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TOPOLOGICAL DEGREE FOR ELLIPTIC OPERATORS IN UNBOUNDED CYLINDERS

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Abstract. Semilinear elliptic operators in unbounded cylinders are considered. It is shown that under some conditions the operators are Fredholm and proper. A topological degree for these operators is defined and shown to be unique. The results are applied to operators describing traveling waves.

1. Introduction. In this work we construct a topological degree for elliptic operators in unbounded cylinders. We consider operators of the form

$$A(u) = a(x)\Delta u + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} + F(x,u) + K(u), \tag{1.1}$$

where $x = (x_1, \ldots, x_m) \in \Omega$, $u = (u_1, \ldots, u_p)$, $F(x,u) = (F_1(x,u), \ldots, F_p(x,u))$ is a vector-valued function, $a(x)$, $b_i(x)$ are $p\times p$ matrices, Δ denotes the Laplace operator, $K(u)$ is a finite dimensional operator, $\Omega \subset R^m$ is an unbounded cylinder, $\Omega = R \times G$, G is a bounded domain in R^{m-1}. We suppose that the axis of the cylinder is along the x_1-direction and denote by $x^{'} = (x_2, ..., x_m)$ the variable in the section of the cylinder. The boundary

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S of the cylinder Ω is supposed to be of class $C^{2+\delta}$ for some positive δ. The boundary operator has the form

$$\Lambda u = \alpha \frac{\partial u}{\partial \nu} + \beta(x) u \mid_S .$$

Here $\alpha = 0,\ \beta = 1$ (Dirichlet problem) or $\alpha = 1,\ \beta(x)$ is a given diagonal matrix.

The operator A is considered as acting from the Banach space E_1 of functions belonging to the weighted Hölder space $C_\mu^{2+\delta}(\bar{\Omega})$ and satisfying the boundary conditions $\Lambda u = 0$, into the space $E_2 = C_\mu^\delta(\bar{\Omega})$.

The operator K in (1.1) can be any arbitrary finite dimensional operator. In particular, if $K = 0$, we have the usual semilinear elliptic operator. We have included the term $K(u)$ in view of some specific applications, such as the investigation of traveling waves (see Section 5).

The weighted Hölder space $C_\mu^\delta(\bar{\Omega})$ is endowed with the norm $\|u\|_\mu^\delta = \|u\mu\|^\delta$, where $\|\cdot\|^\delta$ is the usual Hölder norm $(0 < \delta < 1)$. By $C_\mu^{2+\delta}(\bar{\Omega})$ we denote the space of functions whose derivatives up to second order belong to $C_\mu^\delta(\bar{\Omega})$.

The weight function $\mu(x_1)$ is a sufficiently smooth positive function of the single variable x_1, such that $\mu(x_1) \to \infty$, $x_1 \to \pm\infty$, and the functions

$$\mu_1(x_1) = \frac{\mu'(x_1)}{\mu(x_1)}, \quad \mu_2(x_1) = \frac{\mu''(x_1)}{\mu(x_1)}$$

are bounded in a Hölder norm and tend to 0 as $x_1 \to \pm\infty$. As an example we may take $\mu(x_1) = 1 + x_1^2$. Together with the operator A we consider the operator depending on a parameter $\tau \in [0,1]$:

$$A_\tau(u) = a(x,\tau)\Delta u + \sum_{i=1}^m b_i(x,\tau)\frac{\partial u}{\partial x_i} + F(x,u,\tau) + K(u,\tau), \qquad (1.2)$$

$$\Lambda u = \alpha \frac{\partial u}{\partial \nu} + \beta(x) u \ \mid_S . \qquad (1.3)$$

The matrices $a(x,\tau),\ b_i(x,\tau),\ \beta(x)$, the function $F(x,u,\tau)$, and the operator $K(u,\tau)$ are supposed to be sufficiently smooth with respect to all variables (including τ). More precise conditions which specify the class of operators will be given below. In the corresponding class of operators we will construct a topological degree satisfying the usual properties.

We recall the definition of a topological degree (see, e.g. [4]). Let E_1 and E_2 be two Banach spaces. Suppose we are given a class Φ of operators acting from E_1 to E_2 and a class H of homotopies, i.e., maps $A_\tau(u) : E_1 \times [0,1] \to E_2$, $\tau \in [0,1]$, $u \in E_1$ such that for any $\tau \in [0,1]$ fixed, $A_\tau(u) \in \Phi$. Assume moreover that for any bounded open set $D \subset E_1$ and any operator $A \in \Phi$ such that $A(u) \neq 0$, $u \in \partial D$ (∂D denotes the boundary of the set D) there is an integer $\gamma(A, D)$ satisfying the following conditions:

(i) *Homotopy invariance.* Let $A_\tau(u) \in H$ and $A_\tau(u) \neq 0$, $u \in \partial D$, $\tau \in [0,1]$. Then $\gamma(A_0, D) = \gamma(A_1, D)$.

(ii) *Additivity.* Let $D \subset E_1$ be an arbitrary open set in E_1, and $D_1,\ D_2 \subset D$ be open sets such that $D_1 \cap D_2 = \emptyset$. Suppose that $A \in \Phi$ and

$$A(u) \neq 0, \ \ u \in \bar{D} \backslash (D_1 \cup D_2).$$

Then

$$\gamma(A, D) = \gamma(A, D_1) + \gamma(A, D_2).$$

(Here $\bar{D}$ is the closure of D.)

(iii) *Normalization.* There exists a bounded linear operator $J : E_1 \to E_2$ with a bounded inverse defined on all of E_2 such that for any bounded open set $D \subset E_1$ with $0 \in D$, $\gamma(J, D) = 1$.

The integer $\gamma(A, D)$ is called *topological degree*.

In the case where $E_1 = E_2$, $A = I + K$, I is the identity operator and K is a compact operator, the topological degree was constructed by Leray and Schauder [18]. It is well known that elliptic operators (1.1) in bounded domains can be reduced to completely continuous vector fields and thus this construction is applicable to this case. If the domain is not bounded, the operator cannot be reduced to the form $I + K$ and the Leray-Schauder theory cannot be applied. However there are numerous generalizations of the degree theory (see [4], [9], [10], [11], [14], [15], [19], [21], [23]). In particular, the degree can be defined for operators A acting between a Banach space E and its dual E^* which satisfy the following condition [23], [4]:

If a sequence $u_n \in E$ converges weakly to u_0 and

$$\lim_{n \to \infty} < A(u_n), u_n - u_0 > \leq 0,$$

then the convergence $u_n \to u_0$ is strong. Here $< f, u >$ denotes the action of the functional $f \in E^*$ on the element $u \in E$.

In previous works by two of the authors [24]–[26] the following estimate of elliptic operators in unbounded cylinders was obtained

$$< A(u) - A(u_0), S(u - u_0) > \geq \|u - u_0\|^2 + \phi(u, u_0), \tag{1.4}$$

where the space E here is a weighted Sobolev space, S is a linear symmetric positive definite operator, and $\phi(u, u_0)$ is a functional which tends to 0 when u_n converges to u_0 weakly. This estimate may then be used to prove that the above condition is satisfied, and to apply the construction of the degree for elliptic operators in unbounded domains. However the estimate of the operators requires certain restrictions on the class of operators.

In this work we develop another approach to the degree construction. It is based on the theory of Fredholm operators. For some proper zero-index Fredholm operators (satisfying some additional conditions) the topological degree can be constructed (see [6]–[11], [14], [21] and references there). By definition a nonlinear operator is Fredholm in some domain of a function space if the operator linearized about each point of this domain is Fredholm. We suppose that the coefficients a and b_i have limits at infinity:

$$a^{\pm}(x') = \lim_{x_1 \to \pm\infty} a(x), \;\; b_i^{\pm}(x') = \lim_{x_1 \to \pm\infty} b_i(x), \; i = 1, ..., n$$

and that the limit

$$c^{\pm}(x') = \lim_{\substack{x_1 \to \pm\infty, \\ u \to 0}} F_u'(x, u)$$

exists. Moreover, we assume that as $x_1 \to \pm\infty$, the matrix $\beta(x)$ has limits in the $C^{1+\delta}(\bar{G})$ norm for some δ. Then by a change of variables $u = \gamma v$ with some diagonal matrix γ we can reduce the problem to the case where matrix $\beta(x) \equiv \beta(x')$ does not depend on the variable x_1. This will be useful in what follows, because the space corresponding to this homogeneous boundary condition is invariant under shifts with respect to x_1.

We study the Fredholm property of $A(u)$. To do this, in Section 2 we consider linear operators of the form

$$Lu = a(x)\Delta u + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u, \tag{1.5}$$

where $a(x)$, $b_i(x)$ are the same as for the operator (1.1) and $c(x)$ is a $p \times p$ matrix. We find conditions for this operator to be Fredholm. The operator L acts from $C^{2+\delta}(\bar{\Omega})$ to $C^{\delta}(\bar{\Omega})$. The operator (1.5) is a special case of the operator (1.1), and in this case the matrices $c^{\pm}(x')$ are given by

$$c^{\pm}(x') = \lim_{x_1 \to \pm\infty} c(x).$$

We consider the following operators whose coefficients do not depend on x_1:

$$L^{\pm}u = a^{\pm}(x^{'})\Delta u + \sum_{i=1}^{n} b_i^{\pm}(x^{'})\frac{\partial u}{\partial x_i} + c^{\pm}(x^{'})u.$$

Condition 1. The equation

$$L^{\pm}u = 0, \quad \Lambda u = 0 \tag{1.6}$$

has no nontrivial solution in $C^{2+\delta}(\bar{\Omega})$.

Condition 2. The equation

$$L^{\pm}u - \lambda u = 0, \quad \Lambda u = 0 \tag{1.7}$$

has no nontrivial solution in $C^{2+\delta}(\bar{\Omega})$ for all $\lambda \geq 0$.

Conditions 1 and 2 are formulated for linear operators. We will say that the nonlinear operator $A(u)$ of the form (1.1) satisfies these Conditions if the linearized operator $Lu = A^{'}(u_0)u - K^{'}(u_0)u$ satisfies them for any u_0 from a given domain $D \subset E_1$.

We will show that Condition 1 is necessary and sufficient for the operator L to be normally solvable and for the dimension of its kernel $\alpha(L)$ to be finite. If Condition 2 is satisfied, then the codimension of its image $\beta(L)$ is also finite, i.e., the operator is Fredholm for all $\lambda \geq 0$. Moreover, its index, $\kappa(L) = \alpha(L) - \beta(L)$ is zero.

Conditions 1 and 2 can be written in a different form. If we apply a formal Fourier transform with respect to the variable x_1 to the corresponding equations, we obtain the following conditions:

Condition 1$^{'}$. The equation

$$\tilde{L}_{\xi}^{\pm}\tilde{u} = 0, \;\; \Lambda\tilde{u} = 0$$

has no nontrivial solution in $C^{2+\delta}(G)$ for all real ξ. Here

$$\tilde{L}_{\xi}^{\pm}\tilde{u} = -\xi^2 a^{\pm}(x^{'})\tilde{u} + a^{\pm}(x^{'})\Delta^{'}\tilde{u} + i\xi b_1^{\pm}(x^{'})\tilde{u} + \sum_{k=2}^{m} b_k^{\pm}(x^{'})\frac{\partial \tilde{u}}{\partial x_k} + c^{\pm}(x^{'})\tilde{u},$$

$$\Delta^{'}\tilde{u} = \sum_{k=2}^{m} \frac{\partial^2 \tilde{u}}{\partial x_k^2}.$$

Condition 2$'$. The equation

$$\tilde{L}^{\pm}_{\xi}\tilde{u} - \lambda\tilde{u} = 0, \quad \Lambda\tilde{u} = 0$$

has no nontrivial solution in $C^{2+\delta}(G)$ for all $\lambda \geq 0$ and all real ξ.

Since functions in $C^{2+\delta}(\bar{\Omega})$ may not be integrable, the transition from Conditions 1 and 2 to Conditions 1$'$ and 2$'$ should be justified. We will show in Section 2 that they are equivalent.

In Sections 3 we prove that if Condition 1 is satisfied, then the restriction of the operator $A(u)$ to any bounded domain in E is a proper operator.

In Section 4 we construct a topological degree for the class of operators $A(u)$ satisfying Condition 2 and for the corresponding class of homotopies. This topological degree satisfies conditions (i)–(iii). We also prove uniqueness of the degree in a more general class of Fredholm operators.

Section 5 is devoted to the construction of the degree for traveling waves which have some specific features.

2. Linear Fredholm operators. In this section we show that Condition 1 is a necessary and sufficient condition for the operator L (see (1.5)) to be normally solvable with a finite dimensional kernel. We prove also that Conditions 1 and 1$'$ are equivalent. In some cases it is more convenient to work with the latter one. We show finally that from Condition 2 it follows that the operator is Fredholm and its index is 0.

We recall briefly some definitions and properties of Fredholm operators. Let $L : E_1 \to E_2$ be a bounded linear operator. If its image is closed, the dimension of its kernel $\alpha(L)$ and the codimension of its image $\beta(L)$ are finite, then the operator is called Fredholm. The index of the operator $\kappa(L) = \alpha(L) - \beta(L)$ does not change under deformation in the class of Fredholm operators.

Normal solvability of the operator is equivalent to the requirement that its image is closed.

A closed linear operator L is called Φ^+-operator if it is normally solvable, $\alpha(L)$ is finite, and $\beta(L)$ is infinite. If B is a bounded operator with a sufficiently small norm, then $L + B$ is also a Φ^+-operator, $\alpha(L+B) \leq \alpha(L)$, and $\beta(L+B) = \beta(L)$ [12]. This property of Φ^+-operators allows to prove that an operator is Fredholm without direct computation of $\beta(L)$. Indeed, let L be a normally solvable operator with finite $\alpha(L)$. A priori we do not know whether $\beta(L)$ is finite. Suppose that we can construct a continuous deformation L_τ, $\tau \in [0,1]$ such that $L_0 = L$, L_τ is normally solvable with finite $\alpha(L_\tau)$ for all $\tau \in [0,1]$, and $\beta(L_1)$ is finite. If $\beta(L_0) = \infty$, then there exists $\tau_0 \in [0,1]$ such that in any its neighborhood there are points where $\beta(L_\tau)$ is finite and there are points where it is infinite. On the other hand, if $\beta(L_{\tau_0})$ is finite it remains finite in some neighborhood of τ_0 and if it is infinite it also remains infinite in some neighborhood of τ_0. This contradiction shows that $\beta(L_\tau)$ is finite for all $\tau \in [0,1]$ and $\kappa(L) = \kappa(L_1)$.

As it is mentioned above, we will show that Conditions 1 and $1^{'}$ are necessary and sufficient for an operator to be normally solvable with finite dimensional kernel. Condition $1^{'}$ in some cases can be verified explicitly. It allows to construct the deformation L_τ and to show that the operator is Fredholm.

We note that Condition $1^{'}$ is used in [26] to show that linear elliptic operators are Fredholm considered as acting in Sobolev spaces. In [24], [25] under the condition similar to Condition $2^{'}$ we obtain the estimation (1.4). From this estimation follows Condition α) (see Introduction) and from this condition follows in its turn the Fredholm property for linear operators. On the other hand there exist examples of Fredholm operators that do not satisfy Condition α). It appears that the Fredholm property can be obtained directly without estimations of the operators and it allows to consider a more general class of operators. The approach presented below is close to that in [25] for linear elliptic operators in cylinders and to that in [20] for operators in R^n.

We consider the operator

$$Lu = a(x)\Delta u + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \tag{2.1}$$

acting from the space of functions $u \in C^{2+\delta}(\bar{\Omega})$ satisfying the boundary condition

$$\Lambda u \equiv \alpha\frac{\partial u}{\partial \nu} + \beta(x^{'})u \ \ |_S = 0 \tag{2.2}$$

to the space $C^{\delta}(\bar{\Omega})$. We suppose that the boundary of the cylinder belongs to the class $C^{2+\delta}$ and the coefficients of the operator to the class $C^{1+\delta}(\bar{\Omega})$. We assume also that the ellipticity condition $(a(x)q, q) \geq \sigma \mid q \mid^2$, $x \in \bar{\Omega}$ is satisfied. Here q is a constant vector, $\sigma > 0$, $(\cdot,\cdot)$ denotes the inner product in R^p. We shall use the Schauder estimates for operator L. For any function $u \in C^{2+\delta}(\bar{\Omega})$ satisfying conditions (2.2) the following estimation takes place:

$$\|u\|^{(2+\delta)} \leq k_1\|Lu\|^{(\delta)} + k_2 \max_x \mid u \mid . \tag{2.3}$$

Here the constants k_1 and k_2 do not depend on the function u but depend on the domain Ω and on the coefficients of the operator. In some cases we will use this estimation for operators depending on parameter. In this case the constants k_1 and k_2 can be chosen such that they are independent of it.

Lemma 2.1. *If Condition* 1 *is satisfied, then the restriction of the operator* L *on the closed unit ball is proper.*

Proof. We should verify that for any compact set $D \subset C^\delta(\bar{\Omega})$, $L^{-1}(D) \cap B$ is compact. Here B is the closed unit ball in the space of functions from $C^{2+\delta}(\bar{\Omega})$ satisfying conditions (2.2). For this it is sufficient to show that for any sequences $\{u_n\}$ and $\{f_n\}$ such that

$$Lu_n = f_n, \quad f_n \in D, \quad u_n \in B,$$

if $f_n \to f_0$ in $C^\delta(\bar{\Omega})$, then there exists a converging subsequence of the sequence $\{u_n\}$. Since this sequence is bounded in $C^{(2+\delta)}(\bar{\Omega})$, then for any bounded subset $\Omega_N = \Omega \cap \{-N \le x_1 \le N\}$ we can choose a subsequence $\{u_i^{(N)}\}$ which converges to some limiting function u_0 in $C^2(\Omega_N)$. We can extend the limiting function u_0 on the whole cylinder Ω and choose a subsequence such that the functions $v_{ik} = u_{ik} - u_0$ converge to zero in C^2 on every bounded subset in Ω. Then $Lu_0 = f_0$.

We show first that this convergence is uniform in $C(\Omega)$. Suppose that it is not so. Then there is a subsequence $\{v_{ik}\}$ (we keep for it the same notation as for the original subsequence) such that $|v_{ik}(x^k)| \ge \delta > 0$ for some x^k. The sequence $\{x^k\}$ cannot be bounded because of the uniform convergence on every bounded subset of Ω. Without loss of generality we can assume that the first coordinates of the points x^k go to $+\infty$. Denote them by h_k and consider the shifted functions $w_k(x) = v_{ik}(x + h_k)$. Here the notation $(x + h_k)$ means that we add h_k only to the first coordinate of the vector x.

The functions $w_k(x)$ satisfy the shifted problems

$$\begin{aligned} a(x+h_k)\Delta w + \sum_{j=1}^{m} c_j(x+h_k)\frac{\partial w}{\partial x_j} + b(x+h_k)w \\ = f_n(x+h_k) - f_0(x+h_k), \ \alpha\frac{\partial w}{\partial n} + \beta(x^{'})w \mid_S = 0. \end{aligned} \tag{2.4}$$

We note that $u_0 \in C^{(2+\delta)}(\bar{\Omega})$. Indeed, it is sufficient to verify the inequality

$$\frac{|D^2 u_0(x^{(1)}) - D^2 u_0(x^{(2)})|}{\mid x^{(1)} - x^{(2)} \mid^\delta} \le K, \ \ x^{(1)}, x^{(2)} \in \bar{\Omega}.$$

Here D^2 denotes the second derivatives. It follows from the similar inequality for the functions $u_n(x)$ with the constant K independent of n and from the convergence of the sequence $\{u_n\}$ to u_0 in $C^{(2)}$.

Thus the norms of the functions w_k are uniformly bounded in $C^{(2+\delta)}(\bar{\Omega})$. Then we can choose a subsequence converging in C^2 to some limiting function on every bounded subset of Ω. We can extend this limiting function on the whole Ω and pass to the limit in (2.4). Since $\mid w_k(0, x_2^k, ..., x_n^k) \mid \geq \delta$, we obtain that the problem (1.6) has a nontrivial solution. This contradiction shows that the sequence $\{v_{ik}\}$ converges to zero uniformly in Ω.

To finish the proof of the lemma, it is sufficient to use inequality (2.3) which gives convergence in $C^{(2+\delta)}(\bar{\Omega})$. The lemma is proved.

As immediate consequence of the lemma we obtain that the kernel of the operator is finite dimensional. Indeed, it is sufficient to take $D = \{0\}$ as a compact set in $C^{(\delta)}(\bar{\Omega})$. To show that the operator is normally solvable we consider a sequence $\{f_n\}$ in its image and suppose that $f_n \to f_0$. If the corresponding sequence $\{u_n\}$, $Lu_n = f_n$ is bounded, then it follows from the lemma that we can choose a converging subsequence, $u_{n_k} \to u_0$, $Lu_0 = f_0$. Hence f_0 belongs to the image. If it is not bounded, we represent it in the form $u_n = v_n + w_n$, where v_n belongs to the kernel of the operator and w_n to its supplement. We put

$$\tilde{w}_n = \frac{w_n}{\|w_n\|^{(2+\delta)}}, \quad \tilde{f}_n = \frac{f_n}{\|w_n\|^{(2+\delta)}}.$$

Then $L\tilde{w}_n = \tilde{f}_n$, $\quad \tilde{f}_n \to 0$. Then we can choose a subsequence $\{w_{nk}\}$ which converges in $C^{(2+\delta)}(\bar{\Omega})$ to some limiting function $w_0 \in C^{(2+\delta)}(\bar{\Omega})$, and passing to the limit in the equation, we obtain $Lw_0 = 0$. Hence w_0 belongs to the kernel of the operator. This contradiction proves that the operator is normally solvable.

Theorem 2.1. *If Condition* 2 *is satisfied, then the operator* L *is Fredholm with index* 0*.*

Proof. Since Condition 2 is satisfied, then the operator $L - \lambda I$ satisfies Condition 1 for positive λ, and Lemma 2.1 can be applied for it. We consider now the operator L_τ depending on the parameter $\tau \in [0, 1]$ such that $L_0 = L - \lambda I$, $L_1 = \Delta - \lambda I$, and L_τ satisfies Condition 1. If λ is sufficiently large it can be for example a linear homotopy (see Lemma 2.2 below).

We use now the fact that the problem

$$\Delta u - \lambda u = f, \quad \Lambda u = 0 \tag{2.5}$$

is solvable in $C^{(2+\delta)}(\bar{\Omega})$ for any $f \in C^{(\delta)}(\bar{\Omega})$. Here λ is a large positive number. Then $\alpha(L_1) = \beta(L_1) = 0$. The theorem is proved.

It remains to prove the following lemma.

Lemma 2.2. *Let*

$$L_0^{\pm} u = L^{\pm} u - \rho u, \qquad L_1 u = \Delta u - \rho u, \qquad L_\tau u = (1-\tau) L_0^{\pm} u + \tau L_1 u.$$

There exists ρ such that the problem

$$L_\tau u = 0, \;\; \Lambda u = 0$$

has only one solution in $C^{(2+\delta)}(\bar{\Omega})$.

Proof. By virtue of Theorem 2.2 below it is sufficient to prove that for any τ, $0 \le \tau \le 1$ and any real ξ the problem

$$\hat{L}_\tau v = 0, \;\; \Lambda v = 0$$

does not have nontrivial solutions in the section G of the cylinder Ω. Here (the index $\pm$ is omitted)

$$\begin{aligned}
\hat{L}_\tau v =& (1-\tau)\tilde{L}_0 v + \tau \tilde{L}_1 v, \\
\hat{L}_0 v =& \sum_{k=2}^{m} a(x^{'}) \frac{\partial^2 v}{\partial x_k^2} + \sum_{k=2}^{m} b_k(x^{'}) \frac{\partial v}{\partial x_k} - a(x^{'})\xi^2 v + b_1(x^{'}) i\xi v + c(x^{'})v - \rho v, \\
\hat{L}_1 v =& \Delta^{'} v - \xi^2 v - \rho v
\end{aligned}$$

We estimate $Re(\hat{L}_\tau v, v)$ for functions $v \in C^2(\bar{\Omega}),\ \Lambda v = 0$.

Denote $a_\tau = \tau I + (1-\tau)a$, where I is the identity matrix. We have

$$\begin{aligned}
\int_G (a_\tau(x^{'})\Delta^{'} v, v) dx^{'} =& -\sum_{k=2}^{m} \int_G (a_\tau(x^{'}) \frac{\partial v}{\partial x_k}, \frac{\partial v}{\partial x_k}) dx^{'} \\
& - \sum_{k=2}^{m} \int_G (\frac{\partial a_\tau}{\partial x_k} \frac{\partial v}{\partial x_k}, v) dx^{'} + \int_{\partial G} (a_\tau \frac{\partial v}{\partial \nu}, v) dS.
\end{aligned}$$

Using the ellipticity condition and the imbedding of $H^1(G)$ into $L^2(\partial G)$, we obtain

$$Re \int_G (a_\tau(x^{'})\Delta^{'} v, v) dx^{'} \le -\mu \int_G \sum_{k=2}^{m} \mid \frac{\partial v}{\partial x_k} \mid^2 dx^{'} + M \int_G \mid v \mid^2 dx^{'}$$

for some positive constants μ and M. For the expression

$$T = -\xi^2(1-\tau)\int_G (a(x^{'})v, v)dx^{'} + i\xi(1-\tau)\int_G (b_1(x^{'})v, v)dx^{'}$$

we have the following estimate

$$ReT \le (1-\tau)(-\xi^2\mu + \mid \xi \mid M_2)\int_G \mid v \mid^2 dx^{'} \le M_3 \int_G \mid v \mid^2 dx^{'}.$$

Here M_2 and M_3 are positive constants.

Using the estimate above and the inequality

$$\mid \int_G \sum_{k=2}^{m} b_k(x^{'})\frac{\partial v}{\partial x_k} + c(x^{'})v, v)dx^{'} \mid \le \epsilon \|\nabla v\|^2 + M_4\|v\|^2,$$

where $\epsilon > 0$ can be taken arbitrary small, we finally obtain

$$Re(L_\tau v, v) \le (M_5 - \rho)\|v\|^2.$$

If $\rho \ge M_5 + 1$ and $L_\tau v = 0$, then $v = 0$. The lemma is proved.

Remark. Consider the operator L given by (2.1) which acts from E_1 to E_2, where E_1 and E_2 are the weighted Hölder spaces defined in the Introduction. It is easy to see that if Condition 2 is satisfied, then this operator is Fredholm. Indeed, let T be the operator of multiplication by the function μ. Then TLT^{-1} acts from $C^{2+\delta}(\bar{\Omega})$ to $C^{\delta}(\bar{\Omega})$ and direct calculations show that Condition 2 is satisfied for this operator.

We now prove that Condition 1 (2) is equivalent to Condition $1^{'}$ $(2^{'})$ and then show that Condition 1 is a necessary condition for the operator to be Fredholm. We consider the problem

$$a(x^{'})\Delta u + \sum_{j=1}^{m} b_j(x^{'})\ \frac{\partial u}{\partial x_j} + c(x^{'})u = 0, \tag{2.6}$$

$$\alpha\frac{\partial u}{\partial \nu} + \beta(x^{'})u \mid_{\partial\Omega} = 0. \tag{2.7}$$

We suppose that the matrices a, b_j, and c belong to $C^{\delta}(\bar{G})$ and β to $C^{1+\delta}(\bar{G})$. If we consider $u(x)$ in the form

$$u(x) = e^{\lambda x_1} v(x^{'}), \tag{2.8}$$

we obtain

$$a(x^{'})\Delta^{'} v + \sum_{j=2}^{m} b_j(x^{'}) \frac{\partial v}{\partial x_j} + (a(x^{'})\lambda^2 + b_1(x^{'})\lambda + c(x^{'}))v = 0, \tag{2.9}$$

$$\alpha \frac{\partial v}{\partial \nu} + \beta(x^{'})v \mid_{\partial G} = 0. \tag{2.10}$$

If the problem (2.9), (2.10) has a solution for some $\lambda = i\xi$ with a real ξ, then (2.8) is a bounded solution of (2.6), (2.7). The following theorem establishes the equivalence between existence of solutions of problems (2.6), (2.7) and (2.9), (2.10) for $\lambda = i\xi$. In L^2 spaces the Fourier transform may be used to study this connection (see [26]).

If we consider the problem (2.6), (2.7) in Hölder spaces, then we cannot apply the classical Fourier transform directly.

Theorem 2.2. *The problem* (2.6), (2.7) *has a nontrivial bounded solution if and only if the problem* (2.9), (2.10) *has a nontrivial solution for some* $\lambda = i\xi$.

Proof. Let $u(x)$ be a bounded solution of (2.6), (2.7). Consider the Laplace transform of the function $u(x)$ in the x_1 variable,

$$v(x^{'}) = \int_0^\infty e^{-px_1} u(x) dx_1.$$

Here $Re\ p > 0$. Then

$$\begin{aligned} a(x^{'})\Delta^{'} v + \sum_{j=2}^{m} b_j(x^{'}) \frac{\partial v}{\partial x_j} + (a(x^{'})p^2 + b_1(x^{'})p + c(x^{'}))v \\ = a(x^{'})(u(0,x^{'})p + u^{'}(0,x^{'})) + b_1(x^{'})u(0,x^{'}). \end{aligned} \tag{2.11}$$

Here $u^{'} = \partial u/\partial x_1$. We consider the operator

$$A_p v = a(x^{'})\Delta^{'} v + \sum_{j=2}^{m} b_j(x^{'}) \frac{\partial v}{\partial x_j} + (a(x^{'})p^2 + b_1(x^{'})p + c(x^{'}))v$$

acting from $C^{(2+\delta)}(\bar{G})$ with boundary conditions (2.10) to $C^{(\delta)}(\bar{G})$ and suppose that for any $p = i\xi,\ -\infty < \xi < \infty$ it does not have the eigenvalue zero. Let $p_0 = i\xi_0$. For each $p = p_0 + \delta$ with small nonnegative δ, the equation

$$A_p v = f_p, \tag{2.12}$$

where

$$f_p(x^{'}) = a(x^{'})(u(0,x^{'})p + u^{'}(0,x^{'})) + b_1(x^{'})u(0,x^{'}),$$

is uniquely solvable, and its solution depends continuously on p. Denote it by v_p. Then

$$v_p(x^{'}) = \int_0^\infty e^{-px_1}u(x)dx_1, \quad \delta > 0$$

and

$$\int_0^\infty e^{-px_1}u(x)dx_1 \to v_{i\xi_0}(x^{'}), \quad \delta \to 0. \tag{2.13}$$

Similarly we put

$$w(x^{'}) = \int_{-\infty}^0 e^{qx_1}u(x)dx_1, \qquad q = -i\xi_0 + \delta,$$

$$B_q w = a(x^{'})\Delta^{'} w + \sum_{j=2}^m b_j(x^{'})\,\frac{\partial w}{\partial x_j} + (a(x^{'})q^2 - b_1(x^{'})q + c(x^{'}))w,$$

$$g_q(x^{'}) = a(x^{'})(u(0,x^{'})q - u^{'}(0,x^{'})) - b_1(x^{'})u(0,x^{'}).$$

The equation

$$B_q w = g_q \tag{2.14}$$

has a unique solution for small δ,

$$w_q(x^{'}) = \int_{-\infty}^0 e^{qx_1}u(x)dx_1, \quad \delta > 0$$

and

$$\int_{-\infty}^0 e^{qx_1}u(x)dx_1 \to w_{-i\xi_0}(x^{'}), \quad \delta \to 0. \tag{2.15}$$

We note that $v_{i\xi_0}(x^{'}) = -w_{-i\xi_0}(x^{'})$. Then from (2.13), (2.15) we obtain

$$\int_{-\infty}^0 e^{(-i\xi_0+\delta)x_1}u(x)dx_1 + \int_0^\infty e^{(-i\xi_0-\delta)x_1}u(x)dx_1 \to 0\ , \quad \delta \to 0. \tag{2.16}$$

If we pass to the limit formally in the left hand side of (2.16), we obtain that the Fourier transform of the function $u(x)$ in the x_1 variable is zero since ξ_0 is arbitrary. It would contradict the assumption that $u(x)$ is a nontrivial solution of (2.6), (2.7) and prove that the operator $A_{i\xi}$ (or the same $B_{i\xi}$) has a zero eigenvalue for some ξ. We now justify this passage to the limit.

For some bounded ϕ, and $x^{'} \in G$

$$\int_{-\infty}^{+\infty} u(x)\phi(x_1)dx_1 \neq 0. \tag{2.17}$$

Then

$$\int_G dx^{'} \Big| \int_{-\infty}^{+\infty} u(x)\phi(x_1)dx_1 \Big| \neq 0. \tag{2.18}$$

Denote

$$u_\delta(x) = \begin{cases} u(x)e^{-\delta x_1}, & x_1 > 0 \\ u(x)e^{\delta x_1}, & x_1 < 0. \end{cases}$$

Then

$$2\pi \int_{-\infty}^{+\infty} u_\delta(x)\phi(x_1)dx_1 = \int_{-\infty}^{+\infty} \tilde{u}_\delta(\xi, x^{'})\psi(\xi)d\xi, \tag{2.19}$$

where

$$\tilde{u}_\delta(\xi, x^{'}) = 2\pi \int_{-\infty}^{+\infty} e^{-i\xi} u_\delta(x)dx_1,$$

and ψ is the Fourier transform of ϕ. Then

$$\int_G dx^{'} \Big| \int_{-\infty}^{+\infty} u_\delta(x)\phi(x_1)dx_1 \Big| = \frac{1}{2\pi} \int_G dx^{'} \Big| \int_{-\infty}^{+\infty} \tilde{u}_\delta(\xi, x^{'})\psi(\xi)d\xi \Big|. \tag{2.20}$$

The left hand side in (2.20) tends to the integral in (2.18) as $\delta \to 0$. We now show that the right hand side in (2.20) tends to zero. From (2.16) it follows that $\tilde{u}_\delta(\xi, x^{'}) \to 0$, $\delta \to 0$ for each fixed ξ. Moreover this convergence is uniform in ξ on every bounded interval because the solutions v_p and w_q of the equations (2.12) and (2.14) depend continuously on p and q, respectively. It remains to estimate the behavior of the function $\tilde{u}_\delta(\xi, x^{'})$ for large ξ.

We have $\tilde{u}_\delta(\xi, x^{'}) = 2\pi(v_p(x^{'}) + w_q(x^{'}))$. So we should estimate the solutions v_p and w_q, $p = i\xi + \delta$, $q = -i\xi + \delta$ for large ξ. Substituting $p = i\xi + \delta$ in (2.11), we obtain

$$\begin{aligned} \Delta^{'} v + \sum_{j=2}^{m} a(x^{'})^{-1} b_j(x^{'}) \frac{\partial v}{\partial x_j} - \xi^2 v + B(x^{'}, \xi)v \\ = (u(0, x^{'})(i\xi + \delta) + u^{'}(0, x^{'})) + a(x^{'})^{-1} b_1(x^{'}) u(0, x^{'}), \end{aligned} \tag{2.21}$$

where

$$B(x',\xi) = (2i\xi\delta + \delta^2) + a(x')^{-1}b_1(x')(i\xi + \delta) + a(x')^{-1}c(x').$$

Consider the problem

$$\Delta' v + \sum_{j=2}^{m} a(x')^{-1}b_j(x')\ \frac{\partial v}{\partial x_j} - \xi^2 v = f(x),\ \ \alpha\frac{\partial v}{\partial \nu} + \beta(x')v\ |_{\partial G} = 0. \quad (2.22)$$

Its solution satisfies the estimate

$$\|v(x')\|_{L^2(G)} \le K\frac{\|f(x')\|_{L^2(G)}}{\xi^2} \quad (2.23)$$

for some constant K (for large ξ) because the operator which corresponds to the problem (2.22), is sectorial in L^2. Then for the solution of (2.21)

$$\|v_p(x')\|_{L^2(G)} \le K\left(\frac{\|f_p(x')\|_{L^2(G)}}{\xi^2} + \frac{\|B(\xi,x')v\|_{L^2(G)}}{\xi^2}\right)$$

uniformly in δ, $0 \le \delta \le \delta_0$. Since the norm of the matrix $B(\xi,x')/\xi^2$ becomes less than 1 for large ξ, then

$$\|v_p(x')\|_{L^2(G)} \le \frac{c}{|\ \xi\ |}$$

for some positive c. We obtain a similar estimate for w_p. Hence we have, uniformly in δ

$$\|\tilde{u}_\delta(\xi,x')\|_{L^2(G)} \le \frac{c}{|\ \xi\ |}.$$

Here we use the same notation c for different constants. We have thus

$$\int_G dx' \Big| \int_{-\infty}^{+\infty} \tilde{u}_\delta(\xi,x')\psi(\xi)d\xi \Big| \le \int_G dx' \Big| \int_{|\xi \le N} \tilde{u}_\delta(\xi,x')\psi(\xi)d\xi \Big|$$
$$+ \int_G dx' \Big| \int_{|\xi| \ge N} \tilde{u}_\delta(\xi,x')\psi(\xi)d\xi \Big|.$$

It remains to note that the second integral in the right hand side of the last inequality tends to zero uniformly in δ as N increases, the first integral goes to zero as $\delta \to 0$ for any given N. It proves that the right hand side of

(2.20) goes to zero and contradicts (2.18). This contradiction shows that the operator $A_{i\xi}$ has a zero eigenvalue for some ξ. Thus, we have proved necessity. Sufficiency is obvious, and this completes the proof of the theorem.

In the next theorem we assume that the coefficients of the operator L converge to their limiting values at infinity in the Hölder norm:

$$\lim_{N_\pm \to \pm\infty} \|a(x) - a^+(x')\|^{(\delta)}_{\Omega_{N_\pm}} = 0.$$

Here $\Omega_{N_\pm} = \{x_1 \geq N_+ (x_1 \leq N_-)\} \cap \Omega$. A similar convergence is supposed to hold true for the coefficients b_i and c.

Theorem 2.3. *If the operator L is normally solvable and its kernel is finite dimensional, then Condition 1 is satisfied.*

Proof. Suppose that Condition 1 is not satisfied, i.e. there exists a bounded nontrivial solution to one of the two problems

$$L^+ u = 0, \quad \Lambda u = 0, \qquad L^- u = 0, \quad \Lambda u = 0.$$

We assume for simplicity that it is the first problem. Then from the previous theorem it follows that the same problem has for some real ξ a solution of the form $u(x) = e^{i\xi x_1}\phi(x^{'})$, where $\phi(x^{'})$ is a solution of the corresponding problem in the section of the cylinder. We introduce a partition of unity with sufficiently smooth functions $\psi^-(x_1)$, $\psi_N^0(x_1)$, and $\psi_N^+(x_1)$ such that

$$\psi^-(x_1) + \psi_N^0(x_1) + \psi_N^+(x_1) = 1,$$
$$\psi^-(x_1) = \begin{cases} 1 & \text{if } x_1 < -1 \\ 0 & \text{if } x_1 > 0, \end{cases}$$
$$\psi_N^0(x_1) = \begin{cases} 1 & \text{if } 0 < x_1 < N \\ 0 & \text{if } x_1 < -1, x_1 > N+1, \end{cases}$$
$$\psi_N^+(x_1) = \begin{cases} 1 & \text{if } x_1 > N+1 \\ 0 & \text{if } x_1 < N, \end{cases}$$

Consider the sequence of functions $u_n(x) = e^{i(\xi+\epsilon_n)x_1}\phi(x^{'})$, $v_n(x) = (1 - \psi^-(x_1))(u_n(x) - u(x))$, where $\epsilon_n \to 0$ as $n \to \infty$. Denote $f_n = Lv_n$ and show that $f_n \to 0$ in $C^{(\delta)}(\bar{\Omega})$. We have

$$f_n = (\psi^-(x_1) + \psi_N^0(x_1) + \psi_N^+(x_1)) L(\psi_N^0(x_1) + \psi_N^+(x_1))(u_n(x) - u(x)). \quad (2.24)$$

For any given $\sigma > 0$ we should choose n_0 such that $\|f_n\|^{(\delta)} \le \sigma$ for $n \ge n_0$.

We note that $u_n(x) \to u(x)$ in $C^{(2+\delta)}$ on every bounded subset of Ω. Hence

$$\psi^-(x_1)\, L\, (\psi_N^0(x_1) + \psi_N^+(x_1))(u_n(x) - u(x)) \to 0$$

independently of N and this term is less than $\sigma/3$ for n sufficiently large. We have further

$$\begin{aligned}\psi_N^+(x_1)L\psi_N^+(x_1)(u_n(x) - u(x)) &= \psi_N^+(x_1)L^+\psi_N^+(x_1)(u_n(x) - u(x)) \\ &\quad + \psi_N^+(x_1)(L - L^+)\psi_N^+(x_1)(u_n(x) - u(x)).\end{aligned} \tag{2.25}$$

We estimate the second summand in the right hand-side of the last equality. Since the coefficients of the operator L converge to their limits at infinity in the Hölder norm,

$$\|\psi_N^+(x_1)((a, b_i, c(x)) - (a^+, b_i^+, c^+(x^{'})))\|^{(\delta)} \to 0, \quad N \to \infty,$$

then the norm of the operator $\psi_N^+(x_1)(L - L^+)$ tends to zero as N increases. Since

$$\|u\|^{(2+\delta)} \le M, \quad \|u_n\|^{(2+\delta)} \le M$$

for some positive M, we can choose N sufficiently large such that

$$\|\psi_N^+(x_1)(L - L^+)\psi_N^+(x_1))(u_n(x) - u(x))\|^{(\delta)} \le \frac{\sigma}{3}.$$

We suppose now that N is fixed and estimate the remaining terms. The estimate for the first term in the right hand side of (2.25) follows from the equalities

$$\begin{aligned}L^+(u_n(x) - u(x)) &= L^+u_n(x) \\ &= (L^+u(x))e^{i\epsilon_n x_1} + \epsilon_n(-(\epsilon_n + 2\xi)a^+ + ib_1^+)e^{i(\xi+\epsilon_n)x_1}\phi(x^{'}) \\ &= \epsilon_n(-(\epsilon_n + 2\xi)a^+ + ib_1^+)e^{i(\xi+\epsilon_n)x_1}\phi(x^{'}).\end{aligned}$$

The terms $\psi_N^0(x_1)L(\psi_N^0(x_1) + \psi_N^+(x_1))(u_n(x) - u(x))$ and $\psi_N^+(x_1)L\psi_N^0(x_1)$ $\times(u_n(x) - u(x))$ tend to zero as n increases by virtue of the convergence $u_n \to u$ and their sum becomes less than $\sigma/3$ for large n.

Thus we have shown that f_n converges to zero. If the operator L is normally solvable with a finite dimensional kernel, then it is proper on every bounded set in E. Hence the inverse image of the sequence $\{f_n\}$ should be compact. To obtain a contradiction it remains to show that the sequence $\{w_n = u_n - u\}$ is not compact. Suppose that we can choose a converging subsequence from the sequence $\{w_n\}$ and denote its limit by w_0. By virtue of the uniform convergence $u_n \to u$ on every bounded subset of Ω, the limiting function w_0 is identically zero. It is clear however that for every positive ϵ

$$\sup_{x_1} \mid e^{i\xi x_1} - e^{i(\xi+\epsilon)x_1} \mid = 2.$$

Thus, the convergence $w_n \to 0$ is not uniform in Ω. The theorem is proved.

3. Properness. Consider the semilinear operator $A : C^{2+\delta}(\bar{\Omega}) \to C^{\delta}(\bar{\Omega})$ depending on parameter $\tau \in [0, 1]$

$$A(u,\tau) \equiv a(x,\tau)\Delta u + \sum_{k=1}^{n} b_k(x,\tau)\frac{\partial u}{\partial x_k} + F(x,u,\tau) \tag{3.1}$$

subject to the boundary conditions

$$\alpha\frac{\partial u}{\partial \nu} + \beta(x^{'})u = 0, \quad \text{for} \quad x' \in \partial\Omega', x_1 \in R \tag{3.2}$$

$$\lim_{x_1\to\pm\infty} u(x) = 0 \quad \text{for} \quad x' \in \Omega'. \tag{3.3}$$

We suppose that the coefficients of the operator and the nonlinearity satisfy the following conditions:

1. The functions $a(x,\tau)$, $b_i(x,\tau)$, $i = 1, ..., n$ belong to $C^{(\delta)}(\bar{\Omega})$,

$$F_u^{'}(x,u,\tau) \in C^{(\delta)}(\bar{\Omega}, |u| \le R)$$

for each τ and R,

2. There exist limiting functions $a_\pm(x^{'},\tau)$, $b_{i\pm}(x^{'},\tau)$, $c_\pm(x^{'},\tau)$ such that

$$a(x,\tau) \to a_\pm(x^{'},\tau), \ \ b_i(x,\tau) \to b_{i\pm}(x^{'},\tau), \ \ x_1 \to \pm\infty$$
$$F_u^{'}(x,u,\tau) \to c_\pm(x^{'},\tau), \ \ x_1 \to \pm\infty, \ u \to 0,$$

all limits being uniform in $x^{'}$ for each τ,

3. The following convergence hold true:

$$\|a(x,\tau) - a(x,\tau_0)\|^{(\delta)} \to 0, \ \ \|b_i(x,\tau) - b_i(x,\tau_0)\|^{(\delta)} \to 0,$$
$$\|(F(x,u,\tau) - F(x,u,\tau_0))\mu\|_{C^{(\delta)}(\bar{\Omega},|u|\le R)} \to 0$$

as $\tau \to \tau_0$ for any R.

Here $\mu = \mu(x_1)$ is a weight function as in Section 1. We recall that E_1 is the space of functions from $C^{2+\delta}_\mu(\bar\Omega)$ satisfying the boundary conditions (3.2), (3.3). We also consider the space $E_1^{'} = E_1 \times [0,1]$ and the operator $A(u,\tau)$ acting from $E_1^{'}$ to $E_2 = C^\delta_\mu(\bar\Omega)$.

Theorem 3.1. *Let Condition* 1 *be satisfied for each* $\tau \in [0,1]$. *Then the restriction of* $A(u,\tau)$ *to any bounded set of* $E_1^{'}$ *is a proper operator.*

Proof. Let B be a ball of given radius R (centered at 0) in $E_1^{'}$, D be a given compact set in E_2, and (u_n, τ_n) a sequence in the set $G = A^{-1}(D) \cap B$. We should show that we can choose a converging subsequence. Without loss of generality we can assume that $\tau_n \to \tau_0$ as n increases.

We first show that the sequence $\{u_n\}$ has a cluster point in the space $C^2(\bar\Omega)$. Since G is bounded in E_1, then $\sup_x |u_n(x)\mu(x)| \le K$, where K is a positive constant independent of n, or

$$| \; u_n(x) \; |\le \frac{K}{\mu(x)}, \; n = 1, 2, ... \tag{3.4}$$

A similar estimate holds true for the first and second derivatives of the functions u_n. For any bounded subset Ω^* in $\bar\Omega$, we can choose a subsequence $\{u_{nk}\}$ converging to some limiting function u_0 in $C^2(\Omega^*)$. Passing to a subsequence, we can extend the function u_0 to all of $\bar\Omega$. By virtue of (3.4) and the similar inequalities for the derivatives, this convergence is uniform in $C^2(\bar\Omega)$. We will show below that this convergence also occurs in E_1.

Clearly we have $u_0 \in E_1$; we are now going to show that the last convergence actually takes place in the space $C^{2+\delta}_\mu(\bar\Omega)$. Consider the relation:

$$A(u_n, \tau_n) = f_n \in D. \tag{3.5}$$

Since D is compact without loss of generality we may assume that:

$$f_n \to f_0 \quad \text{in} \quad C^\delta_\mu(\bar\Omega) \tag{3.6}$$

and by the same arguments as above $u_n \to u_0$ in $C^2(\bar\Omega)$. We can pass to the limit in the equation $A(u_n, \tau_n)(x) = f_n(x)$ for any fixed $x \in \Omega$; thus we obtain

$$A(u_0, \tau_0) = f_0. \tag{3.7}$$

We now obtain an equation for the function $\mu(u_n - u_0)$. Let us set

$$v_n = u_n\mu, \; v_0 = u_0\mu, \; g_n = f_n\mu, \; g_0 = f_0\mu, \; w_n = v_n - v_0.$$

We multiply equations (3.5) and (3.7) by μ and subtract one from another. We obtain

$$\begin{aligned} &a(x,\tau_0)\Delta w_n + (b_1(x,\tau_0) - 2\mu_1 a(x,\tau_0))\frac{\partial w_n}{\partial x_1} + \sum_{k=2}^{m} b_k(x,\tau_0)\frac{\partial w_n}{\partial x_k} \\ &\quad + [(2\mu_1^2 - \mu_2)a(x,\tau_0) - \mu_1 b_1(x,\tau_0) + B_n(x)]w_n = h_n - h_0, \\ &\alpha\frac{\partial w_n}{\partial \nu} + \beta(x^{'})w_n = 0, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} B_n(x) &= \int_0^1 F'(x, u_0(x) + t(u_n(x) - u_0(x)), \tau_0)\, dt, \\ h_n - h_0 =& g_n - g_0 + \sum_{k=2}^{m}(b_k(x,\tau_0) - b_k(x,\tau_n))\frac{\partial v_n}{\partial x_k} \\ &- (F(x,u_n,\tau_n) - F(x,u_n,\tau_0))\mu - (b_1(x,\tau_n) - b_1(x,\tau_0))\frac{\partial v_n}{\partial x_1} \\ &+ 2\mu_1(a(x,\tau_n) - a(x,\tau_0))\frac{\partial v_n}{\partial x_1} - (a(x,\tau_n) - a(x,\tau_0))\Delta v_n \\ &- (2\mu_1^2 - \mu_2)(a(x,\tau_n) - a(x,\tau_0))v_n + (b_1(x,\tau_n) - b_1(x,\tau_0))v_n\mu_1. \end{aligned}$$

Suppose that $h_n \to h_0$ in $C^{(\delta)}(\bar{\Omega})$. We show that $w_n \to 0$ in $C^{(2+\delta)}(\bar{\Omega})$.

Since $u_0, u_n \in E_1$ we see that for fixed n, the integrand in the definition of $B_n(x)$ converges to $c_\pm(x^{'},\tau_0)$ as $x_1 \to \pm\infty$, and in view of the inequality

$$| \, u_n(x), u_0(x) \, | \leq \frac{K}{\mu(x_1)},$$

we obtain

$$B_n(x) \to c_\pm(x^{'},\tau_0) \;\; as \;\; x_1 \to \pm\infty, \tag{3.9}$$

this convergence being uniform in n and $x^{'}$.

From the convergence $u_n \to u_0$ in $C^2(\bar{\Omega})$ we obtain that

$$w_n \to 0 \quad \text{in} \quad C^2(\bar{\Omega}^*), \tag{3.10}$$

for any bounded subset Ω^* of Ω. Let us now assume that w_n does not converge to 0 in $C(\bar{\Omega})$.

In what follows we denote subsequences of the sequence $\{w_n\}$ with the same subscript.

There is a sequence $x^{(n)}$ of points of Ω such that

$$|w_n(x^{(n)})| \geq \delta > 0, \quad n = 1, 2, ... \tag{3.11}$$

It is clear in view of (3.10) that the sequence of their first coordinates $\{x_1^{(n)}\}$ cannot be bounded. Without loss of generality we can assume that the whole sequence $\{x_1^{(n)}\}$ converges to $+\infty$. Consider the sequence of functions $\tilde{w}_n$ defined by $\tilde{w}_n(x_1, x') = w_n(x_1 + x_1^{(n)}, x')$. The sequence $\{\tilde{w}_n\}$ is bounded in $C^{2+\delta}(\bar{\Omega})$, and by the same argument as in the previous step we can obtain a subsequence converging to some limiting function $\tilde{w}_0$ (defined on $\bar{\Omega}$) in $C^2(\bar{\Omega}^*)$, for any bounded subset Ω^*. Note that in view of (3.11), $\tilde{w}_0$ is clearly non zero. Using (3.8) at $(x_1 + x_1^{(n)}, x')$ for a given x, we obtain:

$$\begin{aligned} &a(x_1 + x_1^{(n)}, x', \tau_0)\Delta\tilde{w}_n(x) + (b_1(x_1 + x_1^{(n)}, x', \tau_0) \\ &\quad - 2a(x_1 + x_1^{(n)}, x', \tau_0)\mu_1(x_1 + x_1^{(n)}))\frac{\partial \tilde{w}_n}{\partial x_1}(x) \\ &\quad + \sum_{k=2}^{m} b_k(x_1 + x_1^{(n)}, x', \tau_0)\frac{\partial \tilde{w}_n}{\partial x_k}(x) + \Lambda_n(x)\tilde{w}_n(x) \\ &\quad = h_n(x_1 + x_1^{(n)}, x') - h_0(x_1 + x_1^{(n)}, x'), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} \Lambda_n(x) =& (2\mu_1^2(x_1 + x_1^{(n)}) - \mu_2(x_1 + x_1^{(n)}))a(x_1 + x_1^{(n)}, x', \tau_0) \\ & - b_1(x_1 + x_1^{(n)}, x', \tau_0)\mu_1(x_1 + x_1^{(n)}) + B_n(x_1 + x_1^{(n)}). \end{aligned}$$

Let us now take the limit $n \to \infty$ in (3.12). Since the right hand side of (3.12) goes to zero and $\Lambda_n(x) \to c_+(x', \tau)$, thus, we obtain that $\tilde{w}_0$ is a nontrivial solution of the limiting equation, which contradicts Condition 1. Thus we have showed that:

$$w_n \to 0 \quad \text{in} \quad C(\bar{\Omega}). \tag{3.13}$$

To prove the convergence in $C^{2+\delta}(\bar{\Omega})$, we use the estimate (2.3). We rewrite (3.8) in the form

$$a\Delta w_n = -(b_1 - 2\mu_1 a)\frac{\partial w_n}{\partial x_1} - \sum_{k=2}^{n} b_k \frac{\partial w_n}{\partial x_k} + p_n(x), \tag{3.14}$$

where

$$p_n(x) = -[(2\mu_1^2 - \mu_2)a - \mu_1 b_1 + B_n(x)]w_n + (h_n - h_0).$$

Applying (2.3), we obtain

$$\|w_n\|^{(2+\delta)} \le K(\|p_n\|^{(\delta)} + \|w_n\|^{(1+\delta)}).$$

We note that the sequence $\{w_n\}$ is bounded in $C^{2+\delta}(\bar{\Omega})$. This together with (3.13) implies that:

$$w_n \to 0 \quad \text{in} \quad C^2(\bar{\Omega}). \tag{3.15}$$

It remains to check that $\|p_n\|^{(\delta)} \to 0$, but this is an immediate consequence of (3.15), of the conditions on the function F and the fact that $h_n \to h_0$ in $C^\delta(\bar{\Omega})$. To prove this, we put

$$h_n - h_0 = g_n - g_0 + H_1 + H_2,$$

where $H_1 = -(F(x, u_n, \tau_n) - F(x, u_n, \tau_0))\mu$, and H_2 contains all other terms. By construction $g_n \to g_0$ in $C^\delta(\bar{\Omega})$. H_1 converges to zero by virtue of the condition on F. All terms in H_2 have the form $y_n(x)z_n(x)$, where $y_n \to 0$ and $z_n(x)$ is uniformly bounded in $C^\delta(\bar{\Omega})$. Hence the product of these functions converges to zero in this norm. The theorem is proved.

4. Existence and uniqueness of topological degree. We begin with the uniqueness of the degree and consider a more general class of operators than was defined in the Introduction. Then the uniqueness of the degree will be used to show its existence.

4.1. Uniqueness of the degree. Here we present results on uniqueness of the topological degree for a class of Fredholm operators. For some classes of operators in Banach spaces, uniqueness of the degree was proved in [1].

Let Ψ be a set of linear Fredholm operators $L : E_1 \to E_2$. We suppose that Ψ is connected. This means that for any two operators $L_0 \in \Psi$ and $L_1 \in \Psi$ there exists a homotopy $L_\tau : E_1 \times [0,1] \to E_2$ which connects them. We suppose that L_τ is continuous with respect to τ in the operator norm. We also assume that the class Ψ is complete with respect to finite dimensional linear operators, i.e. for any $L \in \Psi$ we have $L + K \in \Psi$, where K is an arbitrary finite dimensional operator from E_1 to E_2. We consider the following classes of nonlinear operators and homotopies.

Class Φ is the set of all proper operators $A(u)$ acting from E_0 to E_2 which are continuous, have Frechet derivative $A^{'}(u)$ at any point $u \in E_0$ and $A^{'}(u) \in \Psi$. Here E_0 is a given open bounded set in E_1.

Class H is the set of all proper operators $A_\tau(u) : E_0 \times [0,1] \to E_2$ which are continuous with respect to u and τ, and belong to Φ for any $\tau \in [0,1]$.

Theorem 4.1. *For the classes* Φ *and* H *and a given normalization operator* J *the topological degree satisfying conditions* (i)–(iii) *is unique.*

We begin with some auxiliary results. Denote Ψ_0 the set of all invertible operators $A \in \Psi$. We use also the following notations. E_1^* is a space dual to E_1, $< u, \phi >$ is the value of the functional $\phi \in E_1^*$ at the element $u \in E_1$.

As usual, the index of an isolated stationary point $u_0 \in E_1$ is defined as the degree on a small ball centered at this point, $ind(A, u_0) = \gamma(A, B)$, where $B = \{u : \|u - u_0\|_1 < r\}$ for r sufficiently small. Here $\| \cdot \|_1$ is the norm in E_1.

Lemma 4.1. *Let* $A \in \Psi_0,\ A - K \in \Psi_0,$ *where*

$$Ku =< u, \phi > e, \ \ u \in E_1, \ \phi \in E_1^*, \ e \in E_1.$$

Let $f(u) = A(u - u_0), \ \tilde{f}(u) = (A - K)(u - u_0),$ *where* $u_0 \in D \subset E_1$. *Then* $< A^{-1}e, \phi >\neq 1$ *and for any topological degree* γ

$$\gamma(\tilde{f}, D) = \gamma(f, D) \quad \text{if} \ \ < A^{-1}e, \phi > < 1, \tag{4.1}$$

$$\gamma(\tilde{f}, D) = -\gamma(f, D) \ \ \text{if} \ \ < A^{-1}e, \phi > > 1. \tag{4.2}$$

Proof. Consider first the case $< A^{-1}e, \phi >= 0$. Let

$$f_\tau(u) = A(u - u_0) - \tau < u - u_0, \phi > e, \ \ 0 \le \tau \le 1.$$

For all $\tau \in [0, 1]$ the equation $f_\tau(u) = 0$ has only one solution $u = u_0$. So $\gamma(f_\tau, D)$ does not depend on τ and consequently $\gamma(f, D) = \gamma(\tilde{f}, D)$. (4.1) is proved.

Suppose now that $< A^{-1}e, \phi >\neq 0$. Define $\beta =< A^{-1}e, \phi >, \ \psi = \phi/\beta$. Then $< A^{-1}e, \psi >= 1$ and $< u - u_0, \phi >= \beta < u - u_0, \psi >$. So $\tilde{f}(u) = A(u-u_0) - \beta < u-u_0, \psi > e$. Consider the function $g(u) = A(u-u_0) - \hat{\mu}(< u - u_0, \psi >)e$, where $\hat{\mu}(\xi)$ is a smooth function of the real variable ξ. It is easy to see that u is a solution of the equation

$$g(u) = 0 \tag{4.3}$$

if and only if $c = \hat{\mu}(< u - u_0, \psi >)$ is a solution of the equation

$$c = \hat{\mu}(c). \tag{4.4}$$

For any solution c of (4.4), $u = u_0 + cA^{-1}e$, is a solution of (4.3). Suppose that $\hat{\mu}(\xi) = \xi^2 + \xi - \alpha$, where α is a real number. Then equation (4.4) has the form $c^2 = \alpha$. If $\alpha > 0$, then equation (4.3) has two solutions $u = u_0 \pm \sqrt{\alpha}A^{-1}e$. Denote by B the ball $\|u - u_0\|_1 < r$, where $r = \sqrt{\alpha}\|A^{-1}e\|_1 + \delta$, $\delta > 0$ is a given number. Then

$$\gamma(g, B) = 0. \tag{4.5}$$

We suppose that α and δ are so small that $B \subset D$. To prove (4.5) consider the function

$$\hat{\mu}_\tau(\xi) = \xi^2 + \xi + (2\tau - 1)\alpha, \quad \tau \in [0, 1].$$

Define

$$g_\tau(u) = A(u - u_0) - \hat{\mu}_\tau(< u - u_0, \psi >)e.$$

The equation $g_\tau(u) = 0$ has the solution

$$u = u_0 \pm \sqrt{(1 - 2\tau)\alpha}A^{-1}e \tag{4.6}$$

for $0 \le \tau \le 1/2$, and does not have any solution for $1/2 < \tau \le 1$. For the solutions (4.6) we have $\|u - u_0\|_1 < r$. Therefore, $\gamma(g_\tau, D)$ does not depend on τ, $\gamma(g, B) = \gamma(g_1, B) = 0$ and (4.5) is proved. Let $u_\pm = u_0 \pm \sqrt{\alpha}A^{-1}e$. From (4.5)

$$ind(g, u_+) = -ind(g, u_-). \tag{4.7}$$

For the derivative $g^{'}(u)$ of g we have

$$g^{'}(u_\pm)u = Au - (1 \pm 2\sqrt{\alpha}) < u, \psi > e.$$

Define

$$\hat{f}_\sigma(u) = Au - \sigma < u, \psi > e.$$

Let T be the translation operator, $T(y)\hat{f}_\sigma(x) = \hat{f}_\sigma(x - y)$. Then we have for the ball $B_+ = \{u : \|u - u_+\|_1 < \rho\}$, where ρ is sufficiently small

$$\begin{aligned} ind(g, u_+) &= \gamma(g, B_+) = \gamma(T(u_+)\hat{f}_{2\sqrt{\alpha}+1}, B_+) \\ &= \gamma(T(u_+)\hat{f}_{2\sqrt{\alpha}+1}, D) = \gamma(T(u_0)\hat{f}_{2\sqrt{\alpha}+1}, D) = \gamma(\tilde{f}, D) \end{aligned} \tag{4.8}$$

if $\beta > 1$. Similarly,

$$ind(g, u_-) = \gamma(T(u_0)\hat{f}_{-2\sqrt{\alpha}+1}, D) = \gamma(f, D). \tag{4.9}$$

We have used the fact that $\gamma(T(u_0)\hat{f}_\sigma, D)$ does not depend on σ for all $\sigma > 1$ and for all $\sigma < 1$. From (4.8), (4.9), and (4.7) we obtain (4.2). The equality (4.1) follows from the fact that $\gamma(T(u_0)\hat{f}_\sigma, D)$ does not depend on σ for all $\sigma < 1$. The lemma is proved.

Lemma 4.2. *Let $A \in \Psi_0,\ A - K \in \Psi_0$, where K is a finite dimensional operator acting from E_1 to E_2. Define*

$$f(u) = A(u - u_0), \ \ \tilde{f}(u) = (A - K)(u - u_0),$$

where $u_0 \in D \subset E_1$. Then

$$\gamma(\tilde{f}, D) = \pm\gamma(f, D), \tag{4.10}$$

where the sign + or - does not depend on choice of the topological degree γ.

Proof. If K is one-dimensional, the assertion of the lemma follows from the previous lemma. We can use induction with respect to the dimension n of K. If K is n-dimensional, we can represent it in the form $K = K_0 + K_1$, where K_0 is $(n-1)$-dimensional and

$$K_1 u =< u, \phi > e, \ \ u \in E_1, \ \phi \in E_1^*, \ e \in E_1.$$

Consider the operator $A_\rho = A - \rho K_0 - K_1$, where ρ is a real number. The operator $A - \rho K_0$ is invertible for all $\rho \in (1-\delta, 1+\delta)$ if δ is positive and sufficiently small, with the possible exception $\rho = 1$. Indeed, $A - \rho K_0 = \rho A(\frac{1}{\rho}I - A^{-1}K_0)$ and the operator $\frac{1}{\rho}I - A^{-1}K_0$ is not invertible only for a discrete set of ρ.

Let $f_\rho(u) = A_\rho(u - u_0),\ \rho \in (1-\delta, 1+\delta)$. Obviously $f_1 = \tilde{f}$ and if δ is sufficiently small, we have

$$\gamma(f_\rho, D) = \gamma(\tilde{f}, D). \tag{4.11}$$

If we choose ρ such that $A - \rho K_0$ is invertible, then Lemma 4.1 implies

$$\gamma(f_\rho, D) = \epsilon\gamma(g_\rho, D), \tag{4.12}$$

where $g_\rho(u) = (A - \rho K_0)(u - u_0)$ and $\epsilon = \pm 1$ does not depend on the choice of γ. By the induction assumption

$$\gamma(g_\rho, D) = \epsilon_1\gamma(f, D), \tag{4.13}$$

where $\epsilon_1 = \pm 1$ does not depend on the choice of γ. From (4.11)–(4.13) follows (4.10). The lemma is proved.

Lemma 4.3. *Let $A \in \Psi_0$. Denote $f(u) = A(u - u_0)$, $u_0 \in E_0$. Then*

$$ind(f, u_0) = \pm 1, \tag{4.14}$$

where the sign + or - does not depend on the choice of the topological degree γ and the point u_0.

Proof. Since Ψ is a connected set, there exists an operator $A_\tau : E_1 \times [0,1] \to E_2$ such that $A_\tau \in \Psi$ for any $\tau \in [0,1]$, A_τ is continuous in τ in the operator norm, and $A_0 = J$, $A_1 = A$, where J is the normalization operator. For any $\tau_0 \in [0,1]$ there exists a finite dimensional operator $K_{\tau_0} : E_1 \to E_2$ such that $A_{\tau_0} + K_{\tau_0}$ is invertible. Consequently, there exists a neighborhood $\Delta(\tau_0)$ of the point τ_0 in $[0,1]$ such that the operator $A_\tau + K_{\tau_0}$ is invertible for $\tau \in \Delta(\tau_0)$. We can cover the interval $[0,1]$ with such neighborhoods and choose a finite subcovering. Define $f_{\tau_0,\tau} u = (A_\tau + K_{\tau_0})(u - u_0)$. Obviously $ind(f_{\tau_0,\tau}, u_0)$ does not depend on $\tau \in \Delta(\tau_0)$. If $\Delta(\tau_0)$ and $\Delta(\tau_1)$ have a common point $\bar{\tau}$, then $A_{\bar{\tau}} + K_{\tau_0}$ and $A_{\bar{\tau}} + K_{\tau_1}$ differ only by a finite dimensional operator. By Lemma 4.2

$$ind(f_{\tau_0,\bar{\tau}}, u_0) = \epsilon \; ind(f_{\tau_1,\bar{\tau}}, u_0),$$

where $\epsilon = \pm 1$ does not depend on the choice of γ. Therefore,

$$ind(f_{\tau_0,\tau_0}, u_0) = \epsilon \; ind(f_{\tau_1,\tau_1}, u_0).$$

It follows that

$$ind(A(u - u_0), u_0) = \epsilon \; ind(J(u - u_0), u_0) = \epsilon,$$

where $\epsilon = \pm 1$ does not depend on the choice of γ. The fact that (4.14) does not depend on u_0 follows from the construction. The lemma is proved.

Lemma 4.4. *If $A \in \Phi$ and u_0 is a regular point of A, $A(u_0) = 0$, then*

$$ind(A, u_0) = \pm 1,$$

where the sign + or - does not depend on the choice of the topological degree.

The proof is obvious.

Proof of the Theorem. Let $D \subset E$ be an open set, $A \in \Phi$,

$$A(u) \neq 0, \quad u \in \partial D. \tag{4.15}$$

Since A is a proper operator, we have

$$A(u) \neq a, \quad u \in \partial D \tag{4.16}$$

for $\|a\| < \epsilon$, if ϵ is sufficiently small. We can suppose that a is a regular value of A (see [22]). Then the equation

$$A(u) = a \tag{4.17}$$

has only finitely many solutions $u = u_k,\ k = 1, ..., n$ in D. Denote $\tilde{A}(u) = A(u) - a,\ A_\tau(u) = A(u) - \tau a,\ \tau \in [0,1]$. Then $A_\tau(u) \neq 0,\ u \in \partial D,\ \tau \in [0,1]$. Then

$$\gamma(A, D) = \gamma(\tilde{A}, D) = \sum_{k=1}^{n} ind(\tilde{A}, u_k)$$

does not depend on the choice of γ by virtue of Lemma 4.4. The theorem is proved.

4.2. Construction of the degree. Here we construct a topological degree for the class of elliptic operators introduced in Section 1. We suppose that the following assumptions are satisfied:

1. Operator $A(u)$ defined by (1.1) and acting from E_1 to E_2 has two Frechet derivatives. The functions $a^{\pm}(x^{'}),\ b_i{}^{\pm}(x^{'}), c^{\pm}(x^{'})$, and $\partial a^{\pm}(x^{'})/\partial x_i$, $i = 1, ..., m$ are continuous,

2. Operator $A_\tau(u)$ defined by (1.2) and acting from $E_1 \times [0,1]$ to E_2 has two Frechet derivatives with respect to u and τ. The functions $a^{\pm}(x^{'}, \tau)$, $b_i{}^{\pm}(x^{'}, \tau),\ c^{\pm}(x^{'}, \tau)$, and $\partial a^{\pm}(x^{'}, \tau)/\partial x_i,\ i = 1, ..., m$ are continuous in $x^{'}$ and τ,

3. Conditions of Sections 2 and 3 used to prove the Fredholm property and the properness are satisfied.

Denote

$$J_k u = \Delta u - ku, \quad k = 1, 2, ... \tag{4.18}$$

We consider J_k as an operator acting from E_1 to E_2 (weighted Hölder spaces, see Introduction). We denote Φ_k the set of all operators (1.1), $A(u) : E_1 \to E_2$ such that the operator

$$A(u) + \sigma J_k u \tag{4.19}$$

satisfies Condition 2 for all $\sigma \geq 0$.

Φ is a class of operators $A(u)$ of the form (1.1) satisfying Condition 2.

Lemma 4.5. *The following relations hold true:*

$$\Phi_k \subset \Phi_{k+1}, \quad k = 1, 2, ... \tag{4.20}$$

$$\Phi = \cup_{k=1}^{\infty} \Phi_k. \tag{4.21}$$

Proof. The inclusion (4.20) is obvious. To prove (4.21) we consider an operator $A \in \Phi$. Condition 2 for the operator $A(u) + \sigma J_k u$ has the form:

The problem

$$\hat{L}\tilde{v} \equiv \tilde{L}_{\xi}^{\pm}\tilde{v} + \sigma(-\xi^2\tilde{v} + \Delta^{'}\tilde{v} - k\tilde{v}) - \lambda\tilde{v} = 0, \quad x \in G \tag{4.22}$$

$$\alpha\frac{\partial\tilde{v}}{\partial\nu} + \beta(x^{'})\tilde{v} \ \ |_{\partial G} = 0 \tag{4.23}$$

has only the zero solution in $C^{(2+\delta)}(\bar{G})$ for any real ξ and $\lambda \geq 0$. We should prove that this condition is satisfied for some k and for any $\sigma \geq 0$.

Consider the operator

$$L_{\rho}u = \tilde{L}_{\xi}^{\pm}u + \sigma(-\xi^2 u + \Delta^{'}u) - \rho u.$$

Define

$$[u, v] = \int_G (u, v) dx^{'}, \quad \|u\|^2 = [u, u].$$

Then for any $u \in C^{(2+\delta)}(G)$ satisfying the condition

$$\Lambda u \equiv \alpha\frac{\partial u}{\partial\nu} + \beta(x^{'})u \ \ |_{\partial G} = 0 \tag{4.24}$$

we have the following estimate

$$Re\ [L_{\rho}u, u] \leq (\kappa + \sigma)(-\xi^2 + K|\xi| + M)\|u\|^2 - \rho\|u\|^2. \tag{4.25}$$

Here $\kappa > 0$ is the constant in the ellipticity condition $(a^{\pm}(x^{'})\eta, \eta) \geq \kappa\|\eta\|^2$, K, M are constants which do not depend on ξ, $\sigma \geq 0$, $\rho \geq 0$, and u.

Suppose that $0 \leq \sigma \leq 1$. Then there exists a number ρ_0 such that for all $\rho \geq \rho_0$ and ξ the inequality

$$Re\ [L_{\rho}u, u] \leq -\|u\|^2 \tag{4.26}$$

holds. Moreover, there exists a number $\xi_0 > 0$ such that for all $\rho \geq 0$, $|\xi| > \xi_0$ the inequality (4.26) holds. This means that for such values of the parameters the problem

$$L_{\rho}u = 0, \quad x \in G \tag{4.27}$$

with the boundary condition (4.24), has only the zero solution. Consider now ρ and ξ:

$$0 \le \rho \le \rho_0, \quad |\xi| \le \xi_0. \tag{4.28}$$

Since for $\sigma = 0$ and for all $\rho \ge 0$ the problem (4.27) has only the zero solution, then there exists $\sigma_0 > 0$ such that for $0 \le \sigma \le \sigma_0$ and for $\rho,\ \xi$ satisfying (4.28) the problem (4.27) has only the zero solution.

Therefore we have proved that for $0 \le \sigma \le \sigma_0$ and all $\xi,\ \lambda \ge 0,\ k \ge 0$ the problem (4.22), (4.23) has only the zero solution.

Let now $\sigma > \sigma_0$. The estimate (4.25) may be rewritten in the form

$$Re\ [\hat{L}u, u] \le (\kappa + \sigma)(-\xi^2 + K|\xi| + M - \frac{k\sigma + \lambda}{\kappa + \sigma})\|u\|^2. \tag{4.29}$$

Obviously, there exists k_0 such that for all $k \ge k_0,\ \lambda \ge 0,\ \xi,\ \sigma > \sigma_0$, we have,

$$Re\ [\hat{L}u, u] \le -(\kappa + \sigma)\|u\|^2,$$

thus the problem (4.22), (4.23) has only the zero solution. The lemma is proved.

We now construct the topological degree for the class Φ_k of operators A. We consider $k \ge r$, where r is number such that for $k \ge r$ the operator J_k (see (4.18) has a bounded inverse defined on all of E_2. Consider the set $\hat{\Phi}_k$ of operators of the form

$$\hat{A} = J_k^{-1} A, \tag{4.20}$$

where $A \in \Phi_k$. These are continuous operators acting in E_1. Moreover for all $\sigma \ge 0$ the operator $\hat{A} + \sigma I$ is Fredholm. Here I is the identity operator in E_1. This follows from the fact the operator (4.19) is Fredholm by virtue of the results of Section 2. From the results of Section 3 it follows that the restriction of $\hat{A}$ to any bounded subset of E_1 is a proper operator. We also consider the following class $\hat{H}_k$ of homotopies: $\hat{A}_\tau \in \hat{H}_k$ if $\hat{A}_\tau$: $E_0 \times [0, 1] \to E_1$ is a differentiable proper map which has two derivatives with respect to u and τ in $E_0 \times [0, 1]$ and for any fixed $\tau \in [0, 1]$ belongs to the class $\hat{\Phi}_k$. Hence for the classes $\hat{\Phi}_k$ and $\hat{H}_k$ of operators and homotopies, the topological degree with properties (i)–(iii) and normalization operator I may be constructed (see [9]).

Denote this topological degree by $\hat{\gamma}_k(\hat{A}, D)$.

We can now introduce a topological degree in the class Φ_k: for any $A \in \Phi_k$ and $D \subset E$ such that $A(u) \ne 0,\ u \in \partial D$ we put $\gamma_k(A, D) = \hat{\gamma}_k(J_k^{-1} A, D)$.

By this definition J_k is the normalization operator. Obviously, operator J_r may also be taken as normalization operator. Indeed, the linear homotopy $(1-\tau)J_r + \tau J_k$ belongs to H_k. By H_k we denote the class of homotopies $A_\tau(u) : E_1 \times [0,1] \to E_2$ such that $A_\tau(u)$ is a differentiable proper map which has two derivatives with respect to u and τ in $E_1 \times [0,1]$ and for any fixed $\tau \in [0,1]$ belongs to the class Φ_k.

Therefore we have constructed the topological degree γ_k in the class Φ_k of operators and the class H_k of homotopies, with normalization operator J_r. We now prove that the class Φ_k satisfies the conditions of the uniqueness theorem proved in Section 4.1. Denote Ψ_k the set of all linear operators of the form $A^{'}(u)$, where $A \in \Phi_k$ and $A^{'}$ is its Frechet derivative. Ψ_k is connected. Indeed, if $A \in \Phi_k$, then from the definition of Φ_k it follows that $(1-\tau)A + \tau J_k \in \Phi_k$ for all $\tau \in [0,1]$. We may therefore use the uniqueness theorem.

Consider now the class Φ. Let $A \in \Phi$. By (4.20), (4.21) $A \in \Phi_k$ for some $k \geq r$. For any $D \subset E_0$ such that $A(u) \neq 0, \quad u \in \partial D$ we denote $\gamma(A,D) = \gamma_k(A,D)$. By the uniqueness theorem it does not depend on k.

We now consider the class H of homotopies. By definition H is a set of operators $A_\tau(u)$ given by (1.2), such that $A_\tau(u) \in \Phi$ for any $\tau \in [0,1]$, $A_\tau(u) : E_0 \times [0,1] \to E_2$ is a proper map, and $A_\tau(u)$ has two derivatives with respect to $u \in E_1$ and $\tau \in [0,1]$. We shall prove the homotopy invariance of the degree γ introduced above. For any $\tau \in [0,1]$ we have $A_{\tau_0}(u) \in \Phi$ and so $A_{\tau_0}(u) \in \Phi_k$ for some $k = k(\tau_0)$. By virtue of (4.20) the value of k can be chosen sufficiently large.

We shall prove that $A_\tau \in \Phi_{k(\tau_0)}$ for all τ in some neighborhood of the point τ_0. We consider the operator

$$\begin{aligned}\tilde{L}^{\pm}_{\xi,\tau} v = & -\xi^2 a_{\pm}(x^{'},\tau)v + a_{\pm}(x^{'},\tau)\Delta^{'} v \\ & - i\xi b_{1\pm}(x^{'},\tau)v + \sum_{k=2}^{n} b_{k\pm}(x^{'},\tau)\frac{\partial v}{\partial x_k} + c_{\pm}(x^{'},\tau)v.\end{aligned}$$

By definition of Φ_k the operator $A_{\tau_0}(u) + \sigma J_k u$ satisfies Condition 2 for all $\sigma \geq 0$. This means that the problem

$$\begin{aligned}\hat{L}v \equiv \tilde{L}^{\pm}_{\xi,\tau} v + \sigma(-\xi^2 v + \Delta^{'} v - k(\tau_0)v) - \lambda v = 0, \\ \alpha \frac{\partial v}{\partial \nu} + \beta(x^{'})v|_{\partial G} = 0\end{aligned} \tag{4.31}$$

has only the zero solution in $C^{(2+\delta)}(\bar{G})$ for any real ξ, $\lambda \geq 0$, and $\tau = \tau_0$.

We prove that the same is true for all τ in some neighborhood Δ of τ_0.

Using the smoothness properties of the coefficients of the operator, we can obtain the estimate

$$Re\ [\hat{L}v, v] \leq (\kappa + \sigma)T\|v\|^2, \tag{4.32}$$

where $T = -\xi^2 + K|\xi| + M - \frac{k\sigma+\lambda}{\kappa+\sigma}$ and the estimate

$$Re\ [\bar{L}v, v] \leq S\|u\|^2, \qquad S = (\kappa + \sigma)(-\xi^2 + K|\xi| + M) - \rho$$

for the operator

$$\bar{L} = \tilde{L}^{\pm}_{\xi,\tau} v + \sigma(-\xi^2 u + \Delta' u) - \rho u.$$

Here the constants $\kappa,\ K,\ M$ do not depend on τ in the ϵ-neighborhood Δ of τ_0 for some $\epsilon > 0$.

Consider first the case $0 \leq \sigma \leq 1$. Then we can find $\xi_0 > 0,\ \rho_0 > 0$ such that for all $|\xi| \geq \xi_0$ or $\rho \geq \rho_0$ we have $S \leq -\kappa$. For these values of parameters the problem

$$\bar{L}v = 0, \quad \Lambda v = 0 \tag{4.33}$$

has only the zero solution. If $0 \leq \sigma \leq 1,\ |\xi| \leq \xi_0,\ \rho \leq \rho_0$, then for ϵ sufficiently small the problem (4.33) has only the zero solution. Thus for all $0 \leq \sigma \leq 1,\ \xi,\ \rho \geq 0$, and $\tau \in \Delta$ the problem (4.33) has only the zero solution. The same is true for all $\xi,\ k,\ \lambda,\ 0 \leq \sigma \leq 1,\ \tau \in \Delta$ for the problem (4.31).

If $\sigma > 1$, then we use the estimate (4.32) and choose k so large that $T \leq -1$ for all $\xi,\ \lambda \geq 0,\ \sigma > 1$. So in this case the problem (4.31) has only the zero solution.

We have proved that $A_\tau(u) \in \Phi_{k(\tau_0)}$ for $\tau \in \Delta$ and some k. The interval $0 \leq \tau \leq 1$ can be covered by a finite number of intervals such that for each of them, a corresponding value of k may be chosen as above. Let m be the maximum of all these k. Then $A_\tau \in \Phi_m$ for all $\tau \in [0, 1]$. So γ is a homotopy invariant.

We have constructed the topological degree γ for the class of operators Φ and homotopies H. This topological degree satisfies conditions (i)–(iii) with normalization operator J_r. From the results of Section 4.1 it follows that the topological degree in classes Φ and H satisfying conditions (i)–(iii) is unique.

We have proved the following theorem.

Theorem 4.2. *Suppose that* Φ *is a class of operators* $A(u)$ *defined by* (1.1) *and satisfying Condition* 2. *Let* H *be a class of homotopies* $A_\tau(u)$ *given by* (1.2). *Then there exists one and only one topological degree for classes* Φ *and* H *and normalization operator* J_r *which satisfies conditions* (i)–(iii).

5. Traveling waves. In this Section we apply the results obtained above to the operators arising in the investigation of traveling waves. We consider the parabolic system of equations

$$\frac{\partial u}{\partial t} = a(x^{'})\Delta u + \sum_{k=1}^{m} b_k(x^{'})\frac{\partial u}{\partial x_k} + F(x^{'}, u) \tag{5.1}$$

in the cylinder Ω with the boundary condition

$$\alpha\frac{\partial u}{\partial \nu} + \beta(x^{'})u \,|_{\partial\Omega} = 0. \tag{5.2}$$

A traveling wave solution of this problem is a solution of the form

$$u(x,t) = w(x_1 - ct, x^{'}).$$

Here c is the wave velocity, which is unknown. The function $w(x)$ is a solution of the problem

$$a(x^{'})\Delta w + (c + b_1(x^{'}))\frac{\partial w}{\partial x_1} + \sum_{k=2}^{m} b_k(x^{'})\frac{\partial w}{\partial x_k} + F(x^{'}, w) = 0 \tag{5.3}$$

$$\alpha\frac{\partial w}{\partial \nu} + \beta(x^{'})w \,|_{\partial\Omega} = 0. \tag{5.4}$$

We consider traveling waves having limits at infinity

$$\lim_{x_1\to\pm\infty} w(x) = w_\pm(x^{'}), \quad w_+(x^{'}) \not\equiv w_-(x^{'}),$$

where the functions $w_\pm(x^{'})$ satisfy the limiting problems

$$a(x^{'})\Delta w_\pm + \sum_{k=2}^{m} b_k(x^{'})\frac{\partial w_\pm}{\partial x_k} + F(x^{'}, w_\pm) = 0 \tag{5.5}$$

$$\alpha\frac{\partial w_\pm}{\partial \nu} + \beta(x^{'})w_\pm \,|_{\partial G} = 0 \tag{5.6}$$

in the section G of the cylinder.

We note that together with the function $w(x)$, the one-parameter family of functions $w_h(x) = w(x_1+h, x^{'})$ satisfies the problem (5.3), (5.4). This family of functions is not uniformly bounded in the weighted Hölder space and the topological degree cannot be defined. Moreover the linearized problem has a zero eigenvalue. To avoid this situation we introduce a functionalized of the parameter c [15], [26]. This means that instead of unknown constant c we consider a given functional $c(w)$ which satisfies the following properties:
1. $c(w)$ satisfies the Lipschitz condition on every bounded set in $C^{(2+\delta)}(\bar{\Omega})$, and has a Frechet derivative $c^{'}(w)$;
2. The function $\tilde{c}(h) = c(w_h(x))$ is a decreasing function of h, $\tilde{c}(-\infty) = \infty$, $\tilde{c}(\infty) = -\infty$. There are different ways to construct a functional satisfying these conditions. We take it in the form

$$c(w) = ln \int_{\Omega} |w(x) - w_+(x^{'})|^2 \sigma(x_1) dx,$$

where $\sigma(x_1)$ is an increasing function, $\sigma(-\infty) = 0$, $\sigma(+\infty) = 1$,

$$\int_{-\infty}^{0} \sigma(x_1) dx_1 < \infty.$$

The functionalized of the parameter moves the zero eigenvalue of the problem linearized about a traveling wave to the left half-plane, and it singles out one particular element of the family of solutions [26]. Moreover it eliminates the unknown constant from the problem.

To include the conditions at ∞ into the operator, we introduce a smooth function $\phi = \phi(x_1)$ such that $\phi(x_1) = 0$ if $x_1 \geq 1$, $\phi(x_1) = 1$ if $x_1 \leq -1$, and put $\psi(x, \tau) = (1 - \phi(x_1))w_+(x', \tau) + \phi(x_1)w_-(x', \tau)$. Then $u = w - \psi$ satisfies the boundary conditions

$$\alpha \frac{\partial u}{\partial \nu} + \beta(x^{'})u = 0, \quad \text{for} \quad x' \in \partial\Omega', \ x_1 \in R \tag{5.7}$$

$$\lim_{x_1 \to \pm\infty} u(x) = 0 \quad \text{for} \quad x' \in \Omega'. \tag{5.8}$$

Let $\mu = \mu(x_1)$ be a weight function as in Section 1. We additionally suppose that $\mu^{-2}(x_1)$ is summable. We denote by E_1 the space of functions in $C_{\mu}^{2+\delta}(\bar{\Omega})$ satisfying the boundary conditions (5.7). If $u \in E_1$ we set

$\tilde{A}(u,\tau) = A(u+\psi,\tau)$, i.e.

$$\begin{aligned}\tilde{A}(u,\tau) \equiv& a(x^{'},\tau)\Delta u + (\tilde{c}(u) + b_1(x^{'},\tau))\frac{\partial u}{\partial x_1}\\ &+ \sum_{k=2}^{m} b_k(x^{'},\tau)\frac{\partial u}{\partial x_k} + \tilde{F}(x^{'},u,\tau) + K(u),\end{aligned} \tag{5.9}$$

where

$$\tilde{F}(x^{'},u,\tau) = a(x^{'},\tau)\Delta\psi(\tau) + \sum_{k=1}^{m} b_k(x^{'},\tau)\frac{\partial\psi(\tau)}{\partial x_k} + F(x^{'},u+\psi(\tau),\tau),$$

or using the form of the function ψ and equations for $w_\pm$

$$\begin{aligned}\tilde{F}(x^{'},u,\tau) &= (a(x^{'},\tau)\phi^{''}(x_1) + b_1(x^{'},\tau)\phi^{'}(x_1))(w_-(x^{'},\tau) - w_+(x^{'},\tau))\\ &+ F(x^{'},u+\psi(\tau),\tau) - (F(x^{'},w_-(x^{'},\tau),\tau)\phi(x_1)\\ &+ F(x^{'},w_+(x^{'},\tau),\tau)(1-\phi(x_1))),\\ \tilde{c}(u) &= c(u+\psi), \quad K(u) = \tilde{c}(u)\frac{\partial\psi(\tau)}{\partial x_1}.\end{aligned}$$

We consider the operator $\tilde{A}$ acting from E_1 to $E_2 = C_\mu^\delta(\bar{\Omega})$ for each fixed τ.

The linearized operator has the form $\tilde{A}^{'}(u,\tau)v = Lv + K^{'}(u)v + Mv$, where

$$Lv = a(x^{'},\tau)\Delta v + (\tilde{c}(u) + b_1(x^{'},\tau))\frac{\partial v}{\partial x_1} + \sum_{k=2}^{m} b_k(x^{'},\tau)\frac{\partial v}{\partial x_k} + c(x^{'},\tau)v,$$

$$c(x^{'},\tau) = \tilde{F}^{'}(x^{'},u,\tau), \quad Mv = \frac{\partial u}{\partial x_1} < \tilde{c}^{'}(u), v > .$$

Here, $< c^{'}(u), v >$ denotes the action of the linear functional $c^{'}(u)$ on the element $v \in C_\mu^{2+\delta}(\bar{\Omega})$. We suppose that the operator L satisfies Condition 2. Since a sum of a Fredholm operator and a finite dimensional operator remains Fredholm, the operator $\tilde{A}^{'}(u,\tau)v - \lambda v$ is Fredholm for all $\lambda \geq 0$.

It was shown in Section 2 that Condition 2 and Condition $2^{'}$ are equivalent. Condition $2^{'}$ for the operator L has the following form: The problem

$$\begin{aligned}&a(x^{'},\tau)\Delta^{'} v + \sum_{k=2}^{m} b_k(x^{'},\tau)\frac{\partial v}{\partial x_k} + (-a(x^{'},\tau)\xi^2 + (\tilde{c}(u) + b_1(x^{'},\tau))i\xi\\ &\quad + F^{'}(x^{'},w_\pm(x^{'}),\tau))v - \lambda v = 0\\ &\alpha\frac{\partial v}{\partial \nu} + \beta(x^{'})v|_{\partial G} = 0\end{aligned}$$

in the section G of the cylinder does not have nonzero solutions for all real ξ, $\lambda \geq 0$. Consider the operators

$$L^{\pm}v = a(x^{'},\tau)\Delta^{'}v + \sum_{k=2}^{m} b_k(x^{'},\tau)\frac{\partial v}{\partial x_k} + F^{'}(w_{\pm}(x^{'}),x^{'},\tau)v$$

which appear as a result of linearization of the left hand side in (5.5) about $w_{\pm}(x^{'})$. The case where this operator with corresponding boundary conditions has all eigenvalues in the left half-plane, is the so-called bistable case [26]. In the case where the matrices a and b_1 are scalar, Condition $2^{'}$ is satisfied in the bistable case and the operator is Fredholm. In the general case where these matrices are arbitrary, Condition $2^{'}$ is more restrictive and may be not satisfied in the bistable case.

Suppose that Condition $2^{'}$ in the form given above is satisfied. Then as was done in Sections 3 and 4 we prove that the operator (5.9) is proper, and construct the topological degree.

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