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LOCAL STRUCTURE OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR N-DIMENSIONAL REACTION-DIFFUSION EQUATIONS IN BOUNDED DOMAINS

Ugur G. Abdulla*

Faculty of Applied Mathematics and Cybernetics
Baku State University, Baku, Azerbaijan
and
Department of Theoretical Mechanics, University of Nottingham, Nottingham UK

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Abstract. We consider the problem

$$\begin{aligned} u_t - \Delta u^m + bu^\beta = 0, &\qquad (x,t) \in Q_T = \Omega \times (0,T] \\ u(x,t) = 0, &\qquad (x,t) \in S_T = \partial\Omega \times (0,T] \\ u(x,0) = u_0(x), &\quad x \in \Omega, \end{aligned}$$

where $m \geq 1$, $b \in R^1$, $\beta > 0$, $T > 0$; Ω is a bounded, connected domain in R^N with compact boundary, which is assumed to be piecewise smooth and to satisfy the exterior sphere condition. Let $u_0 \in C(\bar{\Omega})$, $u_0 > 0$ for $x \in \Omega$ and $u_0 = 0$ for $x \in \partial\Omega$. Assume also that u_0 is smooth near $\partial\Omega$. We show that the small time behaviour of the solution near the boundary $\partial\Omega$ depends on the relative strength of the diffusion and reaction terms near the boundary, that is to say on the function

$$\gamma(\bar{x}) = \lim_{x \to \bar{x}} \big(\Delta u_0^m / bu_0^\beta\big), \quad \bar{x} \in \partial\Omega, \quad x \in \Omega.$$

We are essentially interested in the initial development of the interface between the dead core and the positive $u(x,t)$ field. In all possible cases, the small time behaviour of the interface is found, together with the local solution.

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*Current Address: Fachbereich Mathematik-Informatik, Universität Paderborn, Warburger Str. 100, Paderborn 33098, Germany.

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1. Introduction. In this paper we consider the initial-boundary value problem

$$Lu \equiv u_t - \Delta u^m + bu^\beta = 0, \qquad (x,t) \in Q_T = \Omega \times (0,T] \tag{1.1}$$

$$u(x,t) = 0, \qquad (x,t) \in S_T = \partial\Omega \times (0,T] \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $m \geq 1$, $b \in R^1$, $\beta > 0$, $T > 0$; Ω is a bounded, connected domain in R^N with compact boundary $\partial\Omega$ which is assumed to be piecewise of class C^1 and to satisfy the exterior sphere condition (i.e., each point of $\partial\Omega$ can be touched from without by a sphere of fixed size such that the sphere does not have any points in common with Ω). For example, $\partial\Omega$ may have a finite number of corner points with interior angles from $(0,\pi)$. Let $u_0 \in C(\bar{\Omega})$, $u_0(x) > 0$ for $x \in \Omega$ and $u_0(x) = 0$ for $x \in \partial\Omega$. Suppose that u_0 is a smooth function near $\partial\Omega$, that is to say there exists a bounded, connected domain Ω_1, with compact C^1 boundary $\partial\Omega_1$, such that $\bar{\Omega}_1 \in \Omega$ and $u_0 \in C^2(\Omega\backslash\Omega_1)$.

Equation (1.1) is a widely used model for various physical, chemical and biological problems involving diffusion and a source or sink, such as filtration in porous media, transport of thermal energy in a plasma, flow of a chemically reacting fluid from a flat surface, evolution of populations etc.; see, for example, [6, 9, 28].

In this paper we investigate the local structure of the solution of the problem (1.1)–(1.3) near $\partial\Omega$ for small t. We are essentially interested in the case when there exists a dead core for small t. More precisely, we are interested in the short time behaviour of the interface between the dead core and the positive $u(x,t)$ field as well as in the local structure of solution near the interface.

If $m > 1$ the equation (1.1) degenerates at points (x,t) where $u = 0$ and we cannot expect the problem (1.1)–(1.3) to have a classical solution. Among the different notions of weak solutions we shall follow the one from [8], [10].

The first existence and uniqueness theorems in the theory of degenerate parabolic equations appeared in the paper of Oleinik et al. [29], where the one-dimensional porous medium equation ($b = 0$, $m > 1$, $N = 1$ in (1.1)) was considered. Further results on the problem of existence and uniqueness of solutions to initial-boundary value problems to (1.1) have been obtained in [17], [7], [8], [10], [30] etc. For a general study we can refer to the survey article [23]. It is well known that there exists a unique weak solution of problem (1.1)–(1.3) in any cylinder Q_T, if $b \geq 0$ (see e.g. [10]). Moreover, if $b < 0$ and $\beta \geq 1$ there exists a unique bounded weak solution of problem (1.1)–(1.3), at least for some $T > 0$. Continuity of solutions to the porous medium equation was first proved in [12]. Continuity of bounded solution to problem (1.1)–(1.3) follows from results of [16], [30]–[32].

We shall assume henceforth that $\beta > 0$ if $b > 0$ and $\beta \geq 1$ if $b < 0$. Then there exists a unique continuous weak solution of problem (1.1)–(1.3) in some cylinder Q_T. By using a generalization of the Nash theorem [27, Ch. III, Theorem 10.1] and Friedman's apriori interior estimates [19, Ch. III, Theorem 10], one may show by standard methods that the weak solution is a classical solution in a neighbourhood of any interior point (x_0, t_0), where $u(x_0, t_0) > 0$.

Let $Q_+ = \{(x,t) \in Q_T : u(x,t) > 0\}$. The set $Q_0 = Q_T / Q_+$ is called a dead core. The set $\Gamma = \partial Q_+$ is said to be an interface. Initial development of interfaces in problems for equation (1.1) have been studied in [11], [18], [25], [26], [33], [34], [21], [5], [24], [1]–[4] etc. (see also the survey article [23]). The regularity properties of the interface for N-dimensional porous medium equations have been widely studied in [13]–[15]. Regularity of the interface for diffusion-absorption equations have been studied in [22], [35].

We essentially restrict ourselves to the case $b > 0$, $0 < \beta < 1$, when the dead core may instantaneously appear for small $t > 0$. We perform a rigorous investigation of the local estimation of the interface and of the estimations of the local solution near the interface. From our results it follows that locally the interface may be described by the function $t = \eta(x)$ which is to be defined as follows

$$\eta(x) = \sup\{\tau : u(x,t) > 0 \text{ for } 0 \leq t < \tau\}.$$

Our main result consists of the fact that the short time behaviour of $\eta(x)$ and short time behaviour of the solution $u(x,t)$ near $t = \eta(x)$ depend crucially on the relative strength of the diffusion and reaction terms near the interface on the initial hyperplane, that is to say on the relative strength of terms Δu_0^m and $b u_0^\beta$, as $x \to \partial\Omega$ with $x \in \Omega$. As we shall see, similar results on the local structure of solution near $\partial\Omega$ are also valid for problem (1.1)–(1.3) with the range of parameters $b \leq 0$, $\beta \geq 1$, when a dead core and interface do not appear since an assumption was made that $\operatorname{supp} u_0 = \Omega$

The organization of the paper is as follows. In Section 2 we describe the main results. In Section 3 we prove the theorems from Section 2.

2. Description of main results.

CASE (I): $b > 0,\ m \geq 1,\ 0 < \beta < 1$.

We have different cases according to the sign of number $m + \beta - 2$.

Theorem 2.1. *Let* $m + \beta \geq 2$ *and*

$$\Delta u_0^m = o(u_0^\beta),$$
$$u_0^{m-\beta-2}|\nabla u_0|^2 = o(1) \quad \text{as } x \to \bar{x} \quad \text{for } x \in \Omega, \quad \bar{x}\ \epsilon\ \partial\Omega. \tag{2.1}$$

Then the interface initially shrinks at the boundary $\partial\Omega$ *and for arbitrary sufficiently small* $\varepsilon > 0$ *there exists a* $\delta_\varepsilon > 0$ *and a domain* Ω_ε *with smooth boundary* $\partial\Omega_\varepsilon$ *and with* $\bar{\Omega}_\varepsilon \subset \Omega$ *such that*

$$\left[u_0^{1-\beta}(x) - b(1-\beta)(1+\varepsilon)t\right]_+^{1/(1-\beta)} \leq u(x,t) \tag{2.2}$$
$$\leq \left[u_0^{1-\beta}(x) - b(1-\beta)(1-\varepsilon)t\right]_+^{1/(1-\beta)}, \quad x \in \Omega \backslash \Omega_\varepsilon, \quad 0 \leq t \leq \delta_\varepsilon$$

$$\frac{u_0^{1-\beta}(x)}{b(1-\beta)(1+\varepsilon)} \leq \eta(x) \leq \frac{u_0^{1-\beta}(x)}{b(1-\beta)(1-\varepsilon)}, \qquad x \in \Omega \backslash \Omega_\varepsilon, \tag{2.3}$$

where $(\theta)_+ = \max(\theta, 0)$.

From (2.2), (2.3), it follows that

$$\eta(x) \sim u_0^{1-\beta}/b(1-\beta), \qquad x \to \bar{x}\ \epsilon\ \partial\Omega, \quad x \in \Omega \tag{2.4}$$

$$u(x,t) \sim \left[u_0^{1-\beta}(x) - b(1-\beta)t\right]^{1/(1-\beta)} \quad \text{for } t = \phi_c(x), \quad x \to \partial\Omega, \tag{2.5}$$

where $\phi_c(x) = u_0^{1-\beta}(x)/C$ and C is an arbitrary constant such that $C > b(1-\beta)$.

Theorem 2.2. *Let* $m + \beta > 2$ *and*

$$\lim_{x\to\bar{x}} \left(\Delta u_0^m / b u_0^\beta\right) = \gamma(\bar{x}), \qquad x \in \Omega, \quad \bar{x} \in \partial\Omega, \tag{2.6}$$

$$\rho = \operatorname{sign}\left(u_0 \Delta u_0 - \beta\left|\nabla u_0\right|^2\right) = \text{constant} \qquad \text{for } x \in \Omega \backslash \Omega_1, \tag{2.7}$$

where $\gamma \geq 0$ for $\bar{x} \in \partial\Omega$, $\bar{\gamma} \equiv \sup\limits_{\partial\Omega} \gamma < 1$ *and* Ω_1 *is a bounded, connected domain with compact and smooth boundary* $\partial\Omega_1$ *and with* $\bar{\Omega}_1 \subset \Omega$. *If* $\rho = -1$ *assume also that*

$$\left[(m-1+\beta)(m-2)+\beta^2\right]|\nabla u_0|^2 + (m-1)u_0\Delta u_0 \geq 0, \quad x \in \Omega \backslash \Omega_1. \tag{2.8}$$

Then the interface initially shrinks at the boundary $\partial\Omega$ *and for arbitrary sufficiently small* $\varepsilon > 0$ *there exists a* $\delta_\varepsilon > 0$ *and a domain* Ω_ε *as before* (*see Theorem* 2.1) *such that*

$$\begin{aligned}&\big[u_0^{1-\beta}(x) - b(1-\beta)(1+\varepsilon)t\big]_+^{1/(1-\beta)} \le u(x,t)\\&\qquad \le \big[u_0^{1-\beta}(x) - b(1-\beta)(1-\varepsilon-\bar\gamma)t\big]_+^{1/(1-\beta)}\ 0 \le t \le \delta_\varepsilon, \qquad (2.9)\\&u_0^{1-\beta}(x)/b(1-\beta)(1+\varepsilon) \le \eta(x)\\&\qquad \le u_0^{1-\beta}(x)/b(1-\beta)(1-\bar\gamma-\varepsilon),\ \ x \in \Omega\backslash\Omega_\varepsilon. \qquad (2.10)\end{aligned}$$

From (2.10) it follows that

$$1 \le \underline{\lim}\ \frac{b(1-\beta)\eta(x)}{u_0^{1-\beta}(x)}\ \le\ \overline{\lim}\ \frac{b(1-\beta)\eta(x)}{u_0^{1-\beta}(x)}\ \le\ (1-\bar\gamma)^{-1}, \qquad (2.11)$$

where lim is taken as $x \to \bar x \in \partial\Omega, \quad x \in \Omega$. From (2.9) it follows that

$$\begin{aligned}&\underline{\lim}\ \frac{u(x,t)}{\big[u_0^{1-\beta}(x) - b(1-\beta)t\big]^{1/(1-\beta)}} \ge 1,\\&\overline{\lim}\ \frac{u(x,t)}{\big[u_0^{1-\beta}(x) - b(1-\beta)(1-\bar\gamma)t\big]^{1/(1-\beta)}} \le 1, \qquad (2.12)\end{aligned}$$

where $\underline{\lim}$ ($\overline{\lim}$) is taken as $x \to \partial\Omega$ along the surface

$$t = \phi_c(x) \equiv C^{-1}u_0^{1-\beta}$$

for arbitrary C satisfying

$$C > b(1-\beta) \qquad \big(C > b(1-\beta)(1-\bar\gamma)\big).$$

Remark. It should be noted that the existence of the limit in (2.6) is not essential for estimates (2.9) and (2.10). In fact, the upper estimate of $u(x,t)$ depends on the upper limit, while the lower estimate of $u(x,t)$ depends on the lower limit of the expression $\Delta u_0^m/bu_0^\beta$. For example, (2.6) may be changed by the following condition

$$\begin{aligned}&\lim_{x\to\bar x}\sup\big(\Delta u_0^m/bu_0^\beta\big) = \gamma(\bar x),\\&\lim_{x\to\bar x}\inf\big(\Delta u_0^\beta/bu_0^\beta\big) = \gamma_0(\bar x), \quad x \in \Omega,\ \bar x \in \partial\Omega.\end{aligned}$$

where $\gamma_0 \ge 0$ for $\bar x \in \partial\Omega$, $\bar\gamma \equiv \sup\limits_{\partial\Omega}\ \gamma < 1$. This observation is of a general nature and it relates to all theorems (corollaries) below.

Theorem 2.3. *Let $m+\beta=2$, (2.6), (2.7) be satisfied and*

$$\lim_{x\to\bar{x}} \frac{m(m+\beta)}{2b} u_0^{m-2-\beta}\left|\nabla u_0\right|^2 = \gamma_1(\bar{x}), \qquad x\in\Omega, \quad \bar{x}\in\partial\Omega, \tag{2.13}$$

with $\bar{\gamma}_1 = \sup\limits_{\partial\Omega}\gamma_1 < 1$. Then the interface initially shrinks at the boundary $\partial\Omega$ and for arbitrary sufficiently small $\varepsilon>0$ there exists a $\delta_\varepsilon>0$ and a domain Ω_ε as before, such that

$$\begin{aligned}&\left[u_0^{1-\beta}(x) - b(1-\beta)(1+\varepsilon-p)t\right]_+^{1/(1-\beta)} \le u(x,t)\\ &\quad \le \left[u_0^{1-\beta}(x) - b(1-\beta)(1-\varepsilon-q)t\right]_+^{1/(1-\beta)}, \qquad x\in\Omega\backslash\Omega_\varepsilon, \quad 0\le t\le\delta_\varepsilon,\end{aligned} \tag{2.14}$$

$$\begin{aligned}&u_0^{1-\beta}(x)/\big(b(1-\beta)(1+\varepsilon-p)\big) \le \eta(x)\\ &\quad \le u_0^{1-\beta}(x)/\big(b(1-\beta)(1-\varepsilon-q)\big), \qquad x\in\Omega\backslash\Omega_\varepsilon,\end{aligned} \tag{2.15}$$

where

$$p = \left\{\underline{\gamma}_1,\ \text{if } \rho\ge 0;\ \underline{\gamma}\ \text{if } \rho=-1\right\},$$

$$q = \left\{\bar{\gamma},\ \text{if } \rho\ge 0;\ \bar{\gamma}_1\ \text{if } \rho=-1\right\},$$

and

$$\underline{\gamma}_1 = \inf_{\partial\Omega}\gamma_1, \qquad \underline{\gamma} = \inf_{\partial\Omega}\gamma.$$

Corollary 2.1. *Let the conditions of Theorem 2.3 be valid and $\gamma(\bar{x}) = \gamma_1(\bar{x}) \equiv \gamma$. Then we have*

$$\eta(x) \ \sim u_0^{1-\beta}(x)/\big(b(1-\beta)(1-\gamma)\big), \quad x\to\bar{x}\in\partial\Omega, \quad x\in\Omega, \tag{2.16}$$

$$\begin{aligned}u(x,t) &\ \sim \left[u_0^{1-\beta}(x) - b(1-\beta)(1-\gamma)t\right]^{1/(1-\beta)}\\ &\text{for } t=\phi_c(x), \quad x\to\bar{x}\in\partial\Omega,\end{aligned} \tag{2.17}$$

where C is arbitrary such that $C > b(1-\beta)(1-\gamma)$.

Theorem 2.4. *Let $m+\beta<2$, (2.6), (2.7), (2.13) be valid and*

$$\theta = \operatorname{sign}\big[u_0\Delta u_0 + \big((m-\beta-2)/2\big)\,|\nabla u_0|^2\big] = \text{constant for } \ x\in\Omega\backslash\Omega_1, \quad (2.18)$$

where Ω_1 is a connected domain with compact and smooth boundary $\partial\Omega_1$ and with $\bar\Omega_1\subset\Omega$. If $\rho=-1$, assume also that $\Delta u_0^m\geq 0$ for $x\in\Omega\backslash\bar\Omega_1$. Then the interface initially shrinks at the boundary $\partial\Omega$ and for sufficiently small arbitrary $\varepsilon>0$ there exists a $\delta_\varepsilon>0$ and a domain Ω_ε, as before, such that

$$\begin{aligned}&\big[u_0^{1-\beta}(x) - b(1-\beta)(1-\gamma_2+\varepsilon)t\big]_+^{1/(1-\beta)} \leq u(x,t)\\ &\quad\leq \big[u_0^{(m-\beta)/2} - t/\varepsilon\big]^{2/(m-\beta)}, \quad x\in\Omega\backslash\Omega_\varepsilon, \quad 0\leq t\leq\delta_\varepsilon,\end{aligned} \quad (2.19)$$

$$\begin{aligned}&u_0^{1-\beta}(x)/\big[b(1-\beta)(1-\gamma_2+\varepsilon)\big] \leq \eta(x)\\ &\quad\leq \varepsilon u_0^{(m-\beta)/2} \quad \text{for } x\in\Omega\backslash\Omega_\varepsilon,\end{aligned} \quad (2.20)$$

where $\gamma_2 = \big\{\big[2(m-1+\beta)/(m+\beta)\big]\underline{\gamma}_1, \text{ if } \rho\geq 0;\ \underline{\gamma}, \text{ if } \rho=-1\big\}.$

Let us now apply the results of Theorems 2.1–2.4 in the particular case of power law initial data. Let

$$\Omega = B_R(0) = \big\{x : |x|<R\big\};\quad u_0(x) = C(R-|x|)^\alpha \quad \text{for } x\in\overline{B_R(0)}\backslash B_{R_1}(0), \quad (2.21)$$

for some $R_1\in(0,R)$, where $C>0$, $\alpha>0$ are given numbers. It may easily be checked that Theorems 2.1–2.4 imply that the interface initially shrinks at the boundary $\partial B_R(0)$, either when $\alpha>2/(m-\beta)$ and C is an arbitrary positive constant or when $\alpha=2/(m-\beta)$ and $C\in(0,C_1)$, where

$$C_1 = \big(b(m-\beta)^2/\big(2m(m+\beta)\big)\big)^{1/(m-\beta)}.$$

Obviously, if $N=1$ this condition is necessary for the interface to shrink, since $u_0(x)=C_1(R-x)^{2/(m-\beta)}$ is a stationary solution of equation (1.1) for $x<R$ (see [4], [5], [21]).

From Theorem 2.1 it follows that if $\alpha>2/(m-\beta)$ and C is an arbitrary positive constant, then the interface initially shrinks at the boundary $\partial B_R(0)$ and

$$\begin{aligned}&\eta(x)\sim\big(C^{1-\beta}/b(1-\beta)\big)(R-|x|)^{\alpha(1-\beta)}, \quad x\to\partial B_R(0),\ \ x\in B_R(0),\\ &u(x,t)\sim\big[C^{1-\beta}(R-|x|)^{\alpha(1-\beta)} - b(1-\beta)t\big]^{1/(1-\beta)},\\ &\qquad\qquad t<\eta(x),\ \ x\to\partial B_R(0),\ \ x\in B_R(0).\end{aligned}$$

From Theorem 2.2 it follows that if $m+\beta>2,\ \alpha=2/(m-\beta)$ and $C\in(0,C_1)$, then the interface shrinks at the boundary $\partial B_R(0)$ and for sufficiently small arbitrary $\varepsilon>0$ there exists a $\delta_\varepsilon>0$ and a domain Ω_ε with smooth boundary $\partial\Omega_\varepsilon$ and with $\overline{B_{R_1}(0)}\subset\bar{\Omega}_\varepsilon\subset B_R(0)$ such that

$$\begin{aligned}
&\big[C^{1-\beta}(R-|x|)^{2(1-\beta)/(m-\beta)}-b(1-\beta)(1+\varepsilon)t\big]_+^{1/(1-\beta)} \ \le\ u(x,t)\\
&\quad\le \big[C^{1-\beta}(R-|x|)^{2(1-\beta)/(m-\beta)}-b(1-\beta)(1-\varepsilon-(C/C_1)^{m-\beta})t\big]_+^{1/(1-\beta)},\\
&\qquad\qquad x\in B_R(0)\backslash\Omega_\varepsilon,\quad 0\le t\le\delta_\varepsilon,\\
&\frac{C^{1-\beta}}{b(1-\beta)(1+\varepsilon)}\ (R-|x|)_+^{2(1-\beta)/(m-\beta)}\ \le\ \eta(x)\\
&\quad\le \frac{C^{1-\beta}(R-|x|)_+^{2(1-\beta)/(m-\beta)}}{b(1-\beta)\big(1-(C/C_1)^{m-\beta}-\varepsilon\big)},\quad x\in B_R(0)\backslash\Omega_\varepsilon,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\eta(x)\sim\big[C^{1-\beta}/\big(b(1-\beta)\mu\big)\big](R-|x|)^{2(1-\beta)/(m-\beta)},\ x\to\partial B_R(0),\ x\in B_R(0),\\
&u(x,t)\sim\big[C^{1-\beta}(R-|x|)_+^{2(1-\beta)/(m-\beta)}-b(1-\beta)\mu t\big]_+^{1/(1-\beta)},\\
&\qquad t<\eta(x), x\to\partial B_R(0),\ x\in B_R(0).
\end{aligned}$$

for some constant μ satisfying

$$1-(C/C_1)^{m-\beta}\le\mu\le 1.$$

From Theorem 2.3 and Corollary 2.1 it follows that if $m+\beta=2,\ \alpha=2/(m-\beta),\ C\in(0,C_1)$, then the interface shrinks at the boundary $\partial B_R(0)$ and

$$\begin{aligned}
&\eta(x)\sim\big[C^{1-\beta}/\big(b(1-\beta)\big(1-(C/C_1)^{m-\beta}\big)\big)\big](R-|x|)^{2(1-\beta)/(m-\beta)},\\
&\qquad\qquad x\to B_R(0),\quad x\in B_R(0),\\
&u(x,t)\sim\big[C^{1-\beta}(R-|x|)^{2(1-\beta)/(m-\beta)}-b(1-\beta)\big(1-(C/C_1)^{m-\beta}\big)t\big]^{1/(1-\beta)},\\
&\qquad\qquad t<\eta(x),\quad x\to\partial B_R(0),\quad x\in B_R(0).
\end{aligned}$$

From Theorem 2.4 it follows that if $m+\beta<2$, then the interface shrinks either when $\alpha>2/(m-\beta)$ and C is arbitrary positive constant or when

$\alpha = 2/(m-\beta)$ and $C \in (0, C_1)$. Moreover for sufficiently small arbitrary $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ and a domain Ω_ε, as before, such that

$$\begin{aligned}
&\left[C^{1-\beta}(R-|x|)^{\alpha(1-\beta)} - b(1-\beta)(1-\gamma_2+\varepsilon)t\right]_+^{1/(1-\beta)} \leq\ u(x,t) \\
&\leq \left[C^{(m-\beta)/2}(R-|x|)^{\alpha(m-\beta)/2} - t/\varepsilon\right]_+^{2/(m-\beta)}, \quad x \in B_R(0)/\Omega_\varepsilon,\ 0 \leq t \leq \delta_\varepsilon, \\
&\frac{C^{1-\beta}(R-|x|)^{\alpha(1-\beta)}}{b(1-\beta)(1-\gamma_2+\varepsilon)} \leq \eta(x) \leq \varepsilon C^{(m-\beta)/2}(R-|x|)^{\alpha(m-\beta)/2}, x \in B_R(0)/\Omega_\varepsilon,
\end{aligned}$$

where $\gamma_2 = \big\{0,\ \text{if } \alpha > 2/(m-\beta);\ \dfrac{4m(m-1+\beta)}{b(m-\beta)^2}\, C^{m-\beta},\ \text{if } \alpha = 2/(m-\beta),$ $C \in (0, C_1)\big\}$ It should be noted that the same method which we shall use to prove Theorems 2.1–2.4 is also applicable in the case $\beta \geq 1$, when an interface can not appear.

CASE (II): $b \neq 0,\ m > 1,\ \beta = 1.$

Theorem 2.5. *Let*

$$\lim_{x \to \bar{x}} \big(\Delta u_0^m / b u_0\big) = \gamma(\bar{x}), \quad \bar{x} \in \partial\Omega, \quad x \in \Omega, \tag{2.22}$$

where γ is a bounded function in $\partial\Omega$. Then for arbitrary sufficiently small $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ and a domain Ω_ε, as before, such that

$$\begin{aligned}
&u_0(x)\exp\big[-b\big(1-\underline{\gamma}_b+\varepsilon_b\big)t\big] \leq u(x,t) \\
&\quad \leq u_0(x)\exp\big[-b\big(1-\overline{\gamma}_b-\varepsilon_b\big)t\big], \quad x \in \Omega\backslash\Omega_\varepsilon, \quad 0 \leq t \leq \delta_\varepsilon,
\end{aligned} \tag{2.23}$$

where

$$\varepsilon_b = \varepsilon \text{ sign } b,\ \underline{\gamma}_b = (\text{sign } b)\inf_{\partial\Omega}\ \big[\gamma(\bar{x})\text{sign } b\big],\ \bar{\gamma}_b = (\text{sign } b)\sup_{\partial\Omega}\ \big[\gamma(\bar{x})\text{sign } b\big].$$

From (2.23) it follows that:

$$u(x,t) \sim u_0(x) \text{ as } (x,t) \to (\bar{x}, 0), \qquad x \in \Omega, \quad \bar{x} \in \partial\Omega. \tag{2.24}$$

CASE (III): $b = 0,\ m > 1.$

Theorem 2.6. *Let* (2.22) *be valid* (*we assume* $b = 1$ *in* (2.22)). *Then for arbitrary sufficiently small* $\varepsilon > 0$ *there exists a* $\delta_\varepsilon > 0$ *and a domain* Ω_ε*, as before, such that*

$$u_0(x)\big[1-(m-1)t\big]^{\frac{-\varepsilon+\underline{\gamma}}{1-m}} \le u(x,t) \le u_0(x)\big[1-(m-1)t\big]^{\frac{\varepsilon+\overline{\gamma}}{1-m}}, \qquad (2.25)$$

$x \in \Omega\backslash\Omega_\varepsilon$, $0 \le t \le \delta\varepsilon$, *from which* (2.24) *again easily follows.*

It is simple to check that in the particular case (2.21), the conditions (2.22) imply that $\alpha \ge 2/(m-1)$ and C are arbitrary positive constants. This coincides with the necessary and sufficient condition for the interface to have a rest at the boundary $\partial B_R(0)$ (waiting-time phenomenon) in the Cauchy problem (1.1), (2.21), $x \in R^N$.
CASE (IV): $b \neq 0,\ m > 1,\ \beta > 1.$

Theorem 2.7. *Let* (2.6) *be valid and*

$$u_0^{m-2-\beta}|\nabla u_0|^2 = O(1), \qquad x \to \bar{x} \in \partial\Omega, \quad x \in \Omega. \qquad (2.26)$$

Then for arbitrary sufficiently small $\varepsilon > 0$ *there exists a* $\delta_\varepsilon > 0$ *and a domain* Ω_ε*, as before, such that*

$$\begin{aligned} &\big[u_0^{1-\beta} + b(\beta-1)(1-\underline{\gamma}_b+\varepsilon_b)t\big]^{1/(1-\beta)} \le u(x,t) \qquad (2.27)\\ &\quad \le \big[u_0^{1-\beta} + b(\beta-1)(1-\overline{\gamma}_b-\varepsilon_b)t\big]^{1/(1-\beta)}, \qquad x \in \Omega\backslash\Omega_\varepsilon, \quad 0 \le t \le \delta_\varepsilon, \end{aligned}$$

where ε_b, $\underline{\gamma}_b$, $\overline{\gamma}_b$ *are the same as in Theorem 2.5.*

From (2.27), (2.24) again easily follows.

Theorem 2.6′. *Let the conditions of Theorem* 2.7 *fail, but the conditions of Theorem* 2.6 *be valid. Then for arbitrary sufficiently small* $\varepsilon > 0$ *there exists a* $\delta_\varepsilon > 0$ *and a domain* Ω_ε*, as before, such that* (2.25), (2.24) *are valid.*

In the particular case of (2.21), the conditions of Theorem 2.7 imply that $\alpha \ge 2/(m-\beta)$, $1 < \beta < m$, while that of Theorem 2.6′ implies $2/(m-1) \le \alpha < 2/(m-\beta)$. If $N = 1$, these conditions again coincide with the necessary and sufficient condition for the interface to have a rest at the initial moment in the Cauchy problem (1.1), (2.21).

3. Proofs of the Theorems 2.1–2.7.

Proof of Theorem 2.1. Consider the function

$$g(x,t) = \big[u_0^{1-\beta}(x) - b(1-\beta)(1-C)t\big]_+^{1/(1-\beta)}, \qquad (x,t)\in D, \tag{3.1}$$

where $C < 1$ is an arbitrary number, $D = \big\{(x,t):\ \ x\in\Omega\backslash\bar{\Omega}_1,\ \ \ 0<t\le \delta\big\},\ \ 0<\delta<T$.

The positive number δ is chosen so as to satisfy

$$\delta \le \sup_{\Omega\backslash\Omega_1}\big[u_0^{1-\beta}(x)/\big(b(1-\beta)(1-C)\big)\big]. \tag{3.2}$$

Obviously, g is non-negative in D and

$$\begin{aligned} &g\epsilon C^{2,1}(D\backslash G)\cap C^{1,1}(D)\cap C^{0,1}(\bar{D}), \qquad g^m\in C^{2,1}(D),\\ &G=\big\{(x,t): x\in\Omega\backslash\bar{\Omega}_1, \qquad t=u_0^{1-\beta}(x)/\big(b(1-\beta)(1-C)\big)\big\}. \end{aligned} \tag{3.3}$$

Let us estimate the expression Lg in

$$D_+ = \big\{(x,t): x\in\Omega\backslash\bar{\Omega}_1, \quad 0<t<\min\big(\delta,\ u_0^{1-\beta}(x)/\big(b(1-\beta)(1-C)\big)\big)\big\}.$$

We have

$$\begin{aligned} Lg =& bCg^\beta - m(m-1+\beta)u_0^{-2\beta}|\nabla u_0|^2 g^{m+2\beta-2} - mg^{m+\beta-1}\\ &\times\big(u_0^{-\beta}\Delta u_0 - \beta u_0^{-\beta-1}|\nabla u_0|^2\big) = bg^\beta\{C+S\}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} S =& -\frac{m(m-1+\beta)}{b}u_0^{m-2-\beta}|\nabla u_0|^2\big[1-\frac{b(1-\beta)(1-C)t}{u_0^{1-\beta}}\big]^{\frac{m+\beta-2}{1-\beta}}\\ &-\frac{m}{b}\big(u_0^{m-1-\beta}\Delta u_0 - \beta u_0^{m-2-\beta}|\nabla u_0|^2\big)\big[1-\frac{b(1-\beta)(1-C)t}{u_0^{1-\beta}}\big]^{\frac{m-1}{1-\beta}}. \end{aligned} \tag{3.5}$$

From (2.1) it follows that

$$u_0^{m-1-\beta}\Delta u_0 = o(1), \quad x\to\bar{x}, \quad x\in\Omega, \quad \bar{x}\in\partial\Omega.$$

Hence, from (3.5) it follows that for arbitrary ε with $|\varepsilon|$ sufficiently small there exists a $\delta_\varepsilon^1 > 0$ and a domain Ω_ε with smooth boundary $\partial\Omega_\varepsilon$ and with $\bar{\Omega}_1 \subset \bar{\Omega}_\varepsilon \subset \Omega$ such that

$$|S| < |\varepsilon|/2 \quad \text{for } x \in \Omega \backslash \Omega_\varepsilon, \quad 0 \le t < \min\big(\delta_\varepsilon^1,\ u_0^{1-\beta}(x)/\big(b(1-\beta)(1-C)\big)\big).$$

Now taking $C = \varepsilon$ in (3.1), from (3.4) it follows that

$$\begin{gathered} Lg > (<)\ b(\varepsilon/2)g^\beta \text{ for } \varepsilon > (<)\ 0 \text{ and for } x \in \Omega \backslash \Omega_\varepsilon, \\ 0 \le t < \min\big(\delta_\varepsilon^1,\ u_0^{1-\beta}(x)/\big(b(1-\beta)(1-\varepsilon)\big)\big). \end{gathered} \tag{3.6a}$$

Obviously

$$Lg = 0 \quad \text{for } x \in \Omega \backslash \Omega_\varepsilon, \quad u_0^{1-\beta}(x)/\big(b(1-\beta)(1-\varepsilon)\big) \le t \le \delta_\varepsilon^1, \tag{3.6b}$$

and

$$g(x,0) = u_0(x), \quad g_t(x,0) = -b(1-\varepsilon)u_0^\beta(x), \quad x \in \bar{\Omega} \backslash \Omega_1. \tag{3.7}$$

As mentioned before, the weak solution $u(x,t)$ is a classical solution of equation (1.1) in a neighbourhood of any interior point (x_0, t_0), where $u(x_0,t_0) > 0$. Since $u_0 \in C^2(\Omega \backslash \Omega_1)$, $u_0(x) > 0$ for $x \in \partial\Omega_\varepsilon$ and $\bar{\Omega}_1 \subset \bar{\Omega}_\varepsilon \subset \Omega$ (and $\partial\Omega_\varepsilon \cap \partial\Omega_1 = \emptyset$), the function u is a positive classical solution of equation (1.1) in $\partial\Omega_\varepsilon \times [0,\delta]$ for sufficiently small $\delta > 0$. Moreover

$$u_t(x,0) = \Delta u_0^m(x) - bu_0^\beta(x), \quad x \in \partial\Omega_\varepsilon. \tag{3.8}$$

From (2.1) it follows that

$$u_t(x,0) = -bu_0^\beta + o(u_0^\beta), \quad x \to \bar{x}, \quad x \in \Omega, \quad \bar{x} \in \partial\Omega. \tag{3.9}$$

Hence, from (3.7), (3.9), we have

$$\nu(x,0) = 0, \quad \nu_t(x,0) = b\varepsilon\ u_0^\beta + o(u_0^\beta), \quad x \to \bar{x},\ x \in \Omega, \quad \bar{x} \in \partial\Omega, \tag{3.10}$$

where $\nu = g - u$. Obviously, for fixed ε, the domain Ω_ε may be chosen so as to satisfy

$$\nu_t(x,0) > (<)\ 0 \quad \text{for } \varepsilon > (<)\ 0 \quad \text{and for } x \in \partial\Omega_\varepsilon$$

and hence

$$\nu(x,t) \geq (\leq)\ 0, \quad \text{for } \varepsilon > (<)\ 0 \quad \text{for } (x,t) \in \partial\Omega_\varepsilon \times (0,\delta_\varepsilon], \tag{3.11}$$

for some $\delta_\varepsilon \in (0,\delta_\varepsilon^1)$. Consider now the function g in $D_\varepsilon = \{(x,t):\ x \in \Omega\backslash\bar{\Omega}_\varepsilon, 0 < t \leq \delta_\varepsilon\}$. From (3.1)–(3.5) and (3.11) it follows that

$$\begin{aligned}
&g\epsilon\ C^{2,1}(D_\varepsilon\backslash G_\varepsilon) \cap C^{1,1}(D_\varepsilon) \cap C^{0,1}(\bar{D}_\varepsilon), \quad g^m\epsilon\ C^{2,1}(D_\varepsilon), \quad Lg\epsilon\ L^\infty(D_\varepsilon),\\
&G_\varepsilon = \{(x,t):\ x \in \Omega\backslash\bar{\Omega}_\varepsilon, \quad t = u_0^{1-\beta}(x)/\big(b(1-\beta)(1-\varepsilon)\big)\}\\
&Lg \geq (\leq)\ 0 \quad \text{for } \varepsilon > (<)\ 0 \quad \text{and for } (x,t) \in D_\varepsilon,\\
&g(x,0) = u(x,0) \quad \text{for } x \in \Omega\backslash\bar{\Omega}_\varepsilon,\\
&g \geq (\leq)\ u \quad \text{for } \varepsilon > (<)\ 0, \quad \text{on } S_\varepsilon = S_\varepsilon^1 \cup S_\varepsilon^2,
\end{aligned} \tag{3.12}$$

where $S_\varepsilon^1 = \partial\Omega \times [0,\delta_\varepsilon]$, $S_\varepsilon^2 = \partial\Omega_\varepsilon \times [0,\delta_\varepsilon]$. By using a comparison Theorem [10, Theorem 0.2], (3.12) easily implies the estimates (2.2) and (2.3).

From (2.3) it follows that

$$(1+\varepsilon)^{-1} \leq b(1-\beta)\eta(x)/u_0^{1-\beta}(x) \leq (1-\varepsilon)^{-1}, \quad x \in \Omega\backslash\Omega_\varepsilon. \tag{3.13}$$

Because $\varepsilon > 0$ is arbitrary, (2.4) follows from (3.13).

For $(x,t) \in D_{\varepsilon+} = \{(x,t) : x \in \Omega\backslash\bar{\Omega}_\varepsilon,\ 0 < t < \min(\delta_\varepsilon,\ u_0^{1-\beta}(x)/b(1-\beta))\}$ and let $\varepsilon > 0$ be an arbitrary sufficiently small number, then from (2.2) we have

$$\begin{aligned}
\frac{\big[u_0^{1-\beta} - b(1-\beta)(1+\varepsilon)t\big]^{1/(1-\beta)}}{\big[u_0^{1-\beta} - b(1-\beta)t\big]^{1/(1-\beta)}} &\leq u(x,t)\\
&\leq \frac{\big[u_0^{1-\beta} - b(1-\beta)(1-\varepsilon)t\big]^{1/(1-\beta)}}{\big[u_0^{1-\beta} - b(1-\beta)t\big]^{1/(1-\beta)}}.
\end{aligned}$$

Taking $t = \phi_c(x)$,$x \in \Omega\backslash\Omega_\varepsilon$, $C > b(1-\beta)$, we have

$$\begin{aligned}
\Big(\frac{C - b(1-\beta)(1+\varepsilon)}{C - b(1-\beta)}\Big)^{1/(1-\beta)} &\leq \frac{u(x,t)}{\big[u_0^{1-\beta} - b(1-\beta)t\big]^{1/(1-\beta)}}\\
&\leq \Big(\frac{C - b(1-\beta)(1-\varepsilon)}{C - b(1-\beta)}\Big)^{1/(1-\beta)}.
\end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, (2.5) follows and the theorem is proved.

Proof of Theorem 2.2. As in the previous proof we may obtain (3.4) and (3.5). From (2.6) it follows that

$$S\big|_{t=0} = -\Delta\, u_0^m / b u_0^\beta \to -\gamma(\bar{x}), \quad \text{as } x \to \bar{x}, \quad x \in \Omega, \quad \bar{x} \in \partial\Omega. \tag{3.14}$$

Obviously

$$S = 0 \quad \text{for } t = u_0^{1-\beta}(x)/b(1-\beta)(1-C), \quad x\epsilon\, \Omega\backslash\bar{\Omega}_1. \tag{3.15}$$

Let us compute S_t in D_+:

$$S_t = m(1-C)u_0^{m-3}\big[1 - \frac{b(1-\beta)(1-C)t}{u_0^{1-\beta}}\big]^{\frac{m+2\beta-3}{1-\beta}} B, \tag{3.16a}$$

where

$$\begin{aligned} B = {} & (m-1+\beta)(m+\beta-2)|\nabla u_0|^2 \\ & + (m-1)\big(u_0 \Delta u_0 - \beta|\nabla u_0|^2\big)\big[1 - \frac{b(1-\beta)(1-C)t}{u_0^{1-\beta}}\big]. \end{aligned} \tag{3.16b}$$

Obviously

$$\frac{\partial B}{\partial t} = -\frac{b(1-\beta)(1-C)(m-1)}{u_0^{1-\beta}}\big(u_0\Delta u_0 - \beta|\nabla u_0|^2\big)$$

and in view of (2.7) we have

$$\operatorname{sign}\big(\frac{\partial B}{\partial t}\big) = -\rho \quad \text{for } (x,t) \in D_+. \tag{3.17}$$

On the surface $t = u_0^{1-\beta}(x)/b(1-\beta)(1-C), \quad x \in \Omega\backslash\bar{\Omega}_1$ we have

$$B = (m-1+\beta)(m+\beta-2)|\nabla u_0|^2 > 0.$$

Hence, if $\rho \geq 0$, then from (3.17) it follows that there exists a domain Ω_2 with smooth and compact boundary and with $\bar{\Omega}_1 \subset \bar{\Omega}_2 \subset \Omega$ such that

$$B \geq 0 \quad \text{for } x \in \Omega\backslash\Omega_2, \quad 0 \leq t < u_0^{1-\beta}(x)/b(1-\beta)(1-C), \tag{3.18a}$$

which implies, in view of (3.16), that

$$S_t \geq 0, \quad \text{as } x \in \Omega\backslash\Omega_2, \quad 0 \leq t < u_0^{1-\beta}/b(1-\beta)(1-C). \tag{3.18b}$$

If $\rho = -1$, then from (2.8) and (3.16b) it follows that

$$\begin{aligned} B\big|_{t=0} &= \big[(m-1+\beta)(m+\beta-2) - \beta(m-1)\big]|\nabla u_0|^2 + (m-1)u_0\Delta u_0 \\ &= \big[(m-1+\beta)(m-2)+\beta^2\big]|\nabla u_0|^2 + (m-1)u_0\Delta u_0 \geq 0,\ x \in \Omega\backslash\bar{\Omega}_1, \end{aligned}$$

which again leads us to (3.18), in view of (3.17). From (3.14), (3.15) and (3.18) it follows that for arbitrary ε with $|\varepsilon|$ sufficiently small there exists a $\delta_\varepsilon > 0$ and a domain Ω_ε with smooth and compact boundary and with $\bar{\Omega}_1 \subset \bar{\Omega}_\varepsilon \subset \Omega$ (and $\partial\Omega_\varepsilon \cap \partial\Omega_1 = \emptyset$) such that

$$-\bar{\gamma} - |\varepsilon|/2 \leq S \leq 0 \text{ for } x\ \epsilon\Omega\backslash\Omega_\varepsilon,\ 0 \leq t < \min\big(u_0^{1-\beta}/b(1-\beta)(1-C), \delta_\varepsilon\big). \tag{3.19}$$

If $\varepsilon > 0$, then taking $C = \varepsilon + \bar{\gamma}$ in (3.1), from (3.4) and (3.19) it follows that

$$Lg \geq b(\varepsilon/2)g^\beta \quad \text{for } x \in \Omega\backslash\Omega_\varepsilon, \quad 0 \leq t < \min\big(u_0^{1-\beta}/b(1-\beta)(1-C),\ \delta_\varepsilon\big).$$

Obviously, $C < 1$ for sufficiently small $\varepsilon > 0$. If $\varepsilon < 0$, then taking $C = \varepsilon$ in (3.1), from (3.4) and (3.19) we have

$$Lg \leq 0 \quad \text{for } x \in \Omega\backslash\Omega_\varepsilon, \quad 0 \leq t < \min\big(u_0^{1-\beta}/b(1-\beta)(1-C), \delta_\varepsilon\big).$$

As mentioned in the proof of Theorem 2.1, the solution $u(x,t)$ is a positive classical solution of equation (1.1) in $\partial\Omega_\varepsilon \times [0,\delta]$ for sufficiently small $\delta > 0$. Then from (2.6), (1.1) and (3.8) it follows that

$$u_t(x,0) = b(\gamma(\bar{x})-1)u_0^\beta(x) + o(u_0^\beta), \quad x \to \bar{x}, \quad x \in \Omega, \quad \bar{x} \in \partial\Omega. \tag{3.20}$$

Obviously, for $g(x,t)$ with $C = \bar{\gamma} + \varepsilon,\ \varepsilon > 0$ we have

$$g(x,0) = u_0(x), \quad g_t(x,0) = -b(1-\bar{\gamma}-\varepsilon)u_0^\beta, \quad x \in \bar{\Omega}. \tag{3.21}$$

Hence, from (3.20) and (3.21) it follows that

$$\begin{aligned} &\nu(x,0) = 0 \quad \nu_t(x,0) = b\big(\bar{\gamma}+\varepsilon-\gamma(\bar{x})\big)u_0^\beta(x) + o(u_0^\beta), \\ &\qquad x \to \bar{x}, \quad x \in \Omega, \quad \bar{x} \in \partial\Omega, \end{aligned} \tag{3.22}$$

where $\nu = g - u$. Obviously, for fixed ε, the domain Ω_ε may be chosen so as to satisfy

$$\nu_t(x,0) > 0 \quad \text{for } x \in \partial\Omega_\varepsilon,$$

and hence

$$\nu(x,t) \geq 0 \quad \text{for } (x,t) \in \partial\Omega_\varepsilon \times [0,\delta_\varepsilon], \tag{3.23}$$

for some $\delta_\varepsilon \in (0,\delta_\varepsilon^1)$. If on the contrary $\varepsilon < 0$, then taking $C = \varepsilon$ in (3.1), (3.21) and (3.22) are easily obtained, with $\bar\gamma = 0$ on the right-hand sides. Obviously Ω_ε may be chosen so that

$$\nu(x,t) \leq 0 \quad \text{for } (x,t) \in \partial\Omega_\varepsilon \times [0,\delta_\varepsilon], \tag{3.24}$$

for some $\delta_\varepsilon \in (0,\delta_\varepsilon^1)$. Finally, similar analysis as in the end of the Proof of Theorem 2.1 (see (3.12)) leads us to the estimates (2.9) and (2.10).

From (2.10) we have

$$(1+\varepsilon)^{-1} \ \leq b(1-\beta)\eta(x)/u_0^{1-\beta}(x) \ \leq \ (1-\bar\gamma-\varepsilon)^{-1}, \quad x \in \Omega\backslash\Omega_\varepsilon. \tag{3.25}$$

Because $\varepsilon > 0$ is arbitrary, from (3.25), (2.11) follows.

For $(x,t) \in \{(x,t) : x \in \Omega\backslash\bar\Omega_\varepsilon,\ 0 \leq t < \min(\delta_\varepsilon,\ u_0^{1-\beta}(x)/b(1-\beta)(1-\bar\gamma))\}$ and for $\varepsilon > 0$ an arbitrary sufficiently small number, from the right inequality of (2.9) we have

$$\frac{u(x,t)}{\big[u_0^{1-\beta} - b(1-\beta)(1-\bar\gamma)t\big]^{1/(1-\beta)}} \ \leq \ \frac{\big[u_0^{1-\beta} - b(1-\beta)(1-\bar\gamma-\varepsilon)t\big]^{1/(1-\beta)}}{\big[u_0^{1-\beta} - b(1-\beta)(1-\bar\gamma)t\big]^{1/(1-\beta)}}$$

Taking $t = \phi_c(x)$, $x \in \Omega\backslash\Omega_\varepsilon$, $C > b(1-\beta)(1-\bar\gamma)$ we have

$$\frac{u(x,t)}{\big[u_0^{1-\beta} - b(1-\beta)(1-\bar\gamma)t\big]^{1/(1-\beta)}} \ \leq \ \Big(\frac{C - b(1-\beta)(1-\varepsilon-\bar\gamma)}{C - b(1-\beta)(1-\bar\gamma)}\Big)^{1/(1-\beta)}. \tag{3.26}$$

Because $\varepsilon > 0$ is arbitrary, (3.26) implies the second inequality of (2.12). The first inequality from (2.12) may be proved similarly and the theorem is proved.

Proof of Theorem 2.3. As in the Proof of Theorem 2.1, we may obtain (3.4), where

$$S = -\frac{m}{b}u_0^{-2\beta}|\nabla u_0|^2 - \frac{m}{b}\big(u_0^{1-2\beta}\Delta u_0 - \beta u_0^{-2\beta}|\nabla u_0|^2\big)\Big[1 - \frac{b(1-\beta)(1-C)t}{u_0^{1-\beta}}\Big]. \tag{3.5a}$$

From (2.6) it follows that $S\big|_{t=0}$ satisfies the condition (3.14). Obviously

$$S = -\frac{m}{b}u_0^{-2\beta}|\nabla u_0|^2, \text{ as } t = u_0^{1-\beta}(x)/b(1-\beta)(1-C), \quad x \in \Omega\backslash\bar{\Omega}_1. \quad (3.27)$$

From (3.5a) we have

$$S_t = m(1-\beta)(1-C)u_0^{-\beta-1}\big(u_0\Delta u_0 - \beta|\nabla u_0|^2\big),$$

and hence, in view of (2.7), S_t retains its sign in D_+. Therefore, from (3.14), (3.27) and (2.13) it follows that if $\rho \geq 0$, then for arbitrary ε with $|\varepsilon|$ sufficiently small there exists a $\delta_\varepsilon > 0$ and a domain Ω_ε with smooth and compact boundary and with $\bar{\Omega}_1 \subset \bar{\Omega}_\varepsilon \subset \Omega$ such that

$$\begin{aligned} &-\bar{\gamma} - |\varepsilon|/2 \leq S \leq -\underline{\gamma}_1 + |\varepsilon|/2 \quad \text{for } x \in \Omega\backslash\Omega_\varepsilon, \\ &0 \leq t < \min\big(u_0^{1-\beta}(x)/b(1-\beta)(1-C),\ \delta_\varepsilon\big). \end{aligned} \quad (3.28)$$

If $\varepsilon > 0$, then taking $C = \bar{\gamma} + \varepsilon$ in (3.1), it follows from (3.4), (3.5a), (3.28) that

$$Lg \geq b(\varepsilon/2)g^\beta \text{ for } x \in \Omega\backslash\Omega_\varepsilon, \quad 0 \leq t < \min\big(u_0^{1-\beta}/b(1-\beta)(1-C), \delta_\varepsilon\big). \quad (3.29)$$

If $\varepsilon < 0$, then taking $C = \underline{\gamma}_1 + \varepsilon$ in (3.1), it follows from (3.4), (3.5a) and (3.28) that

$$Lg \leq b(\varepsilon/2)g^\beta \text{ for } x \in \Omega\backslash\Omega_\varepsilon, \quad 0 \leq t < \min\big(u_0^{1-\beta}/b(1-\beta)(1-C), \delta_\varepsilon\big). \quad (3.30)$$

Obviously $C < 1$, for $|\varepsilon|$ sufficiently small. Finally, similar analysis as in the end of Proofs of Theorems 2.1, 2.2 (see (3.20)–(3.24) and (3.12)) leads us to (2.14), (2.15). If on the other hand, $\rho = -1$, then from (3.14), (3.27) and (2.13) it follows that for arbitrary ε with $|\varepsilon|$ sufficiently small there exists a $\delta_\varepsilon > 0$ and a domain Ω_ε, as before, such that

$$\begin{aligned} &-\overline{\gamma}_1 - |\varepsilon|/2 \leq S \leq -\underline{\gamma} + |\varepsilon|/2 \quad \text{for } x \in \Omega\backslash\Omega_\varepsilon, \\ &0 \leq t < \min\big(u_0^{1-\beta}(x)/b(1-\beta)(1-C),\ \delta_\varepsilon\big). \end{aligned} \quad (3.31)$$

If $\varepsilon > 0$, then taking $C = \overline{\gamma}_1 + \varepsilon$ in (3.1), from (3.4), (3.5a) and (3.31) we have (3.29), but if on the contrary $\varepsilon < 0$, then taking $C = \underline{\gamma} + \varepsilon$ in (3.1), (3.30) follows from (3.4), (3.5a) and (3.31). Obviously $C < 1$, for $|\varepsilon|$

sufficiently small. Similar analysis as in the ends of Theorems 2.1 and 2.2 again leads us to (2.14) and (2.15). If $\gamma(\bar{x}) = \gamma_1(\bar{x}) \equiv \gamma,\ x \in \partial\Omega$, then the estimates (2.14) and (2.15) imply (2.16) and (2.17). Proof is similar to the proof of estimates (2.4) and (2.5) made at the end of the Proof of Theorem 2.1. The theorem is proved.

Proof of Theorem 2.4. First we prove the left-hand sides of the estimates (2.19) and (2.20). As in the Proof of Theorem 2.1 we may obtain (3.4) and (3.5). From (3.5) we have

$$S = -\frac{m}{b}u_0^{m-2-\beta}\big[1 - \frac{b(1-\beta)(1-C)t}{u_0^{1-\beta}}\big]^{\frac{m+\beta-2}{1-\beta}}E, \tag{3.32a}$$

$$E = (m-1+\beta)|\nabla u_0|^2 + \big(u_0\Delta u_0 - \beta|\nabla u_0|^2\big)\big[1 - \frac{b(1-\beta)(1-C)t}{u_0^{1-\beta}}\big]. \tag{3.32b}$$

Obviously

$$E\big|_{t=0} = (m-1)|\nabla u_0|^2 + u_0\Delta u_0, \tag{3.33}$$

and on the surface $t = u_0^{1-\beta}(x)/b(1-\beta)(1-C)$, $x \in \Omega\backslash\bar{\Omega}_1$, we have

$$E = (m-1+\beta)|\nabla u_0|^2. \tag{3.34}$$

Since

$$\operatorname{sign}(E_t) = -\rho \quad \text{for } x \in \Omega\backslash\bar{\Omega}_1, \tag{3.35}$$

from (3.33) and (3.34) it follows that if $\rho \geq 0$, then for arbitrary ε with $|\varepsilon|$ sufficiently small there exists a $\delta_\varepsilon > 0$ and a domain Ω_ε with smooth and compact boundary $\partial\Omega_\varepsilon$ and with $\bar{\Omega}_\varepsilon \subset \Omega$ such that

$$E \geq (m-1+\beta)|\nabla u_0|^2 \text{ for } x\epsilon\Omega\backslash\Omega_\varepsilon,\ 0 \leq t < \min\big(\frac{u_0^{1-\beta}(x)}{b(1-\beta)(1-C)}, \delta_\varepsilon\big), \tag{3.36}$$

but if, on the contrary $\rho = -1$, then

$$\begin{aligned} &E \geq (m-1)|\nabla u_0|^2 + u_0\Delta u_0 \quad \text{for } x \in \Omega\backslash\Omega_\varepsilon, \\ &0 \leq t < \min\big(u_0^{1-\beta}/b(1-\beta)(1-C),\ \delta_\varepsilon\big). \end{aligned} \tag{3.37}$$

Hence, if $\rho \geq 0$ the from (3.32) and (3.36) we have

$$\begin{aligned} &S \leq -\frac{m}{b}(m-1+\beta)u_0^{m-2-\beta}|\nabla u_0|^2, \quad x \in \Omega\backslash\Omega_\varepsilon, \\ &0 \leq t < \min\big(u_0^{1-\beta}/b(1-\beta)(1-C),\ \delta_\varepsilon\big), \end{aligned} \tag{3.38}$$

but if $\rho = -1$, then from (3.32) and (3.37) it follows that

$$S \le -(\Delta u_0^m / b u_0^\beta),\ x \in \Omega \backslash \Omega_\varepsilon,\ 0 \le t < \min(u_0^{1-\beta}/b(1-\beta)(1-C), \delta_\varepsilon). \tag{3.39}$$

From (3.38) and (3.39) it follows that for arbitrary ε with $|\varepsilon|$ sufficiently small there exists a $\delta_\varepsilon > 0$ and a domain Ω_ε as before such that

$$\gamma_2 + S < |\varepsilon|/2 \text{ for } x \epsilon \Omega \backslash \Omega_\varepsilon, \quad 0 \le t < \min(u_0^{1-\beta}/b(1-\beta)(1-C), \delta_\varepsilon). \tag{3.40}$$

Let $\varepsilon < 0$. Taking $C = \varepsilon + \gamma_2$ in (3.1), from (3.4), (3.32) and (3.40) it follows that

$$Lg \le b(\varepsilon/2) g^\beta \text{ for } x \epsilon \Omega \backslash \Omega_\varepsilon,\ 0 \le t < \min(u_0^{1-\beta}/b(1-\beta)(1-C), \delta_\varepsilon). \tag{3.41}$$

It should be noted that if $\rho \ge 0$, then we have

$$0 \le (m/b)(m-1+\beta) u_0^{m-2-\beta} |\nabla u_0|^2 \le \Delta u_0^m / b u_0^\beta, \quad x \in \Omega \backslash \bar{\Omega}_1, \tag{3.42}$$

which together with (2.6) implies the finiteness of γ_2. It should also be stressed that the constant C satisfies the condition $C < 1$. Indeed, if $\rho = -1$, then $\gamma_2 = \underline{\gamma} < 1$ as a consequence of (2.6), but if on the contrary $\rho \ge 0$, then $\gamma_2 \le \overline{\gamma} < 1$ as a consequence of (3.42).

Suppose now that $\varepsilon < 0$. Similar analysis as in the proof of Theorems 2.1–2.3 (see e.g. (3.8)–(3.11)) leads to the following estimate for $\nu(x,t) = g(x,t) - u(x,t)$ (we take $C = \varepsilon + \gamma_2$ in (3.1)),

$$\nu(x,t) \le 0 \quad \text{for } (x,t) \in \partial\Omega_\varepsilon \times [0, \delta_\varepsilon],\ \delta_\varepsilon > 0.$$

Thus for arbitrary sufficiently small $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ and a domain Ω_ε, as before, such that the function $g(x,t)$ from (3.1) (with $C = \gamma_2 - \varepsilon$) satisfies the following conditions:

$$\begin{gathered}
g \epsilon C^{2,1}(D_\varepsilon \backslash G_{1\varepsilon}) \cap C^{1,1}(D_\varepsilon) \cap C^{0,1}(\bar{D}_\varepsilon), \quad g^m \in C^{2,1}(D_\varepsilon \backslash G_{1\varepsilon}) \\
G_{1\varepsilon} = \{(x,t) : x \in \Omega \backslash \bar{\Omega}_\varepsilon, \quad t = u_0^{1-\beta}(x) / (b(1-\beta)(1-\gamma_2+\varepsilon))\} \\
Lg \le 0,\ \text{for } x \epsilon \Omega \backslash \bar{\Omega}_\varepsilon,\ 0 < t < \min(\delta_\varepsilon, u_0^{1-\beta}(x) / (b(1-\beta)(1-\gamma_2+\varepsilon))) \\
Lg = 0,\ \text{for } x \epsilon \Omega \backslash \bar{\Omega}_\varepsilon,\ u_0^{1-\beta}(x) / (b(1-\beta)(1-\gamma_2+\varepsilon)) < t \le \delta_\varepsilon \\
g(x,0) = u(x,0),\ \text{for } x \in \bar{\Omega} \backslash \Omega_\varepsilon \\
g \le u,\ \text{on } S_\varepsilon = S_\varepsilon^1 \cup S_\varepsilon^2, \quad S_\varepsilon^1 = \partial\Omega \times [0, \delta_\varepsilon], \quad S_\varepsilon^2 = \partial\Omega_\varepsilon \times [0, \delta_\varepsilon].
\end{gathered} \tag{3.43}$$

By using a comparison Theorem [10, Theorem 0.2], (3.43) easily implies the left-hand sides of (2.19) and (2.20).

Let us now prove the right-hand sides of estimates (2.19) and (2.20). Let ε be an arbitrary positive number. Consider the function

$$g = \big[u_0^{(m-\beta)/2} - t/\varepsilon\big]_+^{2/(m-\beta)}, \quad (x,t) \in D,$$

where the positive number δ is chosen so as to satisfy

$$\delta \le \sup_{\Omega\backslash\Omega_1} \varepsilon u_0^{(m-\beta)/2}.$$

Obviously g is non-negative in D and

$$g \in C^{2,1}(D\backslash G) \cap C^{1,1}(D) \cap C^{0,1}(\bar{D}), \quad g^m \in C^{2,1}(D),$$
$$G = \big\{(x,t): \ x \in \Omega\backslash\bar{\Omega}_1, \quad t = \varepsilon u_0^{(m-\beta)/2}\big\}.$$

Let us estimate the expression Lg in $D_+ = \{(x,t): \ x \in \Omega\backslash\bar{\Omega}_1, \ 0 < t < \min(\delta, \varepsilon u_0^{(m-\beta)/2}\}$

$$Lg = -2/(\varepsilon(m-\beta))g^{(2+\beta-m)/2} - (m/2)(m+\beta)u_0^{m-\beta-2}|\nabla u_0|^2 g^\beta \tag{3.44a}$$
$$- m u_0^{(m-\beta-4)/2}\big(u_0\Delta u_0 + \frac{m-\beta-2}{2}|\nabla u_0|^2\big)g^{(m+\beta)/2} + bg^\beta = bg^\beta S,$$

where

$$\begin{aligned} S =& 1 - \frac{m(m+\beta)}{2b}u_0^{m-\beta-2}|\nabla u_0|^2 \\ & - \frac{2}{\varepsilon b(m-\beta)}u_0^{(2-\beta-m)/2}\big[1 - \frac{t}{\varepsilon u_0^{(m-\beta)/2}}\big]^{\frac{2-m-\beta}{m-\beta}} \\ & - \frac{m}{b}u_0^{m-\beta-2}\big(u_0\Delta u_0 + \frac{m-\beta-2}{2}\ |\nabla u_0|^2\big)\big[1 - \frac{t}{\varepsilon u_0^{(m-\beta)/2}}\big]. \end{aligned} \tag{3.44b}$$

Since

$$S\big|_{t=0} = 1 - \Delta u_0^m/bu_0^\beta - \big(2/(\varepsilon b(m-\beta))\big)u_0^{(2-\beta-m)/2},$$

from (2.6) it follows that

$$S\big|_{t=0} \to 1 - \gamma(\bar{x}) > 0, \quad \text{as } x \to \bar{x}, \quad \bar{x} \in \partial\Omega, \quad x \in \Omega. \tag{3.45}$$

Suppose $\theta = -1$. Then from (3.44b) we have

$$S \geq 1 - \frac{m(m+\beta)}{2b} u_0^{m-\beta-2} |\nabla u_0|^2 - \frac{2}{\varepsilon b(m-\beta)} u_0^{\frac{2-\beta-m}{2}} \big[1 - \frac{t}{\varepsilon u_0^{(m-\beta)/2}}\big]^{\frac{2-m-\beta}{m-\beta}}.$$

Hence, from (2.13) it follows that for arbitrary sufficiently small $\varepsilon > 0$ there exists a $\delta_\varepsilon^1 > 0$ and a domain Ω_ε with smooth and compact boundary and with $\bar{\Omega}_1 \subset \bar{\Omega}_\varepsilon \subset \Omega$, such that

$$S \geq 0, \quad x \in \Omega \backslash \Omega_\varepsilon, \quad 0 \leq t < \min\big(\delta_\varepsilon^1,\ \varepsilon u_0^{(m-\beta)/2}\big). \tag{3.46}$$

Suppose $\theta \geq 0$. Then we have

$$S_t = \frac{2(2-m-\beta)}{\varepsilon^2 b(m-\beta)^2}\ u_0^{1-m} \big[1 - \frac{t}{\varepsilon u_0^{(m-\beta)/2}}\big]^{\frac{2(1-m)}{m-\beta}} + \frac{m}{b\varepsilon} u_0^{\frac{m-\beta-4}{2}}$$
$$\big(u_0 \Delta u_0 + \frac{m-\beta-2}{2} |\nabla u_0|^2\big), \quad (x,t) \in D_+.$$

Obviously $S_t \geq 0$ and hence, in view of (3.45), for arbitrary sufficiently small $\varepsilon > 0$ there exists a $\delta_\varepsilon^1 > 0$ and a domain Ω_ε as before such that (3.46) is valid. From (3.44) and (3.46) it follows that

$$Lg \geq 0, \quad x \in \Omega \backslash \Omega_\varepsilon, \quad 0 < t < \min\big(\delta_\varepsilon^1,\ \varepsilon u_0^{(m-\beta)/2}\big).$$

Obviously

$$Lg = 0, \quad \text{as } x \in \Omega \backslash \Omega_\varepsilon, \quad \varepsilon u_0^{(m-\beta)/2} < t \leq \delta_\varepsilon^1.$$

Let $\partial\Omega_\varepsilon \cap \partial\Omega_1 = \emptyset$. As mentioned in the Proof of Theorem 2.1, the solution $u(x,t)$ is a positive classical solution of equation (1.1) in $\partial\Omega_\varepsilon \times [0,\delta]$ for some $\delta > 0$. It is clear that

$$g(x,0) = u_0(x), \quad g_t(x,0) = -\big(2/\big(\varepsilon(m-\beta)\big)\big) u_0^{(2-m+\beta)/2}, \quad x \in \bar{\Omega} \backslash \Omega_1. \tag{3.47}$$

From (3.20) and (3.47) it follows that

$$\nu(x,0) = 0, \tag{3.48}$$
$$\nu_t(x,0) = -\big(2/\big(\varepsilon(m-\beta)\big)\big) u_0^{(2-m+\beta)/2} + \big(1 - \gamma(\bar{x})\big) b u_0^\beta(x) + o(u_0^\beta),$$
$$x \to \bar{x}, \quad x \in \Omega, \quad \bar{x} \in \partial\Omega,$$

where $\nu = g - u$. Since $m+\beta < 2$, from (3.48) it follows that given $\varepsilon > 0$, the domain Ω_ε may be chosen so as to satisfy $\nu(x,t) \geq 0$ for $(x,t) \in \partial\Omega_\varepsilon \times [0,\delta_\varepsilon]$ and for some $\delta_\varepsilon \in (0, \delta_\varepsilon^1]$.

Thus, for arbitrary sufficiently small $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ and a domain Ω_ε, as before, such that the function g satisfies the following conditions

$$\begin{aligned}
&g \in C^{2,1}\big(D_\varepsilon \backslash G_{2\varepsilon}\big) \cap C^{1,1}(D_\varepsilon) \cap C^{0,1}(\bar{D}_\varepsilon),\\
&g^m \in C^{2,1}\big(D_\varepsilon \backslash G_{2\varepsilon}\big), \quad Lg \in L^\infty(D_\varepsilon),\\
&G_{2\varepsilon} = \big\{(x,t):\ x \in \Omega\backslash\bar{\Omega}_\varepsilon, \quad t = \varepsilon u_0^{(m-\beta)/2}\big\}\\
&Lg \geq 0 \quad \text{for } x \in \Omega\backslash\bar{\Omega}_\varepsilon, \quad 0 < t < \min\big(\delta_\varepsilon,\ \varepsilon u_0^{(m-\beta)/2}\big)\\
&Lg = 0 \quad \text{for } x \in \Omega\backslash\bar{\Omega}_\varepsilon, \quad \varepsilon u_0^{(m-\beta)/2} < t \leq \delta_\varepsilon\\
&g(x,0) = u(x,0) \quad \text{for } x \in \bar{\Omega}\backslash\Omega_\varepsilon\\
&g \geq u, \quad S_\varepsilon = S_\varepsilon^1 \cup S_\varepsilon^2, \quad S_\varepsilon^1 = \partial\Omega \times [0,\delta_\varepsilon], \quad S_\varepsilon^2 = \partial\Omega_\varepsilon \times [0,\delta_\varepsilon].
\end{aligned} \tag{3.49}$$

By using a comparison Theorem [10], (3.49) easily implies the right-hand sides of (2.19) and (2.20) and the theorem is proved.

Proof of Theorem 2.5. Let $\varepsilon > 0$ be an arbitrary sufficiently small number and introduce the function

$$g_\varepsilon(x,t) = u_0(x) \exp\big[-b\big(1 - \bar{\gamma}_b - \varepsilon_b\big)t\big], \quad \varepsilon_b = \varepsilon \text{ sign } b.$$

Obviously, g_ε be non-negative in D, $g_\varepsilon \in C^{2,1}_{x,t}(D) \cap C^{0,1}(\bar{D})$ and $Lg_\varepsilon \in L^\infty(D)$. Let us estimate Lg_ε in D, for some $\delta > 0$

$$\begin{aligned}
Lg_\varepsilon = &- b\big(1 - \bar{\gamma}_b - \varepsilon_b\big)u_0(x) \exp\big[-b\big(1 - \bar{\gamma}_b - \varepsilon_b\big)\ t\big]\\
&- \big(\Delta u_0^m\big) \exp\big[-bm\big(1 - \bar{\gamma}_b - \varepsilon_b\big)\ t\big]\\
&+ bu_0(x) \exp\big[-b\big(1 - \bar{\gamma}_b - \varepsilon_b\big)\ t\big] = bg_\varepsilon\big\{\varepsilon_b + S\big\},
\end{aligned} \tag{3.50}$$

where

$$S = \bar{\gamma}_b - \frac{\Delta u_0^m}{bu_0} \exp\big[-b(m-1)\big(1 - \bar{\gamma}_b - \varepsilon_b\big)\ t\big].$$

From (2.22) it follow that for arbitrary sufficiently small $\varepsilon > 0$ there exists a $\delta_\varepsilon^1 > 0$ and a domain Ω_ε with smooth boundary and with $\bar{\Omega}_1 \subset \bar{\Omega}_\varepsilon \subset \Omega$ such that

$$S \text{ sign } b > -\varepsilon/2, \quad x \in \Omega\backslash\Omega_\varepsilon, \quad 0 \leq t \leq \delta_\varepsilon^1.$$

Hence, from (3.50) it follows that

$$Lg_\varepsilon > b\big(\varepsilon_b/2\big)g_\varepsilon \quad \text{for } x \in \Omega\backslash\Omega_\varepsilon, \quad 0 \le t \le \delta_\varepsilon^1. \tag{3.51}$$

Obviously

$$g_\varepsilon(x,0) = u_0(x), \quad g_{\varepsilon t}(x,0) = b\big(\bar{\gamma}_b + \varepsilon_b - 1\big)u_0(x), \quad x \in \Omega\backslash\Omega_1. \tag{3.52}$$

Let $\partial\Omega_\varepsilon \cap \partial\Omega_1 = \emptyset$. As mentioned before, the solution $u(x,t)$ is a positive classical solution of equation (1.1) in $\partial\Omega_\varepsilon \times [0,\delta]$, for some $\delta > 0$. Moreover

$$u_t(x,0) = \Delta u_0^m - bu_0 = b(\gamma(\bar{x}) - 1)u_0 + o(u_0), \quad x \to \bar{x}, \quad x \in \Omega, \quad \bar{x} \in \partial\Omega. \tag{3.53}$$

From (3.52) and (3.53) it follows that

$$\begin{aligned} &\nu(x,0) = 0 \\ &\nu_t(x,0) = b\big(\bar{\gamma}_b + \varepsilon_b - \gamma(\bar{x})\big)u_0 + o(u_0), \quad x \to \bar{x}, \quad x \in \Omega, \quad \bar{x} \in \partial\Omega, \end{aligned} \tag{3.54}$$

where $\nu = g_\varepsilon - u$. Obviously, for given ε, the domain Ω_ε may be chosen so as to satisfy $\nu_t(x,0) > 0$ for $x \in \partial\Omega_\varepsilon$ and hence

$$\nu(x,t) \ge 0, \quad \text{for } (x,t) \in \partial\Omega_\varepsilon \times [0,\delta_\varepsilon],$$

for some $\delta_\varepsilon \in (0,\delta_\varepsilon^1]$. Finally, similar analysis as in the proof of the previous theorems leads to the right hand side of the estimate (2.23). The left-hand side of the estimate (2.23) may be proved similarly by using a function

$$g_\varepsilon(x,t) = u_0(x)\exp\big[-b\big(1 - \underline{\gamma}_b + \varepsilon_b\big)t\,\big], \quad (x,t) \in D.$$

The theorem is proved.

Proof of Theorem 2.6. Suppose $u_1(x,t)$ is a solution of problem (1.1)–(1.3), with $b = 1$, $\beta = 1$. It may easily be checked that

$$u(x,\tau) = \big[1 - (m-1)\tau\big]^{1/(1-m)} u_1\big(x, (\ln(1-(m-1)\tau))/(1-m)\big)$$

will be solution of the problem

$$\begin{aligned} u_\tau &= \Delta u^m, \quad x \in \Omega, \quad 0 < \tau \le T_1, \\ u(x,0) &= u_o(x), \quad x \in \Omega, \\ u(x,t) &= 0, \quad x \in \partial\Omega, \quad 0 \le \tau \le T_1, \end{aligned}$$

for some positive T_1. Hence, the estimate (2.25) directly follows from (2.23). The theorem is proved.

Proof of Theorem 2.7. Let $\varepsilon > 0$ be an arbitrary sufficiently small number and consider the function

$$g_\varepsilon(x,t) = \big[u_0^{1-\beta}(x) + b(\beta-1)\big(1-\bar{\gamma}_b-\varepsilon_b\big)t\big]^{1/(1-\beta)}. \tag{3.55}$$

Obviously, g_ε be non-negative, continuous and $g_\varepsilon \in C^{2,1}_{x,t}(D) \cap C^{0,1}(\bar{D})$ and $Lg_\varepsilon \in L^\infty(D)$. Let us estimate Lg_ε in D, for some $\delta > 0$. Similar estimations of Lg_ε as in the proof of Theorem 2.1 lead to (3.4) and (3.5), for $(x,t) \in D$ and with $\bar{\gamma}_b + \varepsilon_b$ instead of C in the right-hand sides of (3.4) and (3.5). From (2.6) and (2.26) it follows that

$$\lim_{x\to\bar{x}} S = \lim_{x\to\bar{x}} \big(-\Delta u_0^m / b u_0^\beta\big) = -\gamma(\bar{x}), \quad x \in \Omega, \quad \bar{x} \in \partial\Omega,$$

uniformly in $t \in [0,\delta]$. Hence, from (2.6) it follows that for arbitrary sufficiently small $\varepsilon > 0$ there exists a $\delta_\varepsilon^1 > 0$ and a domain Ω_ε with smooth boundary and with $\bar{\Omega}_1 \subset \bar{\Omega}_\varepsilon \subset \Omega$ such that

$$\big(\bar{\gamma}_b + S\big)\text{sign } b > \varepsilon/2, \quad x \in \Omega\backslash\Omega_\varepsilon, \quad 0 \le t \le \delta_\varepsilon^1.$$

Hence, from (3.4) it follows that

$$Lg_\varepsilon > b(\varepsilon_b/2) g_\varepsilon^\beta, \quad \text{as } x \in \Omega\backslash\Omega_\varepsilon, \quad 0 \le t \le \delta_\varepsilon^1.$$

Obviously,

$$g_\varepsilon(x,0) = u_0(x), \quad g_{\varepsilon t}(x,0) = b\big(\bar{\gamma}_b + \varepsilon_b - 1\big)u_0^\beta(x), \quad x \in \Omega\backslash\Omega_1. \tag{3.56}$$

Moreover,

$$u_t(x,0) = \Delta u_0^m - b u_0^\beta = b\big(\gamma(\bar{x}) - 1\big)u_0^\beta(x) + o(u_0^\beta), \quad x \to \bar{x},\ x \in \Omega,\ \bar{x} \in \partial\Omega. \tag{3.57}$$

From (3.56) and (3.57) it follows that

$$\nu(x,0) = 0, \tag{3.58}$$
$$\nu_t(x,0) = b\big(\bar{\gamma}_b + \varepsilon_b - \gamma(\bar{x})\big)u_0^\beta(x) + o(u_0^\beta), \quad x \to \bar{x},\ x \in \Omega, \quad \bar{x} \in \partial\Omega,$$

where $\nu = g_\varepsilon - u$. As before, Ω_ε may be chosen so as to satisfy

$$\nu(x,t) \geq 0, \quad \text{for } (x,t) \in \partial\Omega_\varepsilon \times [0, \delta_\varepsilon],$$

for some $\delta_\varepsilon \in (0, \delta_\varepsilon^1]$. Finally, similar analysis as in the proof of the previous theorems leads to the right-hand side of the estimate (2.27). The left-hand side may be proved similarly by using a function

$$g_\varepsilon(x,t) = \big[u_0^{1-\beta}(x) + b(\beta-1)\big(1 - \underline{\gamma}_b + \varepsilon_b\big)\ t\big]^{1/(1-\beta)}, \quad (x,t) \in D.$$

The theorem is proved.

Proof of Theorem 2.8. Let $\varepsilon > 0$ be an arbitrary sufficiently small number. Consider the function

$$g_\varepsilon(x,t) = u_0(x)\big[1-(m-1)\ t\big]^{(\varepsilon+\bar\gamma)/(1-m)}, \quad (x,t) \in D, \quad 0 < \delta < (m-1)^{-1}.$$

Obviously, g_ε be non-negative, continuous and $g_\varepsilon \in C^{2,1}(D) \cap C^{0,1}(\bar D)$ and $Lg_\varepsilon \in L^\infty(D)$. Let us estimate the expression Lg_ε in D.

$$Lg_\varepsilon = u_0(x)\big[1-(m-1)\ t\big]^{\frac{\varepsilon+\bar\gamma}{1-m} - 1}\ \big\{\varepsilon + S\big\}, \tag{3.59}$$

where

$$S = \bar\gamma - \frac{\Delta u_0^m}{u_0}\big[1-(m-1)\ t\big]^{1-\varepsilon-\bar\gamma} + b u_0^{\beta-1}(x)\big[1-(m-1)\ t\big]^{1+\frac{(\beta-1)(\varepsilon+\bar\gamma)}{1-m}}. \tag{3.60}$$

Since

$$\lim_{x\to\bar x}\big(\Delta u_0^m/u_0\big) = \gamma(\bar x), \quad x \in \Omega, \quad \bar x \in \partial\Omega,$$

from (3.60) it follows that for arbitrary sufficiently small $\varepsilon > 0$ there exists a $\delta_\varepsilon^1 > 0$ and a domain Ω_ε with smooth boundary and with $\bar\Omega_1 \subset \bar\Omega_\varepsilon \subset \Omega$ such that

$$S > -\varepsilon/2 \quad \text{for } x \in \Omega\backslash\Omega_\varepsilon, \quad 0 \leq t \leq \delta_\varepsilon^1. \tag{3.61}$$

From (3.59) and (3.61) we have

$$Lg_\varepsilon > u_0(x)\big[1-(m-1)t\big]^{\frac{\varepsilon+\bar\gamma}{1-m} - 1}\big(\varepsilon/2\big), \quad x \in \Omega\backslash\Omega_\varepsilon, \quad 0 \leq t \leq \delta_\varepsilon^1.$$

Obviously $g_\varepsilon(x,0)=u_0(x)$, $g_{\varepsilon t}(x,0)=\big(\bar\gamma+\varepsilon\big)u_0(x)$, $x\in\Omega\backslash\Omega_1$. Moreover,

$$u_t(x,0)=\Delta u_0^m-bu_0^\beta=\gamma(\bar x)u_0-bu_0^\beta+o(u_0),\quad x\to\bar x,\quad x\in\Omega,\quad \bar x\in\partial\Omega. \tag{3.62}$$

Hence,

$$\nu(x,0)=0,\quad \nu_t(x,0)=\big(\bar\gamma+\varepsilon-\gamma(\bar x)\big)u_0(x)+o(u_0), x\to\bar x,\ x\in\Omega,\ \bar x\in\partial\Omega,$$

where $\nu=g_\varepsilon-u$. As before, Ω_ε may be chosen so as to satisfy

$$\nu(x,t)\geq 0,\quad \text{for } (x,t)\in\partial\Omega_\varepsilon\times[0,\delta_\varepsilon]$$

for some $\delta_\varepsilon\in(0,\delta_\varepsilon^1]$. Finally, similar analysis as in the proofs of the previous theorems leads to the right-hand side of estimate (2.25). The left-hand side of estimate (2.25) may be proved similarly by using a function

$$g_\varepsilon(x,t)=u_0(x)\big[1-(m-1)t\big]^{\frac{-\varepsilon+\gamma}{1-m}},\quad (x,t)\in D,\ 0<\delta<(m-1)^{-1},\ \varepsilon>0.$$

The theorem is proved.

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