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APPROXIMATE RADIAL SYMMETRY FOR OVERDETERMINED BOUNDARY VALUE PROBLEMS

Amandine Aftalion and Jérôme Busca
DMI, Ecole Normale Supérieure
45, rue d'Ulm, 75230 Paris cedex 05, France

Wolfgang Reichel
Mathematisches Institut, Universität zu Köln
D-50931 Köln, Germany

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Abstract. In this paper, we study the stability of Serrin's classical symmetry result for overdetermined boundary value problems [13]. We prove that if there exists a positive solution of $\Delta u + f(u) = 0$ in Ω with $u = 0$ on $\partial\Omega$ and if $\partial u/\partial\nu$ on $\partial\Omega$ is close to a constant, then the domain Ω is close to a ball. Additionally, we give an explicit estimate for the distance of the domain to a circumscribed and inscribed ball. The proof relies on the method of moving planes and new quantitative versions of the Hopf Lemma and Serrin's corner Lemma.

1. Introduction and main results. In a famous paper [13], J. Serrin proved the following result: if a smooth bounded domain $\Omega \subset \mathbb{R}^N$ is such that there exists a solution of the overdetermined problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = c & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

then the domain Ω is a ball and the solution is radial. Note that we call ν the exterior normal to $\partial\Omega$.

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Recently, the present authors have extended this symmetry result in [1], [2], [9], [10] to some cases where the domain is unbounded. In this paper, we are interested in approximate radial symmetry for bounded domains, which is a kind of stability property of Serrin's result. Namely, we consider a bounded $C^{2,\alpha}$-domain Ω such that there exists a solution of

$$\begin{cases} \Delta u + f(u) = 0 & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

Our aim is to prove that if additionally the normal derivative $\partial u/\partial\nu$ on the boundary is close to a constant, then the domain Ω is close to a ball.

Let us mention that an approximate symmetry result has been established by Rosset [11], but in another context. In [11], the author considers problem (2) when the domain Ω is a priori assumed to be close to a ball (either in the C^0 or the C^1-topology). Then he proves that the solution u is nearly radial.

Our result is of a different kind since we impose an extra Neumann condition which forces the domain to be approximately symmetric. Our proof relies on the moving plane method of Alexandroff [3] and Serrin [13]. More precisely, for any fixed direction, the moving plane method up to some critical position provides a symmetric domain X which is included in Ω. Then, with the help of new versions of the Hopf Lemma and Serrin's corner Lemma, we obtain an estimate of the distance between Ω and X. This allows us to construct two concentric balls (one contained in Ω and the other containing Ω), and to bound the difference between the two radii.

Now we state our main result.

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{2,\alpha}$-domain and let f be a locally Lipschitz continuous function with $f(0) \geq 0$. Consider a solution $u \in C^2(\overline{\Omega})$ of (2) with $|\partial u/\partial\nu| \geq d_0 > 0$ on $\partial\Omega$. Then there exist constants ϵ, $C > 0$ such that the following holds: if $\|\frac{\partial u}{\partial\nu} - d\|_{C^1(\partial\Omega)} \leq \epsilon$ for a constant $d < 0$, then there are two concentric balls B_r, B_R such that*

$$B_r \subset \Omega \subset B_R \tag{3}$$

$$\text{and} \qquad R - r \leq C\Big|\log\Big\|\frac{\partial u}{\partial\nu} - d\Big\|_{C^1(\partial\Omega)}\Big|^{-1/N}. \tag{4}$$

The constants ϵ and C depend only on the $C^{2,\alpha}$-regularity of Ω and on upper bounds for $\operatorname{diam}\Omega$, $\|u\|_\infty$, $\|f\|_{C^{0,1}}$ *and* $1/d_0$.

Remark 1. a) In order to denote the constant dependence above, we shall use the notation $C = C(\Omega, \operatorname{diam}\Omega, \|u\|_\infty, f, d_0)$. The dependence on Ω is meant to be only on the $C^{2,\alpha}$ regularity of its boundary. More precisely, we use the following definition of $C^{2,\alpha}$ domains, see Gilbarg-Trudinger [8] p.94:

there exists a constant K and a radius ρ_0, such that for all x on $\partial\Omega$, there is a ball B of radius ρ_0 and a one-to-one $C^{2,\alpha}$ mapping $\psi = \psi_x$ of B onto $D \subset \mathbb{R}^N$ such that

$$x \in B,\ \psi(B \cap \Omega) \subset \mathbb{R}^N_+,\ \psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^N_+,$$
$$\|\psi\|_{C^{2,\alpha}(B)} \le K,\ \text{and}\ \|\psi^{-1}\|_{C^{2,\alpha}(D)} \le K.$$

In particular a constant $C(\Omega)$ will depend only on upper bounds on K and $1/\rho_0$. The Ω-dependence of the constant C will appear explicitly in Proposition 4 when we use the $C^{2,\alpha}$-Schauder estimates up to the boundary for a solution of (2).

b) Since Ω is a $C^{2,\alpha}$-domain, it satisfies a uniform interior ball condition of radius ρ at each point of the boundary. We may assume that ρ is maximal. Part a) of the Remark yields $1/\rho \le C(\Omega)$. Note that we also have $\rho \le \operatorname{diam}\Omega/2$.

c) If $f(0) > 0$ then the hypothesis $|\partial u/\partial\nu| \ge d_0 > 0$ on $\partial\Omega$ is automatically fulfilled. In fact, we show in Section 5 that $f(0) \le d_0 C(\Omega, \|u\|_\infty, f)$.

The paper is organized as follows: in Section 2, we apply the moving plane method with an arbitrary direction $\eta \in \mathbb{R}^N \setminus \{0\}$. It means that hyperplanes orthogonal to η are moved through the domain Ω until a critical geometrical position is reached, which defines the maximal hyperplane T. As a result of the moving plane method, we obtain the non-negativity of the comparison function w, which is defined in a subset Z of Ω and compares values of u at a given point and at its reflection with respect to T. Then we state and prove new versions of the Hopf Lemma and Serrin's corner Lemma which allow us to estimate the smallness of w in Z depending on the Neumann data of u on the boundary. In Section 3, we construct a symmetric set X as the union of Z and its reflection with respect to T. Thanks to the smallness of w and lower bounds on u by the distance to the boundary, we prove that X nearly fills Ω. This is stated in Proposition 5, which is the main step in the proof. In Section 4, we first choose η as the coordinate-direction $\mathbf{e}_1$ and perform the moving plane method for this direction. Then we repeat the

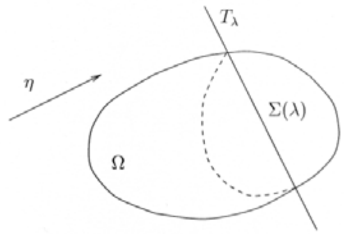

Figure 1: Illustration of the notations.

same process with the remaining standard coordinate-directions $\mathbf{e}_2, \dots, \mathbf{e}_N$. Thus, we obtain a point O as the intersection of the maximal hyperplanes with respect to these N orthogonal directions.

Finally, we prove that O is the center of balls close to Ω in the C^0-topology by means of the moving plane method for another properly chosen direction η. Section 5 is devoted to extra remarks and to a physical and a geometrical application of Theorem 1.

2. Moving planes up to the critical position. We are going to use the moving plane device up to the critical position, developed by Alexandroff [3] and Serrin [13]. For a unit vector $\eta \in \mathbb{R}^N$ we will perform the moving plane method in direction η on the domain Ω. For general reference, we refer to Serrin [13], Gidas, Ni, Nirenberg [7], Berestycki, Nirenberg [4] or Fraenkel [6]. Let us define the following objects:

$T_\lambda = \{x \in \mathbb{R}^N \mid x \cdot \eta = \lambda\}$	the hyperplanes orthogonal to η,
$x^{\lambda,\eta} = x - 2(x \cdot \eta - \lambda)\eta$	the reflection of x about T_λ,
$A^{\lambda,\eta} = \{x \in \mathbb{R}^N \mid x^{\lambda,\eta} \in A\}$	the reflection of a set A,
$\Sigma(\lambda) = \{x \in \Omega \mid x \cdot \eta > \lambda\}$	the right-hand cap,
$\Gamma(\lambda) = \partial\Sigma(\lambda) \setminus T_\lambda$	the right-hand boundary,
$M = \max\{x \cdot \eta \mid x \in \Omega\}$	the extent of Ω in direction η.

It is well-known that for λ close to M, the reflected cap $\Sigma(\lambda)^{\lambda,\eta}$ remains in Ω, and this is true until a critical geometrical position is reached, that is
i) $\Sigma(\lambda)^{\lambda,\eta}$ becomes internally tangent to $\partial\Omega$ at some point, or

ii) T_λ is orthogonal to the boundary of Ω at some point.

The values of λ corresponding to i) and ii) are

$$\alpha = \inf\{ \mu \mid \Sigma(\lambda)^{\lambda,\eta} \subset \Omega \text{ for all } \lambda \in (\mu, M)\},$$

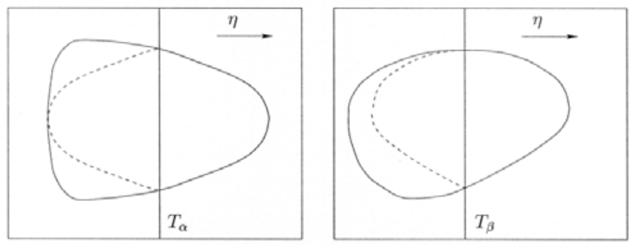

Figure 2: The critical geometrical positions: tangency and orthogonality

$$\beta \;=\; \inf\{\ \mu \mid \nu \cdot \eta > 0 \text{ on } \Gamma(\lambda) \text{ for all } \lambda \in (\mu, M)\}.$$

The hyperplanes T_α and T_β respectively correspond to internal tangency and to orthogonality (see Figure 2). The value $m = \max(\alpha, \beta)$ corresponds to the critical geometrical position.

We denote by p the point where either orthogonality or tangency occurs. In case of tangency, $p \in \Gamma(m)$ and the reflected point $p^{m,\eta} \in \partial\Omega$. In case of orthogonality, $p \in \partial\Omega \cap T_m$, and $\Sigma(m)$ has a right-angled corner at p.

In the following lemma, we prove that the critical position is not reached too close to M. This is due to the fact that Ω satisfies an interior ball condition with radius ρ (see Remark 1).

Lemma 1. *The critical position m and the extent M of Ω in direction η satisfy $M - m \geq \rho$, with $1/\rho \leq C(\Omega)$.*

Proof. At the point p defined earlier, there is an interior ball B_ρ with center $z = p - \rho\nu(p)$ contained in Ω. In case of orthogonality, the claim is obvious. In case of tangency, the point $p^{m,\eta}$ lies on $\partial\Omega$ and hence $|z - p^{m,\eta}| \geq \rho$. A simple computation shows that this means $z \cdot \eta \geq m$, i.e., z is contained in the right-hand half space. □

The key role in the moving plane method is played by the functions

$$v_\lambda(x) = u(x^{\lambda,\eta}), \qquad w_\lambda(x) = v_\lambda(x) - u(x).$$

Notice that both functions are well defined in $\Sigma(\lambda)$ for $\lambda \in [m, M)$. The function w_λ compares the values of the solution u at the point x and at the reflected point $x^{\lambda,\eta}$. It satisfies

$$\Delta w_\lambda + c_\lambda w_\lambda = 0 \text{ in } \Sigma(\lambda), \tag{5}$$

where $\|c_\lambda\|_{L^\infty}$ is bounded by the Lipschitz constant of the function f. Under the hypotheses of Theorem 1, the function w_λ attains nonnegative values on $\partial\Sigma(\lambda)$. The main and most well known result of the moving plane method is the positivity of the comparison function, i.e.,

$$w_\lambda > 0 \text{ on } \Sigma(\lambda) \text{ for } \lambda \in (m, M). \tag{6}$$

In particular, we also have

$$w_m \geq 0 \text{ in } \Sigma(m). \tag{7}$$

Notice that in the following, for simplicity, we will write Σ for the maximal right-hand cap $\Sigma(m)$. It will be convenient to denote by x' the reflection of a point x with respect to the hyperplane T_m (the same notation is used for the reflection A' of a set A).

In Serrin's original proof [13], the symmetry of the underlying domain is derived from the observation that $w_m \equiv 0$ in a connected component of the maximal cap Σ. For the case of internal tangency, $w_m \equiv 0$ is achieved by showing $\partial w_m(p)/\partial\nu = 0$ and using the Hopf Lemma. In case of orthogonality, Serrin shows that w_m has a second-order zero at p, and must therefore vanish by Serrin's corner Lemma, which is an extension of the standard Hopf Lemma to domains with corners.

For our problem, we will derive L^∞-bounds for w_m. In the tangency-case, w_m will be bounded in terms of $\partial w_m(p)/\partial\nu$ by Lemma 3, which is a quantitative version of the Hopf Lemma appropriate for tangency. In case of orthogonality, we will use Proposition 3, a new version of Serrin's corner Lemma, to bound w_m in terms of $\partial^2 w_m(p)/\partial s^2$ for some inward direction s at p. Hence, in our case, the smallness of w_m as expressed in the next Proposition, replaces the fact that w_m vanishes identically in Serrin's case.

We will use the following notation: for an open set $A \subset \mathbb{R}^N$ and a value $\delta > 0$ we define $A_\delta = \{x \in A \mid \operatorname{dist}(x, \partial A) > \delta\}$, which is an open, nonempty parallel subset of A if δ is sufficiently small. We denote by $\widetilde{A}_\delta$ a connected component of A_δ.

Proposition 1. *Let ρ be the radius of a uniform interior ball at each point of the boundary of Ω, and suppose that ρ is maximal with respect to this property. For all δ in $(0, \rho/16)$, the parallel subset Σ_δ of the maximal cap Σ has a connected component $\widetilde{\Sigma}_\delta$ with*

$$\|w_m\|_{L^\infty(\widetilde{\Sigma}_\delta)} \leq C(\Omega, \operatorname{diam}\Omega, \|u\|_\infty, f) C_0^{\operatorname{diam}\Omega^N/\delta^N} \|\frac{\partial u}{\partial\nu} - d\|_{C^1(\partial\Omega)}. \tag{8}$$

This is a first estimate for w_m, but notice that it gets bad when δ gets small. In section 3, we will choose an appropriate δ which will eventually give a measure of how close Ω is to a ball. The rest of this section is devoted to the proof of Proposition 1.

First, we state new quantitative versions of the Hopf Lemma (Proposition 2) and of Serrin's corner Lemma (Proposition 3) in a general framework. These results may be of interest for other problems. Then, in subsection 2.2, we prove these two propositions, as well as another version (Lemma 3) which is adapted to our purposes. Finally, in subsection 2.3, we prove the estimate (8) by applying Proposition 3 and Lemma 3 to the comparison function w_m and the point p defined by the critical geometrical position.

2.1. Quantitative estimates. To start with, we state new versions of the Hopf Lemma and Serrin's corner Lemma. We write $c^-(x) = \min(c(x), 0)$ for the negative part of a function c. For a bounded open set $A \subset \mathbb{R}^N$ we denote by $|A|$ the Lebesgue-measure of A.

Proposition 2. *Let $D \subset \mathbb{R}^N$ be a bounded domain with an interior ball B_ρ at $p \in \partial D$. Let $w \in C^2(D) \cap C^1(D \cup \{p\})$ satisfy*

$$\begin{cases} \Delta w + c(x)w = 0 & \text{in} \quad D, \\ w \geq 0 & \text{in} \quad D, \\ w(p) = 0. \end{cases} \tag{9}$$

Then for any $\delta \in (0, \rho/2)$ there is a connected component $\widetilde{D}_\delta$ of D_δ intersecting B_ρ with

$$w(y) \leq C_0^{|D|/\delta^N} C(\rho, \|c^-\|_\infty) |\frac{\partial w}{\partial \nu}(p)| \text{ for all } y \in \widetilde{D}_\delta, \tag{10}$$

where ν is the exterior normal to ∂B_ρ at p.

Here and in the following, a value $C(\rho, \|c^-\|_\infty)$ is meant to depend only on upper bounds for ρ and $\|c^-\|_\infty$.

Proposition 3. *Let $D \subset \mathbb{R}^N$ be a bounded C^1-domain with an interior ball B_ρ at $p \in \partial D$. Suppose the hyperplane $T = \{x_1 = 0\}$ intersects ∂D orthogonally at p, and let us denote by D^+ the set $D \cap \{x_1 > 0\}$. Let $w \in C^2(\overline{D^+})$ satisfy*

$$\begin{cases} \Delta w + c(x)w = 0 & \text{in} \quad D^+, \\ w \geq 0 & \text{in} \quad D^+, \\ w(p) = \nabla w(p) = 0, \\ w_{|T} \equiv 0. \end{cases} \tag{11}$$

Then for any $\delta \in (0, \rho/8)$, there is a connected component $\widetilde{D}_\delta^+$ of D_δ^+ intersecting B_ρ with

$$w(y) \le C_0^{|D^+|/\delta^N} \frac{C(\rho, \|c^-\|_\infty)}{s_1 \ |s \cdot \nu(p)|} \left|\frac{\partial^2 w}{\partial s^2}(p)\right| \text{ for all } y \in \widetilde{D}_\delta^+,$$

where $s = (s_1, \ldots, s_N)$ is any direction that enters D^+ nontangentially at p.

Note that the classical Hopf Lemma and Serrin's corner Lemma are included in these propositions: if $\partial w(p)/\partial\nu = 0$ or $\partial^2 w(p)/\partial s^2 = 0$, then $w \equiv 0$ in $\widetilde{D}_\delta$, $\widetilde{D}_\delta^+$ respectively. Letting $\delta \to 0$, one gets $w \equiv 0$ in a connected component of D, D^+.

These propositions will be applied to the function w_m on the maximal right hand cap $D = \Sigma$. The point p will be the point where tangency or orthogonality happens. The upper bound on ρ in the constant dependence will eventually be replaced by an upper bound on the diameter of Ω.

2.2. Proof of Proposition 2 and 3. Here we prove the quantitative Hopf Lemmas of the previous section. The proofs can be read independently of the rest of the paper. In Lemma 3, we state another version, which is important for our purpose in the context of moving-planes.

Proof of Proposition 2. The idea is to get an upper bound on w with the help of the comparison function that appears in the proof of the conventional Hopf lemma (see [8] for instance). As w is nonnegative, we have $\Delta w + c^-(x)w \le 0$ in D. Since the statement of Proposition 2 is trivially true for $w \equiv 0$, we may suppose $w > 0$. Let us take as new origin the center of the interior ball at p. Then we have $B_\rho(O) \subset D$ and $\overline{B_\rho(O)} \cap \partial D = \{p\}$. Let us define the comparison function $v(x) = e^{-\alpha r^2} - e^{-\alpha\rho^2}$ in the annular domain $A = \{x \mid \rho/2 < r = |x| < \rho\} \subset D$. A simple calculation yields $\Delta v + c^-(x)v \ge (\alpha^2\rho^2 - 2N\alpha - \|c^-\|_\infty)e^{-\alpha r^2}$. For

$$\alpha = \max(4N/\rho^2, \sqrt{2\|c^-\|_\infty}/\rho) \tag{12}$$

we obtain $\Delta v + c^-(x)v \ge 0$ in A. If we set $\beta = w(q)/(e^{-\alpha(\rho/2)^2} - e^{-\alpha\rho^2})$, where $q \in \partial B_{\rho/2}$ is such that $w(q) = \min_{\partial B_{\rho/2}} w$, then we have $\beta v \le w$ on ∂A. It follows from the Maximum Principle that $0 < \beta v \le w$ in A. Since both βv and w vanish at p, we find $|\beta \frac{\partial v}{\partial \nu}(p)| \le |\frac{\partial w}{\partial \nu}(p)|$. Evaluating $|\frac{\partial v}{\partial \nu}(p)|$ we infer

$$\min_{\partial B_{\rho/2}} w \le \left|\frac{\partial w}{\partial \nu}(p)\right| \frac{(e^{\alpha(\rho^2 - (\rho/2)^2)} - 1)}{2\alpha\rho}.$$

For the choice of α given by (12), one can check that

$$\min_{\partial B_{\rho/2}} w \le C(\rho, \|c^-\|_\infty)|\frac{\partial w}{\partial \nu}(p)|,$$

where $C(\rho, \|c^-\|_\infty)$ is bounded if ρ and $\|c^-\|_\infty$ are bounded from above. If $\delta \in (0, \rho/2)$, then $\partial B_{\rho/2}(O)$ lies in a connected component $\widetilde{D}_\delta$ of D_δ. By Lemma A1, any point in $\widetilde{D}_\delta$ can be connected to q by a path of length at most $L_0|D|/\delta^{N-1}$. An application of Harnack's inequality (see [8]) then results in:

$$\sup_{\widetilde{D}_\delta} w \le C_0^{L_0|D|(1+\frac{1}{\delta})/\delta^{N-1}} \min_{\partial B_{\rho/2}} w \le C_0^{|D|/\delta^N} C(\rho, \|c^-\|_\infty)|\frac{\partial w}{\partial \nu}(p)|,$$

which is the desired result. □

In the sequel, we shall need the following generalization of Proposition 2. Its main feature is that it allows to replace the normal derivative of w at the boundary point by a difference quotient of w.

Lemma 2. *Let $w \in C^2(D) \cap C^0(D \cup \{p\})$ satisfy the hypotheses of Proposition 2. Then the conclusion (10) remains valid if $\frac{\partial w}{\partial \nu}(p)$ is replaced by $w(p - t\nu)/t$ for $t \in (0, \rho/2]$; the constant C does not depend on t.*

Proof. The proof is the same as in Proposition 2, with the only difference that we have $\beta v(p-t\nu)/t \le w(p-t\nu)/t$ instead of $|\beta \frac{\partial v}{\partial \nu}(p)| \le |\frac{\partial w}{\partial \nu}(p)|$. Since there is $\tilde{r} = \tilde{r}(t) \in (\rho/2, \rho)$ such that $v(p - t\nu)/t = |v'(\tilde{r})| = 2\alpha\tilde{r}e^{-\alpha\tilde{r}^2}$, we have a lower bound for $v(p-t\nu)/t$ and we can conclude as in Proposition 2.

Proof of Proposition 3. We take the origin O of our coordinate-system as the center of the interior ball. We may assume $w > 0$ in D since for $w \equiv 0$ the result is trivial. As before, w satisfies $\Delta w + c^- w \le 0$. We introduce the following comparison function which already appears in the proof of Serrin's corner Lemma [13]:

$$v(x) = x_1(e^{-\alpha r^2} - e^{-\alpha\rho^2}),$$

defined in the set $A^+ = \{x \mid \bar{\rho} < r = |x| < \rho \text{ and } x_1 > 0\}$. We choose $\bar{\rho} = \rho/8$, which will be justified later. A simple calculation yields

$$\Delta v + c(x)^- v \ge 2\alpha x_1(-N - 2 + 2\alpha\bar{\rho}^2)e^{-\alpha r^2} - \|c^-\|_\infty x_1 e^{-\alpha r^2} \ge 0$$

if one chooses

$$\alpha = \max(\sqrt{\tfrac{1}{2}\|c^-\|_\infty}/\bar{\rho}, (N+2)/\bar{\rho}^2). \tag{13}$$

Note that v and w both vanish on $T = \{x_1 = 0\}$. Since the standard Hopf Lemma applied to w yields $w \geq \epsilon x_1$ on $\partial B^+_{\bar\rho} = \partial B_{\bar\rho}(O) \cap D^+$, there exists $\beta > 0$ such that $\beta v \leq w$ on $\partial B^+_{\bar\rho}$, and we may take β to be maximal, i.e., $\beta = \sup_{\partial B^+_{\bar\rho}} w/v$. By the Maximum Principle $\beta v \leq w$ in A^+, and since $v(p) = w(p) = 0$, $\nabla v(p) = \nabla w(p) = 0$ we have

$$|\beta \frac{\partial^2 v}{\partial s^2}(p)| \leq |\frac{\partial^2 w}{\partial s^2}(p)|, \tag{14}$$

for any s that enters D^+ nontangentially at p. Evaluating $\frac{\partial^2 v}{\partial s^2}(p)$ and substituting in (14) implies

$$\beta \leq |\frac{\partial^2 w}{\partial s^2}(p)| \frac{e^{\alpha\rho^2}}{4\alpha\rho s_1 |s \cdot \nu(p)|}. \tag{15}$$

Now we distinguish between two different situations that arise from the definition of β:

Case 1: $\dfrac{\partial w}{\partial x_1}(q) = \beta \dfrac{\partial v}{\partial x_1}(q)$ at some point $q \in \partial B_{\bar\rho}(O) \cap T$,

Case 2: $w(q) = \beta v(q)$ at some point $q \in \partial B_{\bar\rho}(O) \cap D^+$.

Case 1: we have

$$|\frac{\partial w}{\partial x_1}(q)| = \beta(e^{-\alpha\bar\rho^2} - e^{-\alpha\rho^2}) \leq |\frac{\partial^2 w}{\partial s^2}(p)| \frac{(e^{\alpha\rho^2} - 1)}{4\alpha\rho s_1 |s \cdot \nu(p)|}. \tag{16}$$

The ball $B_{\rho-\bar\rho}(q)$ is contained in $B_\rho(O)$, and therefore the set D^+ satisfies an interior ball condition at q with a radius $(\rho - \bar\rho)/2$, which is bigger than $\rho/4$ because of the choice $\bar\rho = \rho/8$. Thus, we can apply Proposition 2 for all $\delta \in (0, \rho/8)$ to obtain an upper bound on w in terms of $|\frac{\partial w}{\partial x_1}(q)|$. Thus, (16), the choice of α given by (13) and the observation that $(e^{\alpha\rho^2} - 1)/(\alpha\rho)$ is bounded by $C(\rho, \|c^-\|_\infty)$ yield the desired conclusion.

Case 2: we have

$$w(q) = \beta q_1(e^{-\alpha\bar\rho^2} - e^{-\alpha\rho^2}) \leq |\frac{\partial^2 w}{\partial s^2}(p)| \frac{q_1(e^{\alpha\rho^2} - 1)}{4\alpha\rho s_1 |s \cdot \nu(p)|}, \tag{17}$$

where $q = (q_1, \ldots, q_N)$. Let $\bar q = (0, q_2, \ldots, q_N)$ be the projection of q onto T. As before, the ball $B_{\rho-\bar\rho}(\bar q)$ is contained in $B_\rho(O)$ and hence D^+ satisfies an interior ball condition with a radius bigger than $\rho/4$ at $\bar q$. Moreover, since the distance of $\bar{q}$ to q in the x_1-direction is less than or equal to $\bar{\rho} = \rho/8$, which is at most half of the radius of the interior ball at $\bar{q}$, we can apply Lemma 2 to get an upper bound on w in terms of $w(q)/q_1$. Together with (17), this gives the result. □

Now we prove a version of Proposition 2 that is suitable to give L^∞-bounds in case of tangency. The motivation is the following: the point p, where tangency happens, can be far away from T_m or close to T_m. If p is very close to T_m, then the interior ball B_ρ at p does not lie in Σ. But if we choose a ball of smaller radius, then the estimate in Proposition 2 becomes worse. Therefore we prove an adapted version of Proposition 2.

Lemma 3. *Let $D \subset \mathbb{R}^N$ be a bounded domain with an interior ball B_ρ at $p = (p_1, \ldots, p_N) \in \partial D \cap \{x_1 > 0\}$. Let $D^+ = D \cap \{x_1 > 0\}$ and let us assume that the center of B_ρ lies in D^+. Let $w \in C^2(\overline{D^+})$ satisfy*

$$\begin{cases} \Delta w + c(x)w = 0 & \text{in} \quad D^+, \\ w \geq 0 & \text{in} \quad D^+, \\ w(p) = 0, \quad w_{|T} \equiv 0. \end{cases} \tag{18}$$

Then for any $\delta \in (0, \rho/16)$ there is a connected component $\widetilde{D}_\delta^+$ of D_δ^+ intersecting B_ρ with

$$w(y) \leq C_0^{|D^+|/\delta^N} C(\rho, \|c^-\|_\infty) \frac{\left|\frac{\partial w}{\partial \nu}(p)\right|}{p_1} \text{ for all } y \in \widetilde{D}_\delta^+.$$

Remark 2. Note that the L^∞-estimate for w is of the same quality as in Proposition 2 when p is far away from T. If p is close to T, then the estimate resembles the one in Proposition 3. Hence the previous lemma allows a uniform treatment of all tangency cases.

Proof. The center $z = p - \rho\nu(p)$ of B_ρ lies in the right-hand halfspace $\{x_1 > 0\}$. We distinguish two cases

(i) $z_1 > \rho/12$,

(ii) $0 < z_1 \leq \rho/12$.

On the set $A^+ = (B_\rho(z) \setminus B_{\bar{\rho}}(z)) \cap D^+$, where $\bar{\rho}$ is to be determined, we consider the function

$$v(x) = x_1(e^{-\alpha r^2} - e^{-\alpha\rho^2}) \text{ with } r = |x - z|.$$

For $\alpha = \max(\sqrt{\frac{1}{2}\|c^-\|_\infty}/\bar{\rho}, (N+2)/\bar{\rho}^2)$ we find $\Delta v + c^- v \geq 0$ in A^+. In case (i) we choose $\bar{\rho} = \rho/48$, which implies $B_{\bar{\rho}}(z) \subset D^+$ and $\operatorname{dist}(B_{\bar{\rho}}(z), \partial D^+) = \rho/16$. Then we set $\beta = \min_{\partial B_{\bar{\rho}}(z)} w/v$ and obtain $\beta v \leq w$ in A^+. Now we proceed as in Proposition 2 to get $\beta \leq |\partial_\nu w(p)|/|\partial_\nu v(p)|$ and the conclusion

$$\sup_{\widetilde{D}_\delta^+} w \leq C_0^{|D^+|/\delta^N} C(\rho, \|c^-\|_\infty) \frac{|\partial w(p)/\partial \nu|}{p_1}$$

for $\delta \in (0, \rho/16)$. In case (ii) we choose $\bar{\rho} = \rho/10$. Then $\partial B_{\bar{\rho}}(z)$ and T intersect transversally. Proceeding as in Proposition 3, we find a maximal $\beta > 0$ such that $\beta v \leq w$ on $\partial B_{\bar{\rho}}(z) \cap D^+$. By the Maximum Principle $\beta v \leq w$ in A^+, and since $v(p) = w(p) = 0$, we get similarly to (14) and (15) the estimate

$$\beta \leq C(\rho, \|c^-\|_\infty) \frac{\left|\frac{\partial w}{\partial \nu}(p)\right|}{p_1}. \tag{19}$$

The maximality of β leads to the following two cases:
Case 1: $\dfrac{\partial w}{\partial x_1}(q) = \beta \dfrac{\partial v}{\partial x_1}(q)$ at some point $q \in \partial B_{\bar{\rho}}(z) \cap T$,
Case 2: $w(q) = \beta v(q)$ at some point $q \in \partial B_{\bar{\rho}}(z) \cap D^+$.

Case 1: An evaluation of $\partial v(q)/\partial x_1$ and (19) imply

$$\left|\frac{\partial w}{\partial x_1}(q)\right| \leq C(\rho, \|c^-\|_\infty) \frac{\left|\frac{\partial w}{\partial \nu}(p)\right|}{p_1}. \tag{20}$$

At the point q there is an interior ball or radius at least $(\rho - \bar{\rho})/2 > \rho/8$. Hence, an application of Proposition 2 at q implies the result for $\delta \in (0, \rho/16)$.

Case 2: Now (19) implies

$$w(q) = \beta q_1 (e^{-\alpha \bar{\rho}^2} - e^{-\alpha \rho^2}) \leq q_1 C(\rho, \|c^-\|_\infty) \frac{\left|\frac{\partial w}{\partial \nu}(p)\right|}{p_1}. \tag{21}$$

If $\bar{q} = (0, q_2, \ldots, q_N)$ is the projection of q onto T, then the distance of $\bar{q}$ to z is at most $\rho/12 + \rho/10 < \rho/5$. Therefore the ball $B_{4\rho/5}(\bar{q})$ is contained in $B_\rho(z)$, and hence D^+ satisfies an interior ball condition with a radius bigger than $2\rho/5$ at $\bar{q}$. Moreover, since the distance of $\bar{q}$ to q in the x_1-direction is less than $\rho/12 + \rho/10 < \rho/5$, which is half of the radius of the interior ball at $\bar{q}$, we can apply Lemma 2 for all $\delta \in (0, \rho/5)$ to get an upper bound on w in terms of $w(q)/q_1$. Combining this with (21) implies the result. $\square$

Remark 3. In Proposition 3 and Lemma 3 we have found a connected component $\widetilde{D}_\delta^+$ of D_δ^+ which intersects the interior ball B_ρ. Moreover, we can always assume that a ball of radius $\rho/2-\delta$ is entirely contained in $\widetilde{D}_\delta^+$, which will be important in the proof of Proposition 5. To see this fact, let the center z of the interior ball B_ρ be moved by a distance $\rho/2$ in the x_1-direction away from T. Around this shifted point, a ball of radius $\rho/2-\delta$ is contained in D_δ^+.

2.3. Proof of Proposition 1. The different versions of the Hopf Lemma of the previous section are now applied to the comparison-function w_m in Σ and the point p where tangency or orthogonality happens. This will give the existence of a connected component $\widetilde{\Sigma}_\delta$ of Σ_δ for which (8) is satisfied.

Let us fix $\delta \in (0, \rho/16)$. In case of tangency, we know from Lemma 1 that the center of the interior ball at the tangency point p lies in the right-hand half space. Therefore Lemma 3 applies and gives

$$\|w_m\|_{L^\infty(\widetilde{\Sigma}_\delta)} \leq C(\operatorname{diam}\Omega, \|u\|_\infty, f) C_0^{|\Sigma|/\delta^N} \frac{|\frac{\partial w_m}{\partial \nu}(p)|}{\operatorname{dist}(p, T_m)}, \tag{22}$$

for a component $\widetilde{\Sigma}_\delta$ of Σ_δ. Note that the constants $C(\rho, \|c^-\|_\infty)$ in the Hopf Lemmas are now replaced by $C(\operatorname{diam}\Omega, \|u\|_\infty, f)$ because $\rho \leq \operatorname{diam}\Omega/2$ and $\|c^-\|_\infty$ is bounded by the Lipschitz-constant of f on $[0, \|u\|_\infty]$. Now we need a bound for $|\partial w_m(p)/\partial\nu|/\operatorname{dist}(p, T_m)$. If $|p-p'| = 2\operatorname{dist}(p, T_m) \geq \rho$ then such a bound is easily given by $\frac{4}{\rho}\|\partial u/\partial\nu - d\|_{C^1(\partial\Omega)}$. If $|p-p'| < \rho$, then the path $\gamma: t \in [0,1] \to tp' + (1-t)p$ has distance less then ρ to $\partial\Omega$. It is known, cf. Gilbarg, Trudinger [8] p.355, that the map

$$\Pi : \begin{cases} \Omega \setminus \Omega_\rho & \to \quad \partial\Omega \\ x & \mapsto \quad y \text{ s.t. } x - y = \nu(y)\operatorname{dist}(x, \partial\Omega) \end{cases}$$

is a $C^{1,\alpha}$-map with $\|\Pi\|_{C^{1,\alpha}} \leq C(\Omega)$. The path $\hat\gamma = \Pi \circ \gamma$ connects p to p' on $\partial\Omega$ and its length $l(\hat\gamma)$ is bounded by $C(\Omega)l(\gamma) = C(\Omega)2\operatorname{dist}(p, T_m)$. By the mean value theorem

$$\begin{aligned} |\frac{\partial w_m}{\partial \nu}(p)| &= |\frac{\partial u}{\partial \nu}(p') - \frac{\partial u}{\partial \nu}(p)| \\ &\leq l(\hat\gamma)\|\frac{\partial u}{\partial \nu} - d\|_{C^1(\partial\Omega)} \leq 2\operatorname{dist}(p, T_m)C(\Omega)\|\frac{\partial u}{\partial \nu} - d\|_{C^1(\partial\Omega)}. \end{aligned}$$

Hence (22) implies

$$\|w_m\|_{L^\infty(\widetilde{\Sigma}_\delta)} \leq C(\Omega, \operatorname{diam}\Omega, \|u\|_\infty, f) C_0^{|\Sigma|/\delta^N} \|\frac{\partial u}{\partial \nu} - d\|_{C^1(\partial\Omega)}, \tag{23}$$

which is the estimate of Proposition 1 in case of tangency, if one takes into account that $|\Sigma| \leq C_0 \operatorname{diam}\Omega^N$.

In case of orthogonality, we have $w_m(p) = 0$, $\partial w_m(p)/\partial\eta = 0$ because η is tangential to $\partial\Omega$ at p. Since $p \in T_m$, it is easy to see that also $\partial w_m(p)/\partial\eta^\perp = 0$ for any direction $\eta^\perp$ orthogonal to η. Hence $\nabla w_m(p) = 0$ and Proposition 3 applies and yields

$$\|w_m\|_{L^\infty(\widetilde{\Sigma}_\delta)} \leq \frac{C(\operatorname{diam}\Omega, \|u\|_\infty, f) C_0^{|\Sigma|/\delta^N}}{|(s\cdot\eta)(s\cdot\nu(p))|} |\frac{\partial^2 w_m}{\partial s^2}(p)| \tag{24}$$

for any inward direction s and a component $\widetilde{\Sigma}_\delta$ of Σ_δ. Let us take $s = \frac{1}{\sqrt{2}}(\eta - \nu(p))$. This is an inward direction to Σ at p with $|(s\cdot\eta)(s\cdot\nu(p))| = 1/2$. From the definition of w_m and the fact that $p \in T_m$ we conclude that $\frac{\partial^2 w_m}{\partial\nu^2}(p) = \frac{\partial^2 w_m}{\partial\eta^2}(p) = 0$. Therefore

$$\frac{\partial^2 w_m}{\partial s^2}(p) = -\frac{\partial^2 w_m}{\partial\eta\partial\nu}(p) = 2\frac{\partial^2 u}{\partial\eta\partial\nu}(p) = 2\frac{\partial}{\partial\eta}(\frac{\partial u}{\partial\nu})(p),$$

where the last equality follows from the fact that $u = 0$ on $\partial\Omega$. Hence $|\partial^2 w_m(p)/\partial s^2| \leq 2\|\frac{\partial u}{\partial\nu} - d\|_{C^1(\partial\Omega)}$ and (24) implies the same estimate (23) as in the case of tangency. This finishes the proof of Proposition 1. □

3. Approximate symmetry in direction η. We start this section by extending the estimate (8) of Proposition 1, which describes the smallness of w_m in the set $\widetilde{\Sigma}_\delta$, to a larger set Z_δ. As shown in Figure 3, the set Z_δ extends $\widetilde{\Sigma}_\delta$ up to the hyperplane T_m. This will eventually yield the existence of a symmetric set X_δ which nearly fills Ω in a controlled sense. The precise statement is given in Proposition 5 as the main result in this section.

The extension of the connected component $\widetilde{\Sigma}_\delta$ up to T_m is done in the following way: we denote by ω the trace of $\partial\widetilde{\Sigma}_\delta$ on the hyperplane $T_{m+\delta}$. In the relative-topology on $T_{m+\delta}$ the set ω decomposes into its relative boundary ω^{bd} and its relative interior ω^{i}. Since for each $x \in \widetilde{\Sigma}_\delta$, the straight-line $x - t\eta$ stays in $\widetilde{\Sigma}_\delta$ for all t with $0 \leq t < \operatorname{dist}(x, T_{m+\delta})$ and meets ω^{i} for $t = \operatorname{dist}(x, T_{m+\delta})$, we find the following way of writing the relative interior of ω

$$\omega^{\mathrm{i}} = \{x - \operatorname{dist}(x, T_{m+\delta})\eta \,|\, x \in \widetilde{\Sigma}_\delta\}.$$

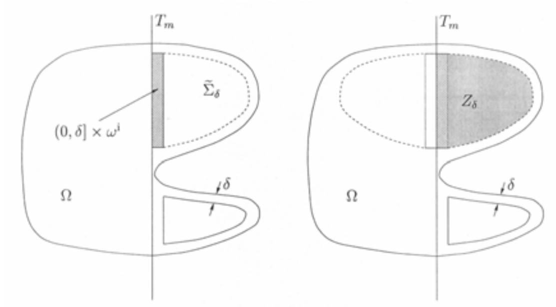

Figure 3: The construction of Z_δ and X_δ.

Let us identify ω as a subset of the orthogonal complement of η. Now we add a cylindrical piece with cross-section ω^{i} to $\widetilde{\Sigma}_\delta$ (see Figure 3)

$$Z_\delta = \widetilde{\Sigma}_\delta \ \cup\ (0,\delta]\eta \times \omega^{\mathrm{i}}.$$

The set Z_δ is an open and connected subset of Σ, which is convex in direction η. Its boundary consists of three parts: $\partial Z_\delta \cap T_m$, $(0,\delta]\eta\times\omega^{\mathrm{bd}}$ and $\partial\widetilde{\Sigma}_\delta \setminus T_{m+\delta}$. It is therefore not hard to see that

$$\mathrm{dist}(x,\partial\Omega) \le 2\delta \text{ for all } x \in \partial Z_\delta \setminus T_m, \tag{25}$$

which will be used in Proposition 5. Moreover, the estimate (8) extends in the following way to Z_δ

$$\begin{aligned}\|w_m\|_{L^\infty(Z_\delta)} \ &\le\ C(\Omega, \mathrm{diam}\,\Omega, \|u\|_\infty, f) C_0^{\mathrm{diam}\,\Omega^N/\delta^N} \|\tfrac{\partial u}{\partial \nu} - d\|_{C^1(\partial\Omega)} \\ &\quad +2\|u\|_{C^1(\Omega)}\delta.\end{aligned} \tag{26}$$

Finally, the symmetrized version of Z_δ is given by (see Figure 3)

$$X_\delta = \widetilde{\Sigma}_\delta \cup \widetilde{\Sigma}'_\delta \ \cup\ [-\delta,\delta]\eta \times \omega^{\mathrm{i}}$$

and it is not difficult to see that X_δ is an open connected subset of Ω.

We want to derive approximate symmetry from the smallness of w_m on the set Z_δ. Therefore we first prove a priori bounds on u from below and above by the distance-function to the boundary. Indeed this means that if $x \in \partial\Omega$ and $w_m(x)$ is small then $u(x') = w_m(x)$ is also small and x' must be close to the boundary of Ω. This will be used in Proposition 5 to show that Ω is almost equal to X_δ.

Proposition 4. *There exist positive constants K, L such that*

$$K\text{dist}(x, \partial\Omega) \le u(x) \le L\text{dist}(x, \partial\Omega) \text{ for all } x \in \Omega. \tag{27}$$

Here $1/K, L$ are bounded by a constant $C(\Omega, \text{diam}\,\Omega, \|u\|_\infty, f, d_0)$.

Proof. The Calderon-Zygmund and Schauder estimates imply $\|u\|_{C^{2,\alpha}(\Omega)} \le C(\Omega, \text{diam}\,\Omega, f)\|u\|_\infty$, and this provides the upper bound in (27). For the lower bound, we use the assumption $\partial u/\partial\nu(x) \le -d_0 < 0$ for every $x \in \partial\Omega$. Let x be in $\Omega \setminus \overline{\Omega}_\delta$ for $0 < \delta \le \rho$. For such x, there exists $\bar{x} \in \partial\Omega$ with $\bar{x} - x = \text{dist}(x, \partial\Omega)\nu(\bar{x})$, and a Taylor argument implies

$$\begin{aligned} u(x) &\ge u(\bar{x}) + \nabla u(\bar{x}) \cdot (x - \bar{x}) - \frac{1}{2}\|u\|_{C^2(\Omega)}|x - \bar{x}|^2 \\ &\ge \text{dist}(x, \partial\Omega)d_0 - \frac{1}{2}\|u\|_{C^2(\Omega)}\text{dist}(x, \partial\Omega)^2. \end{aligned}$$

Choosing $\delta = \min(\frac{d_0}{\|u\|_{C^2(\Omega)}}, \rho)$ we conclude that

$$\frac{u(x)}{\text{dist}(x, \partial\Omega)} \ge \frac{d_0}{2} > 0 \text{ in } \Omega \setminus \overline{\Omega}_\delta, \tag{28}$$

and in particular

$$u(x) \ge \frac{\delta d_0}{4} > 0 \quad \text{if} \quad \frac{\delta}{2} \le \text{dist}(x, \partial\Omega) \le \delta. \tag{29}$$

We derive from (2) the equation

$$\Delta u + c(x)u = -f(0) \le 0 \quad \text{for} \quad x \in \Omega, \tag{30}$$

where $c = (f(u) - f(0))/u$. For equation (30) we can apply the following generalized Harnack-principle, which is a combination of Theorem 8.17 and Theorem 8.18 in Gilbarg, Trudinger [8]. If γ is a path in Ω with length L and distance δ to $\partial\Omega$, then

$$\sup_\gamma u \le C_0^{L(1+\frac{1}{\delta})}(\inf_\gamma u + L\delta|f(0)|).$$

By Lemma A1 we have $L \le L_0|\Omega|\delta^{1-N}$ and hence

$$\frac{\delta d_0}{4} \le \sup_{\Omega_\delta} u \le C_0^{\text{diam}\,\Omega^N/\delta^N}(\inf_{\Omega_\delta} u + |f(0)|\delta^{2-N}). \tag{31}$$

Now we distinguish two cases.

Case 1: $0 \le f(0) \le \frac{d_0}{8}\delta^{N-1}C_0^{-\operatorname{diam}\Omega^N/\delta^N}$. Then it follows immediately from (31) that for all $x \in \Omega_\delta$

$$\frac{u(x)}{\operatorname{dist}(x,\partial\Omega)} \ge \frac{\inf_{\Omega_\delta} u}{\operatorname{diam}\Omega} \ge K_1 := \frac{d_0\delta C_0^{-\operatorname{diam}\Omega^N/\delta^N}}{8\operatorname{diam}\Omega}.$$

Case 2: $f(0) \ge \frac{d_0}{8}\delta^{N-1}C_0^{-\operatorname{diam}\Omega^N/\delta^N}$. For every $x \in \Omega_\delta$ the ball $B_\delta(x)$ lies in Ω. If l is the Lipschitz constant of f on $[0, \|u\|_\infty]$, then the solution of

$$\Delta v - lv = -f(0) \text{ in } B_\delta(x), \quad v = 0 \text{ on } \partial B_\delta(x), \tag{32}$$

is a subsolution to (30). Therefore we have the estimate $u(x) \ge v(0) > 0$ for all $x \in \overline{\Omega}_\delta$. Since $v(0) \ge C_1 f(0)\min(\delta^2, 1/l)$ we get that for all $x \in \Omega_\delta$

$$\frac{u(x)}{\operatorname{dist}(x,\partial\Omega)} \ge K_2 := \frac{C_1 d_0\delta^{N-1}C_0^{-\operatorname{diam}\Omega^N/\delta^N}}{8\operatorname{diam}\Omega}\min(\delta^2, 1/l).$$

Since in both cases $u(x) \ge \frac{d_0}{2}\operatorname{dist}(x,\partial\Omega)$ for all $x \in \Omega \setminus \overline{\Omega}_\delta$ by (28), we get the lower bound in (27) by setting $K = \min(K_1, K_2, d_0/2)$. Recalling the definition of δ we find that $1/\delta \le C(\Omega, \operatorname{diam}\Omega, \|u\|_\infty, f, d_0)$, and therefore we obtain the same estimate for $1/K$ as claimed. $\square$

The next proposition is the key result of our paper. It shows that the symmetric set X_δ, defined at the beginning of this section, fills Ω in a controlled sense.

Proposition 5. *There exist σ, $\delta > 0$ such that*

$$\Omega_\sigma \subset X_\delta \subset \Omega \tag{$*$}$$

and

$$\delta < \sigma < C(\Omega, \operatorname{diam}\Omega, \|u\|_\infty, f, d_0)\left|\log\left\|\frac{\partial u}{\partial\nu} - d\right\|_{C^1(\partial\Omega)}\right|^{-1/N} \tag{33}$$

provided $\|\frac{\partial u}{\partial\nu} - d\|_{C^1(\partial\Omega)}$ is less than a constant $\varepsilon(\Omega, \operatorname{diam}\Omega, \|u\|_\infty, f, d_0)$.

Before we give the proof, let us mention that Ω_σ is connected for $0 < \sigma < \rho/2$. To see this, take x, y in Ω_σ. The connectedness of Ω provides a path connecting x to y in Ω, which can be moved inwards into Ω_σ by the normal field on $\partial\Omega$.

Proof. For this proof, let C_0 and $C = C(\Omega, \operatorname{diam}\Omega, \|u\|_\infty, f)$ be the constants of Proposition 1, and let $K, L = K, L(\Omega, \operatorname{diam}\Omega, \|u\|_\infty, f, d_0)$ denote the constants of Proposition 4. As before, ρ is the maximal ball-radius for which Ω satisfies a uniform interior ball condition. We prove $(*)$ and (33) in two steps:

(i) First we show $\Omega_\sigma \subset X_\delta \subset \Omega$ provided $0 < \sigma, \delta < \rho/16$ and

$$K\sigma > CC_0^{\operatorname{diam}\Omega^N/\delta^N} \|\frac{\partial u}{\partial \nu} - d\|_{C^1(\partial\Omega)} + 2(\|u\|_{C^1(\Omega)} + L)\delta. \tag{34}$$

(ii) Then we choose σ, δ appropriately.

To (i): The second inclusion holds by definition. Since $\widetilde{\Sigma}_\delta$ contains a ball of radius $\rho/4 < \rho - \delta/2$ (see Remark 3), and since $\sigma < \rho/16$ it follows that Ω_σ intersects Z_δ. Let x be a point in $\Omega_\sigma \cap Z_\delta$. Suppose for contradiction that there exists $y \in \Omega_\sigma \setminus X_\delta$. Due to the connectedness of Ω_σ, there is a path γ in Ω_σ from x to y. We take z to be the first point (starting from x) where γ intersects ∂X_δ and find that either $z \in \partial X_\delta^+ = \partial X_\delta \cap \{x \cdot \eta \geq m\}$ or $z \in \partial X_\delta^- = \partial X_\delta \cap \{x \cdot \eta \leq m\}$. In the first case, $\operatorname{dist}(x, \partial\Omega) \leq 2\delta$ by (25), and $u(z) \leq 2L\delta$ follows from Proposition 4. In the second case, recall that $z \in \partial X_\delta^-$ implies that the corresponding z' lies in ∂X_δ^+ and hence $u(z) = w_m(z') + u(z') \leq CC_0^{\operatorname{diam}\Omega^N/\delta^N} \|\frac{\partial u}{\partial \nu} - d\|_{C^1(\partial\Omega)} + 2\|u\|_{C^1(\Omega)}\delta + 2L\delta$ by (26) and the estimate in the first case. In both cases, we have $u(z) \leq CC_0^{\operatorname{diam}\Omega^N/\delta^N} \|\frac{\partial u}{\partial \nu} - d\|_{C^1(\partial\Omega)} + 2(\|u\|_{C^1(\Omega)} + L)\delta$. Since also $z \in \Omega_\sigma$, we obtain $K\sigma \leq u(z)$ again from Proposition 4. Both bounds on $u(z)$ together contradict (34).

To (ii): Now we choose δ such that

$$CC_0^{\operatorname{diam}\Omega^N/\delta^N} \|\frac{\partial u}{\partial \nu} - d\|_{C^1(\partial\Omega)} = 2(\|u\|_{C^1(\Omega)} + L)\delta;$$

i.e.,

$$\delta = \psi\Big(\frac{C\|\frac{\partial u}{\partial \nu} - d\|_{C^1(\partial\Omega)}}{2\|u\|_{C^1(\Omega)} + 2L}\Big),$$

where ψ is the inverse of the function $t \to tC_0^{-\operatorname{diam}\Omega^N/t^N}$ for $t \geq 0$. One can show that $0 \leq \psi(t) \leq C_1|\log t|^{-1/N}$ for $0 \leq t \leq 1$, where $C_1 = C_1(\operatorname{diam}\Omega)$.

Furthermore, we set

$$\sigma = 5(\|u\|_{C^1(\Omega)} + L)\delta/K$$

and this choice of σ, δ satisfies (34) and therefore $(*)$ holds. Moreover, using the estimate for ψ on the interval $[0,1]$, there exists a constant $\epsilon = \epsilon(\Omega, \operatorname{diam}\Omega, \|u\|_\infty, f, d_0)$ such that $\|\frac{\partial u}{\partial \nu} - d\|_{C^1(\partial\Omega)} \leq \epsilon$ implies the estimate (33). $\square$

We will use Proposition 5 in the rest of the paper in the following form

Corollary 1. *For any point $x \in \partial\Omega$ there exists a point $y \in \partial\Omega$ such that $|x' - y| \leq 2\sigma$, where σ is given in Proposition 5 and x' is the reflection of x at T_m.*

Proof. Let x be in $\partial\Omega$. By $(*)$ in Proposition 5 there exists $t \in (0, \sigma]$ such that $z = x + t\nu(x) \in \partial X_\delta$. The symmetry property of X_δ implies that $z' \in \partial X_\delta$ and therefore $\operatorname{dist}(z', \partial\Omega) \leq \sigma$ again by $(*)$ of Proposition 5, i.e., there exists $y \in \partial\Omega$ with $|z' - y| \leq \sigma$. Hence we have

$$|x' - y| \leq |x' - z'| + |z' - y| = |x - z| + |z' - y| \leq 2\sigma,$$

which proves the Corollary.

4. Approximate radial symmetry. In this section, we are going to apply the previous results to the domain Ω, but with different directions η. This will yield approximate symmetry, first in the standard coordinate-directions $\mathbf{e}_1, \ldots, \mathbf{e}_N$ of $\mathbb{R}^N$, and then in any direction. Notice that the value σ in Proposition 5 does not depend on the direction η. Therefore we can choose η to be the standard coordinate-directions $\mathbf{e}_1, \ldots, \mathbf{e}_N$ and find that Corollary 1 holds with respect to reflections in each of the coordinate-directions $\mathbf{e}_1, \ldots, \mathbf{e}_N$. We also define T^i as the maximal hyperplane corresponding to $\mathbf{e}_i$. The next statement follows directly from Corollary 1.

Corollary 2. *Let O be the point of intersection of the N hyperplanes T^i and $s(x)$ denote the reflection of x in O. For any point $x \in \partial\Omega$ there exists a point $y \in \partial\Omega$ such that $|s(x) - y| \leq 2N\sigma$.*

Proof. The reflection in O can be written as the product of the reflections in each T^i:

$$s = s_{T^1} \circ \cdots \circ s_{T^N}.$$

Now we apply Corollary 1 N-times to the directions $\mathbf{e}_i$ and get the existence of y on $\partial\Omega$ such that

$$|y - s(x)| \leq 2N\sigma.$$

We will eventually prove that the point O is the center of two balls which are close to Ω.

4.1. O as an approximate symmetry center.

Proposition 6. *Let η be a direction and let T be the corresponding maximal hyperplane. Then*

$$\text{dist}(O, T) \leq 4N(1 + \text{diam}\,\Omega)\sigma, \tag{35}$$

where σ is given by Proposition 5.

Proof. Let $\tilde{\sigma} = 4N(1 + \text{diam}\,\Omega)\sigma$ and assume for contradiction that $\gamma = \text{dist}(O, T) > \tilde{\sigma}$. Let us write the hyperplane T as $\{x \mid (x - x_0) \cdot \eta = 0\}$ for some $x_0 \in T$, and denote by $H^+ = \{x \mid (x - x_0) \cdot \eta > 0\}$, $H^- = \{x \mid (x - x_0) \cdot \eta < 0\}$ the corresponding half-spaces. We suppose that $O \in H^-$ (if $O \in H^+$ then the proof requires only notational changes). Moreover we may assume that O is the origin of the coordinates, so that $\gamma = x_0 \cdot \eta$. First observe that Lemma 1 implies the existence of a point x_1 on $\partial\Omega \cap H^-$ with $\text{dist}(x_1, T) > \tilde{\sigma}$ provided $\tilde{\sigma}$ is less than ρ (which is always satisfied if $\|\partial u/\partial\nu - d\|_{C^1(\partial\Omega)}$ is small enough).

We define inductively a sequence x_k starting from x_1. Once x_{2k-1} is determined, we apply Corollary 1 and get a point x_{2k} on $\partial\Omega$ with $|x_{2k} - x'_{2k-1}| \leq 2\sigma$. Then we apply Corollary 2 which yields a point x_{2k+1} on $\partial\Omega$ with $|x_{2k+1} - s(x_{2k})| \leq 2N\sigma$. This sequence has the following property

$$|x_{2k+1}|^2 = |x_1|^2 + k(4\gamma\text{dist}(x_1, T) + 2\zeta_k) + 4k(k-1)\gamma(\gamma + \xi_k), \tag{36}$$

where $\xi_k, \zeta_k \in (-\tilde{\sigma}, \tilde{\sigma})$. We are going to prove (36) in several steps:

Step 1: *For any point $x \in H^- \cap \partial\Omega$ with* $\text{dist}(x, T) \geq \tilde{\sigma}$*, let $y \in \partial\Omega$ be a point obtained by Corollary* 1*, i.e., $|x' - y| \leq 2\sigma$. Then $y \in H^+$ and*

$$|y|^2 = |x|^2 + 4\gamma\text{dist}(x, T) + \zeta, \tag{37}$$

$$\text{dist}(y, T) = \text{dist}(x, T) + \xi, \tag{38}$$

where $|\xi|$, $|\zeta| \leq \tilde{\sigma}$.

Since $|x' - y| \leq 2\sigma$ we find $||x'|^2 - |y|^2| \leq 4\sigma \mathrm{diam}\,\Omega \leq \tilde{\sigma}$. This implies

$$\begin{aligned} |x'|^2 &= |x|^2 - 4((x - x_0) \cdot \eta)(x \cdot \eta) + 4((x - x_0) \cdot \eta)^2 \\ &= |x|^2 - 4((x - x_0) \cdot \eta)(x_0 \cdot \eta) = |x|^2 + 4\gamma \mathrm{dist}(x, T). \end{aligned}$$

Hence (37) is satisfied. Recall that $x \in H^-$ and $\mathrm{dist}(x, T) \geq \tilde{\sigma}$. Therefore

$$|(y - x_0) \cdot \eta - \mathrm{dist}(x, T)| = |(y - x_0) \cdot \eta + (x - x_0) \cdot \eta| = |(y - x') \cdot \eta| \leq \tilde{\sigma}.$$

This implies that $y \in H^+$ and that (38) is satisfied.

Step 2: *For any point $x \in H^+ \cap \partial\Omega$, let $y \in \partial\Omega$ be a point obtained by Corollary 2, i.e., $|s(x) - y| \leq 2N\sigma$. Then $y \in H^-$ and*

$$\mathrm{dist}(y, T) = \mathrm{dist}(x, T) + 2\gamma + \xi, \tag{39}$$

$$|y|^2 = |x|^2 + \zeta, \tag{40}$$

where $|\xi|$, $|\zeta| \leq \tilde{\sigma}$. In particular $\mathrm{dist}(y, T) \geq \tilde{\sigma}$.

Since $|y - s(x)| \leq 2N\sigma \leq \tilde{\sigma}$, a proof similar to Step 1 yields $||y|^2 - |s(x)|^2| \leq 4N\sigma \mathrm{diam}\Omega \leq \tilde{\sigma}$ which implies (40). Moreover, $-\tilde{\sigma} \leq (y - x_0) \cdot \eta + (x + x_0) \cdot \eta \leq \tilde{\sigma}$, and since $x \in H^+$ and $(x + x_0) \cdot \eta = \mathrm{dist}(x, T) + 2\gamma$ this means that

$$-\tilde{\sigma} - \mathrm{dist}(x, T) - 2\gamma \leq (y - x_0) \cdot \eta \leq \tilde{\sigma} - \mathrm{dist}(x, T) - 2\gamma. \tag{41}$$

If we recall that $\gamma > \tilde{\sigma}$ by assumption, then (41) implies that $y \in H^-$ and (39) is satisfied.

Step 3: *Conclusion.*

Now we apply the above properties to the sequence x_k. It is easy to see from Step 1 and 2, that $x_{2k-1} \in H^-$ and $x_{2k} \in H^+$ for all k. Then we claim that

$$\mathrm{dist}(x_{2k+1}, T) = \mathrm{dist}(x_1, T) + 2k\gamma + 2k\xi_k, \tag{42}$$

where $\xi_k \in (-\tilde{\sigma}, \tilde{\sigma})$. This can easily be proved by induction using (38) and (39). Then (36) follows by an induction argument from (42) and (37).

Now (36) yields that the sequence x_k is unbounded, because $|\xi_k| \leq \tilde{\sigma} < \gamma$ by assumption. Since Ω is bounded, we have obtained a contradiction.

4.2. Bounds on the radius of the approximating balls. We are now in a position to estimate how close to a ball Ω is. Theorem 1 follows immediately from the next Proposition and the estimate (33) on σ in Proposition 5.

Proposition 7. *We define r and R to be the maximal and minimal distance of O to the boundary of Ω. Then*

$$R - r \leq 8N(1 + \operatorname{diam}\Omega)\sigma. \tag{43}$$

Moreover, $O \in \Omega$ as soon as $\sigma < \rho/(4N(1 + \operatorname{diam}\Omega))$.

Proof. We assume that r and R are respectively attained at the points a, $b \in \partial\Omega$, i.e.,

$$r = |a - O| = \min_{x\in\partial\Omega} |x - O|, \qquad R = |b - O| = \max_{y\in\partial\Omega} |y - O|,$$

and we may assume $a \neq b$. This pair of boundary point allows us to define a direction η, namely $\eta = (b - a)/|b - a|$, to which we apply the results of the previous sections. We denote by T the maximal hyperplane corresponding to this direction η. Recall that we write T as $\{x \mid (x - x_0) \cdot \eta = 0\}$ with some point $x_0 \in T$, which we may choose such that $|O - x_0| = \operatorname{dist}(O, T)$. Next we show that $|a - x_0| \geq |b - x_0|$. So assume the contrary. First note that the definition of η yields $b \cdot \eta > a \cdot \eta$, which means that b lies to the right of a. Together with the assumption $|a - x_0| < |b - x_0|$ this implies that $b \in \{x \mid (x - x_0) \cdot \eta > 0\}$. Moreover, since T corresponds to the critical geometrical position (see beginning of section 3), internal tangency has not been reached before. It means that the reflection of Σ at T lies entirely in Ω so that

$$b - t\eta \in \Omega \text{ for all } t \in (0, 2\operatorname{dist}(b, T)). \tag{44}$$

Since $a \in \partial\Omega$ lies to the left of b, (44) implies

$$\operatorname{dist}(a, T) \geq \operatorname{dist}(b, T), \tag{45}$$

which is equivalent to $|a - x_0| \geq |b - x_0|$, and therefore contradicts the assumption.

Now it is easy to see that $|a - x_0| \geq |b - x_0|$ implies

$$r \geq R - 2|O - x_0|$$

and Proposition 6 gives the desired estimate (43).

Finally, to show that O lies in Ω, observe, that the point $a - t\nu(a)$ is a boundary point for some $t \geq 2\rho$ by the interior ball condition. Since the line $[O, a]$ is parallel to the normal $\nu(a)$ by the definition of a, we find that if $O \in \mathbb{R}^N \setminus \Omega$, then $R \geq |a - O| + 2\rho = r + 2\rho$ in contradiction to (43) and the assumption $\sigma < \rho/(4N(1 + \operatorname{diam}\Omega))$.

5. Remarks and applications.

5.1. Remarks on the hypotheses of Theorem 1. If $f(0) > 0$ then the hypothesis $|\partial u/\partial\nu| \geq d_0 > 0$ on $\partial\Omega$ is automatically satisfied and $d_0^{-1} \leq f(0)^{-1} C(\Omega, \|u\|_\infty, f)$. This can seen by the following comparison argument: recall that u satisfies (30). If B_ρ is an interior ball of radius ρ at $\bar{x} \in \partial\Omega$, then let v be the solution of (32) on B_ρ. Since v is a subsolution to (30) which vanishes at $\bar{x}$, we get $d_0 := |v'(\rho)| \leq |\frac{\partial u}{\partial\nu}(\bar{x})|$. This implies the estimate on d_0 since $|v'(\rho)| \geq C_1 f(0) \min(\rho, 1/\sqrt{l})$.

In the case $f(0) = 0$ the quality of our estimate depends on the upper bound for d_0^{-1}. This can be seen by taking the function ϕ_1/n, where (ϕ_1, λ_1) is the first eigenpair of the Laplacian on an arbitrary smooth bounded domain Ω. The function ϕ_1/n satisfies (2) for $f(t) = \lambda_1 t$. Although $\frac{\partial\phi_1}{n\partial\nu} \to 0$ for $n \to \infty$, we can not derive any information about the shape of the domain.

It is an open question, whether the conclusion of our Theorem remains true for $f(0) < 0$. In this case, u need not be uniformly bounded below by a multiple of the distance-function to the boundary, which is an essential tool in our proof. Consider the following example: let J_0 be the zero-order Bessel function of the first kind, normalized by $J_0(0) = 1$, and let $r_0 > 0$ be the location of the first negative minimum of J_0. Then the function $u(x) = J_0(|x|) - J_0(r_0)$ is positive on the ball $B_{r_0}(0)$. Furthermore, $u(x)$ satisfies $\Delta u + u + J_0(r_0) = 0$ on the ball $B_{r_0}(0)$ and on the boundary we have $u = 0$, $\partial u/\partial\nu = 0$.

However, for some cases where $f(0) < 0$ our main result does hold. The proof of Proposition 4, which is the only place where the sign of $f(0)$ matters, reveals that Case 1 works for $|f(0)| \leq d_0 \delta^{N-1} C_0^{-\operatorname{diam}\Omega^N/\delta^N}/8$, where $\delta = \min(\rho, d_0/\|u\|_{C^2(\Omega)})$.

5.2. Applications. We conclude by explaining the statement of Theorem 1 in the context of two applications.

The torsion problem. If a homogeneous, solid and straight beam of cross-section $\Omega \subset \mathbb{R}^2$ is put under torsion, then the resulting stress is proportional to $(\partial u/\partial x, -\partial u/\partial y)$, where u solves the boundary value problem $\Delta u = -1$ in Ω and $u = 0$ on $\partial\Omega$. This problem has already been studied by many authors, and here Theorem 1 states: if the tangential stress-component $\partial u/\partial\nu$ is almost constant then the cross-section of the beam is almost a disk.

Although our main result is only stated for the Laplace-operator, one can check that it remains valid for uniformly elliptic operators $\mathcal{L}$ of divergence form, i.e. $\mathcal{L}u = \text{div}\,(g(|\nabla u|)\nabla u)$, where the continuous functions $g(t), (g(t)t)' > 0$ are bounded and bounded away from zero on $[0, \infty)$. This allows us to treat the following example:

Surfaces of constant mean-curvature. Consider a surface $\mathcal{S} \subset \mathbb{R}^{N+1}$ of constant mean-curvature $H > 0$ which lies on one side of the hyperplane $\mathbb{R}^N \times \{0\}$. Suppose also that $\mathcal{S}$ is given as a graph $\{(x, u(x)) : x \in \Omega \subset \mathbb{R}^N\}$. Then u must satisfy

$$\frac{1}{N}\text{div}\,\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = -H \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{46}$$

Serrin's fundamental analysis [12] showed that classical solutions to (46) exist if

$$h = \min_{x\in\partial\Omega} H(x) - \frac{N}{N-1}H > 0, \tag{47}$$

where $H(x)$ is the mean curvature of $\partial\Omega$ at x. Under this condition, Serrin proved that $\max_{x\in\partial\Omega}|\nabla u(x)| \le C$, where C depends on an upper bound for $1/h$. Since the function $|\nabla u|$ attains its maximum on the boundary, an a priori gradient bound for ∇u follows, which makes the operator in (46) uniformly elliptic. Hence, for domains that satisfy (47), the generalized version of Theorem 1 applies and shows the following: if the contact angle $\gamma(x)$ at $x \in \partial\Omega$, given by $\cos\gamma(x) = \partial u/\partial\nu(1+|\nabla u|^2)^{-1/2}$, is sufficiently close to the constant value $d < 0$, then the domain Ω lies between two balls B_r, B_R with

$$R - r \le C(\Omega, \text{diam}\,\Omega, h, H)\Big|\log \|\cos\gamma - d\|_{C^1(\partial\Omega)}\Big|^{-1/N},$$

where C depends on upper bounds for $1/h$ and $1/H$. Moreover the surface $\mathcal{S}$ lies between two spherical caps of radius $1/H$, which intersect $\mathbb{R}^N \times \{0\}$ in B_r and B_R.

Appendix.

Lemma A1. *Let $D \subset \mathbb{R}^N$ be a bounded domain, and let $\delta > 0$. Then there exists a constant $L = L_0(N)|D|/\delta^{N-1}$ such that for each connected component $\widetilde{D}_\delta$ of D_δ and any two points $x,\ y \in \widetilde{D}_\delta$ there exists a piecewise linear path $\gamma : [0,1] \to D_{\delta/2}$ satisfying: $\gamma(0) = x$, $\gamma(1) = y$, and the length of γ is bounded by L.*

Proof. Since $\widetilde{D}_\delta$ is connected, there is a path $\gamma_1 : [0,1] \to \widetilde{D}_\delta$ such that $\gamma_1(0) = x$, $\gamma_1(1) = y$. We shall modify γ_1 in order to bound its length. The idea is based on the following intuitive image: a consecutive line of disjoint "pearls" of radius $\delta/2$ is threaded on the "necklace" γ_1. Due to the finite volume of D, only a finite number of pearls can be used and the length of the new path connecting the centers of the "pearls" can be estimated in terms of δ. More formally, we set $t_0 = 0$ and define inductively

$$t_{i+1} = \inf\{t > 0 \mid B_{\delta/2}(\gamma_1(s)) \cap (\cup_{0\le k\le i} B_{\delta/2}(\gamma_1(t_k))) = \emptyset \ \forall s \in [t,1]\ \}.$$

The balls $B_{\delta/2}(\gamma_1(t_i))$ are pairwise disjoint and included in $D_{\delta/2}$. Therefore this process has to stop at some index I_0 such that $I_0(\delta/2)^N \le C(N)|D|$. By the infimum definition of t_i and a continuity argument, one can easily see that the finite family of balls we have constructed has the property

$$\overline{B}_{\delta/2}(\gamma_1(t_i)) \cap (\cup_{0\le k\le i-1} \overline{B}_{\delta/2}(\gamma_1(t_k))) \ne \emptyset$$

for all i. So one can define $\sigma(i)$ to be the largest integer $j > i$ such that

$$\overline{B}_{\delta/2}(\gamma_1(t_i)) \cap \overline{B}_{\delta/2}(\gamma_1(t_j)) \ne \emptyset,$$

and take γ as the path defined by the segments

$$[x\ \gamma_1(t_{\sigma(0)})],\ [\gamma_1(t_{\sigma(0)})\ \gamma_1(t_{\sigma^2(0)})], \ldots,\ [\gamma_1(t_{\sigma^{q-1}(0)})\ \gamma_1(t_{\sigma^q(0)})],\ [\gamma_1(t_{\sigma^q(0)})\ y],$$

where q satisfies $\sigma^q(0) = I_0$. Since the balls $B_{\delta/2}(\gamma_1(t_i))$ are contained in $D_{\delta/2}$, this path lies in $D_{\delta/2}$, and has length $l(\gamma) \le (q+1)\delta \le (I_0+1)\delta \le C(N)|D|/\delta^{N-1}$.

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