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ON A NONLINEAR DISPERSIVE EQUATION WITH TIME-DEPENDENT COEFFICIENTS

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Abstract. As a first step, we consider an evolution linear problem, the symbol of which is a real polynomial of degree three with time-dependent coefficients. We get for this problem smoothing effects known when these coefficients are constant. In particular, by using the theory of Calderón-Zygmund operators and the David and Journé T1 Theorem, we establish a local smoothing effect on the solution of the linear problem. In a second step, we study a nonlinear dispersive equation the linear part of which is the one studied above. We use the previous smoothing properties and a regularization method to establish that the Cauchy problem is locally well-posed in the Sobolev spaces $H^s(\mathbb{R})$ for $s > 3/4$.

1. Introduction

This paper is concerned with the following equation, which appears in nonlinear optics:

$$i\, u_t + i\, a_3\, u_{xxx} + a_2(t)\, u_{xx} = u(g * |u|^2) + i\, \frac{\partial}{\partial x}\big[u(g * |u|^2)\big], \tag{1.1}$$

where $u = u(t, x)$ is a complex-valued function, a_3 belongs to $\mathbb{R}$, $a_2(t)$ is a continuously differentiable real function, $*$ is the convolution with respect to x and g is the sum of a delta function and an integrable one.

This equation is used to model nonlinear propagation of ultrashort light pulses through optical fibers in order to study the generation of solitons due to nonlinear and dispersive effects and based on the Raman self-scattering effect ([24]). If E denotes the complex electric field envelope, z the coordinate along the fiber axis, t the time and $\tau = k_1\, t - k_2\, z$, where k_1 and k_2 are two real

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constants, the third-order nonlinear medium polarization when molecular vibrations are excited by the field intensity can be written in the form

$$E(\tau)\int F(\theta)|E(\tau-\theta)|^2\,d\theta, \tag{1.2}$$

where the response function $F(\theta)$ is determined by the nonlinear properties of the medium. It can be written as the sum of a delta function determined by electrons' contribution and an integrable function determined by the Raman contribution. In this case, the equation for the electric field is in the time domain

$$\begin{aligned} i\,\frac{\partial E}{\partial x} \pm \frac{1}{2}\frac{\partial^2 E}{\partial \tau^2} - i\,\gamma\frac{\partial^3 E}{\partial \tau^3} + E\int F(\theta)|E(\tau-\theta)|^2\,d\theta \\ +i\,\sigma\frac{\partial}{\partial \tau}\big[E\int F(\theta)|E(\tau-\theta)|^2\,d\theta\big] = 0\ . \end{aligned} \tag{1.3}$$

As a particular case of this equation we recover the classical nonlinear Schrödinger equation that was first considered for nonlinear pulse propagation [10]:

$$i\,u_t \pm \frac{1}{2}u_{xx} + |u|^2u = 0. \tag{1.4}$$

Another special case of equation (1.3) is the derivative nonlinear Schrödinger equation

$$i\,u_t + u_{xx} + i\,(|u|^2u)_x = 0 \tag{1.5}$$

for which the well-posedness of the Cauchy problem has been established globally in $H^m(\mathbb{R})$ for $m \in \mathbb{N}^*$ when the initial datum is small enough in $L^2(\mathbb{R})$, by N. Hayashi and T. Ozawa ([11], [12]).

For small initial data too, I. Fukuda and M. Tsutsumi proved that the initial value problem for equation (1.5) is globally well-posed in $H^m(\mathbb{R})$ for $m \geq 2$ and locally for $m > 3/2$ ([6], [7], [8]).

Results on this equation have also been obtained in the more general framework of the mixed nonlinear Schrödinger equation,

$$i\,u_t + u_{xx} = i\,\lambda\,|u|^2u_x + i\,\mu\,u^2\overline{u}_x + f(u), \tag{1.6}$$

where different assumptions are made on f.

Tan Shaobin and Zhang Linghai ([32]) proved that for any $u_0 \in H^1(\mathbb{R})$, equation (1.6) has a unique solution $u \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R})) \cap L^q_{loc}(\mathbb{R}; W^{1,r}(\mathbb{R}))$, with $r \in (2,6)$ and $q = 4\,r/(r-2)$, such that $u(0) = u_0$, while T. Ozawa ([26]) established that the previous problem has a unique solution in $\mathcal{C}(\mathbb{R}; H^1(\mathbb{R})) \cap L^4_{loc}(\mathbb{R}; W^{1,\infty}(\mathbb{R}))$.

The added terms in equation (1.3) take higher-order effects into consideration; in particular, the third-order derivative of E represents higher-order dispersion.

This equation must again be modified when one uses fibers with a slowly decreasing value of the second-order dispersion along the fiber length; then the coefficients of the second derivative of E become a function of z. This is the case when one wants to amplify spectral bandwidth in order to get the shortest possible soliton pulsewidth. And hence comes equation (1.1).

Considering equation (1.1) with a_2 a constant and g the Dirac measure, we recover the equation introduced by A. Hasegawa and Y. Kodama ([9], [20]) taking into consideration higher-order effects than does the NLS equation

$$\begin{aligned} & i\,\frac{\partial E}{\partial z} \pm \frac{1}{2}\frac{\partial^2 E}{\partial \tau^2} + |E|^2 E \\ & + i\,\varepsilon\big(\beta_1 \frac{\partial^3 E}{\partial \tau^3} + \beta_2 \frac{\partial}{\partial \tau}(|E|^2 E) + \beta_3 E \frac{\partial}{\partial \tau}|E|^2\big) = 0. \end{aligned} \tag{1.7}$$

Extending ideas of C. Kenig, G. Ponce and L. Vega ([17]), we were able in [21] and [23] to prove that the Cauchy problem for equation (1.7) is locally well-posed in $H^s(\mathbb{R})$ with $s > 3/4$ and globally well-posed in $H^1(\mathbb{R})$. G. Staffilani improved these results in [28] by establishing local well-posedness in $H^s(\mathbb{R})$ for $s \geq 1/4$.

The present work establishes that the latest local well-posedness result in $H^s(\mathbb{R})$ with $s > 3/4$ still holds for equation (1.1). As a first step, we consider the linear part of this equation, getting smoothing properties on solutions of

$$u_t + L(t)u = 0, \quad u(0,x) = u_0(x),$$

where $L(t)u$ is defined by its Fourier transform with respect to the space variable $\mathcal{F}(L(t)u) = -i\,p(t,\xi)\hat{u}(\xi)$, with p a real polynomial of degree three with coefficients dependent on time. If the coefficients of p are smooth on an interval $[0,T]$ and if its dominant coefficient never vanishes, we get the local smoothing effect

$$\big\|\frac{\partial}{\partial x}\,S(t_0,t)\,u(x)\big\|_{L^\infty_x(\mathbb{R}\,;\,L^2_t(t_0;T))} \leq c\,(T+1)^{\frac{1}{2}}\,\|u_0\|_{L^2(\mathbb{R})} \tag{1.8}$$

where $S(t_0,t)$ $(\,0 \leq t_0 \leq t \leq T\,)$ is the evolution system generated by $L(t)$ in Sobolev spaces. While this result is easily obtained for a time independent polynomial, by using the Plancherel Theorem in the time variable, the method fails in our case, and we have to make use of more delicate tools involving the theory of Calderón-Zygmund operators. In particular, David and Journé's T1 Theorem is used to establish the existence of a bounded extension to $L^2(\mathbb{R})$ for some singular integral operators ([4], [5]). We also establish the following global smoothing effect and the estimate on the maximal function:

$$\left\|S(t_0,t)\,u_0(x)\right\|_{L^2_x(\mathbb{R}\,;\,L^\infty_t(t_0;T))} \le c_s(T+1)^{\frac{1}{2}}\left\|u_0\right\|_{H^s} \tag{1.9}$$

$$\left\|D^{\frac{1}{4}}S(t_0,t)\,u\right\|_{L^4_t(t_0,T\,;\,L^\infty_x(\mathbb{R}))} \le c\,(T+1)^{\frac{1}{4}}\left\|u\right\|_{L^2(\mathbb{R})}\,. \tag{1.10}$$

In a second step, we study the local Cauchy problem for equation (1.1) and establish that it is well-posed in $H^s(\mathbb{R})$ with $s > 3/4$. We use a regularization method together with the smoothing properties of the linear evolution equation. New difficulties arise in the nonlocal nonlinear terms. Part of our results were announced in [22].

Notation 1.1. $\mathcal{F}(u)$ and $\hat{u}$ denote the Fourier transform with respect to the space variable x of a function $u(t,x)$ while $\mathcal{H}u$ will denote its Hilbert transform with respect to x. J^s denotes the Bessel potential: $J^s = (I-\Delta)^{s/2}$ and D^s denotes the Riesz potential: $D^s = (-\Delta)^{s/2}$. $H^s_p(\mathbb{R})$ is the generalized Sobolev space: $H^s_p(\mathbb{R}) = \{f \in \mathcal{S}'(\mathbb{R})\,,\, \left\|J^s\,\right\|_{L^p(\mathbb{R})} < +\infty\}$ and $\left\|f\right\|_{H^s_p(\mathbb{R})} = \left\|J^s\,f\right\|_{L^p(\mathbb{R})}$. BMO denotes the space of bounded mean oscillation functions on $\mathbb{R}$: $BMO(\mathbb{R}) = \{f \in L^1_{loc}(\mathbb{R})\,,\left\|f\right\|_{BMO} < +\infty\}$ with

$$\left\|f\right\|_{BMO} = \sup_I\Big\{\frac{1}{|I|}\int_I |f(x) - f_I|\,dx\Big\},$$

where the supremum is taken over all the intervals $I \subset \mathbb{R}$ and $f_I = \frac{1}{|I|}\int_I f(x)dx$.

We will use subscripts to denote differentiation. By c we will denote different numerical constants and we denote by $c(\star,\dots,\star)$ constants depending on the quantities occurring in brackets.

2. The linear problem

We consider the following linear equation:

$$u_t + L(t)\,u = 0, \tag{2.1}$$

where $u(t,x)$ is a complex-valued function of $t \in \mathbb{R}$ and $x \in \mathbb{R}$, and $L(t)$ is a Fourier multiplier with respect to the space variable x, the symbol of which is given by a real polynomial $p(t,\xi)$ of degree three in ξ with coefficients dependent on time. We write

$$p(t,\xi) \;= \sum_{k=0}^{3} a_k(t)\,\xi^k \tag{2.2}$$

$$\mathcal{F}(L(t)u)(\xi) = -i\,p(t,\xi)\;\hat{u}(\xi)\;. \tag{2.3}$$

We assume that $a_k \in \mathcal{C}^1(0,T_0)$ for some $T_0 > 0$ and that a_3 never vanishes on $[0,T_0]$. We will denote by $a > 0$ a constant such that

$$\forall t \in [0,T_0] \quad |a_3(t)| \geq a \ . \tag{2.4}$$

Under the above assumptions, $L(t)$ generates an evolution system $S(t_0,t)$ $(0 \leq t_0 \leq t \leq T_0\,)$ in the Sobolev spaces, and setting

$$P(t_0,t,\xi) \ = \int_{t_0}^{t} p(\tau,\xi)\,d\tau \tag{2.5}$$

we have

$$S(t_0,t)\,u = \int_{-\infty}^{+\infty} \exp\big[i\,P(t_0,t,\xi)\ + i\,x\,\xi\big]\,\hat{\mathrm{u}}(\xi)\,d\xi. \tag{2.6}$$

To obtain smoothing effects, we will make use of the following notations and results throughout the next three sections.

First, for simplicity of notation, c will denote a constant depending only on T_0 and a_k $(0 \leq k \leq 3)$, and when no confusion can arise, we will not write all the arguments of a function (e.g. $P(\xi)$ for $P(t_0,t,\xi)$ or a_k for $a_k(t)$).

Notation 2.1. For $0 \leq t_0 \leq t \leq T_0$ and $\eta\,,\,\xi \in \mathbb{R}$, let

$$A_k(t_0,t) \ = \int_{t_0}^{t} a_k(\tau)\,d\tau \qquad (0 \leq k \leq 3)$$

$$\varphi(t_0,t,\xi,\eta) = \sum_{k=1}^{3} A_k(t_0,t)\,(\xi^k - \eta^k)$$

$$\psi(t,\xi,\eta) \ = a_3(t)\,(\xi^2 + \eta\,\xi + \eta^2) + a_2(t)\,(\xi + \eta) + a_1(t)$$

$$K(t_0,T,\xi,\eta) \ = \int_{t_0}^{T} \exp\big[i\,\varphi(t_0,t,\xi,\eta)\big]\,\eta\,\xi\,dt \ .$$

Lemma 2.2. *There exist two constants $\sigma \in \mathbb{R}$ and $\lambda > 0$ such that*

$$\forall\, t \in [0,T_0] \ \ \forall\, \xi,\eta \in \mathbb{R} \qquad |\psi(t,\xi,\eta)\ + \sigma| > \lambda\,(\xi^2 + \eta^2 + 1)\,. \tag{2.7}$$

Proof. Write $\psi(t,\xi,\eta) + \sigma - \lambda(\xi^2 + \eta^2 + 1)$ as a polynomial of degree two in ξ of highest-order coefficient $a_3 - \lambda$. Its discriminant is itself a polynomial of degree two in η of highest-order coefficient $(3\,a_3 - 2\,\lambda)(2\,\lambda - a_3)$ and of discriminant $\Delta = 4(2\,\lambda - a_3)(\lambda - a_3)\big[a_2^2 - (a_1 + \sigma - \lambda)(3\,a_3 - 2\lambda)\big]$. To fix the ideas, suppose that a_3 is nonpositive on $[0,T_0]$. If we find $\sigma < 0$ and $\lambda < 0$ such that for any $t \in [0,T_0]$

$$(3\,a_3 - 2\,\lambda)(2\,\lambda - a_3) < 0 \tag{2.8}$$

$$(a_3 - \lambda)\lambda > 0 \tag{2.9}$$

$$\Delta < 0, \tag{2.10}$$

then $\psi(t,\xi,\eta)+\sigma-\lambda(\xi^2+\eta^2+1)$ will be nonpositive and Lemma 2.2 will be proved.

We get (2.8) and (2.9) by requiring that $|\lambda|<\frac{a}{4}$, and as a consequence we also have $(2\lambda-a_3)(\lambda-a_3)>0$. Hence, it remains to obtain

$$a_2^2-(a_1+\sigma-\lambda)(3\,a_3-2\lambda)<0\,. \tag{2.11}$$

Now fix $\sigma_0>0$ and assume that $|\sigma|>\sigma_0+a+\|a_1\|_{L^\infty(0,T_0)}$; then $a_1+\sigma-\lambda<0$ and $|a_1+\sigma-\lambda|\geq\sigma_0$. Therefore (2.11) can be written $|\lambda|<\frac{3}{2}|a_3|-\frac{a_2^2}{2\,|a_1+\sigma-\lambda|}$, and it is sufficient to have $|\lambda|<\frac{3}{2}a-\frac{\|a_2\|^2_{L^\infty(0,T_0)}}{2\sigma_0}$. Fixing $\sigma_0>0$ such that $\frac{3}{2}a-\frac{\|a_2\|^2_{L^\infty(0,T_0)}}{2\sigma_0}>0$ we get the desired result for any $\sigma<0$ and $\lambda<0$ satisfying $|\sigma|>\sigma_0+a+\|a_1\|_{L^\infty(0,T_0)}$, $|\lambda|<\mathrm{Min}\big(\frac{a}{4}\,,\,\frac{3}{2}\,a-\frac{\|a_2\|^2_{L^\infty(0,T_0)}}{2\sigma_0}\big)$.

Notation 2.3. From now on, σ and λ will denote constants satisfying (2.7).

Lemma 2.4. *There exists a constant c_0 such that for $|\xi|\geq c_0$ and $0\leq t_0\leq t\leq T_0$*

$$2\,a\,(t-t_0)\,\xi^2\leq|P_\xi(t_0,t,\xi)\,|\leq c_0\,(t-t_0)\,\xi^2 \tag{2.12}$$

$$a\,(t-t_0)\,|\xi|\leq|P_{\xi\xi}(t_0,t,\xi)\,|\leq c_0\,(t-t_0)\,|\xi|\,. \tag{2.13}$$

Proof. It suffices to compute P_ξ and $P_{\xi\xi}$ and to use the estimates

$$\big|A_k(t_0,t)\,\big|\leq\big\|a_k\big\|_{L^\infty(0,T_0)}|t-t_0|\quad\text{ and }\quad\big|A_3(t_0,t)\big|\geq a\,(t-t_0)\,.$$

3. Local smoothing effect

In this section we shall establish the following result that provides the local smoothing effect

Theorem 3.1. *For $0\leq t_0\leq T\leq T_0$, let $W(t_0,T)$ be the linear operator from $\mathcal{D}(\mathbb{R})$ to $\mathcal{D}'(\mathbb{R})$ defined by*

$$\forall\,f,g\in\mathcal{D}(\mathbb{R})\;\;\langle W(t_0,T)\,f\,,\,g\rangle=\iint K(t_0,T,\xi,\eta)\,f(\xi)\,g(\eta)\;d\xi\,d\eta\,. \tag{3.1}$$

Then (i) $W(t_0,T)$ can be extended to a bounded linear operator on $L^2(\mathbb{R})$.

(ii) There exists a constant c such that

$$\forall\,f,g\in L^2(\mathbb{R})\quad\big|\,\langle W(t_0,T)\,f\,,\,g\rangle\,\big|\leq c\,(T+1)\,\big\|f\big\|_{L^2(\mathbb{R})}\big\|g\big\|_{L^2(\mathbb{R})}\,.$$

Corollary 3.2. *Let* $0 \le t_0 \le T \le T_0$*; then*

$$\Big\| \frac{\partial}{\partial x} S(t_0, t)\, u(x) \Big\|_{L^\infty_x(\mathbb{R}\,;\,L^2_t(t_0;T))} \le c\,(T+1)^{\frac{1}{2}}\, \|u\|_{L^2(\mathbb{R})} \,. \tag{3.2}$$

Proof. It suffices to write

$$\Big\| \frac{\partial}{\partial x} S(t_0, t)\, u(x) \Big\|^2_{L^2_t(t_0,T)} = \Big| \Big\langle W(t_0,T)(e^{ix\xi}\hat{u}(\xi))\,,\, e^{-ix\eta}\overline{\hat{u}(\eta)} \Big\rangle \Big| \,.$$

To establish Theorem 3.1, we decompose K as a sum of singular (Calderón-Zygmund) kernels and a bounded kernel.

Lemma 3.3. *Let* $\xi \ne \eta$*; then*

$$\begin{aligned} K(t_0,T,\xi,\eta) &= \sigma^2 K_5(t_0,T,\xi,\eta) - i\left[\exp\left[i\varphi(t_0,t,\xi,\eta)\right] K_1(t,\xi,\eta)\right]_{t_0}^{T} \\ &- i\int_{t_0}^{T} \exp\left[i\varphi(t_0,t,\xi,\eta)\right] K_2(t,\xi,\eta)\,dt - i\sigma\left[\exp\left[i\varphi(t_0,t,\xi,\eta)\right] K_3(t,\xi,\eta)\right]_{t_0}^{T} \\ &- 2\,i\,\sigma \int_{t_0}^{T} \exp\left[i\,\varphi(t_0,t,\xi,\eta)\right] K_4(t,\xi,\eta)\,dt, \end{aligned}$$

where

$$K_1(t,\xi,\eta) = \frac{\eta\xi}{(\xi-\eta)[\psi(t,\xi,\eta)+\sigma]},\quad K_2(t,\xi,\eta) = \frac{\eta\xi\psi_t(t,\xi,\eta)}{(\xi-\eta)[\psi(t,\xi,\eta)+\sigma]^2}$$

$$K_3(t,\xi,\eta) = \frac{\eta\xi}{(\xi-\eta)[\psi(t,\xi,\eta)+\sigma]^2},\quad K_4(t,\xi,\eta) = \frac{\eta\xi\psi_t(t,\xi,\eta)}{(\xi-\eta)[\psi(t,\xi,\eta)+\sigma]^3}$$

$$K_5(t_0,T,\xi,\eta) = \int_{t_0}^{T} \exp\left[i\varphi(t_0,t,\xi,\eta)\right] \frac{\eta\xi}{[\psi(t,\xi,\eta)+\sigma]^2}\,dt.$$

The proof follows from integrations by parts.

The first step is to check that the antisymmetric kernels K_j ($j \le 4$) define singular integral operators $W_j : \mathcal{D}(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ by:

$$\begin{aligned} \langle W_j\, f\,,\, g\rangle &= \lim_{\varepsilon\to 0} \iint_{|x-y|>\varepsilon} K_j(t,x,y)\, f(y)\, g(x)\, dx\, dy \\ &= \lim_{\varepsilon\to 0} \frac{1}{2} \iint_{|x-y|>\varepsilon} K_j(t,x,y)\left[f(y)\,g(x) - f(x)\,g(y)\right] dx\, dy. \end{aligned} \tag{3.3}$$

Since $|f(y)g(x) - f(x)g(y)| \le c(f,g)|x-y|$, the above integral (3.3) is absolutely convergent thanks to the following lemma:

Lemma 3.4. *There exists a constant c such that for $j \in \{1,2,3,4\}$, K_j satisfies for $0 \leq t \leq T_0$, $x, y \in \mathbb{R}$ and $x \neq y$*

$$|K_j(t,x,y)| \leq \frac{c}{|x-y|} \tag{3.4}$$

$$|\frac{\partial}{\partial x} K_j(t,x,y)\,| \leq \frac{c}{|x-y|^2}\,. \tag{3.5}$$

Proof. Notice that ψ satisfies:

$$|\psi(x,y)| + |\psi_t(x,y)| \leq c\,(x^2+y^2+1) \tag{3.6}$$

$$|\psi_x(x,y)| + |\psi_{tx}(x,y)| \leq c\,(|x|+|y|+1)\,. \tag{3.7}$$

Hence (3.4) is a consequence of Lemma 2.2 and (3.5) is obtained by computing the derivative of the kernels. $\square$

We can now apply David and Journé's T1 Theorem ([1], [4], [5], [25], [30]) to the singular integral operators W_j. Since the antisymmetric kernels K_j satisfy (3.4) and (3.5), to establish that W_j has a bounded extension to $L^2(\mathbb{R})$ it suffices to prove that $W_j(1)$ belongs to BMO.

Let us recall that $W_j(1)$ is defined on zero-mean functions in the following way. Fix $g \in \mathcal{D}(\mathbb{R})$ a zero-mean function. Let $b_1 \in \mathcal{D}(\mathbb{R})$ and $b_2 = 1 - b_1$ be such that b_2 vanishes in a neighborhood of $\operatorname{supp} g$ and let $x_0 \in \operatorname{supp} g$. Set

$$\langle W_j(1), g\rangle = \langle W_j(b_1), g\rangle + \iint \big[K_j(t,x,y) - K_j(t,x_0,y)\big] b_2(y) g(x)\,dx\,dy. \tag{3.8}$$

Thanks to Lemma 3.4, this definition of $\langle W_j(1)\,,\,g\rangle$ does not depend on b_1, b_2 or x_0 and hence is well-defined.

We will establish that $W_j(1)$ belongs to BMO by using the duality between this space and the Hardy space $\mathcal{H}^1$ that we define according to its atomic characterization ([3], [30]), which we recall here.

Definition 3.5. An atom is a function $\theta_k \in L^\infty(\mathbb{R})$ of support enclosed in a bounded interval I_k and such that

$$\|\theta_k\|_{L^\infty(\mathbb{R})} \leq \frac{1}{|I_k|} \quad \text{and} \quad \int_{\mathbb{R}} \theta_k(x)\,dx = 0\,.$$

We can now define $\mathcal{H}^1$ and its norm as:

Definition 3.6. The space $\mathcal{H}^1 = \big\{f = \sum_{k=0}^{+\infty} \lambda_k \theta_k,\ \theta_k$ atom and $(\lambda_k)_{k\in\mathbb{N}} \in l^1\big\}$, and $\|f\|_{\mathcal{H}^1}$ is the infimum of $\sum_{k=0}^{+\infty} |\lambda_k|$ over all atomic decompositions of f.

Set $\mathcal{A} = \{\beta \in L^\infty(\mathbb{R}),\ \operatorname{supp}\beta \subset [-1;1],\ \|\beta\|_{L^\infty(\mathbb{R})} \le 1,\ \int_{\mathbb{R}} \beta(x)\,dx = 0\}$, and notice that for any atom θ there exist $\beta \in \mathcal{A}$, $x_0 \in \mathbb{R}$ and $r > 0$ such that $\forall\, x \in \mathbb{R}$, $\theta(x) = \frac{1}{r}\beta(\frac{x-x_0}{r})$. We argue by duality, and hence it is sufficient, to prove that $W_j(1)$ belongs to BMO, to find a constant c such that

$$\forall\, \beta \in \mathcal{A} \quad \forall\, x_0 \in \mathbb{R} \quad \forall\, r > 0 \quad \Big| \Big\langle W_j(1)\,,\, \frac{1}{r}\beta\big(\frac{x-x_0}{r}\big) \Big\rangle \Big| \le c\,. \tag{3.9}$$

In order to find such a constant, let us fix $\omega \in \mathcal{C}^\infty(\mathbb{R})$ such that $0 \le \omega(x) \le 1$ on $\mathbb{R}$, $\operatorname{supp}\omega \subset [-5;5]$, $\omega(x) = 1$ on $[-4;4]$. Thus, the definition given by (3.8) yields, for any $\beta \in \mathcal{A}$, $x_0 \in \mathbb{R}$ and $r > 0$, the identity

$$\Big\langle W_j(1)\,,\, \frac{1}{r}\beta\big(\frac{x-x_0}{r}\big) \Big\rangle = I_{j,0} + \lim_{\varepsilon \to 0} \big(I^\varepsilon_{j,1} + I^\varepsilon_{j,2}\big), \tag{3.10}$$

where, for $\varepsilon > 0$ sufficiently small,

$$I_{j,0} = \frac{1}{r} \iint \big[K_j(t,x,y) - K_j(t,x_0,y)\big]\big(1 - \omega\big(\frac{y-x_0}{r}\big)\big)\,\beta\big(\frac{x-x_0}{r}\big)\,dx\,dy$$

$$I^\varepsilon_{j,1} = \frac{1}{r} \int\int_{\substack{|x-x_0| \le r \\ h \in [\varepsilon, 6r]}} \big[K_j(t,x,x+h) + K_j(t,x,x-h)\big]\omega\big(\frac{x+h-x_0}{r}\big)\beta\big(\frac{x-x_0}{r}\big)\,dh\,dx$$

$$I^\varepsilon_{j,2} = \frac{1}{r} \int\int_{\substack{|x-x_0| \le r \\ h \in [\varepsilon, 6r]}} K_j(t,x,x-h)\big[\omega\big(\frac{x-h-x_0}{r}\big) - \omega\big(\frac{x+h-x_0}{r}\big)\big]\beta\big(\frac{x-x_0}{r}\big)\,dh\,dx.$$

Hence, to get the estimate (3.9), we are led to consider the above integrals.

Lemma 3.7. *There exists a constant c such that $|I_{j,0}| \le c$, $1 \le j \le 4$.*

Proof. Rewrite $I_{j,0}$ as an integral on $\{(x,y) \in \mathbb{R}^2,\ x \ne y,\ |x-x_0| \le r,\ |y-x_0| > 4r,\ |x-x_0| \le \frac{1}{2}|x-y|\}$ and fix (x,y) in this set. Thanks to (3.5), there exists $x_1 \in (x_0, x)$ such that $|K_j(x,y) - K_j(x_0,y)| \le c\frac{|x-x_0|}{|x_1-y|^2}$, and by the choice of (x,y) we get $|K_j(x,y) - K_j(x_0,y)| \le c\frac{|x-x_0|}{|x-y|^2}$. Therefore,

$$|I_{j,0}| \ \le\ \frac{c}{r} \iint_{\substack{|x-x_0| \le r \\ |y-x_0| > 4r}} \frac{|x-x_0|}{|x-y|^2}\,dx\,dy \le c\,.$$

Lemma 3.8. *For $j \in \{1,2,3,4\}$, $I^\varepsilon_{j,1}$ and $I^\varepsilon_{j,2}$ converge as ε tends to zero, and there exists a constant c such that for $\varepsilon > 0$ sufficiently small*

$$|I^\varepsilon_{j,1}| \ + \ |I^\varepsilon_{j,2}| \ \le \ c\,.$$

Proof. We do not write the dependence on t. The conclusions on $I^{\varepsilon}_{j,2}$ are immediate from the following estimate, which is obtained thanks to (3.4):

$$\Big| K_j(x,x-h)\,\big[\,\omega(\frac{x-h-x_0}{r})-\omega\big(\frac{x+h-x_0}{r})\,\big]\,\beta(\frac{x-x_0}{r})\Big| \le \frac{c}{r}\,\big\|\,\omega'\,\big\|_{L^\infty(\mathbb{R})}\,.$$

For $I^{\varepsilon}_{j,1}$, we consider the different cases successively:

Case $j=1$**.** Let us write

$$\begin{aligned} h\big[\psi(x,x+h)+\sigma\big]\big[\psi(x,x-h)+\sigma\big]\big[K_1(x,x+h)+K_1(x,x-h)\big] \\ = -2\sigma xh - x\Delta\psi(x,h), \end{aligned}$$

where

$$\Delta\psi(x,h) = (x+h)\,\psi(x,x-h) - (x-h)\,\psi(x,x+h).$$

Notice that $|\psi(x,x+h)+\sigma| > \frac{\lambda}{3}(x^2+h^2+1)$ and that

$$\Delta\psi(x,h) = 2\,h\,(a_3\,h^2 + a_2\,x + a_1).$$

Then it follows that

$$\Big|\frac{1}{r}\big[K_1(x,x+h)+K_1(x,x-h)\big]\omega\big(\frac{x+h-x_0}{r}\big)\beta(\frac{x-x_0}{r})\Big| \le c\frac{|x|}{(x^2+h^2+1)}$$

and hence we have Lemma 3.8.

For the other cases, the proof is similar and we just give the equalities which allow us to conclude. The notation $\Delta\psi_t$ is the same as $\Delta\psi$ with ψ_t instead of ψ.

Case $j=2$**.**

$$\begin{aligned} &h\big[\psi(x,x+h)+\sigma\big]^2\big[\psi(x,x-h)+\sigma\big]^2\big[K_2(x,x+h)+K_2(x,x-h)\big] \\ &= \sigma^2 x\Delta\psi_t(x,h) - 2\sigma^2 xh\big[\psi_t(x,x+h)+\psi_t(x,x-h)\big] \\ &- 2\sigma x\psi_t(x,x+h)\Delta\psi(x,h) + 2\sigma x\psi(x,x+h)\Delta\psi_t(x,h) \\ &- 4\sigma xh\psi(x,x+h)\psi_t(x,x-h) - x\psi_t(x,x+h)\psi(x,x-h)\Delta\psi(x,h) \\ &+ x\psi(x,x-h)\psi(x,x+h)\Delta\psi_t(x,h) - x\psi(x,x+h)\psi_t(x,x-h)\Delta\psi(x,h). \end{aligned}$$

Case $j=3$**.**

$$\begin{aligned} &h\big[\psi(x,x+h)+\sigma\big]^2\big[\psi(x,x-h)+\sigma\big]^2\big[K_3(x,x+h)+K_3(x,x-h)\big] \\ &= -2\sigma^2 xh - 2\sigma x\Delta\psi(x,h) - x\big[\psi(x,x-h)+\psi(x,x+h)\big]\Delta\psi(x,h) \\ &+ 2xh\psi(x,x-h)\psi(x,x+h). \end{aligned}$$

Case $j=4$**.**

$$\begin{aligned} &h\big[\psi(x,x+h)+\sigma\big]^3\big[\psi(x,x-h)+\sigma\big]^3\big[K_4(x,x+h)+K_4(x,x-h)\big] \\ &= h\big[\psi(x,x+h)+\sigma\big]^2\big[\psi(x,x-h)+\sigma\big]^3\big[K_2(x,x+h)+K_2(x,x-h)\big] \end{aligned}$$

$$- x\psi_t(x,x-h)\big[\psi(x,x+h)+\sigma\big]^2\Delta\psi(x,h)$$
$$+ 2xh\psi_t(x,x-h)\psi(x,x-h)\big[\psi(x,x+h)+\sigma\big]^2.$$

Proof of Theorem 3.1. The kernel of the integral operator W has been written as the sum of four singular kernels K_j $(1 \leq j \leq 4)$ and a bounded kernel K_5. The kernel K_5 defines a bounded linear operator on $L^2(\mathbb{R})$ since

$$\big|K_5(t_0,T,\xi,\eta)\big| \leq cT\frac{|\xi||\eta|}{(\xi^2+1)(\eta^2+1)}.$$

The singular kernels K_j $(1 \leq j \leq 4)$ define singular integral operators W_j satisfying $W_j(1) \in BMO$ by using duality, (3.10) and Lemmas 3.7 and 3.8. Therefore, David and Journé's T1 Theorem allows us to conclude that W_j has a bounded extension to $L^2(\mathbb{R})$. Moreover $\big\|W_j\big\|_{L^2,L^2}$ can be controlled by the constant occurring in Lemma 3.4 and by $\big\|W_j(1)\big\|_{BMO}$, and hence by a constant.

4. Estimate on the maximal function

As for the local smoothing effect, we establish a stronger result than necessary for the study of the nonlinear problem.

Theorem 4.1. *Let $s_0 > 3/4$ and $0 \leq t_0 \leq T \leq T_0$. For $s \geq s_0$ there exists a constant c_s depending only on s such that*

$$\sum_{j=-\infty}^{+\infty} \sup_{x\in[j;j+1]} \sup_{t\in[t_0;T]} \big|S(t_0,t)\,u_0(x)\big|^2 \leq c_s(T+1)\big\|u_0\big\|_{H^s}^2. \tag{4.1}$$

Moreover, c_s is uniformly bounded for $s \geq s_0$.

Corollary 4.2. *Let $s > 3/4$ and $0 \leq t_0 \leq T \leq T_0$. There exists a constant c_s such that*

$$\big\|S(t_0,t)\,u_0(x)\big\|_{L^2_x(\mathbb{R}\,;\,L^\infty_t(t_0;T))} \leq c_s(T+1)^{\frac{1}{2}}\big\|u_0\big\|_{H^s}. \tag{4.2}$$

The proof of Theorem 4.1 is an adaptation of the one written when the linear operator $L(t)$ does not depend on time. It is based on duality and density arguments and on an estimate on the following oscillatory integral.

Proposition 4.3. *Let $k \in \mathbb{N}$, $0 \leq t_0 \leq T \leq T_0$ and suppose $g \in \mathcal{C}^\infty(\mathbb{R})$ is supported in $[2^{k-1},2^{k+1}]$ or in $[-1,1]$. There exists a constant $c(g)$ depending only on $\big\|g\big\|_{L^\infty}$, $\big\|g'\big\|_{L^1}$ and $\big\|g''\big\|_{L^1}$, a constant c and a function H_T^k on $\mathbb{R}$*

satisfying

$$\sum_{l=-\infty}^{+\infty} H_k^T(l) \le c2^{3k/2}(T+1) \tag{4.3}$$

such that for $t \in [t_0, T]$ *and* $x \in \mathbb{R}$

$$\Big| \int_{-\infty}^{+\infty} \exp\big[iP(t_0,t,\xi) + ix\xi \big] g(\xi)\, d\xi \Big| \le c(g) H_k^T(x). \tag{4.4}$$

The proof of this result requires the following lemmas. The first one is the classical Van der Corput lemma ([29]).

Lemma 4.4. *Let* $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ *and* $\phi \in \mathcal{C}^2(\mathbb{R})$ *be such that* $|\phi''(\xi)| > \lambda > 0$ *on* $\mathrm{supp}\psi$. *Then*

$$\Big| \int_{\mathbb{R}} \exp\big[i\phi(\xi) \big] \psi(\xi)\, d\xi \Big| \le 10\lambda^{-\frac{1}{2}} \big(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1} \big). \tag{4.5}$$

Lemma 4.5. *Let* $0 \le t_0 \le T \le T_0$, $m > 0$ *and* $\beta \in [0,1)$. *There exists a constant* $c(m,\beta)$ *such that for* $t \in [t_0, T]$ *and* $|x| \ge m(T+1)$, *if* $|P_\xi(t_0,t,\xi) + x| \ge |x|/3$, *then*

$$\Big| \frac{|\xi|^\beta P_{\xi\xi}(t_0,t,\xi)}{P_\xi(t_0,t,\xi)\, + x} \Big| \le c(m,\beta) \tag{4.6}$$

and

$$\Big| \frac{|\xi|^\beta P_{\xi\xi\xi}(t_0,t,\xi)}{P_\xi(t_0,t,\xi)\, + x} \Big| \le c(m,\beta). \tag{4.7}$$

Proof. Since $\sup_{\substack{X \in \mathbb{R} \\ 3|X+x| \ge |x|}} \big| \frac{X}{X+x} \big| \le 4$, we get (4.6) and (4.7) for ξ sufficiently large by using Lemma 2.4. From now on ξ is bounded and (4.6), (4.7) are straightforward from the bounds on functions a_j. □

Lemma 4.6. *Let* $x \in \mathbb{R}^*$ *and* $0 \le t_0 \le t \le T_0$. *Suppose that* $U = \{\xi \in \mathbb{R}, |P_\xi(t_0,t,\xi) + x| \le |x|/3\}$ *is not empty. Then there exists* $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ *such that* (i) φ *is supported in* $\Omega = \{\xi \in \mathbb{R}, |P_\xi(t_0,t,\xi) + x| \le \frac{|x|}{2}\}$, (ii) $0 \le \varphi \le 1$ *in* Ω *and* $\varphi = 1$ *in* U, *and* (iii) $\|\varphi'\|_{L^1} \le 4$.

Proof. We do not write the dependence on t. Let ξ_0 be the only zero of $P_{\xi\xi}$ and set $I_1 = (-\infty, \xi_0]$, $I_2 = [\xi_0, +\infty)$. Then P_ξ is one to one from I_k to an interval $J_k (k=1,2)$. Let us write $U = \{\xi \in \mathbb{R}, P_\xi(\xi) \in I_U\}$ and $\Omega = \{\xi \in \mathbb{R}, P_\xi(\xi) \in I_\Omega\}$ where I_U and I_Ω are bounded intervals. Then $U_k = \{\xi \in I_k, P_\xi(\xi) \in I_U\}$ and $\Omega_k = \{\xi \in I_k, P_\xi(\xi) \in I_\Omega\}$ are bounded intervals too, satisfying $\Omega = \Omega_1 \cup \Omega_2$, $U = U_1 \cup U_2$, $U_k \subset \Omega_k$, $\mathrm{Int}(\Omega_1) \cap \mathrm{Int}(\Omega_2) = \emptyset$. If $P_\xi(\xi_0) \in J_k \cap I_U$ for $k=1$ or $k=2$, then $\Omega_1 \cap \Omega_2 = U_1 \cap U_2 = \{\xi_0\}$, and then there exists $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ satisfying (i) and (ii) such that φ' vanishes only once since $U \subset \mathrm{Int}(\Omega)$. Otherwise, $J_k \cap I_U \subset \mathrm{Int}(J_k \cap I_\Omega)$, and there exists $\varphi_k \in \mathcal{C}^\infty(\mathbb{R})$ $(k = 1, 2)$ satisfying (i) and (ii) with Ω, U replaced with Ω_k, U_k and vanishing only once. It suffices then to set $\varphi = \varphi_1 + \varphi_2$. □

Proof of Proposition 4.3. We suppose that g is supported in $[2^{k-1}, 2^{k+1}]$, the proof in the other case is similar, and we shall prove that we can take, for some well-chosen constants c_1 and c_2,

$$H_k^T(x) = \begin{cases} 2^k & \text{if } |x| \le c_1(T+1) \\ 2^{k/2}|x|^{-1/2} & \text{if } c_1(T+1) < |x| \le c_2(T+1)2^{2k} \\ \frac{1}{1+x^2} & \text{if } c_2(T+1)2^{2k} < |x|. \end{cases}$$

As a first step fix c_1, an arbitrary constant; then for $|x| \le c_1(T+1)$, (4.4) is satisfied. Now we define for $|x| > c_1(T+1)$ the sets

$$U_x = \big\{\xi \in [2^{k-1}, 2^{k+1}] \ \ , |P_\xi(\xi) + x| \le \frac{|x|}{3}\big\}$$

$$\Omega_x = \big\{\xi \in [2^{k-1}, 2^{k+1}] \ \ , |P_\xi(\xi) + x| \le \frac{|x|}{2}\big\}.$$

If $U_x \ne \emptyset$, two integrations by parts give

$$\Big|\int_{-\infty}^{+\infty} \exp\big[iP(\xi) + ix\xi\big] g(\xi)\, d\xi\Big| \le \frac{c}{x^2}\big(\|g''\|_{L^1} + \|g'\|_{L^1} + \|g\|_{L^\infty}\big),$$

and hence (4.4) follows.

But there exists a constant c_3 such that if $U_x \ne \emptyset$ then $|x| \le c_3(T+1)2^{2k}$. Fixing $c_1 > 1 + c_0 + c_0^3\frac{T_0}{T_0+1}$—where c_0 is the constant occurring in Lemma 2.4—and $c_2 > c_1 + c_3$, it remains to consider what happens when $U_x \ne \emptyset$ with $c_1(T+1) < |x| \le c_2(T+1)2^{2k}$.

Let φ be a function associated to U_x as in Lemma 4.6, and set $\eta = (1-\varphi)g$. Notice that for $\xi \in \mathrm{supp}\ \eta$, we have $\xi \in [2^{k-1}, 2^{k+1}]$ and $|P_\xi(\xi) + x| \ge |x|/3$. Using again integrations by parts, we get

$$\Big|\int_{-\infty}^{+\infty} \exp\big[iP(\xi) + ix\xi\big] \eta(\xi)\, d\xi\Big| \le c2^{k/2}|x|^{-1/2}\big(\|\eta'\|_{L^1} + \|\eta\|_{L^\infty}\big). \quad (4.8)$$

On the other hand, for $\xi \in \mathrm{supp}(\varphi g)$, we have $|x| \le 2|P_\xi(\xi)| \le 3|x|$. Thus the choice of c_1 and Lemma 2.4 give $|\xi| \ge c_0$ and

$$\Big|P_{\xi\xi}(\xi)\Big| \ge \frac{|x|}{2^{k+2}}\frac{|\xi||P_{\xi\xi}(\xi)|}{|P_\xi(\xi)|} \ge c\frac{|x|}{2^k}.$$

Hence Lemma 4.4 and (4.8) provide (4.4) for $c_1(T+1) < |x| \le c_2(T+1)2^{2k}$ such that $U_x \ne \emptyset$. □

In order to prove Theorem 4.1, we also need the following duality and density result.

Lemma 4.7. *Fix* $T_1, T_2 \in \mathbb{R}$ *such that* $T_1 < T_2$ *and define the sets* $X(T_1,T_2) = \{f \in L^2_x(\mathbb{R}; L^\infty_t(T_1,T_2)); \|f\|_{X(T_1,T_2)} < +\infty\}$, *where*

$$\|f\|^2_{X(T_1,T_2)} = \sum_{j=-\infty}^{+\infty} \sup_{x\in[j;j+1]} \sup_{t\in[T_1;T_2]} |f(t,x)|^2$$

and $Y(T_1,T_2) = \{f$ *measurable function from* $\mathbb{R}$ *to* $L^1(T_1,T_2); \|f\|_{Y(T_1,T_2)} < +\infty\}$, *where*

$$\|f\|^2_{Y(T_1,T_2)} = \sum_{j=-\infty}^{+\infty} \Big(\int_j^{j+1}\int_{T_1}^{T_2} |f(t,x)|\,dt\,dx\Big)^2.$$

Then (i) $\mathcal{D}(\mathbb{R}\times(T_1,T_2))$ *is enclosed and dense in* $Y(T_1,T_2)$. (ii) *If* $f \in X(T_1,T_2)$, *then*

$$\|f\|_{X(T_1,T_2)} = \sup_{\substack{g\in\mathcal{D}(\mathbb{R}\times(T_1,T_2))\\ \|g\|_{Y(T_1,T_2)}\neq 0}} \frac{\Big|\int_{\mathbb{R}}\int_{T_1}^{T_2} f(t,x)g(t,x)\,dt\,dx\Big|}{\|g\|_{Y(T_1,T_2)}}.$$

Proof. The proof is the same as the one in [21], [23] when $T_1 = -T_2$.

Proof of Theorem 4.1. Fix $(\nu_k)_{k\geq -1} \in \mathcal{D}(\mathbb{R})$ such that

$$\begin{cases} \forall x \geq 0 \quad \sum_{k=-1}^{+\infty} \nu_k(x) = 1, \\ \text{supp } \nu_k \subset [2^{k-1}, 2^{k+1}] \ \text{ if } \ k\geq 0, \quad \text{supp } \nu_{-1} \subset [-1,1] \\ \|\nu_k\|_{L^\infty}, \|\nu_k'\|_{L^1} \text{ and } \|\nu_k''\|_{L^1} \text{are bounded independent of } k, \end{cases}$$

and define $S_k(t,\tau)u$ for $t,\tau\in[0,T_0]$ by its Fourier transform,

$$\mathcal{F}\big(S_k(t,\tau)u\big)(\xi) = \exp\big[iP(t,\tau,\xi)\big]\nu_k(|\xi|)\hat{u}(\xi).$$

Thanks to Minkowski's inequality, we are now reduced in proving (4.1) to estimating $\sum_{k=-1}^{+\infty} \|S_k(t_0,t)u_0(x)\|_{X(t_0,T)}$. In order to use Lemma 4.7, let us write for $g \in \mathcal{D}(\mathbb{R}^2)$

$$\Big\|\int_{t_0}^T \exp\big[-iP(t_0,t,\xi)\big]\sqrt{\nu_k(|\xi|)}\hat{g}(t,\xi)\,dt\Big\|^2_{L^2_\xi(\mathbb{R})}$$
$$= \int_{-\infty}^{+\infty}\int_{t_0}^T g(\tau,\xi)\Big(\int_{t_0}^T S_k(t,\tau)g(t,\xi)\,dt\Big)\,d\tau\,d\xi.$$

It follows that

$$\Big|\int_{-\infty}^{+\infty}\int_{t_0}^{T} S_k(t_0,t)u_0(\xi)g(t,x)\,dt\,dx\Big|$$

$$\leq \big\|\hat{u_0}(\xi)\sqrt{\nu_k(|\xi|)}\big\|_{L^2_\xi(\mathbb{R})}\big\|\int_{t_0}^{T} S_k(\tau,t)g(\tau,x)\,d\tau\big\|^{\frac12}_{X(t_0,T)}\big\|g\big\|^{\frac12}_{Y(t_0,T)}. \qquad (4.9)$$

But Proposition 4.3 gives

$$\Big|\int_{t_0}^{T} S_k(\tau,t)g(\tau,x)\,d\tau\Big|$$

$$= \Big|\int_{t_0}^{T}\int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{+\infty}\exp\big[iP(\tau,t,\xi)+iy\xi\big]\nu_k(|\xi|)\,d\xi\Big)g(\tau,x-y)\,dy\,d\tau\Big|$$

$$\leq c\int_{t_0}^{T}\int_{-\infty}^{+\infty} H_k^T(y)|g(\tau,x-y)|dyd\tau$$

and hence, by using Minkowski's inequality, we get

$$\big\|\int_{t_0}^{T} S_k(\tau,t)g(\tau,x)\,d\tau\big\|_{X(t_0,T)}$$

$$\leq c\Big[\sum_{j=-\infty}^{-\infty}\Big(\sum_{l=-\infty}^{-1} H_k^T(l+1)\int_{t_0}^{T}\int_{j-l-1}^{j-l+1} g(t,y)\,dy\,dt$$

$$+\sum_{l=0}^{+\infty} H_k^T(l)\int_{t_0}^{T}\int_{j-l-1}^{j-l+1} g(t,y)\,dy\,dt\Big)^2\Big]^{\frac12} \leq c\Big(\sum_{l=-\infty}^{+\infty} H_k^T(l)\Big)\big\|g\big\|_{Y(t_0,T)}.$$

From Proposition 4.3, Lemma 4.7 and (4.5) we deduce that

$$\|S_k(t_0,t)u_0(x)\|_{X(t_0,T)} \leq c2^{\frac{3k}{4}}(T+1)^{\frac12}\Big(\int_{\xi\in \operatorname{supp}\nu_k}|\hat{u_0}(\xi)|^2\,d\xi\Big)^{\frac12}$$

and hence

$$\sum_{k=-1}^{+\infty}\|S_k(t_0,t)u_0(x)\|_{X(t_0,T)} \leq c(T+1)^{\frac12}\Big(1+\sum_{k=0}^{+\infty}2^{\frac{3k}{4}-(k-1)s}\Big)\big\|u_0\big\|_{H^s(\mathbb{R})};$$

that finishes the proof.

5. Global smoothing effect

To establish the global smoothing effect we will use interpolation of the following analytic family of operators $V_\gamma(t_0,t)$ defined for $0 \le t_0 \le t \le T_0$ by

$$V_\gamma(t_0,t)u(x) = \int_{-\infty}^{+\infty} \exp\big[i\big(P(t_0,t,\xi) + ix\xi\big)\big]\Big|\frac{P_{\xi\xi}(t_0,t,\xi)}{t-t_0}\Big|^{\frac{\gamma}{2}} \hat{u}(\xi)\,d\xi.$$

The smoothing effect can then be written as follows:

Theorem 5.1. *Let $0 \le t_0 < t \le T \le T_0$ and $\gamma \in [0;1]$ and set $q = \frac{4}{\gamma}$, $p = \frac{2}{1-\gamma}$, $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then there exists a constant c such that*

$$\big\|V_{\frac{\gamma}{2}}(t_0,t)u_0\big\|_{L_t^q(t_0,T\,;\,L_x^p(\mathbb{R}))} \le c\big\|u_0\big\|_{L^2(\mathbb{R})} \tag{5.1}$$

$$\Big\|\int_{t_0}^{t} V_\gamma(\tau,t)g(\tau,x)\,d\tau\Big\|_{L_t^q(t_0,T\,;\,L_x^p(\mathbb{R}))} \le c\big\|g\big\|_{L_t^{q'}(t_0,T\,;\,L_x^{p'}(\mathbb{R}))} \tag{5.2}$$

$$\Big\|\int_{t_0}^{T} V_\gamma(\tau,t)g(\tau,x)\,d\tau\Big\|_{L_t^q(t_0,T\,;\,L_x^p(\mathbb{R}))} \le c\big\|g\big\|_{L_t^{q'}(t_0,T\,;\,L_x^{p'}(\mathbb{R}))}. \tag{5.3}$$

Corollary 5.2. *Let $0 \le t_0 \le t \le T \le T_0$ and $\gamma \in [0;1]$. Set $q = \frac{4}{\gamma}$ and $p = \frac{2}{1-\gamma}$. Then, there exists a constant c such that*

$$\big\|D^{\frac{\gamma}{4}} S(t_0,t)\,u\big\|_{L_t^q(t_0,T\,;\,L_x^p(\mathbb{R}))} \le c(1+T)^{\frac{\gamma}{4}}\big\|u\big\|_{L^2(\mathbb{R})}. \tag{5.4}$$

Proof. Let $k(t_0,t,\xi) = \big|\frac{6A_3(t_0,t)}{t-t_0}\big|^{\frac{1}{4}}|\xi|^{\frac{1}{4}} - \big|\frac{P_{\xi\xi}(t_0,t,\xi)}{t-t_0}\big|^{\frac{1}{4}}$. From now on, we do not write the dependence of the functions on time. We have

$$\Big|\frac{6A_3}{t-t_0}\Big|^{\frac{1}{4}} D^{\frac{1}{4}} S(t_0,t)\,u(x) = V_{\frac{1}{2}}(t_0,t)u(x) + J_1(t_0,t)u(x) + J_2(t_0,t)u(x),$$

where

$$J_1(t_0,t)u(x) = \int_{|\xi| \le \frac{|a_2|_{L^\infty(0,T_0)}}{a}} \exp\big[iP(\xi) + ix\xi\big]k(\xi)\hat{u}(\xi)\,d\xi$$

and

$$J_2(t_0,t)u(x) = \int_{|\xi| \ge \frac{|a_2|_{L^\infty(0,T_0)}}{a}} \exp\big[iP(\xi) + ix\xi\big]k(\xi)\hat{u}(\xi)\,d\xi.$$

Thanks to Theorem 5.1, we only have to estimate J_1 and J_2. On the one hand we obviously have $|J_1(t_0,t)u(x)| \le c\big\|u\big\|_{L^2(\mathbb{R})}$. On the second hand Taylor's formula gives

$$\forall |\xi| \ge \frac{|a_2|_{L^\infty(0,T_0)}}{a} \qquad \Big|1 - \Big|1 + \frac{A_2}{3A_3\xi}\Big|^{\frac{1}{4}}\Big| \le c\Big(\frac{1}{|\xi|} + \frac{1}{\xi^2}\Big).$$

Hence $|k(\xi)| \leq c\big(|\xi|^{-\frac{3}{4}} + |\xi|^{-\frac{7}{4}}\big)$ and $|J_2(t_0,t)u(x)| \leq c\|u\|_{L^2(\mathbb{R})}$. That yields (5.4) for $\gamma = 1$. But for $\gamma = 0$, since $S(t_0,t)$ is unitary, we have

$$\|S(t_0,t)\,u\|_{L_t^\infty(t_0,T\,;\,L_x^2(\mathbb{R}))} = \|u\|_{L^2(\mathbb{R})}.$$

Corollary 5.2 follows by interpolation. □

In order to prove Theorem 5.1, let us state the following result:

Proposition 5.3. *Let* $0 \leq t_0 \leq t \leq T \leq T_0$ *and* $\gamma \in \mathbb{R}$*. There exists a constant* c *such that*

$$\Big|\int_{-\infty}^{+\infty} \exp\big[iP(t_0,t,\xi) + ix\xi\big]\Big|\frac{P_{\xi\xi}(t_0,t,\xi)}{t-t_0}\Big|^{\frac{1}{2}+i\gamma} d\xi\Big| \leq c\frac{1+|\gamma|}{|t-t_0|^{\frac{1}{2}}}. \tag{5.5}$$

Proof. Let c_0 be the constant occurring in Lemma 2.4, and define

$$\begin{aligned}
\Omega_1 &= \big\{\xi \in \mathbb{R} : c_0 \leq |\xi| \leq |t-t_0|^{-\frac{1}{3}}\big\},\\
\Omega_2 &= \big\{\xi \in \mathbb{R} : |\xi| \geq c_0 \quad \xi \notin \Omega_1 \quad |P_\xi(\xi) + x| \leq \tfrac{|x|}{2}\big\},\\
\Omega_3 &= \big\{\xi \in \mathbb{R} : |\xi| \geq c_0 \quad \xi \notin \Omega_1 \cup \Omega_2\big\},\\
\Omega_4 &= \big\{\xi \in \mathbb{R} : |\xi| \leq c_0 \quad \big|\xi + \tfrac{A_2}{3A_3}\big| \leq |t-t_0|^{-\frac{1}{3}}\big|\big\},\\
\Omega_5 &= \big\{\xi \in \mathbb{R} : |\xi| \leq c_0 \quad \xi \notin \Omega_4 \quad 2|P_\xi^0(\xi) + A_{1,2} + x| \leq |A_{1,2} + x|\big\},\\
\Omega_6 &= \big\{\xi \in \mathbb{R} : |\xi| \leq c_0 \quad \xi \notin \Omega_4 \cup \Omega_5\big\},
\end{aligned}$$

where $P^0(\xi) = A_3\big(\xi + \frac{A_2}{3A_3}\big)^3$ and $A_{1,2} = -\frac{A_2^2}{3A_3} + A_1$. We denote by $J_k(x)$ the integral occurring in (5.5) on the restricted domain Ω_k $(1 \leq k \leq 6)$ and consider these integrals successively. From Lemma 2.4 we get

$$|J_1(x)| \leq c\int_{\Omega_1} |\xi|^{\frac{1}{2}}\,d\xi \leq c\sup_{\xi\in\Omega_1} |\xi|^{\frac{3}{2}} \leq c|t-t_0|^{-\frac{1}{2}}$$

since Ω_1 is the union of at most two intervals. In the same way, Ω_2 and Ω_3 are finite unions of intervals on which ξ, $P_{\xi\xi}(\xi)$ and $P_{\xi\xi\xi}(\xi)$ have a constant sign, and this finite number is bounded independently on t_0, t and x. Lemma 4.4 leads to

$$\begin{aligned}
|J_2(x)| \leq c|t-t_0|^{-\frac{1}{2}}&\big(\min_{\xi\in\Omega_2}|P_{\xi\xi}(\xi)\,|\big)^{-\frac{1}{2}}\\
&\big[\max_{\xi\in\Omega_2}|P_{\xi\xi}(\xi)\,|^{\frac{1}{2}} + (1+|\beta|)\int_{\Omega_2}|P_{\xi\xi}(\xi)\,|^{-\frac{1}{2}}|P_{\xi\xi\xi}(\xi)\,|\,d\xi\big].
\end{aligned} \tag{5.6}$$

For $\xi \in \Omega_2$, we have $\frac{|x|}{2} \leq |P_\xi(\xi)\,| \leq \frac{3|x|}{2}$, and we deduce from Lemma 2.4 that ξ^2 belongs to an interval whose bounds can be written $c\frac{|x|}{|t-t_0|}$. Hence Lemma

2.4 allows us to conclude that $|J_2(x)| \leq c(1+|\beta|)|t-t_0|^{-\frac{1}{2}}$. Integration by parts on Ω_3 gives the estimate

$$|J_3(x)| \leq c|t-t_0|^{-\frac{1}{2}}\Big[\max_{\xi\in\Omega_3}\frac{|P_{\xi\xi}(\xi)|^{\frac{1}{2}}}{|P_\xi(\xi)+x|} + \int_{\Omega_3}\frac{|P_{\xi\xi}(\xi)|^{\frac{3}{2}}}{|P_\xi(\xi)+x|^2}\,d\xi + (1+|\beta|)\int_{\Omega_3}|P_{\xi\xi}(\xi)|^{-\frac{1}{2}}\frac{|P_{\xi\xi\xi}(\xi)|}{|P_\xi(\xi)+x|}\,d\xi\Big]. \tag{5.7}$$

Notice that for $x, y \in \mathbb{R}$, if $2|x+y| \geq |x|$, then $3|x+y| \geq |y|$. It follows that for $\xi \in \Omega_3$ and $x \in \mathbb{R}$, we have

$$|P_\xi(\xi)+x| \geq \frac{1}{3}|P_\xi(\xi)|, \tag{5.8}$$

and with Lemma 2.4, we get $\max_{\xi\in\Omega_3}\frac{|P_{\xi\xi}(\xi)|^{\frac{1}{2}}}{|P_\xi(\xi)+x|} \leq c$. In the same way, since $|\xi| \geq |t-t_0|^{-\frac{1}{3}}$, Lemma 2.4 gives

$$\int_{\Omega_3}\frac{|P_{\xi\xi}(\xi)|^{\frac{3}{2}}}{|P_\xi(\xi)+x|^2}\,d\xi \leq c$$

and

$$\int_{\Omega_3}|P_{\xi\xi}(\xi)|^{-\frac{1}{2}}\frac{|P_{\xi\xi\xi}(\xi)|}{|P_\xi(\xi)+x|}\,d\xi \leq c|t-t_0|^{-\frac{3}{2}}\int_{\Omega_3}|\xi|^{-\frac{5}{2}}|P_{\xi\xi\xi}(\xi)|\,d\xi \leq c\,.$$

The last estimate results from an integration by parts. To conclude on $J_3(x)$, it suffices to come back to (5.7).

To estimate the last three integrals, we write $P(\xi) = P^0(\xi) + A_{1,2}\xi + A_{0,2}$, where $A_{0,2} = -\frac{A_2^3}{27A_3^2} + A_0$. Then the desired estimate on $J_4(x)$ follows immediately. $J_5(x)$ is treated with the Van der Corput lemma, and as well as for $J_2(x)$ we get on $J_5(x)$ the estimate (5.6). But for $\xi \in \Omega_5$ we obviously have

$$|A_3|^{\frac{1}{2}}|x+A_{1,2}|^{\frac{1}{2}} \leq |P^0_{\xi\xi}(\xi)| \leq 6|A_3|^{\frac{1}{2}}|x+A_{1,2}|^{\frac{1}{2}}$$

since

$$\frac{1}{2}|x+A_{1,2}| \leq |P^0_\xi(\xi)| \leq \frac{3}{2}|x+A_{1,2}|,$$

and so the desired estimate on $|J_5(x)|$ follows.

It remains to estimate $J_6(x)$. We proceed as for $J_3(x)$ by integration by parts and get (5.7) with Ω_3 replaced by Ω_6 and P expressed as function of P^0. In the same way, we have for $\xi \in \Omega_6$

$$|P^0_\xi(\xi) + A_{1,2} + x| \geq \frac{1}{3}|P^0_\xi(\xi)|,$$

and using the fact that $\xi \notin \Omega_4$, we easily get the desired estimate on the first and the third right-hand terms of (5.7). Integration by parts on the second term gives

$$\int_{\Omega_6} |P^0_{\xi\xi}(\xi)|^{-\frac{1}{2}} \frac{|P^0_{\xi\xi\xi}(\xi)|}{|P^0_{\xi}(\xi) + x + A_{1,2}|} d\xi \leq c|A_3|^{-\frac{3}{2}} \int_{\Omega_6} \Big|\xi + \frac{A_2}{3A_3}\Big|^{-\frac{5}{2}} |P^0_{\xi\xi\xi}(\xi)|^{-\frac{1}{2}} d\xi$$

$$\leq c|t-t_0|^{-\frac{3}{2}} \max_{\xi\in\Omega_6}\Big(|A_3|\Big|\xi + \frac{A_2}{3A_3}\Big|^{-\frac{3}{2}}\Big) \leq c,$$

and hence Proposition 5.3 is proved.

Corollary 5.4. *There exists a constant c such that for* $0 \leq t_0 < t \leq T \leq T_0$ *and* $\gamma \in [0;1]$*, we have*

$$\big\|V_{\frac{\gamma}{2}}(t_0,t)u_0\big\|_{L^{\frac{2}{1-\gamma}}(\mathbb{R})} \leq c|t-t_0|^{-\frac{\gamma}{2}} \big\|u_0\big\|_{L^{\frac{2}{1+\gamma}}(\mathbb{R})}.$$

Proof. By using Proposition 5.3 and Young's inequality, we have for $\beta \in \mathbb{R}$

$$\big\|V_{1+i\beta}(t_0,t)u_0\big\|_{L^\infty(\mathbb{R})} \leq c|t-t_0|^{-\frac{1}{2}}(1+|\beta|)\big\|u_0\big\|_{L^1(\mathbb{R})}.$$

On the other hand, Plancherel's theorem gives

$$\big\|V_{i\beta}(t_0,t)u_0\big\|_{L^2(\mathbb{R})} = \big\|u_0\big\|_{L^2(\mathbb{R})}.$$

The result follows by interpolation of the family of analytic operators $V_\gamma(t_0,t)$ ([31]). □

Proof of Theorem 5.1. To simplify notations, we introduce for $0 \leq t_0 \leq t \leq T_0$ the operator $V^-_\gamma(t_0,t)$ defined by

$$V^-_\gamma(t_0,t)u(x) = \int_{-\infty}^{+\infty} \exp\big[i\big(-P(t_0,t,\xi) + x\xi\big)\big]\Big|\frac{P_{\xi\xi}(t_0,t,\xi)}{t-t_0}\Big|^{\frac{\gamma}{2}} \hat{u}(\xi)\, d\xi.$$

The operator V^-_γ clearly satisfies the same properties as V_γ, in particular Corollary 5.4. Thus we can set for $0 \leq t \leq t_0 \leq T_0$, $V_\gamma(t_0,t)u(x) = V^-_\gamma(t,t_0)u(x)$. We argue by duality. Let $g \in L^{q'}_t(t_0,T;L^{p'}_x(\mathbb{R}))$; then

$$|\Big\langle V_{\frac{\gamma}{2}}(t_0,t)u_0, g\Big\rangle| = \int_{t_0}^T \int_{-\infty}^{+\infty} u_0(x) V_{\frac{\gamma}{2}}(t,t_0)g(t,x)\ \ dx\, dt.$$

But

$$\|\int_{t_0}^T V_{\frac{\gamma}{2}}(t,t_0)g(t,x)\,dt\|^2_{L^2_x(\mathbb{R})}$$
$$= \int_{-\infty}^{+\infty}\int_{t_0}^T\int_{t_0}^T V_{\frac{\gamma}{2}}(t,t_0)g(t,x)V_{\frac{\gamma}{2}}(\tau,t_0)g(\tau,x)\,dt\,d\tau\,dx$$

$$= \int_{-\infty}^{+\infty} \int_{t_0}^{T} \Big(\int_{t_0}^{T} V_\gamma(\tau,t) g(\tau,x)\,d\tau \Big) g(t,x)\,dt\,dx.$$

Hence,

$$\begin{aligned} &\Big| \Big\langle V_{\frac{\gamma}{2}}(t_0,t)u_0, g \Big\rangle \Big|^2 \\ &\leq c \|u_0\|^2_{L^2(\mathbb{R})} \|g\|_{L_t^{q'}(t_0,T\,;\,L_x^{p'}(\mathbb{R}))} \Big\| \int_{t_0}^{T} V_\gamma(\tau,t) g(\tau,x)\,d\tau \Big\|_{L_t^{q}(t_0,T\,;\,L_x^{p}(\mathbb{R}))}. \end{aligned} \tag{5.9}$$

Now Corollary 5.4 gives

$$\begin{aligned} &\Big\| \int_{t_0}^{T} V_\gamma(\tau,t) g(\tau,x)\,d\tau \Big\|_{L_t^{q}(t_0,T\,;\,L_x^{p}(\mathbb{R}))} \\ &\leq \Big\| \mathcal{X}_{[-T,T]}(\tau)|\tau|^{-\frac{\gamma}{2}} \star \mathcal{X}_{[t_0,T]}(\tau) \|g(\tau,x)\|_{L_x^{p'}(\mathbb{R})} \Big\|_{L_t^{q}(t_0,T)}, \end{aligned} \tag{5.10}$$

but $\mathcal{X}_{[-T,T]}(\tau)|\tau|^{-\frac{\gamma}{2}} \in L_\omega^{2/\gamma}(\mathbb{R})$ and consequently by using weak Young's inequality, we get in (5.9)

$$\Big| \Big\langle V_{\frac{\gamma}{2}}(t_0,t)u_0, g \Big\rangle \Big| \leq c(\gamma) \|u_0\|_{L^2(\mathbb{R})} \|g\|_{L_t^{q'}(t_0,T\,;\,L_x^{p'}(\mathbb{R}))},$$

and hence (5.1), (5.3) are obtained by writing a weak Young's inequality directly in (5.10). We get (5.2) by integrating in (5.10) on $[t_0,t]$ instead of $[t_0,T]$.

6. The nonlinear problem—smooth solutions

The aim of the last two sections is to establish that the following nonlinear problem is locally well-posed in Sobolev spaces of low index:

$$u_t + L(t)u = F(u) \tag{6.1}$$

$$u(0) = u_0. \tag{6.2}$$

$F(u)$, which abridges $F(u,\overline{u},u_x,\overline{u}_x)$, is a sum of terms of the form

$$\alpha_1 u^{p_1}\overline{u}^{p_2} \qquad \alpha_1 \in \mathbb{C}, p_1+p_2 \geq 1 \tag{6.3}$$

$$\alpha_3 u^{p_3}\overline{u}^{p_4}\overline{u}_x \qquad \alpha_3 \in \mathbb{C}, p_3+p_4 \geq 1 \tag{6.4}$$

$$\alpha_5 |u|^{2p_5} u_x \qquad \alpha_5 \in \mathbb{R}, p_5 \geq 1 \tag{6.5}$$

$$\beta_1 u^{q_1}\overline{u}^{q_2}\overline{u}_x (g_1 * u^{q_3}\overline{u}^{q_4}) \qquad \beta_1 \in \mathbb{C}, q_3+q_4 \geq 1 \tag{6.6}$$

$$\beta_5 |u|^{2q_5} u_x (g_5 * |u|^{2q_6}) \qquad \beta_5 \in \mathbb{C}, q_6 \geq 1 \tag{6.7}$$

$$\beta_7 u|u|^{2q_7} \frac{\partial}{\partial x}(g_7 * |u|^{2q_8}) \qquad \beta_7 \in \mathbb{R}, q_8 \geq 1 \tag{6.8}$$

$$\beta_9 u^{q_9}\overline{u}^{q_{10}}\frac{\partial}{\partial x}(g_9 * u^{q_{11}}\overline{u}^{q_{12}}) \qquad \beta_9\in\mathbb{C}, q_{11}+q_{12}\geq 1, \sum_{j=9}^{j=12} q_j \geq 2 \tag{6.9}$$

$$\beta_{13} u^{q_{13}}\overline{u}^{q_{14}}(g_{13} * u^{q_{15}}\overline{u}^{q_{16}}) \qquad \beta_{13}\in\mathbb{C}, q_{15}+q_{16}\geq 1, \tag{6.10}$$

where $*$ is the convolution with respect to the space variable x, p_j and q_j are positive integers and $g_j \in L^1(\mathbb{R})$.

As a first step, we establish in this section the existence of a smooth solution of the nonlinear problem

$$u_t + \mu(u_t)_{4x} + L(t)u = F(u) \tag{6.11}$$

$$u(0) = u_0 \tag{6.12}$$

for $\mu \geq 0$ and $F(u)$ of the previous form but supposing furthermore that functions g_7 and g_9 belong to $W^{1,1}(\mathbb{R})$.

First, let us recall the commutator estimates [2], [15], [17] and introduce some notations.

Lemma 6.1. *Let $s>0$ and $1<p<+\infty$. There exists a constant c_s such that*

$$\left\|J^s(fg) - fJ^s g\right\|_{L^p} \leq c_s\left(\left\|\nabla f\right\|_{L^{p_1}}\left\|g\right\|_{H^{s-1}_{p_2}} + \left\|f\right\|_{H^s_{p_3}}\left\|g\right\|_{L^{p_4}}\right) \tag{6.13}$$

$$\left\|J^s(fg)\right\|_{L^p} \leq c_s\left(\left\|f\right\|_{L^{p_1}}\left\|g\right\|_{H^s_{p_2}} + \left\|f\right\|_{H^s_{p_3}}\left\|g\right\|_{L^{p_4}}\right), \tag{6.14}$$

where $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}=\frac{1}{p}$.

Corollary 6.2. *Let $s>0$ and $p,p_1,p_2\in\mathbb{N}^*$. There exists a constant c_s such that*

$$\left\|f^p\right\|_{H^s} \leq c_s\left\|f\right\|_{L^\infty}^{p-1}\left\|f\right\|_{H^s} \tag{6.15}$$

$$\left\|f^{p_1}g^{p_2}\right\|_{H^s} \leq c_s\left(\left\|f\right\|_{L^\infty}^{p_1}\left\|g\right\|_{L^\infty}^{p_2-1}\left\|g\right\|_{H^s} + \left\|f\right\|_{L^\infty}^{p_1-1}\left\|g\right\|_{L^\infty}^{p_2}\left\|f\right\|_{H^s}\right) \tag{6.16}$$

$$\textit{If} \quad s\geq 1 \qquad \left\|(J^s - D^s)f\right\|_{L^2} \leq c_s\left\|f\right\|_{H^{s-1}}. \tag{6.17}$$

Notation 6.3. (i) We denote by $f(v)$ the sum of terms occurring in $F(v)$ of the form (6.3), (6.8), (6.9) and (6.10). (ii) Let $s\in\mathbb{R}$ and $R>0$; we denote by $B_s(0,R)$ the open ball in $H^s(\mathbb{R})$ of radius R. For $\mu\geq 0$, we set $\Lambda_\mu = (I+\mu\partial_x^4)^{-1}$. (iii) Let $v\in H^s(\mathbb{R})$. To each term of the form (6.4) through (6.7) in $F(v)$, we associate a linear operator in such a way:

$$u \mapsto \alpha_3 v^{p_3}\overline{v}^{p_4}\overline{u}_x \quad \text{is associated to type (6.4),} \tag{6.18}$$

$$u \mapsto \alpha_5 |v|^{2p_5} u_x \quad \text{is associated to type (6.5),} \tag{6.19}$$

$$u \mapsto \beta_1 v^{q_1}\overline{v}^{q_2}(g_1 * v^{q_3}\overline{v}^{q_4})\overline{u}_x \quad \text{is associated to type (6.6),} \tag{6.20}$$

$$u \mapsto \beta_5 |v|^{2q_5}(g_5 * |v|^{2q_6})u_x \quad \text{is associated to type (6.7).} \tag{6.21}$$

We define $L_1(v)$ as the sum of the previous linear operators and set $Q(t,v) = L(t) - L_1(v)$ for $t \in [0, T_0]$. Hence we can rewrite equation (6.11) as

$$u_t + \Lambda_\mu Q(t,u)u = \Lambda_\mu f(u). \tag{6.22}$$

Thanks to Kato's results ([14]), we can now establish

Theorem 6.4. *Let $\mu \geq 0$, $s \geq 3$, $R > 0$ and $u_0 \in B_s(0,R)$. Then*

(i) *There exists $T_\mu > 0$, depending only on R, s and μ, and a unique solution $u \in \mathcal{C}(0,T_\mu; B_s(0,R)) \cap \mathcal{C}^1(0,T_\mu; L^2(\mathbb{R}))$ of* (6.11)–(6.12).

(ii) *Let $u_{0,n} \in B_s(0,R)$ be a sequence tending to u_0 in $H^s(\mathbb{R})$, and denote by u_n the solution of* (6.11)–(6.12) *found in* (i), *taking $u_{0,n}$ as initial data. Then $u_n(t)$ converges to $u(t)$ in $H^s(\mathbb{R})$, uniformly in $t \in [0, T_\mu]$.*

Proof. The proof of this theorem consists in checking the hypotheses of Theorem 6-7 in [14]. This is the aim of the following five lemmas.

Lemma 6.5. *Let $\mu \geq 0$, $s \geq 3$, $R > 0$. For $t_0 \in [0, T_0]$ and $v \in B_s(0,R)$, the operator $\Lambda_\mu Q(t_0, v)$ generates a semigroup $\mathcal{C}_0$ in $L^2(\mathbb{R})$ denoted by $\exp\big(t\Lambda_\mu Q(t_0,v)\big)$. Moreover there exists a constant $c(R,\mu)$ depending only on R and μ such that*

$$\forall t \geq 0 \quad \big\|\exp\big(t\Lambda_\mu Q(t_0,v)\big)\big\|_{L^2(\mathbb{R})} \leq \exp\big(c(R,\mu)t\big). \tag{6.23}$$

Proof. When $\mu \neq 0$, $\Lambda_\mu Q(t_0,v)$ is a bounded operator on $L^2(\mathbb{R})$ and consequently generates a uniformly continuous semigroup, and there exists a constant $c(R,\mu)$ depending only on $\big\|\Lambda_\mu Q(t_0,v)\big\|_{L^2(\mathbb{R})}$ satisfying (6.23). It remains to prove Lemma 6.5 for $\mu = 0$. For this purpose, we will make use of results in [27]. Let p be the maximum of the terms occurring in $F(u)$ of

$$\Big\{p_1 + p_2, p_3 + p_4, 2p_5, \sum_{j=1}^{j=4} q_j, 2(q_5+q_6), 2(q_7+q_8), \sum_{j=9}^{j=12} q_j, \sum_{j=13}^{j=16} q_j\Big\}$$

and $D(L_1(v))$ be the domain of $L_1(v)$. As a first step, we find a constant $m > 0$ such that $-L_1(v) - mI$ is a dissipative operator: it suffices to find $m > 0$ such that

$$\forall \delta > 0 \ \forall u \in D(L_1(v)) \quad \big\|\big(L_1(v) + mI + \delta I\big)u\big\|_{L^2(\mathbb{R})} \geq \delta \big\|u\big\|_{L^2(\mathbb{R})}. \tag{6.24}$$

We fix $u \in D(L_1(v))$ and estimate each term occurring in $\operatorname{Re}\big(L_1(v)u, u\big)$, and we find a constant c such that

$$\big|\operatorname{Re}\big(L_1(v)u,u\big)\big| \leq c(R+1)^p \big\|u\big\|^2_{L^2(\mathbb{R})}.$$

Now for $m > c(R+1)^p$ we get (6.24) with

$$\big\|\big(L_1(v)+mI+\delta I\big)u\big\|_{L^2(\mathbb{R})}\big\|u\big\|_{L^2(\mathbb{R})} \geq \operatorname{Re}\big(\big(L_1(v)+mI+\delta I\big)u,u\big),$$

and hence $-L_1(v)-mI$ is dissipative.

In a second step, since $L(t_0)$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup of contractions and $-L_1(v)-mI$ is a dissipative operator such that $D(L(t_0)) \subset D(-L_1(v)-mI)$, to prove that $L(t_0)-L_1(v)-mI$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup of contractions, it suffices to find constants $\varepsilon_1 \in [0,1)$ and $\varepsilon_2 \geq 0$ such that for $u \in D(L_1(t_0))$

$$\big\|\big(L_1(v)+mI\big)u\big\|_{L^2(\mathbb{R})} \leq \varepsilon_1\big\|L(t_0)u\big\|_{L^2(\mathbb{R})} + \varepsilon_2\big\|u\big\|_{L^2(\mathbb{R})}. \tag{6.25}$$

If we assume this estimate, since $-mI$ is a bounded operator on $L^2(\mathbb{R})$, $L(t_0)-L_1(v)$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $\exp\big(tQ(t_0,v)\big)$ satisfying (6.23) with $c(R,0)=m$. Let us establish (6.25). By writing $\big\|L(t)u\big\|^2_{L^2(\mathbb{R})}$, we get

$$\frac{a^2}{2}\big\|u_{3x}\big\|^2_{L^2(\mathbb{R})} \leq \big\|L(t_0)u\big\|^2_{L^2(\mathbb{R})} + c\big(\big\|u_{2x}\big\|^2_{L^2(\mathbb{R})} + \big\|u_x\big\|^2_{L^2(\mathbb{R})} + \big\|u\big\|^2_{L^2(\mathbb{R})}\big), \tag{6.26}$$

but ([27]) for any $h>0$, there exists a constant $c(h)$ such that

$$\big\|u_x\big\|_{L^2(\mathbb{R})} \leq h\big\|u_{3x}\big\|_{L^2(\mathbb{R})} + c(h)\big\|u\big\|_{L^2(\mathbb{R})}, \tag{6.27}$$

from which it follows that

$$\big\|u_{2x}\big\|^2_{L^2(\mathbb{R})} \leq h\big\|u_{3x}\big\|^2_{L^2(\mathbb{R})} + c(h)\big\|u\big\|^2_{L^2(\mathbb{R})}. \tag{6.28}$$

Using together (6.26), (6.27) and (6.28), with a well-chosen $h>0$, we get

$$\big\|u_{3x}\big\|^2_{L^2(\mathbb{R})} \leq c\big(\big\|L(t_0)u\big\|^2_{L^2(\mathbb{R})} + \big\|u\big\|^2_{L^2(\mathbb{R})}\big). \tag{6.29}$$

But

$$\big\|\big(L_1(v)+mI\big)u\big\|_{L^2(\mathbb{R})} \leq m\big\|u\big\|_{L^2(\mathbb{R})} + c(R+1)^p\big(\big\|u_x\big\|_{L^2(\mathbb{R})} + \big\|u\big\|_{L^2(\mathbb{R})}\big),$$

and choosing $h>0$ sufficiently small in (6.27), we get (6.25) and hence Lemma 6.5.

Lemma 6.6. *Let $\mu \geq 0$, $s \geq 3$ and $R>0$. There exists a constant $c(R,s)$ depending only on R and s such that for $t \in [0,T_0]$ and $v \in B_s(0,R)$*

$$\big\|J^s\Lambda_\mu Q(t,v)J^{-s} - \Lambda_\mu Q(t,v)\big\|_{L^2,L^2} \leq c(R,s).$$

Proof. It suffices to consider $J^s\Lambda_\mu L_1(v)J^{-s} - \Lambda_\mu \text{Ł}_1(v)$. For $u \in L^2(\mathbb{R})$ we set $w = J^{-s}u$ and we use the commutator estimates of Lemma 6.1 and Corollary 6.2 to estimate terms occurring in $J^s\Lambda_\mu L_1(v)w - \Lambda_\mu \text{Ł}_1(v)J^s w$ by $\|w\|_{H^s(\mathbb{R})}$. We only treat terms of type (6.18) and (6.20); the two other are similar. We have

$$\begin{aligned}\big\|J^s\Lambda_\mu[v^{p_3}\overline{v}^{p_4}\overline{w}_x] - \Lambda_\mu v^{p_3}\overline{v}^{p_4}J^s\overline{w}_x\big\|_{L^2(\mathbb{R})} &\\ \le c\big(\big\|\nabla(v^{p_3}\overline{v}^{p_4})\big\|_{L^\infty(\mathbb{R})}\|w\|_{H^s(\mathbb{R})} + \big\|v^{p_3}\overline{v}^{p_4}\big\|_{H^s(\mathbb{R})}\|w_x\|_{L^\infty(\mathbb{R})}\big) &\\ \le c(R+1)^p\|w\|_{H^s(\mathbb{R})}, &\end{aligned}$$

and in the same way

$$\begin{aligned}\big\|J^s\Lambda_\mu[v^{q_1}\overline{v}^{q_2}(g_1 * v^{q_3}\overline{v}^{q_4})\overline{w}_x] - \Lambda_\mu[v^{q_1}\overline{v}^{q_2}(g_1 * v^{q_3}\overline{v}^{q_4})J^s\overline{w}_x]\big\|_{L^2(\mathbb{R})} &\\ \le c(R+1)^p\|g\|_{L^1(\mathbb{R})}\|w\|_{H^s(\mathbb{R})}. &\end{aligned}$$

Lemma 6.7. *Let $s \ge 3$ and $R > 0$. There exists a constant $c(R,s)$ depending only on R and s such that for any $\mu \ge 0$*
(i) $\forall t \in [0,T_0] \quad \forall v \in B_s(0,R) \quad \forall u \in H^s(\mathbb{R})$

$$\big\|\Lambda_\mu Q(t,v)u\big\|_{L^2(\mathbb{R})} \le c(R,s)\|u\|_{H^s(\mathbb{R})}.$$

(ii) *For $v \in B_s(0,R)$, if $\Lambda_\mu Q(t,v)$ is considered as an operator from $H^s(\mathbb{R})$ to $L^2(\mathbb{R})$, then the function $t \mapsto \Lambda_\mu Q(t,v)$ is continuous on $[0,T_0]$.*
(iii) $\forall t \in [0,T_0] \quad \forall v,w \in B_s(0,R) \quad \forall u \in H^s(\mathbb{R})$

$$\big\|\big(\Lambda_\mu Q(t,v) - \Lambda_\mu Q(t,w)\big)u\big\|_{L^2(\mathbb{R})} \le c(R,s)\|v-w\|_{L^2(\mathbb{R})}\|u\|_{H^s(\mathbb{R})}.$$

Proof. It suffices to consider the case $\mu = 0$. Property (i) is a direct consequence of the definition of $Q(t,v)$, and property (ii) follows from the hypotheses on the functions $a_k \in \mathcal{C}^1(0,T_0)$:

$$\forall t_1,t_2 \in [0,T_0]\ \forall u \in H^s(\mathbb{R}) \quad \big\|\big(L(t_1) - L(t_2)\big)u\big\|_{L^2(\mathbb{R})} \le c|t_1 - t_2|\|u\|_{H^s(\mathbb{R})}.$$

It remains to prove (iii). As in the proof of the previous lemma, we only consider terms of type (6.18) and (6.20). One obtains easily that

$$\big\|\big(v^{p_3}\overline{v}^{p_4} - w^{p_3}\overline{w}^{p_4}\big)\overline{u}_x\big\|_{L^2(\mathbb{R})} \le c(R+1)^{p-1}\|v-w\|_{L^2(\mathbb{R})}\|u\|_{H^s(\mathbb{R})},$$

and the proof is ended by writing

$$\begin{aligned}&v^{q_1}\overline{v}^{q_2}(g_1 * v^{q_3}\overline{v}^{q_4}) - w^{q_1}\overline{w}^{q_2}(g_1 * w^{q_3}\overline{w}^{q_4})\\ &\quad = \big(v^{q_1}\overline{v}^{q_2} - w^{q_1}\overline{w}^{q_2}\big)(g_1 * v^{q_3}\overline{v}^{q_4}) + w^{q_1}\overline{w}^{q_2}\big(g_1 * \big(v^{q_3}\overline{v}^{q_4} - w^{q_3}\overline{w}^{q_4}\big)\big).\end{aligned}$$

Lemma 6.8. *Let $s \geq 3$ and $R > 0$. There exists a constant $c(R,s)$ depending only on R and s such that for any $\mu \geq 0$, $t \in [0, T_0]$ and $v, w \in B_s(0,R)$*

$$\left\| J^s \Lambda_\mu Q(t,v) J^{-s} - \Lambda_\mu Q(t,v) - J^s \Lambda_\mu Q(t,w) J^{-s} + \Lambda_\mu Q(t,w) \right\|_{L^2, L^2} \leq c(R,s) \left\| v - w \right\|_{H^s(\mathbb{R})}.$$

Proof. It is similar to the proof of Lemma 6.6. Let $u \in L^2(\mathbb{R})$ and set $\underline{u} = J^s u$. By using the commutator estimates and a suitable decomposition of $v^{p_3}\overline{v}^{p_4} - w^{p_3}\overline{w}^{p_4}$ we get, for example,

$$\left\| J^s \left[\left(v^{p_3}\overline{v}^{p_4} - w^{p_3}\overline{w}^{p_4} \right) \underline{u}_x \right] - \left(v^{p_3}\overline{v}^{p_4} - w^{p_3}\overline{w}^{p_4} \right) J^s \underline{u}_x \right\|_{L^2(\mathbb{R})} \leq c R^{p_3+p_4-1} \left\| \underline{u} \right\|_{H^s(\mathbb{R})} \left\| v - w \right\|_{H^s(\mathbb{R})}.$$

The other terms to consider can be treated similarly.

Lemma 6.9. *Let $s \geq 3$ and $R > 0$. There exists a constant $c(R,s)$ depending only on R and s such that*

(i) $\forall v \in B_s(0,R)$ $\quad \left\| f(v) \right\|_{H^s(\mathbb{R})} \leq c(R,s)$,

(ii) $\forall v, w \in B_s(0,R)$ $\quad \left\| f(v) - f(w) \right\|_{L^2(\mathbb{R})} \leq c(R,s) \left\| v - w \right\|_{L^2(\mathbb{R})}$,

(iii) $\forall v, w \in B_s(0,R)$ $\quad \left\| f(v) - f(w) \right\|_{H^s(\mathbb{R})} \leq c(R,s) \left\| v - w \right\|_{H^s(\mathbb{R})}$.

Proof. It suffices to consider terms of the form (6.3) and (6.10) since we assume in this section that g_7 and g_9 belong to $W^{1,1}(\mathbb{R})$. Now (i) is obvious since $H^s(\mathbb{R})$ is an algebra and (ii) and (iii) follow from a suitable decomposition of the terms occurring in $f(v) - f(w)$. This ends the proof of Theorem 6.4. □

Now that we have established the existence of a solution of the regularized problem (6.11)–(6.12) in any $H^s(\mathbb{R})$, let us prove that there exists a solution in $H^\infty(\mathbb{R})$ when $\mu = 0$.

Theorem 6.10. *Let $\mu = 0$ and $u_0 \in H^\infty(\mathbb{R})$. Then there exists $T > 0$ and $u \in \mathcal{C}(0,T; H^\infty(\mathbb{R}))$ solution of* (6.11)–(6.12).

The proof is based on a priori estimates, regularization and convergence arguments that will be established in the following lemmas.

Lemma 6.11. *Let $s \geq 3$, $R > 0$ and $u_0 \in B_s(0,R)$. For $\mu > 0$ set $u_{0\mu} = (I - \mu^{\frac{1}{2}}\Delta^2)^{-1} u_0$, and let $u_\mu \in \mathcal{C}(0,T_\mu; H^{s+2}(\mathbb{R})) \cap \mathcal{C}^1(0,T_\mu; L^2(\mathbb{R}))$ be the solution of* (6.11) *satisfying $u_\mu(0) = u_{0\mu}$. Then there exists a constant $c(s)$ depending only on s such that for any $\mu > 0$*

$$\frac{d}{dt}\left(\left\| u_\mu \right\|^2_{H^s(\mathbb{R})} + \mu \left\| (u_\mu)_{xx} \right\|^2_{H^s(\mathbb{R})} \right) \tag{6.30}$$

$$\leq c(s)\big(1+\|u_\mu\|_{L^\infty(\mathbb{R})}\big)^{p-1}\big(1+\|u_\mu\|_{L^\infty(\mathbb{R})}+\|(u_\mu)_x\|_{L^\infty(\mathbb{R})}\big)\|u_\mu\|^2_{H^s(\mathbb{R})}.$$

Proof. By applying J^s to (6.11), multiplying by $J^s\overline{u}_\mu$ and integrating, we get

$$\frac{d}{dt}\big(\|u_\mu\|^2_{H^s(\mathbb{R})}+\mu\|(u_\mu)_{xx}\|^2_{H^s(\mathbb{R})}\big)=2\mathrm{Re}\int J^sF(u_\mu)\cdot J^s\overline{u}_\mu\,dx. \tag{6.31}$$

Now it suffices to apply the commutator estimates to the terms of the right-hand side.

Lemma 6.12. *Let $s\geq 3$, $R>0$ and $u_0\in B_s(0,R)$, and define for $\mu>0$, $u_{0\mu}$ and u_μ as in Lemma 6.11. Suppose there exist two constants $K>0$ and $T_1\in(0,T_0]$ such that, as long as for $t\leq T_1$ the solution u_μ is defined on $[0,t]$,*

$$\int_0^t\big(1+\|u_\mu\|_{L^\infty(\mathbb{R})}\big)^{p-1}\big(1+\|u_\mu\|_{L^\infty(\mathbb{R})}+\|(u_\mu)_x\|_{L^\infty(\mathbb{R})}\big)\,d\tau\leq K. \tag{6.32}$$

Then the solution $u_\mu\in H^{s+2}(\mathbb{R})$ can be extended to a solution on $[0,T_1]$.

Proof. Suppose that the solution $u_\mu\in H^{s+2}(\mathbb{R})$ can only be extended to a solution on a maximal interval with $T_{max}<T_1$. Since $\mu\|(u_{0\mu})_{xx}\|^2_{H^s(\mathbb{R})}\leq\|u_0\|^2_{H^s(\mathbb{R})}$, we get with (6.31) and (6.32)

$$\forall t\in[0,T_{\max}]\quad \|u_\mu\|^2_{H^s(\mathbb{R})}+\mu\|(u_\mu)_{xx}\|^2_{H^s(\mathbb{R})}\leq 2\|u_o\|^2_{H^s(\mathbb{R})}\exp(c(s)K).$$

Hence Theorem 6.4 allows us to extend u_μ on the right-hand side of $T_{\max}$ on an interval whose length depends only on $\|u_o\|^2_{H^s(\mathbb{R})}\exp(c(s)K)$. This contradicts the maximality of $T_{\max}$.

Proposition 6.13. *Let $s\geq 3$, $u_0\in H^s(\mathbb{R})$ and define for $\mu>0$, $u_{0\mu}$ and u_μ as in Lemma 6.11. There exists a constant $T>0$ not depending on $\mu>0$ or $s\geq 3$ such that the solution u_μ belongs to $\mathcal{C}(0,T;H^{s+2}(\mathbb{R}))$.*

Proof. According to Lemma 6.11, we have

$$\frac{d}{dt}\big(\|u_\mu\|^2_{H^s(\mathbb{R})}+\mu\|(u_\mu)_{xx}\|^2_{H^s(\mathbb{R})}\big)\leq c(s)\big(1+\|u_\mu\|^2_{H^s(\mathbb{R})}\big)^{p+2}. \tag{6.33}$$

Using Gronwall's lemma, there exists a constant $T(s)>0$ not depending on $s\geq 3$ or $\|u_0\|_{H^s(\mathbb{R})}$, such that if u_μ is defined for $t\leq T(s)$ then $\|u_\mu\|_{H^s(\mathbb{R})}$ is uniformly bounded in t and $\mu>0$. Hence the hypotheses of Lemma 6.12 are satisfied and $u_\mu\in\mathcal{C}(0,T(s);H^{s+2}(\mathbb{R}))$.

In a second step, suppose that $u_0 \in H^{s'}(\mathbb{R})$ with $s' \geq s$. Since $\|u_\mu\|_{H^s(\mathbb{R})}$ is uniformly bounded we can again apply Lemma 6.12 to conclude that $u_\mu \in \mathcal{C}(0, T(s); H^{s'+2}(\mathbb{R}))$.

Lemma 6.14. *Let $T > 0$ and $u_\mu, v_\mu \in \mathcal{C}(0, T; H^\infty(\mathbb{R}))$. Suppose that for any $s \geq 3$, $\|u_\mu\|_{H^s(\mathbb{R})}$ and $\|v_\mu\|_{H^s(\mathbb{R})}$ are uniformly bounded in $\mu > 0$ and $t \in [0, T]$. Then there exist $u, v \in L^\infty(0, T; H^\infty(\mathbb{R}))$ such that for any $s \geq 3$, u_μ converges to u and v_μ to v in $L^\infty(0, T; H^s(\mathbb{R}))$ weak $\star$. Furthermore, for $g \in L^1(\mathbb{R})$, $g * u_\mu$ converges to $g * u$ and $u_\mu v_\mu$ to uv in the same spaces.* (*The convergences may hold only for a subsequence.*)

Proof. We obtain u and v by considering Sobolev spaces of integer index and diagonal extraction of subsequences. A compactness lemma allows us to suppose that u_μ and v_μ converge strongly in $L^2(0, T; L^4_{loc}(\mathbb{R}))$ and almost everywhere in $[0, T] \times \mathbb{R}$. Let $s \geq 3$. Since $u_\mu v_\mu$ is bounded in $L^\infty(0, T; H^s(\mathbb{R}))$, there exists a subsequence that converges weakly $\star$ to w in this space. Let us prove by duality that $w = uv$ in $L^2(0, T; H^{-1}(\mathbb{R}))$. For $\varphi \in L^2(0, T; H^1(\mathbb{R}))$ with support in a compact subset Ω of $\mathbb{R}$, we get $w = uv$ by writing

$$
\begin{aligned}
|\langle u_\mu v_\mu - uv, \varphi\rangle| \leq \|u_\mu\|_{L^\infty(0,T;L^2(\mathbb{R}))} \| v_\mu \\
- v\|_{L^2(0,T;L^4(\Omega))} \|\varphi\|_{L^2(0,T;L^4(\Omega))} + |\langle u_\mu - u, v\varphi\rangle|.
\end{aligned}
$$

To get the desired convergence for any $s \geq 3$, it suffices to extract diagonal subsequences. We treat $g * u_\mu$ in the same way and conclude by writing

$$
\langle g * u_\mu - g * u, \varphi\rangle = \langle u_\mu - u, \check{g} * \varphi\rangle
$$

where $\check{g}(x) = g(-x)$.

Proof of Theorem 6.10. Set $u_{0\mu} = (I - \mu^{\frac{1}{2}}\Delta)^{-1} u_0$; according to Proposition 6.13 there exists $T > 0$ not depending on μ, s and a unique solution $u_\mu \in \mathcal{C}(0, T; H^{s+2}(\mathbb{R}))$ of (6.11) satisfying $u_\mu(0) = u_{0\mu}$, and hence $u_\mu \in \mathcal{C}(0, T; H^\infty(\mathbb{R}))$. But (6.33) allows us to assume that for $s \geq 3$, $\|u_\mu\|_{H^s(\mathbb{R})}$ is uniformly bounded in $\mu > 0$ and $t \in [0, T]$ and consequently, by using Lemma 6.11, uniformly bounded in $s \geq 3$ too. Then there exists $u \in L^\infty(0, T; H^\infty(\mathbb{R}))$ such that u_μ converges to u in $L^\infty(0, T; H^s(\mathbb{R}))$ weak $\star$ for any $s \geq 3$. Thanks to Lemma 6.14, we can prove that u is solution of (6.11)–(6.12) for $\mu = 0$ and belongs to $\mathcal{C}(0, T; H^\infty(\mathbb{R}))$ according to Theorem 6.4.

7. The nonlinear problem—the Cauchy problem in $H^s(\mathbb{R})$ for $s > 3/4$

We can now establish the local existence of a solution of problem (6.1)–(6.2) in Sobolev spaces of index larger than 3/4.

Theorem 7.1. (i) *Let $s > 3/4$, $R > 0$ and $u_0 \in B_s(0,R)$. Then there exists $T_s = T(s,R) > 0$ depending only on R and s, and a unique solution $u \in \mathcal{C}(0,T_s;H^s(\mathbb{R}))$ of* (6.1)–(6.2) *such that the following norms are finite:*

$$\left\|D^s u_x\right\|_{L^\infty_x(\mathbb{R}\,;\,L^2_t(0;T_s))}, \qquad \left\|J^r u_x\right\|_{L^4_t(0,T_s\,;\,L^\infty_x(\mathbb{R}))} \quad \textit{for} \;\; 0 \le r \le s - \frac{3}{4}$$

$$\left\|J^r u\right\|_{L^2_x(\mathbb{R}\,;\,L^\infty_t(0;T_s))} \quad \textit{for} \quad 0 \le r < s - \frac{3}{4},$$

$$\left\|D^{s+\frac{\theta}{4}} u\right\|_{L^q_t(0,T_s\,;\,L^p_x(\mathbb{R}))} \;\; \textit{for} \;\; \theta \in [0,1] \;\; \textit{and} \;\; (q,p) = (\frac{4}{\theta}, \frac{2}{1-\theta}).$$

(ii) *The solution $u \in \mathcal{C}(0,T_s;H^s(\mathbb{R}))$ depends continuously on the initial data for the norms occurring in* (i). (iii) *Let $s' > s$ and suppose $u_0 \in H^{s'}(\mathbb{R})$. Then the solution $u \in \mathcal{C}(0,T_s;H^s(\mathbb{R}))$ is defined on $[0,T_s]$ in $H^{s'}(\mathbb{R})$. Furthermore* (i) *and* (ii) *are satisfied with s' replacing s and $T_{s'} \ge T_s$.*

We begin with the construction of approximate solutions.

Notation 7.2. Let $s > 3/4$ and $u_0 \in H^s(\mathbb{R})$. Fix $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\varphi \ge 0$ and $\int \varphi(x)\,dx = 1$. For $\varepsilon > 0$, set $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$, $u_0^\varepsilon = u_0 * \varphi_\varepsilon$ and, for functions g_7 and g_9 occurring in (6.8) and (6.9), $g_7^\varepsilon = g_7 * \varphi_\varepsilon$ and $g_9^\varepsilon = g_9 * \varphi_\varepsilon$. Consider now the regularized problem

$$u_t + L(t)u = F_\varepsilon(u) \tag{7.1}$$

$$u(0) = u_0^\varepsilon, \tag{7.2}$$

where $F_\varepsilon(u)$ has the same terms as $F(u)$ except for terms of type (6.8) and (6.9) for which g_7 and g_9 are replaced by g_7^ε and g_9^ε. Since $u_0^\varepsilon \in H^\infty(\mathbb{R})$, Theorem 6.10 gives for each $\varepsilon > 0$ a constant $T_\varepsilon > 0$ and a unique solution $u^\varepsilon \in \mathcal{C}(0,T_\varepsilon;H^\infty(\mathbb{R}))$ of (7.1)–(7.2).

To get the convergence of approximate solutions to the solution of (6.1)–(6.2) of Theorem 7.1, we first need some estimates.

Proposition 7.3. *Fix $s_0 > 3/4$ and $R > 0$ and let $s > s_0$ and $u_0 \in B_s(0,R)$. Then*

(i) *There exist constants $T_1 = T_1(s,R) \le T_0$ and $M_1 = M_1(s,R)$ depending only on s and R and a constant $c_{s_0} = c_{s_0}(s,s_0)$ depending only on s and*

s_0 such that, as long as $u^\varepsilon(t)$ exists on $[0,T]$ with $T \le T_1$

$$\|u^\varepsilon\|_{L_t^\infty(0,T\,;\,H^s(\mathbb{R}))} \le M_1 \tag{7.3}$$

$$\|D^s u_x^\varepsilon\|_{L_x^\infty(\mathbb{R}\,;\,L_t^2(0,T))} \le M_1 \tag{7.4}$$

$$\|J^r u_x^\varepsilon\|_{L_t^4(0,T\,;\,L_x^\infty(\mathbb{R}))} \le M_1 \quad \text{for } 0 \le r \le s - \frac{3}{4} \tag{7.5}$$

$$\|J^r u^\varepsilon\|_{L_x^2(\mathbb{R}\,;\,L_t^\infty(0,T))} \le c_{s_0} M_1 \quad \text{for } 0 \le r < s - s_0 \tag{7.6}$$

$$\|D^{s+\frac{\theta}{4}} u^\varepsilon\|_{L_t^q(0,T\,;\,L_x^p(\mathbb{R}))} \le M_1 \quad \text{for } \theta \in [0,1] \text{ and } (q,p) = (\frac{4}{\theta}, \frac{2}{1-\theta}). \tag{7.7}$$

(ii) *For any r in $[0, s - \frac{3}{4})$, there exists a constant c_r such that*

$$\|J^r u^\varepsilon\|_{L_x^2(\mathbb{R}\,;\,L_t^\infty(0,T))} \le c_r M_1. \tag{7.8}$$

(iii) *The solution u^ε can be extended up to T_1 and consequently can be assumed to belong to $\mathcal{C}(0,T_1;H^\infty(\mathbb{R}))$.* (iv) *If $s' > s$, then* (i) *and* (ii) *written for u_0 belonging to $H^s(\mathbb{R})$ and for u_0 belonging to $H^{s'}(\mathbb{R})$ are satisfied on the same interval $[0,T_1]$—only the bounds change.*

In order to prove Proposition 7.3, on the one hand we notice that, since u^ε belongs to $\mathcal{C}(0,T_\varepsilon;H^\infty(\mathbb{R}))$, proofs similar to the ones of Lemmas 6.11 and 6.12 allow us to get

Lemma 7.4. *Let $s > \frac{3}{4}$. There exists a constant c_s depending only on s such that for $u_0 \in H^s(\mathbb{R})$ and u_0^ε, u^ε given in Notations* 7.2*, for any t in $[0,T_\varepsilon]$*

$$\begin{aligned}\|u^\varepsilon(t)\|_{H^s(\mathbb{R})} \le \|u_0\|_{H^s(\mathbb{R})} \exp\Big[c_s\int_0^t \big(1 + \|u^\varepsilon\|_{L^\infty(\mathbb{R})}\big)^{p-1} \\ \times \big(1 + \|u^\varepsilon\|_{L^\infty(\mathbb{R})} + \|(u^\varepsilon)_x\|_{L^\infty(\mathbb{R})}\big)\,d\tau\Big].\end{aligned}$$

Lemma 7.5. *Let $s > 3/4$, $u_0 \in H^s(\mathbb{R})$ and let u_0^ε, u^ε be as given in Notations* 7.2*. Suppose there exist two constants $K > 0$ and $T_1 \in (0,T_0]$ such that, as long as for $T \le T_1$ the solution u^ε is defined on $[0,T]$,*

$$\int_0^T \big(1 + \|u^\varepsilon\|_{L^\infty(\mathbb{R})}\big)^{p-1}\big(1 + \|u^\varepsilon\|_{L^\infty(\mathbb{R})} + \|(u^\varepsilon)_x\|_{L^\infty(\mathbb{R})}\big)\,dt \le K. \tag{7.9}$$

Then the solution u^ε can be extended to a solution on $[0,T_1]$.

On the other hand, we establish estimates on some special integrals for any functions in $\mathcal{C}(0,T;H^\infty(\mathbb{R}))$.

Lemma 7.6. *Let $g \in L^1(\mathbb{R})$, $T > 0$ and $s > \frac{3}{4}$. For $u \in \mathcal{C}(0,T;H^s(\mathbb{R}))$ define*

$$\mathcal{N}(T,s,u) = \max\Big\{\|u\|_{L_t^\infty(0\,T\,;\,H^s(\mathbb{R}))}, \|D^s u_x\|_{L_x^\infty(\mathbb{R}\,;\,L_t^2(0,T))}, \tag{7.10}$$

$$\|u_x\|_{L^4_t(0,T\,;\,L^\infty_x(\mathbb{R}))}, \|u\|_{L^2_x(\mathbb{R}\,;\,L^\infty_t(0,T))}\Big\}$$

$$\mathcal{B}(T,s) = \Big\{u \in \mathcal{C}(0,T;H^s(\mathbb{R})), \mathcal{N}(T,s,u) < +\infty\Big\}. \tag{7.11}$$

Then for $u, v, w \in \mathcal{B}(T,s)$, we have

$$\int_0^T \|vw_x\|_{H^s(\mathbb{R})}\,dt \le cT^{\frac{1}{2}}(T+1)\mathcal{N}(T,s,v)\mathcal{N}(T,s,w) \tag{7.12}$$

$$\int_0^T \|(g*uv)w_x\|_{H^s(\mathbb{R})}\,dt \le c\|g\|_{L^1(\mathbb{R})}\mathcal{N}(T,s,v)\mathcal{N}(T,s,w)$$

$$\times\big[T^{\frac{1}{2}}(T+1)\|u\|_{L^\infty_t(0\,T\,;\,H^s(\mathbb{R}))} + \int_0^T \|u_x\|_{L^\infty(\mathbb{R})}\,dt\big] \tag{7.13}$$

$$\int_0^T \|v(g*w_x)\|_{H^s(\mathbb{R})}\,dt \le cT^{\frac{1}{2}}(T+1)\mathcal{N}(T,s,v)\mathcal{N}(T,s,w). \tag{7.14}$$

Proof. Estimate (7.12) has already been proved in [21], [23]. Let us prove estimate (7.13). We have

$$\int_0^T \|(g*uv)w_x\|_{H^s(\mathbb{R})}\,dt \le \int_0^T \|J^s(g*uv)w_x - (g*uv)J^s w_x\|_{L^2(\mathbb{R})}\,dt$$
$$+\int_0^T \|(g*uv)(J^s-D^s)w_x\|_{L^2(\mathbb{R})}\,dt + \int_0^T \|(g*uv)D^s w_x\|_{L^2(\mathbb{R})}\,dt.$$

Using the commutator estimates of Lemma 6.1 and Corollary 6.2, we get the following estimate of the left-hand-side term

$$\begin{aligned} c\Big(&\int_0^T \|g\|_{L^1(\mathbb{R})}\|w\|_{H^s(\mathbb{R})}\|u\|_{H^s(\mathbb{R})}\|v\|_{H^s(\mathbb{R})}\,dt \\ &+\int_0^T \|g\|_{L^1(\mathbb{R})}\|w\|_{H^s(\mathbb{R})}\|u_x\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\,dt \\ &+\int_0^T \|g\|_{L^1(\mathbb{R})}\|w\|_{H^s(\mathbb{R})}\|u\|_{L^\infty(\mathbb{R})}\|v_x\|_{L^\infty(\mathbb{R})}\,dt \\ &+\int_0^T \|g\|_{L^1(\mathbb{R})}\|u\|_{H^s(\mathbb{R})}\|v\|_{H^s(\mathbb{R})}\|w_x\|_{L^\infty(\mathbb{R})}\,dt \\ &+T^{\frac{1}{2}}\big[\int_0^T \|(g*uv)D^s w_x\|^2_{L^2(\mathbb{R})}\,dt\big]^{\frac{1}{2}}\Big). \end{aligned} \tag{7.15}$$

The two first integrals in (7.15) are obviously bounded by the right-hand-side term of (7.13). Let us consider the third one. We have

$$\|(g*uv)D^s w_x\|_{L^2(\mathbb{R})} \le \|u\|_{L^\infty(\mathbb{R})}\|(|g|*|v|)D^s w_x\|_{L^2(\mathbb{R})}.$$

But

$$\int_0^T \big\|(|g|*|v|)D^s w_x\big\|_{L^2(\mathbb{R})}^2\,dt \le \big\||g|*|v|\big\|^2_{L^2_x(\mathbb{R}\,;\,L^\infty_t(0,T))}\big\|D^s w_x\big\|^2_{L^\infty_x(\mathbb{R}\,;\,L^2_t(0,T))} \tag{7.16}$$

$$\big\||g|*|v|\big\|_{L^2_x(\mathbb{R}\,;\,L^\infty_t(0,T))}\le \Big\||g|*\big\|v\big\|_{L^\infty_t(0,T)}\Big\|_{L^2(\mathbb{R})}\le \big\|g\big\|_{L^1(\mathbb{R})}\big\|v\big\|_{L^2_x(\mathbb{R}\,;\,L^\infty_t(0,T))}. \tag{7.17}$$

Hence (7.15), (7.16) and (7.17) lead to (7.13). We prove (7.14) in a similar way. Let us write

$$\int_0^T \big\|v(g*w_x)\big\|_{H^s(\mathbb{R})}\,dt \le \int_0^T \big\|J^s\big(v(g*w_x)\big)-vJ^s(g*w_x)\big\|_{L^2(\mathbb{R})}\,dt \tag{7.18}$$

$$+\int_0^T \big\|v\big\|_{L^\infty(\mathbb{R})}\big\|g*(J^s-D^s)w_x\big\|_{L^2(\mathbb{R})}dt + T^{\frac12}\Big[\int_0^T \big\|v(g*D^s w_x)\big\|^2_{L^2(\mathbb{R})}dt\Big]^{\frac12}.$$

We conclude as for the proof of (7.13).

Proof of Proposition 7.3. For simplicity of notation, we do not write in this proof the dependence of u^ε, g_7^ε, g_9^ε and F_ε on ε, but only write u, g_7, g_9 and F. Notice that the norms of g_7^ε and g_9^ε in $L^1(\mathbb{R})$ are smaller than the ones of g_7 and g_9 and that $\|u_0^\varepsilon\|_{H^s(\mathbb{R})}$ is smaller than $\|u_0\|_{H^s(\mathbb{R})}$. Let us write

$$u(t)=S(0,t)u_0+\int_0^t S(\tau,t)F\big(u(\tau)\big)\,d\tau. \tag{7.19}$$

Since $S(\tau,t)$ is unitary, we have

$$\big\|u(t)\big\|_{H^s(\mathbb{R})}\le \big\|u_0\big\|_{H^s(\mathbb{R})}+\int_0^T \big\|F\big(u(\tau)\big)\big\|_{H^s(\mathbb{R})}\,d\tau. \tag{7.20}$$

By using Corollary 3.2 in the equality

$$D^s u_x(t)=\frac{\partial}{\partial x}S(0,t)D^s u_0+\int_0^t \frac{\partial}{\partial x}S(\tau,t)D^s F\big(u(\tau)\big)\,d\tau$$

we get

$$\begin{aligned}\big\|D^s u_x\big\|_{L^2_t(0,T)} &\le c(T+1)^{\frac12}\Big[\big\|u_0\big\|_{H^s(\mathbb{R})}+\int_0^T \Big\|\frac{\partial}{\partial x}S(\tau,t)D^s F\big(u(\tau)\big)\Big\|_{L^2_t(\tau,T)}\,d\tau\Big]\\ &\le c(T+1)^{\frac12}\Big[\big\|u_0\big\|_{H^s(\mathbb{R})}+\int_0^T \big\|F\big(u(\tau)\big)\big\|_{H^s(\mathbb{R})}\,d\tau\Big].\end{aligned} \tag{7.21}$$

In the same way, Corollary 5.2 and the equality

$$J^r u_x(t)=D^{\frac14}S(0,t)D^{\frac34}\mathcal{H}J^r u_0+\int_0^t D^{\frac14}S(\tau,t)D^{\frac34}\mathcal{H}J^r F\big(u(\tau)\big)\,d\tau \tag{7.22}$$

allow us to get the estimate

$$\begin{aligned}&\left\|J^r u_x\right\|_{L_t^4(0,T\,;\,L_x^\infty(\mathbb{R}))}+\left\|D^{s+\frac{\theta}{4}}u\right\|_{L_t^q(0,T\,;\,L_x^p(\mathbb{R}))}\\&\qquad\le c(T+1)^{\frac14}\Big[\left\|u_0\right\|_{H^s(\mathbb{R})}+\int_0^T\left\|F\big(u(\tau)\big)\right\|_{H^s(\mathbb{R})}d\tau\Big].\end{aligned}\tag{7.23}$$

The last estimate that we need results from Corollary 4.2 and a similar equality to (7.22)

$$\left\|J^r u\right\|_{L_x^2(\mathbb{R}\,;\,L_t^\infty(0,T))}\le c_{s_0}(T+1)^{\frac12}\Big[\left\|u_0\right\|_{H^s(\mathbb{R})}+\int_0^T\left\|F\big(u(\tau)\big)\right\|_{H^s(\mathbb{R})}d\tau\Big].\tag{7.24}$$

Now we set for $0<T\le T_0$ and $s>s_0>3/4$

$$\begin{aligned}\mathcal{N}(T,s,s_0,u)=\sup_{\substack{0\le r\le s-\frac34\\0\le r'<s-s_0\\\theta\in[0,1]}}&\Big\{\mathcal{N}(T,s,u),\left\|J^r u_x\right\|_{L_t^4(0,T\,;\,L_x^\infty(\mathbb{R}))},\\&\left\|J^{r'}u\right\|_{L_x^2(\mathbb{R}\,;\,L_t^\infty(0,T))},\left\|D^{s+\frac{\theta}{4}}u\right\|_{L_t^q(0,T\,;\,L_x^p(\mathbb{R}))}\Big\},\end{aligned}$$

where $\mathcal{N}(T,s,u)$ is defined in Lemma 7.6. Thanks to estimates (7.20), (7.21), (7.23), and (7.24), we have

$$\begin{aligned}&\mathcal{N}(T,s,u)\le\mathcal{N}(T,s,s_0,u)\\&\qquad\le c_{s_0}(T+1)\Big[\left\|u_0\right\|_{H^s(\mathbb{R})}+\int_0^T\left\|F\big(u(\tau)\big)\right\|_{H^s(\mathbb{R})}d\tau\Big],\end{aligned}\tag{7.25}$$

where the dependence of c_{s_0} on s_0 disappears if we do not include in $\mathcal{N}(T,s,s_0,u)$ the quantities $\left\|J^{r'}u\right\|_{L_x^2(\mathbb{R}\,;\,L_t^\infty(0,T))}$ for $0<r'<s-s_0$.

Consider now the different terms occurring in $\int_0^T\left\|F\big(u(\tau)\big)\right\|_{H^s(\mathbb{R})}d\tau$. Terms of type (6.3) through (6.5) have already been considered in [21], [23] and lead to an estimate of the corresponding integrals by

$$cT^{\frac12}(T+1)\big[\mathcal{N}(T,s,u)+\big[\mathcal{N}(T,s,u)\big]^{p+1}\big].$$

On the other hand, thanks to Lemma 7.6, it follows that the integrals corresponding to the other terms (6.6) through (6.10) are bounded by

$$c\big(\sum\left\|g_i\right\|_{L^1(\mathbb{R})}\big)T^{\frac12}(T+1)\big[\mathcal{N}(T,s,u)+\big[\mathcal{N}(T,s,u)\big]^{p+1}\big].$$

Hence we get in (7.25)

$$\mathcal{N}(T,s,u)\le c_s(T+1)^2\Big\{\left\|u_0\right\|_{H^s(\mathbb{R})}$$

$$+ \big(1 + \sum \|g_i\|_{L^1(\mathbb{R})}\big) T^{\frac{1}{2}} \big[\mathcal{N}(T,s,u) + \big[\mathcal{N}(T,s,u)\big]^{p+1}\big]\Big\}. \quad (7.26)$$

Let $T \in (0, T_0]$; $t \mapsto \mathcal{N}(t,s,u)$ is an increasing continuous function on $[0,T]$ as long as $u(t)$ is defined. On the one hand, it follows that there exists a constant $c_s(T)$ not depending on ε or R such that for any $t \in (0,T]$ such that $u(t)$ exists,

$$\mathcal{N}(T,s,u) \leq c_s(T)\Big\{R + t^{\frac{1}{2}}\big[\mathcal{N}(T,s,u) + \big[\mathcal{N}(T,s,u)\big]^{p+1}\big]\Big\}. \quad (7.27)$$

On the other hand, either we have $\mathcal{N}(T,s,u) \leq 2c_s(T)R$ for any $t \in (0,T]$, or there exists $t_1 \in (0,T]$ such that $\mathcal{N}(t_1,s,u) = 2c_s(T)R$, and then (7.27) allows us to write $1 \leq 2t_1^{\frac{1}{2}} c_s(T)\big(1 + 2^p c_s(T)^p R^p\big)$. In the first case, we set $T_1 = T$ and in the second case $T_1 = \frac{1}{2\left(1+2^p c_s(T)^p R^p\right)}$, and so, as long as $u(t)$ exists for $t \in (0, T_1]$ we have

$$\mathcal{N}(T,s,u) \leq c_s(T)\big[R + 2T_1^{\frac{1}{2}} c_s(T) R\big(1 + 2^p c_s(T)^p R^p\big)\big],$$

and with (7.25) that finishes the proof of (i) and (ii). Statement (iii) follows readily from Lemma 7.5 and from the bound found in (i). To conclude the proof, let us establish (iv). Fix $s' > s$; thanks to Lemma 7.4, $\|u(t)\|_{H^s(\mathbb{R})}$ is uniformly bounded in $t \in [0, T_1]$ and $\varepsilon > 0$ as long as $u(t)$ is well-defined in $H^{s'}(\mathbb{R})$. Now suppose that (i) and (ii) of the proposition hold on a maximal interval $[0, T_{s'}]$ with $T_{s'} < T_1$. We can use (i) and (ii) on the right of any $T < T_{s'}$ on an interval whose length depends only on this uniform bound, and this contradicts the maximality of $T_{s'}$. □

Before proving Theorem 7.1, let us write terms involving in $F_\varepsilon(u) - F_{\varepsilon'}(u)$ in a useful way.

Lemma 7.7. *Let $\varepsilon, \varepsilon' > 0$ and $u, v \in \mathcal{B}(T,s)$, and set $w = u - v$. Then the terms occurring in $F_\varepsilon(u) - F_{\varepsilon'}(u)$ can be written as*

type (6.3) $(u^{p_1} - v^{p_1})\overline{u}^{p_2} + v^{p_1}(\overline{u}^{p_2} - \overline{v}^{p_2})$

$$= w\overline{u}^{p_2}\sum_{j=0}^{p_1-1} u^j v^{p_1-1-j} + \overline{w} v^{p_1} \sum_{j=0}^{p_2-1} \overline{u}^j \overline{v}^{p_2-1-j}$$

type (6.4) $(u^{p_3} - v^{p_3})\overline{u}^{p_4}\overline{u}_x + v^{p_3}(\overline{u}^{p_4} - \overline{v}^{p_4})\overline{u}_x + v^{p_3}\overline{v}^{p_4}\overline{w}_x$

type (6.5) $(|u|^{2p_5} - |v|^{2p_5})u_x + |v|^{2p_5} w_x$

type (6.6) $(u^{q_1}\overline{u}^{q_2}\overline{u}_x - v^{q_1}\overline{v}^{q_2}\overline{v}_x)(g_1 * u^{q_3}\overline{u}^{q_4})$

$$+ v^{q_1}\overline{v}^{q_2}\overline{v}_x\big[g_1 * (u^{q_3}\overline{u}^{q_4} - v^{q_3}\overline{v}^{q_4})\big]$$

type (6.7) $(|u|^{2q_5}u_x - |v|^{2q_5}v_x)(g_5 * |u|^{2q_6}) + |v|^{2q_5}v_x\big[g_5 * (|u|^{2q_6} - |v|^{2q_6})\big]$

type (6.8) $$(u|u|^{2q_7} - v|v|^{2q_7})\frac{\partial}{\partial x}(g_7^\varepsilon * |u|^{2q_8}) + v|v|^{2q_7}\frac{\partial}{\partial x}\big[g_7^\varepsilon * (|u|^{2q_8} - |v|^{2q_8})\big] + v|v|^{2q_7}\frac{\partial}{\partial x}\big[(g_7^\varepsilon - g_7^{\varepsilon'}) * |v|^{2q_8}\big]$$

type (6.9) $$(u^{q_9}\overline{u}^{q_{10}} - v^{q_9}\overline{v}^{q_{10}})\frac{\partial}{\partial x}(g_9^\varepsilon * u^{q_{11}}\overline{u}^{q_{12}}) + v^{q_9}\overline{v}^{q_{10}}\frac{\partial}{\partial x}\big[g_9^\varepsilon * (u^{q_{11}}\overline{u}^{q_{12}} - v^{q_{11}}\overline{v}^{q_{12}})\big] + v^{q_9}\overline{v}^{q_{10}}\frac{\partial}{\partial x}\big[(g_9^\varepsilon - g_9^{\varepsilon'}) * v^{q_{11}}\overline{v}^{q_{12}}\big]$$

type (6.10) $$(u^{q_{13}}\overline{u}^{q_{14}} - v^{q_{13}}\overline{v}^{q_{14}})(g_{13} * u^{q_{15}}\overline{u}^{q_{16}}) + v^{q_{13}}\overline{v}^{q_{14}}\big[g_{13} * (u^{q_{15}}\overline{u}^{q_{16}} - v^{q_{15}}\overline{v}^{q_{16}})\big].$$

Proof of Theorem 7.1. We first prove the existence of (i). Let $u_0, v_0 \in B_s(0,R)$; construct approximate solutions $u^\varepsilon, v^\varepsilon \in \mathcal{C}(0,T_1;H^\infty(\mathbb{R}))$ thanks to Notations 7.2 and Proposition 7.3. Set, for ε, $\varepsilon' > 0$ fixed, $w_0 = u_0^\varepsilon - v_0^{\varepsilon'}$ and $w = u^\varepsilon - v^{\varepsilon'}$. Then w satisfies

$$w(t) = S(0,t)w_0 + \int_0^t S(\tau,t)\big[F_\varepsilon\big(u^\varepsilon(\tau)\big) - F_{\varepsilon'}\big(v^{\varepsilon'}(\tau)\big)\big]\,d\tau.$$

As in proof of Proposition 7.3, smoothing effects of the linear part give

$$\begin{aligned}\mathcal{N}(T,s,w) &\le \mathcal{N}(T,s,s_0,w)\\ &\le c_{s_0}(T+1)\Big[\big\|w_0\big\|_{H^s(\mathbb{R})} + \int_0^T \big\|F_\varepsilon\big(u^\varepsilon(t)\big) - F_{\varepsilon'}\big(v^{\varepsilon'}(t)\big)\big\|_{H^s(\mathbb{R})}\,dt,\end{aligned}$$

where the dependence of c_{s_0} on s_0 disappears if we do not include in $\mathcal{N}(T,s,s_0,w)$ quantities depending on s_0. Thanks to the writing of Lemma 7.7 and estimates of Lemma 7.6, we get, since $\mathcal{N}(T,s,u^\varepsilon)$ and $\mathcal{N}(T,s,v^{\varepsilon'})$ are bounded by a constant M_1,

$$\begin{aligned}\mathcal{N}(T,s,w) &\le c(T+1)\big\|w_0\big\|_{H^s(\mathbb{R})}\\ &+ cT^{\frac12}(T+1)(M_1+1)^p\Big(\sum\big\|g_i\big\|_{L^1(\mathbb{R})} + 1\Big)\mathcal{N}(T,s,w)\\ &+ cT^{\frac12}(T+1)(M_1+1)^{p+1}\Big(\sum\big\|g_7^\varepsilon - g_7^{\varepsilon'}\big\|_{L^1(\mathbb{R})} + \sum\big\|g_9^\varepsilon - g_9^{\varepsilon'}\big\|_{L^1(\mathbb{R})}\Big).\end{aligned}$$

Now fix $T_2 \in (0,T_1]$ such that

$$cT_2^{\frac12}(T_2+1)(M_1+1)^p\Big(\sum\big\|g_i\big\|_{L^1(\mathbb{R})} + 1\Big) \le \frac12;$$

then for any $T \in (0,T_2]$, we have

$$\mathcal{N}(T,s,w) \le c(T_2+1)\big\|w_0\big\|_{H^s(\mathbb{R})} \tag{7.28}$$

$$+ cT_2^{\frac{1}{2}}(T_2+1)(M_1+1)^{p+1}\big(\sum \big\|g_7^{\varepsilon} - g_7^{\varepsilon'}\big\|_{L^1(\mathbb{R})} + \sum \big\|g_9^{\varepsilon} - g_9^{\varepsilon'}\big\|_{L^1(\mathbb{R})}\big),$$

and hence the existence follows. In a second step, let us prove the uniqueness. Suppose that u and v are two solutions satisfying (i) and set again $w = u - v$. By writing

$$w(t) = \int_0^t S(\tau,t)\big[F\big(u(\tau)\big) - F\big(v(\tau)\big)\big]\,d\tau$$

and choosing $M > 0$ such that $\mathcal{N}(T,s,u) \le M$ and $\mathcal{N}(T,s,v) \le M$, we get as above

$$\mathcal{N}(T,s,w) \le cT^{\frac{1}{2}}(T+1)(M+1)^p\big(\sum \big\|g_i\big\|_{L^1(\mathbb{R})} + 1\big)\mathcal{N}(T,s,w).$$

Hence there exists $T_3 > 0$ depending only on M such that $u(t) = v(t)$ for $t \in [0,T_3]$, and consequently $u = v$ on $[0,T_s]$ and the uniqueness follows. The continuity with respect to initial data of (ii) is a consequence of (7.28) and of the uniqueness. Statement (iii) is obtained in a way similar to the one used for Lemma 7.5.

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