varphi, \psi \in C^3(\overline{\Omega})$ satisfy $\psi \,|\, \mathrm{supp}(\varphi) = \mathbf{1}$ and $\delta\partial_{\nu_a}\varphi = 0$. Given $\sigma \in (-1+1/p, s]$,

$$\|u\|_{s,p} \le c\big(\|\varphi\mathcal{A}u\|_{s-2,p} + \|\gamma\varphi\mathcal{B}u\|_{\partial W_p^s} + \|\psi u\|_{\sigma,p}\big) , \qquad u \in W_p^s ,$$

uniformly for $(\mathcal{A},\mathcal{B})$ in compact subsets of $\mathcal{E}$.

Proof. Observe that

$$\big((\mathcal{A},\mathcal{B}) \mapsto [\mathcal{A}^\sharp,\varphi]\big) \in \mathcal{L}\big(\mathbb{E}, \mathcal{L}(W^t_{p',\mathcal{B}^\sharp}, W^{t-1}_{p',\mathcal{B}^\sharp})\big) , \qquad 1/p' < t < 1+1/p' .$$

Hence the argument leading to (3.10) shows that

$$C_\varphi \in \mathcal{L}\big(\mathbb{E}, \mathcal{L}(W^\sigma_{p,\mathcal{B}}, W^{\sigma-1}_{p,\mathcal{B}})\big) , \qquad -1+1/p < \sigma < 1/p .$$

Since $\delta\partial_{\nu_a}\varphi = 0$ implies that $\varphi v \in W^{2-s}_{p',\mathcal{B}^\sharp}$ whenever $v \in W^{2-s}_{p',\mathcal{B}^\sharp}$, thanks to (3.11), the arguments of the preceding proof imply the statement. □

(b) Suppose that u is a W_p^s-solution of (2.4) and denote by X the complement of $\mathrm{supp}(f) \cup \mathrm{supp}(g)$ in $\overline{\Omega}$. Then $u \,|\, X \in C^2(X)$.

Proof. This follows by an obvious bootstrapping argument from Theorem 3.1. □

Remarks 3.4.

(a) Suppose that δ is constant on Γ. Then the exceptional set $\mathbb{N}+1/p$ can be replaced by $\{\delta+1/p\}$. Thus $s = 1+1/p$ is admissible in the case of a pure Dirichlet boundary condition ($\delta = 0$), and $s = 1/p$ is admissible if we consider pure Neumann type boundary conditions ($\delta = 1$).

(b) For simplicity, we have imposed more regularity for $(\mathcal{A},\mathcal{B})$ than actually needed. For example, it suffices throughout this paper to assume that

$a_0 \in L_\infty(\Omega)$. Moreover, the assumption $\boldsymbol{a} \in C^2$ has only been used in the proof of Theorem 3.1. Theorem 2.3 remains valid for $\boldsymbol{a} \in C^1$. The regularity hypotheses can be further relaxed if we are interested in W_p^s-solutions for s in a given subset of $[0,2]\backslash(\mathbb{N}+1/p)$ only. For details we refer to [5] (also see [6]). $\square$

4. Linear Problems Involving Measures. In this paper we are particularly interested in the case where the right-hand sides in (2.4) are measures. To formulate the optimal regularity results we introduce suitable locally convex spaces.

Suppose that $J \subset \mathbb{R}$ is an interval and $\{\, E_t \ ;\ t \in J \,\}$ is a family of Banach spaces such that $s > t$ implies $E_s \hookrightarrow E_t$. Then, given $s \in \overline{J}$, we put

$$E_{s-} := \bigcap_{t<s} E_t := \varprojlim_{t\uparrow s} E_t \ , \quad E_{s+} := \bigcup_{t>s} E_t := \varinjlim_{t\downarrow s} E_t \ ,$$

provided $s > \inf J$ or $s < \sup J$, respectively. It is obvious that E_{s-} is a Fréchet space.

It follows from the trace theorem that

$$W_{1,\mathcal{B}}^s := \left\{\, u \in W_1^s \ ;\ (1-\delta)\gamma u = 0 \,\right\}$$

is a well-defined closed linear subspace of W_1^s for $1 \le s \le 2$. Hence W_1^{1-} and $W_{1,\mathcal{B}}^{2-}$ are well-defined.

Lemma 4.1.

$$W_1^{1-} = \bigcap_{1<p<1^*} \ \bigcap_{0\le s<1-n/p'} W_p^s$$

and

$$W_{1,\mathcal{B}}^{2-} = \bigcap_{1<p<1^*} \ \bigcap_{1\le s<2-n/p'} W_{p,\mathcal{B}}^s \ .$$

Proof. From Sobolev's embedding theorem we deduce that

$$W_1^t \hookrightarrow W_p^s \qquad \text{if} \quad 1 > 1/p = 1-(t-s)/n > 0 \ . \tag{4.1}$$

If $t-s$ increases from 0 to 1 then $1/p$, defined by (4.1), decreases from 1 to $1/1^*$, that is, p increases from 1 to 1^*. Thus we infer from (4.1) that

$$W_1^{1-} \hookrightarrow \bigcap_{1<p<1^*} \ \bigcap_{0\le s<1-n/p'} W_p^s \tag{4.2}$$

and that

$$W^{2-}_{1,\mathcal{B}} \hookrightarrow \bigcap_{1<p<1^*} \bigcap_{1\le s<2-n/p'} W^s_{p,\mathcal{B}} \ . \tag{4.3}$$

On the other hand, $W^s_p \hookrightarrow W^s_1$ implies that the right-hand side of (4.2) injects continuously in

$$\bigcap_{1<p<1^*} \bigcap_{0\le s<1-n/p'} W^s_1 \ .$$

If $0<t<1$ then we can find $p\in(1,1^*)$ such that $1-n/p'>t$. Hence

$$\bigcap_{1<p<1^*} \bigcap_{0\le s<1-n/p'} W^s_1 \hookrightarrow W^t_1 \ , \qquad 0<t<1 \ .$$

Consequently, the space on the right-hand side of (4.2) injects continuously in W^{1-}_1. Similarly, it follows that the space on the right-hand side of (4.3) injects continuously in $W^{2-}_{1,\mathcal{B}}$. This proves everything. □

Corollary 4.2. $W^{1-}_1 \hookrightarrow L_{1^*-}$ *and* $W^{2-}_{1,\mathcal{B}} \hookrightarrow W^1_{1^*-,\mathcal{B}} := \bigcap_{1<p<1^*} W^1_{p,\mathcal{B}}$.

After these preparations we can prove the following regularity theorem for very weak and weak solutions, respectively, of problem (2.4) in the case where f and g are measures.

Theorem 4.3. *Suppose that* $(\mathcal{A},\mathcal{B})\in\mathcal{E}_0(\Omega)$. *Then*

$$(\mathcal{A},\mathcal{B})^{-1}\in\mathcal{L}\big(\mathcal{M}(\Omega\cup\Gamma_1)\times\mathcal{M}(\Gamma), W^{1-}_1\big)$$

and

$$(\mathcal{A},\mathcal{B})^{-1}\in\mathcal{L}\Big(\mathcal{M}(\Omega\cup\Gamma_1)\times\big(\{0\}\times\mathcal{M}(\Gamma_1)\big), W^{2-}_{1,\mathcal{B}}\Big) \ .$$

Proof. This follows from Theorem 2.3 and Lemmas 2.1 and 4.1. □

In analogy to the definition of $W^s_{p,\mathcal{B}}$ we put

$$C^k_{\mathcal{B}} := C^k_{\mathcal{B}}(\overline{\Omega}) := \begin{cases} \{\, u\in C^1(\overline{\Omega}) \ ; \ (1-\delta)\gamma u = 0 \,\} \ , & k=1 \ , \\ \{\, u\in C^2(\overline{\Omega}) \ ; \ \mathcal{B}u=0 \,\} \ , & k=2 \ . \end{cases}$$

Suppose that $(\mathcal{A},\mathcal{B})\in\mathcal{E}$ and $(\mu,\sigma)\in\mathcal{M}(\Omega\cup\Gamma_1)\times\mathcal{M}(\Gamma)$. Then u is said to be a very weak solution of the elliptic BVP

$$\mathcal{A}u=\mu \text{ in } \Omega \ , \qquad \mathcal{B}u=\sigma \text{ on } \Gamma \ , \tag{4.4}$$

iff $u \in L_1$ and

$$\int_\Omega (\mathcal{A}^\sharp v)u\,dx = \int_\Omega v\,d\mu - \int_{\Gamma_0} \partial_{\nu_a} v\,d\sigma + \int_{\Gamma_1} \gamma v\,d\sigma\ , \qquad v \in C^2_{\mathcal{B}^\sharp}\ . \tag{4.5}$$

If $\sigma\,|\,\Gamma_0 = 0$ then u is called weak solution of (4.4) iff $u \in W^1_{1,\mathcal{B}}$ and

$$\mathfrak{a}(v,u) = \int_\Omega v\,d\mu + \int_{\Gamma_1} \gamma v\,d\sigma\ , \qquad v \in C^1_{\mathcal{B}^\sharp}\ . \tag{4.6}$$

It is easily verified that each weak solution is a very weak one as well.

Using these definitions we can prove the main result of this section.

Theorem 4.4. *Suppose that $(\mathcal{A},\mathcal{B}) \in \mathcal{E}_0(\Omega)$. Then the elliptic BVP (4.4) has for each $(\mu,\sigma) \in \mathcal{M}(\Omega\cup\Gamma_1)\times\mathcal{M}(\Gamma)$ a unique very weak solution u, and u belongs to W_1^{1-}. If $\sigma\,|\,\Gamma_0 = 0$ then u is a weak solution and belongs to $W^{2-}_{1,\mathcal{B}}$. Moreover, given $s \in [0,1)$,*

$$\|u\|_{s,1} \le c\big(\|\mu\|_{\mathcal{M}(\Omega\cup\Gamma_1)} + \|\sigma\|_{\mathcal{M}(\Gamma)}\big)$$

and, if $\sigma\,|\,\Gamma_0 = 0$, then

$$\|u\|_{1+s,1} \le c\big(\|\mu\|_{\mathcal{M}(\Omega\cup\Gamma_1)} + \|\sigma\|_{\mathcal{M}(\Gamma_1)}\big)\ ,$$

where c depends on s, Ω, and $(\mathcal{A},\mathcal{B})$ only. In fact, c can be chosen to be independent of $(\mathcal{A},\mathcal{B})$ if these operators vary in a compact subset of $\mathcal{E}_0(\Omega)$.

Proof. The asserted existence and regularity as well as the stated estimates are easy consequences of Theorems 2.3 and 4.3. As for uniqueness, it suffices to show, since each weak solution is also a very weak one, that $u \in L_1$ and

$$\int_\Omega (\mathcal{A}^\sharp v)u\,dx = 0\ , \qquad v \in C^2_{\mathcal{B}^\sharp}\ , \tag{4.7}$$

imply $u = 0$. Given $\varphi \in \mathcal{D}$, classical elliptic theory guarantees the existence of a unique $v_\varphi \in C^2_{\mathcal{B}^\sharp}$ satisfying $\mathcal{A}^\sharp v = \varphi$. Hence it follows from (4.7) that $\int_\Omega \varphi u\,dx = 0$ for each $\varphi \in \mathcal{D}$, which shows that $u = 0$. $\square$

Corollary 4.5. *Given* $p \in [1, 1^*)$,

$$\|u\|_p \le c\big(\|\mu\|_{\mathcal{M}(\Omega\cup\Gamma_1)} + \|\sigma\|_{\mathcal{M}(\Gamma)}\big)$$

and, if $\sigma\,|\,\Gamma_0 = 0$, *then*

$$\|u\|_{1,p} \le c\big(\|\mu\|_{\mathcal{M}(\Omega\cup\Gamma_1)} + \|\sigma\|_{\mathcal{M}(\Gamma_1)}\big) ,$$

where c *depends on* p, Ω, *and compact subsets of* $\mathcal{E}_0$ *containing* $(\mathcal{A}, \mathcal{B})$, *only.*

Proof. This is a consequence of Corollary 4.2. □

Linear elliptic boundary value problems whose right-hand sides are measures occur in control theory, for example. In this connection the Neumann problem

$$\mathcal{A}u = \mu \text{ in } \Omega , \quad \partial_{\nu_a} u = \sigma \text{ on } \Gamma \tag{4.8}$$

for the operator $\mathcal{A} := -\nabla\cdot(\boldsymbol{a}\nabla\cdot) + a_0$ with $a_0 > 0$ has been studied by Casas [13]. In that paper it is shown, building on a priori estimates of Stampacchia [29], that (4.8) has a unique solution in $W^1_{1^*-}$ and that the a priori estimate of Corollary 4.5 is satisfied in this case.

Remarks 4.6.

(a) It is clear that $\mathcal{M}(\Gamma_1)$ can be identified with the linear subspace of $\mathcal{M}(\Omega\cup\Gamma_1)$ consisting of those measures in $\mathcal{M}(\Omega\cup\Gamma_1)$ whose supports are contained in Γ_1. Thus there is an ambiguity in (4.4) as far as $\sigma_1 \in \mathcal{M}(\Gamma_1)$ is concerned: namely, we can put σ_1 in the equation on Ω or in the boundary condition on Γ_1. However, this is a formal inconsistency only since (4.4) does not have a meaning except the one given in (4.5) or (4.6), respectively. But there is no such ambiguity in (4.5) or (4.6). Nevertheless, we prefer formulation (4.4) for its intuitive appeal.

(b) Everything said up till now remains valid for normally elliptic systems and complex-valued coefficients. For this we also refer to [5]. □

5. Positivity. So far our spaces of distributions could be complex Banach spaces. Now we restrict ourselves to the real setting and discuss order properties. Thus in the remainder of this paper all vector spaces are over the reals. If, in a given formula, there occur explicitly complex numbers or it is referred to the spectrum or the resolvent set of a linear operator, it is always understood that the corresponding complexifications are being considered.

Let E, F, and $E_1,\dots,E_n$ be ordered Banach spaces (OBSs), whose positive cones we denote by E^+, F^+, and $E_1^+,\dots,E_n^+$, respectively. Then the space $E_1\times\cdots\times E_n$ is always given the product order whose positive cone equals the product of the positive cones E_j^+. A bounded linear operator T from E into F is positive iff $T(E^+)\subset F^+$, and we write $T\in\mathcal{L}^+(E,F)$ in this case. The real line is always given the natural order whose positive cone equals $\mathbb{R}^+$. Then $(E')^+:=\mathcal{L}^+(E,\mathbb{R})$ is the dual wedge of E^+. It is a cone, hence E' is an ordered Banach space, iff E^+ is total in E, that is, E^+-E^+ is dense in E. In the following all our positive cones will be total and the dual spaces are always given the dual order induced by the respective dual positive cones. If G is a Banach space such that $G\hookrightarrow E$ then G is given the natural order induced by E whose positive cone equals $G\cap E^+=i^{-1}(E^+)$, where i is the injection $G\hookrightarrow E$. Thus $i\in\mathcal{L}^+(G,E)$.

Let X be a σ-compact metric space. Then $C_0(X)$ is endowed with the natural order whose positive cone equals

$$C_0^+(X):=\left\{\,u\in C_0(X)\ ;\ u(x)\ge 0,\ x\in X\,\right\}\,.$$

It follows that $C_0(X)$ is an OBS, in fact: a Banach lattice. Consequently, $\mathcal{M}(X)$ is a Banach lattice as well. Given $\mu\in\mathcal{M}^+(X)$ and $1\le p<\infty$, the Banach space $L_p(X,\mu)$ is also given its natural order whose positive cone equals

$$L_p^+(X,\mu):=\left\{\,u\in L_p(X,\mu)\ ;\ u(x)\ge 0\ \mu\text{-a.e.}\,\right\}\,.$$

Then $L_p(X,\mu)$ is also a Banach lattice.

By means of the preceding definitions and conventions it follows that W_p^s and $W_{p,\mathcal{B}}^s$ as well as ∂W_p^s are OBSs, whenever they are defined. Moreover, the injections

$$W_{p,\mathcal{B}}^s\hookrightarrow W_{q,\mathcal{B}}^t\ ,\quad \partial W_p^s\hookrightarrow\partial W_q^t\ ,\qquad -2\le t\le s\le 2\ ,\quad 1<q\le p<\infty\ ,$$

are positive, when defined.

Suppose $1<p<\infty$ and $s\in[0,2]\backslash(\mathbb{N}+1/p)$. For $(\mathcal{A},\mathcal{B})\in\mathcal{E}$ we put

$$A_{s-2,p}:=\mathcal{A}\,|\,W_{p,\mathcal{B}}^s\in\mathcal{L}(W_{p,\mathcal{B}}^s,W_{p,\mathcal{B}}^{s-2})\ ,$$

and consider $A_{s-2,p}$ as a linear operator in $W_{p,\mathcal{B}}^{s-2}$. Due to Theorem 2.3 and Remark 2.4(a),

$$A_{t-2,q}\supset A_{s-2,p}\ ,\qquad 0\le t\le s\le 2\ ,\quad 1<q\le p<\infty\ ,\tag{5.1}$$

provided $t \notin \mathbb{N} + 1/q$. Since

$$W^s_{p,\mathcal{B}} \hookrightarrow W^t_{q,\mathcal{B}} , \qquad 0 \le t < s \le 2 , \quad t \notin \mathbb{N} + 1/q , \tag{5.2}$$

where $\hookrightarrow$ denotes compact injection, we infer from Remark 2.4(a) that $A_{s-2,p}$ has compact resolvent. Then it is an easy consequence of (5.1) that the spectrum and the eigenspaces of $A_{s-2,p}$ are independent of s and p. Henceforth we often simply denote $A_{s-2,p}$ by A if no confusion seems likely.

Theorem 5.1. *Suppose that $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}$. Then A possesses a least real eigenvalue $\lambda_0 := \lambda_0(\mathcal{A}, \mathcal{B})$, the* principal eigenvalue of *$(\mathcal{A}, \mathcal{B})$. It is simple and possesses a positive eigenfunction $\varphi \in C^2_{\mathcal{B}}$ satisfying $\varphi(x) > 0$ for $x \in \Omega \cup \Gamma_1$ and $\partial_{\nu_a}\varphi(x) < 0$ for $x \in \Gamma_0$. Moreover, λ_0 is the only eigenvalue of A having a positive eigenfunction, and there is no eigenvalue $\lambda \ne \lambda_0$ with* $\mathrm{Re}\,\lambda \le \lambda_0$. *Lastly, if $\lambda > -\lambda_0$ then*

$$(\lambda + \mathcal{A}, \mathcal{B})^{-1} \in \mathcal{L}^+(W^{s-2}_{p,\mathcal{B}} \times \partial W^s_p, W^s_p) \tag{5.3}$$

for $s \in [0, 2] \backslash (\mathbb{N} + 1/p)$ and $1 < p < \infty$.

Proof. The assertions concerning the spectrum of A and the eigenfunction φ follow from [4, Theorem 12.1], the preceding remarks and standard regularity theory. (In the proof of Theorem 12.1 in [4] it has been referred to [27, appendix] to assert that a positive irreducible compact linear operator on a Banach lattice has a strictly positive spectral radius. However, this does not follow from the results in [27], but is Theorem 3 in [14].) Suppose that $s = 2$ and $p > n$. Then (5.3) is a consequence of [7, Theorem 2.4]. Note that $(W^2_p)^+ \subset (W^2_q)^+$ and that $L_p \times \partial W^2_p$ is dense in $L_q \times \partial W^2_q$ for $p > q$. Hence the continuity of $(\mathcal{A}, \mathcal{B})^{-1}$ entails (5.3) for $s = 2$ and all $p \in (1, \infty)$. In [5, Theorem 8.7] it has been shown that $(\lambda + A_{s-2,p})^{-1}$ is positive for $s \in [0, 2] \backslash (\mathbb{N} + 1/p)$, provided λ is sufficiently large. That proof applies to all $\lambda > -\lambda_0$ to guarantee that

$$(\lambda + \mathcal{A})^{-1} \in \mathcal{L}^+(W^{s-2}_{p,\mathcal{B}}, W^s_{p,\mathcal{B}}) , \qquad s \in [0, 2] \backslash (\mathbb{N} + 1/p) , \quad 1 < p < \infty .$$

By mollifying (in local coordinates) it is not difficult to see that $(\partial W^2_p)^+$ is dense in $(\partial W^s_p)^+$ for $0 \le s < 2$. Fix $\lambda > -\lambda_0$ and $p \in (1, \infty)$. From what we know already it follows that

$$(\lambda + \mathcal{A}, \mathcal{B})^{-1} \,|\, \{0\} \times \partial W^2_p \in \mathcal{L}^+(\partial W^2_p, W^2_p) .$$

Hence, by density and continuity,

$$(\lambda+\mathcal{A},\mathcal{B})^{-1}\,|\,\{0\}\times\partial W_p^s \in \mathcal{L}^+(\partial W_p^s, W_p^s)\ , \qquad s\in[0,2]\backslash(\mathbb{N}+1/p)\ .$$

Thus the decomposition

$$(\lambda+\mathcal{A},\mathcal{B})^{-1}(f,g)=(\lambda+A)^{-1}f+(\lambda+\mathcal{A},\mathcal{B})^{-1}(0,g)$$

implies (5.3). □

Remarks 5.2. Suppose that $(\mathcal{A},\mathcal{B})\in\mathcal{E}$.

(a) Since $(\mathcal{A}^\sharp,\mathcal{B}^\sharp)\in\mathcal{E}$ we can define $A^\sharp$ by $A^\sharp_{s-2,p'}:=\mathcal{A}^\sharp\,|\,W^{s-2}_{p',\mathcal{B}^\sharp}$. Then Theorem 5.1 holds for $(\mathcal{A}^\sharp,\mathcal{B}^\sharp)$ and $A^\sharp$. Since $A_{0,p}$ and $A^\sharp_{0,p'}$ have common points in their resolvent sets one verifies that $A^\sharp_{0,p'}=(A_{0,p})'$. This implies that $\lambda_0(\mathcal{A},\mathcal{B})=\lambda_0(\mathcal{A}^\sharp,\mathcal{B}^\sharp)$ and that there exists a positive eigenfunction $\varphi^\sharp$ of $A^\sharp$ possessing the same properties as φ.

(b) The operator $A_{s-2,p}$ is characterized as follows:

- if $1+1/p<s\le 2$ then $A_{s-2,p}=\mathcal{A}\,|\,W^{s-2}_{p,\mathcal{B}}$;
- if $1/p<s<1+1/p$ then

$$\langle v, A_{s-2,p}u\rangle=\mathfrak{a}(v,u)\ , \qquad (u,v)\in W^s_{p,\mathcal{B}}\times W^{2-s}_{p',\mathcal{B}^\sharp}\ ;$$

- if $0\le s<1/p$ then

$$\langle v, A_{s-2,p}u\rangle=\langle \mathcal{A}^\sharp v,u\rangle\ , \qquad (u,v)\in W^s_{p,\mathcal{B}}\times W^{2-s}_{p',\mathcal{B}^\sharp}\ .$$

Proof. See [5, Section 8]. □

(c) The map

$$L_\infty\times C^1(\Gamma_1)\to\mathbb{R}\ , \quad (b,\beta)\mapsto\lambda_0(\mathcal{A}+b,\mathcal{B}+\delta\beta\gamma)$$

is strictly increasing and continuous.

Proof. The isotonicity follows by an obvious modification of the proof of [23, Proposition 3.2(i)] employing [7, Theorem 2.4].

The proof of Theorem 5.1 shows that $\big[\lambda+\lambda_0(\mathcal{A}+b,\mathcal{B}+\delta\beta\gamma)\big]^{-1}$ is a simple eigenvalue of the compact linear map

$$(\lambda+\mathcal{A}+b,\mathcal{B}+\delta\beta\gamma)^{-1}\,\big|\,L_p\times\{0\}\in\mathcal{L}(L_p)\ ,$$

where λ is a sufficiently large positive number and $p > n$. Hence the asserted continuity is a consequence of the upper semicontinuity of the spectrum. $\square$

6. Semilinear Equations: The General Case. *Throughout Sections 6–13 we suppose that*

$$(\mathcal{A},\mathcal{B}) \in \mathcal{E} \quad \textit{and} \quad \lambda_0 := \lambda_0(\mathcal{A},\mathcal{B}) > 0 \ . \tag{6.1}$$

It follows from Theorem 5.1 that $(\mathcal{A},\mathcal{B}) \in \mathcal{E}_0$ and $(\mathcal{A},\mathcal{B})^{-1}$ is positive.

Henceforth we put

$$\mathcal{M}_j := \begin{cases} \mathcal{M}(\Omega \cup \Gamma_1) \times \mathcal{M}(\Gamma) \ , & j = 0 \ , \\ \mathcal{M}(\Omega \cup \Gamma_1) \times \mathcal{M}(\Gamma_1) \ , & j = 1 \ . \end{cases}$$

We identify $\mathcal{M}_1$ with the closed linear subspace $\mathcal{M}(\Omega \cup \Gamma_1) \times \{0\} \times \mathcal{M}(\Gamma_1)$ of $\mathcal{M} := \mathcal{M}_0$ and write $m = (\mu, \sigma)$ for a generic point of $\mathcal{M}$. Observe that $\mathcal{M}$ is a Banach lattice.

We suppose that $1 < p < 1^*$ and $0 \le s_j < 1 - n/p'$ and put $E_j := W_p^{s_j + j}$ for $j \in \{0, 1\}$. We also suppose that

$$(\mathcal{F}_j, \mathcal{G}_j) \in C(E_j \times \mathcal{M}, \mathcal{M}_j) \ . \tag{6.2}$$

Then, given $m \in \mathcal{M}$, we consider the semilinear elliptic BVP

$$\mathcal{A}u = \mathcal{F}_j(u,m) \text{ in } \Omega \ , \quad \mathcal{B}u = \mathcal{G}_j(u,m) \text{ on } \Gamma \ . \tag{6.3$_{j,m}$}$$

By a **solution** u of $(6.3)_{j,m}$ we mean an element $u \in E_j$ such that u is a very weak or a weak solution, respectively, of the linear BVP $\mathcal{A}u = \mu_u$ in Ω, $\mathcal{B}u = \sigma_u$ on Γ, where $\mu_u := \mathcal{F}_j(u,m)$ and $\sigma_u := \mathcal{G}_j(u,m)$ for $j = 0$ or $j = 1$, respectively.

Lemma 6.1. *Put*

$$\mathcal{T}_j(u,m) := (\mathcal{A},\mathcal{B})^{-1}\big(\mathcal{F}_j(u,m), \mathcal{G}_j(u,m)\big) \ , \qquad (u,m) \in E_j \times \mathcal{M} \ .$$

Then

(i) $\mathcal{T}_j \in C\big(E_j \times \mathcal{M}, W_1^{(1+j)-}\big)$.

(ii) *Suppose that $U \times M \subset E_j \times \mathcal{M}$. If $(\mathcal{F}_j, \mathcal{G}_j)(U \times M)$ is bounded in $\mathcal{M}_j$ then $\mathcal{T}_j(U \times M)$ is bounded in $W_1^{(1+j)-}$ and contained in a compact subset of E_j.*

(iii) *If $(\mathcal{F}_j, \mathcal{G}_j)$ is [strictly] increasing on $U \times M$ then $\mathcal{T}_j$ is also [strictly] increasing on $U \times M$.*

(iv) *If $m \in \mathcal{M}$ then u is a solution of $(6.3)_{j,m}$ iff u is a fixed point of $\mathcal{T}_j(\cdot, m)$ in E_j.*

Proof. (i) and (ii) are easy consequences of $(\mathcal{A},\mathcal{B})\in\mathcal{E}_0$, Theorem 4.3, Corollary 4.2, the fact that $L_1\hookrightarrow\mathcal{M}(\Omega\cup\Gamma_1)$, and the Rellich-Kondrachov theorem.
(iii) follows from the positivity of $(\mathcal{A},\mathcal{B})^{-1}$, and (iv) is obvious. □

As an immediate consequence of this lemma we find the following regularity result.

Corollary 6.2. *If u is a solution of $(6.3)_{j,m}$ for some $m\in\mathcal{M}$ then u belongs to $W_1^{(1+j)-}$, hence to $W_{1^*-}^j$.*

First we prove an almost trivial existence and uniqueness theorem. Here and below we omit the index j everywhere if no confusion seems possible.

Theorem 6.3. *Suppose that $(\mathcal{F},\mathcal{G})\in C^1(E\times\mathcal{M},\mathcal{M})$. Also suppose that $m^*\in\mathcal{M}$ and u^* is a solution of $(6.3)_{m^*}$ such that the linearized problem*

$$\mathcal{A}v=\partial_1\mathcal{F}(u^*,m^*)v \text{ in } \Omega\ ,\quad \mathcal{B}u=\partial_1\mathcal{G}(u^*,m^*)v \text{ on } \Gamma \tag{6.4}$$

has the trivial solution only. Then there exists an open neighborhood $U\times M$ of (u^*,m^*) in $E\times\mathcal{M}$ such that $(6.3)_m$ has for each $m\in M$ exactly one solution $u=u(m)\in U$. The map $m\mapsto u(m)$ is continuously differentiable on M.

Proof. Define $\Phi\in C^1(E\times\mathcal{M},E)$ by $\Phi(u,m):=u-\mathcal{T}(u,m)$. Then Φ vanishes at (u^*,m^*), and $\partial_1\Phi(u^*,m^*)=\mathrm{id}-\partial_1\mathcal{T}(u^*,m^*)$. Since

$$\partial_1\mathcal{T}(u^*,m^*)v=(\mathcal{A},\mathcal{B})^{-1}\bigl(\partial_1\mathcal{F}(u^*,m^*)v,\partial_1\mathcal{G}(u^*,m^*)v\bigr)$$

it follows that $\partial_1\mathcal{F}(u^*,m^*)$ is a compact linear map of E into itself. Thus $\partial_1\Phi(u^*,m^*)$ is a Fredholm operator of index zero. The assumption on the linearized problem (6.4) implies that $\partial_1\Phi(u^*,m^*)$ has a trivial kernel. Hence $\partial_1\Phi(u^*,m^*)\in\mathcal{L}\mathrm{is}(E,E)$, and the assertion follows from the implicit function theorem. □

Remark 6.4. Suppose that $(\mathcal{A},\mathcal{B})\in\mathcal{E}$. Fix $\omega>-\lambda_0(\mathcal{A},\mathcal{B})$. Then problem $(6.3)_{j,m}$ is equivalent to

$$(\mathcal{A}+\omega)u=\mathcal{F}_j(u,m)+\omega u \text{ in } \Omega\ ,\quad \mathcal{B}u=\mathcal{G}_j(u,m) \text{ on } \Gamma\ .$$

Thus, since $\lambda_0(\mathcal{A}+\omega,\mathcal{B})=\lambda_0(\mathcal{A},\mathcal{B})+\omega>0$, it is no loss of generality to assume that $\lambda_0(\mathcal{A},\mathcal{B})>0$. □

7. Sub- and Supersolutions. Suppose that $m \in \mathcal{M}$. Then w is said to be a subsolution of $(6.3)_m$ iff $w \in E$ and

$$\mathcal{A}w \le \mathcal{F}(w,m) \text{ in } \Omega\ , \quad \mathcal{B}w \le \mathcal{G}(w,m) \text{ on } \Gamma\ . \tag{7.1}$$

If both inequality signs in (7.1) are reversed then w is a supersolution of $(6.3)_m$.

Lemma 7.1. *Suppose that $m \in \mathcal{M}$. Then w is a subsolution [resp. supersolution] of $(6.3)_m$ iff $w \le \mathcal{T}(w,m)$ [resp. $\mathcal{T}(w,m) \le w$].*

Proof. This is immediate by Lemma 2.1 and (5.3). □

It is a consequence of Lemmas 6.1 and 7.1 that the well-known techniques of nonlinear functional analysis in OBSs can be applied to problem $(6.3)_m$. As a first result we prove a generalization of the method of sub- and supersolutions to the present setting.

Theorem 7.2. *Suppose that $\overline{w}$ is a subsolution of $(6.3)_{\overline{m}}$ and $\widehat{w}$ is a supersolution of $(6.3)_{\widehat{m}}$ such that $(\overline{w},\overline{m}) \le (\widehat{w},\widehat{m})$. Also suppose that $(\mathcal{F},\mathcal{G})$ is increasing on the order interval $\big[(\overline{w},\overline{m}),(\widehat{w},\widehat{m})\big]$, and that $\big(\mathcal{F}(\cdot,m),\mathcal{G}(\cdot,m)\big)$ is, for each $m \in [\overline{m},\widehat{m}]$, bounded on $[\overline{w},\widehat{w}]$.*

Then $(6.3)_m$ has for each $m \in [\overline{m},\widehat{m}]$ a least solution $\overline{u}(m)$ and a greatest solution $\widehat{u}(m)$ in $[\overline{w},\widehat{w}]$. Furthermore, $\overline{u}(\cdot)$ and $\widehat{u}(\cdot)$ are increasing maps from $[\overline{m},\widehat{m}]$ into E. If $\big(\mathcal{F}(u,\cdot),\mathcal{G}(u,\cdot)\big)$ are strictly increasing on $[\overline{m},\widehat{m}]$ for $u \in [\overline{w},\widehat{w}]$ then $\overline{u}(\cdot)$ and $\widehat{u}(\cdot)$ are also strictly increasing.

Proof. It follows from Lemma 6.1 that $\mathcal{T}$ is increasing on $\big[(\overline{w},\overline{m}),(\widehat{w},\widehat{m})\big]$, and that $\mathcal{T}(\cdot,m)$ is compact on $[\overline{w},\widehat{w}]$ for each $m \in [\overline{m},\widehat{m}]$. Hence

$$\overline{w} \le \mathcal{T}(\overline{w},\overline{m}) \le \mathcal{T}(\overline{w},m)\ , \quad \mathcal{T}(\widehat{w},m) \le \mathcal{T}(\widehat{w},\widehat{m}) \le \widehat{w}\ , \qquad m \in [\overline{m},\widehat{m}]\ .$$

Thus [3, Corollary 6.2] guarantees the existence of the least and the greatest fixed point of $\mathcal{T}(\cdot,m)$ in $[\overline{w},\widehat{w}]$. Suppose that $\overline{m} \le m_0 < m_1 \le \widehat{m}$. Then

$$\overline{u}(m_1) = \mathcal{T}\big(\overline{u}(m_1),m_1\big) \ge \mathcal{T}\big(\overline{u}(m_1),m_0\big)\ ,$$

with a strict inequality sign if $\big(\mathcal{F}(u,\cdot),\mathcal{G}(u,\cdot)\big)$, hence $\mathcal{T}(u,\cdot)$, are strictly increasing. Thus, $\mathcal{T}(\cdot,m_0)$ has a fixed point u in $\big[\overline{w},\overline{u}(m_1)\big]$, which is distinct from $\overline{u}(m_1)$ if $\mathcal{T}(u,\cdot)$ is strictly increasing. Consequently, $\overline{u}(m_0) \le u \le \overline{u}(m_1)$ which shows that $\overline{u}(\cdot)$ is increasing, and strictly increasing if $\mathcal{T}(u,\cdot)$ is strictly increasing. Similarly, we prove the assertion for $\widehat{u}(\cdot)$. □

Remark 7.3. Theorem 7.2 remains valid if it is only assumed that there exists a constant $\omega \geq 0$ such that

$$(u,m) \mapsto \big(\mathcal{F}(u,m) + \omega u, \mathcal{G}(u,m) + \delta\omega\gamma u\big)$$

is increasing on $\big[(\overline{w},\overline{m}),(\widehat{w},\widehat{m})\big]$.

Proof. Clearly, problem $(6.3)_m$ is equivalent to

$$\begin{aligned} (\mathcal{A}+\omega)u &= \mathcal{F}(u,m) + \omega u \qquad &&\text{in } \Omega \ , \\ (\mathcal{B}+\delta\omega\gamma)u &= \mathcal{G}(u,m) + \delta\omega\gamma u \qquad &&\text{on } \Gamma \ . \end{aligned}$$

Since Remark 5.2(c) implies $\lambda_0(\mathcal{A}+\omega, \mathcal{B}+\delta\omega\gamma) \geq \lambda_0(\mathcal{A},\mathcal{B}) > 0$, the assertion follows. $\square$

8. Positive Solutions. Now we restrict our considerations to the study of positive solutions. For this purpose we assume that

$$\left.\begin{array}{l} (\mathcal{F},\mathcal{G})(0,0) = 0 \text{ and} \\ (\mathcal{F},\mathcal{G}) \text{ is increasing on } E^+ \times \mathcal{M}^+ \text{ and} \\ \text{bounded on bounded subsets of } E^+ \times \mathcal{M}^+ \\ \text{and on order intervals in } E^+ \times \mathcal{M}^+ \ . \end{array}\right\} \tag{8.1}$$

We also suppose that

$$m\colon \mathbb{R}^+ \to \mathcal{M} \text{ is continuous and increasing with } m(0) = 0 \ . \tag{8.2}$$

We put

$$\Sigma := \big\{\, (\lambda,u) \in \mathbb{R}^+ \times E^+ \ ;\ (\lambda,u) \text{ is a solution of } (6.3)_{m(\lambda)} \,\big\}$$

and

$$\Lambda := \big\{\, \lambda \in \mathbb{R}^+ \ ;\ \text{there exists } u \in E^+ \text{ with } (\lambda,u) \in \Sigma \,\big\} \ .$$

We also define $\mathcal{S} \in C(\mathbb{R}^+ \times E^+, E^+)$ by

$$\mathcal{S}(\lambda,u) := \mathcal{T}\big(u, m(\lambda)\big) \ , \qquad \lambda \in \mathbb{R}^+ \ , \quad u \in E^+ \ .$$

Then it follows from Lemma 6.1 and assumptions (8.1) and (8.2) that $\mathcal{S}$ is well-defined, increasing, and maps bounded sets and order intervals into compact sets. Moreover, $(\lambda,u) \in \Sigma$ iff $u = \mathcal{S}(\lambda,u)$.

Theorem 8.1. *Λ is an interval (possibly degenerate) containing 0, and problem $(6.3)_{m(\lambda)}$ has for each $\lambda \in \Lambda$ a least positive solution $\overline{u}_\lambda$. The map $\Lambda \to E^+$, $\lambda \mapsto \overline{u}_\lambda$ is increasing and left continuous. If $\mathcal{S}(\cdot, u)$ is strictly increasing on $\mathbb{R}^+$ for $u \in E^+\backslash\{0\}$ then $\lambda \mapsto \overline{u}_\lambda$ is also strictly increasing. If $\lambda^* := \sup \Lambda < \infty$ then $\lambda^* \in \Lambda$ iff the set $\{\, \overline{u}_\lambda \,;\, \lambda^* - \varepsilon < \lambda < \lambda^* \,\}$ is bounded in E_j for some $\varepsilon > 0$.*

Proof. Thanks to the preceding observations this follows from [3, Theorem 20.3], where it suffices to observe that the normality of the positive cone had only been presupposed to guarantee that the map under consideration sends order intervals into compact sets. □

The next proposition gives a sufficient condition for λ^* to be finite. Here and below we denote by $\varphi^\sharp$ a fixed positive eigenfunction of $A^\sharp$. For abbreviation we also set

$$\mathcal{C}^\sharp := \begin{cases} (\delta - 1)\partial_{\nu_a} + \delta\gamma \,, & j = 0 \,, \\ \gamma \,, & j = 1 \,. \end{cases}$$

Proposition 8.2. *Suppose that $\beta \in C(\mathbb{R}^+, \mathbb{R})$ satisfies*

$$\lambda_0 \langle \varphi^\sharp, u \rangle + \beta(\lambda) \le \big\langle \varphi^\sharp, \mathcal{F}\big(u, m(\lambda)\big)\big\rangle + \big\langle \mathcal{C}^\sharp \varphi^\sharp, \mathcal{G}\big(u, m(\lambda)\big)\big\rangle_\partial \tag{8.3}$$

for $(\lambda, u) \in \Sigma$, where $\lambda_0 := \lambda_0(\mathcal{A}, \mathcal{B})$. If $\overline{\lim}_{t\to\infty} \beta(t) > 0$ then $\lambda^* < \infty$.

Proof. By testing $(6.3)_{m(\cdot)}$ at $(\lambda, u) \in \Sigma$ with $\varphi^\sharp$ and by taking into acount Remarks 5.2, the very weak or weak formulation, respectively, of $(6.3)_{m(\cdot)}$, and the hypothesis, we see that $\beta(\lambda) \le 0$ for $(\lambda, u) \in \Sigma$. This proves the assertion since Λ is an interval. □

It should be remarked that the idea behind Proposition 8.2 is the same as the one in [3, Proposition 20.2].

Since $\mathcal{S}(\lambda, \cdot)$ is continuous and maps bounded subsets of E^+ into compact ones the fixed point index $i\big(\mathcal{S}(\lambda, u), U\big) := i\big(\mathcal{S}(\lambda, \cdot), U, E^+\big)$ is well-defined for each open subset U of E^+ with $u \ne \mathcal{S}(\lambda, u)$ on ∂U (cf. [3, Theorem 11.1]). The next proposition evaluates this fixed point index in a simple situation. It is the basis for certain multiplicity results. Here and below $\mathbb{B}$ is the open unit ball in E, and $\mathbb{B}_\rho^+ := \rho\mathbb{B} \cap E^+$ for $\rho > 0$. By $\partial\mathbb{B}_\rho^+$ we mean the boundary of $\mathbb{B}_\rho^+$ *in* E^+.

Proposition 8.3. *Suppose that* $\lambda^\bullet \geq 0$ *and* $\rho > 0$ *are such that*

$$\{\, (\lambda, u) \in \Sigma \; ; \; 0 \leq \lambda \leq \lambda^\bullet \,\} \subset \mathbb{B}_\rho \; . \tag{8.4}$$

Also suppose that, given $T > 0$*, there exists* $R > 0$ *such that each positive solution of*

$$\mathcal{A}u = \mathcal{F}(u, 0) + \tau\varphi \text{ in } \Omega \; , \quad \mathcal{B}u = \mathcal{G}(u, 0) \text{ on } \Gamma \; , \qquad 0 \leq \tau \leq T \; , \tag{8.5}$$

belongs to $\mathbb{B}_R$*. Lastly, let estimate* (8.3) *be satisfied for* $\lambda = 0$ *and each* $u \in E^+$*. Then* $i\big(\mathcal{S}(\lambda, \cdot), \mathbb{B}_\rho\big) = 0$ *for* $0 \leq \lambda \leq \lambda^\bullet$*.*

Proof. By testing (8.5) with $\varphi^\sharp$ and using (8.3) for $\lambda = 0$ and $u \in E^+$ it follows that $\beta(0) + \tau\langle\varphi^\sharp, \varphi\rangle \leq 0$ whenever $\tau \in \mathbb{R}^+$ and u is a solution of (8.5). Since $\langle\varphi^\sharp, \varphi\rangle > 0$ we infer that there exists $\tau^* > 0$ such that (8.5) has no solution for $\tau \geq \tau^*$.

Put

$$\widetilde{\mathcal{S}}(\tau, u) := (\mathcal{A}, \mathcal{B})^{-1}\big(\mathcal{F}(u, 0) + \tau\varphi, \mathcal{G}(u, 0)\big) \; , \qquad u \in E^+ \; , \quad \tau \geq 0 \; .$$

Then $\widetilde{\mathcal{S}}(0, \cdot) = \mathcal{S}(0, \cdot)$, and there exists $\rho > 0$ such that $\mathcal{S}(\lambda, \cdot)$ and $\widetilde{\mathcal{S}}(\tau, \cdot)$ have no fixed points on $\partial\mathbb{B}_\rho^+$ for $0 \leq \lambda \leq \lambda^\bullet$ and $0 \leq \tau \leq \tau^*$, respectively. Thus, by the homotopy invariance of the fixed point index,

$$i\big(\mathcal{S}(\lambda, \cdot), \mathbb{B}_\rho^+\big) = i\big(\mathcal{S}(0, \cdot), \mathbb{B}_\rho^+\big) = i\big(\widetilde{\mathcal{S}}(\tau^*, \cdot), \mathbb{B}_\rho^+\big) = 0$$

for $0 \leq \lambda \leq \lambda^\bullet$, where the last equality holds since $\widetilde{\mathcal{S}}(\tau^*, \cdot)$ has no fixed point at all. $\square$

Suppose that u^* is an isolated fixed point of $\mathcal{S}(\lambda, \cdot)$ in E^+ for some $\lambda \in \Lambda$. Then the local index

$$i\big(\mathcal{S}(\lambda, \cdot), u^*\big) := \lim_{\rho \to 0} i\big(\mathcal{S}(\lambda, \cdot), (u^* + \rho\mathbb{B}) \cap E^+\big)$$

is well-defined.

Corollary 8.4. *Let the hypotheses of Proposition* 8.3 *be satisfied. Suppose that* $\lambda \in \Lambda \cap [0, \lambda^\bullet]$ *and that* $\overline{u}_\lambda$ *is an isolated fixed point of* $\mathcal{S}(\lambda, \cdot)$ *in* E^+

lying in $\mathbb{B}_\rho$. *If* $i\big(\mathcal{S}(\lambda,\cdot),\overline{u}_\lambda\big)\neq 0$ *then* $(6.3)_{m(\lambda)}$ *has at least two solutions in* E^+.

Proof. This follows by the additivity property of the fixed point index. $\square$

It should be remarked that in the literature there seem to be no results on elliptic BVPs involving measures in the same general form as in problem $(6.3)_{j,m}$. In all papers known to us measures are simply added to the nonlinearities as in model problems (1.1) and (1.2), except for [24], where equations containing a term $u\mu$ are being considered.

It is the main purpose of Sections 6–8 to show that the precise results for the linear BVPs given in Section 1 allow a reformulation of the nonlinear BVP as a fixed point equation on a suitable ordered Banach space involving a compact map. Thus we can appeal to well-known general theorems from nonlinear functional analysis to obtain existence theorems etc. Using this observation it is easy to obtain further results by imposing more specific conditions like asymptotic linearity or concavity, for example. We leave this to the interested reader.

9. Semilinear Equations: Local Nonlinearities. In this section we study generalizations of problems (1.1) and (1.2), namely problem

$$\mathcal{A}u = f(x,u,\nabla u)+\mu \text{ in } \Omega\ , \quad \mathcal{B}u=\delta g(x,u)+\sigma \text{ on } \Gamma \tag{9.1_m}$$

and problem

$$\mathcal{A}u = h(x,u)+\mu \text{ in } \Omega\ , \quad \mathcal{B}u=\sigma \text{ on } \Gamma\ , \tag{9.2_m}$$

respectively, where $m:=(\mu,\sigma)\in\mathcal{M}$ and f, g, and h are Carathéodory functions satisfying estimates (1.3) and (1.4).

By increasing r_1, if necessary, we can (and will) assume that

$$\frac{1}{r_1}\leq\Big(\frac{1}{r_0}-\frac{1}{n}\Big)\wedge\Big(1-\frac{1}{n}\Big)\frac{1}{r_2}\ . \tag{9.3}$$

First we show that (9.1) and (9.2) can be cast in form $(6.3)_{j,m}$ so that the results of Sections 6–8 are applicable.

Let X, Y, and Z be nonempty sets and let ψ be a map from $X\times Y$ into Z. Then we set $\psi^\natural(u)(x):=\psi\big(x,u(x)\big)$ for $u\colon X\to Y$ and $x\in X$. Thus $\psi^\natural$ is the Nemytskii operator induced by ψ.

Suppose that $1 < p < 1^*$ and put

$$s_j(p) := \begin{cases} n\left(\frac{1}{p} - \frac{1}{r_0}\right)_+ , & j = 0 , \\ \left[n\left(\frac{1}{p} - \frac{1}{r_1}\right) - 1\right]_+ , & j = 1 . \end{cases}$$

Note that $0 \le s_j(p) < 1 - n/p'$. If $0 \le t < 1 - n/p'$ then, by Sobolev's embedding theorem, $W_p^{t+j} \hookrightarrow L_{p_j}(t)$, where

$$p_j(t) := \frac{pn}{n - p(t+j)} < \frac{n}{n - (1+j)} .$$

Observe that (9.3) implies $p_j(t) \ge r_j$ for $t \ge s_j(p)$. Put $F_0(u) := h^\natural(u)$ and $F_1(u) := f^\natural(u, \nabla u)$. The above considerations, the Krasnosel'skii-Vainberg theorem (eg., [5, Lemma 14.2]), and the commutativity of the diagrams

$$\begin{array}{ccc} W_p^t & \hookrightarrow & L_{p_0}(t) \\ & {\scriptstyle F_0} \searrow \quad \swarrow {\scriptstyle h^\natural} & \\ & L_1 & \end{array} \qquad \begin{array}{ccc} W_p^{t+1} & \xrightarrow{(\mathrm{id}, \nabla)} & W_p^{t+1} \times (W_p^t)^n \\ {\scriptstyle F_1} \downarrow & & \downarrow \\ L_1 & \xleftarrow{f^\natural} & L_{p_1(t)} \times (L_{p_0(t)})^n \end{array}$$

guarantee that

$$F_j \in C(W_p^{t+j}, L_1) , \quad s_j(p) \le t < 1 - n/p' .$$

Similarly, $W_p^{t+1-1/p}(\Gamma) \hookrightarrow L_{p_2(t)}$ with

$$p_2(t) := \frac{p(n-1)}{n - p(t+1)} < \frac{n-1}{n-2} .$$

Set $G(u) := g^\natural(\gamma u)$. Since (9.3) entails $p_2(t) \ge r_2$ for $t \ge s_1(p)$, the commutativity of the diagram

$$\begin{array}{ccc} W_p^{t+1} & \xrightarrow{\gamma} & W_p^{t+1-1/p}(\Gamma) \\ {\scriptstyle G} \downarrow & & \downarrow \\ L_1(\Gamma) & \xleftarrow{g^\natural} & L_{p_2(t)} \end{array}$$

implies

$$G \in C\big(W_p^{t+1}, L_1(\Gamma)\big)\ , \quad s_1(p) \le t < 1 - n/p'\ .$$

Lastly, we put

$$\mathcal{F}_j(u,m) := F_j(u) + \mu$$

and

$$\mathcal{G}_j(u,m) := \begin{cases} \sigma\ , & j=0\ , \\ \delta\big(G(u)+\sigma\big)\ , & j=1\ , \end{cases}$$

for $u \in W_p^{t+j}$ and $m := (\mu,\sigma) \in \mathcal{M}$.

Lemma 9.1. *If $s_j(p) \le t < 1 - n/p'$ then $(\mathcal{F}_j, \mathcal{G}_j) \in C(W_p^{t+j} \times \mathcal{M}, \mathcal{M}_j)$, and this map is bounded on bounded sets. If f is independent of $\eta \in \mathbb{R}^n$ then $(\mathcal{F}_j, \mathcal{G}_j)$ is bounded on order intervals as well. If f is independent of η and f, g, and h are increasing in ξ then $(\mathcal{F}_j, \mathcal{G}_j)$ is also increasing.*

Proof. This follows from the above considerations. □

Now we assume that

$$\left.\begin{aligned} &f(\cdot,0,0),\ g(\cdot,0),\ \text{and } h(\cdot,0) \text{ are bounded a.e. and} \\ &f,\ g, \text{ and } h \text{ are continuously differentiable with respect to} \\ &(\xi,\eta) \in \mathbb{R}\times\mathbb{R}^n, \text{ and condition (1.3) is replaced by} \\ &\qquad |\partial_2 f(x,\xi,\eta)| \le c\big(1 + |\xi|^{r_1-1} + |\eta|^{r_0/r_1'}\big)\ , \\ &\qquad |\partial_3 f(x,\xi,\eta)| \le c\big(1 + |\xi|^{r_1/r_0'} + |\eta|^{r_0-1}\big)\ , \\ &\qquad |\partial_2 g(y,\xi)| \le c\big(1 + |\xi|^{r_2-1}\big)\ , \\ &\qquad |\partial_2 h(x,\xi)| \le c\big(1 + |\xi|^{r_0-1}\big) \\ &\text{for } (\xi,\eta) \in \mathbb{R}\times\mathbb{R}^n \text{ and a.a. } (x,y) \in \Omega\times\Gamma. \end{aligned}\right\} \tag{9.4}$$

It is easily verified that (9.4) implies (1.3).

Lemma 9.2. *Let assumption (9.4) be satisfied. If $s_j(p) \le t < 1 - n/p'$ then $(\mathcal{F}_j, \mathcal{G}_j) \in C^1(W_p^{t+j} \times \mathcal{M}, \mathcal{M}_j)$ and $(\partial\mathcal{F}_j, \partial\mathcal{G}_j)$ is bounded on bounded sets.*

Proof. This follows by an obvious modification of the proof of Lemma 9.1, using the 'differentiable version' of the Krasnosel'skii-Vainberg theorem. □

We can assume that $n/(n-1) < r_1 < n/(n-2)$ and define $p_\bullet \in (1, 1^*)$ by

$$\frac{1}{p_\bullet} := \frac{1}{r_j} + \frac{j}{n}\ , \qquad j = 0, 1\ ,$$

so that $s_j(p) = 0$ for $p_\bullet \le p < 1^*$. Of course, $j = 1$ if we consider problem $(9.1)_m$, and $j = 0$ for $(9.2)_m$. It follows from Lemma 9.1 that the following definitions are meaningful:

u is a weak solution (in $W^1_{p_\bullet}$) of problem $(9.1)_m$ if $u \in W^1_{p_\bullet,\mathcal{B}}$ and

$$\mathfrak{a}(v,u) = \int_\Omega vf(\cdot,u,\nabla u)\,dx + \int_\Omega v\,d\mu + \int_{\Gamma_1} \gamma v g(\cdot,\gamma u)\,d\Gamma + \int_{\Gamma_1} \gamma v\,d\sigma \quad (9.5)$$

for $v \in W^1_{p'_\bullet,\mathcal{B}}$;

u is a very weak solution (in $L_{p_\bullet}$) of problem $(9.2)_m$ if $u \in L_{p_\bullet}$ and

$$\int_\Omega (\mathcal{A}^\sharp v)u\,dx = \int_\Omega vh(\cdot,u)\,dx + \int_\Omega v\,d\mu - \int_{\Gamma_0} \partial_{\nu_a} v\,d\sigma + \int_{\Gamma_1} \gamma v\,d\sigma \quad (9.6)$$

for $v \in W^2_{p'_\bullet,\mathcal{B}^\sharp}$. Of course, a weak or a very weak solution, respectively, of $(9.1)_m$ or $(9.2)_m$, respectively, is just a solution of $(6.3)_{1,m}$ or $(6.3)_{0,m}$, respectively.

For these types of solutions we have the following regularity result.

Theorem 9.3. *Suppose that u is a weak [resp. a very weak] solution of $(9.1)_m$ [resp. $(9.2)_m$]. Then u belongs to W^{2-}_1 [resp. W^{1-}_1], hence to $W^1_{1^*-}$ [resp. L_{1^*-}].*

Proof. Put $p := p_\bullet$. Then (9.5) [resp. (9.6)] and Lemma 9.1 imply that u is a weak [resp. very weak] solution of $(9.1)_m$ [resp. $(9.2)_m$] iff u is a solution (in $W^j_{p_\bullet}$) of problem $(6.3)_{j,m}$ where $j = 1$ [resp. $j = 0$]. Thus Corollary 6.2 entails the assertion. □

Theorem 9.4. *Let assumption (9.4) be satisfied. Suppose that $m^* \in \mathcal{M}$ and u^* is a weak [resp. very weak] solution of $(9.1)_{m^*}$ [resp. $(9.2)_{m^*}$]. Also suppose that the linear BVP*

$$\begin{aligned} \mathcal{A}v &= \partial_2 f(\cdot,u^*,\nabla u^*)v + \partial_3 f(\cdot,u^*,\nabla u^*)\cdot\nabla v && \text{in } \Omega\,, \\ \mathcal{B}v &= \delta\partial_2 g(\cdot,u^*)v && \text{on } \Gamma\,, \end{aligned}$$
$$\big[\text{resp. } \mathcal{A}v = \partial_2 h(\cdot,u^*)v \text{ in } \Omega\,,\quad \mathcal{B}v = 0 \text{ on } \Gamma\big]$$

has the trivial solution only. Then there exists a neighborhood $U \times M$ of (u^*,m^*) in $W^1_{p_\bullet} \times \mathcal{M}$ [resp. $L_{p_\bullet} \times \mathcal{M}$] such that $(9.1)_m$ [resp. $(9.2)_m$] has for each $m \in M$ exactly one weak [resp. very weak] solution in U.

Proof. This is a consequence of Lemma 9.2 and Theorem 6.3. □

Remark 9.5. It should be noted that everything said so far in this section is also true — with the obvious modifications — if $(\mathcal{A}, \mathcal{B})$ is a normally elliptic system. □

Clearly, u is said to be a weak [resp. very weak] subsolution of $(9.1)_m$ [resp. $(9.2)_m$] if $u \in W^1_{p_\bullet}$ [resp. $L_{p_\bullet}$] and u satisfies (9.5) [resp. (9.6)] with the equality sign replaced by $\le$ and only nonnegative test functions v being admitted. Similarly, weak [resp. very weak] supersolutions are defined by replacing the equality signs by $\ge$.

10. The Sub- and Supersolutions Theorem. In the remainder of this paper we consider the case where f is independent of $\eta \in \mathbb{R}^n$. To be more precise, we consider the BVP

$$\mathcal{A}u = f(x,u) + \mu \text{ in } \Omega \ , \quad \mathcal{B}u = \delta g(x,u) + \sigma \text{ on } \Gamma \ , \tag*{$(10.1)_m$}$$

with $m = (\mu, \sigma) \in \mathcal{M}$ and f and g being Carathéodory functions such that

$$\left.\begin{array}{l} |f(x,\xi)| \le c\big(1+|\xi|^r\big) \ , \quad |g(y,\xi)| \le c\big(1+|\xi|^\rho\big) \\ \text{for } \xi \in \mathbb{R} \text{ and a.a. } (x,y) \in \Omega \times \Gamma, \\ \text{where } r \text{ and } \rho \text{ satisfy (1.8) and (1.9),} \\ \text{and } f(x,\cdot) \text{ and } g(y,\cdot) \text{ are increasing for a.a. } (x,y) \in \Omega \times \Gamma. \end{array}\right\} \tag{10.2}$$

We put $j := 0$ if $\sigma \,|\, \Gamma_0 \ne 0$ and $j := 1$ otherwise. Note that now

$$\frac{1}{p_\bullet} = \frac{1}{r} + \frac{j}{n} \ , \qquad j = 0, 1 \ .$$

By a solution of $(10.1)_m$ we mean a weak solution in $W^1_{p_\bullet}$ if $\sigma \,|\, \Gamma_0 = 0$ and a very weak solution in $L_{p_\bullet}$ if $\sigma \,|\, \Gamma_0 \ne 0$. Similar definitions apply to sub- and supersolutions of $(10.1)_m$.

The preceding considerations together with Theorem 7.2 lead directly to the following fundamental existence theorem which generalizes the well-known sub- and supersolutions theorem for classical elliptic BVPs.

Theorem 10.1. *Suppose that $\overline{w}$ is a subsolution of $(10.1)_{\overline{m}}$ and $\widehat{w}$ is a supersolution of $(10.1)_{\widehat{m}}$ such that $(\overline{w}, \overline{m}) \le (\widehat{w}, \widehat{m})$. Then $(10.1)_m$ has for each $m \in [\overline{m}, \widehat{m}]$ a least solution $\overline{u}(m)$ and a greatest solution $\widehat{u}(m)$ in $[\overline{w}, \widehat{w}]$ and the maps $\overline{u}(\cdot)$ and $\widehat{u}(\cdot)$ are strictly increasing.*

Remark 10.2. For Theorem 10.1 to remain valid it suffices to assume that $\big(f(x,\cdot),g(y,\cdot)\big)$ is increasing on $\big[\overline{w}(x),\widehat{w}(x)\big]\times\big[\gamma\overline{w}(y),\gamma\widehat{w}(y)\big]$ for a.a. $(x,y)\in\Omega\times\Gamma$. $\square$

11. Parameter-Dependent Problems. Now we investigate the existence of positive solutions. For this we assume, in addition to (10.2), that

$$\big(f(\cdot,0),g(\cdot,0)\big)=(0,0)\quad\text{and}\quad m=(\mu,\sigma)>0\ . \tag{11.1}$$

Then we consider the parameter-dependent problem

$$\mathcal{A}u=f(x,u)+\lambda\mu\text{ in }\Omega\ ,\quad \mathcal{B}u=\delta g(y,u)+\lambda\sigma\text{ on }\Gamma\ . \tag*{$(11.2)_{\lambda m}$}$$

The next theorem guarantees, in particular, that each solution of $(11.2)_{\lambda m}$ is positive if $\lambda>0$. Recall that $\lambda_0:=\lambda_0(\mathcal{A},\mathcal{B})$.

Theorem 11.1. *Theorem* 8.1 *is valid for problem* $(11.2)_{\lambda m}$. *If there exists* $\beta>0$ *such that*

$$f(x,\xi)\ge\lambda_0\xi-\beta\ ,\qquad \xi\in\mathbb{R}^+\ ,\quad a.a.\ x\in\Omega\ , \tag{11.3}$$

then $\lambda^*<\infty$.

Proof. From our earlier considerations we know that $(11.2)_{\lambda m}$ is a special case of problem $(6.3)_{j,\lambda m}$, where $j=0$ if $\sigma\,|\,\Gamma_0\ne0$, and $j=1$ otherwise. This implies the first assertion.

We deduce from (11.3) that

$$\big\langle\varphi^\sharp,F(u)\big\rangle\ge\lambda_0\langle\varphi^\sharp,u\rangle-\beta\int_\Omega\varphi^\sharp\,dx\ ,\qquad u\in(W^j_{p_\bullet})^+\ .$$

Since $\varphi^\sharp\in C^2$ with $\varphi^\sharp(x)>0$ for $x\in\Omega\cup\Gamma_1$ and $\partial_{\nu_a}\varphi^\sharp(x)<0$ for $x\in\Gamma_0$ we see that $\alpha:=\int_\Omega\varphi^\sharp\,d\mu+\langle\mathcal{C}^\sharp\varphi^\sharp,\sigma\rangle_\partial>0$. Hence

$$\big\langle\varphi^\sharp,F(u)+\lambda\mu\big\rangle+\big\langle\mathcal{C}^\sharp\varphi^\sharp,\delta G(u)+\lambda\sigma\big\rangle\ge\lambda_0\langle\varphi^\sharp,u\rangle+\beta(\lambda)$$

for $(\lambda,u)\in\Sigma$, where $\beta(\lambda):=\lambda\alpha-\beta\int\varphi^\sharp\,dx$. Hence $\beta(\lambda)\to\infty$ as $\lambda\to\infty$, and $\lambda^*<\infty$ follows from Proposition 8.2. $\square$

12. Index Computations. Now we strengthen hypotheses (10.2) by assuming differentiability. More precisely, we suppose that $m = (\mu, \sigma) > 0$ and f and g are Carathéodory functions such that

$$\left.\begin{array}{l} \bigl(f(x,\cdot), g(y,\cdot)\bigr) \in C^1(\mathbb{R},\mathbb{R}) \text{ with} \\ |\partial_2 f(x,\xi)| \le c\bigl(1+|\xi|^{r-1}\bigr)\ , \quad |\partial_2 g(y,\xi)| \le c\bigl(1+|\xi|^{\rho-1}\bigr) \\ \text{for a.a. } (x,y) \in \Omega \times \Gamma \text{ and } \xi \in \mathbb{R}, \\ \text{where } r \text{ and } \rho \text{ satisfy (1.8) and (1.9),} \\ \bigl(f(x,0), g(y,0)\bigr) = (0,0) \text{ and } \bigl(\partial_2 f(x,\cdot), \partial_2 g(y,\cdot)\bigr) \ge (\alpha, 0) \\ \text{for a.a. } (x,y) \in \Omega \times \Gamma \text{ and some } \alpha > 0. \end{array}\right\} \tag{12.1}$$

We set $E := W^j_{p_\bullet}$ and $T(u) := (\mathcal{A},\mathcal{B})^{-1}\bigl(F(u), \delta G(u)\bigr)$ for $u \in E$. Then

$$\mathcal{S}(\lambda, u) = T(u) + \lambda L(m)\ , \qquad (\lambda, u) \in \mathbb{R} \times E\ ,$$

where

$$L(m) := (\mathcal{A},\mathcal{B})^{-1}(\mu,\sigma) \in W_1^{(j+1)-}$$

and $L(m) > 0$. It follows from (12.1) and Lemma 9.2 that $\mathcal{S} \in C^1(\mathbb{R} \times E, E)$ and that

$$\partial_2 \mathcal{S}(\lambda, u)v = \partial T(u)v = (\mathcal{A},\mathcal{B})^{-1}\bigl(\partial F(u)v, \delta \partial G(u)v\bigr)$$

with

$$\bigl(\partial F(u)v, \partial G(u)v\bigr) = \bigl((\partial_2 f)^\natural(u)v, (\partial_2 g)^\natural(\gamma u)\gamma v\bigr)\ .$$

Note that $E \hookrightarrow L_r$ and that

$$\bigl[v \mapsto \bigl(\partial F(u)v, \partial G(u)v\bigr)\bigr] \in \mathcal{L}\bigl(L_r, L_1 \times L_1(\Gamma)\bigr)\ .$$

Hence $\partial T(u)$ is a compact endomorphism of L_r. In this interpretation we denote it by $K(u)$.

Lemma 12.1. *Suppose $u \in E^+$. Then the spectral radius $\kappa(u)$ of $K(u)$ is positive and a simple eigenvalue. There exist an eigenfunction $\psi(u)$ of $K(u)$ which is positive a.e. in Ω and a stricly positive eigenvector $\psi'(u)$ of $\bigl[K(u)\bigr]'$ to the eigenvalue $\kappa(u)$. Moreover, $\kappa(u)$ is the only eigenvalue of $K(u)$ possessing a positive eigenfunction, and the map $u \mapsto \kappa(u)$ is continuous on E^+.*

Proof. Since $u \ge 0$ it follows from (12.1) that $K := K(u)$ is a positive endomorphism of L_r. Note that

$$Kv \ge (\mathcal{A},\mathcal{B})^{-1}(\alpha v, 0) = \alpha A^{-1} v\ , \qquad v \in L_r^+\ .$$

Hence

$$K^k v \geq \alpha^k A^{-k} v \ , \qquad v \in L_r^+ \ , \quad k \in \mathbb{N} \ . \tag{12.2}$$

Since $A^{-1} \in \mathcal{L}(L_q, W_q^2)$ for $q \in (1, \infty)$ and $W_q^2 \hookrightarrow L_{q_1}$ with $1/q_1 \geq 1/q - 2/n$ we can fix $k_0 > 1$ such that $(\alpha^{-1}A)^{-k_0+1} \in \mathcal{L}(L_r, L_q)$ for some $q > n$. Thus, by the maximum principle [4, Theorem 6.1], $w := w(v) := \alpha^{k_0} A^{-k_0} v$ is continuous on $\overline{\Omega}$ and everywhere positive on $\Omega \cup \Gamma_1$, provided $v \in (L_r^+)\backslash\{0\}$. Hence we deduce from (12.2) that $K^{k_0} v \geq w$. Consequently,

$$K(\lambda - K)^{-1} v = \sum_{k=1}^{\infty} \lambda^{-k} K^k v \geq \lambda^{-k_0} K^{k_0} v \geq \lambda^{-k_0} w \ , \qquad v \in (L_r^+)\backslash\{0\} \ ,$$

for every $\lambda > \kappa := \kappa(u)$. This shows that K is irreducible (cf. [27, V.7.7 and App. 3]). Since L_r is a Banach lattice, de Pagter's theorem [14, Theorem 3] guarantees that $\kappa > 0$. Now [27, App. 3.2] implies that κ is a simple eigenvalue of K, that it possesses a positive eigenfunction ψ, and that K' has a strictly positive eigenvector ψ' belonging to κ. Note that

$$\psi = \kappa^{-1} K \psi = \kappa^{-k_0} K^{k_0} \psi \geq \kappa^{-k_0} w(\psi) \tag{12.3}$$

so that ψ is positive a.e. in Ω.

Suppose that t is an eigenvalue of K possessing a positive eigenfunction v. Then, by applying ψ' to $tv = Kv$, it follows that $t\langle \psi', v \rangle = \kappa \langle \psi', v \rangle$. Since ψ' is strictly positive, $\langle \psi', v \rangle > 0$, so that $t = \kappa$.

Lastly, observe that the map $u \mapsto K(u)$ is continuous from E into $\mathcal{L}(L_r)$. Since $K(u)$ is compact, $\kappa_0(u)$ is an isolated simple eigenvalue for $u \in E^+$. Thus the upper continuity of the spectrum entails that $\kappa(\cdot)$ is continuous $\square$

Remark 12.2. It should be observed that (12.3) contains more precise information on $\psi(u)$. For example, if $j = 1$ then (12.3) also implies that $\gamma\psi(u)(y) > 0$ for a.a. $y \in \Gamma_1$. $\square$

It is easily seen that Lemma 12.1 guarantees the existence of a positive eigenvalue, namely $1/\kappa(u)$, and of a corresponding positive eigenfunction, namely $\psi(u)$, for the elliptic eigenvalue problem

$$\mathcal{A}w = \lambda \partial_2 f(\cdot, u) w \text{ in } \Omega \ , \quad \mathcal{B}w = \lambda \delta \partial_2 g(\cdot, u) w \text{ on } \Gamma$$

whenever $u \in E^+$. However, thanks to the low regularity of $\partial_2 f(\cdot, u)$ and $\partial_2 g(\cdot, u)$, the eigenvector $\psi'(u)$ cannot be interpreted as the solution of the 'dual' linear eigenvalue problem which is obtained by replacing $(\mathcal{A}, \mathcal{B})$ by $(\mathcal{A}^\sharp, \mathcal{B}^\sharp)$.

Lemma 12.3. *The set* $\Lambda_0 := \{ \lambda \in \Lambda \; ; \; \kappa(\overline{u}_\lambda) < 1 \}$ *is open in* Λ, *and*

$$(\lambda \mapsto \overline{u}_\lambda) \in C^1(\Lambda_0, E^+) \; .$$

If $\lambda \in \Lambda_0$ *then* $\overline{u}_\lambda$ *is an isolated fixed point of* $\mathcal{S}(\lambda, \cdot)$ *in* E. *If* $\kappa(0) < 1$ *then* $\lambda^* > 0$. *Lastly, if* λ^* *is finite and a limit point of* Λ_0 *in* Λ *then* $\kappa(\overline{u}_{\lambda^*}) = 1$.

Proof. Put $\Psi(\lambda, u) := u - \mathcal{S}(\lambda, u) = u - T(u) - \lambda L(m)$. Then Ψ is continuously differentiable and

$$\partial_2 \Psi(\lambda, u)v = v - \partial T(u)v \; , \qquad (\lambda, u) \in \mathbb{R} \times E \; , \quad v \in E \; . \tag{12.4}$$

Moreover, $\Psi(\lambda, u) = 0$ for $(\lambda, u) \in \Sigma$.

Suppose that $\lambda_0 \in \Lambda_0$. Since $\kappa(\overline{u}_{\lambda_0}) < 1$ and $\kappa(\overline{u}_{\lambda_0})$ is the spectral radius of $K(\overline{u}_{\lambda_0})$, it follows from (12.4) and the fact that $K(\overline{u}_{\lambda_0})$ equals $\partial T(\overline{u}_{\lambda_0})$ as an endomorphism of L_r, that $\partial_2 \Psi(\lambda, \overline{u}_{\lambda_0})$ is an automorphism of E. Hence the implicit function theorem guarantees the existence of a neighborhood $J \times U$ of $(\lambda_0, \overline{u}_{\lambda_0})$ in $\mathbb{R} \times E$ and of a map $u(\cdot) \in C^1(J, U)$ such that $\Psi(\lambda, u) = 0$ holds for $(\lambda, u) \in J \times U$ iff $\lambda \in J$ and $u = u(\lambda)$.

Assume that $0 < \lambda_0 < \lambda^*$. Then the left continuity and isotonicity of $\lambda \mapsto \overline{u}_\lambda$ implies that $u(\lambda) = \overline{u}_\lambda$ for $\lambda \in J \cap \mathbb{R}^+$. Hence suppose that $\lambda_0 = 0$. Since $\kappa(0) < 1$ we can compute $u(\lambda)$ for λ sufficiently small by the iteration scheme

$$u_{n+1} = \mathcal{S}(\lambda, u_n) = T(u_n) + \lambda L(m) \; , \quad n \in \mathbb{N} \; , \qquad u_0 = 0 \; .$$

From this we infer that $u(\lambda) > 0$ for $0 < \lambda < \varepsilon$, which entails that $\lambda^* > 0$ and $u(\lambda) = \overline{u}_\lambda$ for $0 \le \lambda < \varepsilon$ and a suitable $\varepsilon > 0$.

Finally, it follows from $\overline{u}_\lambda = u(\lambda)$ for $\lambda \in \Lambda_0$ that Λ_0 is open in Λ. The last assertion is a consequence of the continuity of $u \mapsto \kappa(u)$ in E^+, which is guaranteed by Lemma 12.1. □

Remarks 12.4. **(a)** Suppose that $f(x, \cdot)$ and $g(y, \cdot)$ are either both convex or both concave on $\mathbb{R}^+$ for a.a. $(x, y) \in \Omega \times \Gamma$. Then $(0, \lambda^*) \subset \Lambda_0$.

Proof. The assumption entails that $\big(\partial_2 f(x, \cdot), \partial_2 g(y, \cdot)\big)$ is either increasing or decreasing on $\mathbb{R}^+$ for a.a. $(x, y) \in \Omega \times \Gamma$. This implies that either $K(u) \le K(v)$ or $K(v) \le K(u)$ for $0 \le u \le v$. Suppose that $0 \le \lambda_0 < \lambda_1$ and $\lambda_1 \in \Lambda$ and set $u_j := \overline{u}_{\lambda_j}$. Since $\lambda \mapsto \overline{u}_\lambda$ is strictly increasing, it follows that

$$\begin{aligned} u_1 - u_0 > T(u_1) - T(u_0) &= \int_0^1 K\big(u_0 + t(u_1 - u_0)\big) \, dt \, (u_1 - u_0) \\ &\ge K(u_0)(u_1 - u_0) \end{aligned}$$

or $u_1 - u_0 > K(u_1)(u_1 - u_0)$, respectively. Now the assertion is obtained by applying either $\psi'(u_0)$ or $\psi'(u_1)$, respectively, to these inequalities. □

(b) Suppose that $\lambda^* < \infty$. Also suppose that $f(x, \cdot)$ is strictly convex and $g(y, \cdot)$ is convex for a.a. $(x, y) \in \Omega \times \Gamma$. If $\lambda^* \in \Lambda$ then problem $(11.2)_{\lambda^* m}$ is uniquely solvable.

Proof. Let $u \neq u^* := \overline{u}_{\lambda^*}$ be a solution of $(11.2)_{\lambda^* m}$. Then $u > u^*$ and, similarly as in the proof of (a), we see that $u - u^* > K(u^*)(u - u^*)$. Thus, by applying $\psi'(u^*)$ to this inequality, it follows that $\kappa(u^*) < 1$ which contradicts Lemma 12.3. □

(c) Assume that $\big(\partial_2 f(\cdot, 0), \partial_2 g(\cdot, 0)\big) \in C(\overline{\Omega}) \times C^1(\Gamma)$. If

$$\lambda_0\big(\mathcal{A} - \partial_2 f(\cdot, 0), \mathcal{B} - \delta\partial_2 g(\cdot, 0)\gamma\big) > 0 \tag{12.5}$$

then $\kappa(0) < 1$.

Proof. Observe that $\kappa(0)\psi = K(0)\psi$ with $\psi > 0$ iff $\psi \in (W^2_{\infty-})^+$ and

$$\begin{aligned} \big[\mathcal{A} - \kappa(0)^{-1}\partial_2 f(\cdot, 0)\big]\psi &= 0 \qquad \text{in } \Omega \ , \\ \big[\mathcal{B} - \kappa(0)^{-1}\delta\partial_2 g(\cdot, 0)\gamma\big]\psi &= 0 \qquad \text{on } \Gamma \ . \end{aligned}$$

Hence it follows from Theorem 5.1 that $\kappa(0)$ is the unique solution of

$$\phi(t) := \lambda_0\big(\mathcal{A} - t^{-1}\partial_2 f(\cdot, 0), \mathcal{B} - t^{-1}\delta\partial_2 g(\cdot, 0)\gamma\big) = 0 \ , \qquad t > 0 \ .$$

Since $\big(\partial_2 f(\cdot, 0), \partial_2 g(\cdot, 0)\big) \geq (\alpha, 0)$, Remark 5.2(c) implies that ϕ is a strictly increasing function of t. Hence assumption (12.5) guarantees that $\phi(t) > 0$ for $t \geq 1$, which proves the assertion. □

Lemma 12.5. $i\big(\mathcal{S}(\lambda, \cdot), \overline{u}_\lambda\big) = 1$ *for* $\lambda \in \Lambda_0$.

Proof. Fix $\lambda \in \Lambda_0$ and put $u_0 := \overline{u}_\lambda$ and $\mathcal{S}(u) := \mathcal{S}(\lambda, u)$. Lemma 12.3 guarantees the existence of $\varepsilon_0 > 0$ such that u_0 is the only fixed point of $\mathcal{S}$ in $u_0 + \varepsilon_0\mathbb{B}$. Note that

$$\mathcal{S}(u) - u_0 = T(u) - T(u_0) = K(u_0)(u - u_0) + r(u, u_0)(u - u_0) \ , \qquad u \in E \ ,$$

where $r(u - u_0) \to 0$ as $u \to u_0$ in E. Fix α and β with $\kappa(u_0) < \alpha < \beta < 1$. Since $\kappa(u_0)$ is the spectral radius of $K(u_0)$ we can fix an equivalent norm — again denoted by $\|\cdot\|$ — on E such that $\|K(u_0)\| \leq \alpha$ (e.g., [20, Lemma 2.2]).

Hence choosing $\varepsilon_1 \in (0, \varepsilon_0)$ so small that $\|r(u, u_0)\| \le \beta - \alpha$ for $u \in u_0 + \varepsilon_1 \mathbb{B}$, it follows that

$$\|\mathcal{S}(u) - u_0\| \le \alpha \|u - u_0\| + (\beta - \alpha) \|u - u_0\| = \beta \|u - u_0\| < \|u - u_0\|$$

for $\|u - u_0\| \le \varepsilon_1$. Thus

$$\left\|\bigl(t\mathcal{S}(u) + (1-t)u_0\bigr) - u_0\right\| = t \|\mathcal{S}(u) - u_0\| < \varepsilon \ , \qquad 0 \le t \le 1 \ ,$$

for $u \in u_0 + \varepsilon \partial \mathbb{B}^+$ and $0 < \varepsilon \le \varepsilon_1$. Hence $t\mathcal{S}(u) + (1-t)u_0$ has no fixed point on $u_0 + \varepsilon \partial \mathbb{B}^+$ for $0 < \varepsilon \le \varepsilon_1$ and $0 \le t \le 1$. Since $t\mathcal{S}(u) + (1-t)u_0$ belongs to E^+ for $u \in E^+$ and $0 \le t \le 1$, the homotopy invariance of the fixed point index guarantees that

$$i\bigl(\mathcal{S}, (u_0 + \varepsilon \mathbb{B}) \cap E^+\bigr) = i\bigl(u_0, (u_0 + \varepsilon \mathbb{B}) \cap E^+\bigr) = 1$$

for $0 < \varepsilon \le \varepsilon_1$, where the last equality sign follows from the normalization property of the fixed point index (see [3, Theorem 11.1]). This proves the assertion. □

13. A Priori Estimates. In this section we establish a priori bounds for solutions of problem $(11.2)_{\lambda m}$. Combined with the results of the preceding section these bounds will imply the existence of multiple solutions.

We begin with a technical lemma for which we observe that $p'_\bullet > n$.

Lemma 13.1. *Let $\varphi, \psi \in C^3(\overline{\Omega})$ be such that ψ equals $\mathbf{1}$ in a neighborhood of* $\operatorname{supp}(\varphi)$. *Fix $p_0 \in (p_\bullet, 1^*)$ and define s_j and t_j by*

$$\frac{1}{s_j} := (r-1)\Bigl(\frac{1}{p_0} - \frac{j}{n}\Bigr) \ , \quad \frac{1}{t_j} := \frac{(\rho - 1)n}{n-1}\Bigl(\frac{1}{p_0} - \frac{j}{n}\Bigr) \ .$$

Also fix $\alpha \in (0, 1 - n/p'_\bullet)$. Let $\bigl((\mathcal{A}_k, \mathcal{B}_k)\bigr)$ be a relatively compact sequence in $\mathcal{E}$ such that $\inf_k \lambda_0(\mathcal{A}_k, \mathcal{B}_k) > 0$ and let $\bigl((a_k, b_k)\bigr)$ be a bounded sequence in $L_{s_j} \times L_{t_j}(\Gamma_1)$. Suppose that $u_k \in W^j_{p_\bullet}$ and

$$\mathcal{A}_k u_k = a_k u_k \ \text{in } \Omega \ , \quad \mathcal{B}_k u_k = \delta_{1,j} b_k \delta \gamma u_k \ \text{on } \Gamma \ , \qquad k \in \mathbb{N} \ . \tag{13.1}$$

Then $u_k \in C^\alpha(\overline{\Omega})$ for $k \in \mathbb{N}$. If the sequence (ψu_k) is bounded in $W^j_{p_\bullet}$ then (φu_k) is bounded in $C^\alpha(\overline{\Omega})$.

Proof. It follows from (13.1) that

$$u_k = (\mathcal{A}_k, \mathcal{B}_k)^{-1}(a_k u_k, \delta_{1,j} b_k \delta \gamma u_k) = P_k u_k + \delta_{1,j} Q_k u_k \ , \tag{13.2}$$

where

$$P_k u := A_k^{-1} a_k u \ , \quad Q_k u := R_k b_k \delta \gamma u \ ,$$

and

$$R_k g := (\mathcal{A}_k, \mathcal{B}_k)^{-1}(0, g) \ , \qquad g \in W_p^{-1-1/p}(\Gamma_1) \ , \quad 1 < p < \infty \ .$$

Sobolev's embedding theorem, Hölder's inequality, and Theorem 2.1 imply the commutativity of the following diagram of continuous linear maps:

$$\begin{array}{ccccc} W_p^{2-j} & \hookrightarrow & L_{q_j} & \xrightarrow{\ a_k \cdot\ } & L_{r_j} \\ & \searrow P_k & & A_k^{-1} \swarrow & \\ & W_{\pi_j}^{2-j} & \hookleftarrow & W_{r_j}^2 & \end{array} \tag{13.3}$$

with

$$\begin{aligned} \frac{1}{\pi_j} \geq \frac{1}{r_j} - \frac{j}{n} \geq \frac{1}{q_j} + \frac{1}{s_j} - \frac{j}{n} \geq \frac{1}{p} - \frac{2-j}{n} + \frac{1}{s_j} - \frac{j}{n} \\ = \frac{1}{p} - \frac{2}{n} + (r-1)\Big(\frac{1}{p_0} - \frac{j}{n}\Big) \end{aligned} \tag{13.4}$$

for $1 < p < \infty$ with $p \geq p_\bullet$ if $j = 1$. Similarly, we see that

$$\begin{array}{ccccc} W_p^1 & \xrightarrow{\ \gamma\ } & W_p^{1-1/p}(\Gamma) & \hookrightarrow & L_q(\Gamma) \\ Q_k \downarrow & & & & \downarrow b_k \delta \cdot \\ W_\pi^1 & \hookleftarrow & W_t^{1+1/t-\varepsilon} & \xleftarrow{\ R_k\ } & L_t(\Gamma_1) \end{array}$$

with

$$\begin{aligned} \frac{1}{\pi} &\geq \Big(1 - \frac{1}{n}\Big)\frac{1}{t} + \frac{\varepsilon}{n} \geq \Big(1 - \frac{1}{n}\Big)\Big(\frac{1}{q} + \frac{1}{t_1}\Big) + \frac{\varepsilon}{n} \\ &\geq \frac{1}{p} - \frac{1}{n} + \Big(1 - \frac{1}{n}\Big)\frac{1}{t_1} + \frac{\varepsilon}{n} = \frac{1}{p} - \frac{1-\varepsilon}{n} + (\rho - 1)\Big(\frac{1}{p_0} - \frac{1}{n}\Big) \end{aligned} \tag{13.5}$$

for $p_\bullet \leq p < \infty$, provided $0 < \varepsilon < 1/t$. The restriction $p \geq p_\bullet$ for $j = 1$ stems from the fact that we have to insure that $r_j > 1$ in (13.4) and $t > 1$ in (13.5).

Since $1/q_j \ge 0$, the second inequality in (13.4) implies $r_j \le s_j$. Now we infer from the first inequality in (13.4) that $1/\pi_j \ge 1/s_j - j/n$. Note that $p_0 > p_\bullet$ entails

$$\frac{1}{s_j} - \frac{j}{n} < 1 - \frac{1}{r} - \frac{j}{n} = 1 - \frac{1}{p_\bullet} = \frac{1}{p_\bullet'} \ .$$

This shows that (13.3) holds for $p \ge p_\bullet$ if $\pi_j \le p_\bullet'$.

Similarly, since $1/q \ge 0$ we deduce from the second inequality in (13.5) that $t \le t_1$ so that the first inequality in (13.5) gives

$$\frac{1}{\pi} \ge \Bigl(1 - \frac{1}{n}\Bigr)\frac{1}{t_1} + \frac{\varepsilon}{n} \ .$$

Since $p_0 > p_\bullet$ implies

$$\Bigl(1 - \frac{1}{n}\Bigr)\frac{1}{t_1} < \frac{\rho - 1}{r} \le 1 - \frac{1}{n} - \frac{1}{r} = \frac{1}{p_\bullet'}$$

(thanks to $j = 1$ in this case), we see that $Q_k \in \mathcal{L}(W_p^1, W_\pi^1)$, provided $1/\pi$ is greater than or equal to $1/p_\bullet' + \varepsilon/n$.

Thanks to $0 < \varepsilon < 1/t$ and $t \le t_1$ we can fix ε with

$$0 < \varepsilon < \frac{n}{2}\Bigl(\frac{1}{r} - 1 + \frac{2}{n}\Bigr) \wedge \frac{1}{t_1} \wedge \Bigl(1 - \frac{n}{p_\bullet'} - \alpha\Bigr)$$

independently of $p \ge p_\bullet$. Observe that

$$-\frac{2}{n} + (r-1)\Bigl(\frac{1}{p_0} - \frac{j}{n}\Bigr) < -\frac{2}{n} + (r-1)\Bigl(\frac{1}{p_\bullet} - \frac{j}{n}\Bigr) = 1 - \frac{2}{n} - \frac{1}{r} < -\frac{2\varepsilon}{n}$$

and

$$\begin{aligned} -\frac{1-\varepsilon}{n} + (\rho - 1)\Bigl(\frac{1}{p_0} - \frac{1}{n}\Bigr) &< -\frac{1-\varepsilon}{n} + \frac{\rho - 1}{r} \\ &\le \frac{\varepsilon}{n} + 1 - \frac{2}{n} - \frac{1}{r} < -\frac{\varepsilon}{n} \ . \end{aligned} \tag{13.6}$$

Hence it follows from (13.4) and (13.5) and the above considerations that

$$P_k \in \mathcal{L}(W_p^{2-j}, W_{q(p)}^{2-j}) \ , \quad Q_k \in \mathcal{L}(W_p^1, W_{q(p)}^1) \ , \tag{13.7}$$

where $1/q(p) \geq (1/p - \varepsilon/n) \vee (1/p'_\bullet + \varepsilon/n)$ for $p \geq p'_\bullet$.

Suppose that $j = 0$. By replacing W_p^{2-j} in (13.3) by L_r it follows that

$$P_k \in \mathcal{L}(L_r, W_{p_1}^2) \ , \qquad \frac{1}{p_1} := \frac{1}{r} + \frac{r-1}{p_0} < 1 \ .$$

Thus we deduce from (13.2) that

$$u_k \in W_{p_1}^{2-j} \ , \quad \frac{1}{p_1} := \begin{cases} 1/r + (r-1)/p_0 \ , & j = 0 \ , \\ 1/p_\bullet \ , & j = 1 \ . \end{cases}$$

An obvious bootstrapping argument, based on (13.2) and (13.7), shows that $u_k \in W_p^{2-j}$ with $1/p \geq 1/p'_\bullet + \varepsilon/n$ for $k \in \mathbb{N}$. Since $1/p'_\bullet + \varepsilon/n < (1-\alpha)/n$, thanks to the choice of ε, we see that u_k belongs to $W^1_{n/(1-\alpha)}$, hence to $C^\alpha(\overline{\Omega})$, for $k \in \mathbb{N}$.

Now we prove the boundedness assertion. First assume that $j = 0$. Let $\varphi_0, \varphi_1 \in C^3(\overline{\Omega})$ satisfy $\varphi_0 \,|\, \mathrm{supp}(\varphi_1) = \mathbf{1}$ and suppose the sequence $(\varphi_0 u_k)$ is bounded in L_p for some $p \geq p_\bullet$. Then the sequence $(\varphi_1 a_k u_k)$ is bounded in L_q, where

$$\frac{1}{q} \geq \frac{1}{p} + \frac{1}{s_0} = \frac{1}{p} + \frac{r-1}{p_0} \ .$$

Since $W^1_{p'} \hookrightarrow L_{q'}$ if $1/q \leq 1/p + 1/n$, and since $(r-1)/p_0 < 1 - 1/r < 1/n$, it follows that $L_q \hookrightarrow W^{-1}_{p,\mathcal{B}}$. Hence Theorem 3.1 implies that $\big((\varphi_1 u_k)\big)$ is bounded in W_p^1, and we infer from $W_p^1 \hookrightarrow L_\pi$ with $1/\pi \geq 1/p - 1/n$ that the boundedness of the sequence $(\varphi_0 u_k)$ in L_p implies the boundedness of $(\varphi_1 u_k)$ in L_π with $1/p - 1/\pi \leq 1/n$.

Set $\varphi_0 := \psi$ and fix an integer $\ell > 2 + n(1/r - 1/s_0)$. Also set $\varphi_\ell := \varphi$ and choose $\varphi_i \in C^3(\overline{\Omega})$ such that φ_{i-1} equals $\mathbf{1}$ in a neighborhood of $\mathrm{supp}(\varphi_i)$ for $1 \leq i \leq \ell - 1$. Since $(\varphi_0 u_k)$ is bounded in L_r we deduce inductively from the previous considerations that $(\varphi_{\ell-1} u_k)$ is bounded in $W^1_{s_0}$. Hence $(\varphi a_k u_k)$ is bounded in L_{s_0} and Theorem 3.1 guarantees that (φu_k) is bounded in $W^2_{s_0}$. Since $2 - n/s_0 > 1$ we see that $W^2_{s_0} \hookrightarrow C^\alpha(\overline{\Omega})$, so that (φu_k) is bounded in $C^\alpha(\overline{\Omega})$.

Next suppose that $j = 1$. Let $\varphi_0, \varphi_1 \in C^3(\overline{\Omega})$ satisfy $\varphi_0 \,|\, \mathrm{supp}(\varphi_1) = \mathbf{1}$ and let $(\varphi_0 u_k)$ be bounded in W_p^1 for some $p \geq p_\bullet$. Then $\big(\gamma(\varphi_0 u_k)\big)$ is bounded in $L_\tau(\Gamma)$, where

$$\frac{1}{\tau} \geq \frac{n}{n-1}\Big(\frac{1}{p} - \frac{1}{n}\Big) \ .$$

Hence $(\gamma\varphi_1 b_k \delta\gamma u_k)$ is bounded in $L_t(\Gamma_1)$ for

$$\frac{1}{t} \ge \frac{1}{\tau} + \frac{1}{t_1} \ge \frac{n}{n-1}\Big(\frac{1}{p} - \frac{1}{n} + (\rho-1)\Big(\frac{1}{p_0} - \frac{1}{n}\Big)\Big) . \tag{13.8}$$

Note that $L_t(\Gamma_1) \hookrightarrow W_p^{s-1-1/p}(\Gamma_1)$ if $W_{p'}^{1+1/p-s}(\Gamma_1) \hookrightarrow L_{t'}(\Gamma_1)$, which is the case if

$$\frac{1}{t} \le \frac{n}{n-1}\Big(\frac{1}{p} - \frac{s-1}{n}\Big) . \tag{13.9}$$

Inequalities (13.8) and (13.9) are compatible iff

$$\frac{s}{n} \le \frac{2}{n} - (\rho-1)\Big(\frac{1}{p_0} - \frac{1}{n}\Big) . \tag{13.10}$$

By (13.6) we see that the term on the right-hand side of this inequality is strictly bigger than $(1+2\varepsilon)/n$.

Since $(\varphi_0 u_k)$ is bounded in W_p^1 it follows that $(\varphi_1 a_k u_k)$ is bounded in L_q for

$$\frac{1}{q} \ge \frac{1}{p} - \frac{1}{n} + \frac{1}{s_1} = \frac{1}{p} - \frac{1}{n} + (r-1)\Big(\frac{1}{p_0} - \frac{1}{n}\Big) . \tag{13.11}$$

Furthermore, $L_q \hookrightarrow W_{p,\mathcal{B}}^{s-2}$ if $W_{p'}^{2-s} \hookrightarrow L_{q'}$, which is the case for

$$\frac{1}{q} \le \frac{1}{p} + \frac{2-s}{n} . \tag{13.12}$$

Inequalities (13.11) and (13.12) are compatible iff

$$\frac{s}{n} \le \frac{3}{n} - (r-1)\Big(\frac{1}{p_0} - \frac{1}{n}\Big) . \tag{13.13}$$

Since $(r-1)(1/p_0 - 1/n) < 1 - 1/r$, the term on the right-hand side of estimate (13.13) is also strictly bigger than $1/r - 1 + 2/n + 1/n > (1+2\varepsilon)/n$. Thus, setting $s := 1+\varepsilon$, both inequalities (13.10) and (13.13) are satisfied. Consequently, $(\varphi_1 a_k u_k)$ is bounded in W_p^{s-2} and $(\gamma\varphi_1 b_k \delta\gamma u_k)$ is bounded in $W_p^{s-1-1/p}(\Gamma_1)$. Since $(\varphi_0 u_k)$ is trivially bounded in W_p^{s-1}, Theorem 3.1 implies that $(\varphi_1 u_k)$ is bounded in W_p^s. Thus we see that $(\varphi_1 u_k)$ is bounded in W_π^1, provided $1/\pi \ge 1/p - \varepsilon/n$. Since $t \le t_1$ and $q \le s_1$ we find that $1/p \ge 1/p'_\bullet + \varepsilon/n$. Hence we see that $(\varphi_1 u_k)$ is bounded in W_π^1 if $1/\pi$ is greater than or equal to $(1/p-\varepsilon/n)\vee 1/p'_\bullet$. Now the boundedness of (φu_k) in $W^1_{p'_\bullet}$, consequently in $C^\alpha(\overline{\Omega})$, follows by a bootstrapping argument which is analogous to the one used for the case $j=0$. □

Let φ be a smooth diffeomorphism of $\mathbb{R}^n$ onto itself. Then φ induces smooth diffeomorphisms of Ω onto $\varphi(\Omega)$ and of Γ_i onto $\varphi(\Gamma_i)$ for $i=0,1$, all of them being denoted by φ again, without fearing confusion. Note that $\varphi(\Gamma)$ is the boundary of $\varphi(\Omega)$, and $\varphi(\Gamma)=\varphi(\Gamma_0)\cup\varphi(\Gamma_1)$. We define the push-forward $(\varphi_*\mathcal{A},\varphi_*\mathcal{B})\in\mathcal{E}\bigl(\varphi(\Omega)\bigr)$ of $(\mathcal{A},\mathcal{B})$ by

$$(\varphi_*\mathcal{A},\varphi_*\mathcal{B}):=\varphi_*\circ(\mathcal{A},\mathcal{B})\circ\varphi^*\ ,$$

where $\varphi^*w:=w\circ\varphi$ is the usual pull-back by φ, and $\varphi_*:=(\varphi^{-1})^*$ is the corresponding push-forward.

We also define the push-forward

$$(\varphi_*\mu,\varphi_*\sigma)\in\mathcal{M}\bigl(\varphi(\Omega)\cup\varphi(\Gamma_1)\bigr)\times\mathcal{M}\bigl(\varphi(\Gamma)\bigr)$$

of $(\mu,\sigma)\in\mathcal{M}(\Omega\cup\Gamma_1)\times\mathcal{M}(\Gamma)$ by

$$\int_{\varphi(\Omega)}v\,d(\varphi_*\mu):=\int_\Omega\varphi^*v\,d\mu\ ,\quad\int_{\varphi(\Gamma)}w\,d(\varphi_*\sigma):=\int_\Gamma\varphi^*w\,d\sigma$$

for $v\in C_0\bigl(\varphi(\Omega)\cup\varphi(\Gamma_1)\bigr)$ and $w\in C\bigl(\varphi(\Gamma)\bigr)$, respectively. It is obvious that

$$\|\varphi_*\mu\|_{\mathcal{M}(\varphi(\Omega)\cup\varphi(\Gamma_1))}=\|\mu\|_{\mathcal{M}(\Omega\cup\Gamma_1)}\ ,\quad\|\varphi_*\sigma\|_{\mathcal{M}(\varphi(\Gamma))}=\|\sigma\|_{\mathcal{M}(\Gamma)}\ .\qquad(13.14)$$

Now we consider the particularly simple case where $\varphi:=(\cdot-z)/\varepsilon$ for some $z\in\mathbb{R}^n$ and $\varepsilon>0$. Denote by ω_n Lebesgue's measure on $\mathbb{R}^n$ and by ω_Γ the Riemann-Lebesgue volume measure on Γ. Then

$$\varphi_*\omega_n=\varepsilon^n\omega_n\ ,\quad\varphi_*\omega_\Gamma=\varepsilon^{n-1}\omega_{\varphi(\Gamma)}\ .\qquad(13.15)$$

We set

$$\mathcal{A}_\varepsilon w:=-\nabla\cdot(\boldsymbol{a}_\varepsilon\nabla w+\varepsilon\vec{b}_\varepsilon w)+\varepsilon\vec{c}_\varepsilon\cdot\nabla w+\varepsilon^2a_{0,\varepsilon}w$$

and

$$\mathcal{B}_\varepsilon w:=\delta\bigl(\partial_{\nu_{a_\varepsilon}}w+\varepsilon(\gamma\vec{b}_\varepsilon\cdot\vec{\nu}+d_\varepsilon)\gamma w\bigr)+(1-\delta)\gamma w$$

where $\boldsymbol{a}_\varepsilon := \varphi_* \boldsymbol{a} = \boldsymbol{a}(z + \varepsilon \cdot)$, etc. It is easily verified, by using the very weak or weak formulation, respectively, that

$$\varphi_* \mathcal{A} = \varepsilon^{-2} \mathcal{A}_\varepsilon \ , \quad \varphi_* \mathcal{B} = \varepsilon^{-1} \delta \mathcal{B}_\varepsilon + (1 - \delta) \gamma \ . \tag{13.16}$$

From this and (13.15) one sees that problem $(11.2)_{\lambda m}$ is transformed into

$$\begin{aligned} \mathcal{A}_\varepsilon w &= \varepsilon^2 f_\varepsilon(\cdot, w) + \varepsilon^{2-n} \lambda \mu_\varepsilon && \text{in } \Omega_\varepsilon := \varepsilon^{-1}(\Omega - z) \ , \\ \mathcal{B}_\varepsilon w &= \delta\big(\varepsilon g_\varepsilon(\cdot, w) + \varepsilon^{2-n} \lambda \sigma_\varepsilon\big) + (1 - \delta) \varepsilon^{1-n} \lambda \sigma_\varepsilon && \text{on } \Gamma_\varepsilon \ , \end{aligned} \tag{13.17}$$

where we have put $f_\varepsilon(\cdot, w) := f(z + \varepsilon \cdot, w)$ and $g_\varepsilon(\cdot, w) := g(z + \varepsilon \cdot, w)$ as well as $(\mu_\varepsilon, \sigma_\varepsilon) := (\varphi_* \mu, \varphi_* \sigma)$ and $\Gamma_\varepsilon := \varphi(\Gamma)$.

Finally, we impose assumption (1.11), in addition to (12.1), of course. Using this precise information on the asymptotic behavior of f — and to some extent of g — we can now establish the following a priori bounds.

Lemma 13.2. *Let assumptions* (1.11) *and* (12.1) *be satisfied and suppose that* $\lambda^\bullet \in \Lambda$. *Then* $\{\, u \in E \ ; \ (\lambda, u) \in \Sigma, \ \lambda \le \lambda^\bullet \,\}$ *is bounded in* E.

Proof. We proceed by contradiction using the ideas of [19].

(i) Assume that $(\lambda_k, u_k) \in \Sigma$ satisfy $\lambda_k \le \lambda^\bullet$ and $\|u_k\| \to \infty$ as $k \to \infty$. Put $v_k := \overline{u}_{\lambda_k}$. Since $\lambda \mapsto \overline{u}_\lambda$ is increasing we see that $0 \le v_k \le v$ where we set $v := \overline{u}_{\lambda^\bullet} \in W_1^{(1+j)-}$. Hence the sequence (v_k) is bounded in L_r and, if $j = 1$, it follows that $0 \le \gamma v_k \le \gamma v$, so that (γv_k) is bounded in $L_\rho(\Gamma)$. This implies that the sequence $\big(\big(f^\natural(v_k), \delta g^\natural(\gamma v_k)\big)\big)$ is bounded in $L_1 \times L_1(\Gamma_1)$. Note that

$$v_k = (\mathcal{A}, \mathcal{B})^{-1} \big(f^\natural(v_k), \delta g^\natural(\gamma v_k)\big) + \lambda_k L(m) \ , \qquad k \in \mathbb{N} \ .$$

Hence we infer from Theorem 4.3 that the sequence (v_k) is bounded in $W_{p_0}^j$.

Observe that

$$\mathcal{A}(u_k - v_k) = a_k(u_k - v_k) \text{ in } \Omega \ , \quad \mathcal{B}(u_k - v_k) = b_k \delta \gamma(u_k - v_k) \text{ on } \Gamma \ , \tag{13.18}$$

where $a_k := a(u_k, v_k)$ with

$$a(u, v) := \int_0^1 (\partial_2 f)^\natural \big(v + s(u - v)\big) \, ds$$

and $b_k := b(u_k, v_k)$ with

$$b(u,v) := \delta \int_0^1 (\partial_2 g)^\natural \big(\gamma v + s\gamma(u-v)\big)\, ds \ ,$$

respectively. Since $u_k \in W_1^{(1+j)-} \hookrightarrow W_{p_0}^j$ by Theorem 9.3 and Corollary 4.2, it follows from (12.1) that

$$(a_k, b_k) \in L_{s_j} \times L_{t_j}(\Gamma_1) \ . \tag{13.19}$$

Lemma 13.1 and (13.19) imply $u_k - v_k \in C^\alpha(\overline{\Omega})$ for a suitable $\alpha > 0$. Thus there exists $x_k \in \overline{\Omega}$ with

$$0 \le (u_k - v_k)(x_k) = \max_{\overline{\Omega}} (u_k - v_k) =: M_k \ , \qquad k \in \mathbb{N} \ .$$

Suppose that the sequence (M_k) is bounded. Then the boundedness of (v_k) in $W_{p_0}^j$ and $u_k = (u_k - v_k) + v_k$ imply that the sequence $\big(v_k + s(u_k - v_k)\big)$ is bounded in $L_{np_0/(n-jp_0)}$ uniformly with respect to $s \in [0,1]$. If $j = 1$ then the sequence (γv_k) is bounded in $L_{(n-1)p_0/(n-p_0)}(\Gamma)$. Hence the sequence $\big(\gamma v_k + s\gamma(u_k - v_k)\big)$ is also bounded in $L_{(n-1)p_0/(n-p_0)}(\Gamma)$, uniformly with respect to $s \in [0,1]$. From this and (12.1) we deduce that $\big((a_k, b_k)\big)$ is bounded in $L_{s_j} \times L_{t_j}(\Gamma_1)$. Now Lemma 13.1 (with $\varphi = \psi = \mathbf{1}$) implies that the sequence $(u_k - v_k)$ is bounded in E which, thanks to the boundedness of (v_k) in $W_{p_0}^j \hookrightarrow E$, contradicts $\|u_k\| \to \infty$.

(ii) Thus we can assume that $M_k \to \infty$ as $k \to \infty$, so that

$$\varepsilon_k := M_k^{(1-t)/2} \to 0 \ \ (k \to \infty) \ .$$

Now we consider the diffeomorphisms $\varphi_k := (\cdot - x_k)/\varepsilon_k$ of $\mathbb{R}^n$. We also put

$$\widetilde{w}_k := \varepsilon_k^{2/(t-1)} \varphi_{k,*} w_k \ , \qquad w_k \in \{u_k, v_k\} \ ,$$

and

$$\widetilde{f}_k(\cdot, \xi) := \varepsilon_k^{2t/(t-1)} f_{\varepsilon_k}(\cdot, \varepsilon_k^{-2/(t-1)} \xi)$$

as well as

$$\widetilde{g}_k(\cdot, \xi) := \varepsilon_k^{(t+1)/(t-1)} g_{\varepsilon_k}(\cdot, \varepsilon_k^{-2/(t-1)} \xi)$$

for $\xi \in \mathbb{R}$. Then it follows from (13.17) that $\widetilde{w}_k$ satisfies

$$\begin{aligned} \mathcal{A}_{\varepsilon_k}\widetilde{w}_k &= \widetilde{f}_k(\cdot,\widetilde{w}_k) + \alpha_k\mu_{\varepsilon_k} && \text{in } \Omega_{\varepsilon_k}\ , \\ \mathcal{B}_{\varepsilon_k}\widetilde{w}_k &= \delta\bigl(\widetilde{g}_k(\cdot,\gamma\widetilde{w}_k) + \alpha_k\sigma_{\varepsilon_k}\bigr) + (1-\delta)\beta_k\sigma_{\varepsilon_k} && \text{on } \Gamma_{\varepsilon_k}\ , \end{aligned} \tag{13.20}$$

where

$$\alpha_k := \varepsilon_k^{-n+2t/(t-1)}\lambda_k\ , \quad \beta_k := \varepsilon_k^{-n+(t+1)/(t-1)}\lambda_k\ .$$

Note that $2t/(t-1) > n$ since $t \le r < n/(n-2)$. Also $(t+1)/(t-1) > n$ since $t \le r < n/(n-1) < (n+1)/(n-1)$ if $j=0$. Thus $\alpha_k \to 0$ and we can also assume that $\beta_k \to 0$, since β_k is only present if $j=0$. Also note that $\widetilde{u}_k - \widetilde{v}_k \in C^\alpha(\overline{\Omega})$ and that

$$0 \le \max_{\overline{\Omega}}(\widetilde{u}_k - \widetilde{v}_k) = (\widetilde{u}_k - \widetilde{v}_k)(0) = 1\ . \tag{13.21}$$

By passing to a suitable subsequence we can assume that $x_k \to x_\infty \in \overline{\Omega}$. Now we have to distinguish between the two cases $x_\infty \in \Omega$ and $x_\infty \in \Gamma$.

(a) Suppose that $x_\infty \in \Omega$ and set $B_R := \{\, x \in \mathbb{R}^n\ ;\ |x| < R \,\}$ for $R > 0$. Then, given $R > 0$, there exists k_R such that $B_{R+1} \subset \Omega_{\varepsilon_k}$ for $k \ge k_R$. Set $p_j := np_0/(n - jp_0)$. Then

$$\|\widetilde{v}_k\|_{L_{p_j}(B_{R+1})} \le \|\widetilde{v}_k\|_{L_{p_j}(\Omega_{\varepsilon_k})} = \varepsilon_k^{m(t)}\,\|v_k\|_{p_j} \to 0$$

since

$$\begin{aligned} m(t) := \frac{2}{t-1} - \frac{n}{p_j} &> \frac{2}{t-1} - \frac{n}{r} \\ &= \frac{n}{t-1}\Bigl(\frac{1}{r} - \frac{t}{r} + \frac{2}{n}\Bigr) \ge \frac{n}{t-1}\Bigl(\frac{1}{r} - 1 + \frac{2}{n}\Bigr) > 0\ . \end{aligned}$$

Thus (13.21) implies that the sequence $(\widetilde{u}_k)$ is bounded in $L_{p_j}(B_{R+1})$.

Denote by $\mathcal{A}_k$ the operator $\mathcal{A}_{\varepsilon_k}$ with its coefficients restricted to $\overline{\mathbb{B}}_{R+1}$ and let $\mathcal{B}_k$ be the trace operator on ∂B_{R+1} for $k \in \mathbb{N}$. Then $\bigl((\mathcal{A}_k, \mathcal{B}_k)\bigr)$ is a relatively compact sequence in $\mathcal{E}(B_{R+1})$ satisfying $\inf_k \lambda_0(\mathcal{A}_k, \mathcal{B}_k) > 0$. It follows from (13.20) that

$$\begin{aligned} \mathcal{A}_k(\widetilde{u}_k - \widetilde{v}_k) &= \widetilde{a}_k(\widetilde{u}_k - \widetilde{v}_k) && \text{in } B_{R+1}\ , \\ \mathcal{B}_k(\widetilde{u}_k - \widetilde{v}_k) &= \gamma(\widetilde{u}_k - \widetilde{v}_k) && \text{on } \partial B_{R+1}\ , \end{aligned} \tag{13.22}$$

where

$$\widetilde{a}_k := \int_0^1 (\partial_2 \widetilde{f}_k)^\natural \big(\widetilde{v}_k + s(\widetilde{u}_k - \widetilde{v}_k)\big)\, ds \ .$$

Observe that

$$\partial_2 \widetilde{f}_k(\cdot,\xi) = \varepsilon_k^2 \partial_2 f_{\varepsilon_k}(\cdot, \varepsilon_k^{-2/(t-1)}\xi) \ , \qquad \xi \in \mathbb{R} \ .$$

Hence we deduce from (1.11) that

$$|\partial_2 \widetilde{f}_k(\cdot,\xi)| \le \varepsilon_k^2 c\big(1 + \varepsilon_k^{-2}\, |\xi|^{t-1}\big) \le c(1 + |\xi|^{r-1}) \ , \qquad x \in \mathbb{R} \ , \tag{13.23}$$

for $k \in \mathbb{N}$. Thus the boundedness of the sequences $(\widetilde{u}_k)$ and $(\widetilde{v}_k)$ in $L_{p_j}(B_{R+1})$ implies that $(\widetilde{a}_k)$ is bounded in $L_{s_j}(B_{R+1})$. Fix $\varphi, \psi \in \mathcal{D}(B_{R+1})$ such that $\psi \,|\, \mathrm{supp}(\varphi) = \mathbf{1}$ and $\varphi \,|\, B_R = \mathbf{1}$. Since $\big(\psi(\widetilde{u}_k - \widetilde{v}_k)\big)$ is bounded in $L_r(B_{R+1})$ we deduce from (13.22) and Lemma 13.1 (by observing that the boundary values do not play any rôle in this argument) that $\big(\varphi(\widetilde{u}_k - \widetilde{v}_k)\big)$ is bounded in $C^\alpha(\overline{B}_{R+1})$. Consequently, the sequence $(\widetilde{u}_k - \widetilde{v}_k)$ is bounded in $C^\alpha(\overline{B}_R)$. Since the latter space is compactly embedded in $C(\overline{B}_R)$ we can assume, by passing to a suitable subsequence, that $(\widetilde{u}_k - \widetilde{v}_k)$ converges in $C(\overline{B}_R)$ to some w. Since $\widetilde{v}_k \to 0$ in $L_r(B_R)$ it follows that $\widetilde{u}_k \to w$ in $L_r(B_R)$, hence in $L_t(B_R)$. This implies, thanks to (1.11), that

$$\widetilde{f}_k(\cdot, \widetilde{u}_k) = \frac{f(x_k + \varepsilon_k \cdot, M_k \widetilde{u}_k)}{(M_k \widetilde{u}_k)^t}\, \widetilde{u}_k^t \to \ell(x_\infty) w^t$$

in $L_1(B_R)$. Thus, by passsing to the limit in the very weak, resp. weak, formulation of

$$\mathcal{A}_k \widetilde{u}_k = \widetilde{f}_k(\cdot, \widetilde{u}_k) + \alpha_k \mu_{\varepsilon_k}$$

using test functions $\varphi \in \mathcal{D}(B_R)$ only, we see that

$$\mathcal{A}_\infty w = \ell(x_\infty) w^t \tag{13.24}$$

in $\mathcal{D}'(B_R)$, where $\mathcal{A}_\infty v := -\nabla \cdot \big(\boldsymbol{a}(x_\infty)\nabla v\big)$. Letting $R \to \infty$ it follows that (13.24) holds in $\mathcal{D}'(\mathbb{R}^n)$. It also follows that $w \in C(\mathbb{R}^n)$ with $w \ge 0$ and $w(0) = 1$. Thus w is a positive solution of (13.24) belonging to $C^2(\mathbb{R}^n)$ which, after employing a linear change of variables, contradicts [18, Theorem 1.1].

(b) Suppose that $x_\infty \in \Gamma$. By a smooth change of variables we can assume that $x_\infty = 0$ and that $\overline{\Omega}$ is a neighborhood of zero in the closure of the half-space $\mathbb{H}^n := \{\, x \in \mathbb{R}^n \ ;\ x^n > 0 \,\}$. Set

$$Q_R := \big\{\, (y, x^n) \in \mathbb{R}^{n-1} \times \mathbb{R} \ ;\ |y| < R,\ 0 < x^n < R \,\big\} \ , \qquad R > 0 \ .$$

Also put $\vec{e} := (0, \dots, 0, 1) \in \mathbb{R}^n$ and $\eta_k := x_k^n / \varepsilon_k \geq 0$. Let $R > 0$ be fixed, choose a smooth domain Ω_R such that

$$\overline{Q}_{R+1} \cap \mathbb{H}^n \subset \Omega_R \subset Q_{R+2} \ ,$$

and set $\Omega_{R,k} := \Omega_R - \eta_k \vec{e}$. By extending $d \in C^1(\mathbb{R}^{n-1})$ to a C^1-function on $\mathbb{R}^n$ we can assume that $\mathcal{B}_{\varepsilon_k}$ is well-defined on $\partial\Omega_{R,k}$. Fix $d_0 \in C^1(\partial\Omega_R)$ with $d_0 > 0$ and having its support contained in $\partial\Omega_R \cap \mathbb{H}^n$. Define $\mathcal{B}_k$ to be $\mathcal{B}_{\varepsilon_k} + \delta d_0(\cdot + \eta_k \vec{e})\gamma$ and observe that $(\mathcal{A}_k, \mathcal{B}_k) \in \mathcal{E}(\Omega_{R,k})$ for $k \in \mathbb{N}$. Define a linear automorphism ψ_k of $\mathbb{R}^n$ by $\psi_k(x) := x + \eta_k \vec{e}$ and note that $\varphi_k(\Omega_{R,k}) = \Omega_R$. It is easily verified that $\big((\psi_{k,*}\mathcal{A}_k, \psi_{k,*}\mathcal{B}_k)\big)_{k\in\mathbb{N}}$ is a relatively compact sequence in $\mathcal{E}(\Omega_R)$ satisfying $\inf_k \lambda_0(\psi_{k,*}\mathcal{A}_k, \psi_{k,*}\mathcal{B}_k) > 0$.

There exists k_R such that $\widetilde{u}_k$ and $\widetilde{v}_k$ are well-defined on $\Omega_{R,k}$ for $k \geq k_R$ and satisfy (13.20) on $\Omega_{R,k}$.

(i) Suppose that $\underline{\lim}\, \eta_k = 0$. Then, by passing to a suitable subsequence, we can assume that $\eta_k \to 0$. Thus $\Omega_{R,k} \to \Omega_R$ as $k \to \infty$.

Assume that $x_\infty = 0 \in \Gamma_0$. In this case an obvious modification of the arguments of step (a) shows that

$$\mathcal{A}_\infty u = \ell(0) u^t \text{ in } \mathbb{H}^n \ , \quad u = 0 \text{ on } \partial\mathbb{H}^n$$

possesses a positive solution in $C^2(\overline{\mathbb{H}}^n)$. Hence we arrive once more at a contradiction to [18, Theorem 1.1].

Thus suppose that $0 \in \Gamma_1$. Since (13.20) is valid on $\Omega_{R,k}$ it follows that

$$\begin{aligned} \mathcal{A}_k(\widetilde{u}_k - \widetilde{v}_k) &= \widetilde{a}_k(\widetilde{u}_k - \widetilde{v}_k) && \text{in } \Omega_{R,k} \ , \\ \mathcal{B}_k(\widetilde{u}_k - \widetilde{v}_k) &= \widetilde{b}_k \gamma(\widetilde{u}_k - \widetilde{v}_k) && \text{on } \partial\Omega_{R,k} \cap \partial\mathbb{H}^n \ , \end{aligned} \tag{13.25}$$

where

$$\widetilde{b}_k := \int_0^1 (\partial_2 \widetilde{g}_k)^\natural \big(\gamma\widetilde{v}_k + s\gamma(\widetilde{u}_k - \widetilde{v}_k)\big)\, ds \ .$$

Observe that

$$\partial_2\widetilde{g}_k(\cdot,\xi)=\varepsilon_k\partial_2 g_{\varepsilon_k}(\cdot,\varepsilon_k^{-2/(t-1)}\xi)\ , \qquad \xi\in\mathbb{R}\ .$$

Hence we deduce from (12.1) that

$$|\partial_2\widetilde{g}(\cdot,\xi)|\le\varepsilon_k c\bigl(1+\varepsilon_k^{-\frac{2(\rho-1)}{t-1}}\,|\xi|^{\rho-1}\bigr)\le c(1+|\xi|^{\rho-1})\ , \qquad \xi\in\mathbb{R}\ , \quad (13.26)$$

thanks to $\rho<(t+1)/2$.

First suppose that $j=1$. Since (v_k) is bounded in $W_{p_0}^1$ it follows that (γv_k) is bounded in $L_{(\rho-1)t_1}(\Gamma)$. Note that

$$\|\gamma\widetilde{v}_k\|_{L_{(\rho-1)t_1}(\partial\Omega_{R,k}\cap\partial\mathbb{H}^n-\eta_k\vec{e})}\le\varepsilon_k^{n(t)}\,\|\gamma v_k\|_{L_{(\rho-1)t_1}(\Gamma)}$$

with

$$n(t):=\frac{2}{t-1}-\frac{n-1}{(\rho-1)t_1}=\frac{2}{t-1}-n\Bigl(\frac{1}{p_0}-\frac{1}{n}\Bigr)>\frac{2}{t-1}-\frac{n}{r}>0\ .$$

Consequently, $\bigl(\psi_{k,*}(\gamma\widetilde{v}_k)\bigr)$ is bounded in $L_{(\rho-1)t_1}(B_{R+1}\cap\partial\mathbb{H}^n)$. It follows from (13.21) that $\bigl(\psi_{k,*}(\gamma\widetilde{u}_k)\bigr)$ is bounded in $L_{(\rho-1)t_1}(B_{R+1}\cap\partial\mathbb{H}^n)$ as well. Thus we deduce from (13.26) that $(\psi_{k,*}\widetilde{b}_k)$ is bounded in $L_{t_1}(B_{R+1}\cap\partial\mathbb{H}^n)$. The arguments of step (a) show that $(\psi_{k,*}\widetilde{a}_k)$ is bounded in $L_{s_1}(\Omega_R)$.

Fix $\varphi,\psi\in C^3(\overline{\Omega}_R)$ such that $\psi\,|\,\mathrm{supp}(\varphi)=\mathbf{1}$ and $\varphi\,|\,Q_R=\mathbf{1}$, and such that φ vanishes on $\partial\Omega_R\cap\mathbb{H}^n$. Then $\bigl(\varphi\psi_{k,*}\bigl(\widetilde{a}_k(\widetilde{u}_k-\widetilde{v}_k)\bigr)\bigr)$ is bounded in $L_{s_1}(\Omega_R)$, hence in $W_{p,\mathcal{B}_k}^{-1}(\Omega_R)$, provided

$$\frac{1}{p}\ge(r-1)\Bigl(\frac{1}{p_0}-\frac{1}{n}\Bigr)-\frac{1}{n}\ . \quad (13.27)$$

Similarly, it follows that $\bigl(\varphi\psi_{k,*}\bigl(\widetilde{b}_k\gamma(\widetilde{u}_k-\widetilde{v}_k)\bigr)\bigr)$ is bounded in $L_{t_1}(\partial\Omega_R)$, hence in $W_p^{-1/p}(\partial\Omega_R)$, provided

$$\frac{1}{p}\ge\frac{n-1}{n}\,\frac{1}{t_1}=(\rho-1)\Bigl(\frac{1}{p_0}-\frac{1}{n}\Bigr)\ . \quad (13.28)$$

Note that

$$(r-1)\Bigl(\frac{1}{p_0}-\frac{1}{n}\Bigr)-\frac{1}{n}<1-\frac{1}{r}-\frac{1}{n}<\frac{1}{n}$$

and that

$$(\rho-1)\Big(\frac{1}{p_0}-\frac{1}{n}\Big)<\frac{\rho-1}{r}\le 1-\frac{1}{n}-\frac{1}{r}<\frac{1}{n}\ .$$

Thus we can fix $p>n$ satisfying (13.27) and (13.28). Now (13.25) and Theorem 3.1 imply that $\big(\varphi\psi_{k,*}(\widetilde{u}_k-\widetilde{v}_k)\big)$ is bounded in $W_p^1(\Omega_R)$, thus in $C^\beta(\overline{\Omega}_R)$ with $\beta:=1-n/p>0$. Consequently, $\big(\psi_{k,*}(\widetilde{u}_k-\widetilde{v}_k)\big)$ is bounded in $C^\beta(\overline{Q}_R)$. By selecting a subsequence we can assume that $\big(\psi_{k,*}(\widetilde{u}_k-\widetilde{v}_k)\big)$ converges in $C(\overline{Q}_R)$ to some w. Now we see, as in step (a), that w satisfies (13.24) in $\mathcal{D}'(Q_R)$.

Observe that

$$|\widetilde{g}_k(\cdot,\gamma\widetilde{u}_k)|\le c\Big(\varepsilon_k^{(t+1)/(t-1)}+\varepsilon_k^{(t+1-2\rho)/(t-1)}\,|\gamma\widetilde{u}_k|^\rho\Big)\ . \tag{13.29}$$

Also note that $\big(\psi_{k,*}(\gamma\widetilde{u}_k)\big)$ is bounded in $L_{(\rho-1)t_1}(B_{R+1}\cap\partial\mathbb{H}^n)$, hence in $L_\rho(B_{R+1}\cap\partial\mathbb{H}^n)$. Thus, since $2\rho<t+1$, we infer from (13.29) and the weak formulation of (13.20) with $\widetilde{w}_k:=\widetilde{u}_k$, pushed forward to Ω_R, using test functions which vanish in a neighborhood of $\partial\Omega_R\cap\mathbb{H}^n$ that

$$\mathcal{B}_\infty w=0 \text{ on } Q_R\cap\partial\mathbb{H}^n\ ,$$

where $\mathcal{B}_\infty:=\vec{\nu}\cdot\gamma\big(\boldsymbol{a}(0)\nabla\cdot\big)$. Finally, by letting $R\to\infty$ we see that

$$\mathcal{A}_\infty w=\ell(0)w^t \text{ in } \mathbb{H}^n\ ,\quad \mathcal{B}_\infty w=0 \text{ on } \partial\mathbb{H}^n\ . \tag{13.30}$$

By standard regularity theory $w\in C^2(\overline{\mathbb{H}}^n)$, and $w\ge 0$ with $w(0)=1$. Suppose that $j=0$. In this case $g=0$ and it is obvious that we are led to problem (13.30) as well.

By employing an obvious linear change of coordinates in (13.30) we arrive at a contradiction to [10, Corollary 2.1].

(ii) Suppose that $\overline{\lim}\,\eta_k=\infty$. By selecting a suitable subsequence we can assume that $\eta_k\to\infty$. In this case $\Omega_{R,k}$ contains B_{R+1} for all sufficiently large k. Now we reach a contradiction by modifying the arguments of (a) in an obvious way.

(iii) Lastly, assume that $0<\underline{\lim}\,\eta_k\le\overline{\lim}\,\eta_k<\infty$. By going to a subsequence we can assume that $\eta_k\to\eta$ for some $\eta>0$. Then, by modifying the arguments of step (i) suitably, we once more obtain a positive classical solution of (13.30), hence a contradiction.

This shows that our hypothesis that $\|u_k\|\to\infty$ cannot be sustained. Hence the lemma has been proven. □

Lemma 13.3. *Let assumptions* (1.11) *and* (12.1) *be satisfied and fix* $T > 0$. *Then the set of all positive solutions of*

$$\mathcal{A}u = f(x,u) + \tau\varphi \text{ in } \Omega\ , \quad \mathcal{B}u = \delta g(x,u) \text{ on } \Gamma\ , \qquad 0 \le \tau \le T\ , \tag{13.31$_\tau$}$$

is bounded in E.

Proof. By Sobolev's embedding and the Krasnosel'skii-Vainberg theorem we infer from the commutativity of the diagrams

$$\begin{array}{ccc} E & \hookrightarrow & L_r \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle f^\natural} \\ \mathcal{M}(\Omega \cup \Gamma_1) & \hookleftarrow & L_1 \end{array}$$

and

$$\begin{array}{ccccc} W^1_{p_\bullet} & \xrightarrow{\ \gamma\ } & W^{1-1/p_\bullet}_{p_\bullet}(\Gamma) & \hookrightarrow & L_{r/1^*}(\Gamma) \\ {\scriptstyle \delta G}\big\downarrow & & & & \big\downarrow{\scriptstyle \delta g^\natural} \\ \mathcal{M}(\Gamma_1) & \hookleftarrow & L_1(\Gamma_1) & \hookleftarrow & L_{r/1^*\rho}(\Gamma_1) \end{array}$$

and from Theorem 4.3 that

$$(\mathcal{A},\mathcal{B})^{-1}(F + \tau\varphi, \delta G) \in C(E, W_1^{2-})\ , \tag{13.32}$$

and that this map is bounded on bounded sets, uniformly with respect to $\tau \in [0,T]$. Now suppose that $p_\bullet < p < n$ and put $p^* := np/(n-p)$. Then

$$\begin{array}{ccc} W^1_p & \hookrightarrow & L_{p^*} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle f^\natural} & \\ & L_{p^*/r} & \end{array} \qquad \begin{array}{ccccc} W^1_p & \xrightarrow{\ \gamma\ } & W^{1-1/p}_p & \hookrightarrow & L_{p^*/1^*}(\Gamma) \\ & {\scriptstyle \delta G}\searrow & & \swarrow{\scriptstyle \delta g^\natural} & \\ & L_{p^*/r}(\Gamma_1) & \hookleftarrow & L_{p^*/1^*\rho}(\Gamma_1) & \end{array}$$

and Theorem 2.3 imply that $(\mathcal{A},\mathcal{B})^{-1}(F, \delta G)$ maps W^1_p continuously and boundedly into $W^s_{p^*/r}$ for $s < 1 + r/p^*$. From this we infer that

$$(\mathcal{A},\mathcal{B})^{-1}(F + \tau\varphi, \delta G) \in C(W^1_p, W^1_\pi) \tag{13.33}$$

whenever $1/\pi > r/1^*p^*$, and that this map is bounded on bounded sets, uniformly with respect to $\tau \in [0,T]$.

Note that

$$\begin{aligned}\frac{1}{p} - \frac{r}{1^*p^*} &= \frac{1}{p} - \Big(1 - \frac{1}{n}\Big)\Big(\frac{1}{p} - \frac{1}{n}\Big)r \\ &= \frac{r}{p}\Big(\frac{1}{r} - 1 + \frac{2}{n}\Big) - \frac{r}{n}\Big(\frac{1}{p} - 1 + \frac{1}{n}\Big) \\ &> \frac{r}{n}\Big[\Big(\frac{1}{r} - 1 + \frac{2}{n}\Big) - \Big(\frac{1}{p} - 1 + \frac{1}{n}\Big)\Big]\ .\end{aligned}$$

Put $\beta := r(1/r - 1 + 2/n)/2n$ and $\varepsilon_0 := (1/r - 1 + 2/n)/2$. Then it follows from the last estimate that we can choose $1/\pi > r/1^*p^*$ such that

$$\frac{1}{\pi} < \frac{1}{p} - \beta\ , \qquad \frac{1}{p} := 1 - \frac{1}{n} + \varepsilon\ , \quad \frac{1}{n} - 1 < \varepsilon \le \varepsilon_0\ . \tag{13.34}$$

From (13.32) and Corollary 4.2 we infer that each solution of $(13.31)_\tau$ belongs to $W^1_{p_0}$, where $1/p_0 := 1 - 1/n + \varepsilon_0 < 1/p_\bullet$. Hence we deduce from (13.33) and (13.34) in finitely many steps that each solution of $(13.31)_\tau$ belongs to W^1_q for a suitable $q > n$, hence to $C^\alpha(\overline{\Omega})$ for an appropriate $\alpha > 0$. Furthermore, if the solution set is bounded in $C(\overline{\Omega})$, uniformly with respect to $\tau \in [0,T]$, then its image under $(f^\natural, \delta g^\natural)$ is bounded in $L_1 \times L_1(\Gamma_1)$, also uniformly with respect to $\tau \in [0,T]$. Then the arguments leading to (13.32) show that the solution set is bounded in E, uniformly with respect to $\tau \in [0,T]$.

Now we can proceed by contradiction. Thus suppose that the assertion is false. Then there exist $(u_k, \tau_k) \in E \times [0,T]$ such that u_k is a positive solution of $(13.31)_\tau$ and $\|u_k\| \to \infty$ as $k \to \infty$. By the preceding considerations we know that $u_k \in C(\overline{\Omega})$ and we can assume that $M_k := \max_{\overline{\Omega}} u_k \to \infty$ as $k \to \infty$. Now an obvious modification of part (ii) of the proof of Lemma 13.2 leads to a contradiction, thus proving the assertion. □

14. General Existence and Multiplicity Results. Finally, we prove the main result of this paper concerning semilinear parameter-dependent BVPs whose prototypes are problems (1.1) and (1.2).

We suppose that $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}$ and $(\mu, \sigma) \in \mathcal{M}$ with $(\mu, \sigma) > 0$. Then we consider the problem

$$\mathcal{A}u = f(x,u) + \lambda\mu \text{ in } \Omega\ , \quad \mathcal{B}u = \delta g(y,u) + \lambda\sigma \text{ on } \Gamma \tag{14.1$_\lambda$}$$

for $\lambda \in \mathbb{R}^+$, where f and g are Carathéodory functions being continuously differentiable with respect to the second variable and satisfying the growth restrictions

$$|\partial_2 f(x,\xi)| \le c\big(1+|\xi|^{r-1}\big)\ , \quad |\partial_2 g(y,\xi)| \le c\big(1+|\xi|^{\rho-1}\big)\ , \qquad \xi \in \mathbb{R}\ , \tag{14.2}$$

for a.a. $(x,y) \in \Omega \times \Gamma$, where r and ρ obey (1.8) and (1.9). We suppose that

$$\big(f(x,0), g(y,0)\big) = (0,0)\ , \qquad \text{a.a. } (x,y) \in \Omega \times \Gamma\ , \tag{14.3}$$

and that there exists $(\omega, \omega_\Gamma) \ge (0,0)$ such that

$$\partial_2 f(x,\cdot) \ge -\omega\ , \quad \partial_2 g(y,\cdot) \ge -\omega_\Gamma\ , \qquad \text{a.a. } (x,y) \in \Omega \times \Gamma\ . \tag{14.4}$$

Lastly, we suppose that

$$\lambda_0\big(\mathcal{A} - \partial_2 f(\cdot,0), \mathcal{B} - \delta\partial_2 g(\cdot,0)\gamma\big) > 0\ , \tag{14.5}$$

and we put $\alpha_0 := \lambda_0(\mathcal{A}, \mathcal{B} + \delta\omega_\Gamma\gamma)$.

Theorem 14.1. *Let the above hypotheses be satisfied. Then the assertions of Theorem* 1.2 *are true for problem* $(14.1)_\lambda$.

If, in addition, assumption (1.11) *is fulfilled then the assertions of Theorem* 1.3 *hold for problem* $(14.1)_\lambda$ *as well.*

Proof. Fix $\widetilde{\omega} > \big(-\lambda_0(\mathcal{A},\mathcal{B})\big) \vee \omega$ and put $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}) := (\mathcal{A} + \widetilde{\omega}, \mathcal{B} + \delta\omega_\Gamma\gamma)$. Also set $\widetilde{f}(\cdot,\xi) := f(\cdot,\xi) + \widetilde{\omega}\xi$ and $\widetilde{g}(\cdot,\xi) := g(\cdot,\xi) + \omega_\Gamma\xi$ for $\xi \in \mathbb{R}$. Then problem $(14.1)_\lambda$ is equivalent to

$$\widetilde{\mathcal{A}}u = \widetilde{f}(x,u) + \lambda\mu \text{ in } \Omega\ , \quad \widetilde{\mathcal{B}}u = \delta\widetilde{g}(x,u) + \lambda\sigma \text{ on } \Gamma\ , \tag{14.6$_\lambda$}$$

and

$$\lambda_0(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}) = \lambda_0(\mathcal{A}, \mathcal{B} + \delta\omega_\Gamma\gamma) + \widetilde{\omega} \ge \lambda_0(\mathcal{A},\mathcal{B}) + \widetilde{\omega} > 0\ , \tag{14.7}$$

thanks to Remark 5.2(c). It is an immediate consequence of (14.2)–(14.4) that $(\widetilde{f}, \widetilde{g})$ satisfies condition (12.1) with $\alpha := \widetilde{\omega} - \omega$. If $f(\cdot,\xi) \ge \alpha_0\xi - \beta$ then (14.7) implies

$$\widetilde{f}(\cdot,\xi) \ge \lambda_0(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})\xi - \beta\ , \qquad \xi \in \mathbb{R}^+\ . \tag{14.8}$$

Thus the validity of the assertions of Theorem 1.2 for problem $(14.1)_\lambda$ follows by applying Theorem 11.1, Lemma 12.3, and Remark 12.4(c) to problem (14.6), taking into consideration that $\widetilde{\mathcal{A}} - \partial_2 \widetilde{f}(\cdot,0) = \mathcal{A} - \partial_2 f(\cdot,0)$ and $\widetilde{\mathcal{B}} - \delta\partial_2\widetilde{g}(\cdot,0)\gamma = \mathcal{B} - \delta\partial_2 g(\cdot,0)\gamma$.

If f satisfies (1.11) then it is clear that $\widetilde{f}$ satisfies (1.11) as well. Hence we infer from Lemma 13.2 that, given $\lambda^\bullet \in \Lambda$, the set of all solutions of $(14.6)_\lambda$ with $0 \le \lambda \le \lambda^\bullet$ is bounded in E. Lemma 13.3 guarantees that, given $T > 0$, the set of all positive solutions of

$$\widetilde{\mathcal{A}}u = \widetilde{f}(x,u) + \tau\varphi \text{ in } \Omega \ , \quad \widetilde{\mathcal{B}}u = \delta\widetilde{g}(x,u) \text{ on } \Gamma \ , \qquad 0 \le \tau \le T \ ,$$

is bounded in E. Clearly, hypothesis (1.11) implies the validity of (14.8). Hence we infer from the proof of Theorem 11.1 that the assumptions of Proposition 8.3 are satisfied in the present situation. Now the second part of the assertion follows from Corollary 8.4, Lemma 12.3, Remarks 12.4, and Lemma 12.5. □

Remark 14.2. It is clear from the above proof that the first part of Theorem 14.1 remains valid if (14.2) is replaced by

$$|f(x,\xi)| \le c\bigl(1+|\xi|^r\bigr) \ , \quad |g(y,\xi)| \le c\bigl(1+|\xi|^\rho\bigr) \ , \qquad \xi \in \mathbb{R} \ ,$$

for a.a. $(x,y) \in \Omega \times \Gamma$, and if (f,g) is only supposed to be differentiable at $\xi = 0$. □

Proof of Theorem 1.2. Put $(\mathcal{A},\mathcal{B}) := \bigl(-\Delta, \delta\partial_{\nu_a} + (1-\delta)\gamma\bigr)$. Then

$$\lambda_0\bigl(\mathcal{A} - \partial_2 f(\cdot,0), \mathcal{B} - \delta\partial_2 g(\cdot,0)\gamma\bigr) > \lambda_0(\mathcal{A} - \alpha_0, \mathcal{B}) = 0$$

by the last part of assumption (1.10). Thus condition (1.10) implies the validity of (14.3)–(14.5). Consequently, the assertion follows from the first part of Theorem 14.1. □

15. Existence of Solutions for $\lambda = \lambda^*$. Lastly, we consider the question whether λ^* belongs to Λ. It is well-known from the theory of classical elliptic BVPs that this is not the case, in general (cf. [3]).

For convenience, we restrict ourselves to a simple situation and leave it to the reader to study more general cases.

Theorem 15.1. *Suppose that* $g = 0$*, and let assumptions* (14.3)–(14.5) *and* (1.11) *be satisfied. Also suppose that* $\sigma\,|\,\Gamma_0 = 0$*, and let* $t < 2/(n-2)$*. Then* $\lambda^* \in \Lambda$*. If* $f(x,\cdot)$ *is strictly convex for a.a.* $x \in \Omega$ *then problem* $(14.1)_{\lambda^*}$ *is uniquely solvable.*

Proof. By passing to the equivalent problem $(14.6)_\lambda$ we can assume that $\lambda_0(\mathcal{A},\mathcal{B}) > 0$. Put

$$w := (\mathcal{A},\mathcal{B})^{-1}(\mu,\sigma) \in W_1^{2-}\ . \tag{15.1}$$

Then u is a solution of $(14.1)_\lambda$ iff

$$\mathcal{A}(u-\lambda w) = f(x,u) \text{ in } \Omega\ , \quad \mathcal{B}(u-\lambda w) = 0 \text{ on } \Gamma\ . \qquad (15.2)_\lambda$$

If $u \in E$ then we infer from $t < 2/(n-2)$ that $f^\natural(u) \in L_p$ for some $p > n/2$. Hence $(15.2)_\lambda$, $A^{-1} \in \mathcal{L}(L_p, W_p^2)$, and $W_p^2 \hookrightarrow C^\alpha$, where $\alpha := 2 - n/p$, imply that $u - \lambda w \in C^\alpha(\overline{\Omega})$ whenever u is a solution of $(14.1)_\lambda$.

Recall that Theorem 14.1 guarantees $\lambda^* < \infty$. Let (λ_k) be a sequence in Λ. Put $u_k := \overline{u}_{\lambda_k}$ and $M_k := \max_{\overline{\Omega}}(u_k - \lambda_k w) \ge 0$ for $k \in \mathbb{N}$ and suppose that $\|u_k\| \to \infty$. Then $\|u_k\|_r \to \infty$ and it follows from

$$\|u_k\|_r \le \|u_k - \lambda_k w\|_r + \|\lambda_k w\|_r \le |\Omega|^{1/r} M_k + \lambda^* \|w\|_r$$

that $M_k \to \infty$. Now part (ii) of the proof of Lemma 13.2 leads to contradictions. This proves that $\{\,\overline{u}_\lambda\ ;\ 0 < \lambda < \lambda^*\,\}$ is bounded in E, and Theorem 8.1 guarantees that $\lambda^* \in \Lambda$.

The uniqueness assertion follows from Remark 12.4(b). $\square$

Remark 15.2. There are other conditions guaranteeing that $\lambda^* \in \Lambda$. For example, if Ω is convex and $\operatorname{supp}(\mu) \subset\subset \Omega$ then problem (1.12) is solvable for $\lambda = \lambda^*$ if $1 < r < n/(n-2)$.

Proof. Thanks to Theorem 8.1 we have to establish the boundedness of $\{\,\overline{u}_\lambda\ ;\ \lambda \in \Lambda\,\}$ in E. Due to Theorem 4.3, it is sufficient to find L_r-estimates for the solutions $u := \overline{u}_\lambda$. Testing (1.12) with φ yields $L_r(K)$-estimates for any compact set $K \subset \Omega$. Since u is a solution of the equation $-\Delta u = u^r$ in $\Omega_\varepsilon := \{\, x \in \Omega\ ;\ \operatorname{dist}(x,\Gamma) < \varepsilon\,\}$ for $\varepsilon > 0$ small, the results of Gidas, Ni and Nirenberg [17] imply $\partial_\nu u(x - t\nu) < 0$ for $t < \varepsilon/2$ and $x \in \Gamma$. Hence

$$\int_{\Omega_{\varepsilon/4}} u^r\,dx \le c \int_{\Omega\setminus\Omega_{\varepsilon/4}} u^r\,dx\ .$$

We refrain from giving details. □

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