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# OPTIMALITY PRINCIPLES AND REPRESENTATION FORMULAS FOR VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS I. EQUATIONS OF UNBOUNDED AND DEGENERATE CONTROL PROBLEMS WITHOUT UNIQUENESS

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**Abstract.** We prove general optimality principles for semicontinuous viscosity solutions of Hamilton-Jacobi equations. We also characterize the minimal nonnegative supersolution and the maximal subsolution null on a closed given set for a class of equations without uniqueness, including the degenerate eikonal equation and the Bellman equation of the linear quadratic control problem.

**1. Introduction.** One of the main achievements of Crandall-Lions [6] theory of viscosity solutions, was to give a notion of weak solution for first order (and next also second order, see Crandall-Ishii-Lions [5] and the references therein) fully nonlinear partial differential equations providing existence and uniqueness in a wide class of problems. Yet there are non trivial examples of equations interesting for the applications for which uniqueness does not hold. This is well known for discontinuous solutions of boundary value problems, but it is actually true even for continuous ones and also classical solutions of some equations in the whole space. Good examples for this kind of phenomena are the eikonal equation and the Bellman equation of the classical linear quadratic control problem. Both cases fit into the following general framework

$$H(x,u(x),Du(x)) = \sup_{a\in A}\{-f(x,a)\cdot Du(x) - h(x,a) + k(x,a)u(x)\} = 0,\ \mathbb{R}^N, \tag{1.1}$$

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where $A$ may be unbounded as well as the data functions $f, h, k$ in the parameter $a$. For a more detailed discussion about classes of equations that can be written in the form (1.1), we refer to the examples in the next section.

Our purpose here is to develop some new insights about multiplicity of viscosity solutions of (1.1). As promptly recognized, in the case $k \equiv 0$ adding a constant to a solution gives another solution, but the nonuniqueness problem is far less trivial as we will see in some examples, in particular uniqueness may not hold when we ask our solutions to vanish on a given set. We start proving new general optimality principles for semicontinuous super and subsolutions of (1.1) which hold despite of the well posedness of (1.1). Next, assuming $h$ nonnegative, from these we derive representation formulas and so a characterization of the minimal nonnegative supersolution and the maximal subsolution which is null on some given set, as certain value functions of control problems. We will also show that there is at most one continuous solution once we prescribe its values on the set

$$\mathcal{Z} = \{x : H(x, 0, 0) = 0\},$$

though existence may fail in general if some compatibility condition on the data is not satisfied, and if the whole space is not reachable from $\mathcal{Z}$ by means of the vector field $-f(x, a)$. We do not deal with the existence problem in full details however. Some results in this direction, for more special classes of equations can be found in Lions [9] and Lions, Rouy, Turin [10]. Relating viscosity solutions to value functions of optimal control problems (and differential games) is a matter long studied in the literature, see e.g. Lions [9], Lions, Souganidis [11], Ishii [8], and Evans, Ishii [7].

Most of the results we prove can be generalized, with some technical effort, to more complicated hamiltonians than $H$ in (1.1), including those with inf-sup or sup-inf type nonlinearity, using differential games arguments similar to the ones in the paper by the author [13], see also [14]. We will therefore avoid doing this here for the sake of simplicity.

The outline of the paper is as follows. In Section 2 we discuss some examples that motivate our study. In Section 3 we prove optimality principles for viscosity solutions of equation (1.1). In Section 4 we study the nonuniqueness problem of equation (1.1) and revisit the examples of Section 2.

**2. Preliminaries and examples.** We start setting precise some assumptions on the equation (1.1). We will assume that: $A \subset \mathbb{R}^M$ is a closed set, $f : \mathbb{R}^N \times A \to \mathbb{R}^N$ and $h, k : \mathbb{R}^N \times A \to \mathbb{R}$ are continuous functions,

$1 \le q < p$, and for any $R > 0$ there are real numbers $C_R, L_R, L > 0$ such that

$$\begin{cases} |f(x,a) - f(z,a)| \le L(1 + |a|^q)|x - z|, \\ |f(x,a)| \le L(1 + |x| + |a|^q), \\ |h(x,a) - h(z,a)| \le L_R(1 + |a|^p)|x - z|, \quad \text{for all } x, z \in \mathbb{R}^N,\ a \in A, \\ C_R|a|^p - L_R \le h(x,a) \le L_R(1 + |a|^p), \\ |k(x,a) - k(z,a)| \le L_R(1 + |a|^q)|x - z|, \\ |k(x,a)| \le L_R(1 + |a|^q), \quad \text{for all } |x|, |z| \le R,\ a \in A. \end{cases} \tag{2.1}$$

Under the assumptions in (2.1), the hamiltonian $H$ in (1.1) is locally Lipschitz continuous. This fact can be easily proved after observing that the coercivity of $h$ in (2.1) implies that for $R > 0$ fixed, there is a compact $A_R \subset\subset A$ such that for all $x, p \in \mathbb{R}^N$, $u \in \mathbb{R}$, $|x|, |p|, |u| \le R$, we have

$$H(x,u,p) = \max_{a \in A_R} \{-f(x,a) \cdot p - h(x,a) + k(x,a)u\}.$$

Such regularity of the hamiltonian, however, is not enough to ensure a comparison theorem for viscosity solutions of equation (1.1), and in fact uniqueness does not hold in general (for the definition of viscosity solution we refer, for the reader's convenience, to the end of this section). Under certain conditions on the data, it looks interesting for the applications to characterize certain particular solutions and get a-priori bounds for the whole class of solutions. To do this, we will restrict ourselves to the case where

$$h(x,a) \ge 0, \quad \text{for all } x \in \mathbb{R}^N,\ a \in A. \tag{2.2}$$

When (2.2) holds, it is reasonable to look for the minimal nonnegative (super) solution (the constant zero is in fact a subsolution so we expect at least one nonnegative solution). Moreover, note that if some nonnegative solution satisfies $U(x) = 0$ then necessarily $0 \in \mathcal{Z} = \{x : H(x,0,0) = 0\}$ (since 0 is a viscosity subdifferential of $U$ at $x$, check this by the definition of viscosity solution). Another interesting problem is therefore to characterize the maximal (sub) solution vanishing on some closed subset of $\mathcal{Z}$. The set $\mathcal{Z}$ is a sort of degenerate set for the equation (1.1) and we will implicitly assume throughout the paper that $\mathcal{Z} \neq \emptyset$.

We proceed with a short collection of examples of equations having the structure (1.1) and multiple solutions. We will revisit them shortly again at the end of Section 4.

**Example 2.1.** Consider the following equation arising in linear quadratic control theory (an example of unbounded control problem)

$$\lambda u - bx \cdot Du + |Du|^2 - \beta|x|^2 = 0, \tag{2.3}$$

where $\lambda, \beta \geq 0$ and $2b > \lambda$. This equation can be easily reduced to the form (1.1) by observing that

$$|p|^2 = \max_{a \in \mathbb{R}^N} \{-2a \cdot p - |a|^2\}. \tag{2.4}$$

However, if we look for solutions of (2.3) of the particular form $u(x) = c|x|^2$, we will find two of them, one nonnegative and one nonpositive if $\beta > 0$, the null function and a nonpositive one if instead $\beta = 0$. We can see in particular that the minimal nonnegative solution does not depend continuously on the parameter $\beta$ as this tends to zero.

**Example 2.2.** Consider the degenerate eikonal equation (we use here the summation convention)

$$a_{ij}(x) u_{x_i} u_{x_j} + 2b_i(x) u_{x_i} - h^2(x) = 0, \tag{2.5}$$

where $a_{ij} = \sigma_{ik}\sigma_{jk}$ and $\sigma$ is symmetric. Also this class can be reduced to the form (1.1) again by (2.4). Moreover if $b(x) = 2\sigma(x)\overline{b}(x)$, for some vector field $\overline{b}(\cdot)$, then the parameter set $A$ can be also chosen to be compact, since now the equation is equivalent to

$$|\sigma(x)Du + \overline{b}(x)| = (h(x)^2 + |\overline{b}|^2(x))^{1/2}.$$

Nevertheless we cannot in general expect uniqueness of solutions vanishing at the origin as the following special case shows. Consider the equation

$$|u'|^2 = x^2(1 - x^2)^2, \quad x \in \mathbb{R},$$

and observe that $\mathcal{Z} = \{-1, 0, 1\}$. Two nonnegative viscosity solutions of this equation vanishing at the origin are

$$u_1(x) = \begin{cases} g(x), & |x| \leq 1, \\ 1/2 - g(x), & |x| > 1, \end{cases}$$

$$u_2(x) = \begin{cases} g(x) \wedge (1/4 - g(x)), & |x| \leq 1, \\ 1/4 - g(x), & |x| > 1, \end{cases},$$

where $g(x) = x^2/2 - x^4/4$. The solution $u_1$ assumes the value zero only at zero, while $u_2$ is zero on $\mathcal{Z}$. As a matter of fact it is easily shown that there are infinitely many nonnegative viscosity solutions $u$ null at the origin and satisfying $u_2 \leq u \leq u_1$.

**Example 2.3.** A similar example is the following equation

$$I(x)\psi(Du) - b(x) \cdot Du - h(x)^2 = 0,$$

where $I$ is nonnegative and $\psi$ is convex with $\psi(0) = 0$. Therefore its Legendre transform $\psi^*$ is nonnegative and we can rewrite the equation as

$$\max_{a \in \mathcal{D}om(\psi^*)} \{-(b(x) - I(x)a) \cdot Du - (h(x)^2 + I(x)\psi^*(a))\} = 0.$$

A particular case is the following equation which appears in shape from shading problems, see Lions, Rouy, Turin [10], where nonuniqueness features are pointed out and some results similar to ours are derived for this equation

$$I(x)(1 + |Du|^2)^{1/2} - 1 = 0,$$

where $I(x) \in [0, 1]$. This equation can in fact be rewritten as

$$\max_{|a| \leq 1} \{-I(x)a \cdot Du - 1 + I(x)(1 - |a|^2)^{1/2}\} = 0.$$

**Example 2.4.** The next example comes from Fuller's problem in control theory. The equation is the following

$$|yD_x u| + |D_y u| - x^2 = 0, \quad \mathbb{R}^2.$$

Again this problem has at least two nonnegative viscosity solutions vanishing at the origin. One is null only at the origin, while the other is zero on $\mathcal{Z} = \{(x, y) : x = 0\}$. If instead we change the equation to

$$-yD_x u + |D_y u| - x^2 = 0, \quad \mathbb{R}^2,$$

then there is only one continuous, nonnegative viscosity solution attaining the value zero at the origin and it is positive otherwise. Note that the set $\mathcal{Z}$ remains unchanged.

**Remark 2.5.** Assume that in (1.1) $A$ is compact and $h \equiv k$, then our equation is very much related to the one with $k \equiv 0$ by means of the change of variables $v = 1 - \exp(-u)$. This change, known as Kruzkov transform, is often helpful since nonnegative, extended real valued functions, become bounded nonnegative functions. When we consider equations where $k \equiv 0$ and require solutions to vanish at a certain point, then they may blow up to $+\infty$ and therefore they may exist only locally. This problem becomes irrelevant after the change of variables. This applies to all of the examples above (the first one only for $\lambda = 0$).

**Remark 2.6.** We have chosen to prove our results here only for equations of the type (1.1) to simplify the presentation. With some additional technical effort, the results of this paper can be extended to equations arising in differential games theory, where inf-sup (or sup-inf) type of nonlinearities appear in (1.1). The class of equations for which our results hold therefore contains those arising in nonlinear $\mathcal{H}_\infty$ control theory, and for this side of the problem we refer the reader to the paper by the author [14], where most of the game theoretic arguments can be found (see also [13]).

We end this section recalling the definition of viscosity solution. Let $w : \Omega \to \mathbb{R}, \Omega \subset \mathbb{R}^N$ open, be a locally bounded function. We define its lower and upper semicontinuous envelopes as, respectively

$$w_*(x) = \lim_{r\to 0^+} \inf \{w(y) : |x-y| \le r\},$$
$$w^*(x) = \lim_{r\to 0^+} \sup \{w(y) : |x-y| \le r\}.$$

**Definition 2.7.** Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function. The lower (resp. upper) semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity super- (resp. sub-) solution of

$$F(x, u, Du) = 0, \qquad \text{in } \Omega, \tag{2.6}$$

if for all $\varphi \in \mathcal{C}^1(\Omega)$ and $x \in argmin_{x\in\Omega}(u-\varphi)$, (resp. $x \in argmax_{x\in\Omega}(u-\varphi)$), we have

$$F(x, u(x), D\varphi(x)) \ge 0, \qquad (\text{resp. } F(x, u(x), D\varphi(x)) \le 0).$$

We also say that $D\varphi(x) \in D^-u(x)$, the subdifferential of u at x (resp. $D\varphi(x) \in D^+u(x)$, the superdifferential). A locally bounded function u is a viscosity solution of (2.6) if $u_*$ is a supersolution and $u^*$ is a subsolution.

In "convex" problems, i.e., when the hamiltonian $F$ is convex in the "Du" variable as it happens in (1.1) (and therefore not for genuine game problems), a different notion of solution has been formulated and proved to be very effective, mainly in the case of discontinuous solutions (if the solution is continuous, it is actually equivalent to the one of viscosity solution, otherwise it is more restrictive, see e.g. Barron-Jensen [3] and Barles [1]). This idea was proposed by Barron-Jensen [3], who proved new uniqueness results for discontinuous solutions, and then simplified and developed by Barles [1] and the author [12].

**Definition 2.8.** We say that a lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a bilateral supersolution of (2.6) if for all $\varphi \in \mathcal{C}^1(\Omega)$ and $x \in \arg\min_{x\in\Omega}(u-\varphi)$, we have

$$F(x, u(x), D\varphi(x)) = 0.$$

**3. Optimality principles for viscosity solutions.** We start this section recalling the control problem underlying equation (1.1). We consider the controlled dynamical system

$$\dot{y} = f(y, a),\ y(0) = x \in \mathbb{R}^N, \tag{3.1}$$

and by our growth conditions, we will choose the admissible controls $a(\cdot)$ in the set

$$\mathcal{A} = L^p_{\text{loc}}(\mathbb{R}_+; A).$$

In a given time interval $[0, t]$, to any trajectory $y_x(\cdot; a) \equiv y(\cdot)$, solution of (3.1), we associate the following payoff functional

$$J(t, x, a(\cdot)) = \int_0^t \exp\,(-\int_0^s k(y, a)dr)h(y, a)ds, \tag{3.2}$$

that we will use to define various value functions.

When $A$ is compact, we are also interested in a wider class of controls, namely relaxed controls

$$\mu \in \mathcal{A}^r = L^\infty(\mathbb{R}_+; A^r),$$

where $A^r = \{\mu : \text{Radon probability measure on } A\}$, and extend our system by setting, for $\mu \in \mathcal{A}^r$,

$$\nu^r(x, \mu) = \int_A \nu(x, a)d\mu(a), \quad \nu \in \{f, k, h\}.$$

It is easy to recognize that, when $A$ is compact, the functions $f^r, h^r, k^r$ satisfy conditions analogous to (2.1). Moreover the hamiltonian associated to such new functions coincides with $H$ in (1.1). For any $\mu \in \mathcal{A}^r$, we indicate by $y^r_x(\cdot;\mu) \equiv y^r_x(\cdot)$ the trajectory solution of

$$\dot{y^r} = f^r(y^r, \mu), \quad y^r(0) = x.$$

For any control $\mu(\cdot) \in \mathcal{A}^r$, we will consider the relaxed payoff given by

$$J^r(t, x, \mu) = \int_0^t \exp\left(-\int_0^s k^r(y, \mu)dr\right) h^r(y, \mu)ds.$$

We observe that if $A$ is compact, then $A^r$ is metrizable and compact in the weak* topology as a subset of the dual of $C(A;\mathbb{R})$. We will identify any control $a \in A$ with the Dirac measure $\delta_a \in A^r$.

**Definition 3.1.** Let $u : \mathbb{R}^N \to \mathbb{R}$ be a function, we say that $u$ satisfies the upper optimality principle with respect to the optimal control problem (3.1), (3.2) if

$$u(x) = \inf_{a \in \mathcal{A}} \sup_{t \in \mathbb{R}_+} \{J(t, x, a) + \exp\left(-\int_0^t k(y, a)ds\right) u(y(t))\}. \tag{3.3}$$

We say that u satisfies the relaxed upper optimality principle if

$$u(x) = \inf_{\mu \in \mathcal{A}^r} \sup_{t \in \mathbb{R}_+} \{J^r(t, x, \mu) + \exp\left(-\int_0^t k^r(y, \mu)ds\right) u(y^r(t))\}. \tag{3.4}$$

We say that $u$ satisfies the lower optimality principle if

$$u(x) = \inf_{a \in \mathcal{A}} \inf_{t \in \mathbb{R}_+} \{J(t, x, a) + \exp\left(-\int_0^t k(y, a)ds\right) u(y(t))\}. \tag{3.5}$$

Finally we say that $u$ satisfies the relaxed lower optimality principle if

$$u(x) = \inf_{\mu \in \mathcal{A}^r} \inf_{t \in \mathbb{R}_+} \{J^r(t, x, \mu) + \exp\left(-\int_0^t k^r(y, \mu)ds\right) u(y^r(t))\}.$$

The purpose of this section is to prove optimality principles for viscosity solutions of equation (1.1) in the generality of assumption (2.1). The localized version of these results is presented in [15]. Precisely we prove the following result.

**Theorem 3.2.** *Assume* (2.1) *and let* $u : \mathbb{R}^N \to \mathbb{R}$ *be a function. The following hold.*

(i) *If* $u$ *is an upper semicontinuous subsolution of* (1.1)*, then it satisfies the lower optimality principle. It also satisfies the relaxed lower optimality principle if* $A$ *is compact.*

(ii) *If* $u$ *is a continuous supersolution of* (1.1)*, then it satisfies the upper optimality principle.*

(iii) *If* $A$ *is compact and* $u$ *is a lower semicontinuous supersolution of* (1.1)*, then it satisfies the relaxed upper optimality principle. If the sets* $\{(f(x,a), h(x,a), k(x,a)) : a \in A\}$ *are convex for all* $x \in \mathbb{R}^N$*, then the upper optimality principle holds.*

(iv) *If* $u$ *is a lower semicontinuous bilateral supersolution of* (1.1) *then both* $u$ *and* $u^*$ *satisfy the lower optimality principle* (*and the statements* (ii)*,* (iii) *apply to* $u$ *as well*)*.*

**Remark 3.3.** It is clear by the statement that the problem for super and subsolutions is not symmetric. As it is well known in control theory, dealing with subsolutions of (1.1) is quite easier than with supersolutions, less regularity is required and a stronger result can be obtained. Such a symmetry does not occur for example in differential games problems.

Observe that bilateral supersolutions are lower semicontinuous and nonetheless satisfy the lower optimality principle, which by (i) is expected to be attained by upper semicontinuous functions.

**Proof.** 1. We start considering (iii) and a lower semicontinuous supersolution $U : \mathbb{R}^N \to \mathbb{R}$ of (1.1). We will point out during the proof how to simplify the argument if $U$ is continuous and prove (ii). We introduce a change of variables. Let $\rho : \mathbb{R} \to \mathbb{R}_+$ be a smooth, bounded function such that $0 < \dot\rho \le M$ and $\rho(s) \to 0$ as $s \to -\infty$. Consider the function

$$u(z) = u(x, r, s) = \rho(\exp(-s)U(x) + r),$$

then by the usual rules of change of variables (as for example in Crandall-Lions [6]), we can easily prove that $u$ is a viscosity supersolution of

$$\sup_{a\in A}\{-F(z,a)\cdot D_z u(z)\} \ge 0, \quad \mathbb{R}^{N+2}, \tag{3.6}$$

where $F = (f, e^{-s}h, k)/(1+|a|^p)$ (note that we renormalize the extended vector field by the factor $(1+|a|^p)$).

2. We continue the proof under a simplifying assumption, namely we suppose that

$$|F(z,a) - F(z',a)| \le L|z - z'|, \quad z, z' \in \mathbb{R}^{N+2}, \quad a \in A. \tag{3.7}$$

We also consider an increasing sequence of continuous, nonnegative functions $\varphi_n : \mathbb{R}^{N+2} \to \mathbb{R}$ such that

$$u = \sup_n \varphi_n.$$

(We agree to choose the constant sequence, i.e. $\varphi_n \equiv u$, if $u$ is continuous). For any fixed $\lambda > 0$, since $u$ is nonnegative, it is also a supersolution of the equation (quasi variational inequality)

$$\lambda u + \min\{\sup_{a\in A}\{-F(z,a)\cdot D_z u\}, u - (1+\lambda)\varphi_n\} \ge 0, \quad \mathbb{R}^{N+2}, \tag{3.8}$$

for all $n \in \mathbb{N}$. Holding (3.7), equation (3.8) has a unique viscosity solution which is in fact continuous. As a matter of fact the assumptions of the comparison theorem for viscosity solutions are satisfied (see e.g. Crandall, Ishii, Lions [5]). Moreover by more or less standard dynamic programming arguments, as we will outline in the appendix, such solution can be proven to be the value function

$$V^\lambda(z) = \inf_{a\in\mathcal{A}} \sup_{\tau\in\mathbb{R}_+} \exp(-\lambda\tau)\varphi_n(z(\tau)), \tag{3.9}$$

where $z(\cdot) \equiv z_z(\cdot, a)$ is the trajectory corresponding to the following dynamical system

$$\dot z(\tau) = F(z(\tau), a(t(\tau))),\ z(0) = z,$$

where $a \in \mathcal{A}$ and $t(\cdot) = \tau^{-1}(\cdot)$, $\tau(t) = \int_0^t (1 + |a|^p)ds$. By our construction, if $z = (x, 0, 0)$ such a trajectory is provided by the following formula

$$z(\tau) = (y_x(t(\tau); a), J(t(\tau), x, a(\cdot)), \int_0^{t(\tau)} k(y,a)dt). \tag{3.10}$$

Comparison theorem applied to (3.8) then gives, for all $T > 0$

$$\begin{aligned} u(z) \ge V^\lambda(z) &\ge \inf_{a\in\mathcal{A}} \sup_{\tau\in[0,T]} \exp(-\lambda\tau)\varphi_n(z(\tau)) \\ &\ge \exp(-\lambda T) \inf_{a\in\mathcal{A}} \sup_{\tau\in[0,T]} \varphi_n(z(\tau)). \end{aligned}$$

As $\lambda \to 0^+$ we therefore obtain,

$$u(z) \geq \inf_{a\in\mathcal{A}} \sup_{\tau\in[0,T]} \varphi_n(z(\tau)), \quad \text{for all } T>0, n\in\mathbb{N}. \tag{3.11}$$

If $u$ is continuous, last formula (which holds with $\varphi_n \equiv u$) is good for now and we can skip to part 3 of this proof. Otherwise we have to take the limit as $n \to +\infty$. In order to do so we will assume $A$ compact and use relaxed controls. We observe that, as easily checked

$$u(z) = \liminf_{n\to+\infty, z'\to z} \varphi_n(z'). \tag{3.12}$$

For fixed $n$ by (3.11) we can find $a^n \in \mathcal{A}$ such that, indicating $z_n(\cdot) \equiv z_z(\cdot; a^n)$,

$$u(z) + 1/n \geq \varphi_n(z_n(\tau)), \quad \text{for all } \tau \in [0,T].$$

If we use the weak* compactness of $\mathcal{A}^r$, we can find a sequence $a^{n_k} \to \mu \in \mathcal{A}^r$ as $n_k \to +\infty$, weak* in $L^\infty([0,T]; A^r)$ and then using (3.12) and the fact that $z_{n_k}(\cdot; a^{n_k}) \to z^r(\cdot;\mu)$ uniformly in $[0,T]$, we conclude that

$$u(z) \geq \inf_{\mu\in\mathcal{A}^r} \sup_{\tau\in[0,T]} u(z(\tau)). \tag{3.13}$$

If moreover the sets $\{(f(x,a), h(x,a), k(x,a)) : a \in A\}$ are convex for all $x \in \mathbb{R}^N$, then by Filippov's Theorem, see Castaing-Valadier [4], it is well known that relaxed trajectories can be also obtained by means of usual measurable controls, so we can even obtain the corresponding of (3.13) with $\mathcal{A}$ instead of $\mathcal{A}^r$.

3. So far we proved that

$$u(z) \geq \inf_{a\in\mathcal{B}} \sup_{\tau\in[0,1]} u(z(\tau)), \tag{3.14}$$

where $\mathcal{B} = \mathcal{A}$ or $\mathcal{A}^r$ according to the fact that $U$ is continuous or not (a notation that we use hereafter in the proof). Let now $\varepsilon > 0$, from (3.14) we can find $b_0 \in \mathcal{B}$ so that

$$u(z) + \varepsilon/2 \geq \sup_{\tau\in[0,1]} u(z^0(\tau)).$$

Let $z^1 = z^0(1; b_0)$, then again from (3.14) we can find $b_1 \in \mathcal{B}$ so that

$$u(z^1) + \varepsilon/2^2 \geq \sup_{\tau\in[0,1]} u(z^1(\tau)).$$

We proceed recursively and by an induction argument easily conclude that if we define

$$b(t) = b_n(t), \quad \text{for } [t] = n,$$

then $b \in \mathcal{B}$ and the corresponding trajectory satisfies $u(z) + \varepsilon \geq u(z_z(\tau; b))$, for all $\tau \geq 0$. Therefore, as $\varepsilon$ is arbitrary, we conclude

$$u(z) = \inf_{a\in\mathcal{B}} \sup_{\tau\in\mathbb{R}_+} u(z(\tau)),$$

since the other inequality is obvious by choosing $\tau = 0$. By the definition of $u$ and the representation (3.10) of the trajectories $z(\cdot)$, we then easily conclude (ii) and (iii) by computing $u(x, 0, 0)$.

4. We now study the general case where (3.7) does not hold and use a localization argument. For $n \in \mathbb{N}$, we consider the family of smooth functions $\zeta_n : \mathbb{R}^{N+2} \to \mathbb{R}$ such that $0 \leq \zeta_n \leq 1$, $\zeta_n \equiv 1$ in $B(0, n) \subset \mathbb{R}^{N+2}$, $\zeta_n \equiv 0$ in $B(0, n+1)^c$, $|D\zeta| \leq 2$ and define $F_n = \zeta_n F$. Then it is clear that from (3.6) our function $u$ is also a supersolution of

$$\sup_{a\in A}\{-F_n(z, a) \cdot Du\} \geq 0, \quad \mathbb{R}^{N+2}, \tag{3.15}$$

and that $F_n$ satisfies (3.7). Therefore by what we just proved in part 3, we can state that

$$u(z) = \inf_{b\in\mathcal{B}} \sup_{\tau\in\mathbb{R}_+} u(z^n(\tau)),$$

where $z^n$ is the trajectory corresponding to the vector field $F_n$, i.e.

$$\dot{z}^n(\tau) = F_n(z^n(\tau), b(t(\tau))), \quad z^n(0) = z$$

for $b \in \mathcal{B}$. Then we get, for all $n \in \mathbb{N}$,

$$u(z) = \inf_{b\in\mathcal{B}} \sup_{\tau\in[0,\tau_z^n)} u(z(\tau)), \quad z \in B(0, n)$$

where $\tau_z^n(b) = \inf\{\tau \geq 0 : |z^n(\tau)| = n\}$. We can easily conclude from here using an induction argument similar to the one in part 3, which used (3.14) instead.

5. If $u$ is an upper semicontinuous subsolution, then the proof proceeds similarly as in parts 1 and 2, except that in this case there is no difficulty to pass to the limit as $n \to +\infty$ in the corresponding inequality of (3.11) which

holds for any control and not only for optimal ones, and there is no need of using relaxed controls to conclude (i).

6. Part (iv) of the statement remains to be shown. Let $U : \mathbb{R}^N \to \mathbb{R}$ be a lower semicontinuous bilateral supersolution of (1.1), then it is easily seen that also $u$ constructed as in part 1 is a bilateral supersolution of (3.15) for any $n \in \mathbb{N}$. Now we use the inf-convolution as in Barles [1], and for $\varepsilon, K > 0$, we define

$$u_\varepsilon(z, \tau) = \inf_{y \in \mathbb{R}^{N+2}} \{u(y) + \exp(-K\tau)|y - z|^2/\varepsilon^2\}.$$

By standard arguments, we can prove that, if K is sufficiently large but independent of $\varepsilon$ (actually $K$ can be choosen as twice the best Lipschitz constant of $F_n$), $u_\varepsilon$ is a continuous subsolution of

$$v_\tau + \sup_{a \in A}\{-F_n(z, a) \cdot D_z v\} \leq 0, \quad \mathbb{R}^{N+3}. \tag{3.16}$$

If we apply (i) to such equation, we get

$$u_\varepsilon(z, \tau_0) = \inf_{a \in \mathcal{A}} \inf_{\tau \in \mathbb{R}_+} u_\varepsilon(z^n(\tau), \tau_0 - \tau).$$

Therefore since $u_\varepsilon(z^n(\tau), \tau) \leq u(z^n(\tau))$, we conclude as $\varepsilon \to 0^+$ that

$$u(z) = \sup_{\varepsilon > 0} u_\varepsilon(z, 0) \leq \inf_{a \in \mathcal{A}} \inf_{\tau \in \mathbb{R}_+} u(z^n(\tau)),$$

the first equality being a well known property of nonlinear convolution. Then the result for $u$ follows as in part 4. On the other hand, if we pass to the limit in (3.16) as $\varepsilon \to 0$, using the fact that $u^*(x) = \limsup_{y \to x, \varepsilon \to 0} u_\varepsilon(y)$, for all $\tau$, and the stability of viscosity solutions as in Barles, Perthame [2], see also Crandall, Ishii, Lions [5] and the author [12], we obtain that $u^*$ is a subsolution of (1.1) so we can apply part (i) to get the rest of the statement.

**4. Representation formulas for viscosity solutions.** In this section we apply the results of the previous one and get representation formulas for solutions of the Hamilton-Jacobi equation (1.1). Let us consider the two value functions

$$V_\infty(x) = \inf_{a \in \mathcal{A}} \int_0^{+\infty} \exp(-\int_0^t k(y, a) ds) h(y, a) dt,$$

$$V_\infty^r(x) = \inf_{\mu \in \mathcal{A}^r} \int_0^{+\infty} \exp(-\int_0^t k^r(y, \mu) ds) h^r(y, \mu) dt.$$

Note that by our assumptions (2.1) and (2.2), $V_\infty(x)$ (or $V_\infty^r(x)$) is nonnegative but not necessarily finite.

**Theorem 4.1.** *Assume* (2.1) *and* (2.2).

(i) *If* $U : \mathbb{R}^N \to \mathbb{R}$ *is a nonnegative, continuous viscosity supersolution of* (1.1), *then* $U(x) \geq V_\infty(x)$. *Therefore if* $V_\infty$ *is continuous in* $\mathbb{R}^N$, *then it is the minimal continuous, nonnegative viscosity* (*super*)*solution of* (1.1).

(ii) *If* $A$ *is compact and* $U : \mathbb{R}^N \to \mathbb{R}$ *is a nonnegative, lower semicontinuous viscosity supersolution of* (1.1), *then* $U(x) \geq V^r_\infty(x)$. *Therefore if such a supersolution exists, then* $V^r_\infty$ *is lower semicontinuous and the minimal nonnegative, lower semicontinuous* (*super*)*solution of* (1.1). *If moreover the sets* $\{(f(x,a), h(x,a), k(x,a)) : a \in A\}$ *are convex for all* $x \in \mathbb{R}^N$, *then the result holds for* $V_\infty = V^r_\infty$.

**Proof.** The comparisons in (i) and (ii) are an immediate consequence of Theorem 3.2 (ii) and (iii), respectively. The fact that $V_\infty$ and $V^r_\infty$, if locally bounded, are viscosity solutions of (1.1) can be obtained by rather standard methods, directly for $V^r_\infty$ or if $A$ is compact, or after using the reparametrization of trajectories as in the proof of Theorem 3.2 otherwise. Indeed considering the system

$$y'(\tau) = f(y(\tau), a(t(\tau)))/(1 + |a(t(\tau))|^p),$$

using the dynamic programming principle, one first shows that $V_\infty$ is a viscosity solution of

$$\sup_{a \in A}\{(-f(x,a) \cdot Du - h(x,a) + k(x,a)u)/(1 + |a|^p)\} = 0, \quad \mathbb{R}^N.$$

To conclude we then use the coercivity assumption on $h$ in (2.1) which implies that for $R > 0$ fixed, there is a compact $A_R \subset\subset A$ such that for all $x, p \in \mathbb{R}^N$, $u \in \mathbb{R}$, $|x|, |p|, |u| \leq R$, we have

$$H(x,u,p) = \max_{a \in A_R}\{-f(x,a) \cdot p - h(x,a) + k(x,a)u\}. \qquad \square$$

The previous result shows that the value function of the infinite horizon control problem provides the smallest nonnegative solution of our equation. Moreover $V_\infty$ and $V^r_\infty$ locally bounded is a necessary condition for existence of global nonnegative continuous or lower semicontinuous supersolutions, respectively. When $k \equiv 0$, this roughly amounts to stability properties of the set $\mathcal{Z}$ with respect to the vector field $f$.

In the following we will consider a closed subset

$$\mathcal{T} \subset \mathcal{Z} = \{x : H(x,0,0) = 0\},$$

and, for $a \in \mathcal{A}$, define the exit-time from $\mathcal{T}^c$, i.e.

$$t_x(a) = \inf\{t \geq 0 : y_x(t;a) \in \mathcal{T}\}.$$

We also consider for $x \in \mathbb{R}^N$ the set of admissible controls

$$\mathcal{A}_x = \{a \in \mathcal{A} : t_x(a) < +\infty\}.$$

We need to remark that in general the set $\mathcal{A}_x$ might be empty. As a matter of fact, the condition $\mathcal{A}_x \neq \emptyset$ for all $x \in \mathbb{R}^N$ is sometimes called global attainability of the set $\mathcal{T}$ with respect to the vector field $f$.

**Theorem 4.2.** *Assume* (2.1) *and* (2.2).

(i) *If $U : \mathbb{R}^N \to \mathbb{R}$ is an upper semicontinuous viscosity subsolution of* (1.1)*, which is null on $\mathcal{T}$, then*

$$U(x) \leq V_{\mathcal{T}}(x) = \inf_{\mathcal{A}_x} \int_0^{t_x(a)} \exp\left(-\int_0^t k(y,a)ds\right)h(y,a)dt.$$

*Therefore if $V_{\mathcal{T}}$ is finite, locally bounded and continuous on $\mathcal{T}$, then it is upper semicontinuous in $\mathbb{R}^N$ and the maximal subsolution of* (1.1)*, null on $\mathcal{T}$.*

(ii) *If $U : \mathbb{R}^N \to \mathbb{R}$ is a lower semicontinuous bilateral supersolution of* (1.1)*, which is null on $\mathcal{T}$, then $U \leq V_{\mathcal{T}}$. Therefore if $V_{\mathcal{T}}$ is finite and locally bounded, then $(V_{\mathcal{T}})_*$ is the maximal viscosity supersolution of $-H(x,u(x),Du(x)) \geq 0$ in $\mathbb{R}^N$, vanishing on $\mathcal{T}$.*

**Proof.** Everything follows similarly to the proof of Theorem 4.1 as a consequence of Theorem 3.2. The function $U$ in the statement is not necessarily nonnegative. When $U$ is nonnegative, the assumptions in (i) imply that $U$ is continuous at the points of $\mathcal{T}$, while this is not necessarily true for (ii). In order to prove that, when real valued and locally bounded, $(V_{\mathcal{T}})_*$ is a bilateral supersolution, one can use the backward dynamic programming principle, introduced by the author [12]. (For some more details see also the proof of Proposition 4.1 in [15]). □

As a consequence of the previous result, the following verification theorem holds.

**Corollary 4.3.** *Let $x \in \mathbb{R}^N$ and assume* (2.1), (2.2). *Suppose that there are $\overline{a} \in \mathcal{A}$ and a function $U : \mathbb{R}^N \to \mathbb{R}$, null on $\mathcal{T}$, which is either an upper semicontinuous subsolution of* (1.1) *or a lower semicontinuous supersolution of $-H(x, u(x), Du(x)) \geq 0$, in $\mathbb{R}^N$, such that*

$$\int_0^{t_x(\overline{a})} \exp\,(-\int_0^t k(y, \overline{a})ds)h(y, \overline{a})dt \leq U(x).$$

*Then $\overline{a}$ is an optimal control for $V_{\mathcal{T}}$.*

In the following we prescribe the values of our solution on $\mathcal{Z}$ by means of a continuous function $g : \mathcal{Z} \to \mathbb{R}_+$. We start with the following Lemma.

**Lemma 4.4.** *Assume* (2.1), (2.2) *and that for all $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that*

$$h(x, a) \geq C_\varepsilon(1 + |a|^q) > 0,$$

*for all $x \in \mathbb{R}^N$ satisfying $dist(x, \mathcal{Z}) \geq \varepsilon$, and $a \in A$. Suppose moreover that there is $\sigma > 0$ such that $|f(x, a)| \leq L(1 + |a|^q)$, for all $x$ satisfying $dist(x, \mathcal{Z}) \leq \sigma$, and $a \in A$ (this in particular can be deduced from the other assumptions if $\mathcal{Z}$ is bounded). Then for all $a \in \mathcal{A}$*

$$\int_0^\infty h(y, a)dt < +\infty,$$

*implies $dist(y(t), \mathcal{Z}) \to 0$, as $t \to +\infty$.*

**Proof.** It is clear by the assumption that $\liminf_{t\to+\infty} dist(y(t), \mathcal{Z}) = 0$. We then argue by contradiction and suppose that there are $\varepsilon > 0$ $2\varepsilon \leq \sigma$, and an increasing sequence $t_n \to +\infty$ such that

$$dist(y(t_{2n}), \mathcal{Z}) = 2\varepsilon,\ dist(y(t_{2n+1}), \mathcal{Z}) = \varepsilon,$$
$$dist(y(t), \mathcal{Z}) \in [\varepsilon, 2\varepsilon] \text{ for } t \in [t_{2n}, t_{2n+1}].$$

Therefore we get

$$\varepsilon \leq |y(t_{2n}) - y(t_{2n+1})| \leq L \int_{t_{2n}}^{t_{2n+1}} (1 + |a|^q)dt.$$

By (2.2) and the assumption, this implies that

$$\int_0^\infty h(y, a)dt \geq \sum_n \int_{t_{2n}}^{t_{2n+1}} h(y, a)dt \geq C_\varepsilon \sum_n \int_{t_{2n}}^{t_{2n+1}} (1 + |a|^q)dt = +\infty,$$

a contradiction.

**Theorem 4.5.** *Suppose that all the assumptions of Lemma* 4.4 *hold true,* $k \equiv 0$ *and let* $g$ *be as above. Then there is at most one continuous, nonnegative viscosity solution* $U : \mathbb{R}^N \to \mathbb{R}$ *of* (1.1) *such that* $U(x) = g(x)$ *on* $\mathcal{Z}$ *and* $U$ *is continuous, uniformly for* $x \in \mathcal{Z}$. *By* $U$ *continuous, uniformly for* $x \in \mathcal{Z}$ *we mean that for any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $x \in \mathcal{Z}$, $y \in \mathbb{R}^N$ *and* $|x - y| \leq \delta$ *implies* $|U(x) - U(y)| \leq \varepsilon$. *This is always satisfied if* $\mathcal{Z}$ *is bounded). If moreover* $A$ *is compact and* $U, V$ *are viscosity solutions of* (1.1) *and continuous, uniformly for* $x \in \mathcal{Z}$, *then* $U, V$ *are continuous in* $\mathbb{R}^N$.

**Proof.** Let $U, V$ be two continuous solutions, the proof is similar if $A$ is compact. By Theorem 3.2 (i) we have

$$U(x) = \inf_{a\in\mathcal{A}} \inf_{t\in\mathbb{R}_+} \{J(t,x,a)+U(y(t))\} \leq \inf_{a\in\mathcal{A}} \{\int_0^\infty h(y,a)dt+\liminf_{t\to+\infty} U(y(t))\}.$$

By Theorem 3.2 (ii) we also get that

$$V(x) = \inf_{a\in\mathcal{A}} \sup_{t\in\mathbb{R}_+} \{J(t,x,a) + V(y(t))\}.$$

Fix $\varepsilon > 0$ and let $a \in \mathcal{A}$ be such that

$$V(x) + \varepsilon \geq \int_0^\infty h(y,a)dt + \limsup_{t\to+\infty} V(y(t)).$$

Therefore by Lemma 4.4 and the fact that $V$ is nonnegative, we obtain

$$dist(y(t), \mathcal{Z}) \to 0, \qquad \text{as } t \to +\infty.$$

Let us also consider an increasing sequence $t_n \to +\infty$ such that

$$V(y(t_n)) \to \limsup_{t\to+\infty} V(y(t)).$$

Then passing to a subsequence if necessary, we either get $y(t_n) \to \overline{y} \in \mathcal{Z}$, or $|y(t_n))| \to +\infty$ as $n \to +\infty$. In both cases, using the fact that $U, V$ are continuous, uniformly for $x \in \mathcal{Z}$ and coincide on $\mathcal{Z}$, we obtain $V(x) + \varepsilon \geq U(x)$ for arbitrary $\varepsilon > 0$. Exchanging the roles of $U, V$, we conclude.

**Remark 4.6.** The existence issue for the problem in Theorem 4.5 is technically more complicated and, from the point of view of this paper, less interesting. Two questions are involved. One is the real valuedness and the continuity of the appropriate value function candidate to solve it, and this is nontrivial if the discount factor $k$ is degenerate. The second is the need of some compatibility condition on the given function $g$, because, as Theorems 4.1 and 4.2 and the construction of the minimal and maximal solutions show, we have to fulfill a priori necessary conditions for existence.

With the same proof of the previous result, we can prove the following statement which explains why the second equation of Example 2.4 has a unique solution vanishing at the origin.

**Corollary 4.7.** *In the assumptions of Theorem* 4.5*, suppose moreover that the closed subset* $\mathcal{T} \subset \mathcal{Z}$ *is attractive for the vector field* $f$ *in the following sense: if* $a \in \mathcal{A}$ *is such that* $dist(y(t), \mathcal{Z}) \to 0$*, as* $t \to +\infty$*, then in fact* $dist(y(t), \mathcal{T}) \to 0$ *as* $t \to +\infty$*. Then there is at most one continuous, nonnegative viscosity solution* $U$*, which is continuous, uniformly for all* $x \in \mathcal{T}$ *and such that* $U = g$ *on* $\mathcal{T}$*.*

The next interesting property of propagation holds specifically for bilateral supersolutions.

**Theorem 4.8.** *Assume* (2.1), (2.2) *and let* $U$ *be lower semicontinuous and a bilateral supersolution of* (1.1)*. Assume that there is an optimal control* $\overline{a} \in \mathcal{A}$*:* (i) *optimal for the value function*

$$\inf_{a\in\mathcal{A}} \sup_{\mathbb{R}_+} J(t,x,a) + \exp\,(-\int_0^t k(y,a)ds)U(y(t))\ (= U(x)),$$

*if either* $U$ *is continuous or* $A$ *is compact and the sets*

$$\{(f(x,a), h(x,a), k(x,a)) : a \in A\}$$

*are convex for all* $x \in \mathbb{R}^N$*; or*
(ii) *optimal for the value function*

$$\inf_{\mu\in\mathcal{A}^r} \sup_{\mathbb{R}_+} J^r(t,x,\mu) + \exp\,(-\int_0^t k^r(y,\mu)ds)U(y(t)),$$

*if* $A$ *is compact. Then the map* $t \to J(t,x,\overline{a}) + \exp\,(-\int_0^t k(y,\overline{a})ds)U(y(t))$ *is constant.*

**Proof.** By Theorem 3.2 (iii) and the assumption (if $A$ is compact for example), for the given $\overline{a}$ we have

$$\begin{aligned} U(x) &= \inf_{\mu\in\mathcal{A}^r}\sup_{t\in\mathbb{R}_+} J^r(t,x,\mu)+\exp\left(-\int_0^t k^r(y,\mu)ds\right)U(y(t)) \\ &= \sup_{t\in\mathbb{R}_+} J(t,x,\overline{a})+\exp\left(-\int_0^t k(y,\overline{a})ds\right)U(y(t)). \end{aligned}$$

Moreover by Theorem 3.2 (iv) we also have

$$\begin{aligned} U(x) &= \inf_{a\in\mathcal{A}}\inf_{t\in\mathbb{R}_+} J(t,x,a)+\exp\left(-\int_0^t k(y,a)ds\right)U(y(t)) \leq \\ &\leq \inf_{t\in\mathbb{R}_+} J(t,x,\overline{a})+\exp\left(-\int_0^t k(y,\overline{a})ds\right)U(y(t)), \end{aligned}$$

from which the result follows. □

**Remark 4.9.** The previous statement shows once again why optimal trajectories of control problems have to be considered generalized characteristics of the corresponding Bellman equation. Recall also that if $U$ is continuous, then $U$ bilateral supersolution is equivalent to viscosity solution (as for example follows from our optimality principles). The existence of optimal trajectories can be automatically obtained, as in the proof of Theorem 3.2, by classical methods and Filippov's Theorem if $A$ is compact and the sets $\{(f(x,a),h(x,a),k(x,a)) : a\in A\}$ are convex for all $x\in\mathbb{R}^N$.

**Conclusions.** We now go back and discuss our examples in view of the results we proved. In the following we indicate $\mathcal{T}=\{0\}$. Note that all of the dynamical systems we consider below are globally controllable to the origin, meaning that there are a nonnegative, continuous function $\omega:\mathbb{R}_+\to\mathbb{R}$, null at the origin, and a locally bounded function $C:\mathbb{R}^N\to\mathbb{R}_+$, such that for all $x\in\mathbb{R}^N$ there are $a_x\in\mathcal{A}$, $t_x\geq 0$ satisfying $y_x(t_x)=0$, $t_x\leq\omega(|x|)$, $||a_x||_{L^p(0,t_x)}\leq C(x)$. Moreover there is a control $\overline{a}\in A$ such that $f(0,\overline{a})=0$ and $h(0,\overline{a})=0$. A consequence of these facts is that the value functions $V_\infty$ and $V_{\mathcal{T}}$ introduced above in this section are locally bounded, vanishing and are continuous at the origin, as easily checked.

Specifically, in Example 2.1, by the particular structure of the problem with $f(x,a)=2a+bx$ linear and $h(x,a)=|a|^2+\beta|x|^2$ quadratic, one immediately realizes that both value functions are quadratic functions, i. e. satisfy $V(x)=|x|^2V(x/|x|)$ if $x\neq 0$. Then by Theorem 4.5, $V_\infty=V_{\mathcal{T}}$ when

$\beta \neq 0$, while they are obviously different when $\beta = 0$ (note that in this case $\mathcal{Z} = \mathbb{R}^N$).

In the special case presented in Example 2.2, taking the square root of both terms, one obtains the equivalent equation $|u'| = |x||1 - x^2|$. Since we can rewrite this equation in the form (1.1) with the choice of $A = [-1, 1]$, Theorem 4.5 implies that $V_\infty$ and $V_\mathcal{T}$ are continuous and characterized by their values on $\mathcal{Z}$. Clearly $V_\infty \equiv 0$ on $\mathcal{Z}$, therefore must coincide with $u_2$, while $V_\mathcal{T}$ is null only at the origin. Moreover choosing the control $a(t) = -1$ for $t \in [0, 1]$, $a(t) = 0$ afterwards, it follows that $V_\mathcal{T}(1) \leq 1/4$. Similarly $V_\mathcal{T}(-1) \leq 1/4$ and being $V_\mathcal{T}$ the maximal solution vanishing on $\mathcal{T}$ by Theorem 4.2, we deduce that it must coincide with $u_1$.

The special case of Example 2.3 coming from shape from shading problems has the properties we mentioned at the beginning of the remark only if $I(x) \in (0, 1]$, which is some limitation to the result. In this case $\mathcal{Z} = \{x : I(x) = 1\}$, $A = \{x : |x| \leq 1\}$ is compact, and Theorem 4.5 applies if we assume moreover that for all $\varepsilon > 0$ we have $I(x) \leq 1 - C_\varepsilon < 1$ for all $x$ satisfying $dist(x, \mathcal{Z}) > \varepsilon$, for some $C_\varepsilon > 0$, and $I$ is Lipschitz continuous. For this special structure of the equations the results in Theorem 4.5 were known and can be found in Lions, Rouy, Tourin [10].

For the first equation of Example 2.4, one easily sees directly that $V_\infty$ and $V_\mathcal{T}$ are different. As much as the second equation is concerned, it does not fit into the assumptions of Lemma 4.4, but that statement still holds, due to the special structure of the system, again by a direct computation we skip. Moreover we can apply Corollary 4.7 with $A$ compact to deduce that $V_\infty = V_\mathcal{T}$ and they are continuous.

**Appendix.** In this section we outline how to prove that value functions like

$$\begin{aligned} V(z) &= \inf_{a \in \mathcal{A}} \sup_{\mathbb{R}_+} \exp(-\lambda\tau)\psi(z(\tau)), \\ W(z) &= \inf_{a \in \mathcal{A}} \inf_{\mathbb{R}_+} \exp(-\lambda\tau)\varphi(z(\tau)), \end{aligned} \tag{A.1}$$

where $\dot{z} = F(z, a)$, $z(0) = z$, $\psi, \varphi : \mathbb{R}^N \to \mathbb{R}$ are continuous, $F$ is also continuous and $|F(z, a) - F(z', a)| \leq L|z - z'|$, satisfy certain Hamilton-Jacobi equations in the viscosity sense, precisely

$$\begin{aligned} &\min\{\lambda V + \sup_{a \in A}\{-F(z, a) \cdot DV\}, V - \psi\} = 0, \\ &\max\{\lambda W + \sup_{a \in A}\{-F(z, a) \cdot DW\}, V - \varphi\} = 0, \end{aligned}$$

respectively. Let us consider the value function in (A.1) as an example. The arguments are similar for the other. This is the usual discounted obstacle problem, where the obstacle is given by the function $\varphi$. The key fact is a proper dynamic programming principle.

**Proposition A.1.** *The value function in* (A.1) *satisfies the following: for all* $\tau \geq 0$

$$W(z) \leq \inf_{a \in \mathcal{A}} \exp(-\lambda\tau) W(z_z(\tau)). \tag{A.2}$$

*Moreover, if* $W_*(z) < \varphi(z)$*, there is* $\varepsilon > 0$ *such that for all sequences* $z_n \to z$ *satisfying* $W(z_n) \to W_*(z)$ *the equality holds in* (A.2) *for all* $z_n, \tau$ *such that* $|z_n - z| \leq \varepsilon,\ \tau \in [0, \varepsilon]$.

**Proof.** The first statement is almost trivial. We outline the proof of the second using a contradiction argument. Assume that $W_*(z) < \varphi(z)$ and we can find sequences $z_n \to z$, $W(z_n) \to W_*(z)$, $\tau_n \to 0$, $0 < \varepsilon_n \to 0$ such that

$$W(z_n) + \varepsilon_n \leq \inf_{a \in \mathcal{A}} \exp(-\lambda\tau_n) W(z(\tau_n)).$$

Then by definition of $W(z_n)$ there is $a_n \in \mathcal{A}$ such that

$$\inf_{\mathbb{R}_+} \exp(-\lambda\tau)\varphi(z_{z_n}(\tau)) < W(z_n) + \varepsilon_n \leq \exp(-\lambda\tau_n) W(z_{z_n}(\tau_n)).$$

Using the control $a_n(\cdot + \tau_n)$, in the righthandside we then get

$$\begin{aligned}\inf_{\tau \in \mathbb{R}_+} \exp(-\lambda\tau)\varphi(z(\tau)) &< W(z_n) + \varepsilon_n \\ &\leq \exp(-\lambda\tau_n) \inf_{\tau \in \mathbb{R}_+} \exp(-\lambda\tau)\varphi(z_{z(\tau_n)}(\tau)),\end{aligned}$$

and therefore there is $s_n \in [0, \tau_n]$ such that

$$\exp(-\lambda s_n)\varphi(z(s_n)) \leq W(z_n) + \varepsilon_n.$$

Using Gronwall estimates we can prove that $|z_{z_n}(s_n; a_n) - z| \leq o(1)$ as $n \to +\infty$. By the continuity of $\varphi$ and passing to the limit as $n \to +\infty$, we conclude

$$\varphi(z) \leq W_*(z),$$

a contradiction. $\square$

From Proposition A.1, standard arguments will lead to the conclusion, see for example Ishii [8], Barles-Perthame [2] or the author [13].

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