Advances in Differential Equations Volume 6, Number 5, May 2001, pp. 613–640

SEMIGROUPS OF LIPSCHITZ OPERATORS

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(Submitted by: J.A. Goldstein)

Abstract. The notion of semigroups of Lipschitz operators is introduced and a characterization of continuous infinitesimal generators of such semigroups is obtained. An application of the abstract theory obtained here is given to the Cauchy problem for quasi-linear wave equation with damping.

Introduction. After a pioneering work by Kōmura [6] the generation theorem of semigroups of quasi-contractions in Banach spaces has been studied intensively and applied to the well-posedness of Cauchy problems for porous medium equations, Hamilton-Jacobi equations and scalar first-order equations and so on. (Some of the main points of theory of quasi-contractive semigroups have been outlined in a review paper by Crandall [3].) However, Temple [12] showed that the theory of quasi-contractive semigroups could not be applied to solve genuinely nonlinear symmetric hyperbolic systems with initial data dense in the whole underlying Banach space based on the space of integrable functions. It is expected that solution operators of the Cauchy problem for first-order systems of conservation laws are Lipschitz continuous. In this case, it is conjectured that such solution operators form a semigroup of Lipschitz operators. In fact, an attempt has recently been made by Bressan, Liu and Yang [1] to prove that a family of solution operators of the Cauchy problem for strictly hyperbolic systems of conservation laws is a semigroup of Lipschitz operators in the space of integrable functions if its domain is defined by the set of integrable functions whose total variations are sufficiently small.

Received for publication June 2000.
AMS Subject Classifications: 34G20; 47H15.

Throughout this paper X denotes a real Banach space with norm $\|\cdot\|$ and D a closed subset of X. By a *semigroup of Lipschitz operators on* D we mean a one-parameter family $\{T(t)\colon t \geq 0\}$ of Lipschitz operators from D into itself with the following three conditions:

(S1) $T(0)x = x$ and $T(t)T(s)x = T(t+s)x$ for $x \in D$ and $t, s \geq 0$.

(S2) For each $x \in D$, the function $t \to T(t)x$ is continuous from $[0, \infty)$ into X.

(S3) For each $\tau > 0$ there exists $M_\tau \geq 1$ such that

$$\|T(t)x - T(t)y\| \leq M_\tau \|x - y\| \quad \text{for } x, y \in D \text{ and } t \in [0, \tau].$$

We are here interested in studying a basic property of semigroups of Lipschitz operators and a characterization of infinitesimal generators of such semigroups which are stated roughly as follows:

(i) A nonlinear analogue of Feller's theorem for semigroups of class (C_0) (Theorem 4.1): A semigroup of Lipschitz operators is a quasi-contractive semigroup with respect to a certain metric-like functional.

(ii) A characterization of infinitesimal generators of semigroups of Lipschitz operators (Theorem 4.2): A continuous operator A from D into X is the infinitesimal generator of a semigroup of Lipschitz operators on D if and only if it satisfies the subtangential condition and a general type of dissipative condition that there is a metric-like functional with respect to which A is dissipative.

Our discussion is restricted to a special case in which infinitesimal generators are continuous operators. However this does not mean that the abstract theory obtained here cannot be applied to any partial differential equations. In fact, in Section 5 we give an application of our results to the Cauchy problem for the quasi-linear wave equation with damping. For wider application, it is strongly desired to extend our results to the case where infinitesimal generators are not always continuous. The final section 6 contains the statement of a result in this direction.

The problem of characterizing continuous infinitesimal generators of semigroups of Lipschitz operators is closely related to the abstract Cauchy problem

$$u'(t) = Au(t) \quad \text{for } t \geq 0, \qquad \text{and} \qquad u(0) = x \qquad \text{(ACP;}A, x\text{)}$$

where A is a continuous operator from D into X satisfying the dissipative condition defined by means of metric-like functionals. The problem of existence and uniqueness of solutions has been already solved by Lakshmikantham *et al.* [7] if the directional derivative of a metric-like functional is assumed to be upper semicontinuous. The elimination of this condition is required to establish our purpose. In Section 1 we introduce a class of functionals to define a general type of dissipative condition. Some results concerning the uniqueness, continuous dependence on initial data and continuation of solutions to the abstract Cauchy problem are obtained under such a dissipative condition. Section 2 provides the construction of approximate solutions for the abstract Cauchy problem where the forward Euler difference scheme is used instead of the method of Cauchy polygons. Section 3 discuss the convergence of approximate solutions whose proof needs a new idea, and the central part of this section is devoted to this argument (Propositions 3.1 and 3.2).

1. Properties of uniqueness and continuation of solutions. This section is devoted to the study of some properties of solutions of the abstract Cauchy problem (ACP;A, x) where A is a continuous operator from D into X and $x \in D$.

Let J be an interval of the form $[0, \tau)$ or $[0, \tau]$ with τ such that $0 < \tau < \infty$. A continuous function u from J into X is said to be a *solution to* (ACP;A, x) *on* J if $u(0) = x$, $u(t) \in D$ for $t \in J$, u is differentiable on J, and u satisfies $u'(t) = Au(t)$ for $t \in J$. A solution to (ACP;A, x) on $[0, \infty)$ is called a *global solution to* (ACP;A, x).

We introduce a functional V from $X \times X$ into $[0, \infty)$ satisfying the following three properties (V1) through (V3) to define a general dissipative-type condition:

(V1) There exists $L > 0$ such that

$$|V(x, y) - V(\hat{x}, \hat{y})| \leq L(\|x - \hat{x}\| + \|y - \hat{y}\|) \quad \text{for } (x, y), (\hat{x}, \hat{y}) \in X \times X.$$

(V2) $V(x, x) = 0$ for $x \in D$.

(V3) If a sequence $\{(x_n, y_n)\}$ in $D \times D$ satisfies $V(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Throughout this section we assume that $A \colon D \to X$ satisfies the following dissipative condition, which will guarantee the uniqueness of solutions.

(D) There exists a real number ω such that

$$D_+V(x, y)(Ax, Ay) \leq \omega V(x, y) \quad \text{for } x, y \in D,$$

where

$$D_+V(x, y)(\xi, \eta) = \liminf_{h \downarrow 0} \, (V(x + h\xi, y + h\eta) - V(x, y))/h$$

for $(x, y), (\xi, \eta) \in X \times X$.

The above-mentioned notion of dissipativity has already been introduced in [10], [11] and [15] in order to investigate the problem of uniqueness of solutions of Cauchy problems of ordinary differential equations. Throughout this paper we use the notation $a \vee b = \max(a, b)$ for real numbers a and b, and set $\omega_0 = \omega \vee 0$. We are now in a position to state a result of the uniqueness and continuous dependence on initial data of solutions.

Proposition 1.1. *Let $\tau > 0$ and $x_i \in D$ for $i = 1, 2$. For $i = 1, 2$, let u_i be a solution to* (ACP;A, x_i) *on $[0, \tau]$. Then we have*

$$V(u_1(t), u_2(t)) \leq e^{\omega t} V(x_1, x_2)$$

for all $t \in [0, \tau]$. In particular, (ACP;A, x) *has at most one solution for each $x \in D$.*

Proof. We set $w(t) = V(u_1(t), u_2(t))$ for $t \in [0, \tau]$. From property (V1) it easily follows that $w \in C([0, \tau] \colon [0, \infty))$. Let $t \in [0, \tau)$. Since $\lim_{h \downarrow 0}(u_i(t + h) - u_i(t))/h = Au_i(t)$ for $i = 1, 2$ we have, by property (V1),

$$\liminf_{h \downarrow 0} (w(t + h) - w(t))/h \leq D_+V(u_1(t), u_2(t))(Au_1(t), Au_2(t)),$$

and condition (D) implies $D_+w(t) \leq \omega w(t)$ for $t \in [0, \tau)$, where $D_+w(t)$ denotes the lower right derivative of $w(t)$. Since $w(0) = V(x_1, x_2)$, the first assertion is proved. The second assertion follows immediately from the first assertion and properties (V2) and (V3). □

Proposition 1.2. *If* (ACP;A, x) *has a local solution for every $x \in D$, then* (ACP;A, x) *has a global solution for every $x \in D$.*

Proof. If the proposition were not true, there would exist $x \in D$ such that $\overline{\tau} := \sup\{\tau \in [0, \infty) \colon \text{(ACP;}A, x\text{) has a solution on } [0, \tau]\} < \infty$. By the uniqueness of solutions we see that there exists a solution u to (ACP;A, x) on $[0, \overline{\tau})$. Let $t, \hat{t} \in [0, \overline{\tau})$. Then we have, by Proposition 1.1,

$$V(u(t), u(\hat{t})) \leq \exp(\omega_0(t \wedge \hat{t}))V(u(|t - \hat{t}|), x),$$

which implies that the limit $\lim_{t \uparrow \overline{\tau}} u(t)$ exists and is in D. By assumption the function u can be extended beyond $\overline{\tau}$, which contradicts the definition of $\overline{\tau}$. □

2. Construction of approximate solutions. In this section we discuss the construction of approximate solutions for the abstract Cauchy problem (ACP;A, x) where A is a continuous operator from D into X satisfying the dissipative condition (D). Although most results in this section are essentially shown by [4], [7] and [9], we state the results with proof from the viewpoint of the forward Euler difference approximation.

In order that (ACP;A, x) has a local solution for $x \in D$, the operator A must clearly satisfy the so-called subtangential condition

(T) $\qquad \liminf_{h\downarrow 0} d(x + hAx, D)/h = 0 \quad$ for $x \in D$,

where $d(z, D)$ denotes the distance between $\{z\}$ and D.

Throughout this section the subtangential condition (T) is imposed on the operator A, and the symbol $B[x_0, r]$ stands for the closed ball with center x_0 and radius r. As an approximation of solutions to (ACP;A, x), the forward Euler difference scheme will be used in this paper instead of the method of Cauchy polygons.

We begin by studying some properties of approximate solutions. The following is easily proved by induction.

Lemma 2.1. *Let $x_0 \in D$, and assume $r > 0$ and $M > 0$ to be such that $\|Ax\| \le M$ for $x \in B[x_0, r] \cap D$. Let $\varepsilon > 0$ and $\sigma \in (0, r/(M + \varepsilon)]$. If a sequence $\{(s_i, z_i)\}_{i=0}^n$ in $[0, \sigma] \times D$ satisfies*

$$0 = s_0 < s_1 < \cdots < s_n \le \sigma, \tag{2.1}$$

$$\|z_{i-1} + (s_i - s_{i-1})Az_{i-1} - z_i\| \le \varepsilon(s_i - s_{i-1}) \text{ for } i = 1, 2, \dots, n, \tag{2.2}$$

where $z_0 = x_0$, then $\|z_i - z_j\| \le (M + \varepsilon)(s_i - s_j)$ for $0 \le j \le i \le n$, and $\|Az_i\| \le M$ for $0 \le i \le n$.

Lemma 2.2. *Let $x_0 \in D$, and assume $r > 0$, $M > 0$ and $\eta > 0$ to be such that $\|Ax\| \le M$ and $\|Ax - Ax_0\| \le \eta$ for $x \in B[x_0, r] \cap D$. Let $\varepsilon > 0$ and $\sigma \in (0, r/(M + \varepsilon)]$. Then the following assertions hold:*

(i) *If a sequence $\{(s_i, z_i)\}_{i=0}^n$ in $[0, \sigma] \times D$ satisfies* (2.1) *and* (2.2), *then*

$$\|x_0 + s_nAx_0 - z_n\| \le (\varepsilon + \eta)s_n. \tag{2.3}$$

(ii) *If a sequence $\{(s_i, z_i)\}_{i=0}^\infty$ in $[0, \sigma) \times D$ satisfies*

$$0 = s_0 < s_1 < \cdots < s_i < \cdots < \sigma, \quad \text{and} \quad \lim_{i\to\infty} s_i = \sigma, \tag{2.4}$$

$$\|z_{i-1} + (s_i - s_{i-1})Az_{i-1} - z_i\| \le \varepsilon(s_i - s_{i-1}) \quad \text{for } i = 1, 2, \dots, \text{ where } z_0 = x_0, \tag{2.5}$$

then the limit $z = \lim_{i\to\infty} z_i$ *exists and is in* D*, and*

$$\|x_0 + \sigma A x_0 - z\| \le (\varepsilon + \eta)\sigma. \tag{2.6}$$

Proof. Let $\{(s_i, z_i)\}_{i=0}^{n}$ be a sequence in $[0,\sigma]\times D$ satisfying (2.1) and (2.2). By Lemma 2.1, we have $\|z_i - x_0\| \le (M+\varepsilon)\sigma$ for $i = 0, 1, 2, \dots, n$, which implies $\|Az_i - Ax_0\| \le \eta$ for $i = 0, 1, 2, \dots, n$, by the choice of σ. It follows that

$$\|z_{i-1} + (s_i - s_{i-1})Ax_0 - z_i\| \le (s_i - s_{i-1})(\eta + \varepsilon)$$

for $i = 1, 2, \dots, n$. Adding these inequalities we find the desired inequality (2.3).

To prove (ii), let $\{(s_i, z_i)\}_{i=0}^{\infty}$ be a sequence in $[0,\sigma)\times D$ satisfying (2.4) and (2.5). By Lemma 2.1, $\|z_i - z_j\| \le (M+\varepsilon)(s_i - s_j)$ for $0 \le j \le i$, which shows that the limit $z = \lim_{i\to\infty} z_i$ exists and is in D, and the desired inequality (2.6) is obtained by passing to the limit in (2.3) as $n \to \infty$. □

Proposition 2.3. *Let* $x_0 \in D$*, and assume* $r > 0$*,* $M > 0$ *and* $\eta > 0$ *to be such that* $\|Ax\| \le M$ *and* $\|Ax - Ax_0\| \le \eta$ *for* $x \in B[x_0, r] \cap D$*. If we set* $\tau = r/M$*, then for any* $\sigma \in [0,\tau]$*,*

$$d(x_0 + \sigma A x_0, D) \le \eta\sigma. \tag{2.7}$$

Proof. Since $z \to d(z, D)$ is continuous, it suffices to prove that (2.7) holds for $\sigma \in (0,\tau)$. Now, let $\sigma \in (0,\tau)$ be fixed, and choose $\varepsilon \in (0, (r - \sigma M)/\sigma]$ arbitrarily so that $\sigma \le r/(M+\varepsilon)$. Once it is proved that there is a sequence $\{(s_i, z_i)\}_{i=0}^{\infty}$ in $[0,\sigma)\times D$ satisfying (2.4) and (2.5) we have, by (ii) of Lemma 2.2, $d(x_0 + \sigma A x_0, D) \le (\varepsilon + \eta)\sigma$, and so the desired inequality (2.7) holds since ε is arbitrarily chosen.

Now, let $k \ge 1$ and assume that a sequence $\{(s_i, z_i)\}_{i=0}^{k-1}$ in $[0,\sigma)\times D$ has been chosen so that (2.4) and (2.5) may hold for $0 \le i \le k-1$. Then we denote by $\bar{h}_k$ the supremum of all $h \ge 0$ such that $h < \sigma - s_{k-1}$ and $d(z_{k-1} + hAz_{k-1}, D) \le (\varepsilon/2)h$. Since $\bar{h}_k > 0$ by condition (T), we choose $h_k > 0$ so that $\bar{h}_k/2 < h_k < \sigma - s_{k-1}$ and $d(z_{k-1} + h_k A z_{k-1}, D) \le (\varepsilon/2)h_k$. If we define $s_k = s_{k-1} + h_k$ then $s_{k-1} < s_k < \sigma$ and there exists $z_k \in D$ satisfying (2.5) with $i = k$.

It remains to prove that $\lim_{i\to\infty} s_i = \sigma$. For this purpose, assume $\bar{s} = \lim_{i\to\infty} s_i < \sigma$. By Lemma 2.1, $\|z_i - z_j\| \le (M+\varepsilon)(s_i - s_j)$ for $0 \le j \le i$, which implies that the limit $z = \lim_{i\to\infty} z_i$ exists and is in D. By condition

(T) one finds a number $h > 0$ such that $h < \sigma - \bar{s}$ and $d(z + hAz, D) \le (\varepsilon/3)h$. It is clear that $h < \sigma - s_{i-1}$ for all $i \ge 1$. Since $\bar{h}_i < 2h_i = 2(s_i - s_{i-1}) \to 0$ as $i \to \infty$, there is an $i_0 \ge 1$ such that $\bar{h}_i < h$ for all $i \ge i_0$. The definition of $\bar{h}_i$ implies $d(z_{i-1} + hAz_{i-1}, D) > (\varepsilon/2)h$ for all $i \ge i_0$, which contradicts the choice of h. □

The following result follows immediately from the continuity of A.

Proposition 2.4. *For $x_0 \in D$ there are $r > 0$ and $M > 0$ such that $\|Ax\| \le M$ for $x \in B[x_0, r] \cap D$.*

The existence of a forward difference approximate solution for (ACP;A, x) is established by our next proposition.

Proposition 2.5. *Let $x_0 \in D$, and assume $r > 0$ and $M > 0$ to be such that $\|Ax\| \le M$ for $x \in B[x_0, r] \cap D$. Let $\varepsilon \in (0, 1)$ and $\tau \in (0, r/(M+1)]$. Then there is a sequence $\{(t_j, x_j)\}_{j=0}^{\infty}$ in $[0, \tau) \times D$ such that*

(i) $0 = t_0 < t_1 < \cdots < t_j < \cdots < \tau$, *and* $\lim_{j\to\infty} t_j = \tau$;

(ii) $t_j - t_{j-1} \le \varepsilon$ *for* $j = 1, 2, \dots$;

(iii) $\|x_{j-1} + (t_j - t_{j-1})Ax_{j-1} - x_j\| \le (\varepsilon/2)(t_j - t_{j-1})$ *for* $j = 1, 2, \dots$;

(iv) *if* $x \in B[x_{j-1}, (M+1)(t_j - t_{j-1})] \cap D$, *then*

$$\|Ax - Ax_{j-1}\| \le \varepsilon/4 \quad \text{for } j = 1, 2, \dots.$$

Proof. Let i be a positive integer and assume that a sequence $\{(t_j, x_j)\}_{j=0}^{i-1}$ is defined so that properties (i) through (iv) hold for $0 \le j \le i-1$, and then consider a nonnegative number $\bar{h}_i$ defined by the supremum of all $h \in [0, \varepsilon]$ such that $h < \tau - t_{i-1}$ and that $\|Ax - Ax_{i-1}\| \le \varepsilon/4$ for $x \in B[x_{i-1}, (M+1)h] \cap D$. By the continuity of A we have $\bar{h}_i > 0$, and hence there is a number $h_i \in (0, \varepsilon]$ so that $\bar{h}_i/2 < h_i < \tau - t_{i-1}$ and that

$$\|Ax - Ax_{i-1}\| \le \varepsilon/4 \quad \text{for } x \in B[x_{i-1}, (M+1)h_i] \cap D. \tag{2.8}$$

If we put $t_i = t_{i-1} + h_i$, then $t_{i-1} < t_i < \tau$, and (ii) and (iv) hold for $j = i$. To check (iii) with $j = i$, we first note by Lemma 2.1 that $\|Ax_j\| \le M$ for $0 \le j \le i-1$. The inequality (2.8) implies $\|Ax\| \le M + 1$ for $x \in B[x_{i-1}, (M+1)h_i] \cap D$. By this fact and (2.8) we apply Proposition 2.3 with $x_0 = x_{i-1}$, $r = (M+1)h_i$ and $\eta = \varepsilon/4$ to find $d(x_{i-1} + h_iAx_{i-1}, D) \le (\varepsilon/4)h_i$, which ensures the existence of an element $x_i \in D$ satisfying (iii) with $j = i$.

Finally, the fact that $\lim_{j\to\infty} t_j = \tau$ will be proved. Assume, for contradiction, that $\bar{t} = \lim_{j\to\infty} t_j < \tau$. Since $\|x_i - x_j\| \le (M + \varepsilon/2)(t_i - t_j)$ for

$0 \le j \le i$ (by Lemma 2.1) and D is closed, the limit $\bar{x} = \lim_{j\to\infty} x_j$ exists and is in D. The continuity of A enables us to choose an $h \in (0, \varepsilon]$ such that $h < \tau - \bar{t}$ and $\|Ax - A\bar{x}\| \le \varepsilon/8$ for $x \in B[\bar{x}, 2(M+1)h] \cap D$. If we choose an integer $j_0 \ge 1$ so that $\bar{t} - t_{j-1} \le h$ and $\|\bar{x} - x_{j-1}\| \le (M+1)h$ for $j \ge j_0$, then we have $B[x_{j-1}, (M+1)h] \subset B[\bar{x}, 2(M+1)h]$ for $j \ge j_0$. It follows by the choice of h that if $j \ge j_0$ then $\|Ax - Ax_{j-1}\| \le \varepsilon/4$ for $x \in B[x_{j-1}, (M+1)h] \cap D$. Since $h < \tau - t_{j-1}$ for all $j \ge 1$, the definition of $\bar{h}_j$ implies $h \le \bar{h}_j < 2h_j = 2(t_j - t_{j-1})$ for all $j \ge j_0$, and the right-hand side tends to zero as $j \to \infty$. This contradicts the fact that h is positive.

3. Convergence of approximate solutions and existence of solutions. This section is devoted to the study of convergence of approximate solutions for (ACP;A,x) constructed in the previous section, where A is a continuous operator from D into X satisfying the dissipative condition (D) and the subtangential condition (T).

The following is needed for the comparison between two approximate solutions to (ACP;A,x).

Proposition 3.1. *Let $(x_0, \hat{x}_0) \in D \times D$, and assume $r > 0$, $M > 0$, and $\eta, \hat{\eta} \in (0,1)$ to be such that*

$$\|Ax\| \le M \quad \text{and} \quad \|Ax - Ax_0\| \le \eta/4 \quad \text{for } x \in B[x_0, r] \cap D;$$
$$\|A\hat{x}\| \le M \quad \text{and} \quad \|A\hat{x} - A\hat{x}_0\| \le \hat{\eta}/4 \quad \text{for } \hat{x} \in B[\hat{x}_0, r] \cap D.$$

Let $\sigma \in (0, r/(M+1)]$. Then there is a pair $(y_0, \hat{y}_0) \in D \times D$ such that

$$\|x_0 + \sigma Ax_0 - y_0\| \le \eta\sigma, \tag{3.1}$$
$$\|\hat{x}_0 + \sigma A\hat{x}_0 - \hat{y}_0\| \le \hat{\eta}\sigma, \tag{3.2}$$
$$V(y_0, \hat{y}_0) \le \exp(\omega_0\sigma)(V(x_0, \hat{x}_0) + L(\eta + \hat{\eta})\sigma). \tag{3.3}$$

Proof. We first show that there exist a sequence $\{s_j\}_{j=0}^{\infty}$ in $[0, \sigma)$ and a sequence $\{(z_j, \hat{z}_j)\}_{j=0}^{\infty}$ in $D \times D$ such that

$$0 = s_0 < s_1 < \cdots < s_j < \cdots < \sigma \quad \text{and} \quad \lim_{j\to\infty} s_j = \sigma, \tag{3.4}$$
$$\|z_{j-1} + (s_j - s_{j-1})Az_{j-1} - z_j\| \le (3\eta/4)(s_j - s_{j-1}) \text{ where } z_0 = x_0, \tag{3.5}$$
$$\|\hat{z}_{j-1} + (s_j - s_{j-1})A\hat{z}_{j-1} - \hat{z}_j\| \le (3\hat{\eta}/4)(s_j - s_{j-1}) \text{ where } \hat{z}_0 = \hat{x}_0, \tag{3.6}$$
$$(V(z_j, \hat{z}_j) - V(z_{j-1}, \hat{z}_{j-1}))/(s_j - s_{j-1}) \le \omega V(z_{j-1}, \hat{z}_{j-1}) + L(\eta + \hat{\eta}) \tag{3.7}$$

for $j = 1, 2, \ldots$. For this purpose, we set $(z_0, \hat{z}_0) = (x_0, \hat{x}_0)$, and define inductively $s_j \in [0, \sigma)$ and $(z_j, \hat{z}_j) \in D \times D$ in the following manner: If a sequence $\{s_j\}_{j=0}^{i-1}$ in $[0, \sigma)$ and a sequence $\{(z_j, \hat{z}_j)\}_{j=0}^{i-1}$ in $D \times D$ where $i \geq 1$ are defined so that the first half of (3.4) and (3.5) through (3.7) hold for $0 \leq j \leq i-1$, then we denote by $\bar{h}_i$ the supremum of all $h \geq 0$ such that $h < \sigma - s_{i-1}$, and

$$V(z_{i-1} + hAz_{i-1}, \hat{z}_{i-1} + hA\hat{z}_{i-1}) - V(z_{i-1}, \hat{z}_{i-1}) \\ \leq (\omega V(z_{i-1}, \hat{z}_{i-1}) + (L/4)(\eta + \hat{\eta}))h.$$

Since $\bar{h}_i > 0$ by the dissipative condition (D), we choose a number $h_i > 0$ so that $\bar{h}_i/2 < h_i < \sigma - s_{i-1}$ and

$$(V(z_{i-1} + h_i Az_{i-1}, \hat{z}_{i-1} + h_i A\hat{z}_{i-1}) - V(z_{i-1}, \hat{z}_{i-1}))/h_i \\ \leq \omega V(z_{i-1}, \hat{z}_{i-1}) + (L/4)(\eta + \hat{\eta}), \tag{3.8}$$

and then define $s_i = s_{i-1} + h_i$. It is obvious that $s_{i-1} < s_i < \sigma$. Since $\|z_{i-1} - x_0\| \leq (M + 3\eta/4)s_{i-1}$ by Lemma 2.1, we have, by the choice of σ, $B[z_{i-1}, (M+1)h_i] \subset B[x_0, r]$. It follows that $\|Ax\| \leq M$ and $\|Ax - Ax_0\| \leq \eta/4$ for $x \in B[z_{i-1}, (M+1)h_i] \cap D$. By the second inequality we have $\|Ax - Az_{i-1}\| \leq \eta/2$ for $x \in B[z_{i-1}, (M+1)h_i] \cap D$. We therefore apply Proposition 2.3 to obtain $d(z_{i-1} + h_i Az_{i-1}, D) \leq (\eta/2)h_i$, which ensures the existence of an element $z_i \in D$ such that (3.5) holds for $j = i$. It is shown similarly that there exists $\hat{z}_i \in D$ satisfying (3.6) with $j = i$. By virtue of property (V1) the desired inequality (3.7) with $j = i$ is obtained by the inequality (3.8) combined with (3.5) and (3.6) with $j = i$.

It remains to prove the second half of (3.4). To do so, assume to the contrary that $\bar{s} = \lim_{j\to\infty} s_j < \sigma$. Lemma 2.1 asserts that $\{z_j\}$ and $\{\hat{z}_j\}$ are Cauchy sequences, and so $z = \lim_{j\to\infty} z_j$ and $\hat{z} = \lim_{j\to\infty} \hat{z}_j$ exist and are in D. By the dissipative condition (D) we choose a number $h > 0$ so that $h < \sigma - \bar{s}$ and

$$(V(z + hAz, \hat{z} + hA\hat{z}) - V(z, \hat{z}))/h \leq \omega V(z, \hat{z}) + (L/8)(\eta + \hat{\eta}). \tag{3.9}$$

Since $\bar{h}_j < 2h_j = 2(s_j - s_{j-1}) \to 0$ as $j \to \infty$, there is an integer $j_0 \geq 1$ such that $\bar{h}_j < h$ for all $j \geq j_0$. By the definition of $\bar{h}_j$ we have

$$(V(z_{j-1} + hAz_{j-1}, \hat{z}_{j-1} + hA\hat{z}_{j-1}) - V(z_{j-1}, \hat{z}_{j-1}))/h \\ > \omega V(z_{j-1}, \hat{z}_{j-1}) + (L/4)(\eta + \hat{\eta})$$

for all $j \geq j_0$, which is a contradiction to (3.9).

Now, we turn to the proof of the existence of a pair $(y_0, \hat{y}_0) \in D \times D$ satisfying (3.1) through (3.3). We apply (ii) of Lemma 2.2 to show that $y_0 = \lim_{j\to\infty} z_j$ and $\hat{y}_0 = \lim_{j\to\infty} \hat{z}_j$ exist and are in D and that they satisfy (3.1) and (3.2). By (3.7) we find inductively

$$V(z_j, \hat{z}_j) \leq \exp(\omega_0 s_j)(V(x_0, \hat{x}_0) + L(\eta + \hat{\eta})s_j) \tag{3.10}$$

for $j \geq 0$. Here we have used the fact that $1+t \leq e^t$ for $t \geq 0$. The inequality (3.3) is obtained by taking the limit in (3.10) as $j \to \infty$. □

Proposition 3.2. *Let $x_0 \in D$, and assume $R > 0$ and $M > 0$ to be such that $\|Ax\| \leq M$ for $x \in B[x_0, R] \cap D$. Let $\tau \in (0, R/(M+1)]$ and $\lambda, \mu \in (0, 1/2)$, and suppose that for each $\varepsilon = \lambda, \mu$, a sequence $\{(t_i^\varepsilon, x_i^\varepsilon)\}_{i=0}^\infty$ in $[0, \tau) \times D$ satisfies the following conditions:*

(i) $0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_i^\varepsilon < \cdots < \tau$, *and* $\lim_{i\to\infty} t_i^\varepsilon = \tau$;
(ii) $t_i^\varepsilon - t_{i-1}^\varepsilon \leq \varepsilon$ *for* $i = 1, 2, \ldots$;
(iii) $\|x_{i-1}^\varepsilon + (t_i^\varepsilon - t_{i-1}^\varepsilon)Ax_{i-1}^\varepsilon - x_i^\varepsilon\| \leq (\varepsilon/2)(t_i^\varepsilon - t_{i-1}^\varepsilon)$ *for* $i = 1, 2, \ldots$, *where* $x_0^\varepsilon = x_0$;
(iv) *if* $x \in B[x_{i-1}^\varepsilon, (M+1)(t_i^\varepsilon - t_{i-1}^\varepsilon)] \cap D$, *then*

$$\|Ax - Ax_{i-1}^\varepsilon\| \leq \varepsilon/4 \quad \text{for } i = 1, 2, \ldots.$$

Let $\{s_k\}_{k=0}^\infty$ be a sequence such that $s_k < s_{k+1}$ for $k = 0, 1, 2, \ldots$ and that

$$\{s_k : k = 0, 1, 2, \ldots\} = \{t_i^\lambda : i = 0, 1, 2, \ldots\} \cup \{t_j^\mu : j = 0, 1, 2, \ldots\}.$$

Then there exists a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^\infty$ in $D \times D$ with the three properties listed below.

(a) *If* $s_k = t_i^\lambda$, *then* $z_k^\lambda = x_i^\lambda$; *and if* $s_k = t_j^\mu$ *then* $z_k^\mu = x_j^\mu$.
(b) *For each* $\varepsilon = \lambda, \mu$, *we have*

$$\sum_{j=q}^{k} \|z_{j-1}^\varepsilon + (s_j - s_{j-1})Az_{j-1}^\varepsilon - z_j^\varepsilon\| \leq 2\varepsilon(s_k - s_{q-1}) + 3\varepsilon \sum_{t_i^\varepsilon \in \{s_q, \ldots, s_k\}} (t_i^\varepsilon - t_{i-1}^\varepsilon)$$

for $1 \leq q \leq k$ *and* $k = 1, 2, \ldots$.
(c) $V(z_k^\lambda, z_k^\mu) \leq \exp(\omega_0 s_k)(2L(\lambda + \mu)s_k + \eta_k(\lambda, \mu))$ *for* $k = 0, 1, 2, \ldots$.

Here the symbol $\eta_k(\lambda,\mu)$ is defined by

$$\eta_k(\lambda,\mu) = 3L\Big(\lambda \sum_{t_i^\lambda \in \{s_1,\dots,s_k\}} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu \in \{s_1,\dots,s_k\}} (t_j^\mu - t_{j-1}^\mu)\Big).$$

Proof. Set $z_0^\varepsilon = x_0$ for each $\varepsilon = \lambda,\mu$, and let $l \geq 1$ and assume that a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^{l-1}$ in $D \times D$ has been defined so that properties (a) through (c) may hold for $0 \leq k \leq l-1$. Let i and j be positive integers such that $t_{i-1}^\lambda < s_l \leq t_i^\lambda$ and $t_{j-1}^\mu < s_l \leq t_j^\mu$. Since $\|x_{i-1}^\lambda - x_0\| \leq (M+\lambda/2)t_{i-1}^\lambda$ by Lemma 2.1, we have $B[x_{i-1}^\lambda, (M+1)(t_i^\lambda - t_{i-1}^\lambda)] \subset B[x_0, R]$, by the choice of τ. It follows that

$$\|Ax\| \leq M \quad \text{for } x \in B[x_{i-1}^\lambda, (M+1)(t_i^\lambda - t_{i-1}^\lambda)] \cap D. \tag{3.11}$$

We shall prove that for each $\varepsilon = \lambda,\mu$,

$$\|Ax\| \leq M \quad \text{and} \quad \|Ax - Az_{l-1}^\varepsilon\| \leq \varepsilon/2 \tag{3.12}$$

for $x \in B[z_{l-1}^\varepsilon, (M+1)(s_l - s_{l-1})] \cap D$. The definition of $\{s_k\}$ implies $t_{i-1}^\lambda \leq s_{l-1} < s_l \leq t_i^\lambda$ and $t_{j-1}^\mu \leq s_{l-1} < s_l \leq t_j^\mu$, and so $t_{i-1}^\lambda = s_p$ for some p with $0 \leq p \leq l-1$, and $t_{j-1}^\mu = s_q$ for some q with $0 \leq q \leq l-1$. By the hypothesis (a) of induction we have $z_p^\lambda = x_{i-1}^\lambda$ and $z_q^\mu = x_{j-1}^\mu$. If $0 \leq p < l-1$ then the set $\{s_{p+1},\dots,s_{l-1}\}$ contains no points t_i^λ. We therefore have, by the hypothesis (b) of induction,

$$\|z_{k-1}^\lambda + (s_k - s_{k-1})Az_{k-1}^\lambda - z_k^\lambda\| \leq 2\lambda(s_k - s_{k-1}) \tag{3.13}$$

for $k = p+1,\dots,l-1$. By (3.11) we use Lemma 2.1 with $x_0 = x_{i-1}^\lambda$ and $r = (M+1)(t_i^\lambda - t_{i-1}^\lambda)$ to obtain $\|z_{l-1}^\lambda - z_p^\lambda\| \leq (M+2\lambda)(s_{l-1} - s_p)$. This is evidently also valid for $p = l-1$. By this inequality we find

$$B[z_{l-1}^\lambda, (M+1)(s_l - s_{l-1})] \subset B[x_{i-1}^\lambda, (M+1)(t_i^\lambda - t_{i-1}^\lambda)]. \tag{3.14}$$

The claim (3.12) with $\varepsilon = \lambda$ follows from (3.11) and condition (iv). We apply the above argument again, with p and i replaced by q and j, to show that (3.12) holds for $\varepsilon = \mu$.

By virtue of (3.12) we deduce from Proposition 3.1 that there exists a pair (y_l^λ, y_l^μ) in $D \times D$ satisfying

$$\|z_{l-1}^\varepsilon + (s_l - s_{l-1})Az_{l-1}^\varepsilon - y_l^\varepsilon\| \le 2\varepsilon(s_l - s_{l-1}) \quad \text{for } \varepsilon = \lambda, \mu, \tag{3.15}$$

$$V(y_l^\lambda, y_l^\mu) \le \exp(\omega_0(s_l - s_{l-1}))(V(z_{l-1}^\lambda, z_{l-1}^\mu) + 2L(\lambda+\mu)(s_l - s_{l-1})). \tag{3.16}$$

Now, we define (z_l^λ, z_l^μ) in $D \times D$ by

$$z_l^\lambda = \begin{cases} y_l^\lambda & \text{if } s_l < t_i^\lambda \\ x_i^\lambda & \text{if } s_l = t_i^\lambda, \end{cases} \qquad \text{and} \qquad z_l^\mu = \begin{cases} y_l^\mu & \text{for } s_l < t_j^\mu \\ x_j^\mu & \text{for } s_l = t_j^\mu. \end{cases}$$

If $s_l = t_i^\lambda$ then we have, by condition (iii),

$$\|x_{i-1}^\lambda + (s_l - t_{i-1}^\lambda)Ax_{i-1}^\lambda - z_l^\lambda\| \le (\lambda/2)(s_l - t_{i-1}^\lambda),$$

while in view of (3.11) and condition (iv) we find, by applying (i) of Lemma 2.2 to (3.13) and (3.15),

$$\|x_{i-1}^\lambda + (s_l - t_{i-1}^\lambda)Ax_{i-1}^\lambda - y_l^\lambda\| \le (2\lambda + \lambda/4)(s_l - t_{i-1}^\lambda).$$

These inequalities together give

$$\|z_l^\lambda - y_l^\lambda\| \le 3\lambda \sum_{t_i^\lambda = s_l} (t_i^\lambda - t_{i-1}^\lambda). \tag{3.17}$$

Similarly we have

$$\|z_l^\mu - y_l^\mu\| \le 3\mu \sum_{t_j^\mu = s_l} (t_j^\mu - t_{j-1}^\mu). \tag{3.18}$$

Combining (3.17) and (3.18) with (3.15), and adding the resultant inequality to the inequality (b) with $k = l-1$, we conclude that the desired property (b) holds for $k = l$.

The proof will be completed by induction on k if we show that (c) is true for $k = l$. Using (3.16) through (3.18) we have, by property (V1),

$$\begin{aligned} V(z_l^\lambda, z_l^\mu) \le{}& \exp(\omega_0(s_l - s_{l-1}))(V(z_{l-1}^\lambda, z_{l-1}^\mu) + 2L(\lambda+\mu)(s_l - s_{l-1})) \\ &+ 3L\Big(\lambda \sum_{t_i^\lambda = s_l} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu = s_l} (t_j^\mu - t_{j-1}^\mu)\Big). \end{aligned} \tag{3.19}$$

The claim that (c) holds for $k = l$ is proved by substituting the estimate (c) with $k = l-1$ into (3.19). $\square$

Theorem 3.3. *Let $x_0 \in D$, and assume $R > 0$ and $M > 0$ to be such that $\|Ax\| \le M$ for $x \in B[x_0, R] \cap D$. Let $\tau \in (0, R/(M+1)]$, and suppose that for $\varepsilon \in (0, 1/2)$, a sequence $\{(t_i^\varepsilon, x_i^\varepsilon)\}_{i=0}^\infty$ in $[0,\tau) \times D$ satisfies conditions* (i) *through* (iv) *in Proposition* 3.2*. If we define the function $u^\varepsilon : [0,\tau) \to X$ by*

$$u^\varepsilon(t) = x_i^\varepsilon \quad \text{for } t \in [t_i^\varepsilon, t_{i+1}^\varepsilon) \text{ and } i = 0, 1, 2, \dots ,$$

then there is a solution u to (ACP;A, x_0) *on $[0,\tau]$ such that*

$$\sup_{t \in [0,\tau)} \|u^\varepsilon(t) - u(t)\| \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{3.20}$$

Proof. Let $\lambda, \mu \in (0, 1/2)$ and $\{s_k\}_{k=0}^\infty$ be the sequence defined as in Proposition 3.2. Then there exists a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^\infty$ in $D \times D$ satisfying properties (a) through (c) of Proposition 3.2.

Let $t \in [0,\tau)$, and let $k \ge 1$ be an integer such that $t \in [s_{k-1}, s_k)$. If i and j are positive integers such that $t_{i-1}^\lambda \le s_{k-1} < s_k \le t_i^\lambda$ and $t_{j-1}^\mu \le s_{k-1} < s_k \le t_j^\mu$, then we have, in a way similar to the derivation of (3.14), $\|z_{k-1}^\lambda - x_{i-1}^\lambda\| \le (M+1)(t_i^\lambda - t_{i-1}^\lambda)$ and $\|z_{k-1}^\mu - x_{j-1}^\mu\| \le (M+1)(t_j^\mu - t_{j-1}^\mu)$. Combining these estimates with (c) of Proposition 3.2 we find, by property (V1),

$$V(u^\lambda(t), u^\mu(t)) \le 5L \exp(\omega_0 \tau)(\lambda + \mu)\tau + L(M+1)(\lambda + \mu).$$

We thus deduce from condition (V3) that the sequence $\{u^\varepsilon(t)\}$ converges to a function $u(t)$ uniformly on $[0,\tau)$ as $\varepsilon \downarrow 0$. By Lemma 2.1 we have $\|u^\varepsilon(t) - u^\varepsilon(s)\| \le (M + \varepsilon/2)(|t-s| + 2\varepsilon)$ for $t, s \in [0,\tau)$ and sufficiently small $\varepsilon \in (0, 1/2)$; hence $\|u(t) - u(s)\| \le M|t-s|$ for $t, s \in [0,\tau)$. This fact shows that there is a continuous function u defined on $[0,\tau]$ satisfying (3.20). Finally, it is easily seen that u is a solution to (ACP;A, x_0) on $[0,\tau]$, by condition (iii) of Proposition 3.2. $\square$

4. Semigroups of Lipschitz operators. In this section we discuss a characterization of continuous infinitesimal generators of semigroups of Lipschitz operators.

Let $\{T(t)\colon t \ge 0\}$ be a semigroup of Lipschitz operators on D. Then we define an operator A_0 in X by

$$\begin{cases} A_0 x = \lim_{h \downarrow 0} (T(h)x - x)/h & \text{for } x \in D(A_0), \\ D(A_0) = \{x \in D\colon \lim_{h \downarrow 0} (T(h)x - x)/h \text{ exists in } X\}. \end{cases}$$

The operator A_0 is called the *infinitesimal generator of* $\{T(t)\colon t \ge 0\}$.

We first give a nonlinear analogue of Feller's theorem for semigroups of class (C_0).

Theorem 4.1. *Let* $\{T(t)\colon t \ge 0\}$ *be a one-parameter family of Lipschitz operators from* D *into itself satisfying two conditions* (S1) *and* (S2). *Then the following statements are mutually equivalent:*

(i) $\{T(t)\colon t \ge 0\}$ *is a semigroup of Lipschitz operators on* D.

(ii) *There exist* $M \ge 1$ *and* $\omega \in \mathbb{R}$ *such that*

$$\|T(t)x - T(t)y\| \le Me^{\omega t}\|x - y\| \quad \text{for } x, y \in D \text{ and } t \ge 0.$$

(iii) *There exist* $\omega \in \mathbb{R}$ *and a nonnegative, Lipschitz-continuous functional* V *on* $X \times X$, *satisfying the property*

(V) *there exist* $M \ge m > 0$ *such that* $m\|x - y\| \le V(x, y) \le M\|x - y\|$ *for* $x, y \in D$, *such that*

$$V(T(t)x, T(t)y) \le e^{\omega t}V(x, y) \quad \text{for } x, y \in D \text{ and } t \ge 0. \tag{4.1}$$

Proof. The implication "(iii) $\Rightarrow$ (i)" is obvious. The fact that (i) implies (ii) is elementary. Let $\{M, \omega\}$ be a pair of constants appearing in statement (ii). A nonnegative functional V_0 on $D \times D$ defined by

$$V_0(x, y) = \sup_{t \ge 0} e^{-\omega t}\|T(t)x - T(t)y\|$$

for $x, y \in D$ is a metric in D and satisfies

$$\|x - y\| \le V_0(x, y) \le M\|x - y\| \quad \text{for } x, y \in D. \tag{4.2}$$

Moreover, it is easy to prove that $V_0(T(t)x, T(t)y) \le e^{\omega t}V_0(x, y)$ for $t \ge 0$ and $x, y \in D$.

If $x, y \in X$ then we have, by (4.2), $V_0(x', y') - M(\|x - x'\| + \|y - y'\|) \le M\|x - y\|$ for every $x', y' \in D$. This fact enables us to define a nonnegative functional V on $X \times X$ by

$$V(x, y) = \sup\{V_0(x', y') - M(\|x - x'\| + \|y - y'\|)\colon x', y' \in D\} \vee 0.$$

By (4.2) and the fact that V_0 is a metric in D we have $V_0(x', y') \le V_0(x, y) + M(\|x - x'\| + \|y - y'\|)$ for $(x, y), (x', y') \in D \times D$, which implies $V(x, y) \le$

$V_0(x,y)$ for $x,y \in D$. The converse inequality follows readily from the definition of V, and hence $V(x,y) = V_0(x,y)$ for $x,y \in D$. It is thus seen that the functional V satisfies property (V) and (4.1) holds. The Lipschitz continuity of V follows immediately from the inequality

$$V_0(x',y') - M(\|x-x'\| + \|y-y'\|) - (V_0(x',y') - M(\|\hat{x}-x'\| + \|\hat{y}-y'\|)) \\ \leq M(\|x-\hat{x}\| + \|y-\hat{y}\|)$$

for $(x,y), (\hat{x},\hat{y}) \in X \times X$ and $(x',y') \in D \times D$. $\square$

The following characterization of continuous infinitesimal generators of semigroups of Lipschitz operators on D is an extension of Martin's result [9, Theorem 5].

Theorem 4.2. *Suppose that A is a continuous operator from D into X. Then A is the infinitesimal generator of a semigroup $\{T(t)\colon t \geq 0\}$ of Lipschitz operators on D if and only if it satisfies the following two conditions.*
(A1) $\liminf_{h \downarrow 0} d(x + hAx, D)/h = 0$ *for all* $x \in D$.
(A2) *There exist $\omega \in \mathbb{R}$ and a nonnegative, Lipschitz-continuous functional V on $X \times X$ satisfying property* (V) *of Theorem* 4.1 *such that*

$$D_+V(x,y)(Ax,Ay) \leq \omega V(x,y) \quad \text{for } x,y \in D.$$

In this case, for each $x \in D$ the (ACP;A,x) *has a unique global solution $u(t;x)$ given by $u(t;x) = T(t)x$ for $t \geq 0$.*

Proof. Suppose that A satisfies two conditions (A1) and (A2). All assumptions of Theorem 3.3 are satisfied by Propositions 2.4 and 2.5. It follows that for every $x \in D$, there exists a local solution to (ACP;A,x). This fact and Proposition 1.2 together imply that (ACP;A,x) has a global solution $u(t;x)$ for every $x \in D$. The uniqueness and continuous dependence on initial data of solutions follow from Proposition 1.1; namely $V(u(t;x),u(t;y)) \leq e^{\omega t}V(x,y)$ holds for $t \geq 0$ and $x,y \in D$. It is shown by a standard argument that a one-parameter family $\{T(t)\colon t \geq 0\}$ defined by $T(t)x = u(t;x)$ for $t \geq 0$ and $x \in D$ is a semigroup of Lipschitz operators on D and that its infinitesimal generator is A.

Conversely, let A be the infinitesimal generator of a semigroup $\{T(t)\colon t \geq 0\}$ of Lipschitz operators on D. It is seen routinely that condition (A1) is satisfied. Finally we check condition (A2). By Theorem 4.1 there exist $\omega \in \mathbb{R}$ and a nonnegative, Lipschitz-continuous functional V on $X \times X$ with

Lipschitz constant L such that $V(T(t)x, T(t)y) \leq e^{\omega t} V(x,y)$ for $x, y \in D$ and $t \geq 0$. This estimate and the Lipschitz continuity of V together imply

$$D_+V(x,y)(Ax,Ay) \leq \liminf_{h\downarrow 0} (V(T(h)x, T(h)y) - V(x,y))/h$$
$$+ \lim_{h\downarrow 0} L(\|Ax - (T(h)x - x)/h\| + \|Ay - (T(h)y - y)/h\|) = \omega V(x,y)$$

for $x, y \in D$. □

5. Application to quasi-linear wave equations. This section is devoted to an application of our abstract theory to the Cauchy problem for quasi-linear wave equation with damping

$$\begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \partial_x \varphi'(u) - \nu v. \end{cases} \tag{5.1}$$

Here $\nu > 0$, and we assume that $\varphi \in C^4(\mathbb{R})$ satisfies $\varphi(0) = \varphi'(0) = 0$ and has the property that there is a $c_0 > 0$ such that $\varphi''(r) \geq c_0$ for $r \in \mathbb{R}$. Let us mention the main result in this section.

Theorem 5.1. *There is an $r_0 > 0$ such that for each $(u_0, v_0) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ with $\|(u_0, v_0)\|_{H^2 \times H^2} \leq r_0$, problem* (5.1) *has a unique solution (u, v) in the class*

$$C^1([0,\infty); L^2(\mathbb{R}) \times L^2(\mathbb{R})) \cap L^\infty(0,\infty; H^2(\mathbb{R}) \times H^2(\mathbb{R}))$$

satisfying the initial condition $(u,v)|_{t=0} = (u_0, v_0)$. Moreover, there exist constants $M \geq 1$ and $\omega \geq 0$ such that if (u,v) and $(\hat{u}, \hat{v})$ are solutions of (5.1) *with initial data (u_0, v_0) and $(\hat{u}_0, \hat{v}_0) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ satisfying $\|(u_0, v_0)\|_{H^2 \times H^2} \leq r_0$ and $\|(\hat{u}_0, \hat{v}_0)\|_{H^2 \times H^2} \leq r_0$ respectively, then we have*

$$\|(u(t,\cdot), v(t,\cdot)) - (\hat{u}(t,\cdot), \hat{v}(t,\cdot))\|_{L^2 \times L^2} \leq M e^{\omega t} \|(u_0, v_0) - (\hat{u}_0, \hat{v}_0)\|_{L^2 \times L^2}$$

for $t \geq 0$.

Remark 5.1. The problem of existence and uniqueness of global solutions of (5.1) has been studied in different ways, by several authors such as [8] and [13]. Our approach is operator-theoretic and based on the results in the previous sections.

For the proof of Theorem 5.1 we introduce a functional H defined by

$$H(u,v) = \frac{1}{2}\int_{-\infty}^{\infty} v^2 + |\nu u + \partial_x v|^2 + |\nu\partial_x u + \partial_x^2 v|^2\,dx \\ + \frac{1}{2}\int_{-\infty}^{\infty} \varphi''(u)(|\partial_x u|^2 + |\partial_x^2 u|^2)\,dx + \int_{-\infty}^{\infty}\varphi(u)\,dx$$

for $(u,v) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$. It holds that $c_0 r^2/2 \le \varphi(r) \le M_2(|r|)r^2/2$ for $r \in \mathbb{R}$ where for each $k = 2, 3$, we set $M_k(r) = \sup\{|\varphi^{(k)}(s)| \colon |s| \le r\}$ for $r \in [0,\infty)$. By this inequality and the fact

$$H^1(\mathbb{R}) \subset L^\infty(\mathbb{R}) \quad \text{and} \quad \|u\|_{L^\infty} \le \|u\|_{H^1} \quad \text{for } u \in H^1(\mathbb{R}), \tag{5.2}$$

there exist $C_\nu \ge c_\nu > 0$ such that

$$c_\nu\,\|(u,v)\|^2_{H^2\times H^2} \le H(u,v) \le C_\nu\,(1 \vee M_2(\|u\|_{H^1}))\,\|(u,v)\|^2_{H^2\times H^2} \tag{5.3}$$

for $(u,v) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$.

Now, let X be a real Hilbert space $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with norm $\|(u,v)\| = (\|u\|^2_{L^2} + \|v\|^2_{L^2})^{1/2}$, and define a functional V from $X \times X$ to $[0,\infty)$ by

$$V\left((u,v),(\hat{u},\hat{v})\right) = \Big(\int_{-\infty}^{\infty}\Big(\int_u^{\hat{u}} \sqrt{\varphi''(r)} \wedge M_0\,dr\Big)^2 + (\hat{v}-v)^2\,dx\Big)^{1/2}$$

for $(u,v),(\hat{u},\hat{v}) \in X$. Here M_0 $(\ge \sqrt{c_0})$ will be determined in Proposition 5.5. It is easily seen that $V(\cdot,\cdot)$ is a metric in X and the inequality

$$(1 \wedge c_0)^{1/2}\,\|(u,v)-(\hat{u},\hat{v})\| \le V\left((u,v),(\hat{u},\hat{v})\right) \le (1 \vee M_0)\,\|(u,v)-(\hat{u},\hat{v})\| \tag{5.4}$$

holds for $(u,v),(\hat{u},\hat{v}) \in X$. It follows that the functional V is Lipschitz continuous on $X \times X$. To prove Theorem 5.1 it suffices to show the following theorem by virtue of Theorem 4.2.

Theorem 5.2. *There are $r_0 > 0$ and M_0 $(\ge \sqrt{c_0})$ such that a nonlinear operator A in X defined by*

$$\begin{cases} A(u,v) = (\partial_x v, \partial_x\varphi'(u) - \nu v) & \text{for } (u,v) \in D \\ D = \big\{(u,v) \in H^2(\mathbb{R}) \times H^2(\mathbb{R}) \colon H(u,v) \le r_0\big\} \end{cases}$$

satisfies the following four properties:

(a) *The set D is closed in X.*

(b) *The operator $A\colon D \to X$ is continuous.*

(c) *There exists $\omega \in \mathbb{R}$ such that A is ω-dissipative with respect to V; namely*

$$D_+V\left((u,v),(\hat{u},\hat{v})\right)\left(A(u,v),A(\hat{u},\hat{v})\right) \le \omega V\left((u,v),(\hat{u},\hat{v})\right)$$

for $(u,v),(\hat{u},\hat{v}) \in D$.

(d) $\liminf_{\lambda\downarrow 0} d\left((u,v)+\lambda A(u,v),D\right)/\lambda = 0$ *for $(u,v) \in D$.*

The proof will be divided into a sequence of propositions. Throughout Propositions 5.3, 5.4 and 5.5, let $r_0 > 0$ be an arbitrary but fixed positive number, and the number r_0 will be determined in Proposition 5.9.

Proposition 5.3. *The set $D = \left\{(u,v) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})\colon H(u,v) \le r_0\right\}$ is closed in X.*

Proof. Let $\{(u_n,v_n)\}$ be a sequence in D which converges to (u,v) in X as $n \to \infty$. Since $H(u_n,v_n) \le r_0$ for $n \ge 1$, it follows from (5.3) that $\|(u_n,v_n)\|_{H^2\times H^2}$ is bounded as $n\to\infty$, and we find

$$\begin{aligned} r_0 \ge \frac{1}{2}\int_{-\infty}^{\infty} \{&|v_n|^2 + |\nu u_n + \partial_x v_n|^2 + |\nu\partial_x u_n + \partial_x^2 v_n|^2 \\ &+ \varphi''(u)(|\partial_x u_n|^2 + |\partial_x^2 u_n|^2)\}\,dx \\ &+ \int_{-\infty}^{\infty}\varphi(u_n)\,dx - \frac{1}{2}\int_{-\infty}^{\infty}|\varphi''(u)-\varphi''(u_n)|(|\partial_x u_n|^2 + |\partial_x^2 u_n|^2)\,dx \end{aligned} \tag{5.5}$$

for $n \ge 1$. The reflexivity of $H^2(\mathbb{R})\times H^2(\mathbb{R})$ then implies that $(u,v) \in H^2(\mathbb{R})\times H^2(\mathbb{R})$ and $\{(u_n,v_n)\}$ converges weakly to (u,v) in $H^2(\mathbb{R})\times H^2(\mathbb{R})$ as $n\to\infty$. It remains to show $H(u,v)\le r_0$. We note here that

$$\|\partial_x w\|_{L^2} \le \|w\|_{L^2}^{1/2}\|\partial_x^2 w\|_{L^2}^{1/2} \tag{5.6}$$

for $w \in H^2(\mathbb{R})$, which shows that the sequence $\{(u_n,v_n)\}$ converges to (u,v) in $H^1(\mathbb{R})\times H^1(\mathbb{R})$ as $n\to\infty$, and moreover it converges to (u,v) in $L^\infty(\mathbb{R})\times L^\infty(\mathbb{R})$ as $n\to\infty$, by (5.2). It follows that the third term on the right-hand side of (5.5) vanishes as $n\to\infty$. Since $\varphi(u) = u\int_0^1\varphi'(\theta u)\,d\theta$ and $|\int_0^1\varphi'(\theta u)\,d\theta| \le M_2(\|u\|_{L^\infty})|u| \in L^2(\mathbb{R})$, we have $\varphi(u)\in L^1(\mathbb{R})$, and similarly $\varphi(u_n) \in L^1(\mathbb{R})$ for $n\ge 1$. Noting $\|\int_0^1\varphi'(\theta u)\,d\theta - \int_0^1\varphi'(\theta u_n)\,d\theta\|_{L^2} \le$

$M_2(\|u\|_{L^\infty} \vee \|u_n\|_{L^\infty})\|u - u_n\|_{L^2} \to 0$ and $u_n \to u$ in $L^2(\mathbb{R})$ as $n \to \infty$ we obtain the fact that $\|\varphi(u) - \varphi(u_n)\|_{L^1}$ vanishes as $n \to \infty$. Taking the lim inf in (5.5) as $n \to \infty$ we obtain the desired claim. $\square$

Proposition 5.4. *The operator $A\colon D \to X$ is continuous in X.*

Proof. Let $(u, v), (\hat{u}, \hat{v}) \in D$. By Young's inequality we have

$$\|A(u,v) - A(\hat{u}, \hat{v})\|^2 \leq \|\partial_x(v - \hat{v})\|_{L^2}^2 + 2\|\partial_x(\varphi'(u) - \varphi'(\hat{u}))\|_{L^2}^2 + 2\nu^2\|v - \hat{v}\|_{L^2}^2.$$

The second term on the right-hand side is bounded by

$$2(M_2(\|u\|_{H^1})\|\partial_x(u - \hat{u})\|_{L^2} + M_3(\|u\|_{H^1} \vee \|\hat{u}\|_{H^1})\|\hat{u}\|_{H^2}\|u - \hat{u}\|_{L^2})^2$$

because $\partial_x(\varphi'(u) - \varphi'(\hat{u})) = \varphi''(u)\partial_x(u - \hat{u}) + (\int_0^1 \varphi'''(\theta u + (1-\theta)\hat{u})\, d\theta)(u - \hat{u})\partial_x \hat{u}$. The continuity of A follows from (5.6), since the set D is bounded in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$. $\square$

From (5.2) and the boundedness of the set D in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ it follows that there exists M_0 $(\geq \sqrt{c_0})$ such that

$$\sup\{\sqrt{\varphi''(r)}\colon |r| \leq \|u\|_{L^\infty}\} \leq M_0 \quad \text{for all } (u, v) \in D. \tag{5.7}$$

Proposition 5.5. *Let M_0 $(\geq \sqrt{c_0})$ be a real number satisfying* (5.7). *Then the operator A is ω-dissipative with respect to V for some ω.*

Proof. Let $(u, v), (\hat{u}, \hat{v}) \in D$ and $(\xi, \eta), (\hat{\xi}, \hat{\eta}) \in X$. Noting the choice of M_0 we find, by an easy computation,

$$\begin{aligned} &D_+ V\left((u, v), (\hat{u}, \hat{v})\right)\Big((\xi, \eta), (\hat{\xi}, \hat{\eta})\Big) \cdot V\left((u, v), (\hat{u}, \hat{v})\right) \\ &\quad = \int_{-\infty}^{\infty} (\sqrt{\varphi''(u)}\xi - \sqrt{\varphi''(\hat{u})}\hat{\xi})\Big(\int_{\hat{u}}^{u} \sqrt{\varphi''(r)}\, dr\Big) + (\eta - \hat{\eta})(v - \hat{v})\, dx. \end{aligned}$$

We substitute $(\xi, \eta) = A(u, v)$ and $(\hat{\xi}, \hat{\eta}) = A(\hat{u}, \hat{v})$ into this identity, and use the relation $\int_{-\infty}^{\infty} \partial_x(\varphi'(u) - \varphi'(\hat{u}))(v - \hat{v})\, dx = -\int_{-\infty}^{\infty} \left(\int_{\hat{u}}^{u} \varphi''(r)\, dr\right) \partial_x(v - \hat{v})\, dx$. This yields

$$\begin{aligned} &D_+ V\left((u, v), (\hat{u}, \hat{v})\right)\left(A(u, v), A(\hat{u}, \hat{v})\right) \cdot V\left((u, v), (\hat{u}, \hat{v})\right) \\ &\quad = \int_{-\infty}^{\infty} \partial_x v \int_{\hat{u}}^{u} (\sqrt{\varphi''(u)}\sqrt{\varphi''(r)} - \varphi''(r))\, dr\, dx \\ &\qquad + \int_{-\infty}^{\infty} \partial_x \hat{v} \int_{u}^{\hat{u}} (\sqrt{\varphi''(\hat{u})}\sqrt{\varphi''(r)} - \varphi''(r))\, dr\, dx - \nu \int_{-\infty}^{\infty} |v - \hat{v}|^2\, dx. \end{aligned}$$

By making use of the inequality

$$\Big|\int_s^t \sqrt{\varphi''(t)}\sqrt{\varphi''(r)} - \varphi''(r)\,dr\Big| \le M_3(|t|\vee|s|)|t-s|^2 \quad \text{for } t, s \in \mathbb{R},$$

the equality above implies the dissipativity of A by virtue of (5.2) and (5.4), since the set D is bounded in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$. □

The following slightly modified results of Corollaries VIII.9 and VIII.10 in Brezis [2] are needed for later arguments.

Lemma 5.6. *The following assertions hold:*

(i) *If $u \in W^{1,\infty}(\mathbb{R})$ and $v \in H^1(\mathbb{R})$, then $uv \in H^1(\mathbb{R})$ and $\partial_x(uv) = \partial_x u \cdot v + u \cdot \partial_x v$.*

(ii) *If $G \in C^1(\mathbb{R})$ and $u \in W^{1,\infty}(\mathbb{R})$, then $G(u) \in W^{1,\infty}(\mathbb{R})$ and $\partial_x G(u) = G'(u)\partial_x u$.*

It remains to prove assertion (d) of Theorem 5.2. For this purpose we need the following fundamental result for a linear problem.

Proposition 5.7. *Let $p \in W^{1,\infty}(\mathbb{R})$ be such that $p(x) \ge c_0$ for $x \in \mathbb{R}$, and define a linear operator L in X by*

$$\begin{cases} L(u,v) = (\partial_x v, p(x)\partial_x u - \nu v) & \text{for } (u,v) \in D(L) \\ D(L) = H^1(\mathbb{R}) \times H^1(\mathbb{R}). \end{cases} \tag{5.8}$$

Then, $L - \beta I$ is m-dissipative in X with $\beta = \|\partial_x p\|_{L^\infty}/2\sqrt{c_0}$, if $X = L^2(\mathbb{R}) \times L^2(\mathbb{R})$ is equipped with the following inner product equivalent to the original one:

$$((u,v),(\hat{u},\hat{v})) = \int_{-\infty}^{\infty} p(x)u\hat{u} + v\hat{v}\,dx \qquad \text{for } (u,v),(\hat{u},\hat{v}) \in X.$$

Proof. It is easily seen that $L - \beta I$ is dissipative in the Hilbert space $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ equipped with inner product $(\cdot,\cdot)$. Now, let $V = H^1(\mathbb{R})$, $\lambda > \beta$, and $f, g \in L^2(\mathbb{R})$. A bounded bilinear form $a\colon V \times V \to \mathbb{R}$ defined by $a(u,w) = \int_{-\infty}^{\infty} \lambda(\lambda+\nu)uw + p(x)\partial_x u \partial_x w\,dx$ is coercive on $V \times V$. A functional F on V defined by $F(w) = \int_{-\infty}^{\infty} (\lambda+\nu)fw - g\partial_x w\,dx$ is linear and bounded. By Lax-Milgram's theorem there exists $u \in V = H^1(\mathbb{R})$ such that $a(u,w) = F(w)$ for all $w \in V$. If we put $v = (p(x)\partial_x u + g)/(\lambda+\nu) \in L^2(\mathbb{R})$, then $v \in H^1(\mathbb{R})$ and $(\lambda I - L)(u,v) = (f,g)$. □

Proposition 5.8. *Let $(u_0, v_0) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$. Then, there is a $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0]$, the problem*

$$u_\lambda - u_0 = \lambda \partial_x v_\lambda \tag{5.9}$$

$$v_\lambda - v_0 = \lambda \varphi''(u_0) \partial_x u_\lambda - \lambda \nu v_\lambda \tag{5.10}$$

has a solution $(u_\lambda, v_\lambda) \in H^3(\mathbb{R}) \times H^3(\mathbb{R})$ satisfying the following properties:

(a) *The sequence $\{(u_\lambda, v_\lambda)\}$ converges to (u_0, v_0) in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ as $\lambda \downarrow 0$.*

(b) *There is a nonnegative, nondecreasing, continuous function $\rho \colon \mathbb{R} \to \mathbb{R}$, depending only on ν and φ, which satisfies the property $\rho(0) = 0$ such that*

$$\begin{aligned}&(H(u_\lambda, v_\lambda) - H(u_0, v_0))/\lambda \\ &\le (1 + \lambda^2)\rho\left(\|(u_\lambda, v_\lambda)\|_{H^2 \times H^2} \vee \|(u_0, v_0)\|_{H^2 \times H^2}\right)(\|\partial_x u_\lambda\|_{H^1} \vee \|\partial_x u_0\|_{H^1})^2 \\ &\qquad - \nu c_0 \|\partial_x u_\lambda\|_{H^1}^2\end{aligned}$$

for $0 < \lambda \le \lambda_0$.

Proof. Let $(u_0, v_0) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$. By (5.2) we have $u_0 \in W^{1,\infty}(\mathbb{R})$; hence $\varphi''(u_0) \in W^{1,\infty}(\mathbb{R})$ by (ii) of Lemma 5.6, and $\varphi''(u_0(x)) \ge c_0$ for $x \in \mathbb{R}$. We now consider a linear operator L defined by (5.8) where $p(x) = \varphi''(u_0(x))$. Set $\beta = \|\partial_x p\|_{L^\infty}/2\sqrt{c_0}$, and choose $\lambda_0 > 0$ so that $\lambda_0 \beta < 1$. Then, it is seen by Proposition 5.7 that for $0 < \lambda \le \lambda_0$, $(u_\lambda, v_\lambda) := (I - \lambda L)^{-1}(u_0, v_0)$ satisfies (5.9) and (5.10). A simple computation yields $D(L^k) = H^k(\mathbb{R}) \times H^k(\mathbb{R})$ for $k = 2, 3$, by using Lemma 5.6. Since $(u_0, v_0) \in D(L^2)$ it follows that $(u_\lambda, v_\lambda) \in H^3(\mathbb{R}) \times H^3(\mathbb{R})$ and $L^k(u_\lambda, v_\lambda) = (I - \lambda L)^{-1} L^k(u_0, v_0)$ for $k = 0, 1, 2$, which implies that for each $k = 0, 1, 2$, the sequence $\left\{L^k(u_\lambda, v_\lambda)\right\}$ converges to $L^k(u_0, v_0)$ in X as $\lambda \downarrow 0$. Noting that the graph norm $(\|L(u, v)\|^2 + \|(u, v)\|^2)^{1/2}$ of L is equivalent to the norm $\|(u, v)\|_{H^1 \times H^1}$, we deduce that the sequence $\{(u_\lambda, v_\lambda)\}$ converges to (u_0, v_0) in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ as $\lambda \downarrow 0$, by the identity $\partial_x\left(L(u_\lambda, v_\lambda)\right) = L(\partial_x u_\lambda, \partial_x v_\lambda) + (0, \partial_x p(x) \partial_x u_\lambda)$.

To prove (b), from now on, ρ stands for a nonnegative, nondecreasing, continuous function, depending only on ν and φ, which satisfies $\rho(0) = 0$. Since $\varphi'(0) = 0$, we have $\varphi'(u_\lambda) \in H^1(\mathbb{R})$ and $\partial_x \varphi'(u_\lambda) = \varphi''(u_\lambda)\partial_x u_\lambda$, by which (5.10) is rewritten as

$$v_\lambda - v_0 = \lambda \partial_x \varphi'(u_\lambda) - \lambda \nu v_\lambda + \lambda(\varphi''(u_0) - \varphi''(u_\lambda))\partial_x u_\lambda. \tag{5.11}$$

We multiply (5.9) and (5.11) by $\varphi'(u_\lambda)$ and v_λ respectively. The sum of these two equations gives

$$\begin{aligned}&\varphi'(u_\lambda)(u_\lambda - u_0) + v_\lambda(v_\lambda - v_0)\\&\quad = \lambda\partial_x(v_\lambda\varphi'(u_\lambda)) - \lambda\nu|v_\lambda|^2 + \lambda(\varphi''(u_0) - \varphi''(u_\lambda))v_\lambda\partial_x u_\lambda.\end{aligned}$$

Integrating this equality over $\mathbb{R}$ and using Young's inequality we find

$$\int_{-\infty}^{\infty} \varphi'(u_\lambda)(u_\lambda - u_0) + v_\lambda(v_\lambda - v_0)\,dx \le \frac{\lambda}{4\nu}\int_{-\infty}^{\infty} |\varphi''(u_0) - \varphi''(u_\lambda)|^2|\partial_x u_\lambda|^2\,dx. \tag{5.12}$$

Since $\varphi''(u_\lambda) - \varphi''(u_0) = \left(\int_0^1 \varphi'''(\theta u_\lambda + (1-\theta)u_0)\,d\theta\right)(u_\lambda - u_0)$ we have, by (5.2) and (5.9),

$$\|\varphi''(u_\lambda) - \varphi''(u_0)\|_{L^\infty} \le \lambda M_3(\|u_0\|_{H^1} \vee \|u_\lambda\|_{H^1})\|v_\lambda\|_{H^2}. \tag{5.13}$$

The fact that φ is convex and the inequality (5.13) are used to estimate the left and right-hand sides of (5.12) respectively. This yields

$$\begin{aligned}&\Big(\int_{-\infty}^{\infty} \varphi(u_\lambda) - \varphi(u_0) + \frac{1}{2}(|v_\lambda|^2 - |v_0|^2)\,dx\Big)/\lambda \qquad (5.14)\\&\quad \le \lambda^2\rho\left(\|(u_\lambda, v_\lambda)\|_{H^2\times H^2} \vee \|(u_0, v_0)\|_{H^2\times H^2}\right)\|\partial_x u_\lambda\|_{L^2}^2.\end{aligned}$$

By (5.9) we have

$$\partial_x u_\lambda - \partial_x u_0 = \lambda\partial_x(\partial_x v_\lambda), \tag{5.15}$$

$$\partial_x^2 u_\lambda - \partial_x^2 u_0 = \lambda\partial_x(\partial_x^2 v_\lambda). \tag{5.16}$$

Since $u_0 \in W^{1,\infty}(\mathbb{R})$ and $\varphi \in C^4(\mathbb{R})$, the fact that $\varphi''(u_0), \varphi'''(u_0) \in W^{1,\infty}(\mathbb{R})$ follows from (ii) of Lemma 5.6. This together with the fact that $u_\lambda \in H^3(\mathbb{R})$ implies $\varphi''(u_0)\partial_x u_\lambda \in H^2(\mathbb{R})$ by (i) of Lemma 5.6. By (5.10) we have

$$\partial_x v_\lambda - \partial_x v_0 = \lambda\partial_x(\varphi''(u_0)\partial_x u_\lambda) - \lambda\nu\partial_x v_\lambda, \tag{5.17}$$

into which we substitute (5.9) to get

$$\nu u_\lambda + \partial_x v_\lambda - (\nu u_0 + \partial_x v_0) = \lambda\partial_x(\varphi''(u_0)\partial_x u_\lambda). \tag{5.18}$$

Differentiating (5.18) we obtain

$$\nu\partial_x u_\lambda + \partial_x^2 v_\lambda - (\nu\partial_x u_0 + \partial_x^2 v_0) = \lambda\partial_x(\varphi''(u_0)\partial_x^2 u_\lambda) + \lambda\partial_x(\partial_x\varphi''(u_0)\cdot\partial_x u_\lambda). \tag{5.19}$$

We multiply (5.15) and (5.18) by $\varphi''(u_0)\partial_x u_\lambda$ and $\nu u_\lambda + \partial_x v_\lambda$ respectively, and sum up these two equations. Integrating the resultant equality over $\mathbb{R}$, we find by integration by parts

$$\int_{-\infty}^{\infty}\varphi''(u_0)\partial_x u_\lambda(\partial_x u_\lambda - \partial_x u_0) + (\nu u_\lambda + \partial_x v_\lambda)(\nu u_\lambda + \partial_x v_\lambda - (\nu u_0 + \partial_x v_0))dx$$
$$= -\lambda\nu\int_{-\infty}^{\infty}\varphi''(u_0)|\partial_x u_\lambda|^2\,dx;$$

hence,

$$\Big(\int_{-\infty}^{\infty}\varphi''(u_0)\frac{1}{2}(|\partial_x u_\lambda|^2 - |\partial_x u_0|^2) + \frac{1}{2}(|\nu u_\lambda + \partial_x v_\lambda|^2 - |\nu u_0 + \partial_x v_0|^2)\,dx\Big)/\lambda$$
$$\leq -\nu c_0\|\partial_x u_\lambda\|_{L^2}^2, \tag{5.20}$$

where we have used the fact that $\varphi''(r) \geq c_0$ for $r \in \mathbb{R}$. Multiplying (5.16) and (5.19) by $\varphi''(u_0)\partial_x^2 u_\lambda$ and $\nu\partial_x u_\lambda + \partial_x^2 v_\lambda$ respectively we find, in a way similar to the derivation of (5.20),

$$\Big(\int_{-\infty}^{\infty}\varphi''(u_0)\tfrac{1}{2}(|\partial_x^2 u_\lambda|^2 - |\partial_x^2 u_0|^2) + \tfrac{1}{2}(|\nu\partial_x u_\lambda + \partial_x^2 v_\lambda|^2 - |\nu\partial_x u_0 + \partial_x^2 v_0|^2)dx\Big)/\lambda$$
$$\leq \rho\left(\|(u_0,v_0)\|_{H^2\times H^2} \vee \|(u_\lambda,v_\lambda)\|_{H^2\times H^2}\right)(\|\partial_x u_0\|_{H^1} \vee \|\partial_x u_\lambda\|_{H^1})^2$$
$$- \nu c_0\|\partial_x^2 u_\lambda\|_{L^2}^2. \tag{5.21}$$

Combining (5.14), (5.20) and (5.21) we have

$$(H(u_\lambda,v_\lambda) - H(u_0,v_0))/\lambda$$
$$\leq (1+\lambda^2)\rho\left(\|(u_0,v_0)\|_{H^2\times H^2} \vee \|(u_\lambda,v_\lambda)\|_{H^2\times H^2}\right)(\|\partial_x u_0\|_{H^1} \vee \|\partial_x u_\lambda\|_{H^1})^2$$
$$+ \frac{1}{\lambda}\int_{-\infty}^{\infty}(\varphi''(u_\lambda) - \varphi''(u_0))\frac{1}{2}(|\partial_x u_\lambda|^2 + |\partial_x^2 u_\lambda|^2)\,dx - \nu c_0\|\partial_x u_\lambda\|_{H^1}^2$$

for $\lambda \in (0,\lambda_0]$. By (5.13) we conclude that the problem (5.9) and (5.10) has a solution $(u_\lambda, v_\lambda) \in H^3(\mathbb{R}) \times H^3(\mathbb{R})$ satisfying two properties (a) and (b). $\square$

Proposition 5.9. *There is an* $r_0 > 0$ *such that*

$$\liminf_{\lambda\downarrow 0} d\left((u,v)+\lambda A(u,v), D\right)/\lambda = 0$$

for $(u,v) \in D$*, where* $D = \left\{(u,v) \in H^2(\mathbb{R}) \times H^2(\mathbb{R}) \colon H(u,v) \le r_0\right\}$*.*

Proof. Let ρ be a nonnegative, nondecreasing, continuous function as in (b) of Proposition 5.8. By (5.3) we choose an $r_0 > 0$ such that $H(u,v) \le r_0$ implies

$$\rho\left(\|(u,v)\|_{H^2\times H^2}\right) < \nu c_0. \tag{5.22}$$

Now, let $(u_0,v_0) \in D$; namely $(u_0,v_0) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ and $H(u_0,v_0) \le r_0$. Then the problem (5.9) and (5.10) has a solution $(u_\lambda, v_\lambda) \in H^3(\mathbb{R}) \times H^3(\mathbb{R})$ satisfying two properties (a) and (b) of Proposition 5.8. Since

$$\limsup_{\lambda\downarrow 0}(H(u_\lambda,v_\lambda) - H(u_0,v_0))/\lambda \le \left(\rho\left(\|(u_0,v_0)\|_{H^2\times H^2}\right) - \nu c_0\right)\|\partial_x u_0\|_{H^1}^2,$$

we have $H(u_\lambda,v_\lambda) \le r_0$ for sufficiently small λ, unless $\|\partial_x u_0\|_{H^1} = 0$. If $\|\partial_x u_0\|_{H^1} = 0$, then $u_0(x) = 0$ for $x \in \mathbb{R}$, and we have by (b)

$$\begin{aligned}&(H(u_\lambda,v_\lambda) - H(u_0,v_0))/\lambda \\ &\quad \le \left((1+\lambda^2)\rho\left(\|(u_\lambda,v_\lambda)\|_{H^2\times H^2} \vee \|(u_0,v_0)\|_{H^2\times H^2}\right) - \nu c_0\right)\|\partial_x u_\lambda\|_{H^1}^2.\end{aligned}$$

By (5.22) and property (a), there exists $\lambda_1 > 0$ such that

$$(1+\lambda^2)\rho\left(\|(u_\lambda,v_\lambda)\|_{H^2\times H^2} \vee \|(u_0,v_0)\|_{H^2\times H^2}\right) - \nu c_0 \le 0$$

for $\lambda \in (0,\lambda_1]$; hence $H(u_\lambda,v_\lambda) \le r_0$ for $\lambda \in (0,\lambda_1]$. It has been shown that $(u_\lambda,v_\lambda) \in D$ for sufficiently small λ. Since (u_λ,v_λ) satisfies (5.9) and (5.10) we have $\lim_{\lambda\downarrow 0}\left\|\lambda^{-1}\left((u_\lambda,v_\lambda)-(u_0,v_0)\right) - A(u_0,v_0)\right\| = 0$ by property (a) of Proposition 5.8, and finally the subtangential condition is proved. □

6. Toward a generation theory in the discontinuous "generator" case. In this section we discuss, without proof, a generation theorem of a semigroup of Lipschitz operators associated with the differential inclusion

$$u'(t) \in Au(t) \quad \text{for } t > 0 \quad \text{and} \quad u(0) = u_0, \tag{DI}$$

where A is a multivalued operator in X satisfying the dissipativity condition

(H1) for $\varepsilon > 0$ there exists $h_0 > 0$ such that $\lambda, \mu \in (0, h_0]$ implies

$$(V(x,y) - V(x-\lambda\xi, y))/\lambda + (V(x,y) - V(x, y-\mu\eta))/\mu \leq \omega V(x,y) + \varepsilon$$

for $(x,\xi), (y,\eta) \in A$.

Here the functional V is assumed to satisfy conditions (V1) through (V3).

In general, the differential inclusion (DI) does not admit solutions in the sense of Section 1 even though the initial value lies in $D(A)$. This is the reason why we adopt a notion of solution which refers directly to the approximation method used to establish the existence of solutions, the so-called *method of discretization in time*.

Let $\varepsilon > 0$. A piecewise-constant function $v(t) = v_0$ for $t = 0$, and v_i for $t \in (t_{i-1}, t_i] \cap [0,\tau]$ is called an *ε-approximate solution* of (DI) on $[0,\tau]$, if there exists a partition $\{0 = t_0 < t_1 < \cdots < t_N\}$ of the interval $[0, t_N]$ and a finite sequence $\{(v_i, w_i)\}_{i=1}^N$ with the three properties below:

(i) $(v_i - v_{i-1})/(t_i - t_{i-1}) \in Av_i + w_i$ for $i = 1, 2, \ldots, N$.

(ii) $t_i - t_{i-1} \leq \varepsilon$ for $i = 1, 2, \ldots, N$ and $\tau \leq t_N < \tau + \varepsilon$.

(iii) $\sum_{i=1}^N (t_i - t_{i-1})\|w_i\| \leq \varepsilon t_N$.

A continuous function $u\colon [0,\tau] \to X$ is said to be a *mild solution* of (DI) on $[0,\tau]$, provided that for each $\varepsilon > 0$ there is an ε-approximate solution v^ε of (DI) on $[0,\tau]$ such that $\|u(t) - v^\varepsilon(t)\| \leq \varepsilon$ for $t \in [0,\tau]$.

The uniqueness theorem for mild solutions of (DI) is given by

Theorem 6.1. *Let $u\colon [0,\tau] \to X$ and $v\colon [0,\tau] \to X$ be mild solutions of* (DI) *on $[0,\tau]$ respectively. Then we have*

$$V(v(t), u(t)) \leq e^{\omega t} V(v(0), u(0)) \quad \text{for } t \in [0,\tau].$$

If $v(0) = u(0)$ in particular, then we have $v(t) = u(t)$ for $t \in [0,\tau]$, under the assumption that $\overline{D(A)} = D$.

To establish an existence theorem, the following tangency range condition is imposed on A:

(H2) For $\varepsilon > 0$ and $x \in D$ there exist $\delta \in (0,\varepsilon]$, $x_\delta \in D(A)$ and $z_\delta \in X$ such that $(x_\delta - x)/\delta \in Ax_\delta + z_\delta$ and $\|z_\delta\| \leq \varepsilon$.

We are now in a position to state an existence theorem.

Theorem 6.2. *Assume that $\overline{D(A)} = D$ and let $u_0 \in D$. Then there exists a global mild solution u of the differential inclusion* (DI).

In a special case where the nonnegative, Lipschitz-continuous functional V on $X \times X$ satisfies property (V) of Theorem 4.1, we obtain the following generation theorem:

Theorem 6.3. *Assume that* $\overline{D(A)} = D$. *Then there exists a semigroup* $\{T(t)\colon t \geq 0\}$ *of Lipschitz operators on* D *such that for each* $u_0 \in D$, *the function* $u(\cdot) = T(\cdot)u_0$ *gives a unique global mild solution of* (DI).

This proof is based on the following convergence theorem, which is an extension of [5, Theorem 2.1]:

Theorem 6.4. *Assume that* $\overline{D(A)} = D$, *and let* $\tau > 0$ *and* $u_0 \in D$. *Suppose that there exists an* $\varepsilon_0 > 0$ *such that for each* $\varepsilon \in (0, \varepsilon_0]$, *the differential inclusion* (DI) *has an* ε*-approximate solution* u^ε *on* $[0, \tau]$. *If* $\lim_{\varepsilon \downarrow 0} u^\varepsilon(0) = u_0$ *then there exists a unique mild solution* u *of* (DI) *on* $[0, \tau]$ *such that*

$$\lim_{\varepsilon \downarrow 0} (\sup\{\|u^\varepsilon(t) - u(t)\|\colon t \in [0, \tau]\}) = 0.$$

Finally, we give an application of the abstract theory in this section to the following quasi-linear wave equation of Kirchhoff type:

$$\begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \beta'(\|u\|_{L^2}^2)\partial_x u - \nu v. \end{cases} \tag{6.1}$$

Here $\nu > 0$ and $\beta \in C^2([0, \infty)\colon \mathbb{R})$ satisfies $\beta(0) = 0$ and the property that there exists $m_0 > 0$ such that $\beta'(r) \geq m_0$ for $r \in [0, \infty)$.

Let X be the real Hilbert space $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ equipped with the norm $\|(u, v)\| = (\|u\|_{L^2}^2 + \|v\|_{L^2}^2)^{1/2}$. Let $R_0 > 0$ and set $W = \{(u, v) \in X\colon \|u\|_{L^2} < R_0\}$. Then, there exists a nonnegative functional V on $X \times X$ satisfying conditions (V1) through (V3) such that

$$V((u, v), (\hat{u}, \hat{v})) = (\beta'(\|u\|_{L^2}^2)\|u - \hat{u}\|_{L^2}^2 + \|v - \hat{v}\|_{L^2}^2)^{1/2}$$

for $(u, v), (\hat{u}, \hat{v}) \in W$. Let us consider the closed subset D of X in the form

$$D = \{(u, v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})\colon H(u, v) \leq r_0\},$$

where the functional H is defined by

$$H(u, v) = \|v\|_{L^2}^2 + \|\nu u + \partial_x v\|_{L^2}^2 + \beta(\|u\|_{L^2}^2) + \beta'(\|u\|_{L^2})\|\partial_x u\|_{L^2}^2$$

for $(u, v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Then, we obtain the following theorem.

Theorem 6.5. *There exists $r_0 > 0$ such that the operator A in X defined by*

$$A(u,v) = (\partial_x v, \beta'(\|u\|_{L^2}^2)\partial_x u - \nu v) \quad \text{for } (u,v) \in D \ (\subset W)$$

satisfies conditions (H1) *and* (H2).

By virtue of Theorem 6.5 the following theorem is deduced from Theorem 6.4.

Theorem 6.6. *There is an $r_0 > 0$ such that for each $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ with $\|(u_0, v_0)\|_{H^1 \times H^1} \leq r_0$, the problem* (6.1) *has a unique solution (u, v) in the class*

$$W^{1,1}_{\text{loc}}(0, \infty; L^2(\mathbb{R}) \times L^2(\mathbb{R})) \cap L^\infty(0, \infty; H^1(\mathbb{R}) \times H^1(\mathbb{R}))$$

satisfying the initial condition $(u,v)|_{t=0} = (u_0, v_0)$. Moreover, there exist constants $M \geq 1$ and $\omega \geq 0$ such that if (u, v) and $(\hat{u}, \hat{v})$ are solutions of (6.1) *with initial data (u_0, v_0) and $(\hat{u}_0, \hat{v}_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ satisfying $\|(u_0, v_0)\|_{H^1 \times H^1} \leq r_0$ and $\|(\hat{u}_0, \hat{v}_0)\|_{H^1 \times H^1} \leq r_0$ respectively, then we have*

$$\|(u(t,\cdot), v(t,\cdot)) - (\hat{u}(t,\cdot), \hat{v}(t,\cdot))\|_{L^2 \times L^2} \leq Me^{\omega t} \|(u_0, v_0) - (\hat{u}_0, \hat{v}_0)\|_{L^2 \times L^2}$$

for $t \geq 0$.

Remark 6.1. Another approach to the problem (6.1) is found in [14].

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