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GLOBAL SOLUTIONS OF A SEMILINEAR PARABOLIC EQUATION

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Abstract. Radial solutions of the Gelfand equation on an N–dimensional ball are studied for $3 \le N \le 9$. It is shown that global classical solutions are uniformly bounded while unbounded global L^1–solutions are constructed for some parameter range.

1. Introduction. In this paper we study the large time behavior of solutions of the problem

$$\text{(P)} \qquad \begin{cases} u_t = \Delta u + \lambda e^u, & x \in B_1(0),\ t > 0, \\ u = 0, & x \in \partial B_1(0),\ t > 0, \\ u(x,0) = u_0(|x|), & x \in B_1(0), \end{cases}$$

where $B_1(0) = \{x \in \mathbb{R}^N : |x| < 1\}$, u_0 is a radial continuous function on $\overline{B}_1(0)$ vanishing on $\partial B_1(0)$ and λ is a positive parameter.

This initial-boundary value problem has attracted much attention in the research literature. It is relevant as a model of solid fuel ignition (see [2]) and has also gained theoretical significance due to a variety of interesting phenomena exhibited by its solutions (see [2] for references and a survey of results obtained up to 1988; a classical work on blow-up of solutions of (P) is [11]).

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For results on the corresponding stationary problem

$$\text{(SP)}\qquad \begin{cases} \phi_{rr} + \frac{N-1}{r}\phi_r + \lambda e^{\phi} = 0, & r \in (0,1), \\ \phi_r(0) = 0, \quad \phi(1) = 0, \end{cases}$$

we refer to [13, 16] and [25]. Let us recall the following facts about (SP). There is a constant $\lambda_1 > 0$ such that (SP) has a solution for $0 < \lambda < \lambda_1$ but not for $\lambda > \lambda_1$. For $N > 2$ there exists $\lambda_2 > 0$ such that (SP) has a unique solution for $0 < \lambda < \lambda_2$. Moreover, $\lambda_2 < \lambda_1$ if $3 \le N \le 9$ and $\lambda_1 = \lambda_2$ if $N \ge 10$. Also, for $3 \le N \le 9$ there exists $\lambda_3 \in (\lambda_2, \lambda_1)$ such that (SP) has at most two solutions for $\lambda > \lambda_3$.

The main aim of this paper is to show that every global classical solution of (P) is uniformly bounded if $3 \le N \le 9$. For $N = 1, 2$ this is known even for general domains and initial data (see [10]). On the other hand, for $N \ge 10$ and $\lambda = \lambda_1$, global unbounded classical solutions do exist (cf. [19, 28]). For $3 \le N \le 9$ it is known that global unbounded L^1–solutions (cf. Definition 3.1) exist for $0 < \lambda < \lambda_2$ (see [21]), which makes the problem of existence of unbounded global classical solutions even more interesting. It was conjectured (see [21] and [31]) that such solutions might exist for some dimensions $3 \le N \le 9$ and $\lambda > 0$. Our result rules out this conjecture, as far as radial solutions are concerned. The method we use does not apply to nonradial solutions because it depends, to a large extent, on the fact that the solution of (P) can also be viewed as a solution of the following problem with a one–dimensional space variable:

$$\text{(PR)}\qquad \begin{cases} u_t = u_{rr} + \frac{N-1}{r}u_r + \lambda e^{u}, & r \in (0,1),\ t > 0 \\ u_r(0,t) = 0, \quad u(1,t) = 0, & t > 0. \end{cases}$$

The main idea of our proof of boundedness of global classical solutions is to employ the notion of "intersection–comparison" with a particular self–similar solution. We were inspired by the proof of Theorem 15.1 in [12] where a similar result was shown for the problem

$$\begin{aligned} u_t &= \Delta u^m + u^p, && x \in B_R(0),\ t > 0, \\ u &= 0, && x \in \partial B_R(0),\ t > 0, \\ u(x,0) &= u_0(|x|) \ge 0, && x \in B_R(0), \end{aligned}$$

where $m \ge 1$, $B_R(0) = \{x \in \mathbb{R}^N : |x| < R\}$, $N \ge 3$ and $p \in (p_s, p_p)$, with $p_s = m(N+2)/(N-2)$ and

$$p_p = \begin{cases} \infty & \text{if } N \le 10, \\ 1 + \frac{3m + [(m-1)^2(N-10)^2 + 2(m-1)(5m-4)(N-10) + 9m^2]^{\frac{1}{2}}}{N-10} & \text{if } N > 10. \end{cases}$$

Note, however, that unlike the case in [12], the nonincrease of the number of zeros of the difference of two solutions, by itself, is not sufficient for our proof. We also employ the more subtle "zero number diminishing property" (see Section 2).

Another ingredient of our proof is the stabilization of L^1–solutions. More specifically, we observe that if u is a global L^1–solution emanating from u_0, and u is a monotone increasing limit of classical solutions, then $u(\cdot,t)$ converges as $t \to \infty$ to a steady state solution which may be singular at the origin. This generalizes a result from [31] where it is assumed that $\lambda < \lambda_2$ and that u is a classical unbounded solution.

Having ruled out unbounded classical solutions, our next concern is the problem of existence of global unbounded L^1–solutions. In [21] it is proved that such solutions exist for $N \geq 3$ and $0 < \lambda < \lambda_2$. We show that for $3 \leq N \leq 9$ there is an $\varepsilon > 0$ such that global unbounded L^1–solutions exist for $0 < \lambda < \lambda_3 + \varepsilon$. For $\lambda \in (\lambda_2, \lambda_3]$ we actually construct a connecting (heteroclinic) orbit between an unstable equilibrium ϕ and the minimal equilibrium ϕ_0, which blows up in finite time but continues to exist in the L^1–sense and converges to ϕ_0 pointwise (possibly not at the origin). It would be interesting to know whether or not this connecting orbit can become bounded again after blow-up. We also leave open the question whether or not global unbounded L^1–solutions exist for all $\lambda \in (0, \lambda_1]$. The nonexistence of such solutions for $\lambda > \lambda_1$ follows from [18] and [20].

The paper is organized as follows. In Section 2 we prove the boundedness of global classical solutions of Problem (P) for $3 \leq N \leq 9$. In Section 3 global unbounded L^1–solutions are constructed for $3 \leq N \leq 9$ and $0 < \lambda < \lambda_3 + \varepsilon$. A discussion of the structure of connecting orbits of (P) is also included in the section.

Throughout the paper we use several results on the stationary problem (SP) and its linearization. For the reader's convenience we have collected these results in the appendix.

Throughout the paper, X stands for the state space for problem (P) (i.e., (P) defines a dynamical system on X). Although the specific choice of X is not very relevant, due to the smoothing effect of the parabolic equation, it will be convenient to have X chosen such that it is imbedded in $C^1(\overline{B}_1(0))$:

$$X \hookrightarrow C^1(\overline{B}_1(0)).$$

Thus we define X to be the space of all radial functions in a fractional power space Y^α associated with the realization of the Laplacian on $Y = L^p(B_1(0))$, under Dirichlet boundary conditions (cf. [15]). If $p > N$ and $1 > \alpha > (N+p)/(2p)$, we have the desired imbedding, that is (P) is well–posed on X and all functions in X vanish on $\partial B_1(0)$.

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2. Global classical solutions. In this section we prove boundedness of global classical solutions of (P).

Throughout the section we assume $3 \le N \le 9$.

Theorem 2.1. *If the classical solution $u(|x|,t)$ of* (P) *is global, then it is uniformly bounded, that is*

$$\sup_{t>0, r\in[0,1]} |u(r,t)| < \infty.$$

Recall that global classical solutions exist only if $\lambda \le \lambda_1$, see [18], where λ_1 is the last bifurcation value of (SP).

We prepare the proof of the theorem by several lemmas. The first two contain the basic ingredients of our method: properties of the "zero number" and existence of a self–similar solution with an asymptotic profile.

For a continuous function ψ defined on an interval J, we put

$$z_J(\psi) = \#\{r \in J \colon \psi(r) = 0\}.$$

The subscript J is usually omitted if $J = [0,1]$ or if it is clear from the context which interval we have in mind.

Consider the linear equation

$$w_t = \Delta w + a(|x|,t)w, \quad t \in (T_1,T_2), x \in \Omega, \tag{2.1}$$

where Ω is either a ball, $\Omega = B_R(0)$, or an annulus, $\Omega = B_R(0) \setminus \overline{B}_{R_0}(0)$ with $R > R_0 > 0$, and a is a radial continuous function on $\overline{\Omega} \times (T_1,T_2)$.

Lemma 2.2. *Let $w(|x|,t)$ be a (classical) solution of (2.1) which is not identical to zero. Assume that $w(r,t)$ is continuous on $J\times(T_1,T_2)$ ($J=[0,R]$ for the ball and $J=[R_0,R]$ for the annulus) and either*

$$w(R,t)\equiv 0 \quad (t\in(T_1,T_2)) \tag{2.2}$$

or else

$$w(R,t)\neq 0 \quad (t\in(T_1,T_2)). \tag{2.3}$$

In case $\Omega=B_R(0)\setminus\overline{B}_{R_0}$ also assume that either

$$w(R_0,t)\equiv 0 \quad (t\in(T_1,T_2))$$

or else

$$w(R_0,t)\neq 0 \quad (t\in(T_1,T_2)).$$

Then the following properties are satisfied:

(i) *$z_J(w(\cdot,t))$ is finite for any $t\in(T_1,T_2)$,*
(ii) *$t\to z_J(w(\cdot,t))$ is monotone nonincreasing,*
(iii) *(Diminishing Property) if $w_r(r_0,t_0)=w(r_0,t_0)=0$ for some $r_0\in J$, $t_0\in(T_1,T_2)$, then*

$$z_J(w(\cdot,t))>z_J(w(\cdot,s)) \quad (T_1<t<t_0<s<T_2).$$

Proof. For $\Omega=B_R(0)$ and condition (2.2), properties (i)–(iii) are proved in [5, Theorem 2.1]. The proof given there also works under condition (2.3) (cf. the extension lemma, Lemma 2.4, in [5]). If $\Omega=B_R(0)\setminus\overline{B}_{R_0}(0)$ then $w(r,t)$ solves the (nonsingular) one-dimensional equation

$$w_t=w_{rr}+\frac{N-1}{r}w_r+a(r,t)w=0, \quad r\in(R_0,R),\ t\in(t_1,t_2).$$

In this case, (i) and (iii) are consequences of more general results proved in [1, Theorem C] and (ii) is well known (cf. references in [1], for example).

Observe that if $u_i(|x|,t)$, $i=1,2$, are two solutions of

$$u_t=\Delta u+\lambda e^u \tag{2.4}$$

on $\Omega\times(T_1,T_2)$ then $w=u_1-u_2$ is a solution of (2.1) with

$$a(r,t)=\int_0^1 \lambda e^{u_1(r,t)+s(u_2(r,t)-u_1(r,t))}(u_2(r,t)-u_1(r,t))ds.$$

This observation will frequently be used below, with no further notice.

Lemma 2.3. *There exists a solution v of the equation*

$$v_{\eta\eta} + \left(\frac{N-1}{\eta} - \frac{\eta}{2}\right) v_\eta + \lambda e^v - 1 = 0, \quad \eta > 0,$$

with the following properties:

(i) $v \in C^1([0,\infty))$ *and* $v'(0) = 0$, $v'(\eta) < 0$ $(\eta > 0)$,

(ii) *there is a constant* $c < 0$ *such that*

$$v(\eta) - \left(c + \ln \frac{2(N-2)}{\lambda \eta^2}\right) \to 0 \quad \text{as } \eta \to \infty,$$

(iii) *for any* $\gamma \in \mathbb{R}$ *the function* $v(\eta) - \left(\gamma + \ln \frac{2(N-2)}{\lambda\eta^2}\right)$ *has at most two zeros, counting multiplicity.*

Proof. For $\lambda = 1$ and $N = 3$, the existence of a solution v with properties (i), (ii) is proved in [21]. The case $3 \le N \le 9$ and $\lambda = 1$ can be treated in much the same way (see also [8] for a result in that direction). From $\lambda = 1$ one gets to the general case by adding the constant $-\ln \lambda$ to both v and $\ln \frac{2(N-2)}{\eta^2}$. Property (iii) for $\gamma = 0$ is obtained in the process of construction of v (see [21]). The proof of (iii) for any γ is a little involved and is postponed until the end of this section.

Note that (i) implies that $v(|x|)$ is a weak solution of

$$\Delta v - \frac{1}{2}(x \cdot \nabla) v + \lambda e^v - 1 = 0.$$

By elliptic regularity, v is of class C^∞.

Corollary 2.4. *Let v be as in Lemma* 2.3. *For $T > 0$ set*

$$u^*(r,t) = -\ln(T-t) + v\left(\frac{r}{\sqrt{T-t}}\right) \quad (t < T, r \ge 0).$$

Then $u^*(|x|,t)$ is a solution of (2.4) *on $\mathbb{R}^N \times (-\infty, T)$ with the following properties*:

(i) $u^*_r(0,t) = 0$ *and* $u^*_r(r,t) < 0$ $(r > 0,\ t < T)$,

(ii) $u^*(r,t) \to c + \ln \frac{2(N-2)}{\lambda r^2}$ *as $t \to T$, where the convergence is uniform on any r-interval $[\delta, \infty)$ with $\delta > 0$,*

(iii) *the function*

$$t \to u^*(1,t) \colon (-\infty, T) \to \mathbb{R}$$

has at most two zeros, counting multiplicity.

Proof. The fact that u^* is a solution of (2.4) and that properties (i), (ii) hold is obtained directly from Lemma 2.3 on setting $\eta = r/\sqrt{T-t}$. Property (iii) is a consequence of Lemma 2.3(iii) and the relation

$$u^*(1,t) = v\left(\frac{1}{\sqrt{T-t}}\right) - \left(\ln\lambda - \ln 2(N-2) + \ln\left(2(N-2)\frac{T-t}{\lambda}\right)\right). \tag{2.5}$$

Lemma 2.5. *Let $u_0 \in X$ be such that the classical solution u of* (P) *is global. Let u^* be as in Corollary* 2.4, *with T chosen so large that*

$$u^*(r,0) < \max\{0, u(r,0)\} \quad (r \in [0,1]). \tag{2.6}$$

Then there exists $T_0 \in (0,T)$ such that for any $t \in (T_0, T)$ one has

$$u^*(0,t) > u(0,t), \tag{2.7}$$

and $u^*(\cdot,t) - u(\cdot,t)$ has at most one zero (*counting multiplicity*) *in* $[0,1]$.

Proof. First note that the inequality $u^*(0,t) > u(0,t)$ is satisfied for $t < T$, $t \approx T$, because $u^*(r,t)$ blows up at $T < \infty$ and is decreasing in r.

By Lemma 2.3(i), $u^*(t,1) \to -\infty$ as $t \to -\infty$. Using this in conjunction with (2.6) and Corollary 2.4(iii), we see that one of the following possibilities must occur:

(a) $u^*(1,t) \le 0 \quad (t \in (0,T))$.
(b) There is $t_1 \in (0,T)$ such that

$$u^*(1,t) < 0 \ \ (t \in (0,t_1)), \quad u^*(1,t) > 0 \ \ (t \in (t_1,T)) \quad \text{and} \quad u_t^*(1,t_1) > 0. \tag{2.8}$$

(c) There are $0 < t_1 < t_2 < T$ such that

$$u^*(1,t) < 0 \quad (t \in (0,t_1) \cup (t_2,T)), \qquad u^*(1,t) > 0 \quad (t \in (t_1,t_2)) \\ \text{and} \quad u_t^*(1,t_1) > 0 > u_t^*(1,t_2). \tag{2.9}$$

We show that (a) is impossible and that the conclusion of the lemma holds in cases (b) and (c).

Define

$$w(r,t) = u^*(r,t) - u(r,t) \quad (t \in (0,T), r \in [0,1]).$$

Assume (a) holds. Then

$$w(r,0) < 0 \quad (r \in [0,1]) \quad \text{and} \quad w(1,t) \le 0 \quad (t \in (0,t)).$$

The maximum principle implies $w(r,t) < 0$ $(r \in [0,1), t \in (0,T))$, a contradiction to (2.7).

We proceed assuming that (b) or (c) holds. In both cases, the maximum principle gives

$$w(r,t) < 0 \quad (r \in [0,1), t \in [0,t_1]). \tag{2.10}$$

Moreover, the relation $w(1,t_1) = 0$ and the Hopf boundary point principle imply

$$w_r(1,t_1) > 0. \tag{2.11}$$

Further, $w(1,t) > 0$ for $t > t_1$, $t \approx t_1$. By the intermediate value theorem, for any such t the function $w(\cdot,t)$ has a zero in $[0,1]$. By (2.10), (2.11) and the implicit function theorem, this zero is unique, thus

$$z(w(\cdot,t)) = 1 \quad (t > t_1, t \approx t_1). \tag{2.12}$$

If (b) holds, the nonincrease of $z(w(\cdot,t))$ together with its diminishing property (Lemma 2.2) imply the conclusion of Lemma 2.5.

Assume (c). By (2.12), (2.9) and Lemma 2.2, we have $z(w(\cdot,t)) \le 1$ $(t \in (t_1,t_2))$. More specifically, either

(c1) $$z(w(\cdot,t)) = 0 \quad (t < t_2, t \approx t_2)$$

or else

(c2) $$z(w(\cdot,t)) \equiv 1 \quad (t \in (t_1,t_2)).$$

Consider (c1). By (2.9), we necessarily have

$$w(r,t) > 0 \quad (r \in [0,1], t < t_2, t \approx t_2).$$

The arguments leading to (2.12) above are easily adapted to this case to yield

$$z(w(\cdot,t)) = 1 \quad (t > t_2, t \approx t_2).$$

Using (2.9) and Lemma 2.2, we obtain the conclusion of Lemma 2.5.

Next consider (c2). Let $s(t)$ be the unique zero of $w(\cdot,t)$ $(t \in (t_1,t_2))$. By the diminishing property, this zero is simple. In particular, $s(t) > 0$ for $w_r(0,t) = 0$ (cf. (PR) and Corollary 2.4). Also $s(t) < 1$, by (2.9). Thus

$$w(r,t) \begin{cases} < 0 \;\; (0 \le r < s(t)), \\ > 0 \;\; (s(t) < r \le 0). \end{cases} \tag{2.13}$$

We claim that it is not possible for $s(t)$ to satisfy

$$\lim_{t \nearrow t_2} s(t) = 1. \tag{2.14}$$

Indeed, if it was true then $w(r,t_2) \le 0$ $(r \in [0,1])$. Using this together with (2.9), an application of the maximum principle gives a contradiction as in case (a) above.

We next claim that

$$\varrho = \limsup_{t \nearrow t_2} s(t) < 1.$$

To see this, recall that $w_t(1,t_2) < 0$. Thus, if the upper limit equals 1, then (2.14) necessarily holds, a contradiction.

It follows that for a sufficiently small $\epsilon > 0$ one has

$$w(r,t) > 0 \quad (r \in [\varrho+\epsilon, 1],\, t \in [t_2-\epsilon, t_2)).$$

Consequently, by the maximum principle,

$$w(r,t_2) > 0 \quad (r \in (\varrho+\epsilon,1)) \quad \text{and} \quad w_r(1,t_2) < 0.$$

A local analysis similar to the one carried out above for $t = t_1$ now shows that, for some $\delta > 0$,

$$z_{[\varrho+\epsilon,1]}(w(\cdot,t)) = 1 \quad (t \in (t_2, t_2+\delta)). \tag{2.15}$$

Fix $r_0 \in (\varrho+\epsilon, 1)$. We have, making δ smaller if necessary,

$$w(r_0,t) \ne 0 \quad (t \in (t_2-\delta, t_2+\delta)).$$

Applying Lemma 2.2 (with domain $B_{r_0}(0)\times(t_2-\delta,t_2+\delta)$) and using (2.13), we see that either

$$z_{[0,r_0]}(w(\cdot,\bar{t}))=0 \tag{2.16}$$

for some $\bar{t}\in(t_2-\delta,t_2+\delta)$, or else

$$z_{[0,r_0]}(w(\cdot,t))\equiv 1 \quad (t\in(t_2-\delta,t_2+\delta)). \tag{2.17}$$

In the former case, a combination of (2.15) and (2.16) yields

$$z(w(\cdot,t))=1 \quad (t>t_2,\, t\approx t_2).$$

Lemma 2.2 and (2.9) then imply the conclusion of Lemma 2.5.

In the latter case, we see from (2.15), (2.17) that the inequality

$$z(w(\cdot,t))\le 2$$

is valid for $t>t_2$, $t\approx t_2$. By Lemma 2.2 and (2.19) it remains valid for any $t\in(t_2,T)$. Identity (2.17) in conjunction with Lemma 2.2(iii) also implies $w(0,t)\neq 0$ ($t\in(t_2-\delta,t_2+\delta)$) (otherwise $r=0$ is a double zero). By (2.13), $w(0,t)<0$ on this interval. In view of (2.7), there must exist a $t_3\in(t_2,T)$ such that $w(0,t_3)=0$. By the diminishing property,

$$z(w(\cdot,t))\le 1 \quad (t\in(t_3,T)).$$

Thus, starting from some time $t=T_0$, $z(w(\cdot,t))$ remains constant and not greater than 1. This and Lemma 2.2 imply the conclusion of Lemma 2.5. The proof is complete.

Lemma 2.6. *Let $u_0\in X$ and suppose the classical solution u of* (P) *is global. Then, as $t\to\infty$,*

$$u(\cdot,t)\to\phi \quad \text{in } C^1_{loc}((0,1]),$$

where $\phi\in C^2((0,1])$ satisfies

$$\phi_{rr}+\frac{N-1}{r}\phi_r+\lambda e^{\phi}=0, \quad (r\in(0,1)), \tag{2.18}$$

$\phi(1)=0$ and either $\phi(0)=\infty$ or $\phi\in C^1([0,1))$ and $\phi_r(0)=0$.

Proof. If the solution u is bounded, then convergence follows from standard results since the set of equilibria is discrete (see e.g. [15]). We claim that if u is unbounded and global, then for any $\bar{u}_0 < u_0$ the solution $\bar{u}$ with the initial condition $\bar{u}_0$ is bounded and converges to the stable equilibrium ϕ_0. Indeed, the fact that $\bar{u}$ is bounded follows directly from the results of [31, Section 3]. If it converges to an unstable equilibrium ϕ then, using $\bar{u}_0 < u_0$ and a comparison argument with a linearization of (P) (see [23]), one can show that there is a value t such that $u(\cdot, t) > \phi$. But this is not possible for a global solution (see [18]). The claim is proved. This implies that u satisfies the assumptions of Lemma 3.2 and Lemma 2.6 now becomes a special case of Theorem 3.3 which is proved in the next section.

Remark 2.7. Note that $\phi(0) = \infty$ means that $\phi = \phi_\infty$ is the singular steady state of (P). It is unique and can be calculated explicitly, namely

$$\phi_\infty(r) = \ln \frac{2(N-2)}{\lambda r^2}$$

(see [16, 25]). Thus a singular steady state satisfying $\phi(1) = 0$ exists if and only if $\lambda = 2(N-2)$. This is the value of λ for which (SP) has an infinite sequence of solutions ϕ_k, $k = 0, 1, \ldots$, satisfying

$$\phi_k \to \phi_\infty \quad \text{in } C^1_{loc}((0,1]), \qquad z_{(0,1]}(\phi_k - \phi_\infty) \to \infty \tag{2.19}$$

(see the appendix).

Lemma 2.8. *Let $u_0 \in X$ and let u be a global classical solution of* (P). *Let $\phi(r)$ be the limit of $u(r,t)$ as in Lemma* 2.6. *Assume that ϕ is a solution of* (SP) (*i.e., ϕ is a classical steady state*). *Then there exist constants $r_1 \in (0,1)$ and $t_1 > 0$ such that*

$$u(r_1, t) < \ln \frac{2(N-2)}{\lambda r_1^2} + c \quad (t \ge t_1), \tag{2.20}$$

$$u(r, t_1) \le \ln \frac{2(N-2)}{\lambda r^2} + c \quad (r \in [0, r_1)), \tag{2.21}$$

where $c < 0$ is given in Lemma 2.3.

Proof. Let

$$\zeta(r) = \ln \frac{2(N-2)}{\lambda r^2} + c.$$

Fix $r_1 \in (0,1)$ such that $\phi(r_1) < \zeta(r_1)$. Then for a sufficiently large constant τ one has

$$u(r_1,t) < \zeta(r_1) \quad (t \geq \tau). \tag{2.22}$$

To the solution $u(r, t+\tau)$, $t \geq 0$, we find the self–similar solution $u^*(r,t)$, $t \in (0,T)$, as in Lemma 2.5. Let T_0 be as in the conclusion of that lemma. Then

(s1)

$u^*(\cdot,t) - u(\cdot,t+\tau)$ has at most one zero (counting multiplicity) in $[0,1]$,

(s2) $$u^*(0,t) - u(0,t+\tau) > 0 \quad (t \in (T_0,T)).$$

Since $u^*(r_1,t) \to \zeta(r_1)$ as $t \nearrow T$ (see Corollary 2.4), (2.22) in addition yields

(s3) $$u^*(r_1,t) - u(r_1,t+\tau) > 0 \quad (t \in (T_0,T)),$$

possibly after increasing T_0. Relations (s2) and (s3) imply that $u^*(\cdot,t) - u(\cdot,t+\tau)$ has no zero in $[0,r_1]$, for otherwise there would be two zeros or a multiple zero, in contradiction to (s1). Therefore,

$$u^*(r,t) - u(r,t+\tau) > 0 \quad (r \in [0,r_1],\, t \in [T_0,T)).$$

Taking the limit as $t \nearrow T$ we obtain (2.21) with $t_1 = T_0 + \tau$. Property (2.20) follows from (2.22).

We are in position to prove our main theorem.

Proof of Theorem 2.1. Let ϕ be the limit of the classical global solution u, that is

$$\phi(\cdot) = \lim_{t\to\infty} u(\cdot,t) \;\text{ in } C^1_{loc}((0,1]) \tag{2.23}$$

(cf. Lemma 2.6). First assume that $\phi(0) < \infty$ so that $\phi \in C^1([0,1])$ and solves (SP). By Lemma 2.8, the estimates (2.20), (2.21) are satisfied. It follows that, for $\epsilon > 0$ sufficiently small, one has

$$u(r_1,t) < \psi(r_1) \quad (t \geq t_1) \quad \text{and} \quad u(r,t_1) < \psi(r) \quad (r \in [0,r_1]), \tag{2.24}$$

where

$$\psi(r) = \ln \frac{2(N-2)}{\lambda(r+\epsilon)^2}.$$

A straightforward calculation yields

$$\psi_{rr} + \frac{N-1}{r}\psi_r + \lambda e^{\psi} < 0, \quad (r > 0), \qquad \psi_r(0) = -\frac{2}{\epsilon} < 0.$$

Relations (2.24) therefore imply, via a standard comparison argument, that

$$u(r,t) \le \psi(r) \quad (t \ge t_1,\, r \in [0, r_1]).$$

This and (2.23) guarantee that $u(r,t)$ is uniformly bounded.

We now complete the proof by showing that the possibility $\phi(0) = \infty$ cannot occur. Suppose the contrary. By Remark 2.7, we have $\phi = \phi_\infty$ and there is a sequence ϕ_k of (classical) solutions of (SP) such that (2.19) holds. Fix positive constants r_0, t_1 so small that

$$u(r,t) < \phi(r) \quad (r \in (0, r_0],\, t \in [0, t_1]). \tag{2.25}$$

Applying Lemma 2.2 to the function

$$v(r,t) = \phi(r) - u(r,t), \quad ((r,t) \in [r_0, 1] \times (0, t_1)),$$

we find that there exists $t_0 \in (0, t_1)$ such that all zeros of $v(\cdot, t_0)$ in $[r_0, 1]$ are simple. Therefore,

$$z_{[r_0,1]}(\phi_k(\cdot) - u(\cdot, t_0)) = n = z_{[r_0,1]}(\phi(\cdot) - u(\cdot, t_0))$$

for all k sufficiently large. Furthermore, since each ϕ_k is decreasing in r (see [25, 16]), (2.19) and (2.25) imply that

$$u(r, t_0) < \phi_k(r) \quad (r \in [0, r_0])$$

for k sufficiently large. Combining the above properties, we obtain

$$z(\phi_k(\cdot) - u(\cdot, t_0)) = n$$

for all k sufficiently large (here $z = z_{[0,1]}$). Lemma 2.2 now implies

$$z(\phi_k(\cdot) - u(\cdot, t)) \le n \quad (k \text{ sufficiently large}, \quad t \ge t_0).$$

On the other hand, (2.23) and the fact that $\phi_k - \phi$ has only simple zeros (the functions ϕ_k and ϕ solve the same second order ODE), imply that one can choose $t_k > t_0$ such that

$$z(\phi_k(\cdot) - u(\cdot, t_k)) \geq z(\phi_k - \phi).$$

Hence, by (2.19),

$$z(\phi_k(\cdot) - u(\cdot, t_k)) \to \infty \quad \text{as} \quad k \to \infty.$$

This contradiction rules out the possibility $\phi(0) = \infty$, as desired.

Proof of Lemma 2.3(iii). For the purpose of this proof let $z^c(w)$ denote the number of zeros, counting multiplicities, of a smooth function $w : (0, \infty) \to \mathbb{R}$. Let

$$v^*(\eta) = \ln \frac{2(N-2)}{\lambda \eta^2}.$$

Observe that v^* is a solution of

$$v_{\eta\eta} + \left(\frac{N-1}{\eta} - \frac{\eta}{2} \right) v_\eta + \lambda e^v - 1 = 0, \quad \eta > 0. \tag{2.26}$$

From statements (i), (ii) of Lemma 2.3, and their proof, recall that v is also a solution of (2.26), with the following properties:

(a) $v \in C^1([0, \infty))$,

(b) there is a constant $c < 0$ such that

$$v(\eta) - (c + v^*(\eta)) \to 0 \quad \text{as } \eta \to \infty,$$

(c) the function $v(\eta) - v^*(\eta)$ has at most two zeros, and each of them is simple (the latter follows from (2.26)).

We want to show that for any γ the function

$$w_\gamma = v - (\gamma + v^*)$$

satisfies $z^c(w_\gamma) \leq 2$. By (a) we have

$$w_\gamma(\eta) \to -\infty \quad \text{as } \eta \searrow 0. \tag{2.27}$$

This, together with (b) and (c), implies that $z^c(w_\gamma) \le 2$ for $\gamma \approx 0$. Let

$$\gamma^+ = \sup\{\gamma_0 > 0\colon z^c(w_\gamma) \le 2 \text{ for any } \gamma \in (0, \gamma_0)\}.$$

Suppose for contradiction $\gamma^+ < \infty$.

Assume first that w_{γ^+} has at least two different zeros. We claim there are exactly two of them and both are simple. To see this, observe that w_γ is a solution of

$$y_{\eta\eta} + \left(\frac{N-1}{\eta} - \frac{\eta}{2}\right) y_\eta = -\lambda(e^v - e^{v^*}) = \frac{2(N-2)}{\eta^2}(1 - e^\gamma e^y). \tag{2.28}$$

Since $\gamma^+ > 0$, we have $w''_\gamma(\eta_0) < 0$ if $w'_\gamma(\eta_0) = w_\gamma(\eta_0) = 0$ and $\gamma = \gamma^+ > 0$. Thus any multiple zero of w_{γ^+} is a double zero and a local maximum point of w_{γ^+}. Clearly, for $\gamma < \gamma^+$, $\gamma \approx \gamma^+$, w_γ has two zeros near any such η_0. Of course, simple zeros persist when γ is perturbed slightly. As w_γ is allowed to have at most two zeros for $\gamma < \gamma^+$, $\gamma \approx \gamma^+$, by definition of γ^+, we see that the zeros of w_{γ^+} are simple and there are exactly two of them. But then, by (b) and (2.26), the same is true for $\gamma > \gamma^+$, $\gamma \approx \gamma^+$, in contradiction to the definition of γ^+.

If w_{γ^+} has no zero or exactly one (multiple, hence double), then (2.27) and (b) imply that $w_{\gamma^+} \le 0$. Clearly then $w_\gamma < 0$ for any $\gamma > \gamma^+$, contradicting the definition of γ^+.

We next prove that

$$\gamma^- = \inf\{\gamma_0 < 0\colon z^c(w_\gamma) \le 2 \ \text{ for any } \gamma \in (\gamma_0, 0)\} = -\infty.$$

Suppose to the contrary that $\gamma^- > -\infty$. As $\gamma^- < 0$, (2.28) this time implies that at each multiple zero η_0 of w_{γ^-} one has $w''_{\gamma^-}(\eta_0) > 0$. It therefore follows from (2.27) that w_{γ^-} has a zero to the left of each multiple zero. It follows, in view of (2.27), that the first zero of w_{γ^-} is simple, and so persists under small perturbations of γ. Obviously, near a double zero of w_{γ^-}, the function w_γ with $\gamma > \gamma^-$, $\gamma \approx \gamma^-$, would have two additional zeros, which is impossible. We infer that all zeros of w_{γ^-} are simple and there are at most two of them. Now consider the three cases

$$z^c(w_{\gamma^-}) = 2, \qquad z^c(w_{\gamma^-}) = 1, \qquad z^c(w_{\gamma^-}) = 0.$$

If $z^c(w_{\gamma^-}) = 2$ then by (2.27) we have $w_{\gamma^-}(\eta) < 0$ for large η. If

$$c < \gamma^-$$

then $z^c(w_\gamma) = 2$ for $\gamma > \gamma^-$, $\gamma \approx \gamma^-$ (recall that $c - \gamma = \lim_{\eta\to\infty} w_\gamma$). If, on the other hand,

$$c = \gamma^-,$$

then $z^c(w_\gamma) \geq 3$ for $\gamma < \gamma^-$, $\gamma \approx \gamma^-$. In both cases we have a contradiction with the definition of γ^-.

If w_{γ^-} has a unique zero (which is simple) then the same is true for w_γ with $\gamma < \gamma^-$, $\gamma \approx \gamma^-$, a contradiction.

Finally suppose $z^c(w_{\gamma^-}) = 0$, that is $w_{\gamma^-} < 0$. We immediately obtain a contradiction if $c - \gamma^- < 0$, for then $w_\gamma < 0$ ($\gamma < \gamma^-$, $\gamma \approx \gamma^-$). Suppose

$$c = \gamma^-. \tag{2.29}$$

Using (2.28), we see that for all η sufficiently large the function w_{γ^-} satisfies

$$w''_{\gamma^-}(\eta) + \left(\frac{N-1}{\eta} - \frac{\eta}{2}\right) w'_{\gamma^-}(\eta) > 0.$$

For large η the coefficient of w'_{γ^-} is negative, and thus w''_{γ^-} is positive as long as w'_{γ^-} remains positive. We conclude that for all sufficiently large η one has

$$w'_{\gamma^-}(\eta) > d$$

with a positive constant d, which contradicts (2.29).

3. Global L^1–solutions. We introduce the notion of L^1–solutions, following [21]:

Definition. By an L^1*–solution* of (P) on $[0,T]$ we mean a function $u \in C([0,T]; L^1(B_1(0)))$ such that $e^u \in L^1(Q_T)$ with $Q_T = B_1(0) \times (0,T)$, and the equality

$$\int_{B_1(0)} [u\psi]_\tau^t \, dx - \int_\tau^t \int_{B_1(0)} u\psi_t \, dx \, ds = \int_\tau^t \int_{B_1(0)} (u\Delta\psi + \lambda\psi e^u) \, dx \, ds$$

holds for any $0 \leq \tau < t \leq T$ and $\psi \in C^2(\overline{Q}_T)$ with $\psi = 0$ on $\partial B_1(0) \times [0,T]$. By a global L^1–solution we mean an L^1–solution which exists on $[0,T]$ for every $T > 0$.

The main goal of this section is to prove that there is $\epsilon > 0$ such that nonclassical global solutions exist for any $\lambda \in (0, \lambda_3 + \epsilon)$. For $\lambda \in (0, \lambda_2)$ the existence of L^1–solutions has been established earlier by Lacey and Tzanetis (see [21]). In Theorem 3.4 and Proposition 3.9 we give a proof for $\lambda \in (\lambda_2, \lambda_3 + \epsilon)$ and $\lambda = \lambda_2$, respectively. A convenient way to find global L^1–solutions is to take initial conditions on the boundary of the domain of attraction of the stable equilibrium (cf. Lemma 3.2 below). However, it is nontrivial to prove that the choice can be made such that the solution is not classical. We still do not know whether this is possible for all $\lambda \in (0, \lambda_1]$. This section also contains a description of asymptotics of L^1–solutions (cf. Theorem 3.3).

In order to avoid possible confusions, we make the convention that a *solution* (with no adjective) always refers to a classical solution; it will be explicitly specified if a solution is to be understood in the L^1–sense. Sometimes, when we need to stress the dependence on the initial condition, we write $u(\cdot, t; u_0)$, $t \in [0, T(u_0))$ $(T(u_0) \leq \infty)$ for the maximally defined (classical) solution of (P).

As a preparation for the main results of this section, we introduce some notation and recall a few known properties of domains of attraction.

Let $\phi_0, \phi_1, \dots$ denote the sequence (finite or infinite, cf. the appendix) of solutions of (SP) ordered so that their values at 0 are increasing. The minimal equilibrium ϕ_0 is asymptotically stable for $\lambda \in (0, \lambda_1)$ and it is the only stable equilibrium of (P). If $\lambda = \lambda_1$ then it is stable from below and unstable from above. Let D_A denote the domain of attraction of ϕ_0, that is

$$D_A = \{u_0 \in X \colon \text{the solution of (P) is global, bounded and converges to } \phi_0 \text{ in } X\},$$

and let ∂D_A be the boundary of D_A in X. By $v \leq w$ we denote the standard pointwise order relation between functions $v, w \in X$. We also introduce the following strong order relation:

$$v \ll w \quad \text{if and only if} \quad v(x) < w(x) \quad (x \in B_1(0)) \quad \text{and } \frac{\partial v(x)}{\partial \nu} > \frac{\partial w(x)}{\partial \nu} \ (x \in \partial B_1(0)).$$

Here $\nu(x) = \nu$ is the unit outward normal vector field on $\partial B_1(0)$. Note that $\ll$ is an open relation on X since X is imbedded in $C^1(\overline{B}_1(0))$.

Lemma 3.1. *Assume $\lambda < \lambda_1$. Then D_A is open in X and the following properties hold*:

(a1) *For $u_0, \bar{u}_0 \in X$ the relations $\bar{u}_0 \leq u_0$ and $u_0 \in D_A$ imply $\bar{u}_0 \in D_A$.*

(a2) *$v \in \partial D_A$ if and only if $v \notin D_A$ and there is an increasing sequence $\{v_n\} \subset D_A$ such that $v_n \to v$ in X.*

(a3) *∂D_A is positively invariant under the flow of* (P)*: if $u_0 \in \partial D_A$ then the solution $u(\cdot, t)$ of* (P) *satisfies $u(\cdot, t) \in \partial D_A$ as long as it exists* (*in the classical sense*).

(a4) *If $v, w \in \partial D_A$ are distinct then they are not ordered: neither $v \leq w$ nor $w \leq v$.*

Proof. The fact that D_A is open is a standard consequence of asymptotic stability of ϕ_0. Property (a1) follows from the comparison principle and the fact that ϕ_0 is globally asymptotically stable below (any solution below ϕ_0 converges to ϕ_0). Property (a2) is a consequence of general results on domains of attraction for equations admitting strong comparison principle (cf. [29], for example). In fact, any point in the boundary can be approximated from below or from above by a monotone sequence of elements of D_A. In the present case, approximation from above is prevented by (a1), thus we have (a2).

Positive invariance of D_A follows easily from the obvious positive invariance of D_A and continuity with respect to initial conditions. Property (a4) is a consequence of positive invariance and the strong comparison principle (see the proof of Theorem 5.4(v) in [29]).

Lemma 3.2. *Assume $\lambda \leq \lambda_1$. If $u_0 \in \partial D_A$ and $\{u_{0,n}\} \subset D_A$ is an increasing sequence such that $u_{0,n} \to u_0$, then*

$$u(r,t) = \lim_{n\to\infty} u(r,t;u_{0,n})$$

is a global L^1–solution.

Proof. The result is proved in [21], although it is not stated there in this form. Using the estimates for global solutions from Theorem 2.3 in [21], one checks easily that parts (i) and (ii) of the proof of Theorem 2.5 in [21], repeated without any change, prove Lemma 3.2.

Theorem 3.3. *If $\lambda \leq \lambda_1$ and u is a global L^1–solution as in Lemma* 3.2, *then it is a classical solution of* (P) *on $(B_1(0) \setminus \{0\}) \times (0, \infty)$ and there is a $\varphi \in C^2((0,1])$ with the following properties*:

$$u(\cdot,t) \to \varphi, \text{ in } C^1_{loc}((0,1]) \quad \text{as } t \to \infty,$$

$$\varphi_{rr} + \frac{N-1}{r}\varphi_r + \lambda e^{\varphi} = 0, \quad r \in (0,1), \tag{3.1}$$

$$\varphi(1) = 0$$

and either $\varphi(0) = \infty$ *or* $\varphi \in C^1([0,1])$ *and* $\varphi_r(0) = 0$.

Proof. The proof is very similar to the proof of (0.10), Theorem 4 in [27]. Therefore, we only give a brief sketch of it.

Let $\{u_{0,n}\}$ be a sequence as in Lemma 3.2. Let T^* be the blow-up time of the solution of $y' = \lambda e^y, y(0) = \min u_0$. Then for $t > T^*$ and all n sufficiently large we have $u_r(r,t;u_{0,n}) < 0$ for $r \in (0,1)$, see the proof of Theorem 1 in [26]. Since u is an L^1–solution, we have that $u(r,t;u_0)$ is finite for $r \in (0,1]$ and $t > T^*$. Also, $u(r,t;u_0)$ is a classical solution of

$$\begin{aligned} &u_t = u_{rr} + \frac{N-1}{r}u_r + \lambda e^u, && t > T^*,\ r \in (0,1),\\ &u(1,t) = 0, && t > T^*, \end{aligned}$$

and $u_r(r,t;u_0) < 0$ in $(0,1) \times (T^*,\infty)$.

Further, $u(r,t;u_0)$ is bounded for $(r,t) \in (r_0,1) \times (T^*,\infty)$, $r_0 \in (0,1)$ (cf. (3.1)–(3.6) in [27]). Following (3.7)–(3.12) in [27], we obtain that for any sequence $t_n \to \infty$ there exists a subsequence, denoted again by t_n, and a $\varphi \in C^2((0,1])$ such that

$$u(\cdot,t_n;u_0) \to \varphi(\cdot) \quad \text{in } C^1_{loc}((0,1]), \tag{3.2}$$

where φ satisfies (3.1) and $\varphi(1) = 0$. Next, either $\varphi(0) = \infty$ or $\varphi(0) < \infty$. In the latter case, φ must satisfy $\varphi_r(0) = 0$ by Lemma 1.1 in [27] and elliptic regularity ([22]). Now, by standard arguments, the set of limit points of $u(\cdot,t;u_0)$ must be connected in $C([r_0,1])$ $(r_0 \in (0,1))$. Thus, as the solutions of (SP) are isolated in $C([r_0,1])$, there is exactly one limit point, hence the conclusion of the theorem follows.

Theorem 3.4. *There exists* $\epsilon > 0$ *such that for any* $\lambda \in (\lambda_2, \lambda_3 + \epsilon)$ *there is* $u_0 \in X$ *such that the solution* $u(\cdot,t)$ *of* (P) *has the following properties*:

(i) $u(\cdot,t)$ *blows up in finite time,*
(ii) $u(\cdot,t)$ *extends to a global* L^1*–solution,*
(iii) *after the extension, the* L^1*–solution* $u(\cdot,t)$ *approaches the minimal equilibrium* ϕ_0 *in the topology of* $C^1_{loc}((0,1])$, *as* $t \to \infty$.

If $\lambda \in (\lambda_2, \lambda_3]$, *then* u_0 *can be chosen such that the following additional property holds*:

(iv) $u(\cdot,t)$ *is defined* (*as a classical solution of* (P)) *on the interval* $(-\infty, T)$ *for* $T = T(u_0)$, *and* $u(\cdot,t) \to \phi_2$ *in* X, *as* $t \to -\infty$.

The outline of the proof is as follows. In Lemma 3.5 we give sufficient conditions for u_0 to satisfy (i)-(iii) of Theorem 3.3. Then we consider the case $\lambda \in (\lambda_2, \lambda_3]$ and prove that there is an arc in the strong unstable manifold of ϕ_2 that necessarily contains some u_0 satisfying the conditions from Lemma 3.5. Property (iv) is automatically satisfied for such u_0. Finally, we prove that the arc for $\lambda = \lambda_3$ also contains some u_0 that yields an L^1–solution for any $\lambda > \lambda_3$ sufficiently close to λ_3.

Recall that for a function v defined on $[0,1]$, $z(v)$ denotes the number of zeros of v in $[0,1]$. In this section it is convenient to allow for v in this definition to take an infinite value at 0.

Lemma 3.5. *Let $\lambda \in (\lambda_2, \lambda_1)$, $u_0 \in \partial D_A$, and $u(\cdot,t) = u(\cdot,t;u_0)$, $0 \le t < T = T(u_0)$. Assume the following conditions are satisfied for some $t_0 > 0$:*

$$\begin{aligned} z(u_t(\cdot,t_0)) &= 2, \\ u_t(0,t_0) &> 0, \\ u(0,t_0) &> \phi_1(0). \end{aligned} \tag{3.3}$$

Then properties (i) *and* (ii) *of Theorem* 3.3 *hold. If, in addition, $\lambda \in (\lambda_2, \lambda_3]$ and $u_r(1,t_0) > \partial_r\phi_2(0)$, or $\lambda \in (\lambda_3, \lambda_1)$ and $u(0,t_0) > \partial_r\phi_1(0)$, then property* (iii) *holds as well.*

Proof. By Theorem 2.1, in order to show that $u(\cdot,t)$ blows up in finite time, we only need to verify that it is not bounded. To prepare an argument for the latter and also for other purposes in this proof, we first establish the following property:

$$z(u_t(\cdot,t)) \equiv 2 \quad (t \in (t_0,T)). \tag{3.4}$$

To prove (3.4), observe that $w = u_t$ is a solution of

$$w_t = \Delta w + \lambda e^{u(|x|,t)} w, \qquad t \in (0,T),\ x \in B_1(0), \tag{3.5}$$

$$w = 0, \qquad 0 < t < T,\ x \in \partial B_1(0), \tag{3.6}$$

thus Lemma 2.2 applies to it. We have $z(u_t(\cdot,t)) \ge 1$ as $u_t(1,t) = 0$, due to the boundary condition. If the zero number equals 1 for some t, then it is identical to 1 thereafter. This further implies, by zero number diminishing property, that the only zero $r = 1$, of $u_t(\cdot,t)$ is simple, hence either $u_t(\cdot,t) \ll 0$ or $u_t(\cdot,t) \gg 0$ for all t near T. These relations imply that the trajectory $\{u(\cdot,t) : 0 < t < T\}$ contains two different ordered points.

This, however, is not possible by (a3) and (a4) of Lemma 3.1. Therefore, $z(u_t(\cdot,t)) \geq 2$. Hypotheses (3.3) and the nonincrease of $z(u_t(\cdot,t))$ now give (3.4).

We next prove by contradiction that $u(\cdot,t)$ is not bounded. Suppose it is bounded. Then in particular $T=\infty$. Let ϕ_k be the limit equilibrium of $u(\cdot,t)$, that is

$$u(\cdot,t) \to \phi_k \quad \text{as } t \to \infty. \tag{3.7}$$

As $u_0 \notin D_A$ (recall that D_A is open), we have $k>0$. Property (3.4) implies that $u_t(\cdot,t)$ has only simple zeros for $t \in (t_0,\infty)$. In particular, $u_t(0,t)$ is different from zero, because $u_{rt}(0,t)=0$, thus $u_t(0,t)>0$ by (3.3). This and (3.3) now imply that u cannot converge to ϕ_1, thus $k \geq 2$.

We now invoke a result on asymptotics of solutions of asymptotically autonomous equations (note that the potential in (3.5) converges to $\exp(\phi_k)$). We claim that

$$\frac{u_t(\cdot,t)}{\|u_t(\cdot,t)\|_X} \to \psi \quad \text{in } X \text{ as } t \to \infty, \tag{3.8}$$

where ψ is a normalized eigenfunction of

$$\Delta\psi + \lambda e^{\phi_k(|x|)}\psi + \mu\psi = 0, \quad x \in B_1(0), \tag{3.9}$$

$$\psi \in X, \tag{3.10}$$

corresponding to a positive eigenvalue μ. One can show that this follows from results of [4, Appendix B]. Using [4, Corollary B.3] one has

$$\|u_t(\cdot,t)\|_X^{\frac{1}{t}} \to e^{-\mu} \quad \text{as } t \to \infty,$$

where μ is an eigenvalue of (3.9), (3.10). Since $\|u_t(\cdot,t)\|_X$ is bounded, we have $\mu \leq 0$. Now, as all eigenvalues of (3.9), (3.10) are simple, (3.8) follows from [4, Theorem B.5]. (We remark that the results in [4, Appendix B] are formulated for discrete time equations. Straightforward modifications are needed in order to adapt them to the present situation.)

Using (3.8) in conjunction with the facts that ψ has only simple zeros in $[0,1]$ and, since $k \geq 2$ and $\mu \geq 0$,

$$z(\psi) \geq 3$$

(see the appendix), we obtain

$$z(u_t(\cdot,t)) \geq 3$$

for all t sufficiently large. This contradicts (3.4), however. Thus $u(\cdot,t)$ cannot be bounded, which proves Property (i) of Theorem 3.3.

Let $u_0^n \in D_A$ be an increasing sequence such that $u_0^n \to u_0$. This sequence gives rise to an L^1–solution, as in Lemma 3.2, which, by continuous dependence of (classical) solutions on initial conditions, is an extension of $u(\cdot,t)$. This proves (ii).

Denote the extension of u to a global L^1–solution by u again. Assume that the last hypothesis of the lemma is satisfied. We know, by Theorem 3.3, that $u(\cdot,t)$ approaches a solution, possibly the singular one, of (SP). We want to show that the limit is ϕ_0. This will follow once we prove that $u_r(1,t)$ is a monotone nondecreasing function on the t-interval (t_0,∞). Indeed, suppose the latter is true. Since the value $\partial_r\phi_2(1)$ (or $\partial_r\phi_1(1)$) is maximal in

$$\{\partial_r\phi(1) : \phi \text{ is a solution of (SP) greater than } \phi_0\}$$

if $\lambda \in (\lambda_2,\lambda_3]$ (or $\lambda \in (\lambda_3,\lambda_1)$, respectively) (cf. Appendix, Theorem A.1(g)) the last hypothesis of Lemma 3.5 implies that $u_r(1,t)$ cannot converge to any value in this set, so the limit equilibrium is necessarily ϕ_0.

Observe that (3.4) and $u_t(0,t_0) > 0$ imply $u_{tr}(1,t) > 0$ for $t \in (t_0,T)$. Thus $u_r(1,t)$ is increasing on this interval. Suppose, for a contradiction, that it is not monotone nondecreasing on (t_0,∞). Then there exist $t_3 > t_2 > t_1 > t_0$ such that

$$u_r(1,t_1) = u_r(1,t_3) < u_r(1,t_2). \tag{3.11}$$

Consider the solutions $u^n(\cdot,t) = u(\cdot,t;u_0^n) \in D_A$ that give rise to the L^1–solution u, that is

$$u^n(r,t) \to u(r,t) \quad (r \in (0,1],\ t \geq 0).$$

Note that the convergence here is in the $C^1([0,1])$ topology for any fixed $t \in (0,T)$ (on this interval $u(\cdot,t)$ is still classical). Pick any $\bar{t} > t_0$ with $\bar{t} < \min\{t_1,T\}$. From (3.4) it follows that $z(u_t(\cdot,\bar{t})) = 2$ and that $u_t(\cdot,\bar{t})$ has only simple zeros in $[0,1]$. Therefore, for any n sufficiently large one also has $z(u_t^n(\cdot,\bar{t})) = 2$ and, consequently,

$$z(u_t^n(\cdot,t) \leq 2 \quad (t \in (\bar{t},\infty)).$$

Next, using (3.11) we obtain that for all sufficiently large n, the function $t \mapsto u_{rt}^n(1,t)$ has a zero in (t_1,t_3). This means that $r \mapsto u_r^n(r,t)$ has a double zero at $r = 1$ for some $t \in (t_1, t_3)$ and therefore $z(u^n_t(\cdot, t))$ drops at this value of t. Moreover, due to the boundary condition, the zero number is at least one. Therefore, there is precisely one dropping time in the whole interval $(\bar{t}, \infty)$. In particular, $u^n_t(0, t)$ can never equal zero in this interval for we already have $u^n_{tr}(0, t) \equiv 0$ (it can be checked easily that $u^n_t(0, t)$ cannot be zero at the dropping time either since that would cause dropping by two at least which is not possible). Thus $u^n_t(0, t)$ has a definite sign on $(\bar{t}, \infty)$ and that sign must be positive for n sufficiently large, because $u^n_t(0, t) \to u_t(0, t)$ ($t \in [\bar{t}, T)$) and $u(0, t)$ is increasing on some time interval before it blows up. However, the increase of $u^n(0, t)$ just established, combined with the property

$$u^n(0, \bar{t}) \approx u(0, \bar{t}) > \phi_1(0) > \phi_0(0),$$

(cf. the hypotheses) imply that $u^n(0, t)$ cannot converge to $\phi_0(0)$, contradicting the choice of u^n. This contradiction proves that $u_r(1, t)$ is monotone nondecreasing, as needed, and thereby completes the proof.

Our next aim is to find u_0 that satisfies the hypotheses of Lemma 3.5. We do it first for $\lambda \in (\lambda_2, \lambda_3]$ by looking for u_0 in the unstable manifold of ϕ_2. Recall that the (strong) unstable manifold of ϕ_2, $W^u(\phi_2)$, is defined as the set of all $u_0 \in X$ such that the solution $u(x, t; u_0)$ of (P) extends to a classical solution $u(\cdot, t)$ of (P) that is defined on $(-\infty, T(u_0))$ and such that

$$\|u(\cdot, t) - \phi_2)\|_X \leq c e^{\epsilon t} \quad (t \leq 0),$$

for some positive constants c and ϵ. This set is an (imbedded) submanifold of X (see [7, 4] and [15 , Theorem 6.1.10]) and its dimension is equal to the number of negative eigenvalues of (3.9), (3.10), which in the case of ϕ_2 is 2 (cf. Theorem A.1.e). Let us remark, that even if the flow of (P) is not restricted to the space of radially symmetric functions, $W^u(\phi_2)$ still consists of radially symmetric functions only (see [30]).

In the next lemma, when considering a solution $u(\cdot, t; u_0)$ with $u_0 \in W^u(\phi_2)$ it is understood that the interval of definition of $u(\cdot, t; u_0)$ is extended to $(-\infty, T(u_0))$.

Lemma 3.6. *Let $\lambda \in (\lambda_2, \lambda_3]$. There is a compact arc Γ in $W^u(\phi_2)$ with the following properties*:

(g1) *For the boundary points u_0^+ , u_0^- of Γ one has*

$$u(\cdot, t; u_0^+) \gg \phi_2, \quad u_t(\cdot, t; u_0^+) \gg 0,$$
$$u(\cdot, t; u_0^-) \ll \phi_2, \quad u_t(\cdot, t; u_0^+) \ll 0,$$

for all $t \le 0$.

(g2) *For any* $u_0 \in \Gamma \setminus \{u_0^+, u_0^-\}$ *one has*

$$z(u_t(\cdot,t;u_0)) \equiv 2, \quad u(0,t;u_0) > \phi_2(0) \quad \text{and} \quad u_t(0,t;u_0) > 0$$

for all sufficiently large negative t.

Proof. Let ψ_1, ψ_2 denote the eigenfunctions corresponding to the first two eigenvalues of (3.9), (3.10) with $k = 2$. We assume the eigenfunctions are normalized in X, that is

$$\|\psi_1\| = \|\psi_2\| = 1.$$

Without loss of generality, we may also assume that $\psi_i(0) > 0$, which for ψ_1 gives

$$\psi_1 \gg 0.$$

Recall that the first two eigenvalues are the only two positive eigenvalues, ψ_1, ψ_2 have only simple zeros in $[0,1]$, and

$$z(\psi_i) = i \quad (i = 1, 2).$$

We next quote the following result that was proved by Brunovský and Fiedler in [3] in a more general context: There is a 1–dimensional submanifold C of $W^u(\phi_2)$ containing ϕ_2 that is tangent to $\operatorname{span}\{\psi_1\}$ at ϕ_2 and negatively invariant (that is, $u(\cdot,t;u_0) \in C$ $(t \le 0)$ if $u_0 \in C$). Also, the following holds:

$$\begin{aligned} \text{If} \quad u_0 \in C \setminus \{\phi_2\} \quad \text{then} \quad & \lim_{t\to-\infty} \frac{u(\cdot,t;u_0) - \phi_2}{\|u(\cdot,t;u_0) - \phi_2\|} = \pm\psi_1, \\ \text{and} \quad & z(u(\cdot,t;u_0) - \phi_2) \equiv 1 \quad (t \le 0). \end{aligned} \tag{3.12}$$

$$\begin{aligned} \text{If} \quad u_0 \in W^u(\phi_2) \setminus C \quad \text{then} \quad & \lim_{t\to-\infty} \frac{u(\cdot,t;u_0) - \phi_2}{\|u(\cdot,t;u_0) - \phi_2\|} = \pm\psi_2, \\ \text{and} \quad & z(u(\cdot,t;u_0) - \phi_2) \equiv 2 \quad (t \text{ large negative}). \end{aligned} \tag{3.13}$$

We remark that although the hyperbolicity of the equilibrium is assumed and solutions on the unstable manifold are considered in [3], the same proof works for equilibria which are not necessarily hyperbolic if one considers solutions on the strong unstable manifold.

Take a small neighborhood D of ϕ_2 in $W^u(\phi_2)$ such that $\bar{D}$ is diffeomorphic to a closed disk, $\bar{D}$ is negatively invariant and $\partial D = \bar{D} \setminus D$ intersects C at exactly two points u_0^-, u_0^+. Existence of such a neighborhood follows from the fact that the flow of (P) on $W^u(\phi_2)$ near ϕ_2 is represented by a planar vector field for which ϕ_0 is a nondegenerate unstable node (cf. [3, Proof of Theorem 2.1 and Lemma 1.1]). Interchanging u_0^+, u_0^-, if necessary, we may assume

$$u_0^+ \gg \phi_2 \gg u_0^-$$

(note that (3.12) implies that $u_0^\pm - \phi_2$ has a unique zero, namely $r = 1$, and this zero is simple). Now $\bar{D} \setminus C$ consists of elements u_0 for which (3.13) holds. By negative invariance of $\bar{D}$, C divides $\bar{D}$ into two parts, $\bar{D}^+$, $\bar{D}^-$, such that (3.13) holds with the plus sign on $\bar{D}^+$ and with the minus sign on $\bar{D}^-$. Let

$$\Gamma = \partial D \cap (\mathrm{cl}\, \bar{D}^+).$$

Clearly, Γ is a compact arc with the boundary points u_0^+, u_0^-. Using (3.13) and $\psi_2(0) > 0$, we see that for any $u_0 \in \Gamma \setminus \{u_0^+, u_0^-\}$ one has $u(0, t; u_0) > \phi_2(0)$ if t is sufficiently large negative, which is one of the requirements in (g2).

We further claim that for $u_0 \in \Gamma \setminus \{u_0^+, u_0^-\}$ one has

$$\lim_{t \to -\infty} \frac{u_t(\cdot, t; u_0)}{\|u_t(\cdot, t; u_0)\|} = \psi_2. \tag{3.14}$$

Indeed, the limit (in X) exists and equals one of the functions $\pm\psi_1$, $\pm\psi_2$. This follows from properties of solutions of the asymptotically autonomous equation (3.5), this time considered on the interval $(-\infty, 0)$, see [4, Appendix B]. The limit cannot equal $\pm\psi_1$ because that would imply that either $u_t(\cdot, t; u_0) \gg 0$ or $u_t(\cdot, t; u_0) \ll 0$, hence $u(\cdot, t)$ is monotone in t, for large negative t. This and the fact that $u(\cdot, t; u_0)$ approaches ψ_2 imply that either $u(\cdot, t) \ll \phi_2$ or $u(\cdot, t) \gg \phi_2$, in contradiction to (3.13). Thus the limit is ψ_2 or $-\psi_2$. In either case, as $\psi_2(0) \neq 0$, $u_t(0, t; u_0)$ has a definite sign for all large negative t and the sign is the same as the sign of the limit $\pm\psi(0)$. By definition of Γ we know that $u(0, t; u_0) \to \phi_2(0)$ and $u(0, t; u_0) > \phi_2(0)$ for t near $-\infty$, therefore $u_t(0, t; u_0)$ cannot be negative. We have thus proved (3.14) and, at the same time, we have verified another property in (g2):

$u_t(0,t;u_0) > 0$ for t near $-\infty$. As ψ_2 has precisely two zeros in $[0,1]$, both simple, (3.14) also implies the last property in (g2).

Using arguments similar to the ones above one proves that

$$\lim_{t\to-\infty} \frac{u_t(\cdot,t;u_0)}{\|u_t(\cdot,t;u_0)\|} = \psi_1,$$

which implies (g1).

Lemma 3.7. *Let $\lambda \in (\lambda_2,\lambda_3]$ and let Γ be as in Lemma* 3.6. *Then $\Gamma \cap \partial D_A \neq \emptyset$, and if $u_1 \in \Gamma \cap \partial D_A$ then there is a $t_0 > 0$ such that $u_0 = u(\cdot,-t_0;u_1)$ satisfies all hypotheses of Lemma* 3.5, *hence it also satisfies* (i)–(iv) *of Theorem* 3.4.

Proof. Since $u_0^+ \gg \phi_2 \gg u_0^-$, we have $u_0^+ \notin D_A$, $u_0^- \in D_A$. This clearly implies that the arc Γ contains a $u_1 \in \partial D_A$. By (g2), $u_0 = u(\cdot,-t_0;u_1)$ satisfies all hypotheses of Lemma 3.5. By that lemma, (i)–(iii) of Theorem 3.3 hold. As u_0 is an element of the unstable manifold of ϕ_2, (iv) holds as well.

At this stage, we have completed the proof of Theorem 3.4 for $\lambda \in (\lambda_2,\lambda_3]$. In the next lemma we settle the case $\lambda \in (\lambda_3,\lambda_3+\epsilon)$.

Lemma 3.8. *There exists $\epsilon > 0$ such that for any $\lambda \in (\lambda_3,\lambda_3+\epsilon)$ there is $u_0 \in X$ that satisfies the hypotheses of Lemma* 3.5, *hence it also satisfies* (i)–(iii) *of Theorem* 3.4.

Proof. For $\lambda = \lambda_3$ consider the arc Γ, as in Lemma 3.6. Take a smaller compact arc Γ_1 such that $\Gamma_1 \subset \Gamma \setminus \{u_0^+, u_0^-\}$ and the boundary points u_1^+, u_1^- of Γ_1 are so close to u_0^+, u_0^-, that one still has

$$u_t(\cdot,0,u_1^+) \gg 0 \gg u_t(\cdot,0,u_1^-).$$

By definition of Γ and (3.14),

$$\frac{u_t(\cdot,s;u_1)}{\|u_t(\cdot,s;u_1\|} \to \psi_2 \quad \text{as} \quad s\to-\infty \tag{3.15}$$

for each $u_1 \in \Gamma_1$. This implies that for each $u_1 \in \Gamma_1$ there exists $\bar{s} < 0$ such that

$$z(u_t(\cdot,\bar{s};u_1)) = 2, \tag{3.16}$$

$$u_t(\cdot,\bar{s};u_1) \text{ has only simple zeros in } [0,1], \tag{3.17}$$

$$u_t(0,\bar{s};u_1) > 0. \tag{3.18}$$

We claim that $\bar{s}$ can be chosen independently of u_1. Indeed, since the backward flow of (P) is continuous on the unstable manifold, that is

$$u_t(\cdot,\bar{s};\widetilde{u}_1) \approx u_t(\cdot,\bar{s};u_1) \quad \text{if} \quad \widetilde{u}_1 \approx u_1 \quad (\widetilde{u}_1, u_1 \in W^u(\phi_2))$$

(cf. [3], e.g.), $\bar{s}$ can be obviously chosen locally independently of u_1. Now if (3.16)–(3.18) hold for some value of $\bar{s}$ then they hold for any smaller value of $\bar{s}$, due to the properties of the zero number and (3.14). Using compactness of Γ_1, one easily obtains that $\bar{s}$ can be chosen independently of $u_1 \in \Gamma_1$.

Note that (3.15) and (3.16) imply that $u_{rt}(1,t;u_1) > 0$ $(t < \bar{s})$ because $r = 1$ is a simple zero of $u_t(\cdot,t;u_1)$. Since $u(\cdot,t;u_1)$ approaches ϕ_2 in X, one has

$$u_r(1,\bar{s};u_1) < \partial_r\phi_2(1) < \partial_r\phi_1(1).$$

Let

$$\widetilde{\Gamma}_1 = \{u(2\bar{s},\cdot;u_1)\colon u_1 \in \Gamma_1\}.$$

By continuity of the backward flow on $W^u(\phi_2)$, $\widetilde{\Gamma}_1$ is a compact arc. For $u_0 \in \widetilde{\Gamma}_1$ one has

$$u(\cdot,-\bar{s};u_0) = u(\cdot,\bar{s};u_1)$$

with

$$u_1 = u(\cdot,-2\bar{s};u_0) \in \Gamma_1.$$

Hence, by (3.16)-(3.18), any $u_0 \in \widetilde{\Gamma}_1$ has the following properties:

$$\begin{gathered} z(u_t(\cdot,-\bar{s};u_0)) = 2, \\ u_t(\cdot,-\bar{s};u_0) \text{ has only simple zeros in } [0,1], \\ u_t(0,-\bar{s};u_0) > 0. \end{gathered} \tag{3.19}$$

For the boundary points $\widetilde{u}_0^{\pm} = u(2\bar{s};u_1^{\pm})$ one also has

$$u_t(\cdot,-2\bar{s};\widetilde{u}_0^{+}) \gg 0 \gg u_t(\cdot,-2\bar{s};\widetilde{u}_0^{-}). \tag{3.20}$$

Here $u(\cdot,t;u_0)$ refers to the (classical) solution of (P) with $\lambda = \lambda_3$. Since (3.19) and (3.20) are open properties and $\widetilde{\Gamma}_1$ is compact, the continuity of solutions with respect to parameters implies that they are also satisfied for solutions of (P) with $\lambda > \lambda_3$ sufficiently close to λ_3. For any such λ, property (3.20) guarantees that $\widetilde{\Gamma}_1$ contains an element u_0 of ∂D_A (cf. the proof of Lemma 3.7), and (3.19) implies that the hypotheses of Lemma 3.5 are satisfied for such u_0, with $t_0 = -\bar{s}$.

By Lemma 3.8 we have completed the proof of Theorem 3.4. We next consider the case $\lambda = \lambda_2$.

Proposition 3.9. *For $\lambda = \lambda_2$ there exists $u_0 \in X$ such that the solution $u(\cdot, t)$ satisfies* (i) *and* (ii) *of Theorem* 3.4.

Remark. We show that the solution $u(\cdot, t)$ has the additional property that it is defined on $(\infty, T(u_0))$ and converges to ϕ_1 in X as $t \to -\infty$. One can also prove that $u(\cdot, t) \to \phi_0$ in C^1_{loc} as $t \to \infty$. This could be done by modifying the arguments used above for $\lambda > \lambda_2$. Although the necessary modifications are not difficult, they would require quite a bit of extra technical work. We did not find the result worth it.

Proof of Proposition 3.9. We only need to prove that ∂D_A contains some u_0 such that the solution of (P) is not bounded.

We will use a local *center–unstable manifold* of ϕ_1. It will be useful to recall a few facts first. For any $\delta > 0$ sufficiently small there is a C^1–submanifold $W^{cu}_{loc}(\phi_1)$ of $B^X_\delta = \{v \in X : \|v\|_X < \delta\}$ with the following properties:

(c1) $W^{cu}_{loc}(\phi_1)$ is locally invariant for the flow of (P), that is, $W^{cu}_{loc}(\phi_1)$ is covered by local trajectories of (P),
(c2) $W^{cu}_{loc}(\phi_1)$ contains ϕ_1 and is tangent at ϕ_1 to span $\{\psi_1, \psi_2\}$, the eigenspace of the linearization corresponding to the nonpositive eigenvalues, see the appendix.
(c3) If $u_0 \in B^X_\delta$ is such that $u(\cdot, t; u_0)$ extends to a solution of (P) defined on $(-\infty, T(u_0))$ with $u(\cdot, t; u_0) \in B^X_\delta$ $(t \le 0)$ then $u_0 \in W^{cu}_{loc}(\phi_1)$.

To prove the existence of $W^{cu}_{loc}(\phi_1)$ one modifies the equation in (P), using an appropriate cut off function (see e.g. [17]), so that it becomes a small perturbation (in the C^1–sense) of the linearization at ϕ_1. One then finds a global center manifold whose intersection with B^X_δ has the above properties (see [7] for details).

The manifold $W^{cu}_{loc}(\phi_1)$ contains the local part of the unstable manifold of ϕ_1, $W^u_{loc}(\phi_1) = W^u(\phi_1) \cap B^X_\delta$, as a connected (one–dimensional) submanifold. This follows, for a sufficiently small $\delta > 0$ at least, from (c3) and the fact that $W^u(\phi_1)$ is an imbedded submanifold of X. Assuming $\delta > 0$ sufficiently small, we then see that $W^{cu}_{loc}(\phi_1) \setminus W^u_{loc}(\phi_1)$ consists of precisely two components D^-, D^+. The behavior of solutions in these components is as follows: if $u_0 \in D^-$ then $u(\cdot, t; u_0) \to \phi_1$ in X as $t \to \infty$; if $u_0 \in D^+$ then $u(\cdot, t; u_0)$ extends to a solution of (P) defined for all $t \le 0$ and $u(\cdot, t; u_0) \to \phi_1$ as $t \to -\infty$. This behavior of trajectories follows from the facts that the flow on $W^{cu}_{loc}(\phi_1)$ is represented by a planar ODE and that (SP) undergoes a generic saddle–node bifurcation at $\lambda = \lambda_2$ (the latter follows from the analysis in [16, 25]; for analysis of saddle–node bifurcations see [24, 6]).

Take any compact arc Γ in $D^+ \cup W^u_{loc}(\phi_1)$ with the following properties: Γ is entirely contained in D^+ except for its boundary points, u_0^+, u_0^-, which are contained in $W^u_{loc}(\phi_1)$ and satisfy $u_0^+ \gg \phi_1 \gg u_0^-$. The latter implies, that Γ contains some $u_0 \in \partial D_A$. We claim that $u(\cdot, t; u_0)$ is the sought unbounded solution. Indeed, if it is bounded then it converges to ϕ_1, as $t \to \infty$, because ∂D_A is invariant and ϕ_1 is the only equilibrium in ∂D_A. Then, by definition of $\Gamma \subset D^+$, $u(\cdot, t; u_0)$ is homoclinic to ϕ_1, that is

$$u(\cdot, t; u_0) \to \phi_1 \quad \text{as } t \to \pm\infty.$$

This, however, is impossible, as the standard energy functional

$$V(\varphi) = \int_{B_1(0)} \left(\frac{|\nabla \varphi|^2}{2} - \lambda_2 e^{\varphi} \right)$$

strictly decreases along nonconstant solutions. Thus $u(\cdot, t; u_0)$ is unbounded, as claimed, and Proposition 3.9 is proved.

We conclude this section with a few remarks on connecting orbits. By definition, a connection (or connecting or heteroclinic orbit) from an equilibrium ϕ^- to an equilibrium ϕ^+ is a solution $u(\cdot, t)$ of (P) that is defined on $(-\infty, \infty)$ and satisfies

$$u(\cdot, t) \to \phi^{\pm} \text{ as } t \to \pm\infty.$$

(We do not specify the topology here as that is a part of the discussion.) We write

$$\phi^- \rightsquigarrow \phi^+$$

if such a connection exists.

In qualitative theory of one–dimensional parabolic equations, much effort has been devoted to the study of the connection problem - determining which equilibria are connected by heteroclinic solutions (see [9] for recent developments and references). For problem (P) it is rather easy to establish the *classical* connecting orbits:

(CO) Let λ be different from all the bifurcation values $\lambda_1, \lambda_2, \ldots$. Given any j, k, a connection from ϕ_k to ϕ_j exists if and only if $k > j$.

Let us sketch an argument for (CO). Using local analysis near the turning point (λ_k, ϕ_k) (cf. Theorem A.1 statements (a) and (c)) and reduction to one–dimensional center manifolds one shows that $\phi_k \rightsquigarrow \phi_{k-1}$. Due to the Morse–Smale structure of (P) (see [30]), the connection persists as λ is moved away form the bifurcation value. This way one shows existence of the connections

$$\phi_k \rightsquigarrow \phi_{k-1} \rightsquigarrow \ldots \rightsquigarrow \phi_0,$$

where k is such that the equilibria exist for a particular λ. Existence of this chain of connections, in conjunction with the Morse–Smale structure, further implies that the connection $\phi_i \rightsquigarrow \phi_j$ exists if $i > j$ (cf. [9]). Finally, the Morse–Smale structure implies that the connection $\phi_k \rightsquigarrow \phi_j$ can exist only if the Morse index of ϕ_k is greater than the Morse index of ϕ_j (cf. [9]), thus $k > j$ is necessary (cf. Theorem A.1(e)).

The results of this section suggest the following problem: Determine which equilibria are connected by heteroclinic L^1–solutions. For $\lambda \in (\lambda_2, \lambda_3)$, we have found L^1–connections from from ϕ_2 to ϕ_0, but we have not pursued the study of L^1–connections any further.

Appendix. In this section we consider the problem for stationary solutions of (P), that is

$$\Delta\phi + \lambda e^{\phi} = 0, \quad x \in B_1(0), \tag{A.1}$$

$$\phi = 0, \quad x \in \partial B_1(0). \tag{A.2}$$

Note that (A.1), (A.2) is equivalent to (SP) as all solutions of (A.1), (A.2) are positive and therefore radially symmetric [14]. For any solution $\phi = \phi(|x|)$ we also consider the eigenvalue problem

$$\Delta\psi + \lambda e^{\phi(|x|)}\psi + \mu\psi = 0, \quad x \in B_1(0), \tag{A.3}$$

$$\psi \in X. \tag{A.4}$$

Note that (A.4) requires that ψ satisfies the Dirichlet boundary condition and that it is radially symmetric. (Let us remark that the symmetry is no restriction if $\mu \le 0$ since all solutions of (A.3) satisfying the boundary condition are radially symmetric, see [5] for a more general result and references to earlier results of this kind). Problem (A.3), (A.4) is equivalent to

$$\text{(L)} \qquad \begin{cases} \psi_{rr} + \frac{N-1}{r}\psi_r + \lambda e^{\phi(r)}\psi + \mu\psi = 0, \quad r \in (0,1), \\ \psi_r(0) = \psi(1) = 0. \end{cases}$$

By Sturm–Liouville theory all eigenvalues of (L) are simple and the eigenfunction of the j-th eigenvalue has exactly j zeros in $[0,1]$ $(j = 1, 2, \dots)$. Moreover all zeros of any eigenfunction are simple. (This is obvious for $r > 0$; for any $r \in [0,1]$ this property follows directly from Lemma 2.2.) We say that a solution ϕ of (A.1), (A.2) is *hyperbolic* if $\mu = 0$ is not an eigenvalue of (L).

Denote by S the solution set of the parametrized problem (A.1), (A.2):

$$S = \{(\phi, \lambda) : \lambda \in \mathbb{R}^+ \text{ and } \phi \text{ is a solution of (A.1), (A.2)}\}.$$

Theorem A.1. *There exists a smooth curve*

$$s \mapsto (\phi(s), \lambda(s)) \colon \mathbb{R}^+ \to X \times \mathbb{R}^+$$

such that $\phi(s)$ is the solution of (A.1), (A.2) *with $\lambda = \lambda(s)$,*

$$\sup_{x \in B_1(0)} \phi(s)(x) = \phi(s)(0) = s$$

and $S = \{(\phi(s), \lambda(s)) : s > 0\}$. Moreover the following properties are satisfied:

(a) *$\lambda(s)$ is a Morse function, that is*

$$\textit{if} \;\; \lambda'(s) = 0 \;\; \textit{then} \;\; \lambda''(s) \neq 0.$$

(b) *The critical values of $\lambda(\cdot)$ form an infinite sequence $\lambda_1, \lambda_2, \dots$ with the following properties:*

$$\lambda_1 > \lambda_3 > \dots > \lambda_{2j+1} \searrow \lambda_\infty = 2(N-2),$$
$$\lambda_2 < \lambda_4 \dots < \lambda_{2j+2} \nearrow \lambda_\infty.$$

(c) *For each $\lambda \leq \lambda_1$ define*

$$\phi_i^\lambda = \phi(s_i) \;\; (i = 0, 1, \dots),$$

where $s_0 < s_1 < \dots$ is the sequence of all points s with $\lambda(s) = \lambda$. This sequence is finite if $\lambda \neq \lambda_\infty$ and infinite if $\lambda = \lambda_\infty$. In the latter case we have

$$\phi_i(r) \to \phi_\infty = \ln \frac{2(N-2)}{\lambda r^2} \quad \textit{in} \quad C^1_{loc}((0,1]).$$

(d) *$\phi(s)$ is hyperbolic if and only if $\lambda'(s) \neq 0$.*

(e) *Let $m(\phi(s))$ denote the number of negative eigenvalues of* (A.3), (A.4) *with $\phi = \phi(s)$ and $\lambda = \lambda(s)$. Then*

$$m(\phi(s)) = \#\{\bar{s} \in \mathbb{R}^+ : \bar{s} < s \text{ and } \lambda'(\bar{s}) = 0\}.$$

In other words, $m(\phi_j^\lambda) = j$ for any j such that ϕ_j^λ is defined. In particular, ϕ_0^λ is asymptotically stable for $\lambda < \lambda_1$ and it is the only stable equilibrium of (P).

(f) *If $\lambda < \lambda_1$ and $k > j$ are such that ϕ_k^λ and ϕ_j^λ are both defined then*

$$\phi_k^\lambda - \phi_j^\lambda \text{ has exactly } j+1 \text{ zeros in } [0,1], \text{ all of them simple.} \tag{A.5}$$

(g) *For any $\lambda \in (\lambda_2, \lambda_1)$ and any j such that ϕ_j^λ is defined one has*

$$\partial_r \phi_j^\lambda(1) < \partial_r \phi_1^\lambda(1).$$

Proof. All statements, except possibly for (f) and (g) are well–known and their proofs can be found in [13, 16, 25]. We prove (f). The fact that $\phi_k^\lambda - \phi_j^\lambda$ has only simple zeros follows directly from Lemma 2.2. Fix a positive integer j and consider first the zero number $z(\phi_{j+1}^\lambda - \phi_j^\lambda)$. For definiteness we assume that j is even, the proof for j odd is analogous. For λ near the bifurcation value λ_{j+1} the zero number is equal to the number of zeros of the eigenfunction corresponding to the eigenvalue $\mu = 0$ of (A.3), (A.4) with $\lambda = \lambda_{j+1}$, $\phi = \phi_j^\lambda$ (cf. [25]). This number is equal to $j+1$ because $\mu = 0$ is the $(j+1)$–th eigenvalue by (e). By simplicity of zeros and continuation in X, we further have

$$z(\phi_{j+1}^\lambda - \phi_j^\lambda) = j+1 \quad (\lambda \in [\lambda_{j+2}, \lambda_{j+1})).$$

Now, for $\lambda \in (\lambda_{j+2}, \lambda_{j+3}]$, $\lambda \approx \lambda_{j+2}$, one has $\phi_{j+1}^\lambda \approx \phi_{j+2}^\lambda$ and therefore

$$z(\phi_{j+2}^\lambda - \phi_j^\lambda) = z(\phi_{j+1}^\lambda - \phi_j^\lambda) = j+1$$

for such λ. This again can be continued for any $\lambda \in [\lambda_{j+2}, \lambda_{j+3}]$. For $\lambda \approx \lambda_{j+3}$ we may replace ϕ_{j+2}^λ by ϕ_{j+3}^λ and so on. An induction argument completes the proof of (A.5).

Property (g) follows from (f) and the relation

$$\phi_1(0) < \phi_j(0) \quad \text{if } j > 1.$$

REFERENCES

[1] S. Angenent, *The zeroset of a solution of a parabolic equation*, J. reine angew. Math. **390** (1988), 79-96.

[2] J. Bebernes and D. Eberly, *Mathematical Problems from Combustion Theory*, Springer-Verlag, New York, 1989.

[3] P. Brunovský and B. Fiedler, *Number of zeros on invariant manifolds in reaction-diffussion equations*, Nonlinear Anal. **10** (1986), 179-193.

[4] M. Chen, X-Y. Chen and J.K. Hale, *Structural stability for time-periodic one-dimensional parabolic equations*, J. Differential Equations **96** (1992), 355-418.

[5] X.-Y. Chen and P. Poláčik, *Asymptotic periodicity of positive solutions of reaction-diffusion equations on a ball*, J. reine angew. Math. **472** (1996), 17-51.

[6] S.-N. Chow and J.K. Hale, *Methods of Bifurcation Theory*, Springer, New York, 1982.

[7] S.-N. Chow and K. Lu, *C^k center unstable manifolds*, Proc. Royal Soc. Edinburgh **108A** (1988), 303-320.

[8] D. Eberly and W.C. Troy, *Existence of logaritmic type solutions to the Kapila-Kassoy problem in dimensions 3 through 9*, J. Differential Equations **70** (1987), 309-324.

[9] B. Fiedler and C. Rocha, *Heteroclinic orbits of scalar semilinear parabolic equations*, J. Differential Equations **125** (1996), 239-281.

[10] M. Fila, *Boundedness of global solutions of nonlinear diffusion equations*, J. Diff. Equations **98** (1992), 226–240.

[11] H. Fujita, *On the nonlinear equations $\Delta u + e^u = 0$ and $v_t = \Delta v + e^v$*, Bull. Amer. Math. Soc. **75** (1969), 132–135.

[12] V. Galaktionov and J.L. Vázquez, *Continuation of blow-up solutions of nonlinear heat equations in several space dimensions*, Comm. Pure Applied Math. **50** (1997), 1–67.

[13] I.M. Gelfand, *Some problems in the theory of quasilinear equations*, Amer. Math. Soc. Transl. **29** (1963), 295–381.

[14] B. Gidas, W. Ni and L. Nirenberg, *Symmetry and related properties by the maximum principle*, Comm. Math. Phys. **68** (1979), 209 - 243.

[15] D. Henry, *Geometric theory of semilinear parabolic equations*, Springer, New York, 1981.

[16] D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rat. Mech. Anal. **49** (1973), 241–269.

[17] H. Koch, *On center manifolds*, Nonlinear Anal. **28** (1997), 1227-1248.

[18] A.A. Lacey, *Mathematical analysis of thermal runaway for spatially inhomogeneous reactions*, SIAM J. Appl. Math. **43** (1983), 1350–1366.

[19] A.A. Lacey and D. Tzanetis, *Global existence and convergence to a singular steady state for a semilinear heat equation*, Proc. Roy. Soc. Edinburgh Sect. A **105** (1987), 289–305.

[20] A.A. Lacey and D. Tzanetis, *Complete blow-up for a semilinear diffusion equation with a sufficiently large initial condition*, IMA J. Appl. Math. **41** (1988), 207–215.

[21] A.A. Lacey and D.E. Tzanetis, *Global, unbounded solutions to a parabolic equation*, J. Diff. Equations **101** (1993), 80–102.

[22] O.A. Ladyzenskaya and N.N. Ural'ceva, *Linear and Quasilinear Equations of Elliptic type*, Academic Press, New York and London, 1968.

[23] P. L. Lions, *Structure of steady state solutions and asymptotic behavior of semilinear heat equation*, J. Diff. Equations **53** (1984), 362-386.

[24] M. Medved', *Fundamentals of dynamical systems and bifurcation theory*, Adam Hilger, Philadelphia, 1992.

[25] K. Nagasaki and T. Suzuki, *Spectral and related properties about the Emden-Fowler equation $-\Delta u = \lambda e^u$ on circular domains*, Math. Ann. **299** (1994), 1-15.

[26] W.-M. Ni and P. Sacks, *The number of peaks of positive solutions of semilinear parabolic equations*, SIAM J. Math. Anal. **16** (1985), 460–471.

[27] W.-M. Ni and P. Sacks, *Singular behavior in nonlinear parabolic equations*, Trans. Amer. Math. Soc. **287** (1985), 657–671.

[28] I. Peral and J.L. Vázquez, *On the stability or instability of the singular solution of the semilinear heat equation with exponential reaction term*, Arch. Rat. Mech. Anal. **129** (1995), 201-224.

[29] P. Poláčik, *Domains of attraction of equilibria and monotonicity properties of convergent trajectories in parabolic systems admitting strong comparison principle*, J. reine angew. Math. **400** (1989), 32-56.

[30] P. Poláčik, *Transversal and nontransversal intersections of stable and unstable manifolds in reaction diffusion equations on symmetric domains*, Diff. Int. Equations **7** (1994), 1527-1545.

[31] D. Tzanetis, *Asymptotic behaviour and blow-up of some unbounded solutions for a semilinear heat equation*, Proc. Edinb. Math. Soc. **39** (1996), 81-96.