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SOLUTION SURFACES FOR SEMILINEAR ELLIPTIC EQUATIONS ON ROTATED DOMAINS

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Abstract. For the problem

$$\begin{aligned} \Delta u + \lambda f(u) = 0 \quad & \text{in } \ \Omega, \ \ \lambda \in \mathbf{R}^+, \\ u = 0 \quad & \text{on } \ \partial\Omega, \end{aligned}$$

if Ω and f satisfy certain hypotheses, a parameterized curve of positive solutions $\alpha \mapsto (u(\alpha), \lambda(\alpha))$ has been shown to exist, where $\alpha = max_\Omega u$. If $\Omega \subset \mathbf{R}^n$ is translated by $1/\epsilon$ and then rotated about a coordinate axis to obtain a new domain $\Omega_\epsilon \subset \mathbf{R}^{n+1}$, it can be shown that a surface of positive rotationally invariant solutions $(\alpha, \epsilon) \mapsto (\hat{u}(\alpha, \epsilon), \hat{\lambda}(\alpha, \epsilon))$ exists for the resulting problem

$$\begin{aligned} \Delta_\epsilon \hat{u} + \hat{\lambda} f(\hat{u}) = 0 \quad & \text{in } \ \Omega_\epsilon, \ \hat{\lambda} \in \mathbf{R}^+ \\ \hat{u} = 0 \quad & \text{on } \ \partial\Omega_\epsilon, \end{aligned}$$

where Δ_ϵ is the Laplacian in the new variables and $(\hat{u}(\alpha, 0), \hat{\lambda}(\alpha, 0)) = (u(\alpha), \lambda(\alpha))$. From this, we can give various examples of problems on domains with a large hole for which the structure of solutions can be well described.

Introduction. Our investigation begins with some well-known results regarding the problem

$$\begin{aligned} \Delta u + \lambda f(u) = 0 \quad & \text{in } \ \Omega, \ \lambda \in \mathbf{R}^+ \\ u = 0 \quad & \text{on } \ \partial\Omega \\ u > 0 \quad & \text{in } \ \Omega. \end{aligned} \tag{1}$$

This problem has been studied in numerous papers for various domain types and nonlinearities. We will refer to some of these results below that especially

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apply to our problem. (For a good summary of results regarding solution continua generated by (1), see [10].) Our approach is by way of *dimension breaking*: We start with a domain (say Ω) in $\mathbf{R}^n$ on which certain information can be gathered regarding the solutions of (1), after which Ω is translated and rotated to obtain a new domain (say Ω') in $\mathbf{R}^{n+1}$. This is done in order to investigate the properties of solutions of (1) on Ω' as they relate to solutions on Ω. In this work our attention is focused on solutions that do not vary with the rotation of Ω; solutions that vary with rotation will be considered in a later paper.

The results obtained by this approach are restricted to domains Ω' that have undergone a large translation, or in other words, domains that have a large hole. Roughly speaking, the results will show that there is a differentiable solution *surface* that can be parameterized over $||u||_{L^\infty(\Omega')}$, λ and ϵ, the variable on which the translation of Ω depends. For a special type of nonlinearity, we can show as well that the curve continues uniformly in α for some small $\epsilon > 0$. Furthermore, it will be proven that the cross-sections of solutions on Ω' are not symmetric; to a certain degree we can describe the amount of variation from symmetry a solution has undergone after rotation of the domain. The main results as described are found in Theorems 3, 4, and 5.

The derivation of the solution surface begins with results that guarantee the existence of differentiable solution curves on a domain Ω; on the rotated domain, under proper hypotheses it will be shown that a surface of solutions results. If Ω is a ball, the solution set always gives solution *curves* instead of just continua. Properties of some interesting curves were derived by Joseph and Lundgren (in [9]) some time ago that are especially applicable to our setup. For Ω a ball domain they derive multiplicity results for radial solutions of (1) if $f(u) = e^u$, $(1-u)^{-p}$, or $(1+u)^p$. This is done by first transforming to polar coordinates, so that radial solutions of (1) satisfy the ordinary differential equation

$$\frac{d^2u}{dr^2} + \frac{n-1}{r}\frac{du}{dr} + \lambda f(u) = 0, \quad u'(0) = u(R) = 0.$$

This equation is then rescaled and phase plane analysis applied. We sketch the solution curve $\max_\Omega u =: \alpha \mapsto \lambda(\alpha)$ for $f(u) = e^u$ based on the properties derived in [9]. The shape of the graph varies depending on whether the spacial dimension n satisfies $n \leq 2$, $3 \leq n \leq 9$ or $n \geq 10$. These different cases are represented in Figures 1(a), 1(b) and 1(c), respectively.

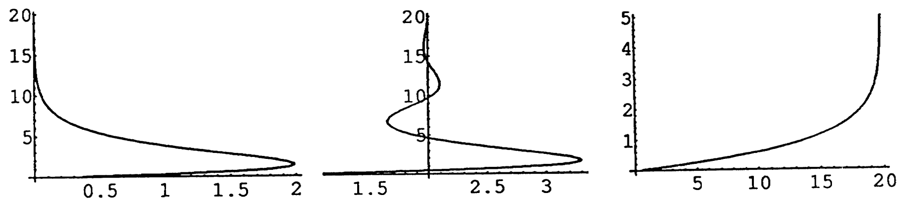

Figure 1: Solution curves for $f(u) = e^u$

An interesting case is where $f(u) = (1-u)^{-p}$ in (1). Here, the branch resembles the shape of the branch (b) in Figure 1 already for $n = 2$.

An important breakthrough came when all solutions of (1) (under appropriate assumptions on f) were shown to have the same reflection symmetries as Ω, with the maximum of u found at the center of the domain (see [5]). This implies that all solutions of (1) for Ω a ball are radially symmetric, and hence the Joseph and Lundgren-type characterization of solutions is complete for problem (1) with the given nonlinearities: no other solutions could exist on a ball domain. As we will use this result extensively, it is restated in Theorem 1 for convenience.

Recently, global uniqueness and smoothness of the branch $\{(u, \lambda)\}$ of solutions of (1) has been established on more general domain types in $\mathbf{R}^2$, along with the parameterization of these solutions over their amplitude $||u||_{L^\infty}$. This was first derived in a paper by Dancer [4], and then generalized by Holzmann and Kielhöfer [7]. The limitation is that the nodal sets of eigenfunctions have not been sufficiently described except in the case where $n = 2$. In higher dimensions, the nodal sets are much more difficult to characterize. These nodal set properties have been the key to proving the existence of the solution curves on non-radial domains, and hence the result has not yet been extended to higher dimensions. In this work we rely heavily on the Holzmann-Kielhöfer result as given in Theorem 2 for the two-dimensional case. For dimensions $n \neq 2$, we are still restricted to ball domains for the result.

1. Assumptions and foundations. As stated, the surface representing all rotationally invariant solutions on the rotated domain will be parameterized in terms of $\alpha := max_\Omega u$, and the variable that describes the distance of translation of Ω before it is rotated. Consequently, we start with an

α-dependent *base* branch of solutions of (1). By this we mean the parameterized solution curve $\alpha \mapsto (u(\alpha), \lambda(\alpha))$ of solutions of (1). To prove the existence of the surface containing *all* rotationally invariant solutions on Ω' we start with a curve that represents all solutions of (1) on Ω, then establish bifurcation from this curve. For $\Omega \subset \mathbf{R}^n$ with $n \neq 2$, a suitable base branch has only been shown to exist if Ω is a ball. However, results in [4] and [7] allow us to start with a complete base branch derived on a more general domain type in $\mathbf{R}^2$.

The assumptions needed for the domain Ω in their work are the following, which we will assume for the two-dimensional case throughout this paper:

$$\begin{aligned}
&\Omega \subset \mathbf{R}^2 \text{ is bounded and has boundary } \partial\Omega \text{ of class } C^3,\\
&\Omega \text{ is symmetric with respect to the } x- \text{ and } y- \text{ axis},\\
&\text{and } \Omega \text{ is partially convex; that is,}\\
&\text{if } (x_1, y), (x_2, y) \in \Omega \text{ then } (tx_1 + (1-t)x_2, y) \in \Omega,\\
&\text{if } (x, y_1), (x, y_2) \in \Omega \text{ then } (x, ty_1 + (1-t)y_2) \in \Omega\\
&\text{for all } t \in [0,1],\ x, x_i,\ y, y_i \in \Omega,\ i = 1, 2.
\end{aligned} \tag{2}$$

For the nonlinearity f we will assume

$$\begin{aligned}
&f\colon \mathbf{R}^+ = [0,\infty) \ \to \ \mathbf{R}^+ \text{ is in } C^2(\mathbf{R}^+, \mathbf{R}^+),\\
&\text{has at most isolated zeros, and } f(0) \geq 0.
\end{aligned} \tag{3}$$

We now state the result in [GNN] that is necessary for our setup in $\mathbf{R}^2$:

Theorem 1. **([5])** *Let* $(\lambda, u) \in \mathbf{R} \times C^{2,\mu}(\overline{\Omega})$ *be any solution of* (1). *Then* u *has the same symmetries as* Ω, *and*

$$u_x < 0 \text{ on } \{(x, y) \in \Omega : x > 0\}, \quad u_y < 0 \text{ on } \{(x, y) \in \Omega : y > 0\}.$$

Moreover, $u_\nu < 0$ *on* $\partial\Omega$, *where* u_ν *is the derivative in the direction of the outward normal unit vector* ν.

Note that Theorem 1 implies that solutions of (1) in $\mathbf{R}^2$ satisfy

$$u(-x, y) = u(x, y), \qquad u(x, -y) = u(x, y).$$

We will call a function with such symmetry *axially symmetric*. Theorem 1 applies to higher-dimensional symmetric domains as well.

Now starting with the domain $\Omega \subset \mathbf{R}^n$, we are ready to define the rotated, torus-like domain $\Omega' \subset \mathbf{R}^{n+1}$. The domain begins with a translation of $1/\epsilon$, where $\epsilon > 0$ is sufficiently small, followed by a rotation about the x_n axis. We will change notation from Ω' to Ω_ϵ to emphasize the ϵ-dependence of the rotation. The new domain is

$$\begin{aligned}\Omega_\epsilon :=&\{(x_1', \ldots, x_{n+1}') : (x_1', x_{n+1}') = (x_1 + \frac{1}{\epsilon})(\cos\theta, \sin\theta), \\ x_2' =& x_2, \ldots, x_n' = x_n, (x_1, \ldots, x_n) \in \Omega, \, 0 \le \theta < 2\pi\}.\end{aligned} \tag{4}$$

With the new domain in hand, we define the Laplacian in the new variables, which we will denote by $\tilde{\Delta}_\epsilon$. After some computation we find that

$$\tilde{\Delta}_\epsilon v = \Delta v + \frac{1}{(x_1 + \frac{1}{\epsilon})^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{(x_1 + \frac{1}{\epsilon})} \frac{\partial v}{\partial x_1}, \tag{5}$$

where, of course, Δ is the normal Laplacian in $x_1, \cdots, x_n$.

We note here that the definition of this transformed Laplacian suggests that solutions of

$$\tilde{\Delta}_\epsilon u + \lambda f(u) = 0 \;\text{ in }\; \Omega \times \mathbf{S}^1, \; \lambda \in \mathbf{R}, \qquad u|_{\partial(\Omega \times \mathbf{S}^1)} = 0$$

may be “close” to rotated solutions of (1) for small ϵ, since $\tilde{\Delta}_\epsilon$ “approaches” Δ as ϵ approaches 0. This will be made precise later for *rotationally invariant* solutions (that is, solutions where the θ derivative term is 0); in fact the differentiable parameterization of these solutions is the main focus of this paper. For this purpose we now define our problem for the transformed Laplacian in a way that reflects the rotational invariance of solutions we seek. Since the θ-derivative term vanishes in (5), we get

$$\begin{aligned}\Delta_\epsilon u + \lambda f(u) = 0 \;\text{ in }\; \Omega_\epsilon, \; \lambda \in \mathbf{R}, \\ u = 0 \;\text{ on }\; \partial\Omega_\epsilon, \qquad u > 0 \;\text{ on }\; \Omega_\epsilon,\end{aligned} \tag{6}$$

where

$$\Delta_\epsilon u := \Delta u + \frac{1}{(x_1 + \frac{1}{\epsilon})} \frac{\partial u}{\partial x_1}.$$

As mentioned before, our procedure begins with a base curve. The existence of a differentiable base branch is well-established on ball domains for many types of nonlinearities, some examples of which are given in the introduction. The base branch is well established for all nonlinearities which we consider.

In this case, the analysis only needs to be done for solutions of the ODE given there. Under assumptions (2) and (3), we are ready to state the result of [7], which guarantees solution curves for a larger class of domains in $\mathbf{R}^2$. We will use the common convention $C_0^{2,\mu}(\overline{\Omega}) = C^{2,\mu}(\overline{\Omega}) \cap \{u|_{\partial\Omega} = 0\}$, where $C^{2,\mu}$ is the usual Hölder space.

Theorem 2. ([7]) *Let f as given in* (6) *satisfy $f(r) \geq 0$ for $r \in [0, b), 0 < b \leq \infty$. Then for each $\alpha \in (0, b)$ with $f(\alpha) \neq 0$ there is a unique solution $(\lambda, u) = (\lambda(\alpha), u(\alpha)) \in \mathbf{R} \times C_0^{2,\mu}(\overline{\Omega})$ of* (1) *with amplitude $\|u\|_\infty = \alpha$. The corresponding parameter $\lambda(\alpha)$ is positive. The set $\{(\lambda(\alpha), u(\alpha)) \,|\, \alpha \in (0, b)$ where $f(\alpha) \geq 0\}$ gives all solutions of* (1) *with amplitudes $\|u\|_\infty < b$ and is composed of (at most countably many) curves of class C^1. To each maximal interval $(r_1, r_2) \subset (0, b)$ where $f(\alpha) > 0$ corresponds one maximal solution curve. If $f(r_0) = 0$ for $r_0 \in (0, b)$ and $f(\alpha) > 0$ for $\alpha \in (r_0 - \delta, r_0)$ (or $\alpha \in (r_0, r_0 + \delta)$) for some $\delta > 0$, then $\lambda(\alpha) \to +\infty$ as $\alpha \to r_0^-$ (r_0^+).*

In this paper, we are considering functions f for which $f(0) \geq 0$; the case where f is allowed to change sign and $f(0) < 0$ will be considered in a later paper.

We would like to point out that the solution curves guaranteed by this theorem have turning points as well as the ones shown in the introduction. These are the only points that give any difficulty in showing that the solution surface exists for all α, as will be seen later.

The derivation of our main result uses the implicit function theorem in a suitable setting. For this purpose, we now define the function

$$F : C_0^{2,\mu}(\overline{\Omega}) \times \mathbf{R} \times \mathbf{R} \to C^\mu(\overline{\Omega})$$

by

$$F(u, \lambda, \epsilon) := \Delta_\epsilon u + \lambda f(u). \tag{7}$$

Then if $T(\alpha) \in L(C_0^{2,\mu}(\overline{\Omega}), C^\mu(\overline{\Omega}))$ is the u-linearization of F at $\epsilon = 0$, $\lambda = \lambda(\alpha)$, and $u = u(\alpha)$,

$$T(\alpha)v = F_u(u, \lambda, 0)v = \Delta v + \lambda f'(u)v. \tag{8}$$

With these preliminaries, the main theorem of this paper (Theorem 3) can be stated.

Let Ω *satisfy the conditions given in* (2)*, or be a ball domain if* $\Omega \subset \mathbf{R}^n$ *for* $n \geq 3$*, and let* f *satisfy* (3)*. Then all positive solutions of* (6) *near the base curve form a two-dimensional surface* $(\alpha, \epsilon) \mapsto (u(\alpha, \epsilon), \lambda(\alpha, \epsilon))$*, that is, continuously differentiable in* α *and* ϵ*. Moreover,* $max_{\Omega_\epsilon} u(\alpha, \epsilon) = \alpha$*, and* $\frac{\partial \lambda}{\partial \epsilon}(\alpha, 0) = 0$.

Theorem 2 and its analogue for ball domains establish the existence of solution curves for $\epsilon = 0$. We will show that a solution *surface* is obtained that emanates from the $\epsilon = 0$ curve. From this point on, we will use heavily the fact that solutions (u, λ) of (1) are α-dependent, where $\alpha = max_\Omega u$, and that the curve $\alpha \mapsto (u(\alpha), \lambda(\alpha))$ is continuously differentiable for $f \in C^2$.

2. Some facts on nodal sets of eigenfunctions and a fundamental lemma for $\Omega \subset \mathbf{R}^2$. As we have mentioned, difficulty in establishing the existence of the surface occurs mainly at the turning points. As we shall see, the reason this is so is that $T(\alpha)$ has a nontrivial kernel there. Hence, Theorem 3 will follow a sequence of lemmas that establish the properties of functions in the kernel of $T(\alpha)$. All results in this section will be for the case $\Omega \subset \mathbf{R}^2$. We first need to know something of the nature of nodal sets of solutions of the linearized equation on a domain Ω satisfying the assumptions in (2). These properties are given in [7]; we restate them here for convenience.

The results refer to nodal sets and nodal domains of nontrivial solutions $v \in C^2(\overline{\Omega})$ of a linear problem

$$\Delta v + cv = 0 \text{ in } \Omega, \tag{9}$$

where $c : \overline{\Omega} \mapsto \mathbf{R}$ is a C^1 function, and are of interest to us here because of the linearization (8). (Actually, for the following results the symmetry of Ω is not necessary.) We define

$$N_v := \{(x, y) \in \overline{\Omega} \mid v(x, y) = 0\} \text{ is the nodal set and}$$
$$\text{any component in } \Omega \setminus N_v \text{ is a nodal domain of } v.$$

There are two facts that will be used to establish results in the lemmas. These are stated in the following propositions, where in each it is assumed that v is a solution of (9).

Proposition 1. *The set* $N_v \cap \Omega$ *consists of smooth curves which intersect equiangularly at singular points which do not cluster in the interior of* Ω*.*

(Here, a singular point $(x, y) \in \Omega$ satisfies $v(x, y) = 0$ and $\nabla v(x, y) = \mathbf{0}$.) For the arguments which prove Proposition 1, see [8] and [1]. We also have the following facts about N_v at the boundary of Ω:

Proposition 2. *Let $(x_0, y_0) \in N_v \cap \Omega$. Then (x_0, y_0) is on a maximal smooth curve $\Gamma \subset N_v$ of finite length which is either totally in Ω and closed or meets the boundary $\partial\Omega$ in two well-defined points. In the second case it is assumed that $v = 0$ on $\partial\Omega$, whereas in the first case the behavior of v on $\partial\Omega$ plays no role.*

The proof of the main lemma of this section (Lemma 2) will be simplified if we eliminate the possibility of a nodal domain of a solution of the linearization lying entirely on one side of either the x or the y axis. Based on this goal, we define subsets of Ω with respect to the coordinate axes as follows: $\Omega_x^+ := \{x \in \Omega \,|\, x > 0\}$, $\Omega_x^- := \{x \in \Omega \,|\, x < 0\}$, with similar definitions for Ω_y^+ and Ω_y^-. Furthermore, we note that for f satisfying (3), Theorem 1 guarantees that a solution u to (1) satisfies $u_x < 0$ on $(\partial\Omega \setminus \{(x, y) \in \partial\Omega \,|\, x \neq 0\}) \cap \partial\Omega_x^+$ and $u_x > 0$ on $(\partial\Omega \setminus \{(x, y) \in \partial\Omega \,|\, x \neq 0\}) \cap \partial\Omega_x^-$, with similar (strict) inequalities holding for u_y. These facts will be used in the proof of the following lemma.

Lemma 1. *Let Ω satisfy (2), and let v be a nontrivial solution to $T(\alpha)v = 0$. Then v has no nodal domain contained entirely in Ω_x^+, Ω_x^-, Ω_y^+, or Ω_y^-.*

Proof. Let v be a nontrivial element of $\ker(T(\alpha))$, and suppose without loss of generality that v has a nodal domain D lying entirely in Ω_x^+. Let $u = u(\alpha)$ be a solution to (1). Then we get the two sets of equations

$$\begin{aligned} \Delta v + \lambda f'(u)v = 0 \ \text{ in } D \\ v = 0 \ \text{ on } \partial D \end{aligned} \tag{10}$$

and

$$\begin{aligned} \Delta u_x + \lambda f'(u)u_x = 0 \ \text{ in } D \\ u_x \leq 0 \ \text{ on } \partial D, \end{aligned} \tag{11}$$

where the second is obtained by differentiating $\Delta u + \lambda f(u) = 0$ with respect to x, and the inequality for u_x on ∂D follows from Theorem 1. Multiplying (10) by u_x and (11) by $\hat{v}$ and subtracting, an application of Green's theorem gives

$$0 = \int_D (u_x \Delta v - v \Delta u_x) d(x, y) = \int_{\partial D} (u_x \partial_\eta v - v \partial_\eta u_x) \; dS = \int_{\partial D} u_x \partial_\eta v \; dS.$$

However, by the maximum principle $\partial_\eta v$ is one sign on ∂D. Furthermore, by comments preceding Lemma 1, whether part of ∂D coincides with $\partial\Omega$ or not, $u_x \leq 0$ on ∂D by Theorem 1, and is only equal to zero where ∂D coincides with the y axis. Hence we arrive at a contradiction. An analogous argument gives the result for the other subsets of Ω, so the result is established.

We will need information about the boundary behavior of the normal derivatives of a solution v to $T(\alpha)v = 0$. Under the hypothesis of axial symmetry of Ω, we will eliminate the possibility of the existence of a nodal set meeting the boundary of Ω in two well-defined points as given as an alternative in Proposition 2. This is done in

Lemma 2. *Let Ω satisfy* (2). *Then a nontrivial solution v to $T(\alpha)v = 0$ is axially symmetric. If η is the outward normal function of Ω defined on $\partial\Omega$, then either $\partial_\eta v < 0$ or $\partial_\eta v > 0$ on $\partial\Omega$. Furthermore, for each α, dim*: $\ker(T(\alpha)) \leq 1$.

Note: The basic idea for the second part of this lemma is found in [7].

Proof. To see that v is axially symmetric, assume the contrary. Then without loss of generality, we can assume that $v(x,y) \neq v(-x,y)$. But $0 \neq \bar{v}(x,y) := v(x,y) - v(-x,y) \in \ker(T(\alpha))$, and $\bar{v}$ is odd in x. But $\bar{v}(0,y) = 0$ and $\bar{v} = 0$ on $\partial\Omega$ taken together contradict Lemma 1. This establishes the axial symmetry of v.

We next turn to the question of the boundary behavior of v. Using a result in [3], there are at most finitely many nodal domains composing $\Omega \setminus N_v$. If $N_v \setminus \partial\Omega$ consisted entirely of closed curves (of finite length) that did not intersect $\partial\Omega$, the maximum principle would give us the second part of the result. But (in light of Proposition 2) if we were to suppose that there is a curve $\Gamma \subset N_v$ that meets $\partial\Omega$ in two well-defined points, the axial symmetry of v would imply the existence of a nodal domain entirely in Ω_x^+, contradicting Lemma 1. So N_v is entirely in Ω and closed, and an application of the maximum principle as previously stated gives the second part of the lemma.

Finally, suppose that the dimension of $\ker(T(\alpha))$ is greater than 1, then there exist linearly independent functions v_1 and v_2 in $\ker(T(\alpha))$. We claim that there exists a function in $\ker(T(\alpha))$ that is zero at $(0,0)$. For, if neither v_1 nor v_2 is zero at $(0,0)$, we can find (c_1, c_2) orthogonal (in $\mathbf{R}^2$) to $(v_1(0,0), v_2(0,0))$ so that $c_1v_1 + c_2v_2 \in \ker(T(\alpha))$ *is* zero at the origin. Let $\tilde{v} \in \ker(T(\alpha))$ be such that $\tilde{v}(0,0) = 0$. Since the point $(0,0)$ cannot be an isolated zero of $\tilde{v}$ by the maximum principle, invoking the symmetry of

$\tilde{v}$ established earlier, there must exist a nodal domain on one half of Ω, contradicting Lemma 1. This proves the lemma.

We note here that there are α values for which $\ker(T(\alpha))$ is nontrivial. To see this, suppose that α satisfies $\lambda'(\alpha) = 0$. (Notice that the solution curves in Figure 1 has such values. Some of the curves given in Theorem 2 do as well, as we commented earlier.) Differentiating (1) with respect to α (where $'$ means $\frac{d}{d\alpha}$) we have

$$\Delta u'(\alpha) + \lambda(\alpha) f'(u(\alpha)) u'(\alpha) = -\lambda'(\alpha) f(u(\alpha)) = 0;$$

that is, at the values of α where $\lambda'(\alpha) = 0$, $u'(\alpha)$ solves $T(\alpha)u'(\alpha) = 0$. Since $u(\alpha)$ is a solution to (1), and $h(\alpha) := u(\alpha)(0,0) = \alpha$, $h'(\alpha) = u'(\alpha)(0,0) = 1$, showing that nontrivial solutions exist if the solution curve has α values where $\lambda'(\alpha) = 0$. Later we will see that such values are the *only* ones for which $\ker(T(\alpha))$ is nontrivial.

3. The case where Ω is a ball. We will first show that any solution of the linearized equation is radially symmetric in case Ω is a ball, as well as establishing the nonzero boundary condition as in Lemma 2 that will be important for the proof of Lemma 4. We give these results in

Lemma 3. *Let Ω be a ball domain. If $v \in \ker(T(\alpha))$ is nontrivial, then v is radially symmetric and $\partial_\eta v \neq 0$ on $\partial\Omega$.*

Proof. If v is not radially symmetric, there exists a vector τ such that $v \neq v_\tau$, where v_τ is the reflection of v in the plane $\langle x, \tau \rangle = 0$. Since Δ is equivariant with respect to reflection and rotation, and $f'(u(\alpha))$ is symmetric, we can assume that $\tau = e_1 = (1, 0, 0, \dots)$ without restriction. Then we get

$$\hat{v}(x) := v(x_1, x^*) - v(-x_1, x^*) \neq 0,$$

but $\hat{v} \in \ker(T(\alpha))$ with $\hat{v}(x_1, x^*) = -\hat{v}(-x_1, x^*)$. It follows that $\hat{v}(0, x^*) = 0$, so 0 is an eigenvalue of $T(\alpha)$ on the domain $\Omega^+ := \{x \in \Omega \mid x_1 = \langle x, e_1 \rangle > 0\}$. Let $\dots \leq \mu_3 \leq \mu_2 < \mu_1 < \infty$ be the eigenvalues of $T(\alpha)$ on Ω^+. Then it follows that $\mu_1 \geq 0$. Let v^+ be the eigenfunction corresponding to μ_1 for $T(\alpha)$ on Ω^+. Then by the Krein-Rutman theorem (see [11]), $v^+ > 0$ on Ω^+, and the reflection

$$v_r := \begin{cases} v^+(x_1, x^*) & \text{if } x_1 \geq 0 \\ -v^+(-x_1, x^*) & \text{if } x_1 < 0 \end{cases}$$

is an eigenfunction of $T(\alpha)$ with eigenvalue $\mu_1 \geq 0$.

Now by symmetry,

$$\partial_1 u(\alpha)(x_1, x^*) = -\partial_1 u(\alpha)(-x_1, x^*),$$

and $u_1 := \partial_1 u(\alpha)$ solves the *equation*

$$\Delta u_1 + \lambda(\alpha) f'(u(\alpha)) u_1 = 0 \tag{12}$$

(but not boundary conditions). Multiplying (12) by v_r and integrating by parts, we get

$$0 = -\int_{\partial\Omega} u_1 \partial_\eta v_r + \mu_1 \int_\Omega u_1 v_r = -2\int_{\Gamma^+} u_1 \partial_\eta v_r + 2\mu_1 \int_{\Omega^+} u_1 v_r$$

since both u_1 and v_r are antisymmetric, where $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$ and η is the outward normal vector function to Ω. Because $v_r > 0$ on Ω^+, we get that $\partial_\eta v_r < 0$ on Γ^+ by the strong maximum principle. Then by Theorem 1, $u_1 < 0$ on $\Omega^+ \cup \Gamma^+$, so

$$2\mu_1 \int_{\Omega^+} u_1 v_r = 2\int_{\Gamma^+} u_1 \partial_\eta v_r$$

implies that $\mu_1 < 0$, which gives a contradiction. Hence v is radial.

If we assume that $\partial_\eta v = 0$ on $\partial\Omega$, then using the radial symmetry of v, changing over to $r = |x|$, the equation $T(\alpha)v = 0$ becomes a second-order linear ODE with initial and boundary conditions $v'(0) = 0$, $v(R) = 0$ and $v'(R) = 0$, contradicting the nontrivial nature of v. This proves Lemma 3.

4. The kernel of the linearization. To apply the implicit function theorem in the proof of the main theorem, we need some more information about the linearization $T(\alpha)$ and its kernel. Before giving the proof, we recall that both in Figure 1 and in Figure 2, there are *turning points*, or points at which $\lambda'(\alpha) = 0$. These points end up giving the difficulty in showing that the solution surface exists, since at the turning points the kernel of $T(\alpha)$ is nontrivial. It turns out that these are the only points at which it is so; we state this result in

Lemma 4. $\ker(T(\alpha)) \neq \{0\}$ *if and only if* $\lambda'(\alpha) = 0$. *If* $\lambda'(\alpha) = 0$, *then* $\ker(T(\alpha)) = \mathrm{sp}\{u'(\alpha)\}$, *the span of* $u'(\alpha)$.

Proof. The key to the proof is the nonzero boundary behavior of solutions of the linearized equation established in Lemmas 2 and 3. So we are treating the axial case in $\mathbf{R}^2$ as well as the radial case in any dimension here.

Suppose first that $\lambda'(\alpha) = 0$. Then the comments following the proof of Lemma 2 establish that $\ker(T(\alpha)) \neq \{0\}$.

Now let $\mathbf{x} = (x_1, \dots, x_n)$, and let $u_1(\mathbf{x}) := \nabla u(\alpha)(\mathbf{x}) \cdot \mathbf{x}$; then u_1 is also symmetric. Then (replacing $u(\alpha)$ and $\lambda(\alpha)$ by u and λ)

$$\Delta u_1 + \lambda f'(u) u_1 = -2\lambda f(u).$$

Multiplying by v and integrating by parts gives

$$-\int_{\partial\Omega} u_1 \partial_\nu v = -2\lambda \int_\Omega f(u) v.$$

But by Lemmas 2 and 3, $\partial_\nu v$ does not change sign on $\partial\Omega$. Then $u_1 < 0$ on $\partial\Omega$ gives

$$\int_\Omega f(u) v \neq 0.$$

On the other hand, $T(\alpha)u'(\alpha) = -\lambda'(\alpha) f(u(\alpha))$ implies

$$\lambda'(\alpha) \int_\Omega f(u(\alpha)) v = 0.$$

This completes the proof of the first part of the lemma. The second part follows from Lemma 2 and the comments following it.

5. The existence of the solution surface. Under the same assumptions as in the previous sections, we have that

$$T(\alpha) \text{ is a bounded Fredholm operator of index 0.} \tag{13}$$

(This can be shown using the compactness of Δ^{-1}, and the arguments are summarized in [6]). With this in hand, we are ready to prove the main theorem.

Theorem 3. *Let Ω satisfy the conditions given in* (2) *for $\Omega \subset \mathbf{R}^2$ or be a ball domain in $\mathbf{R}^n$ for any n. Let f be given by* (3). *Then all positive solutions of* (6) *near the base curve form a two-dimensional surface*

$$(\alpha, \epsilon) \mapsto (u(\alpha, \epsilon), \lambda(\alpha, \epsilon))$$

that is continuously differentiable in α and ϵ. Any continuum of positive solutions that stays in a bounded α-interval as $\epsilon \to 0$ is on this surface. Moreover, $max_{\Omega_\epsilon} u(\alpha, \epsilon) = \alpha$, and $\frac{\partial \lambda}{\partial \epsilon}(\alpha, 0) = 0$ for any α.

Proof. Throughout, we will let the base curve be represented by $\alpha \mapsto (u_0(\alpha), \lambda_0(\alpha))$; that is, for each α, $\{u_0(\alpha), \lambda_0(\alpha)\}$ solves (1). The proof proceeds via the implicit function theorem. To apply this, we want to verify that the kernel of the total derivative operator is two-dimensional. For this purpose, remembering that

$$F(u, \lambda, \epsilon) = \Delta u + \frac{1}{(x_1 + \frac{1}{\epsilon})} u_{x_1} + \lambda f(u),$$

let

$$\begin{aligned} G(u, \lambda, \epsilon) &:= DF(u_0(\alpha), \lambda_0(\alpha), 0)(u, \lambda, \epsilon) \\ &= F_u(u_0(\alpha), \lambda_0(\alpha), 0)u + F_\lambda(u_0(\alpha), \lambda_0(\alpha), 0)\lambda + F_\epsilon(u_0(\alpha), \lambda_0(\alpha), 0)\epsilon \\ &= \Delta u + \lambda_0(\alpha) f'(u_0(\alpha))u + \lambda f(u_0(\alpha)) + \epsilon u_0(\alpha)_{x_1} \\ &= T(\alpha)u + \lambda f(u_0(\alpha)) + \epsilon u_0(\alpha)_{x_1}. \end{aligned}$$

Differentiating the equation

$$\Delta u_0(\alpha) + \lambda_0(\alpha) f(u_0(\alpha)) = 0$$

with respect to α (we now use $\cdot = \frac{d}{d\alpha}$) shows that $(\dot{u}_0, \dot{\lambda}_0, 0) \in \ker(G)$ for any α. Lemma 4 shows that no more dimensions are added to $\ker(G)$ by any other triplet of the form $(v, 0, 0)$. We now claim that there is exactly one more dimension to $\ker(G)$, and this is generated by $(\hat{u}, 0, 1)$; that is, $\hat{u}$ is the unique solution to the equation

$$T(\alpha)\hat{u} + u_0(\alpha)_{x_1} = 0.$$

If $T(\alpha)$ is invertible, then on the orthogonal space

$$X := \{w \in C_0^{2,\mu}(\overline{\Omega}) : \int_\Omega w \dot{u}_0 = 0\}$$

of $\dot{u}$ index L^2, it is clear that this extra kernel element is the only extra addition to $\ker(G)$, hence G is a bounded Fredholm operator of index two.

We show that the same is true for α values where $\lambda'(\alpha) = 0$. Suppose that α satisfies this condition. Since the kernel is two-dimensional if

$$f(u_0) \notin R(T(\alpha)) \text{ but } \partial_{x_1} u_0 \in R(T(\alpha)), \tag{14}$$

(where $R(A)$ means the range of the operator A), then by orthogonality, the conditions on $f(u_0)$ and $\partial_{x_1} u_0$ as given in (14) are equivalent to the conditions

$$\int_\Omega f(u_0)\dot{u}_0 \neq 0 \text{ and } \int_\Omega \partial_{x_1} u_0 \dot{u}_0 = 0. \tag{15}$$

Hence, establishing the properties in (15) will show that the kernel of G is indeed two-dimensional in either case. That the second integral in (15) is zero is clear from Theorem 1 together with Lemmas 2 and 3.

The proof that the integral in question is nonzero in case $\dot{\lambda}(\alpha) = 0$ proceeds via the Pohozaev identity (see [10]). Recall that this identity states that solutions $u_0 = u_0(\alpha)$, $\lambda_0 = \lambda_0(\alpha)$ of

$$\begin{aligned} \Delta u_0 + \lambda_0 f(u_0) = 0 \text{ in } \Omega \subset \mathbf{R}^n \\ u_0 = 0 \text{ on } \partial\Omega \end{aligned}$$

satisfy the integral identity

$$\begin{aligned} 2n\lambda_0(\alpha) \int_\Omega \int_0^{u_0(\alpha)} f(s)\, ds\, dx - (n-2)\lambda_0(\alpha) \int_\Omega f(u_0(\alpha))u_0(\alpha)\, dx \\ = \int_{\partial\Omega} |\nabla u_0(\alpha)|^2 (x \cdot \eta)\, dS. \end{aligned} \tag{16}$$

Differentiating both sides of (16) with respect to α, and remembering that u_0 and λ_0 depend on α, we get that

$$\begin{aligned} 2n\lambda_0 \int_\Omega f(u_0)\dot{u}_0\, dx - (n-2)\lambda_0 \int_\Omega (f(u_0)\dot{u}_0 + f'(u_0)u_0\dot{u}_0)\, dx \\ = 2\int_{\partial\Omega} (\nabla u_0 \cdot \nabla \dot{u}_0)(x \cdot \eta)\, dS. \end{aligned} \tag{17}$$

Applying the identity $-|\nabla u_0|\eta = \nabla u_0$ and collecting terms in (17) yields

$$\lambda_0(2n-(n-2))\int_\Omega f(u_0)\dot{u}_0\,dx = -\int_{\partial\Omega}|\nabla u_0|(\nabla\dot{u}_0\cdot\eta)(x\cdot\eta)\,dS + \lambda_0(n-2)\int_\Omega f'(u_0)u_0\dot{u}_0\,dx. \tag{18}$$

Assuming that the left-hand integral in (15) *is* zero, we see that both sides of (18) are zero. But since each of $|\nabla u_0| > 0$ and $(x\cdot\eta) > 0$ on $\partial\Omega$, and by Lemmas 2 and 3 $\nabla\dot{u}_0\cdot\eta = \partial_\eta\dot{u}_0$ does not change sign on $\partial\Omega$, the middle integral in (18) is nonzero. This completes the proof for the $n = 2$ case. If $n \neq 2$, we have that the right-most integral in (18) is nonzero as well as the middle since the sum is zero. But then

$$0 \neq \int_\Omega f'(u_0)u_0\dot{u}_0\,dx = -\int_\Omega \Delta\dot{u}_0 u_0\,dx = -\int_\Omega \Delta u_0\dot{u}_0\,dx = \int_\Omega f(u_0)\dot{u}_0\,dx,$$

a contradiction. The proof in one dimension is a trivial generalization of this case, so we omit it. This establishes (15).

We have already seen that G is a bounded Fredholm operator of index two if $T(\alpha)$ is invertible. We claim that (15) gives that G is a bounded Fredholm operator of index two also in case $T(\alpha)$ is not invertible. This follows from (13) and (15) in the following way. Recalling that X is the orthogonal space of $\dot{u}$ in L^2, the only question becomes whether we can find a triplet (u,λ,ϵ) that solves $G(u,\lambda,\epsilon) = v$ for $v \notin X$. But using (13) and (15), we see that $v = kf(u_0)$ for some k, so the triplet $(\hat{u},k,1)$ (where $\hat{u}$ is defined above) solves the equation. This shows that the deficiency of G is zero again; hence G has Fredholm index two regardless of the invertibility of $T(\alpha)$.

With this result in hand, remembering that we are considering a point $(u_0(\alpha),\lambda_0(\alpha))$ on the solution curve of (1) where $\dot{\lambda}(\alpha) = 0$, we can write any $(u,\lambda,\epsilon) \in C_0^{2,\mu}(\overline{\Omega}) \times \mathbf{R} \times \mathbf{R}$ uniquely as

$$(u,\lambda,\epsilon) = (u_0,\lambda_0,0) + \beta(\dot{u}_0,\dot{\lambda}_0,0) + \epsilon(\hat{u},0,1) + (w,\mu,0), \tag{19}$$

where β and ϵ are real parameters and $w \in X$ where X is defined above. Now let $\hat{G} : \mathbf{R}\times\mathbf{R}\times X\times\mathbf{R} \to C^\mu(\overline{\Omega})$ be given by

$$\hat{G}(\beta,\epsilon,w,\mu) := G(u,\lambda,\epsilon) = G(u_0+\beta\dot{u}_0+\epsilon\hat{u}+w,\lambda_0+\beta\dot{\lambda}_0+\mu,\epsilon).$$

Then $\hat{G}(0,0,0,0) = 0$ and

$$\hat{G}_{(w,\mu)}(0,0,0,0)(w,\mu) = T(\alpha)w + \mu f(u_0),$$

which is invertible in $X \times \mathbf{R}$ as we have already seen. Then using the implicit function theorem, we have locally that w and μ are C^1-functions of β and ϵ on the solution curve of (6) near (0,0), with $w(0,0) = 0$ and $\mu(0,0) = 0$. But then u and λ are C^1 functions of β and ϵ as well near $(u_0(\alpha), \lambda_0(\alpha))$, with $u(0,0) = u_0(\alpha)$ and $\lambda(0,0) = \lambda_0(\alpha)$.

We next show that the parameter β can be replaced by α. We denote

$$u(\beta,\epsilon) := u_0 + \beta\dot{u}_0 + \epsilon\hat{u} + w(\beta,\epsilon),$$
$$\lambda(\beta,\epsilon) := \lambda_0 + \beta\dot{\lambda}_0 + \mu(\beta,\epsilon),$$

and we claim that $\frac{\partial w}{\partial \beta}(\beta,\epsilon) = \frac{\partial w}{\partial \epsilon}(\beta,\epsilon) = 0$ in a neighborhood of $(\beta,\epsilon) = (0,0)$. The implicit function theorem has shown for $\mathbf{a} := (\beta,\epsilon)$ and $\mathbf{b} := (w,\mu)$, in a neighborhood U of $\mathbf{a} = \mathbf{0}$, that $\mathbf{b} = \mathbf{B}(\mathbf{a})$, where $\mathbf{B}$ is of class C^1, and $\hat{G}(\mathbf{a}, \mathbf{B}(\mathbf{a})) = 0$. Then also $\hat{G}_{\mathbf{a}} + \hat{G}_{\mathbf{b}}\mathbf{B}_{\mathbf{a}} = 0$ in U. But $\hat{G}_{\mathbf{a}}$ is zero in U since it gives the kernel elements of G. But this implies that $\mathbf{B}_{\mathbf{a}} = 0$ in U. This is exactly the desired result on the derivatives of w, so we get that $\frac{\partial u}{\partial \beta}(\mathbf{a}) = \dot{u}_0$ in U. In the same way, we see that $\frac{\partial \mu}{\partial \epsilon}(\mathbf{a}) = 0$ locally, which yields $\frac{\partial \lambda}{\partial \epsilon}(\mathbf{a}) = 0$ as well.

Now let

$$g(\beta,\epsilon) := u(\beta,\epsilon)(\mathbf{m}) = \alpha,$$

where $\mathbf{m} \in \Omega$ is the point at which $max_\Omega u(\beta,\epsilon)$ is obtained. We claim that this point is unique, which will be the case if the maximum is nondegenerate, or if $u(\beta,0)$ has a unique nondegenerate maximum. To see that this is true, remembering that $u(\beta,0)$ solves (1), we suppose without loss of generality that the maximum is degenerate in the x_1 direction. Differentiating (1) with respect to x_1 then gives

$$\Delta\partial_1 u(\beta,0) + \lambda f'(u(\beta,0))\partial_1 u(\beta,0) = 0 \text{ on } \Omega^+ := \{\mathbf{x} \in \Omega \,|\, x_1 > 0\},$$
$$\partial_1 u(\beta,0) < 0 \text{ on } \partial\Omega^+,$$

where $\partial_1 u(\beta,0)$ has a maximum along the x_1-axis. But then using the strong maximum principle, $\partial_\nu \partial_1 u(\beta,0) > 0$ there, contradicting the degeneracy.

Since $u(\beta,0)$ has a unique nondegenerate maximum at $\mathbf{0}$, this gives more: the matrix of second partial derivatives is invertible, so that by the implicit function theorem $\mathbf{m}$ is a C^1-function of β and ϵ as well, with $\mathbf{m}(0,0) = \mathbf{0}$. Then differentiating $g(\beta,\epsilon) := u(\beta,\epsilon)(\mathbf{m}(\beta,\epsilon))$ with respect to β and evaluating at $\epsilon = \beta = 0$ gives

$$\begin{aligned}\frac{\partial g}{\partial \beta}(0,0) &= \frac{\partial u}{\partial \beta}(0,0)(\mathbf{m}(0,0)) + \nabla u(0,0)(\mathbf{m}(0,0)) \cdot \frac{\partial \mathbf{m}}{\partial \beta} \\ &= \frac{\partial u}{\partial \beta}(0,0)(\mathbf{m}(0,0)) = \dot{u}_0(\mathbf{m}(0,0)) = 1\end{aligned}$$

using the comments following Lemma 2. By the continuity of $\partial g/\partial\beta$, this implies the existence of an ϵ neighborhood about 0 for which $\partial g/\partial \beta \neq 0$. Hence g is invertible and β is everywhere locally a function of α.

We now show that for sufficiently small ϵ, all positive solutions to (6) that stay in a bounded α region as $\epsilon \to 0$ are found on this solution surface. This follows directly from compactness arguments. Suppose there exists a sequence of points (α_n, ϵ_n) such that $\alpha_n \in [\alpha_1, \alpha_2]$ for all n, $\epsilon_n \to 0$ as $n \to \infty$, and $(u(\alpha_n,\epsilon_n), \lambda(\alpha_n,\epsilon_n))$ is a solution to (6) for each n. Now let $u_n := u(\alpha_n,\epsilon_n)$, $\lambda_n := \lambda(\alpha_n,\epsilon_n)$. Then (u_n,λ_n) is a bounded sequence in $L^\infty \times \mathbf{R}$. Now for each n,

$$\Delta_\epsilon u_n + \lambda_n f(u_n) = \Delta u_n + \frac{1}{x_1 + \frac{1}{\epsilon}} \partial_{x_1} u_n + \lambda_n f(u_n) = 0.$$

Since Δ is invertible, so is Δ_ϵ for small ϵ by the maximum principle. By standard elliptic regularity theory, we get that $||\Delta_\epsilon^{-1}||_* \leq c < \infty$ for some c independent of ϵ (where $||\cdot||_*$ is the $L(L^q, W^{2,q})$-norm) for arbitrary q. Then there exists a subsequence of (u_n,λ_n) that converges in C^1 to a solution of (6) as $\epsilon_n \to 0$. Since all solutions for $\epsilon = 0$ are found on the base curve by hypothesis, the sequence after some $n = N$ must be on the surface. This completes the proof of Theorem 3.

We have depicted the solution surface in the Figure 2.

6. Continuation on the surface for nonzero ϵ. A question that naturally arises in this analysis is whether the curve continues in α for some fixed $\epsilon > 0$. It turns out that this question can be answered depending on the nonlinearity f: If f is linear at infinity with bounded second derivative,

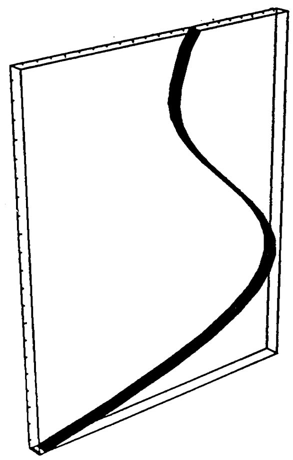

Figure 2. Depiction of the solution surface for $f(u) = (1-u)^3$

a curve of positive solutions on the solution surface can be continued for any ϵ in a region about 0 for all α. We state this result as a theorem.

Theorem 4. *Let the nonlinear function f in (6) satisfy $f'(u) \to c \in \mathbf{R}$ as $u \to \infty$, and suppose that f'' is uniformly bounded. Then there exists an $\bar{\epsilon} > 0$ such that for all $\epsilon_0 \in (-\bar{\epsilon}, \bar{\epsilon})$, the curve $\alpha \mapsto u(\alpha, \epsilon_0)$ exists for all $\alpha > 0$.*

Proof. We consider the operator $F_{(u,\lambda)}(u_0, \lambda_0, \epsilon_0)$. We need to show this operator is invertible in a uniform neighborhood of $(u_0, \lambda_0, 0)$ independent of α in the space $X \times \mathbf{R}$, where X is defined in the proof of Theorem 3. Using the mean value theorem and Taylor expansion, with our assumption on f'' it is enough to show that $L_\alpha := F_{(u,\lambda)}(u_0, \lambda_0, 0)$ has a uniformly bounded inverse in $X \times \mathbf{R}$. Recall that L_α is given by

$$L_\alpha(w, \mu) = \Delta w + \lambda_0 f'(u_0) w + \mu f(u_0);$$

we have already seen that L_α is invertible in $X \times \mathbf{R}$ in the proof of Theorem 3. Showing that its inverse is uniformly bounded is equivalent to showing that

$$\inf_{||(w,\mu)||_C = 1} ||L_\alpha(w, \mu)||_{L_\infty} \geq k > 0,$$

where $|| \cdot ||_C$ is the norm in $C^1 \times \mathbf{R}$.

Using bifurcation from infinity results as in [10], we get

$$\frac{u_0(\alpha,0)}{\alpha} \to \phi_1 \text{ where } \Delta\phi_1 + \sigma_1\phi_1 = 0,$$
$$\lambda_0(\alpha,0) \to \frac{\sigma_1}{c}, \text{ and } \dot{u}(\alpha,0) \to \phi_1 \text{ by L'Hôpital, so } \frac{f(u_0(\alpha))}{\alpha} \to c\phi_1$$

as $\alpha \to \infty$, where convergence is C^1 where applicable. Here, σ_1 is the first eigenvalue of $-\Delta$ on Ω. Then it makes sense to define $T(\infty)\cdot := \Delta \cdot + \sigma_1 \cdot$, since $T(\alpha_n)\phi_k \to T(\infty)\phi_k$ in L^∞ for any eigenfunction ϕ_k of $-\Delta$.

Now assume by way of contradiction that there are sequences (α_n) and (w_n, μ_n) such that $\alpha_n \to \infty$, (w_n, μ_n) satisfies $||(w_n, \mu_n)||_C = 1$, while $L_{\alpha_n}(w_n, \mu_n) \to 0$ as $n \to \infty$. Then we have

$$\begin{aligned}
&\langle L_{\alpha_n}(w_n,\mu_n),\phi_1\rangle \\
&= \langle T(\alpha_n)w_n,\phi_1\rangle + \alpha_n\mu_n\langle\frac{f(u_0(\alpha_n))}{\alpha_n},\phi_1\rangle \\
&= \langle w_n, T(\alpha_n)\phi_1\rangle + \alpha_n\mu_n\langle\frac{f(u_0(\alpha_n))}{\alpha_n},\phi_1\rangle \\
&= \langle w_n,(T(\alpha_n)-T(\infty))\phi_1\rangle + \langle w_n,T(\infty)\phi_1\rangle + \alpha_n\mu_n\langle\frac{f(u_0(\alpha_n))}{\alpha_n},\phi_1\rangle.
\end{aligned} \tag{20}$$

Since $\langle L_{\alpha_n}(w_n,\mu_n),\phi_1\rangle \to 0$ as $\alpha_n \to \infty$, as well as the first two terms on the right-hand side of (20), we have that $\mu_n\alpha_n \to 0$. We now replace ϕ_1 with any other eigenvalue of $-\Delta$ on Ω. Since $\langle L_{\alpha_n}(w_n,\mu_n),\phi_1\rangle$, $(T(\alpha_n)-T(\infty))\phi_k$, and $\mu_n\alpha_n$ go to zero still as $n \to \infty$, we have that $\langle w_n, T(\infty)\phi_k\rangle \to 0$ as well. Since $T(\infty)\phi_k = \Delta\phi_k + \sigma_1\phi_k = (-\sigma_k+\sigma_1)\phi_k$, and since all other eigenvalues are bounded away from σ_1, this implies that $\langle w_n,\phi_k\rangle \to 0$ as $n \to \infty$. Then using the fact that w_n is bounded in C^1 together with the L^p-convergence of w_n to zero, we get that $w_n \to 0$ in C^1. But we have already seen that $\mu_n \to 0$, contradicting $||(w_n,\mu_n)||_C = 1$. Hence the constant k exists as desired.

Example. Here we do an example of where the uniqueness of rotationally invariant solutions can be established on a torus by our method. Let $f(u) = 1 + u$ in (1), and let Ω be a ball, say $\Omega = B_1(0)$. Setting $v = 1 + u$, problem (1) becomes

$$\begin{aligned}
\Delta v + \lambda v = 0 \text{ in } B_1(0), \quad \lambda \in \mathbf{R}^+ \\
v = 1 \text{ on } \partial B_1(0), \quad v > 1 \text{ in } B_1(0).
\end{aligned}$$

Now v corresponds to a rescaling of the first eigenfunction ϕ_1 of $-\Delta$ on $B_R(0)$ for some fixed $R > 1$. Let $d > 1$ and $0 < c < R$. We can write

$$v(\mathbf{x}) = d\phi_1(c\mathbf{x})|_{B_1(0)} \text{ where } v(\mathbf{x})|_{\partial B_1(0)} = d\phi_1(c\mathbf{x})|_{\partial B_1(0)} = 1.$$

Let σ_1 be the eigenvalue corresponding to ϕ_1. We have

$$\Delta v(\mathbf{x}) = c^2 d(\Delta\phi_1)(c\mathbf{x}) = -c^2 d\sigma_1\phi_1(cx) = -\sigma_1 c^2 v(\mathbf{x}).$$

This gives that $\lambda = c^2\sigma_1$ and

$$\alpha = \max_{B_1(0)} u = \max_{B_1(0)}(v-1) = d - 1 = \frac{1}{\phi_1(y_c)} - 1,$$

where y_c is any point with norm c.

We know there is a unique solution u for each α. Since c is a one-to-one function of $\max u = \alpha$, the above implies that there is a unique solution for every $0 < c < R$. This means that there is a unique one for each λ. By continuation in ϵ, we have shown uniqueness of rotationally invariant solutions of (1) for $f(u) = 1 + u$ on tori with large holes.

By the same reasoning, we can show that there is continuation in α of the solution curve for ϵ fixed in other cases of f' constant at ∞. As long as we start with a curve on which exact multiplicity results are known for a symmetric domain satisfying our conditions, and which satisfies $\lambda''(\alpha) \neq 0$ whenever $\lambda'(\alpha) = 0$, we can get the same multiplicity results for the problem on the domain obtained by rotation of the original; the only difference is that the λ domain may shrink or stretch slightly. If we know there is uniqueness of solutions of (1) on some α interval of the form $[\alpha_0, \infty)$, the result will follow in the linear at infinity case as well. As the reader may have guessed, we are not completely satisfied with the result for such a restricted class of nonlinearities. The continuation of the curve for more general f will be investigated in future work.

7. The shape of solutions on rotated ball domains. With the existence of the solution surface in hand, we now turn to questions regarding the general shape of rotationally invariant solutions. Recall that these are solutions of the transformed equation (6) given by

$$\Delta u + \frac{1}{x_1 + \frac{1}{\epsilon}}\frac{\partial u}{\partial x_1} + \lambda f(u) = 0 \text{ in } \Omega_\epsilon, \quad \lambda \in \mathbf{R}$$
$$u|_{\partial\Omega} = 0.$$

We can obtain our results for ball domains in $\mathbf{R}^n$ for $n = 2, 3, \dots$. We will suppress in our notation for the moment the α-dependence of $\lambda(\alpha, \epsilon)$ and $u(\alpha, \epsilon)$ imagining ourselves as fixed at some α along the solution curve. Since the solution surface obtained in the Theorem 3 is C^1 in ϵ, we can write the solution to (6) as

$$u(\epsilon) = u(0) + \epsilon \frac{\partial}{\partial \epsilon} u(0) + O(\epsilon^2). \tag{21}$$

Here, $u(0)$ solves (6) for $\epsilon = 0$ (or (1)) and $\frac{\partial}{\partial \epsilon} u(0)$ solves (6) after the equations undergoes a differentiation with respect to ϵ and is evaluated at $\epsilon = 0$:

$$\Delta \frac{\partial}{\partial \epsilon} u(0) + u(0)_{x_1} + \lambda(0) f'(u(0)) \frac{\partial}{\partial \epsilon} u(0) = 0 \text{ in } \Omega_\epsilon, \quad \lambda(0) \in \mathbf{R}$$
$$\frac{\partial}{\partial \epsilon} u(0)|_{\partial \Omega} = 0.$$

Now let us write $u(0)$ as u, $\frac{\partial}{\partial \epsilon} u(0)$ as w and $\lambda(0)$ as λ. We represent the cross-sectional piece of Ω_ϵ as given in (4) for fixed θ as $\Omega_{\epsilon,\theta}$, and the corresponding cross section of a solution u to (6) as u_θ. Of course, due to our hypothesis of rotational invariance, u_θ and u_γ have the same shape for any two angles θ and γ in $[0, 2\pi)$, so from now on we will fix ourselves at $\theta = 0$. Rotating the corresponding solution u_θ at $\theta = 0$ will give the solution to (6).

Now at $\epsilon = 0$ in (6), u_θ at $\theta = 0$ is symmetric about $x_1 = \frac{1}{\epsilon}$. We claim, however, that for $\epsilon > 0$, the solutions are no longer symmetric; in fact, for a cross-section of a solution, $max_{\Omega_{\epsilon,\theta}} v(\epsilon)_\theta$ is attained to the left of $x_1 = \frac{1}{\epsilon}$ in the x_1-direction. Note that this makes sense for our setting in terms of temperature: The torus-like domain Ω_ϵ has more boundary on the "outside" (the part that lies outside of the rotation of the plane $x_1 = \frac{1}{\epsilon}$) than on the "inside", and since we have "no heat" conditions on the boundary, the maximum temperature should move toward the inside of the torus. We state our result concerning this change of shape of a rotated solution as a theorem.

Theorem 5. *Let $u = u(\alpha, \epsilon)$ be a solution of (6) for any α. Then $max_{\Omega_{\epsilon,\theta}} u_\theta$ for $\theta = 0$ is attained at $(1/\epsilon - x_0, \mathbf{0})$, where x_0 satisfies $0 < x_0 < \epsilon + O(\epsilon^2)$.*

Proof. Continuing to suppress the α-dependence, we let $u(\epsilon)$ be a solution to (6) and $w := \frac{\partial u(0)}{\partial \epsilon}$ as before. As stated we let $\Omega \subset \mathbf{R}^n$ be a ball. By the symmetry of u, (1) reduces to

$$u'' + \frac{n-1}{r}u' + \lambda f(u) = 0, \tag{22}$$

where $r = |x|$. We claim that the equation for w becomes

$$w'' + \frac{n-1}{r}w' - \frac{n-1}{r^2}w + u'\omega_1 + \lambda f'(u)w = 0, \tag{23}$$

where we have changed $\mathbf{x} \in \mathbf{R}^n$ to $r\omega$, $\omega = (\omega_1, \dots, \omega_n) \in \mathbf{S}^{n-1}$. This will be the case if the second eigenvalue of the Laplacian on the $(n-1)$-sphere is $-(n-1)$. Let $\Delta_{\mathbf{x}}$ represent the Laplacian in $(x_1, \dots, x_n)$ and $\Delta_{\mathbf{S}^{n-1}}$ the Laplacian on the $(n-1)$-sphere. Then we have

$$\Delta_{\mathbf{x}} = \frac{\partial}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathbf{S}^{n-1}}.$$

Let $\bar{u}(\mathbf{x}) := x_1$, and $\hat{u}(r, \omega) := \bar{u}(r\omega) = r\omega_1$. Then $\Delta_{\mathbf{x}}\bar{u} = 0$, and

$$0 = \frac{\partial \hat{u}}{\partial r^2} + \frac{n-1}{r}\frac{\partial \hat{u}}{\partial r} + \frac{1}{r^2}\Delta_{\mathbf{S}^{n-1}}\hat{u} = \frac{n-1}{r}\omega_1 + \frac{1}{r}\Delta_{\mathbf{S}^{n-1}}\omega_1.$$

Hence, the ω_i are all second eigenfunctions of $\Delta_{\mathbf{S}^{n-1}}$, each with corresponding eigenvalue $-(n-1)$. This shows that (23) is correct. Furthermore, we have that

$$u_{x_1} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x_1} = \frac{\partial u}{\partial r}\omega_1.$$

So looking for $w = \hat{w}(r)\omega_1$ in (23), differentiating (22) for comparison purposes, dividing (23) by ω_1, and multiplying both by r^2, the equations (22) and (23) become

$$\begin{aligned} (r^2u'')' + (-(n-1) + \lambda r^2 f'(u))u' &= 0, \\ (r^2\hat{w}')' + \left(-(n-1) + \frac{r^2u'}{\hat{w}} + \lambda r^2 f'(u)\right)\hat{w} &= 0. \end{aligned} \tag{24}$$

We can choose the angles in ω_1 so that $\omega_1 = 1$. Then the x_1-axis on $(0, R)$ corresponds to r on $(0, R)$ and w coincides with $\hat{w}$. We assume by way of contradiction that $w > 0$ on $(r_1, r_2) \subset (0, 1)$, where $w(r_1) = w(r_2) = 0$. If we can show that u' has a zero in (r_1, r_2), we will have the contradiction since $u' < 0$ on $(0,1)$ by Theorem 1. By Sturm-Liouville comparison theory (see e.g. [2]), since

$$-(n-1) + \lambda r^2 f'(u) > -(n-1) + \frac{r^2 u'}{\hat{w}} + \lambda r^2 f'(u)$$

on (r_1, r_2) by assumption, we have arrived at the desired contradiction. Hence $w \leq 0$ on $(0,1)$, and the strong maximum principle gives $w < 0$ on $(0,1)$. Combining this fact with (21), this shows that the maximum moves to the negative x-axis.

We still need to demonstrate the bound given in the proposition. To this end, let $\theta = (\theta_1, \ldots, \theta_n)$ be the angles in the coordinates of ω. Note that we want to estimate $r(\epsilon)$ in the spherical coordinate equation

$$\nabla_{(r,\theta)} u(\epsilon)(r(\epsilon), \mathbf{0}) = \mathbf{0}, \tag{25}$$

where u solves (6) as before, and we are again letting w be the ϵ-derivative of u. (We know that the maximum can only to the x_1 direction.) We may instead compute $\frac{\partial r(0)}{\partial \epsilon}$, since $r(\epsilon) = \epsilon \frac{\partial r(0)}{\partial \epsilon} + O(\epsilon^2)$. Now differentiating (25) and evaluating at $\epsilon = 0$ gives ($' = \frac{d}{dr}$ as before)

$$\begin{aligned} &\nabla_{(r,\theta)} w(0, \mathbf{0}) + D^2_{(r,\theta)} u(0, \mathbf{0}) \begin{pmatrix} \frac{\partial r(0)}{\partial \epsilon} \\ \mathbf{0} \end{pmatrix} = \mathbf{0}, \\ &\begin{pmatrix} \hat{w}'(0) \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} u''(0) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \frac{\partial r(0)}{\partial \epsilon} \\ \mathbf{0} \end{pmatrix} = \mathbf{0}, \\ &\frac{\partial r(0)}{\partial \epsilon} = -\frac{\hat{w}'(0)}{u''(0)}, \end{aligned} \tag{26}$$

where $w(r, \theta) = \hat{w}(r)\omega_1$. Hence we can get the desired bounds if we can estimate $\hat{w}'(0)$ and $u''(0)$.

Now $\hat{w}$ is a solution to the second equation in (24) which is zero at $r = 0$ and $r = 1$. Once again we can use comparison with the first equation in (21) together with uniqueness to show that $u' < \hat{w}$ throughout (0,1). This implies that $u''(0) < \hat{w}'(0)$. Since $u''(0) < 0$, we have that $\hat{w}'(0)/u''(0) < 1$, so in lieu of (26), $\partial r(0)/\partial \epsilon > -1$. Then we arrive at $r(\epsilon) > \epsilon + O(\epsilon^2)$, or $0 < x_0 < \epsilon + O(\epsilon^2)$.

We have included a picture of a solution on a two-dimensional cross section below.

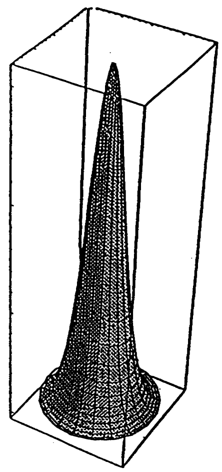

Figure 3. A two-dimensional cross section of a rotationally invariant solution for $f(u) = (1-u)^2 - 2$.

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