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# EXISTENCE OF SOLUTIONS FOR A CLASS OF SEMILINEAR POLYHARMONIC EQUATIONS WITH CRITICAL EXPONENTIAL GROWTH

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**Abstract.** The author considers the semilinear elliptic equation

$$(-\Delta)^m u = g(x,u),$$

subject to Dirichlet boundary conditions $u = Du = \cdots = D^{m-1}u = 0$, on a bounded domain $\Omega \subset \mathbb{R}^{2m}$. The notion of nonlinearity of critical growth for this problem is introduced. It turns out that the critical growth rate is of exponential type and the problem is closely related to the Trudinger embedding and Moser type inequalities. The main result is the existence of non trivial weak solutions to the problem.

## 1. Introduction.

Consider the following problem

$$\begin{cases} -\Delta u = g(x,u) & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

In (P), $\Omega \subset \mathbb{R}^n$ indicates an open bounded set with sufficiently regular boundary, $n \geq 2$, $\Delta$ stands for the Laplace operator and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function which satisfies some regularity and growth conditions.

An important and widely studied example for problem (P) is the following 'model problem'

$$\begin{cases} -\Delta u = |u|^{p-1} u & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{M}$$

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where $p > 1$.

Suppose that $n \geq 3$. It is well known that if $1 < p < (n+2)/(n-2)$ then problem (M) admits infinitely many solutions which can be obtained as critical points of the energy functional

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} \, dx.$$

defined on the Sobolev space $H_0^1(\Omega)$. Indeed, $J$ is an even and continuously (Fréchet) differentiable functional and, in view of the Sobolev embedding $H_0^1(\Omega) \overset{\text{comp}}{\hookrightarrow} L^{p+1}(\Omega)$, satisfies the global Palais-Smale condition. This leads, via the symmetric Mountain Pass Lemma, to the existence of infinitely many solutions (see [16] for a detailed exposition).

If $p \geq (n+2)/(n-2)$, then problem (M), generally, does not admit any non trivial solution. (Notice that $H_0^1(\Omega) \overset{\text{comp}}{\hookrightarrow} L^{p+1}(\Omega)$ is false.) This fundamental result has been proved by Pohožaev [13]. The real number $p^\# = (n+2)/(n-2)$ is called the *critical exponent* for problem (M) and we say that the nonlinearity

$$|t|^{p^\# - 1} t = |t|^{4/(n-2)} t$$

has *critical growth* for problem (M).

Let us turn to problem (P) with $n \geq 3$. The associated energy functional is given by

$$J(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 - G(x,u) \, dx,$$

where $G(x,t) := \int_0^t g(x,s)\, ds$. We say that the nonlinearity $g$ has *critical growth* (briefly $g$ is critical) if it satisfies

$$0 < \liminf_{t \to \infty} \frac{\inf_{x \in \Omega} |g(x,t)|}{|t|^{p^\#}} \leq \limsup_{t \to \infty} \frac{\sup_{x \in \Omega} |g(x,t)|}{|t|^{p^\#}} < \infty. \qquad (1)$$

We say also that $g$ has *subcritical growth* if it satisfies

$$\limsup_{t \to \infty} \frac{\sup_{x \in \Omega} |g(x,t)|}{|t|^{p^\#}} = 0. \qquad (2)$$

The motivation for the above definition is due to the fact that the energy functional (with some additional assumptions on $g$) does satisfies the global

Palais-Smale condition in the subcritical case but it does not in the critical case. In the latter case, existence and non existence of solutions relies on the 'lower order terms' that appear in the nonlinearity. See for example [4], for a treatment of this question.

Suppose now that $n = 2$. It is well known that $H_0^1(\Omega) \overset{\text{comp}}{\hookrightarrow} L^{p+1}(\Omega)$, hence (M) admits infinitely many solutions, for every $p > 1$. By analogy with the case $n \geq 3$ we may ask: "Are there any critical growth functions $g$ (in some appropriate sense) for problem (P) in the case $n = 2$?"

The answer is positive and is closely related to the Trudinger-Moser inequality (see [1, 12, 17]). This inequality is the $n = 2$ analog of the Sobolev inequality for $n \geq 3$. Recent contributions in this direction have been obtained by Adimurthi [3], de Figueiredo and Ruf [5], McLeod and McLeod [11]. The "criticality" of the critical growth functions is seen by the following fact: among critical growth functions there are some for which a non trivial solution for problem (P) exists and others for which there are no such solutions.

One may ask what happens if polyharmonic operators are considered instead of Laplace operator. For instance, consider the following problem

$$\begin{cases} (-\Delta)^m u = g(x,u) & \text{in } \Omega \subset \mathbb{R}^n, \\ u = Du = \cdots = D^{m-1}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{D$_m$}$$

Here $\Omega$ is a bounded open subset of $\mathbb{R}^n$ regular enough (say of class $\mathcal{C}^{m-1,0}$), $n \geq 2$, $m \geq 1$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$.

Many authors have studied problem (D$_m$), specially in the case of the biharmonic operator $\Delta^2$. See for example, in [7, 8, 9, 14], where the authors treat problem (D$_m$), in the case $2m < n$. In [14] the authors discovered unexpected features of polyharmonic operators which show that the behavior of these operators cannot be foreseen completely from the behavior of the Laplacian. It is worthy of notice that generally, polyharmonic operators $(-\Delta)^m$ with Dirichlet boundary conditions do not satisfy the maximum principle if $m \geq 2$ e.g., if $E$ is an ellipse with a sufficiently large eccentricity there exists a changing sign function $u$ for which $\Delta^2 u > 0$ on $E$ and $u = Du = 0$ on $\partial E$ (see [8]).

Let $2m < n$. If we consider $|u|^{p-1}u$ as the second member of the first equation in (D$_m$), the critical exponent is seen to be the number $p^\# = (n+2m)/(n-2m)$. In other words, if we introduce an adequate notion of critical growth function (like (1)), then there are functions of critical growth for which a non trivial solution exists and others for which there exist no non trivial solutions for problem ($\mathrm{D}_m$) (see [14]).

Consider now $2m = n$. From the Trudinger embedding theorem (see Theorem 2), it is natural to expect that the critical growth functions are, roughly speaking, the nonlinearities $g(x,t)$ that behave like $\exp(t^2)$ at infinity. For the case of the Laplace operator (that is $m = 1$ and $n = 2$), the answer has been given in [5] as we mentioned previously.

In this paper, we provide an existence result for problem ($\mathrm{D}_m$), with $2m = n$, and $g$ of critical growth. It is still an open problem to find an example of a smooth critical growth function $g$ for which there exist no non trivial solution in the case $m \geq 2$.

The paper is organized as follows:

In Section 2, we introduce the problem and give the definition of critical growth functions. We state the main result (Theorem 1) which gives the existence of one non trivial weak solution for problem ($\mathrm{D}_m$) where $g$ has critical growth and satisfies some additional conditions related to its behavior near 0 and near $\infty$. We sketch also the leading idea for the proof of Theorem 1.

In Section 3 we give the lemmata needed for the proof of Theorem 1. This section has been divided into four logical subsections: subsection 3.1 states some known results (Trudinger embedding theorem, Moser-Adams inequality and a result of P.-L. Lions); in subsection 3.2 we describe some properties of functionals and sets that are introduced in Section 2; in subsection 3.3 we introduce a special class of functions called the *Adams functions* and use such functions to prove a useful estimate (Lemma 6); the last subsection 3.4 consists of a compactness result.

Finally, in Section 4 we give the proof of Theorem 1.

## 2. Position of the problem.

**2.1. The main result.** Consider problem ($\mathrm{D}_m$), with $2m = n$. We say that a function $g \in \mathcal{C}^1(\bar{\Omega} \times \mathbb{R})$ has *critical growth* for problem ($\mathrm{D}_m$) if there exist $K_1 > 0$, $\gamma \in [0,1)$, and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ such that, for any $t > 0$, $x \in \Omega$:

$$g(x,0) = 0, \quad g(x,t) > 0, \quad g(x,-t) = -g(x,t); \tag{$\mathrm{G}_1$}$$

$$g'(x,t) := \frac{\partial g}{\partial t}(x,t) > \frac{g(x,t)}{t}; \tag{$\mathrm{G}_2$}$$

$$G(x,t) \leq K_1(1 + g(x,t)t^\gamma), \tag{$\mathrm{G}_3$}$$

here $G$ stands for the *primitive* of $g$ given by

$$G(x,t) := \int_0^t g(x,\tau)\,d\tau;$$

and there exists $b > 0$, such that, for all $\varepsilon > 0$

$$\lim_{t\to\infty} \sup_{x\in\bar\Omega} g(x,t)\exp\big((-b-\varepsilon)t^2\big) = 0, \tag{G$_4$}$$

$$\lim_{t\to\infty} \inf_{x\in\bar\Omega} g(x,t)\exp\big((-b+\varepsilon)t^2\big) = \infty. \tag{G$_5$}$$

We shall make use of the function $h : \Omega \times \mathbb{R} \to \mathbb{R}$, given by

$$g(x,t) = h(x,t)\exp(bt^2).$$

Some examples of functions with critical growth for $(\mathrm{D}_m)$ are given by:

$$\operatorname{sgn}(t)\big(\exp(bt^2)-1\big);\ b>0,$$

$$h(x,t)\exp(bt^2);\ b>0,\ h(x,t) = \sum_{r=1}^{d} c_r(x)\,|t|^{r-1}\,t,\ c_r \in \mathcal{C}_+(\bar\Omega),\ d\ge 1;$$

$$\operatorname{sgn}(t)(\exp(a\,|t|^{\alpha})-1)\exp(bt^2);\ 0<a,\ 0<\alpha<2.$$

Recall that a function $u \in H_0^m(\Omega)$ is a *weak solution* of problem $(\mathrm{D}_m)$ if

$$g(x,u) \in H^{-m}(\Omega) \quad \text{and} \quad (\nabla^m u, \nabla^m v) = \langle g(x,u), v\rangle,\ \text{for all } v \in H_0^m(\Omega), \tag{3}$$

where $(u,v)$ is the usual $L^2(\Omega)$ inner product, $\langle\cdot,\cdot\rangle$ is the canonical map for the duality $(H^{-m}(\Omega); H_0^m(\Omega))$ and

$$\nabla^m u = \begin{cases} \operatorname{grad}\Delta^{(m-1)/2}u & \text{if } m \text{ is odd,}\\ \Delta^{m/2}u & \text{if } m \text{ is even.}\end{cases}$$

Observe that, in $H_0^m(\Omega)$, the semi norm

$$\|u\| := \sqrt{(\nabla^m u, \nabla^m u)}$$

is a norm that agrees with the usual norm $\sqrt{\sum_{|\alpha|=m} C_\alpha (D^\alpha u, D^\alpha u)}$, where $C_\alpha := m!/\alpha!$

Introduce the following notations:

$$\begin{aligned} &\beta_0 := m!(4\pi)^m, && \Lambda := \inf\left\{\|u\|^2 :\ u \in H_0^m(\Omega),\ \|u\|_2^2 = 1\right\}, \\ &h_0(t) := \inf_{x\in\bar\Omega} h(x,t), && H_0 := \sup_{t\geq 0} h_0(t)t, \\ &R_0 := \sup_{x\in\Omega} \operatorname{dist}(x,\partial\Omega), && \kappa_0 := \begin{cases} \frac{n\beta_0}{\sigma_n R_0^n H_0}, & \text{if } H_0 < \infty, \\ 0, & \text{if } H_0 = \infty. \end{cases} \end{aligned}$$

We shall refer to $\beta_0$ as the Moser-Adams constant (see Theorem 3).

**Theorem 1.** *Suppose that*

$$g(x,t) = h(x,t)\exp(bt^2)$$

*is a function with critical growth for problem* ($D_m$). *If* $g$ *satisfies*

$$\sup_{x\in\Omega} g'(x,0) < \Lambda, \tag{H$_1$}$$

*and*

$$\kappa_0 < b. \tag{H$_2$}$$

*then there exists a nontrivial weak solution for problem* ($D_m$).

Observe that the assumption ($H_2$) holds if $H_0 = \infty$. In the case $H_0 < \infty$, instead this condition means that $H_0 > cb^{-1}$, where $c$ is a constant which depends only on $\Omega$ and $b$ and tends to infinity as diam$\Omega$ tends to zero. Note that in the example of non existence, given in [5], assumption ($H_2$) does not hold.

**2.2. The variational method.** Throughout the rest of the paper, we shall suppose that $2m = n$ and that $g(x,t) = h(x,t)\exp(bt^2)$ is a critical growth function for problem ($D_m$).

Put

$$\begin{aligned} &J(u) := \frac{1}{2}\|u\|^2 - \int_\Omega G(x,u)\,dx, \\ &F(u) := \|u\|^2 - \int_\Omega g(x,u)u\,dx, && I(u) := J(u) - \frac{1}{2}F(u), \\ &S := \left\{u \in H_0^m(\Omega) :\ u \neq 0,\ F(u) = 0\right\}, && s := \sqrt{2\inf_{u\in S} J(u)}. \end{aligned}$$

Using the Trudinger embedding theorem (see 3.1), one can show that for every $\mu \geq 0$, the Nemickiĭ operator

$$u \in H_0^m(\Omega) \to g(x,u)\,|u|^\mu \in L^2(\Omega)$$

is continuous, that $F$ and $I$ are continuous functionals on $H_0^m(\Omega)$, and that $J$ is a continuously Fréchet-differentiable functional whose derivative is given by

$$\langle J'(u), v\rangle = \int_\Omega \nabla^m u \nabla^m v - g(x,u)v\,dx. \tag{4}$$

Problem $(\mathrm{D}_m)$ has a variational structure; indeed (3) is the Euler-Lagrange equation of the functional

$$J(u) := \int_\Omega \frac{1}{2}\,|\nabla^m u|^2 - G(x,u)\,dx \tag{5}$$

which is well defined on the space $H_0^m(\Omega)$. Suppose that $u \in H_0^m(\Omega)$ is a weak solution of problem $(\mathrm{D}_m)$, by letting $v = u$ in (3) one obtains

$$F(u) := \|u\|^2 - \int_\Omega g(x,u)u\,dx = 0. \tag{6}$$

Thus any non trivial solution of problem $(\mathrm{D}_m)$ belongs to the constraint

$$S := \{v \in H_0^m(\Omega) \setminus \{0\} : \; F(v) = 0\}.$$

The method we shall adopt in solving the problem consists first in the minimization of the functional $J$ on the constraint $S$, that is to find $u^* \in S$ such that

$$\inf_{v\in S} J(v) = \min_{v \in S} J(v) = J(u^*).$$

We show, then, that $u^*$ is actually a critical point for $J$.

The main difficulty in the variational approach to the critical growth problem is the lack of compactness, precisely the global Palais-Smale condition does not hold. Eventually some partial Palais-Smale condition still holds under a given level. Since the use of the Palais-Smale condition is not explicit in the proof of Theorem 1, we did not mention it there. Nevertheless, for the sake of completeness, we have included in Section 3.4 a discussion about this condition and its important relationship with the Moser-Adams constant $\beta_0$

(see Theorem 4). The reader which is interested only in skimming this paper can omit the technicalities of the proof of Theorem 4.

## 3. Preliminary results.

**3.1. Trudinger embedding and Moser-Adams inequality.** The following result known as the Trudinger embedding theorem; actually this is a more general version of Trudinger's.

**Theorem 2. (Pohožaev-Trudinger-Strichartz).** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Let $1 < p < \infty$ and $m \in \mathbb{N}^+$ be such that $mp = n$ and $\Theta(t) = \exp(t^{n/(n-m)}) - 1$. Then*

$$W_0^{m,p}(\Omega) \hookrightarrow E_\Phi(\Omega), \tag{7}$$

*for every $N$-function $\Phi$ which is dominated by $\Theta$ at infinity.*

($E_\Phi(\Omega)$ is a particular closed subset of the wider Orlicz space $L_\Phi(\Omega)$ relative to $\Phi$ and $\Omega$, see [2] for the details.) A proof of the previous theorem can be found in several works, we mention [1, 2, 6, 15, 17].

Trudinger embedding theorem relies on an estimate known as Trudinger inequality. Moser [12] gave a sharp form of this estimate in the case $m = 1$, $n > 1$. Adams [1] extended Moser inequality for the general case $0 < m < n$.

**Theorem 3. (Moser-Adams inequality, [1]).** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $n, m \in \mathbb{N}$ such that $0 < m < n$. Let $p = n/m$ and*

$$\beta_0 = \beta_0(m,n) := \begin{cases} \frac{n}{\sigma_n}\Big[\frac{\pi^{n/2}2^m\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n-m+1}{2}\right)}\Big]^{\frac{n}{n-m}}, & \text{if } m \text{ is odd },\\ \frac{n}{\sigma_n}\Big[\frac{\pi^{n/2}2^m\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)}\Big]^{\frac{n}{n-m}}, & \text{if } m \text{ is even.}\end{cases}$$

*Then there exists a constant $c_0 = c_0(m,n) > 0$ such that:*

(a) *If $\beta \le \beta_0$ and $u \in \mathcal{C}_c^m(\Omega)$ with $\|\nabla^m u\|_p \le 1$,*

$$\int_\Omega \exp\left(\beta\,|u(x)|^{p'}\right) dx \le c_0\,|\Omega|\,. \tag{8}$$

(b) *If $\beta > \beta_0$, then for all $C > 0$ there exists a function $u = u_C \in \mathcal{C}_c^m(\Omega)$ with $\|\nabla^m u\|_p \le 1$ and such that*

$$\int_\Omega \exp\left(\beta\,|u(x)|^{p'}\right) dx > C.$$

The proof of the second part of the previous result, in [1], is based on an asymptotic estimate of the $(m,p)$-*conductor capacity*. In the sequel we shall be concerned with this estimate in the construction of what will be referred to as *Adams functions*.

To conclude this paragraph, we mention the following

**Lemma 1.** *Let $u \in H_0^m(\Omega) \setminus \{0\}$ and $\{u_k\}$ a sequence in $H_0^m(\Omega)$ such that*

$$\|u_k\| = 1, \text{ for all } k \geq 1,$$
$$u_k \rightharpoonup u, \quad u_k \to u \text{ a.e. on } \Omega.$$

*Then for every $p < \bar{p} := (1 - \|u\|^2)^{-1}$, the sequence $\{\exp(\beta_0 {u_k}^2)\}$ is bounded in $L^p(\Omega)$.*

This lemma is a straightforward generalization of P.-L. Lions result [10] which deals with the case $m = 1$ and $n = 2$. Note that Trudinger embedding $H_0^m(\Omega) \hookrightarrow E_\Theta(\Omega)$, where $\Theta(t) = \exp(t^2) - 1$, is not compact (see [10] for example). Lemma 1 states that Moser-Adams constant, $\beta_0$, can be improved for sequences that keep 'far enough' from zero in the weak topology of $H_0^m(\Omega)$.

**3.2. The functionals and the constraint.** It is easy to check that if $g$ is a function with critical growth and $p \geq 1$, then $|g|^{p-1} g$ is also a function with critical growth. In the sequel we shall make a wide use of the following fact:

Let $g(x,t) = h(x,t)\exp(bt^2)$ be a function with critical growth. Then, for every $\varepsilon > 0$,there exists $C_0(\varepsilon, b) > 0$ such that

$$|g(x,t)| \leq C_0(\varepsilon, b) \exp\left((b+\varepsilon)\,|t|^2\right), \text{ for } t \in \mathbb{R}. \tag{9}$$

This is due to $(\mathrm{G}_4)$ and the continuity of $h$.

**Lemma 2.** *Suppose that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function of critical growth. The following holds:*

(i) *If $u \in H_0^m(\Omega)$, then $g(x,u) \in L^r(\Omega)$ for all $r \in [1, \infty)$.*

(ii) $\beta_0 / b = \sup\left\{c^2 : \sup_{\|w\|=1} \int_\Omega g(x, cw) w \, dx < \infty\right\}$.

(iii) *If $\{u_k\}$, $\{v_k\} \subset H_0^m(\Omega)$ and*

$$u_k \rightharpoonup u, \quad v_k \rightharpoonup v,$$
$$u_k \to u, \quad v_k \to v, \text{ a.e. on } \Omega,$$
$$\limsup_{k\to\infty} \|u_k\|^2 < \beta_0 / b,$$

*then, for every integer* $\ell \geq 1$,

$$\lim_{k\to\infty} \int_\Omega \frac{g(x,u_k)}{u_k} v_k^\ell \, dx = \int_\Omega \frac{g(x,u)}{u} v^\ell \, dx.$$

(iv) *If* $u \in H_0^m(\Omega)$, $\{u_k\} \subset H_0^m(\Omega)$ *and*

$$u_k \rightharpoonup u,$$
$$u_k \to u \text{ a.e. on } \Omega,$$
$$\sup_k \int_\Omega g(x,u_k) u_k < \infty,$$

*then, for all* $\tau \in [0,1)$,

$$\lim_{k\to\infty} \int_\Omega g(x,|u_k|)\,|u_k|^\tau \; dx = \int_\Omega g(x,|u|)\,|u|^\tau \; dx, \tag{10}$$
$$\lim_{k\to\infty} \int_\Omega G(x,u_k)\, dx = \int_\Omega G(x,u)\, dx. \tag{11}$$

(v) *For all* $u \in H_0^m(\Omega) \setminus 0$, $I(u) > 0$, $I(0) = 0$*; moreover there exists a constant* $M_1 > 0$*, independent of* $u$*, such that*

$$\int_\Omega g(x,u) u \, dx \leq M_1(1 + I(u)). \tag{12}$$

**Proof.** Let $\varepsilon_0 > 0$. From (9) we get

$$|g(x,t)| \leq C_0(\varepsilon_0, b) \exp((b+\varepsilon_0)t^2), \text{ for all } (x,t) \in \Omega \times \mathbb{R}.$$

From Theorem 2 it follows that, for all $r > 0$, we have

$$\int_\Omega |g(x,u)|^r \; dx \leq [C_0(\varepsilon_0, b)]^r \int_\Omega \exp\left(r(b+\varepsilon_0)u^2\right) dx < \infty,$$

Thus (i) is proved.

From (9) it follows that, for all $\tilde{\varepsilon} > 0$,

$$\left|h(x,t)\exp(bt^2)t\right| \leq C_0(\tilde{\varepsilon}, b) \exp((b+\tilde{\varepsilon})t^2) \tilde{C}(\tilde{\varepsilon}) \exp(\tilde{\varepsilon}t^2),$$

where $\tilde{C}(\tilde{\varepsilon}) = (2\tilde{\varepsilon}e)^{-1/2}$.

Given $\varepsilon > 0$, put $\tilde{\varepsilon} := b\varepsilon/2$ and $C_1(\varepsilon) = \tilde{C}(\tilde{\varepsilon})C_0(\tilde{\varepsilon}, b)$. With these notations we obtain

$$g(x,t)t = |g(x,t)t| \leq C_1(\varepsilon)\exp(b(1+\varepsilon)t^2), \text{ for all } t \in \mathbb{R}. \tag{13}$$

On the other hand, for every $\varepsilon > 0$, there exists $C_2(\varepsilon) > 0$ such that

$$C_2(\varepsilon)\exp(b(1-\varepsilon)t^2) \leq g(x,t)t, \text{ for } |t| \geq 1. \tag{14}$$

Indeed, $(\mathrm{G}_5)$ implies that, for $\varepsilon > 0$ fixed, there exists $t_\varepsilon > 0$ such that

$$h(x,t)\exp((\varepsilon/b)t^2) \geq \inf h(x,t)\exp((\varepsilon/b)t^2) \geq 1, \text{ for all } t > t_\varepsilon.$$

Put

$$C_2(\varepsilon) := \begin{cases} \min\left(1, \inf_{\substack{1\leq t\leq t_\varepsilon \\ x\in\bar{\Omega}}} h(x,t)\exp((\varepsilon/b)t^2)\right) > 0 & \text{if } t_\varepsilon > 1, \\ 1 & \text{if } t_\varepsilon \leq 1, \end{cases}$$

in view of $(\mathrm{G}_1)$, we obtain

$$g(x,t)t = g(x,|t|)\,|t| \geq g(x,|t|) = h(x,|t|)\exp(bt^2) \geq C_2(\varepsilon)\exp(b(1-\varepsilon)t^2),$$

for $|t| \geq 1$.

Suppose now that $c > 0$ is such that

$$\sup_{\|w\|\leq 1} \int_\Omega g(x,cw)w\,dx < \infty.$$

From (14) it follows that for every $\varepsilon > 0$,

$$C_2(\varepsilon) \sup_{\|w\|\leq 1} \int_\Omega \exp(b(1-\varepsilon)c^2w^2) < \infty.$$

In view of Theorem 3 we get

$$b(1-\varepsilon)c^2 \leq \beta_0, \text{ for all } \varepsilon > 0.$$

That is $c^2 \leq \beta_0/b$.

Conversely, suppose that $c^2 < \beta_0/b$. Let $0 < \varepsilon_0 < \beta_0/bc^2 - 1$, then (13) and Theorem 3 imply that

$$\begin{aligned}\sup_{\|w\|\le 1}\int_\Omega g(x,cw)w\,dx &\le \frac{C_1(\varepsilon_0)}{c}\sup_{\|w\|\le 1}\int_\Omega \exp(b(1+\varepsilon_0)c^2w^2)\\ &\le \frac{C_1(\varepsilon_0)}{c}\sup_{\|w\|\le 1}\int_\Omega \exp(\beta_0 w^2) < \infty.\end{aligned}$$

This proves (ii).

In order to prove (iii), observe first that if $w$, $z \in H_0^m(\Omega)$ then

$$(g(x,w)/w)z^\ell \in L^1(\Omega).$$

This follows from ($G_2$), the Sobolev embedding $H_0^m(\Omega) \hookrightarrow L^{\ell p'}(\Omega)$ and Hölder inequality. Now, from the assumptions we have $\limsup_{k\to\infty}\|u_k\|^2 < \beta_0/b$. Hence there exist $\zeta > 0$, $\varepsilon > 0$ and $p > 1$ such that

$$\limsup_{k\to\infty}\|u_k\|^2 < \zeta^2 < \beta_0/(bp+\varepsilon) < \beta_0/b.$$

It follows that, for some sufficiently large $k_0 \in \mathbb{N}$,

$$\|u_k\| < \zeta, \text{ for all } k \ge k_0.$$

The function $|g|^p$ has critical growth for ($D_m$). Thus there exists $C_0(\varepsilon, bp) > 0$ such that

$$|g(x,t)|^p = |g^p(x,|t|)| \le C_0(\varepsilon,bp)\exp((bp+\varepsilon)t^2).$$

Theorem 3 implies that

$$\sup_{k\ge k_0}\int_\Omega |g(x,u_k)|^p\,dx \le C_0(\varepsilon,bp)\sup_{\|w\|\le\zeta}\int_\Omega \exp\big((bp+\varepsilon)\zeta^2\big(\frac{w}{\zeta}\big)^2\big)\,dx < \infty.$$

Put $c_1^p := \sup_{k\ge 1}\int_\Omega |g(x,u_k)|^p\ dx$.

On the other hand since $v_k \rightharpoonup v$ in $H_0^m(\Omega)$, by Rellich-Kondrašov we have

$$v_k^\ell \to v^\ell, \text{ in } L^{p'}(\Omega) \text{ for all } \ell \in \mathbb{N}.$$

(Here $p' = p/(p-1)$.) Put $c_2^{p'} := \sup_{k\geq 1} \left\| v_k^\ell \right\|_{p'}^{p'}$.

Fixing $N > 0$, in view of Hölder inequality, we get

$$\int_{|u_k|>N} \frac{g(x,u_k)}{u_k} v_k^\ell \, dx \leq \frac{c_1 c_2}{N}, \text{ for all } k \geq 1.$$

Hence

$$\int_\Omega \frac{g(x,u_k)}{u_k} v_k^\ell \, dx = \int_\Omega \varphi_{N,k} \, dx + O(1/N), \text{ for all } k \geq 1,$$

where

$$\varphi_{N,k}(x) = \big(g(x,u_k(x))/u_k(x)\big) v_k(x)^\ell \mathbf{1}_{|u_k|\leq N}(x).$$

($\mathbf{1}_A$ indicates the characteristic function of the set $A$.)

Observe that $\varphi_{N,k} \to \varphi_N$, almost everywhere on $\Omega$, where

$$\varphi_N(x) = \frac{g(x,u(x))}{u(x)} v(x)^\ell \mathbf{1}_{|u|\leq N}.$$

From dominated convergence first ($k \to \infty$) and monotone convergence second ($N \to \infty$), it follows that

$$\lim_{k\to\infty} \int_\Omega \frac{g(x,u_k)}{u_k} v_k^\ell \, dx = \int_\Omega \frac{g(x,u)}{u} v^\ell \, dx.$$

We prove now (iv). Let $K := \sup_k \int_\Omega g(x,u_k) u_k \, dx$. Using the same arguments as the one used in the proof of (iii), we can cut off at an arbitrary level $N > 0$:

$$\int_{|u_k|>N} g(x,|u_k|) \left|u_k\right|^\tau \, dx \leq \frac{1}{N^{1-\tau}} \int_\Omega g(x,|u_k|) \left|u_k\right| \, dx \leq \frac{K}{N^{1-\tau}}.$$

We get

$$\int_\Omega g(x,|u_k|) \left|u_k\right|^\tau \, dx = \int_\Omega \varphi_{N,k}(x) \, dx + O(1/N),$$

where $\varphi_{N,k}$ is given by

$$\varphi_{N,k} := g(x,|u_k|) \left|u_k\right|^\tau \, \mathbf{1}_{|u_k|\leq N}.$$

We can then conclude as above, that is that

$$\lim_{k\to\infty}\int_\Omega g(x,|u_k|)\,|u_k|^\tau\;dx=\int_\Omega g(x,|u|)\,|u|^\tau\;dx,$$

as long as $g(x,|u|)\,|u|^\tau$ is integrable on $\Omega$. The latter is easily shown to be satisfied, indeed from Fatou Lemma ($\mathrm{G}_1$) we obtain

$$\begin{aligned}\int_\Omega g(x,|u|)\,|u|^\tau &\le \int_{|u|\ge 1}+\int_{|u|\le 1} g(x,|u|)\,|u^\tau|\;dx\\ &\le \int_\Omega g(x,|u|)\,|u| + \int_\Omega g(x,|u|)\,dx\\ &\le \liminf_k \int_\Omega g(x,|u_k|)\,|u_k|\;dx + \|g(x,|u|)\|_1\\ &< \infty.\end{aligned}$$

This proves (10). In order to complete the proof of (iv), it is enough to observe that ($\mathrm{G}_3$), (10) and the dominated convergence imply that

$$\lim_{k\to\infty}\int_\Omega G(x,u_k)\,dx=\int_\Omega G(x,u)\,dx.$$

That is (11).

Finally, let us show (v). Let $\rho:\Omega\times\mathbb{R}\to\mathbb{R}$ be defined as

$$\rho(x,t):=\frac{1}{2}g(x,t)t-G(x,t).$$

Then

$$I(u)=\int_\Omega \rho(x,u)\,dx.$$

From of ($\mathrm{G}_2$) we have that for all $t>0$,

$$\frac{\partial\rho}{\partial t}(x,t)=\frac{1}{2}\big[g'(x,t)t-g(x,t)\big]>0.$$

Since $g(x,0)=0$, $g(x,t)t$ and $G(x,t)$ are even functions with respect to $t$ (for all $x\in\Omega$), we get

$$\rho(x,t)>0,\ \text{if}\ |t|>0 \text{ and } \rho(x,0)=0,\ \text{for all } x\in\Omega.$$

Henceforth $I$ is positive on $H_0^m(\Omega)\setminus\{0\}$ and $I(0)=0$.

Next, since $\gamma<1$ in ($\mathrm{G}_3$), there exists $t_0>0$ such that

$$2K_1\big(1+g(x,t)t^\gamma\big)\le \frac{1}{2}g(x,t)t,\ \text{ for all } t>t_0.$$

Put $C_3:=|\Omega|\max\Big(0,-\inf_{\substack{x\in\bar\Omega\\0\le t\le t_0}}\big[g(x,t)t-2K_1(1+g(x,t)t^\gamma)\big]\Big)$, we obtain

$$\begin{aligned}2I(u)&=\int_\Omega g(x,u)u-2G(x,u)\,dx\\&\ge\int_\Omega g(x,u)u-2K_1\big(1+g(x,|u|)\,|u|^\gamma\big)\,dx\\&\ge C_3+\frac{1}{2}\int_{|u|>t_0}g(x,u)u\,dx.\end{aligned}$$

Hence

$$\begin{aligned}\int_\Omega g(x,u)u\,dx&=\int_{|u|\le t_0}+\int_{|u|>t_0}g(x,u)u\,dx\\&\le|\Omega|\sup_{\substack{x\in\bar\Omega\\0\le t\le t_0}}g(x,t)t+4I(u)-2C_3\le M_1(1+I(u)),\end{aligned}$$

with $M_1>0$ independent from $u$.

**Lemma 3.** *Suppose that $g$ is a function of critical growth for which* ($\mathrm{H}_1$) *holds. Then*

(i) *The constraint $S$ is a closed set in $H_0^m(\Omega)$ with the norm topology.*
(ii) *For all $u\in H_0^m(\Omega)\setminus\{0\}$, there exists one and only one $\lambda>0$ such that $\lambda u\in S$.*
(iii) *If $F(u)\ge0$, then $\lambda\le1$.*
(iv) $S=\{u\in H_0^m(\Omega)\setminus\{0\}:\ J(u)=\sup_{\lambda>0}J(\lambda u)\}$.
(v) $\inf\{J(u):\ u\in S\}>0$.

**Proof.** From the continuity of $F$, $F^{-1}\{0\}$ is closed in $H_0^m(\Omega)$. Suppose by contradiction, that $S=F^{-1}\{0\}\setminus\{0\}$ is not closed. Then there exists a sequence $\{u_k\}\subset S$ such that $u_k\to0$ in $H_0^m(\Omega)$. Define $v_k:=u_k/\|u_k\|$. The sequence $\{v_k\}$ is bounded. Since $H_0^m(\Omega)$ is a Hilbert space and $H_0^m(\Omega)\overset{\text{comp}}{\hookrightarrow}L^1(\Omega)$, without loss of generality we may assume that $v_k\rightharpoonup v$ in $H_0^m(\Omega)$ and

$v_k \to v$ a.e. on $\Omega$. From (iii) in Lemma 2 and the fact that $u_k \in S$ it follows that

$$\begin{aligned} 1 &= \lim_{k\to\infty} \frac{\int_\Omega g(x,u_k)u_k\,dx}{\|u_k\|^2} = \lim_{k\to\infty}\int_\Omega \frac{g(x,u_k)}{u_k} v_k^2\,dx \\ &= \int_\Omega g'(x,0)v^2\,dx < \Lambda \|v\|_2^2 \le \liminf_{k\to\infty}\|v_k\|^2 \le 1, \end{aligned} \tag{15}$$

a contradiction. Hence $S$ is a closed set in $H_0^m(\Omega)$ with respect to the norm topology and (i) is proved.

Next, we show (ii). Fix $u \in H_0^m(\Omega)\backslash\{0\}$ and define the function $\psi : \mathbb{R} \to \mathbb{R}$ as

$$\psi(\lambda) := \begin{cases} \frac{1}{\lambda}\int_\Omega g(x,\lambda u)u\,dx, & \text{if } \lambda \in \mathbb{R}\setminus\{0\}; \\ \int_\Omega g'(x,0)u^2\,dx, & \text{if } \lambda = 0. \end{cases}$$

Notice that $\psi$ is continuous on $\mathbb{R}$. The non trivial fact is the continuity at 0: if we consider the sequences $\{u_k\}$ and $\{v_k\}$, where $v_k = u$ and $u_k = \lambda_k u$ with $\lambda_k \to 0$, an application of (iii) in Lemma 2 gives

$$\lim_{\lambda\to 0}\psi(\lambda) = \int_\Omega g'(x,0)u^2\,dx = \psi(0).$$

From ($\mathrm{H}_1$) we have

$$\psi(0) < \Lambda\|u\|_2^2 \le \|u\|^2. \tag{16}$$

On the other hand there exists $\varepsilon_0 > 0$ such that $E_0 := \{x \in \Omega :\ |u(x)| > \varepsilon_0\}$ is a set of positive measure. From (14), we get for any $\lambda > 1/\varepsilon_0$,

$$\begin{aligned} \psi(\lambda) &\ge \frac{1}{\lambda}\int_{E_0} g(x,\lambda u)u\,dx \ge \frac{1}{\lambda}\int_{E_0} C_2\exp\left(\frac{b}{2}\lambda^2|u|^2\right)dx \\ &\ge \frac{C_2\exp(\frac{b}{2}\varepsilon_0^2\lambda^2)\,|E_0|}{\lambda}. \end{aligned}$$

Thus

$$\lim_{\lambda\to\infty}\psi(\lambda) = \infty. \tag{17}$$

By a continuity argument, from (16) and (17) it follows that there exists $\lambda_0 > 0$ such that $\psi(\lambda_0) = \|u\|^2$. That is

$$\int_\Omega g(x,\lambda_0 u)(\lambda_0 u) = \|\lambda_0 u\|^2.$$

Observe that conditions ($G_1$) and ($G_2$) imply that expression $\frac{g(x,\lambda t)}{\lambda}t$ is an increasing function with respect to $\lambda > 0$ if $t \neq 0$; this means that $\psi$ is increasing. This proves the uniqueness of $\lambda_0$, and completes the proof of (ii).

In order to show (iii), let $u \in H_0^m(\Omega)$ be such that $F(u) \geq 0$. With the same notation as above we obtain that $\|u\|^2 \leq \int_\Omega g(x,u)u\,dx$. That is $\|u\|^2 \leq \psi(1)$. But $\psi$ is increasing, so it must be $\lambda_0 \leq 1$.

Let us show now (iv). To do it fix $u \in H_0^m(\Omega) \setminus \{0\}$ and consider $j(\lambda) = j_u(\lambda) := J(\lambda u)$, for $\lambda \geq 0$ . Differentiating $j$ with respect to $\lambda$ we get

$$j'(\lambda) = \lambda\,\|u\|^2 - \int_\Omega g(x,\lambda u)u\,dx = \lambda\{\|u\|^2 - \psi(\lambda)\}.$$

If $\lambda > 0$, then $j'(\lambda) = 0$ if and only if

$$\|u\|^2 - \int_\Omega \frac{g(x,\lambda u)}{\lambda}u\,dx = \frac{j'(\lambda)}{\lambda} = 0.$$

The function $j'(\lambda)/\lambda$ is a decreasing function, it tends to $-\infty$ as $\lambda$ tends to $\infty$. On the other hand $\lim_{\lambda\to 0} j'(\lambda)/\lambda > 0$ (since $g'(x,0) < \Lambda$), hence, $j'$ vanishes at only one point on the positive semi axis. Such point must necessarily be a maximum for $j$ since $j(0) = 0$ and $\lim_{\lambda\to\infty} j(\lambda) = -\infty$. So $u \in S$ if and only if $j'(1) = F(u) = 0$, that is $J(u) = \max_{\lambda\in\mathbb{R}_0^+} J(\lambda u_0)$. And this proves (iv).

Finally we check (v). From (v) in Lemma 2 we know that $I(u) > 0$. Since $I \equiv J$ on $S$ it follows that $\inf\{J(u) :\ u \in S\} \geq 0$. Let $s = \sqrt{2\inf_S J}$. Suppose that $s = 0$. There would be a sequence $\{u_k\} \subset S$ such that $J(u_k) \to 0$. Since $I(u_k) = J(u_k)$, in view of (v) in Lemma 2, we get

$$\sup_k \|u_k\|^2 = \sup_k \int_\Omega g(x,u_k)u_k\,dx \leq \sup_k M_1\big(1 + I(u_k)\big) < \infty.$$

Without loss of generality we may assume that $\{u_k\}$ converges weakly in $H_0^m(\Omega)$ and a.e. on $\Omega$ to $u \in H_0^m(\Omega)$. From Fatou Lemma and (iv) in Lemma 2 it follows that

$$\begin{aligned} 0 \leq & I(u) = \int_\Omega \frac{1}{2}g(x,u)u - G(x,u)\,dx \\ \leq & \liminf_{k\to\infty} \int_\Omega \frac{1}{2}g(x,u_k)u_k dx - \lim_{k\to\infty} G(x,u_k)\,dx = \liminf_{k\to\infty} I(u_k) \\ = & \liminf_{k\to\infty} J(u_k) = 0. \end{aligned}$$

This means that $u = 0$ a.e. in $\Omega$ (from (v) in Lemma 2). Applying once more (iv) in Lemma 2, we obtain

$$\lim_{k\to\infty} \|u_k\|^2 = 2 \lim_{k\to\infty} \left\{ J(u_k) + \int_\Omega G(x,u_k)\,dx \right\} = 0.$$

But $S$ is closed, it follows that $0 \in S$: a contradiction. So it must be $s > 0$. This proves (v). $\square$

In order to conclude this paragraph we give the following lemma which is the most important property relating the functional $J$ to the constraint $S$: it guarantees that a minimizer of $J$ restricted to $S$ is a critical point of $J$.

**Lemma 4.** *Suppose that $u_0 \in S$ satisfies $J'(u_0) \neq 0$, then*

$$J(u_0) > \inf \{J(u) : \ u \in S\} \left( = \frac{s^2}{2} \right).$$

**Proof.** Since $J'(u_0) \neq 0$, there exists $w_0 \in H_0^m(\Omega)$ such that $\langle J'(u_0), w_0 \rangle = 1$. Given $\alpha,\ t \in \mathbb{R}$ we define

$$\begin{aligned} v(\alpha,t) &:= \alpha u_0 - t w_0 \in \mathcal{C}^1(\mathbb{R}^2 \to H_0^m(\Omega)); \\ \Psi(\alpha,t) &:= J(v(\alpha,t)) = \frac{1}{2}\|v(\alpha,t)\|^2 - \int_\Omega G(x, v(\alpha,t))\,dx; \\ \Phi(\alpha,t) &:= F(v(\alpha,t)) = \|v(\alpha,t)\|^2 - \int_\Omega g(x,v(\alpha,t))v(\alpha,t)\,dx. \end{aligned}$$

Obviously $\Psi \in \mathcal{C}^1(\mathbb{R}^2)$ and $\Phi \in \mathcal{C}(\mathbb{R}^2)$. Moreover

$$\lim_{\substack{t\to 0\\ \alpha\to 1}} \frac{\partial \Psi}{\partial t}(\alpha,t) = \lim_{\substack{t\to 0\\ \alpha\to 1}} \langle J'(v(\alpha,t)), -w_0 \rangle = -\langle J'(u_0), w_0 \rangle = -1.$$

There exist, thus, $\varepsilon' > 0$ and $\delta' > 0$ such that

$$J(v(\alpha,t)) < J(v(\alpha,0) = J(\alpha u_0), \text{ for all } (\alpha,t) \in (1-\varepsilon', 1+\varepsilon') \times (0,\delta'). \quad (18)$$

On the other hand $\Phi(1,0) = 0$. Put $\varphi(\alpha) = \Phi(\alpha,0)/\alpha^2$. Observing that $g(x,\alpha u_0)/u_0$ is an increasing function with respect to $\alpha$, we deduce that $\varphi$ is decreasing. Since $\varphi(1) = 0$, there exists $\varepsilon'' > 0$ such that $\varphi(1-\varepsilon'') > 0$

and $\varphi(1+\varepsilon'') < 0$. By a continuity argument, there exists $\delta'' > 0$ such that, for all $t$, $0 < t < \delta''$,

$$\Phi(1-\varepsilon'', t) > 0 \text{ and } \Phi(1+\varepsilon'', t) < 0.$$

(We may assume that $\varepsilon'' < \varepsilon'$ and $\delta'' < \delta'$.) It follows that for every $t \in (0, \delta'')$, there corresponds $\alpha_t \in (1-\varepsilon'', 1+\varepsilon'')$ such that $\Phi(\alpha_t, t) = 0$. Fix $t \in (0, \delta'')$. In view of (18) and Lemma 3 we obtain

$$\inf_{u\in S} J(u) \leq J(v(\alpha_t, t)) < J(v(\alpha_t, 0) = J(\alpha_t u_0) \leq \sup_{\alpha\in\mathbb{R}} J(\alpha u_0) = J(u_0).$$

This proves the lemma.

### 3.3. Adams functions.

**Lemma 5.** *Suppose that $g$ is a function of critical growth which satisfies* (H$_2$). *If $a > 0$ is such that*

$$\sup_{\|w\|\leq 1} \int_\Omega g(x, aw) w \, dx \leq a$$

*then $a^2 < \beta_0/b$.*

To prove the Lemma 5 we shall first construct particular functions, namely the *Adams functions*. Denote by $B$ the unit ball in $\mathbb{R}^n$ and by $B_\ell := B(0; \ell)$ whenever $\ell \in (0,1)$.

**Claim 1.** *For all $\ell \in (0,1)$ there exists*

$$u_\ell \in \omega := \{u \in H_0^m(B) : \; u|_{B_\ell} \equiv 1\}$$

*such that*

$$\|u_\ell\|^2 = C_{m,2}(B_\ell; B).$$

$C_{m,2}(K, E)$ denotes the $(m,2)$-*conductor capacity* of $K$ in $E$, whenever $E$ is an open set and $K$ a relatively compact subset of $E$; it is defined as

$$C_{m,2}(K, E) := \inf\left\{ \|\nabla^m u\|_2^2 : \; u \in \mathcal{D}(E), \; u|_K \equiv 1 \right\}.$$

To show the claim 1, consider a minimizing sequence $\{w_k\} \subset \omega$,

$$\|w_k\|^2 \to C_{m,2}(B_\ell; B).$$

Without loss of generality we may assume that there exists $u_\ell \in H_0^m(\Omega)$ such that $w_k \rightharpoonup u_\ell$. Since $H_0^m(\Omega) \overset{\text{comp}}{\hookrightarrow} L^2(\Omega)$ it follows that $w_k \to u_\ell$ in the mean, we may hence assume that $w_k \to u_\ell$ a.e. on $\Omega$. Thus $u_\ell|_{B_\ell} \equiv 1$ a.e., that is $u_\ell \in \omega$. Moreover

$$C_{m,2}(B_\ell; B) = \|u_\ell\|^2 .$$

This proves the claim.

Observe that from the proof of Theorem 3 (refer to [1]) the following inequality holds

$$\|u_\ell\|^2 = C_{m,2}(B_\ell, B) \leq \frac{\beta_0}{n \log(1/\ell)}. \tag{19}$$

**Definition 1.** Let $x_0 \in \Omega$, $R \leq \operatorname{dist}(x_0, \partial\Omega)$ and $0 < r < R$. The *Adams function* (relative to $x_0$, $r$, $R$) is the function

$$A_{x_0,r,R}(x) := \begin{cases} \sqrt{\frac{n \log(R/r)}{\beta_0}}\, u_{r/R}\left(\frac{x-x_0}{R}\right) & \text{if } |x - x_0| < R, \\ 0 & \text{if } x \in \Omega \text{ and } |x - x_0| \geq R, \end{cases}$$

where $u_\ell$ is as in Claim 1.

Inequality (19) implies that Adams functions satisfy $\|A_{x_0,r,R_0}\| \leq 1$. This is easily seen through a variable change and a reduction of the integral to the unit ball of $\mathbb{R}^n$, observing that $2m = n$.

**Proof of Lemma 5.** From Lemma 2 we know that $a^2 \leq \beta_0/b$; we have therefore to show that it cannot be $a^2 = \beta_0/b$.

Since $R_0 = \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$, by a continuity argument, there exists $x_0 \in \Omega$ such that $B(x_0, R_0) \subset \bar{\Omega}$. We proceed by contradiction. Let $a^2 = \beta_0/b$. Fix $t > 0$ and let $r \in (0, R_0)$ such that $r := R_0 \exp(-\beta_0 t^2/a^2 n)$. (Put $A_r(x) := A_{x_0,r,R_0}(x)$ for simplicity.) We have $aA_r|_{B(x_0;r)} \equiv t$. From the assumptions it follows that

$$\begin{aligned} a &\geq \int_\Omega g(x, aA_r) A_r \, dx \geq \int_{B(x_0;r)} \frac{h(x,t)t}{a} \exp(ba^2 A_r^2) \, dx \\ &\geq \int_{B(x_0;r)} \frac{h(x,t)t}{a} \frac{R_0^n}{r^n} dx = \frac{R_0^n h_0(t) t}{r^n a} \int_{B(x_0;r)} dx = \frac{R_0^n h_0(t) t \sigma_n}{an}. \end{aligned}$$

Since $t > 0$ is arbitrary, we get

$$b \leq \frac{n\beta_0}{\sigma_n R_0^n \sup_{t>0} (h_0(t)t)} = \kappa_0,$$

which contradicts $(\mathrm{H}_2)$.

**Lemma 6.** *Suppose that $g$ is a function of critical growth which satisfies both* ($\mathrm{H}_1$) *and* ($\mathrm{H}_2$). *Then*

$$s^2 < \frac{\beta_0}{b}.$$

**Proof.** Let $w \in H_0^m(\Omega)$ be such that $\|w\| = 1$. In view of Lemma 3 there exists $\lambda_0 > 0$ such that $\lambda_0 w \in S$. Hence

$$\frac{s^2}{2} \leq J(\lambda_0 w) = \frac{1}{2}\left\|\lambda_0 w\right\|^2 - \int_\Omega G(x, \lambda_0 w) \leq \frac{\lambda_0^2}{2}\left\|w\right\|^2 = \frac{\lambda_0^2}{2},$$

that is $s < \lambda_0$. Recalling that $\frac{g(x,\lambda t)}{\lambda} t$ is an increasing function with respect to $\lambda$ and that $\lambda_0 w \in S$ we obtain

$$\int_\Omega \frac{g(x, sw)}{s} w\, dx \leq \int_\Omega \frac{g(x, \lambda_0 w)}{\lambda_0} w\, dx = 1.$$

Then

$$\sup_{\|w\| \leq 1} \int_\Omega g(x, sw) w\, dx \leq s.$$

From Lemma 5 it follows that $s^2 < \beta_0/b$.

**3.4. Compactness.** We shall use now Lemma 1 to show the following compactness result.

**Lemma 7.** *Let $u \in H_0^m(\Omega) \setminus \{0\}$ and $\{u_k\} \subset H_0^m(\Omega)$. If*

$$u_k \to u \text{ a.e. on } \Omega, \quad u_k \rightharpoonup u,$$

$$\lim_{k\to\infty} J(u_k) =: C \in \Big(0, \frac{\beta_0}{2b}\Big], \tag{a}$$

$$\left\|u\right\|^2 \geq \int_\Omega g(x, u) u\, dx, \tag{b}$$

$$\sup_k \int_\Omega g(x, u_k) u_k\, dx < \infty, \tag{c}$$

*then*

$$\lim_{k\to\infty} \int_\Omega g(x, u_k) u_k\, dx = \int_\Omega g(x, u) u\, dx.$$

**Proof.** We proceed by steps.

Step 1. In view of (iv) in Lemma 2 and assumption (c) we have

$$\lim_{k\to\infty} \|u_k\|^2 = 2 \lim_{k\to\infty} \big\{ J(u_k) + \int_\Omega G(x, u_k)\, dx \big\} = 2\big(C + \int_\Omega G(x, u)\, dx\big). \quad (20)$$

On the other hand, from (v) in Lemma 2, (b) and the fact that $u \neq 0$ we obtain

$$J(u) = I(u) > 0.$$

Then (a) implies

$$C - J(u) < \frac{\beta_0}{2b}. \quad (21)$$

Step 2. Case $C \leq J(u)$. (20) implies that

$$\lim_{k\to\infty} \|u_k\|^2 \leq 2\Big( J(u) + \int_\Omega G(x, u)\, dx \Big) = \|u\|^2 .$$

Since $H_0^m(\Omega)$ is a Hilbert space and $u_k \rightharpoonup u$, we get

$$u_k \to u \quad \text{in} \quad H_0^m(\Omega).$$

It follows that

$$g(x, u_k)u_k \to g(x, u)u \text{ in } L^2(\Omega).$$

and thus in $L^1(\Omega)$.

Step 3. Case $C > J(u)$. Put $\ell := \lim_{k\to\infty} \|u_k\|^2 \geq \|u\|^2 > 0$. From the preceding step we may assume that $\|u\|^2 < \ell$. Hence there exists $k_0 \geq 1$ such that $\|u_k\| > 0$ for all $k \geq k_0$. From (20) it follows that, for all $\eta \in \mathbb{R}$, $0 < \eta < 1$, there exists $k_1(\eta) \geq k_0$ such that, for every $k \geq k_1(\eta)$,

$$\|u_k\|^2 \leq \frac{2\big(C + \int_\Omega G(x, u)\, dx\big)}{1 - \eta}. \quad (22)$$

From (21) it follows that there exist $\varepsilon_0$, $\eta_0 > 0$ such that

$$(1 + \varepsilon_0)\frac{b}{\beta_0} < \frac{1}{2(C - J(u))}(1 - \eta_0)^2.$$

Thus, for all $k \geq k_0$ we have

$$(1 + \varepsilon_0)\frac{b}{\beta_0} \|u_k\|^2 < \frac{1}{2(C - J(u))}(1 - \eta_0)^2 \|u_k\|^2 . \quad (23)$$

Put $\bar{k} := k_1(\eta_0)$, (23) and (22) imply

$$\begin{aligned}(1+\varepsilon_0)\frac{b}{\beta_0}\|u_k\|^2 &< \frac{C+\int_\Omega G(x,u)\,dx}{C-J(u)}(1-\eta_0)\\ =&\Big(1-\frac{\|u\|^2}{2(C+\int_\Omega G(x,u)\,dx)}\Big)^{-1}(1-\eta_0) < \Big(1-\frac{\|u\|^2}{\ell}\Big)^{-1}.\end{aligned} \tag{24}$$

Define $v := v/\ell$ and $v_k := u_k/\|u_k\|$, for $k \geq \bar{k}$. Obviously $v_k \rightharpoonup v$, $v \neq 0$ and $\|v_k\| = 1$. We can apply now Lemma 1 and deduce that, for all $p < (1-\|v\|^2)^{-1}$,

$$\sup_k \int_\Omega \exp(p\beta_0 |v_k|^2)\,dx < \infty. \tag{25}$$

Let $p_0 > 0$ be such that

$$\max(1, \frac{C+\int_\Omega G(x,u)\,dx}{C-J(u)}) < p_0 < \Big(1-\frac{\|u\|^2}{\ell}\Big)^{-1}. \tag{26}$$

From (24), (25) and (26) we get

$$\begin{aligned}K' := \sup_k \int_\Omega \exp\left[(1+\varepsilon_0)bu_k^2\right]\,dx < \sup_k \int_\Omega \exp\left(p\beta_0\frac{u_k^2}{\|u_k\|^2}\right)\,dx\\ < \infty.\end{aligned}$$

Moreover from assumption ($G_4$) we have

$$K'' := \sup_{\substack{x\in\bar{\Omega}\\ t\in\mathbb{R}}} |h(x,t)t| \exp\big(-\varepsilon_0\frac{b}{2}|t|^2\big) < \infty.$$

If $N > 0$, then

$$\begin{aligned}\int_{|u_k|\geq N} g(x,u_k)u_k\,dx &= \int_{|u_k|\geq N} h(x,u_k)u_k \exp(b|u_k|^2)\,dx\\ &\leq K''\int_{|u_k|\geq N}\exp(-\varepsilon_0\frac{b}{2}|u_k|^2)\exp(b(1+\varepsilon_0)|u_k|^2)\,dx\\ &< K''\exp(-\varepsilon_0\frac{b}{2}N^2)K' = o(1/N).\end{aligned}$$

Using arguments like those in the proof of Lemma 2, from dominated and monotone convergence we deduce that

$$\lim_{k\to\infty}\int_\Omega g(x,u_k)u_k = \int_\Omega g(x,u)u$$

and this concludes the proof. □

An important application of the preceding lemma is the following

**Theorem 4. (Palais-Smale Condition).** *Suppose that $g(\cdot,\cdot)$ is a function with critical growth for problem $(\mathrm{D}_m)$. Then the functional $J$ satisfies the following local Palais-Smale condition: for any $C \in (-\infty, \beta_0/2b)$, every Palais-Smale sequence $\{u_k\} \in H_0^m(\Omega)$, i.e.,*

$$\lim_{k\to\infty} J(u_k) = C \tag{27}$$

*and*

$$\lim_{k\to\infty} J'(u_k) = 0, \text{ in } H^{-m}(\Omega); \tag{28}$$

*admits a strongly convergent subsequence in $H^m(\Omega)$.*

**Proof.** Let us divide the proof in several steps.

Step 1. From (27) and (28) it follows that the sequences $\{J(u_k)\}$ and $\{J'(u_k)\}$ are bounded in $\mathbb{R}$ and $H^{-m}(\Omega)$ respectively; so

$$|I(u_k)| \le |J(u_k)| + \frac{1}{2}\,\|J'(u_k)\|\,\|u_k\| \le C_1(1+\|u_k\|).$$

In view of Lemma 2 (i) we have

$$\int_\Omega g(x,u_k)u_k\,dx \le C_2(1+\|u_k\|),$$

and assumption $(\mathrm{G}_3)$ implies that

$$\int_\Omega G(x,u_k)\,dx \le C_3(1+\|u_k\|).$$

From the boundedness of $\{J(u_k)\}$ we get

$$\|u_k\|^2 \le 2\,|J(u_k)| + \left|\int_\Omega G(x,u)\,dx\right| \le C_4(1+\|u_k\|),$$

hence $\sup_k \|u_k\| < \infty$, that is $\{u_k\}$ is a bounded sequence in $H_0^m(\Omega)$. We also have that

$$\sup_k \int_\Omega g(x,u_k)u_k\,dx < \infty. \tag{29}$$

Step 2. From the reflexivity of $H_0^m(\Omega)$ and the boundedness of $\{u_k\}$ there exist $u_0 \in H_0^m(\Omega)$ and a subsequence (which we denote again by) $\{u_k\}$ such that

$$\lim_{k\to\infty} u_k = u_0, \text{ in the weak topology of } H_0^m(\Omega).$$

In view of Rellich-Kondrašov Theorem, $H_0^m(\Omega) \overset{\text{comp}}{\hookrightarrow} L^2(\Omega)$, we may assume also that $u_k \to u_0$ a.e. on $\Omega$.

Since $H_0^m(\Omega)$ is Hilbert (and thus uniformly convex); in order to show that $\lim_{k\to\infty} u_k = u_0$ it is sufficient to check that

$$\limsup_{k\to\infty} \|u_k\| \le \|u_0\|.$$

Without loss of generality, we shall assume that

$$\lim_{k\to\infty} \|u_k\| = \liminf_{k\to\infty} \|u_k\|.$$

Step 3. Suppose $C \le 0$, then by Fatou Lemma and (v) in Lemma 2 we obtain

$$\begin{aligned}0 \le I(u_0) &= \int_\Omega g(x,u_0)u_0 - G(x,u_0)\,dx\\ &= \liminf_{k\to\infty} \int_\Omega g(x,u_k)u_k\,dx - \lim_{k\to\infty}\int_\Omega G(x,u_k)\,dx\\ &= \liminf_{k\to\infty} I(u_k) = \liminf_{k\to\infty}\big(J(u_k) - \frac{1}{2}\langle J'(u_k),u_k\rangle\big) = C \le 0.\end{aligned}$$

That is $I(u_0) = 0$ and hence, by (v) in Lemma 2, $u_0 = 0$. Therefore, (iv) in Lemma 2 implies that

$$\lim_{k\to\infty} \|u_k\|^2 = 2\lim_{k\to\infty}\big(J(u_k) + \int_\Omega G(x,u_k)\,dx\big) = 0,$$

whence $u_k \to u_0$, strongly in $H_0^m(\Omega)$. This concludes for case $C \le 0$. Observe that in this case the level $C$ is *necessarily* 0 (i.e., there are no Palais-Smale sequences $\{u_k\}$ such that $\lim_{k\to\infty} J(u_k) < 0$).

Step 4. Suppose now that $C \in (0, \beta_0/2b)$; we shall show first that $u_0 \neq 0$, second that $u_0 \in S$ and last that

$$\limsup_{k\to\infty} \|u_k\|^2 = \|u_0\|^2 .$$

If it were $u_0 = 0$, from (iv) in Lemma 2, we would have

$$\lim_{k\to\infty} \int_\Omega G(x, u_k)\, dx = \int_\Omega G(x, 0)\, dx,$$

thus

$$\lim_{k\to\infty} \|u_k\|^2 = 2 \liminf_{k\to\infty} \big( J(u_k) + \int_\Omega G(x, u_k)\, dx \big) = 2C < \beta_0/b;$$

and (iii) in Lemma 2 would then imply that

$$\lim_{k\to\infty} \int_\Omega g(x, u_k) u_k\, dx = \int_\Omega g(x, u_0) u_0\, dx = 0.$$

Therefore,

$$I(u_k) = \int_\Omega \frac{1}{2} g(x, u_k) u_k - G(x, u_k)\, dx \to 0,$$

as $k \to 0$; from the boundedness of $\{u_k\}$ we would get then that

$$\begin{aligned} 0 < C = \lim_{k\to\infty} J(u_k) &= \lim k \to \infty \big( I(u_k) + \frac{1}{2} F(u_k) \big) \\ &= \lim_{k\to\infty} \big( I(u_k) + \frac{1}{2} \langle J'(u_k), u_k \rangle \big) = 0, \end{aligned}$$

a contradiction, thus it must be $u_0 \neq 0$.

Step 5. We prove that $u \in S$. For any $w \in H_0^m(\Omega)$ we have

$$\langle J'(u_0), w \rangle = \int_\Omega \nabla^m u_0 \nabla^m w\, dx - \int_\Omega g(x, u_0) w\, dx.$$

But (28) implies that $u_k \rightharpoonup u_0$, thus (applying (iii) in Lemma 2, with $v_k = w u_k$) we obtain

$$\begin{aligned} 0 = \lim_{k\to\infty} \langle J'(u_k), w \rangle &= \lim_{k\to\infty} \int_\Omega \nabla^m u_k \nabla^m w\, dx - \lim_{k\to\infty} \int_\Omega \frac{g(x, u_k)}{u_k} w u_k\, dx \\ &= \int_\Omega \nabla^m u_0 \nabla^m w - g(x, u_0) w\, dx. \end{aligned}$$

It follows, putting $u_0$ instead of $w$ in the above, that

$$\|u_0\|^2 = \int_\Omega g(x, u_0)u_0\,dx; \tag{30}$$

that is $u_0 \in S$.

Step 6. Finally, observe that (27), $C < \beta_0/2b$, (29) and (30) allow us to apply Lemma 7 in order to obtain

$$\begin{aligned}\limsup_{k\to\infty} \|u_k\|^2 &= \lim_{k\to\infty} \|u_k\|^2 = \lim_{k\to\infty} (F(u_k) + \int_\Omega g(x, u_k)u_k\,dx) \\ &= \lim_{k\to\infty} (\langle J'(u_k), u_k\rangle + \int_\Omega g(x, u_k)u_k\,dx) \\ &= 0 + \int_\Omega g(x, u_0)u_0\,dx = \|u_0\|^2 .\end{aligned}$$

This concludes the proof.

**4. Proof of Theorem 1.** In view of Lemma 4 it is enough to prove that there exists $u^* \in S$ that minimizes the restriction of $J$ on $S$; recalling the definition of $s$, this means that

$$J(u^*) = \frac{s^2}{2}.$$

Consider a minimizing sequence $\{u_k\} \subset S$,

$$\lim_{k\to\infty} J(u_k) = \frac{s^2}{2}.$$

Since $u_k \in S$, we have $I(u_k) = J(u_k)$. In view of (v) in Lemma 2 we get

$$\sup_k \int_\Omega g(x, u_k)u_k\,dx < \infty, \tag{31}$$

$$\sup_k \|u_k\| < \infty. \tag{32}$$

Without loss of generality we may assume that

$$u_k \rightharpoonup u^* \text{ e } u_k \to u^* \text{ a.e. on } \Omega. \tag{33}$$

**Claim 2.** *$u^* \neq 0$ and*

$$\|u^*\|^2 \leq \int_\Omega g(x, u^*)u^* \, dx. \tag{34}$$

Indeed, suppose that $u^* = 0$. Then (31) and (iv) in Lemma 2 would imply

$$\lim_{k\to\infty} \|u_k\|^2 = 2 \lim_{k\to\infty} \{J(u_k) + \int_\Omega G(x, u_k) \, dx\} = s^2. \tag{35}$$

From Lemma 6 we know that $0 < s^2 < \beta_0/b$. Thus (35) implies that $\lim_k \|u_k\|^2 < \beta_0/b$. In view of (iii) in Lemma 2 we obtain

$$\begin{aligned}\lim_{k\to\infty} \int_\Omega g(x, u_k)u_k \, dx &= \lim_{k\to\infty} \int_\Omega \frac{g(x, u_k)}{u_k} u_k^2 \, dx \\ &= \int_\Omega g(x, u^*)u^* \, dx = 0.\end{aligned}$$

It follows that

$$\begin{aligned}0 < \frac{s^2}{2} &= \lim_{k\to\infty} J(u_k) = \lim_{k\to\infty} I(u_k) \\ &= \lim_{k\to\infty} \{ \int_\Omega g(x, u_k)u_k \, dx - \int_\Omega G(x, u_k) \, dx\} = 0,\end{aligned}$$

which is impossible. It is thus necessary that $u^* \neq 0$.

Let us show next (34). By contradiction assume that

$$\|u^*\|^2 > \int_\Omega g(x, u^*)u^* \, dx. \tag{36}$$

This inequality, (31), and that fact that $J(u_k) \to s^2/2$ allow us to apply the compactness Lemma 7, obtaining

$$\lim_{k\to\infty} \int_\Omega g(x, u_k)u_k \, dx = \int_\Omega g(x, u^*)u^* \, dx.$$

From (33) it follows that

$$\|u^*\|^2 \leq \liminf_{k\to\infty} \|u_k\|^2 = \lim_{k\to\infty} \int_\Omega g(x, u_k)u_k \, dx = \int_\Omega g(x, u^*)u^* \, dx,$$

which contrasts with (36). Hence (34) is verified and the claim is true.

Observe, now, that (34) means $F(u^*) \leq 0$. From Lemma 3, there exists $\lambda_0 \in (0,1]$ such that $\lambda_0 u^* \in S$. With a similar argument as that in the proof of Lemma 3, it is easily seen that $I(\lambda u^*)$ is an increasing function of $\lambda \geq 0$. Then

$$\frac{s^2}{2} \leq J(\lambda_0 u^*) = I(\lambda_0 u^*) \leq I(u^*) \leq \liminf_{k\to\infty} I(u_k)$$
$$= \liminf_{k\to\infty} J(u_k) = \frac{s^2}{2}.$$

We deduce, hence, that $I(\lambda_0 u^*) = I(u^*)$, $\lambda_0 = 1$ and $u^* \in S$. Moreover, we get $\inf_S J = s^2/2 = J(\lambda_0 u^*)$, and this concludes the proof of the theorem.

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