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LOWER-BOUND GRADIENT ESTIMATES FOR FIRST-ORDER HAMILTON-JACOBI EQUATIONS AND APPLICATIONS TO THE REGULARITY OF PROPAGATING FRONTS

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Abstract. This paper is concerned with first-order time-dependent Hamilton-Jacobi equations. Exploiting some ideas of Barron and Jensen [9], we derive lower-bound estimates for the gradient of a locally Lipschitz-continuous viscosity solution u of equations with a convex Hamiltonian. Using these estimates in the context of the level-set approach to front propagation, we investigate the regularity properties of the propagating front of u, namely $\Gamma_t = \{x \in \mathbb{R}^n : u(x,t) = 0\}$ for $t \geq 0$. We show that, contrary to the smooth case, such estimates do not guarantee, in general, any expected regularity for Γ_t even if u is semiconcave.

1. Introduction

We consider the first-order time-dependent Hamilton-Jacobi Equation

$$\frac{\partial \omega}{\partial t} + H(x,t,D_x\,\omega) = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \tag{1.1}$$

with the initial condition

$$\omega(x,0) = u_0(x) \quad \text{in } \mathbb{R}^n \tag{1.2}$$

where $H \in C(\mathbb{R}^n \times [0,T] \times \mathbb{R}^n)$, $u_0 \in C(\mathbb{R}^n)$ and the solution ω is a real-valued function. Under suitable assumptions on the Hamiltonian H, if the initial condition u_0 is in $W^{1,\infty}(\mathbb{R}^n)$, it is well-known that there exists a unique viscosity solution $u \in W^{1,\infty}(\mathbb{R}^n \times [0,T])$ of (1.1)–(1.2). Many works are related to this problem: Crandall and Lions [14], Lions [22], Ishii [19], Barles [5, 7], Bardi and Capuzzo-Dolcetta [4], ... and references therein.

Here, we are given a Hamiltonian $H(x,t,p)$ which is assumed to be Lipschitz continuous with respect to x and p and which satisfies

(H1-β) $\left|\frac{\partial H}{\partial x}(x,t,p)\right| \leq C_1(\beta+|p|)$ for almost every $(x,t,p) \in \mathbb{R}^n\times[0,T]\times\mathbb{R}^n$

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(in the sequel, β will be chosen equal to 0 or 1) and

(H2) $\left|\frac{\partial H}{\partial p}(x,t,p)\right| \le A_2|x| + B_2$ for almost every $(x,t,p) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$.

Assumption **(H2)** implies a property of "finite speed of propagation": the value of the solution u at the point (x,t) does not depend on the initial condition u_0 in the whole space $\mathbb{R}^n$ but only in a ball $B(x, r(x,t))$ where $r(x,t) = \mathrm{e}^{(A_2+B_2+A_2|x|)t} - 1$, the base of the so-called *domain of dependence* $\mathcal{D}(x,r) \subset \mathbb{R}^n \times [0,T]$ (see Remark 6.1). This property allows us to localize many arguments and to compare unbounded solutions, regardless of their growth at infinity. This fact was first proved by Ishii in [19]. To be as self-contained as possible, we give in the Appendix a simplified proof of this comparison theorem (Theorem 6.1).

As an application, if the initial data u_0 is locally Lipschitz continuous in $\mathbb{R}^n$, we obtain upper-bound estimates for the gradient of a solution u of (1.1)–(1.2) which imply that u is locally Lipschitz continuous in $\mathbb{R}^n \times [0,T]$ (Theorem 4.1). In the same way, if u_0 is semiconcave in $\mathbb{R}^n$, then so is $u(\cdot,t)$ for $t \in [0,T]$ (Theorem 5.2).

On the other hand, when H satisfies **(H1-**β**)** and

(H3) $H(x,t,p)$ is convex in the p-variable,

Barron and Jensen showed in [9] that a continuous viscosity solution of (1.1) satisfies stronger properties than the ones needed to be a viscosity solution. In particular, the inf-convolution of a solution u of (1.1) is an approximate subsolution of (1.1) in addition to being, as is well-known, an approximate supersolution (see Lemma 3.2). Exploiting this fact with the help of the comparison theorem, we prove a lower-bound estimate for the gradient which is an unusual result in the theory of partial differential equations: suppose that, for some positive $\eta > 0$ and some ball $B(x_0, r) \subset \mathbb{R}^n$, we have

$$|Du_0| \ge \eta \quad \text{in } B(x_0,r) \quad \text{in the viscosity sense.} \tag{1.3}$$

Then the solution u of (1.1)–(1.2) satisfies the following similar lower bound in the viscosity sense, namely

$$|D_x u| \ge \tilde{\eta}\, \mathrm{e}^{-\gamma t/2} \quad \text{in the domain of dependence } \mathcal{D}(x_0,r). \tag{1.4}$$

If u_0 is smooth, from the implicit function theorem, (1.3) implies that $\Gamma_0 \cap B(x_0,r)$ is a smooth hypersurface where $\Gamma_0 = \{x \in \mathbb{R}^n \,:\, u_0(x) = 0\}$ is the 0-level set of u_0. In the same way, if the solution u is smooth, (1.4) implies that, for any $t \in [0,T]$, $\Gamma_t \cap \mathcal{D}(x_0,r)$ is a smooth hypersurface where

$$\Gamma_t = \{x \in \mathbb{R}^n \,:\, u(x,t) = 0\} \tag{1.5}$$

is the 0-level set of u. Now, the question is, what can we say about the regularity of Γ_t if u_0 and u are only Lipschitz continuous or if they are semiconcave?

This question is especially relevant in the theory of the level-set approach to weak propagation of fronts developed by Evans and Spruck [17] and Chen, Giga and Goto [11] (see also Barles, Soner and Souganidis [8]). It can be described in the following way. We consider equation (1.1) with a Hamiltonian which is homogeneous of degree 1 in the gradient variable; i.e.,

(H4) $H(x,t,\lambda p) = \lambda H(x,t,p)$ for all $x \in \mathbb{R}^n$, $t \in [0,T]$, $p \in \mathbb{R}^n$ and $\lambda \geq 0$.

(Note that it implies that the equation is invariant by changes $u \to \varphi(u)$, $\varphi' > 0$). Given an open set Ω_0 whose boundary is Γ_0, we first choose a function u_0 such that

$$\Gamma_0 = \{x \in \mathbb{R}^n \; : \; u_0(x) = 0\} \quad \text{and} \quad \Omega_0 = \{x \in \mathbb{R}^n \; : \; u_0(x) < 0\}. \qquad (1.6)$$

We define the front Γ_t at time t by (1.5) where u is the unique viscosity solution of (1.1)–(1.2). Then it is shown in [17] and [11] that the propagation of Γ_t depends only on the sets Γ_0 and Ω_0 but not on the choice of u_0 satisfying (1.6).

In this context, we consider a Hamiltonian which satisfies **(H1-β)**, **(H2)**, **(H3)** and **(H4)**. Then Γ_t can be seen as a front which propagates in the sense of the level-set approach described above. The most typical example is the well-known Eikonal equation

$$\frac{\partial \omega}{\partial t} + c\,|D_x\,\omega| = 0,$$

which is related to a front propagating with a constant normal velocity c (see Examples 4.1 and 5.3).

In order to investigate the regularity properties of the front Γ_t, we have to assume some regularity properties of the initial hypersurface Γ_0. We suppose that

(H5) there exists a positive constant η such that

$$|u_0| + |Du_0| \geq 2\eta > 0 \quad \text{in } \mathbb{R}^n \;\; \textit{in the viscosity sense}.$$

If u_0 is smooth, **(H5)** is nothing but saying that the graph $\{z = u_0(x)\}$ intersects the hyperplane $\{z = 0\}$ transversely. If u_0 is only Lipschitz continuous, this condition means that, for every $x \in \mathbb{R}^n$, either $|u_0(x)| \geq \eta$ or $|p| \geq \eta$ where $p \in D^- u_0(x)$. Let us mention that if $D^- u_0(x) = \emptyset$, then **(H5)** is fulfilled at x even if $u_0(x) = 0$. We refer to Examples 5.3, 5.4 and Theorem 5.5 for some illustrations.

With these assumptions on H and on the initial data u_0, we obtain the lower-bound estimate

$$|u| + \frac{e^{\gamma t}}{4}|D_x u|^2 \geq C(\eta) > 0 \quad \text{in } \mathbb{R}^n \times [0,T] \text{ in the viscosity sense} \tag{1.7}$$

for some positive constants $C(\eta)$ and γ (Theorem 4.2).

Now, we want to investigate the regularity of a front associated to a semiconcave function. Semiconcave functions arise naturally in the theory of viscosity solutions (see Lions [22], Ishii [19], Cannarsa *et al.* [10, 3, 1] and Theorem 5.2). In general, they are the most regular solutions one can hope for time-dependent Hamilton-Jacobi equations. Roughly speaking, they are "nearly C^1." If the semiconcave function u satisfies (1.7), which can be seen as an expected generalization of a transversality condition for nonsmooth functions, a natural question arises: does estimate (1.7) imply regularity for Γ_t?

We unfortunately answer in a negative way. On the one hand, surprisingly, even if u is semiconcave, the front can be nasty (see the example developed in Theorem 5.5). On the other hand, if we start with a smooth set Γ_0, "non-Lipschitz" singularities can occur (see Example 5.3).

Nevertheless, we obtain some partial positive results. As an immediate corollary of Theorem 4.2, we recover a known result of Barles *et al.* [8] about the so-called *nonempty interior difficulty*: the front has zero Lebesgue measure in this case.

For Lipschitz-continuous solutions, the sharpest regularity we can hope would be a front which is locally Lipschitz (i.e., locally the graph of a Lipschitz-continuous function; see Definition 5.2). In fact, the front for semiconcave solutions is "regular from one side," which means that the front has locally smooth support hypersurfaces which lie in $\bar{\Omega}_0 = \{x \in \mathbb{R}^n : u_0(x) \leq 0\}$ (see Theorem 5.3).

A sufficient condition for Γ_t to be locally Lipschitz at x is

$$0 \notin \partial_C u(x,t) \tag{1.8}$$

where $\partial_C u(\cdot,t)$ denotes the Clarke subdifferential. In this case, we can apply a nonsmooth inversion Theorem of Clarke [12] (see Theorem 5.4). But condition (1.8) does not characterize such points of regularity (cf. Example 5.4) and we cannot hope for regularity everywhere.

The $(n-1)$-Hausdorff measure $\mathcal{H}^{n-1}$ can be viewed as a measure on the level sets. The co-area formula implies that for almost every level set of a semiconcave function, the $\mathcal{H}^{n-1}$-measure of the set of the singular points is 0 (Corollary 5.2). But this gives no result to particular level sets like the front. Theorem 5.5 gives an example of a semiconcave function whose front contains a positive $(n-1)$-Hausdorff measure set of singular points, which shows that we cannot have more regularity.

The paper is organized as follows. In Section 3, we give a simplified proof of a theorem of [9], using some ideas of Barles [6]. In Section 4, we prove the upper and lower-bound estimates for the gradient of a solution u of (1.1)–(1.2). Finally, Section 5 is devoted to the study of the regularity of the front Γ_t.

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2. The notion of viscosity solution

Throughout the paper, Ω will denote an open subset of $\mathbb{R}^n$ and $|\cdot|$ is the standard Euclidean norm.

We recall for convenience the definition of viscosity solutions for equations like (1.1) (see Crandall, Ishii and Lions [15], Barles [7], Bardi and Capuzzo-Dolcetta [4], Fleming and Soner [18], ...).

Definition 2.1. A function $u \in C(\Omega \times (0,T))$ is a viscosity solution of (1.1) if and only if u is a viscosity subsolution; i.e., for all $\varphi \in C^1(\Omega \times (0,T))$, at each local maximum point $(\bar{x},\bar{t}) \in \Omega \times (0,T)$ of $u-\varphi$, we have

$$\frac{\partial \varphi}{\partial t}(\bar{x},\bar{t}) + H(\bar{x},\bar{t},D_x\varphi(\bar{x},\bar{t})) \leq 0,$$

and u is a viscosity supersolution; i.e., for all $\varphi \in C^1(\Omega \times (0,T))$, at each local minimum point $(\bar{x},\bar{t}) \in \Omega \times (0,T)$ of $u-\varphi$, we have

$$\frac{\partial \varphi}{\partial t}(\bar{x},\bar{t}) + H(\bar{x},\bar{t},D_x\varphi(\bar{x},\bar{t})) \geq 0.$$

If one considers initial conditions, $u \in C(\Omega \times [0,T])$ is a viscosity solution of (1.1)–(1.2) if and only if u is a viscosity subsolution of (1.1) and $u(x,0) \leq u_0(x)$ for all $x \in \Omega$, and u is a supersolution of (1.1) and $u(x,0) \geq u_0(x)$ for all $x \in \Omega$. Therefore, in the case of continuous viscosity solutions, the initial conditions are taken in a classical sense: $u(x,0) = u_0(x)$ for all $x \in \Omega$.

Viscosity solutions can also be characterized in terms of sub- and superdifferential, the definitions of which are recalled.

Definition 2.2. Let $u \in C(\Omega \times (0,T))$. The superdifferential (respectively the subdifferential) of u at the point $(x,t) \in \Omega \times (0,T)$ is the closed convex set

$$D^+_{x,t}u(x,t) = \Big\{(p,q) \in \mathbb{R}^n \times (0,T) : \limsup_{\substack{y \to x, y \in \Omega \\ \tau \to t, \tau \in (0,T)}} \frac{u(y,\tau) - u(x,t) - \langle p, y-x \rangle - q \cdot (\tau - t)}{|y-x| + |\tau - t|} \leq 0 \Big\},$$

respectively,

$$D^-_{x,t}u(x,t) = \Big\{(p,q) \in \mathbb{R}^n \times (0,T) : \liminf_{\substack{y \to x, y \in \Omega \\ \tau \to t, \tau \in (0,T)}} \frac{u(y,\tau) - u(x,t) - \langle p, y-x \rangle - q \cdot (\tau - t)}{|y-x| + |\tau - t|} \geq 0 \Big\}.$$

3. Barron-Jensen solutions to first-order Hamilton-Jacobi equations

We recall the concept of Barron-Jensen viscosity solutions which are defined for equations with a convex Hamiltonian.

Definition 3.1. Let u be lower-semicontinuous in $\Omega \times (0,T)$. We say that u is a Barron-Jensen solution of (1.1) if it satisfies the condition that for all $\varphi \in C^1(\Omega \times (0,T))$, at each local minimum point $(\bar{x}, \bar{t}) \in \Omega \times (0,T)$ of $u - \varphi$, we have

$$\frac{\partial \varphi}{\partial t}(\bar{x}, \bar{t}) + H(\bar{x}, \bar{t}, D_x\varphi(\bar{x}, \bar{t})) = 0.$$

This concept of solution was introduced in 1990 by Barron and Jensen [9]. It was also studied for example in Barles [6, 7] and Bardi and Capuzzo-Dolcetta [4] (in the last reference they call such solutions *bilateral supersolutions*). The link with the classical notion of viscosity solution is given by the following theorem:

Theorem 3.1. *Assume that* **(H1-β)** *and* **(H3)** *hold. Let* $u \in C(\Omega \times (0,T))$. *Then u is a viscosity solution of* (1.1) *if and only if u is a Barron-Jensen solution of* (1.1).

This theorem was first stated by Barron and Jensen in [9]. It says that in the continuous case, if the Hamiltonian is convex in p and Lipschitz continuous in x, then viscosity solutions and Barron-Jensen solutions coincide. We present here a simplified proof due to Barles [6] which we adapt in the case of an evolution equation. We point out that contrary to the original proof, we are not required to have a Hamiltonian which is Lipschitz continuous in the t-variable.

We start by giving two fundamental lemmas, proving them and using them to prove Theorem 3.1. We introduce some notation: for $(x_0, t_0) \in \Omega \times (0,T)$, we define $\mathcal{A} = B(x_0, \sigma) \times (t_0 - \sigma, t_0 + \sigma)$ and $\mathcal{A}_\rho = B(x_0, \sigma - \rho) \times (t_0 - \sigma + \rho, t_0 + \sigma - \rho)$ where $\sigma > 0$ is taken small enough such that $\bar{\mathcal{A}} \subset \Omega \times (0,T)$ and $0 < \rho < \sigma$. If u is continuous we define $M_{\mathcal{A}} = (2 \sup_{(y,s)\in\bar{\mathcal{A}}} |u(y,s)|)^{1/2}$.

Lemma 3.1. *Assume* **(H3)**. *If $u \in C(\Omega \times (0,T))$ is a locally Lipschitz-continuous subsolution of* (1.1) *in $\Omega\times(0,T)$, then u is a viscosity supersolution of*

$$-\frac{\partial \omega}{\partial t} - H(x,t,D_x\,\omega) = 0 \quad \text{ in } \Omega \times (0,T).$$

Proof of Lemma 3.1. Let $(x_0, t_0) \in \Omega \times (0,T)$ and define $\mathcal{A}, \mathcal{A}_\rho$ as above, taking $\sigma > 0$ small enough such that u is Lipschitz continuous in $\bar{\mathcal{A}}$ with Lipschitz constant ℓ. By Rademacher's theorem, u is differentiable almost everywhere in $\mathcal{A}$, and then, by Corollary 2.1 of Barles [7], we have

$$\frac{\partial u}{\partial t}(x,t) + H(x,t,D_x\,u(x,t)) \leq 0 \quad \text{almost everywhere in } \mathcal{A}. \tag{3.1}$$

Let $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}), \phi \geq 0$, such that $\operatorname{supp}\phi \subset B(0,1) \times (0,1)$ and $\int_{\mathbb{R}^n\times\mathbb{R}} \phi(y,s)\,dy\,ds = 1$. We set $\phi_\tau(x,t) = \tau^{-n-1}\phi\left(x/\tau, t/\tau\right)$. Let $\tau < \rho/2$. Then the classical convolution

$$u_\tau(x,t) = (u * \phi_\tau)(x,t) = \int_{\mathbb{R}^n\times\mathbb{R}} u(y,s)\,\phi_\tau(x-y,t-s)\,dy\,ds$$

is well-defined for every (x,t) in $\mathcal{A}_\rho$ since $\phi_\tau(x-y,t-s) = 0$ for (y,s) outside $\mathcal{A}$. From (3.1), we get

$$\frac{\partial u_\tau}{\partial t}(x,t) + \int_{\mathbb{R}^n\times\mathbb{R}} H(y,s,D_xu(y,s))\phi_\tau(x-y,t-s)\,dy\,ds \leq 0.$$

To deal with the integral, we define a modulus of continuity $\mu_{\mathcal{A}}$ of H in the compact set $\bar{\mathcal{A}} \times \bar{B}(0,\ell)$. Recall that $\phi_\tau(x-y,t-s) = 0$ for (y,s) outside $\mathcal{A}$ and that $\int_{\mathbb{R}^n\times\mathbb{R}} \phi(y,s)\,dy\,ds = 1$. We have

$$\frac{\partial u_\tau}{\partial t}(x,t) + \int_{\mathbb{R}^n\times\mathbb{R}} H(x,t,D_xu(y,s))\phi_\tau(x-y,t-s)\,dy\,ds \leq \mu_{\mathcal{A}}(\tau). \tag{3.2}$$

Since H is convex in p, from Jensen's inequality, we get

$$H\Big(x,t,\underbrace{\int_{\mathbb{R}^n\times\mathbb{R}} D_x\,u(y,s)\phi_\tau(x-y,t-s)\,dy\,ds}_{D_xu_\tau(x,t)}\Big) \tag{3.3}$$

$$\leq \int_{\mathbb{R}^n\times\mathbb{R}} H(x,t,D_x\, u(y,s))\phi_\tau(x-y,t-s)\,dy\,ds.$$

Combining (3.2) and (3.3), we have

$$\frac{\partial u_\tau}{\partial t}(x,t) + H\left(x,t,D_x u_\tau(x,t)\right) \leq \mu_{\mathcal{A}}(\tau). \tag{3.4}$$

Now, since $u_\tau \in C^\infty(\mathcal{A}_\rho)$, (3.4) means that u_τ is a classical subsolution of the equation $\partial\omega/\partial t + H(x,t,D_x\,\omega) - \mu_{\mathcal{A}}(\tau) = 0$ in $\mathcal{A}_\rho$; thus it is a classical supersolution of the equation $-\partial\omega/\partial t - H(x,t,D_x\,\omega) = -\mu_{\mathcal{A}}(\tau)$ in $\mathcal{A}_\rho$. Applying the discontinuous stability result (see Theorem 4.1 of Barles [7]), we obtain that $(\liminf_{\tau\to 0}{}_*\, u_\tau)(x,t) = u(x,t)$ is a viscosity supersolution of

$$-\frac{\partial\omega}{\partial t} - H(x,t,D_x\,\omega) \geq 0 \quad \text{ in } \mathcal{A}_\rho. \tag{3.5}$$

Since $\cup_{\Omega\times(0,T)}\mathcal{A}_\rho = \Omega\times]0,T[$, we conclude that u is a viscosity supersolution of (3.5) in $\Omega\times(0,T)$ as desired. □

Lemma 3.2. *Assume* **(H1-β)** *and* **(H3)** *and take $\gamma \geq 5C_1/2$. Let $u \in C(\Omega\times[0,T])$ be a Barron-Jensen solution of (1.1) and $(x_0,t_0)\in\Omega\times(0,T)$. If $\varepsilon \leq \rho/2M_{\mathcal{A}}$ and $\alpha \leq \rho/2M_{\mathcal{A}}$, then the inf-convolution*

$$u_{\varepsilon,\alpha}(x,t) = \inf_{(y,s)\in\bar{\mathcal{A}}} \left\{ u(y,s) + \mathrm{e}^{-\gamma t}\frac{|x-y|^2}{\varepsilon^2} + \frac{|t-s|^2}{\alpha^2} \right\}$$

is a viscosity subsolution of

$$\frac{\partial\omega}{\partial t} + H(x,t,D_x\,\omega) - \mu_{\varepsilon,\mathcal{A}}(M_{\mathcal{A}}\alpha) - \frac{\beta C_1 \mathrm{e}^{\gamma T}}{2}\varepsilon^2 = 0 \qquad in\ \mathcal{A}_\rho$$

where $\mu_{\varepsilon,\mathcal{A}}(\cdot)$ is a modulus of continuity for H in the compact set $\bar{\mathcal{A}}\times\bar{B}(0,2M_{\mathcal{A}}/\varepsilon)$.

Proof of Lemma 3.2. Let $\varphi\in C^1(\Omega\times(0,T))$, $(\bar{x},\bar{t})\in\mathcal{A}_\rho$ be a local minimum of $u_{\varepsilon,\alpha}-\varphi$ and $(\bar{y},\bar{s})\in\bar{\mathcal{A}}$ be a point where the infimum in $u_{\varepsilon,\alpha}(\bar{x},\bar{t})$ is achieved. Define

$$f(x,t,y,s) = u(y,s) + \mathrm{e}^{-\gamma t}\frac{|x-y|^2}{\varepsilon^2} + \frac{|t-s|^2}{\alpha^2} - \varphi(x,t).$$

We see easily that $(\bar{x},\bar{t},\bar{y},\bar{s})$ is a minimum for f in $\mathcal{A}_\rho\times\bar{\mathcal{A}}$. From classical estimates for inf-convolutions, we have that $|\bar{y}-\bar{x}|\leq M_{\mathcal{A}}\varepsilon$ and $|\bar{s}-\bar{t}|\leq M_{\mathcal{A}}\alpha$. Hence, if $\varepsilon\leq\rho/2M_{\mathcal{A}}$ and $\alpha\leq\rho/2M_{\mathcal{A}}$, then it forces $(\bar{y},\bar{s})$ to lie in the open set $\mathcal{A}$; thus $(\bar{x},\bar{t},\bar{y},\bar{s})$ is a local minimum for f.

On the one side, $(\bar{y},\bar{s})$ is a local minimum of $f(\bar{x},\bar{t},y,s)$; hence taking $-\mathrm{e}^{-\gamma\bar{t}}|\bar{x}-y|^2/\varepsilon^2-|\bar{t}-s|^2/\alpha^2+\varphi(\bar{x},\bar{t})$ as a test function, since u is a Barron-Jensen of (1.1), we get

$$2\,\frac{\bar{t}-\bar{s}}{\alpha^2}+H\Big(\bar{y},\bar{s},2\,\mathrm{e}^{-\gamma\bar{t}}\frac{\bar{x}-\bar{y}}{\varepsilon^2}\Big)=0. \tag{3.6}$$

On the other side, $(\bar{x},\bar{t})$ is a local minimum of the C^1-function $f(x,t,\bar{y},\bar{s})$; hence

$$\frac{\partial\varphi}{\partial t}(\bar{x},\bar{t})=-\gamma\mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{y}-\bar{x}|^2}{\varepsilon^2}+2\,\frac{\bar{t}-\bar{s}}{\alpha^2}\quad\text{and}\quad D_x\varphi(\bar{x},\bar{t})=2\,\mathrm{e}^{-\gamma\bar{t}}\frac{\bar{x}-\bar{y}}{\varepsilon^2}. \tag{3.7}$$

From (3.6) and (3.7), we have

$$\frac{\partial\varphi}{\partial t}(\bar{x},\bar{t})+\gamma\mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{y}-\bar{x}|^2}{\varepsilon^2}+H\left(\bar{y},\bar{s},D_x\varphi(\bar{x},\bar{t})\right)=0. \tag{3.8}$$

We want to replace $(\bar{y},\bar{s})$ by $(\bar{x},\bar{t})$ in H in the equality (3.8). Since

$$|D_x\varphi(\bar{x},\bar{t})|=2\,\mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{y}-\bar{x}|}{\varepsilon^2}\leq\frac{2M_{\mathcal{A}}}{\varepsilon},$$

we define a modulus of continuity $\mu_{\varepsilon,\mathcal{A}}(\cdot)$ for H in the compact set $\bar{\mathcal{A}}\times\bar{B}\left(0,2M_{\mathcal{A}}/\varepsilon\right)$. We have

$$H\left(\bar{y},\bar{s},D_x\varphi(\bar{x},\bar{t})\right)\geq H\left(\bar{y},\bar{t},D_x\varphi(\bar{x},\bar{t})\right)-\mu_{\varepsilon,\mathcal{A}}(M_{\mathcal{A}}\alpha). \tag{3.9}$$

Note that we do not replace $\bar{y}$ by $\bar{x}$ using the uniform continuity of H not to get a term $\mu_{\varepsilon,\mathcal{A}}(M_{\mathcal{A}}\varepsilon)$ we cannot control; we need **(H1-β)**, which implies

$$H\left(\bar{y},\bar{t},D_x\varphi(\bar{x},\bar{t})\right)\geq H\left(\bar{x},\bar{t},D_x\varphi(\bar{x},\bar{t})\right)-C_1\left(\beta+|D_x\varphi(\bar{x},\bar{t})|\right)|\bar{x}-\bar{y}| \tag{3.10}$$

and

$$\begin{aligned}\beta C_1|\bar{x}-\bar{y}|&+C_1|D_x\varphi(\bar{x},\bar{t})||\bar{x}-\bar{y}|\\&\leq\frac{\beta C_1\mathrm{e}^{\gamma\bar{t}}\varepsilon^2}{2}+\beta C_1\mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{y}-\bar{x}|^2}{2\varepsilon^2}+2C_1\mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{y}-\bar{x}|^2}{\varepsilon^2}.\end{aligned}$$

Using (3.9) and (3.10) in (3.8), we get

$$\begin{aligned}\frac{\partial\varphi}{\partial t}(\bar{x},\bar{t})+\big(\gamma-\frac{\beta C_1}{2}-2C_1\big)\mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{y}-\bar{x}|^2}{\varepsilon^2}+H\left(\bar{x},\bar{t},D_x\varphi(\bar{x},\bar{t})\right)&\\ \leq\mu_{\varepsilon,\mathcal{A}}(M_{\mathcal{A}}\alpha)+\frac{\beta C_1\mathrm{e}^{\gamma T}\varepsilon^2}{2}.&\end{aligned} \tag{3.11}$$

If $\gamma\geq 5C_1/2$, the "bad term" $\big(\gamma-\frac{\beta C_1}{2}-2C_1\big)\mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{y}-\bar{x}|^2}{\varepsilon^2}$ is positive then we can get rid of it in (3.11). Since the inf-convolution $u_{\varepsilon,\alpha}$ is Lipschitz continuous in $\mathcal{A}_\rho$, the inequality (3.11) holds almost everywhere in $\mathcal{A}_\rho$. Then we proceed as in the proof of Lemma 3.1, regularizing $u_{\varepsilon,\alpha}$ by classical convolution, using the uniform continuity of H, Jensen's inequality and the discontinuous stability result. We obtain that $u_{\varepsilon,\alpha}$ is a subsolution of the desired equation, which completes the proof. □

Remark 3.1. The fact that we can write an equality in (3.6) is essential for the proof; if we use only that u is a viscosity supersolution, we will obtain that $u_{\varepsilon,\alpha}$ is a supersolution (instead of a subsolution as desired) of a perturbed Hamilton-Jacobi equation, which is a classical result (for classical results about sup-convolution, we refer to Lasry and Lions [21], Barles [7] and Bardi and Capuzzo-Dolcetta [4]). Here, in a sense, the concept of Barron-Jensen solutions allows us to "reverse the inequality."

Remark 3.2. In fact, we can conclude directly from (3.11) by using a result of Lions [22] since H is convex.

We now turn to the proof of the theorem.

Proof of Theorem 3.1. We start by showing that a continuous viscosity solution of (1.1) is a Barron-Jensen solution of the same equation. Let $(x_0, t_0) \in \Omega \times (0,T)$ and $\mathcal{A}, \mathcal{A}_\rho$ be defined as above. The sup-convolution

$$u^{\varepsilon,\alpha}(x,t) = \sup_{(y,s)\in\bar{\mathcal{A}}} \left\{ u(y,s) - \mathrm{e}^{Kt}\frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\alpha^2} \right\}$$

is a locally Lipschitz-continuous solution of $\partial\omega/\partial t + H(x,t,D_x\omega) - \beta C_1\varepsilon^2/2 - \mu_{\varepsilon,\mathcal{A}}(M_{\mathcal{A}}\alpha) = 0$ in $\mathcal{A}_\rho$ for $K \geq 5C_1/2$, $\varepsilon \leq \rho/2M_{\mathcal{A}}$ and $\alpha \leq \rho/2M_{\mathcal{A}}$. This result is classical and does not require H to be convex. Applying Lemma 3.1, $u^{\varepsilon,\alpha}$ is a viscosity supersolution of

$$-\frac{\partial\omega}{\partial t} - H(x,t,D_x\,\omega) + \frac{\beta C_1\varepsilon^2}{2} + \mu_{\varepsilon,\mathcal{A}}(M_{\mathcal{A}}\alpha) = 0 \quad \text{in } \mathcal{A}_\rho.$$

Letting α go to 0 first, then letting ε go to 0, we obtain, from the discontinuous stability result, that u is a supersolution of $-\partial\omega/\partial t - H(x,t,D_x\,\omega) = 0$ in $\mathcal{A}_\rho$ thus in $\Omega \times (0,T)$ since $\cup_{\Omega\times(0,T)}\mathcal{A}_\rho = \Omega \times (0,T)$. Adding the fact that u is a viscosity solution–thus a supersolution of (1.1), we can conclude that u is a Barron-Jensen solution of (1.1) in $\Omega \times (0,T)$.

Conversely, if u is a Barron-Jensen solution of (1.1) in $\Omega \times (0,T)$, we have to show that u is a viscosity solution. It is clear that u is a supersolution; it suffices to prove that u is a subsolution of (1.1). Let $(x_0,t_0) \in \Omega \times (0,T)$. From Lemma 3.2, the inf-convolution $u_{\varepsilon,\alpha}$ is a subsolution of $\partial\omega/\partial t + H(x,t,D_x\omega) - \mu_{\varepsilon,\mathcal{A}}(M_{\mathcal{A}}\alpha) - \beta C_1\mathrm{e}^{\gamma T}\varepsilon^2/2 = 0$ in $\mathcal{A}_\rho$. Letting $\alpha \to 0$ first, then letting $\varepsilon \to 0$, and thanks to the discontinuous stability theorem once more, we get that u is a viscosity subsolution of (1.1) in every $\mathcal{A}_\rho$ thus in $\Omega \times (0,T)$, which ends the proof. □

Remark 3.3. Theorem 3.1 is true for a slightly more general equation

$$\frac{\partial \omega}{\partial t} + H(x,t,\omega,D_x\omega) = 0 \quad \text{in } \Omega \times (0,T)$$

if we assume, as in Barron and Jensen [9], that for all $x \in \mathbb{R}^n$, $t \in [0,T]$, $u, v \in \mathbb{R}$ and $p \in \mathbb{R}^n$, $|H(x,t,u,p) - H(x,t,v,p)| \le \mu_H(|u-v|)$ where $\mu_H(\alpha) \to 0$ when $\alpha \to 0$.

4. A lower-bound estimate for the gradient

We start by stating a result which provides a locally Lipschitz-continuous solution.

Theorem 4.1. (*Upper bound for the gradient*) *Assume* **(H1-β)**, **(H2)** *and that u_0 is locally Lipschitz continuous in $\mathbb{R}^n$. Let $u \in C(\mathbb{R}^n \times [0,T])$ be a viscosity solution of* (1.1)–(1.2). *Then u is locally Lipschitz continuous in $\mathbb{R}^n \times [0,T]$. Moreover, if u_0 is Lipschitz continuous in $\mathbb{R}^n$, then u is Lipschitz continuous in the x-variable in $\mathbb{R}^n$, uniformly with respect to $t \in [0,T]$, and*

$$\|D_x u(\cdot,t)\|_\infty \le \ell \quad \text{where} \quad \ell = 2\mathrm{e}^{5C_1T/4}\Big(\|Du_0\|_\infty^2 + \frac{\beta C_1 T}{2}\Big)^{1/2}.$$

Upper bounds for the gradient are classical estimates for the solutions of PDEs. Now, if the Hamiltonian is convex, we obtain a surprising result: some lower-bound estimates also hold.

Theorem 4.2. (*Lower bound for the gradient*) *We suppose that the assumptions of Theorem 4.1 and* **(H3)** *hold.*

(*i*) *Let $x_0 \in \mathbb{R}^n$ and $r > 0$. If $|Du_0| \ge \eta$ in $B(x_0,r)$ in the viscosity sense for some positive η, then there exist some positive constants $\tilde{\eta}, \gamma$ and $0 < t_0 \le T$ such that*

$$|D_x u| \ge \mathrm{e}^{-\gamma t/2}\tilde{\eta} \quad \text{in} \quad \mathcal{D}(x_0,r) \cap (\mathbb{R}^n \times (0,t_0)) \quad \text{in the viscosity sense,}$$

where $\mathcal{D}(x_0,r) = \{(x,t) \in B(x_0,r) \times (0,T) : e^{(A_2+B_2+A_2|x_0|)t}(1+|x-x_0|) \le r+1\}$.

(*ii*) *If $\beta = 0$ in* **(H1-β)**, **(H5)** *and* **(H4)** *hold, then there exist positive constants C and γ such that*

$$|u| + \frac{\mathrm{e}^{\gamma t}}{4}|D_x u|^2 \ge C > 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T) \quad \text{in the viscosity sense.}$$

Before giving the proofs of the theorems, we give some examples of Hamiltonians which satisfy the assumptions of Theorems 4.1 and 4.2.

Example 4.1. The most typical example in our context is $H(x,t,p) = a(x,t)|p|$ with $a(x,t) \geq 0$ since it leads to equations which are related to a front propagating with a normal velocity $a(x,t)$. If $a \in C(\mathbb{R}^n \times [0,T])$ is Lipschitz continuous in x (uniformly with respect to t), then H satisfies **(H1-0)** and **(H2)**. Moreover, H satisfies **(H3)** and **(H4)**. An example of a front propagating with constant normal velocity is developed in Example 5.3.

Example 4.2. From optimal control problems, we get some Hamiltonians of the form $H(x,t,p) = \sup_{v\in V}\{\langle b(x,t,v),p\rangle + f(x,t,v)\}$ where V is a compact set. H is clearly convex in p. If b, f are continuous and Lipschitz continuous in x (uniformly with respect to (t,v)) and b is bounded (which is usually supposed in classical control theory), then H satisfies **(H1-0)** and **(H2)**. If $f \equiv 0$, then H satisfies **(H4)**.

Proof of Theorem 4.1. Let $x_0 \in \mathbb{R}^n$ and $r > 0$. We define

$$M_r = \sup_{\bar{B}(x_0,r)\times[0,T]} (2|u|)^{1/2} \quad \text{and}$$

$$||Du_0||_r = \sup\left\{\frac{|u_0(x) - u_0(y)|}{|x-y|} : x, y \in \bar{B}(x_0,r), x \neq y\right\}.$$

Notice that $||Du_0||_r < +\infty$ since u_0 is locally Lipschitz continuous. We start by showing that u is locally Lipschitz continuous in the x variable, uniformly with respect to $t \in [0,T]$. From classical results about sup-convolutions, for $0 < \rho < r, \varepsilon \leq \rho/2M_r$ and $K \geq 5C_1/2$, the sup-convolution

$$u^\varepsilon(x,t) = \sup_{y\in\bar{B}(x_0,2R)} \left\{u(y,t) - \mathrm{e}^{Kt}\frac{|x-y|^2}{\varepsilon^2}\right\}$$

is a viscosity subsolution of $\partial\omega/\partial t + H(x,t,D_x\omega) - \beta C_1\varepsilon^2/2 = 0$ in $B(x_0, r-\rho) \times (0,T)$. From now on, we fix $L = A_2 + B_2 + A_2|x_0|$ and we take $r - \rho \geq 2\mathrm{e}^{LT}$. Then, since u is a supersolution of (1.1), the comparison theorem 6.1 yields

$$u^\varepsilon(x,t) - u(x,t) \leq \sup_{y\in\bar{B}(x_0,r)} \{u^\varepsilon(y,0) - u(y,0)\} + \int_0^t \frac{\beta C_1}{2}\varepsilon^2\, ds \qquad (4.1)$$

for all $(x,t) \in \bar{B}(x_0, \mathrm{e}^{-LT}(r-\rho)/2) \times [0,T]$. We estimate the supremum $\sup_{y\in\bar{B}(x_0,r)}\{u^\varepsilon(y,0) - u(y,0)\}$. Let $y \in \bar{B}(x_0,r)$ and suppose that $\bar{y} \in \bar{B}(x_0,r)$ is a point where the supremum in $u^\varepsilon(y,0)$ is achieved. Then

$$0 \leq u^\varepsilon(y,0) - u(y,0) = u_0(\bar{y}) - \frac{|y-\bar{y}|^2}{\varepsilon^2} - u_0(y); \qquad (4.2)$$

hence, $\frac{|y-\bar{y}|^2}{\varepsilon^2} \le u_0(\bar{y}) - u_0(y) \le \|Du_0\|_r |y-\bar{y}|$. This implies

$$|y-\bar{y}| \le \|Du_0\|_r \varepsilon^2. \tag{4.3}$$

Using (4.2) and (4.3) we obtain the desired estimate: $u^\varepsilon(y,0) - u(y,0) \le \|Du_0\|_r^2 \varepsilon^2$ for all $y \in \bar{B}(x_0, r)$. From (4.1), it follows that

$$u^\varepsilon(x,t) - u(x,t) \le \lambda\varepsilon^2 \quad \text{where} \quad \lambda = \|Du_0\|_r^2 + \frac{\beta C_1 T}{2}.$$

Hence, for every $x, y \in \bar{B}(x_0, \mathrm{e}^{-LT}(r-\rho)/2)$ and $t \in [0,T]$, we have

$$u(y,t) - u(x,t) \le \lambda\varepsilon^2 + \mathrm{e}^{KT}\frac{|y-x|^2}{\varepsilon^2}.$$

The C^1-function $f(\varepsilon) = \lambda\varepsilon^2 + \mathrm{e}^{KT}|y-x|^2/\varepsilon^2$ achieves its minimum at $\bar{\varepsilon} = \mathrm{e}^{KT/4}\lambda^{-1/4}|y-x|^{1/2}$. Taking $|y-x|$ small enough such that $\bar{\varepsilon} < \rho/2M_r$, we obtain that

$$u(y,t) - u(x,t) \le f(\bar{\varepsilon}) = 2\mathrm{e}^{KT/2}\lambda^{1/2}|y-x|,$$

which implies that u is locally Lipschitz continuous in x in $\bar{B}(x_0, \mathrm{e}^{-LT}(r-\rho)/2)$, uniformly with respect to $t \in [0,T]$, with Lipschitz constant

$$\ell_r = 2\mathrm{e}^{KT/2}\Big(\|Du_0\|_r^2 + \frac{\beta C_1 T}{2}\Big)^{1/2}.$$

This implies that u is Lipschitz continuous in x in $\bar{B}(x_0, \mathrm{e}^{-LT}(r-\rho)/2)$ with the same Lipschitz constant as desired. Moreover, if u_0 is Lipschitz continuous in $\mathbb{R}^n$, then u is clearly Lipschitz continuous in x with Lipschitz constant $\ell = 2\mathrm{e}^{KT/2}\Big(\|Du_0\|_\infty^2 + \frac{\beta C_1 T}{2}\Big)^{1/2}$.

It remains to prove that u is Lipschitz continuous in the t variable in $[0,T]$, uniformly with respect to $x \in \bar{B}(x_0, r)$. Define

$$S_r = \sup_{\substack{(y,s) \in \bar{B}(x_0,r) \times [0,T] \\ p \in \bar{B}(0,\ell_r)}} |H(y,s,p)|.$$

Let $x \in \bar{B}(x_0, r-\rho)$ where $0 < \rho < r$ and $t \in [0,T)$. The function

$$\Psi(y,s) = u(y,s) - C(s-t) - \frac{|y-x|^2}{\varepsilon^2}$$

achieves its maximum in $\bar{B}(x_0,r) \times [t,T]$ at a point $(\bar{y},\bar{s})$. If $\varepsilon \le \rho/2M_r$, then $\bar{y} \in B(x_0,r)$. If $C > 2M_r/(T-t)$, then the maximum cannot be achieved at $\bar{s} = T$; otherwise, we have $\Psi(x,t) \le \Psi(\bar{y},T)$, and it follows that

$$-2M_r \le u(x,t) - u(\bar{y},T) \le -C(T-t),$$

which leads to a contradiction. If $t < \bar{s} < T$, since u is a viscosity subsolution of (1.1), we have

$$C + H\Big(\bar{y}, \bar{s}, 2\frac{\bar{y}-x}{\varepsilon^2}\Big) \leq 0.$$

But $2|\bar{y}-x|/\varepsilon^2 \leq \ell_r$ since u is locally Lipschitz continuous in x, and if $C > S_r$, this inequality cannot hold. Finally, if $C = C_{r,t} = \max\{2M_r/(T-t), S_r\} + 1$, then the maximum has to be achieved at $\bar{s} = t$. Letting ε go to 0, we obtain

$$u(x,s) - u(x,t) \leq C_{r,t}(s-t) \tag{4.4}$$

for every $s \geq t$. In the same way, using the function $u(y,s) + C_{r,t}(s-t) + |y-x|^2/\varepsilon^2$ and the fact that u is a supersolution of (1.1), we obtain

$$u(x,s) - u(x,t) \geq -C_{r,t}(s-t) \tag{4.5}$$

for every $s \geq t$. From (4.4) and (4.5), it follows that u is locally Lipschitz continuous in t in $[0,T)$. To conclude, let us show that the Lipschitz constant is independent of t. Since u is locally Lipschitz continuous in $\bar{B}(x_0,r)\times[0,T)$, from Rademacher's theorem, u is differentiable almost everywhere; thus,

$$\frac{\partial u}{\partial t}(x,t) + H(x,t,D_x\, u(x,t)) = 0 \quad \text{almost everywhere in } \ B(x_0,r)\times[0,T).$$

Hence,

$$\Big|\frac{\partial u}{\partial t}(x,t)\Big| \leq S_R \quad \text{almost everywhere in } \ B(x_0,R)\times[0,T),$$

which implies that u is Lipschitz continuous in the t variable in $[0,T]$, uniformly with respect to $x \in \bar{B}(x_0,r)$, with Lipschitz constant S_r, which completes the proof of the theorem. □

Proof of Theorem 4.2. We start by (i); we aim at showing that u is a viscosity supersolution of $|D_x u| - \tilde{\eta}\mathrm{e}^{-\gamma t/2} = 0$ in $\mathcal{D}(x_0,r)\cap(\mathbb{R}^n\times(0,t_0))$ where $\gamma, \tilde{\eta}$ and t_0 have to be specified. We use the notation introduced in the proof of Theorem 4.1.

From Lemma 3.2, if $\varepsilon \leq \rho/2M_r$ and $\gamma = 5C_1/2$, then the inf-convolution

$$u_\varepsilon(x,t) = \inf_{y\in\bar{B}(x_0,2R)} \Big\{u(y,t) + \mathrm{e}^{-\gamma t}\frac{|x-y|^2}{\varepsilon^2}\Big\}$$

is a viscosity subsolution of

$$\frac{\partial \omega}{\partial t} + H(x,t,D_x\omega) - \frac{\beta C_1 \mathrm{e}^{\gamma T}}{2}\varepsilon^2 = 0 \quad \text{ in } B(x_0, r-\rho)\times(0,T).$$

Since u is a supersolution of (1.1), from Theorem 6.1, we get

$$u_\varepsilon(x,t) - u(x,t) \leq \sup_{y\in\bar{B}(x_0,r)} \{u_\varepsilon(x,0) - u(x,0)\} + \int_0^t \frac{\beta C_1 \mathrm{e}^{\gamma T}}{2}\varepsilon^2\, ds \quad (4.6)$$

for all $(x,t) \in \{(x,t) \in \bar{B}(x_0, r-\rho) \times [0,T] : \mathrm{e}^{Lt}(1+|x-x_0|) \leq r+1\} = \bar{\mathcal{D}}(x_0,r) \cap (\bar{B}(x_0,r-\rho)\times[0,T])$. The estimate of the supremum in (4.6) is a fundamental step in the proof of this theorem and is given in the following proposition:

Proposition 4.1. *If* $|Du_0| \geq \eta$ *in* $B(x_0,r)$ *in the viscosity sense, then*

$$\sup_{y\in\bar{B}(x_0,r)} \{u_\varepsilon(x,0) - u(x,0)\} \leq -\frac{\eta^2}{4}\varepsilon^2.$$

The proof of the proposition is postponed for clarity and uses as a main tool the following lemma:

Lemma 4.1. (*Viscosity decrease principle*) *Let* Ω *be an open subset of* $\mathbb{R}^n$ *and* v *be a lower-semicontinuous function satisfying* $|Dv| \geq \eta > 0$ *in* Ω *in the viscosity sense. Then, for all* $x_0 \in \mathbb{R}^n$ *and for all* $r > 0$ *such that* $\bar{B}(x_0,r) \subset \Omega$, *we have*

$$\inf_{y\in\bar{B}(x_0,r)} v(y) \leq v(x_0) - \eta r.$$

The proof of this lemma and some comments about it are postponed.

We now return to the proof of the theorem. Using the above proposition, we get

$$u_\varepsilon(x,t) - u(x,t) \leq -\frac{\eta^2}{4}\varepsilon^2 + \frac{\beta C_1 \mathrm{e}^{\gamma T} t}{2}\varepsilon^2 \quad (4.7)$$

for all $(x,t) \in \bar{\mathcal{D}}(x_0,r) \cap (\bar{B}(x_0,r-\rho)\times[0,T])$. Let $(\bar{x},\bar{t}) \in \mathcal{D}(x_0,r) \cap (B(x_0,r-\rho)\times[0,T])$ and $(p,q) \in D^-_{x,t}u(\bar{x},\bar{t})$. By definition of the subdifferential, $u(y,\bar{t}) \geq u(\bar{x},\bar{t}) + \langle p, y-\bar{x}\rangle + |y-\bar{x}|\mu(y-\bar{x})$ for all $y \in \mathbb{R}^n$ where $\mu(\cdot)$ is a function with limit 0 at 0. Writing this inequality at a point $\bar{y}$ where the infimum is achieved in $u_\varepsilon(\bar{x},\bar{t})$, we obtain

$$\begin{aligned} u_\varepsilon(\bar{x},\bar{t}) &= u(\bar{y},\bar{t}) + \mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} \\ &\geq u(\bar{x},\bar{t}) + \langle p, \bar{y}-\bar{x}\rangle + \mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + |\bar{y}-\bar{x}|\mu(\bar{y}-\bar{x}). \end{aligned} \quad (4.8)$$

On the one side, the C^1-function $g(y) = \langle p, y-\bar{x}\rangle + \mathrm{e}^{-\gamma\bar{t}}|\bar{x}-y|^2/\varepsilon^2$ achieves its minimum at the point $\hat{y} = \bar{x} - p\mathrm{e}^{\gamma\bar{t}}\varepsilon^2/2$ thus,

$$g(\bar{y}) \geq g(\hat{y}) = -\mathrm{e}^{\gamma\bar{t}}\frac{|p|^2}{4}\varepsilon^2. \tag{4.9}$$

On the other side, we want to obtain an estimate for the term $|y-\bar{x}|\mu(y-\bar{x})$ in (4.8). We have

$$u_\varepsilon(\bar{x},\bar{t}) = u(\bar{y},\bar{t}) + \mathrm{e}^{-\gamma\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} \leq u(\bar{x},\bar{t}),$$

which implies

$$|\bar{x}-\bar{y}|^2 \leq \mathrm{e}^{\gamma T}\varepsilon^2\left(u(\bar{x},\bar{t}) - u(\bar{y},\bar{t})\right).$$

But, from Theorem 4.1, u is locally Lipschitz continuous, uniformly with respect to $t \in [0,T]$; thus,

$$u(\bar{x},\bar{t}) - u(\bar{y},\bar{t}) \leq \ell_r|\bar{x}-\bar{y}|$$

where ℓ_r is the Lipschitz constant of $u(\cdot,t)$ in $\bar{B}(x_0,r)$. Defining $M = \mathrm{e}^{\gamma T}\ell_r$ (note that M does not depend on $\bar{y}$), it follows that

$$|\bar{y}-\bar{x}|\mu(\bar{y}-\bar{x}) \leq M\varepsilon^2\mu(M\varepsilon^2). \tag{4.10}$$

Now, using the information (4.9) and (4.10) in (4.8), we obtain

$$u_\varepsilon(\bar{x},\bar{t}) \geq u(\bar{x},\bar{t}) - \frac{\mathrm{e}^{\gamma\bar{t}}}{4}|p|^2\varepsilon^2 - M\varepsilon^2\mu(M\varepsilon^2). \tag{4.11}$$

Finally, combining (4.11) with the inequality (4.7) and dividing by ε^2, we get

$$\mathrm{e}^{\gamma\bar{t}}|p|^2 \geq \eta^2 - 2\beta C_1\mathrm{e}^{\gamma T}\bar{t} - 4M\mu(M\varepsilon^2).$$

Letting ε go to 0 and fixing $t_0 \in (0,T]$ small enough such that $\eta^2 - 2\beta C_1\mathrm{e}^{\gamma T}t_0 = \tilde{\eta}^2 > 0$ (note that we can take $\tilde{\eta} = \eta$ and $t_0 = T$ if $\beta = 0$), we obtain

$$|p| \geq \mathrm{e}^{-\gamma\bar{t}/2}\tilde{\eta} \qquad \text{for every } (p,q) \in D^-_{x,t}(\bar{x},\bar{t}).$$

Since $\tilde{\eta}$ does not depend on ρ, we can let ρ go to 0, which shows that u is a viscosity supersolution of $|D_x u| - \mathrm{e}^{-\gamma t/2}\tilde{\eta} = 0$ in $\mathcal{D}(x_0,r)\cap(B(x_0,r)\times(0,t_0))$.

Now, we turn to the proof of (ii). We have to show that u is a viscosity supersolution of

$$|\omega| + \frac{\mathrm{e}^{\gamma t}}{4}|D_x\omega|^2 - C = 0 \ \text{ in } \mathbb{R}^n \times (0,T),$$

where γ and C have to be specified. Let $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and $r > 0$. As in the proof of (i), for any $0 < \rho < r$, if $\gamma = 5C_1/2$ and $\varepsilon \leq \rho/2M_r$, then the inf-convolution u_ε is a viscosity subsolution of $\partial\omega/\partial t + H(x,t,D_x\omega) = 0$ in $B(x_0, r-\rho) \times (0,T)$ (here $\beta = 0$). We need the following lemma.

Lemma 4.2. *Assume* **(H4)**. *Let* $0 < \varepsilon < 1$ *and* $\Psi_\varepsilon(u)(x,t) = u(x,t) + \varepsilon^2|u(x,t)|$. *If* u *is a viscosity solution of* (1.1), *then* $\Psi_\varepsilon(u)$ *is still a viscosity solution of* (1.1).

This lemma is classical in the theory of viscosity solutions (see Barles [7] for example). The proof is based on **(H4)** since, under this assumption, the equation (1.1) is invariant by increasing changes of variable.

From now on, we take $r-\rho \geq 2\mathrm{e}^{LT}$ where $L = A_2+B_2+A_2|x_0|$. Applying the lemma and the comparison theorem 6.1 for the subsolution u_ε and the supersolution $\Psi_\varepsilon(u)$, we get

$$u_\varepsilon(x,t) - \Psi_\varepsilon(u)(x,t) \leq \sup_{y\in\bar{B}(x_0,r)} \left\{u_\varepsilon(x,0) - u(x,0) - \varepsilon^2|u(x,0)|\right\}.$$

The estimate of the supremum is given by

Proposition 4.2. *Assume* **(H5)**. *Then*

$$\sup_{y\in\bar{B}(x_0,r)} \{u_\varepsilon(x,0) - u(x,0) - \varepsilon^2|u(x,0)|\} \leq -\bar{C}\varepsilon^2 \quad \text{where} \quad \bar{C} = \min\big(\frac{\eta}{2}, \frac{\eta^2}{4}\big).$$

The proof of this proposition is a straightforward adaptation of the proof of Proposition 4.1 so we skip it. Applying this proposition, the end of the proof can be obtained using the same arguments as in the proof of (i). □

We turn to the proof of Proposition 4.1 and Lemma 4.1.

Proof of Proposition 4.1. Let $\bar{x} \in \bar{B}(x_0,r)$ and $0 < \tilde{r} \leq r$. Using the viscosity decrease principle (Lemma 4.1) in $B(x_0,\tilde{r})$, we have

$$u_\varepsilon(\bar{x},0) = \inf_{y\in\bar{B}(x_0,r)}\{u_0(y)+\frac{|y-\bar{x}|^2}{\varepsilon^2}\} \leq \inf_{y\in\bar{B}(\bar{x},\tilde{r})} u_0(y)+\frac{\tilde{r}^2}{\varepsilon^2} \leq u_0(\bar{x})-\tilde{r}\eta+\frac{\tilde{r}^2}{\varepsilon^2}.$$

Taking $\tilde{r} = \varepsilon^2\eta/2$, it follows that $u_\varepsilon(\bar{x},0) - u(\bar{x},0) \leq -\eta^2\varepsilon^2/4$, which ends the proof. □

Proof of Lemma 4.1. We argue by contradiction. Suppose that

$$\exists \tilde{r} > 0, \quad \inf_{y\in\bar{B}(x_0,\tilde{r})} v(y) > v(x_0) - \eta\tilde{r}. \tag{4.12}$$

Since the inequality in (4.12) is strict, we can find $\tilde{\eta}, \hat{\eta} > 0$ such that $\tilde{\eta} < \hat{\eta} < \eta$ and (4.12) holds with $\tilde{\eta}$. For $\theta > 0$, define $f(y) = v(y) + \hat{\eta}|y-x_0|^{1+\theta}/\tilde{r}^\theta$. If $S(x_0,\tilde{r})$ is the sphere $\{x \in \mathbb{R}^n : |x-x_0| = \tilde{r}\}$, then from (4.12) with $\tilde{\eta}$, we have

$$\inf_{y\in S(x_0,\tilde{r})} f(y) \geq v(x_0) - \tilde{\eta}\tilde{r} + \hat{\eta}\tilde{r} > v(x_0). \tag{4.13}$$

Hence, the minimum of the lower-semicontinuous function f in the compact set $\bar{B}(x_0,\tilde{r})$ is achieved at a point $\bar{y}$ which lies in the open ball $B(x_0,\tilde{r})$ since

(4.13) shows that it cannot be achieved on the boundary $S(x_0, \tilde{r})$. Thus $\bar{y}$ is a local minimum of f. Since v is a viscosity supersolution of $|D\omega| - \eta = 0$ in Ω, it follows that

$$\left| D_y\Big(-\frac{\hat{\eta}}{r^\theta}|y-x_0|^{1+\theta}\Big)\right| \geq \eta.$$

But

$$\eta \leq \left| D_y\Big(-\frac{\hat{\eta}}{r^\theta}|y-x_0|^{1+\theta}\Big)\right| = \frac{\hat{\eta}(1+\theta)}{r^\theta}|\bar{y}-x_0|^\theta \leq \hat{\eta}(1+\theta),$$

which leads to a contradiction for θ small enough since $\hat{\eta} < \eta$. This ends the proof. □

Remark 4.1. The *viscosity decrease principle* is an adaptation of a result of Clarke [12] in the context of lower-semicontinuous viscosity solutions (see Clarke *et al.* [13] for a similar result with proximal gradients). Roughly speaking, it means that, if the gradient of a function is large enough, then the function has to decrease enough if we move away enough from x_0 in the right direction.

5. Regularity properties of the level sets for semiconcave solutions

5.1. **Semiconcave solutions.** For properties of semiconcave functions, we refer to Cannarsa *et al.* [10, 2, 3].

Definition 5.1. A function $v \in C(\mathbb{R}^n)$ is said to be semiconcave if for every convex, compact subset $\mathcal{C}$ of $\mathbb{R}^n$, there exists $\theta \in (0,1]$ and $\Lambda > 0$ such that $v(y+h)-2v(y)+v(y-h) \leq \Lambda|h|^{1+\theta}$ for every y, h such that $y, y-h, y+h \in \mathcal{C}$.

We say that v is uniformly semiconcave if θ and Λ can be chosen independent of $\mathcal{C}$.

Note that the definition implies that a semiconcave function can be seen locally as the sum of a concave and a smooth function.

We give below some results which provide that a solution u of (1.1)–(1.2) is semiconcave under an additional assumption on H. For a convex, compact subset $\mathcal{C} \subset \mathbb{R}^n$ and $l > 0$, we state

(H6-$\mathcal{C}$-l) There exist some constants $\ell' = \ell'(l) > l$, $\theta = \theta(l) \in (0,1]$ and $a = a(\mathcal{C}, l)$, $b = b(\mathcal{C}, l) > 0$ such that $H(x+h,t,p+k) - 2H(x,t,p) + H(x-h,t,p-k) \geq -a|h|^{1+\theta} - b|h||k|$ for all x, h such that $x, x+h, x-h \in \mathcal{C}$, $t \in [0,T]$ and p, k in the ball $\bar{B}(0,\ell')$.

Remark 5.1. If H is smooth and convex in p, then we see easily that **(H6-$\mathcal{C}$-l)** is fulfilled for every convex, compact set $\mathcal{C} \subset \mathbb{R}^n$ and every $l > 0$. This example shows that **(H6-$\mathcal{C}$-l)** is not too restrictive.

Theorem 5.1. *Let u be a solution of* (1.1)–(1.2) *which is Lipschitz continuous in the x variable with a Lipschitz constant ℓ. Assume* **(H2)** *and* **(H6-$\mathcal{C}$-ℓ)** *for every convex, compact subset $\mathcal{C}$. If u_0 is uniformly semiconcave in $\mathbb{R}^n$, then $u(\cdot,t)$ is semiconcave in $\mathbb{R}^n$, uniformly with respect to $t \in [0,T]$.*

Theorem 5.2. *Assume* **(H1-β)**, **(H2)** *and* **(H6-$\mathcal{C}$-l)** *for every convex, compact subset $\mathcal{C}$ and every $l > 0$. If u_0 is semiconcave in $\mathbb{R}^n$, then $u(\cdot,t)$ is semiconcave in $\mathbb{R}^n$, uniformly with respect to $t \in [0,T]$.*

Theorem 5.1 is an adaptation of a theorem of Ishii [19] in the case of an evolution equation. The proof is similar, so we skip it. Theorem 5.2 is a straightforward application of previous results. At first, notice that u_0, being semiconcave, is locally Lipschitz continuous in $\mathbb{R}^n$. Thus the solution u of (1.1)–(1.2) is locally Lipschitz continuous from Theorem 4.1. Moreover, Theorem 6.1 allows us to localize the arguments of the proof of Theorem 5.1. This implies the result.

Let us give some examples of Hamiltonians which satisfy **(H6-$\mathcal{C}$-l)**:

Example 5.1. Coming back to Example 4.1, if a is locally Lipschitz continuous and uniformly semiconvex (i.e., $-a$ is uniformly semiconcave), then the Hamiltonian $H(x,t,p) = a(x,t)|p|$ with $a(x,t) \geq 0$ satisfies **(H6-$\mathcal{C}$-l)** for every convex compact subset $\mathcal{C}$ and for every $l > 0$.

Example 5.2. If b is $C^{1,\theta}$ (in (x,t) uniformly with respect to v) for some $\theta \in (0,1]$ and $-f$ is uniformly semiconcave in (x,t), then the Hamiltonian H of Example 4.2 satisfies **(H6-$\mathcal{C}$-l)** for every convex, compact subset $\mathcal{C}$ and for every $l > 0$.

5.2. **Regularity of the level sets.** We start by introducing some notation and giving some definitions. In this section, $\mathcal{L}^p$ denotes the Lebesgue measure in $\mathbb{R}^p$ and $\mathcal{H}^p$ is the Hausdorff measure (for a definition, see Morgan [23] or Evans and Gariepy [16]). Let $u \in C(\mathbb{R}^n \times [0,T])$. We denote by $\Gamma_t^\alpha = \{x \in \mathbb{R}^n : u(x,t) = \alpha\}$ the α-level set of u at time t. To simplify the notation, we write Γ_t for the 0-level set and we call it a *front of u at time t*; finally, if u does not depend on t, we denote by Γ^α the α-level set and Γ the front. If $(e_1, e_2, \ldots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$, we denote by $(y_1, y_2, \ldots, y_n) = (\hat{y}, y_n)$ the coordinates of a vector y in this basis.

Definition 5.2. We say that Γ_t^α is locally Lipschitz at $x \in \Gamma_t^\alpha$ if Γ_t^α is the graph of a Lipschitz-continuous function near x; i.e., there exists a neighborhood $V(x)$ of x in $\mathbb{R}^n$, an orthonormal basis $(e_1, e_2, \ldots, e_n)$ and a Lipschitz-continuous function $\zeta : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\Gamma_t^\alpha \cap V(x) = \{(\hat{y}, \zeta(\hat{y})) : \hat{y} \in V(\hat{x})\}$ where $V(\hat{x}) = V(x) \cap \mathrm{Span}(e_1, e_2, \ldots, e_{n-1})$ is a neighborhood of $\hat{x}$.

From now on, we consider a solution u of (1.1)–(1.2) under the assumptions of Theorem 4.2 (ii). Then for any $t \in [0,T]$,

$$|D_x u(\cdot,t)| \geq c > 0 \quad \text{in } \Omega \ \text{ in the viscosity sense,} \tag{5.1}$$

where $c = \min\{\eta, (2C)^{1/2}e^{-\gamma T/2}\}$ and $\Omega = \{x \in \mathbb{R}^n : |u(x,t)| < \min\{\eta, C/2\}\}$ is a neighborhood of Γ_t. Notice that Ω depends on t, but for the sake of simplicity of notation, we will omit writing, here and below, the subscript t as much as possible. We aim at proving some properties of regularity of the front provided by (5.1).

Corollary 5.1. *Under the assumptions of Theorem* 4.2, $\mathcal{L}^n(\Gamma_t) = 0$ *for any* $t \geq 0$.

Proof of Corollary 5.1. We use a classical theorem which is given in Kavian [20], page 85 (another reference is Evans and Gariepy [16], page 84): for any function $v \in W^{1,p}_{loc}(\Omega)$, $1 \leq p \leq \infty$, we have $\mathbf{1}_{\{v=0\}} \cdot Dv = 0$ almost everywhere in Ω where $\mathbf{1}_A$ is the characteristic function of the set A. We argue by contradiction, supposing that $\mathcal{L}^n(\Gamma_t) > 0$. Since u is locally Lipschitz continuous, $u \in W^{1,\infty}_{loc}$ and $\mathbf{1}_{\Gamma_t} \cdot D_x u = 0$ almost everywhere in Ω. There exists a point $x_0 \in \Gamma_t$ such that u is differentiable at x_0 and $D_x u(x_0,t) = 0$. But (5.1) has to be satisfied in the classical sense at such a point; hence, $|D_x u(x_0,t)| > 0$, which leads to a contradiction. □

Note that the above result is an immediate corollary of Theorem 4.2 and is true even if u is not semiconcave. Now, suppose in addition that u_0 is semiconcave and that **(H6-$\mathcal{C}$-l)** holds for every convex, compact subset $\mathcal{C}$ and for every $l > 0$. Then from Theorem 5.2, $u(\cdot,t)$ is semiconcave and Γ_t satisfies the following property for any $t \in [0,T]$:

Theorem 5.3. *Let* $x = (\hat{x}, x_n) \in \Gamma_t$. *There exists an orthonormal basis* $(e_1, e_2, \ldots, e_n)$, *a neighborhood* $V(x) = V(\hat{x}) \times V(x_n)$ *of* x *and a* C^1*-function* $\zeta : V(\hat{x}) \to \mathbb{R}$ *such that* $D\zeta(\hat{x}) = 0$ *and* $\{(\hat{y}, z) \in \Omega : z \geq \zeta(\hat{y})\} \subset \{y \in \Omega : u(y,t) \leq 0\}$.

Remark 5.2. This theorem means that the front has locally C^1-support hypersurfaces "to one side." This comes from the fact that a semiconcave function admits at each point C^1 upper-support functions.

Proof of Theorem 5.3. We start by a general fact: let x be in Ω; there exists p in $D^+_x u(x,t)$ such that $|p| \geq c > 0$. Let us prove this assertion: the function u is differentiable almost everywhere in Ω; thus, from (5.1) we get a sequence $x_k \to x$ of points of Ω such that u is differentiable at x_k and $|D_x u(x_k,t)| \geq c$. We conclude, letting $k \to \infty$, by using the upper-semicontinuity of $D^+_x u(\cdot,t)$ (see Cannarsa and Soner [10]).

Now, we turn to the proof of the theorem. Let $x \in \Gamma_t$. There exists $p \in D_x^+ u(x,t)$ such that $|p| \geq c$. As u is a viscosity subsolution of (1.1), there exists $\varphi \in C^1(\Omega \times (0,T))$ such that $u - \varphi$ has a local maximum at (x,t) and $D_x\varphi(x,t) = p$ (see Barles [7], page 18). Moreover, we can assume that $u(x,t) - \varphi(x,t) = 0$, which implies $\varphi(x,t) = 0$ since $x \in \Gamma_t$. We fix an orthonormal basis $(e_1, e_2, \ldots, e_n)$ with $e_n = -p/|p|$. From the implicit function theorem, there exists a neighborhood $V(\hat{x})$ of $\hat{x}$ and a C^1-function $\zeta : V(\hat{x}) \to \mathbb{R}$ such that $\varphi(\hat{y}, \zeta(\hat{y}), t) = 0$ for all $\hat{y} \in V(\hat{x})$. It follows that $u(\hat{y}, \zeta(\hat{y}), t) \leq 0$ for all $\hat{y} \in V(\hat{x})$ and $D\zeta(\hat{x}) = 0$. Moreover,

$$
\begin{aligned}
u(\hat{y}, y_n, t) \leq \varphi(\hat{y}, y_n, t) &= \varphi(x,t) + \langle D_x\varphi(x,t), y - x\rangle + o(y-x) \\
&= -|p| \cdot (y_n - x_n) + o(y - x) < 0
\end{aligned}
$$

for y sufficiently close to x and $y_n - x_n > 0$. The proof ends by noting that $\{(\hat{y}, z) \in \Omega : z \geq \zeta(\hat{y})\} \subset \{y \in \Omega : u(y,t) \leq 0\}$. □

Remark 5.3. The regularity property of Theorem 5.3 does not hold for the front if u is not semiconcave. Let us give a counterexample: let $w(x,y) = |x| - y$. The function w is Lipschitz continuous and is convex but not semiconcave. The estimate (5.1) is clearly satisfied. We compute the front $\Gamma = \{(x,y) \in \mathbb{R}^2 : y = |x|\}$ and $\{(x,y) \in \mathbb{R}^2 : w(x,y) \leq 0\} = \{(x,y) \in \mathbb{R}^2 : y \geq |x|\}$ and, from this, it is clear that the regularity property of Theorem 5.3 is not fulfilled at $(0,0)$.

We need some definitions. The *Clarke subdifferential* or *Clarke generalized gradient* (see Clarke [12]) is well adapted for Lipschitz continuous functions:

Definition 5.3. Let u be a Lipschitz-continuous function in $\mathbb{R}^n$ and denote by $\Sigma(u)$ the subset of $\mathbb{R}^n$ of Lebesgue measure zero where u is not differentiable. The Clarke subdifferential of u at the point x in $\mathbb{R}^n$ is the nonempty, compact, convex set

$$
\begin{aligned}
\partial_C u(x) &= \Big\{ \xi \in \mathbb{R}^n : \limsup_{y \to x,\, t \downarrow 0} \frac{u(y+tp) - u(y)}{t} \geq \langle \xi, p\rangle \ \text{ for all } p \in \mathbb{R}^n \Big\} \\
&= \operatorname{co}\Big\{ \lim_{k \to +\infty} Du(x_k) \ : \ x_k \to x,\, x_k \notin \Sigma(u) \Big\}.
\end{aligned}
$$

Definition 5.4. We set $\mathcal{S} = \{x \in \Omega : \Gamma_t^{u(x,t)}$ is not locally Lipschitz at $x\}$ and $\mathcal{S}_0 = \{x \in \Omega : 0 \in \partial_C u(x,t)\}$.

Theorem 5.4. *Let $u \in C(\mathbb{R}^n \times [0,T])$ be locally Lipschitz continuous. If $x \notin \mathcal{S}_0$, then $\Gamma_t^{u(x,t)}$ is locally Lipschitz at x.*

Remark 5.4. This theorem means that for every $x \in \mathbb{R}^n$ such that the Clarke subdifferential of the function does not contain 0, the $u(x,t)$-level set is locally Lipschitz at x. The converse is not true. The function defined in Example 5.4 satisfies $0 \in \partial_C u(x)$ for every x in the 0-level set of u though the front is locally Lipschitz.

Proof of Theorem 5.4. The proof consists in showing that we can use the nonsmooth inversion theorem of Clarke [12]. Let $x \in \Omega$ such that $0 \notin \partial_C u(x,t)$ and let $\alpha = u(x,t)$. The set $\partial_C u(x)$ is convex and compact and does not contain 0; thus, by the separation theorem of convex sets we have

$$\exists q \in \mathbb{R}^n,\ |q| = 1 \text{ such that } \forall p \in \partial_C u(x),\ \langle q, p\rangle \geq \nu > 0. \tag{5.2}$$

From now on, we fix an orthonormal basis $(e_1, e_2, \dots, e_n)$ of $\mathbb{R}^n$ such that $e_n = q$. Let $F_t : \mathbb{R}^n \to \mathbb{R}^n$ be the function $F_t(\hat{y}, y_n) = (\hat{y}, u(\hat{y}, y_n, t))$. Since F_t is locally Lipschitz continuous, we have

$$DF_t(\hat{y}, y_n) = \left(\begin{array}{c|c} & \frac{\partial u}{\partial y_1}(\hat{y}, y_n, t) \\ Id_{n-1} & \vdots \\ & \frac{\partial u}{\partial y_{n-1}}(\hat{y}, y_n, t) \\ \hline 0 & \frac{\partial u}{\partial y_n}(\hat{y}, y_n, t) \end{array}\right)$$

for almost every $y \in \mathbb{R}^n$. The generalized Jacobian of F_t at x is

$$\begin{aligned}\partial_C F_t(\hat{x}, x_n) &= \mathrm{co}\left\{ \lim_{k\to+\infty} DF_t(y^{(k)}),\ y^{(k)} \underset{k\to+\infty}{\longrightarrow} x \text{ and } F_t \text{ is differentiable at } y^{(k)} \right\} \\ &= \left\{ \left(\begin{array}{c|c} & p_1 \\ Id_{n-1} & \vdots \\ & p_{n-1} \\ \hline 0 & p_n \end{array}\right),\ \ p = (p_1, \dots, p_n) \in \partial_C u(x) \right\}.\end{aligned}$$

By definition, $\partial_C F_t(\hat{x}, x_n)$ is of maximal rank provided every matrix in $\partial_C F_t(\hat{x}, x_n)$ is of maximal rank. This condition is clearly fulfilled since, from (5.2), $p_n = \langle p, e_n\rangle \geq \nu > 0$. Thus we can apply the nonsmooth inverse function theorem: there exists a neighborhood $V = V(\hat{x}) \times V(x_n)$ of $(\hat{x}, x_n)$, a neighborhood $V' = V'(\hat{x}) \times V'(\alpha)$ of $F_t(\hat{x}, x_n) = (\hat{x}, u(x,t)) = (\hat{x}, \alpha)$ and a Lipschitz-continuous function $G_t : V' \to V$ such that $\forall (\hat{y}', y'_n) \in V'$, $F_t \circ G_t(\hat{y}', y'_n) = (\hat{y}', y'_n)$ (and $\forall (\hat{y}, y_n) \in V$, $G_t \circ F_t(\hat{y}, y_n) = (\hat{y}, y_n)$). We obtain that for all $y' = (\hat{y}', y'_n) \in V'$,

$$F_t \circ G_t(\hat{y}', y'_n)$$

$$= \Big(\underbrace{\langle e_1, G_t(y')\rangle, \ldots, \langle e_{n-1}, G_t(y')\rangle}_{\hat{y}'}, \underbrace{u\left(\langle e_1, G_t(y')\rangle, \ldots, \langle e_n, G_t(y')\rangle, t\right)}_{y'_n} \Big);$$

hence, for all $(\hat{y}', y'_n) \in V'(\hat{x}) \times V'(\alpha)$, $u\left(\hat{y}', \langle e_n, G_t(\hat{y}', y'_n)\rangle, t\right) = y'_n$. We take $y'_n = \alpha$ and define $\zeta(\hat{y}') = \langle e_n, G_t(\hat{y}', \alpha)\rangle$. The function ζ is Lipschitz continuous since G_t is Lipschitz continuous. Note that ζ has the same Lipschitz constant as G_t, which is $1/\delta$, where δ is the distance from 0 to the convex compact set $\partial_C u(x,t)$. We get the desired result: for all $\hat{y}' \in V'(\hat{x})$,

$$u(\hat{y}', \zeta(\hat{y}'), t) = \alpha$$

and $\zeta(\hat{y}')$ is the unique solution of $u(\hat{y}', \cdot, t) = \alpha$ in V'. □

Corollary 5.2. *Under the assumptions of Theorem* 4.2 *and Theorem* 5.2,
(i) $\mathcal{L}^n(\mathcal{S}) = 0$;
(ii) for any $t \geq 0$, $\mathcal{H}^{n-1}(\mathcal{S} \cap \Gamma_t^\alpha) = 0$ *for* $\mathcal{L}^1$*-almost-every* $\alpha \in \mathbb{R}$.

Remark 5.5. (i) means that for a semiconcave function which satisfies (5.1), the front is locally Lipschitz almost everywhere. $\mathcal{H}^{n-1}$ can be seen as a measure on the α-level set Γ_t^α. Hence, (ii) means that almost every α-level set is locally Lipschitz almost everywhere in the sense of the $\mathcal{H}^{n-1}$ measure. But, note that this theorem gives no information about any special level set, particularly the front (see the counter-example developed in Theorem 5.5).

Proof of Corollary 5.2. We start by proving (i). If $u(\cdot,t)$ is differentiable at a point $x_0 \in \mathbb{R}^n$, then $D_x^+ u(x_0,t) = D_x^- u(x_0,t) = \{D_x u(x_0,t)\}$ and $|D_x u(x_0,t)| \geq c > 0$ by (5.1). Thus $0 \notin D_x^+ u(x_0,t)$. Moreover, since $u(\cdot,t)$ is semiconcave, $D_x^+ u(\cdot,t) = \partial_C u(\cdot,t)$ for every $t \in [0,T]$ (see Cannarsa and Soner [10] for instance). Since $u(\cdot,t)$ is differentiable almost everywhere, it follows that $\mathcal{L}^n(\mathcal{S}_0) = 0$, which implies the result since $\mathcal{S} \subset \mathcal{S}_0$.

Now, we turn to the proof of (ii). We apply the coarea formula (see Evans and Gariepy [16]) with the set $\mathcal{S}$ and the locally Lipschitz-continuous function $u(\cdot,t), t \geq 0$:

$$\int_{\mathcal{S}} |D_x u(x,t)|\, dx = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\mathcal{S} \cap \Gamma_t^\alpha)\, d\alpha.$$

Since $\mathcal{L}^n(\mathcal{S}) = 0$, we obtain the desired result. □

We end by giving some examples and a theorem which develops an example of a semiconcave function, satisfying (5.1), whose front is however very nasty.

Example 5.3. *A smooth front which develops a "non-Lipschitz singularity."* Let $u_0 : \mathbb{R}^2 \to \mathbb{R}$ be defined by $u_0 = \min\{u_1, u_2\}$ where $u_1(x,y) = (x^2 + (y-5)^2)^{1/2} - 4$ and $u_2(x,y) = (x^2 + (y+5)^2)^{1/2} - 4$. On the one side, the function

u_1 (respectively u_2) is Lipschitz continuous and its graph is a cone centered at the point $(0,5)$ (respectively $(0,-5)$). Thus u_0 is Lipschitz continuous. On the other side u_0 satisfies **(H5)**: $|u_0|+|Du_0|\geq 1$ since
— for every $(x,y)\notin\{(0,-5),(0,5)\}$ such that $u_1(x,y)\neq u_2(x,y)$, u_0 is differentiable and $|Du_0(x,y)|=1$;
— at each point (x,y) such that $u_1(x,y)=u_2(x,y)$, $D^-u_0(x,y)=\emptyset$;
— $|u_0(0,-5)|\geq 1$ and $|u_0(0,5)|\geq 1$.

We consider the Eikonal equation whose associated front propagates with a normal constant velocity equal to 1:

$$\begin{cases}\frac{\partial\omega}{\partial t}+|D_x\omega|=0 & \text{in } \mathbb{R}^2\times(0,2]\\ \omega(x,0)=u_0(x) & \text{in } \mathbb{R}^2.\end{cases} \tag{5.3}$$

The Hamiltonian $H(p)=|p|$ satisfies the assumptions of Theorem 4.2 (with **(H1-0)**); thus, the lower-bound inequality (5.1) holds for $t\in[0,2]$. Moreover, the solution of (5.3) is unique and given by the Oleinik-Lax formula (to go into the details, see e.g. Lions [22] or Barles [7]):

$$u(x,y,t)=\inf\{u_0(x',y'),\ ((x-x')^2+(y-y')^2)^{1/2}\leq t\}.$$

Then, one computes easily $\Gamma_t=\partial(B_{1,t}\cup B_{2,t})$ for $t\in[0,2]$ where $\partial(B_{1,t}\cup B_{2,t})$ is the boundary of the union of the balls $B_{1,t}=\{(x,y)\in\mathbb{R}^2: x^2+(y-5)^2<(4+t)^2\}$ and $B_{2,t}=\{(x,y)\in\mathbb{R}^2: x^2+(y+5)^2<(4+t)^2\}$. The front is smooth for $t\in[0,1)$ (union of 2 disjoint circles), is not locally Lipschitz at $(0,0)$ for $t=1$ (the circles are tangent at $(0,0)$) and is locally Lipschitz for $t\in(1,2]$. This illustrates the case of a front which is initially smooth but develops a non-Lipschitz singularity. Note that from the result of uniqueness of the propagation of the front (see the Introduction), up to regularize u_0, we can even start with a smooth function u_0 (thus a semiconcave function) whose front Γ_0 is $\partial(B_{1,0}\cup B_{2,0})$. The propagation of the front will be the same.

Example 5.4. *The "roof."* Let $v:\mathbb{R}^2\to\mathbb{R}$ be defined by $v(x,y)=-|y|$. The function v is Lipschitz continuous, concave and $\Gamma=\{(x,y)\in\mathbb{R}^2: y=0\}$. The function v satisfies the lower-bound estimate of Theorem 4.2 since $D^-v(x,0)=\emptyset$ and $|D^-v(x,y)|=1$ if $y\neq 0$. And $D^+v(x,0)=\partial_C v(x,0)=\{0\}\times[-1,1]$. In this case, the front is locally Lipschitz everywhere but $0\in\partial_C v(x,y)$ for every $(x,y)\in\Gamma$; thus, $\mathcal{S}$ is not equal to $\mathcal{S}_0$. Moreover, coming back to Example 5.3, we see that $(0,0)\in\Gamma_1(u)$, $D_x^-u(0,0,1)=D^-v(0,0)$ and $D_x^+u(0,0,1)=D^+v(0,0)$ while $\Gamma(v)$ is smooth and $\Gamma_1(u)$ is singular at the point $(0,0)$. It shows that sub- and superdifferentials do not characterize the points of the front which are not locally Lipschitz.

Theorem 5.5. *Let K be a Cantor set in $[0,1]$ such that $\mathcal{L}^1(K) > 0$. Define $v : [0,1] \times [-1,1] \to \mathbb{R}$ by $v(x,y) = -|y| + d_K^2(x)$ where $d_K(x)$ is the distance of x to K. Then*

(i) v is Lipschitz continuous, semiconcave and satisfies the lower-bound estimate (5.1);

(ii) $\mathcal{H}^1(\Gamma \cap \mathcal{S}) > 0$ where $\Gamma = \{(x,y) \in [0,1] \times [-1,1] : |y| = d_K^2(x)\}$ is the front of v.

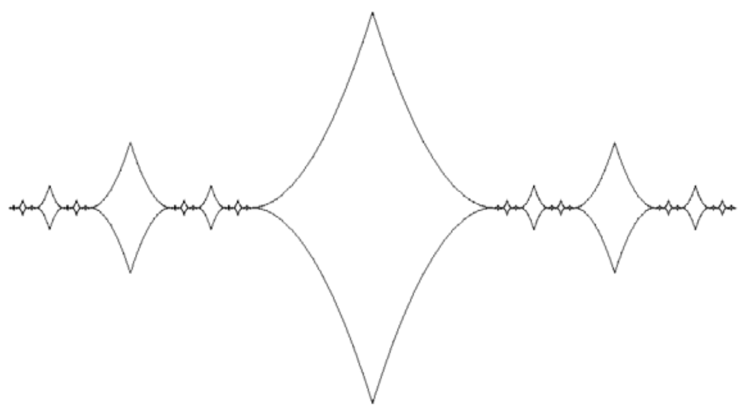

Figure 1. $\Gamma = \{(x,y) \in [0,1] \times [-1,1] : |y| = d_K^2(x)\}$

Proof of Theorem 5.5. The function v is clearly Lipschitz continuous since $[0,1] \times [-1,1]$ is bounded. Writing

$$v(x,y) = -|y| + \inf_{x' \in K} \{|x'|^2 - 2\langle x, x' \rangle\} + |x|^2,$$

we see that v is semiconcave. The function v satisfies (5.1) since for every $p \in D^- v(x,y)$, $|p| \geq |\langle p, (0,1) \rangle| \geq 1$ (notice that if $D^- v(x,y) = \emptyset$, (5.1) is trivially fulfilled).

Now we look more precisely at the front Γ of v (depicted in Figure 1). We define $\tilde{\Gamma}$ by

$$\tilde{\Gamma} = \Gamma \cap ([0,1] \times \{0\}) = \{(x,0) \in [0,1] \times \{0\} : d_K^2(x) = 0\} = K \times \{0\}$$

since K is closed. We want to prove that every point of $\tilde{\Gamma}$ is singular (i.e., lies in $\mathcal{S}$). We start by an important remark: if $(x_0, 0) \in \tilde{\Gamma}$ (i.e., $x_0 \in K$), then $\Gamma \cap \{(x_0, y) : y \in [-1,1]\} = \{(x_0, 0)\}$ since in this case, a point $(x_0, y), y \neq 0$ cannot satisfy $|y| = d_K^2(x_0) = 0$ (roughly speaking, if $(x_0, 0) \in \Gamma$, then there cannot be another point of Γ above or below). Recalling that there is no isolated point in the Cantor set K, we can divide the points of $\tilde{\Gamma}$ in two sets:

— the first set is countable and contains every point $(x_1, 0) \in \tilde{\Gamma}$ which is an extremity of a segment $((x_1, 0), (x_1 + \lambda, 0))$, $\lambda > 0$, which was removed in order to construct the Cantor set. An easy computation shows that, in this case, the piece of curve $\{(x, y) \in [x_1, x_1 + \lambda] \times [-1, 1] : x = \sqrt{|y|} + x_1\}$ is contained in Γ and is not locally Lipschitz at $(x_1, 0)$; thus, Γ is not locally Lipschitz at $(x_1, 0)$;
— the second set consists of points $(x_2, 0) \in \tilde{\Gamma}$ such that there exist two sequences $(x_k^+, 0) \to_{k\to\infty} (x_2, 0)$, $x_k^+ \in K$, $x_k^+ > x_2$ and $(x_k^-, 0) \to_{k\to\infty} (x_2, 0)$, $x_k^- \in K$, $x_k^- < x_2$.

From the remark, there cannot exist any piece of continuous curve which passes through $(x_2, 0)$ and lies in Γ. Thus the connected component of $(x_2, 0)$ in Γ is $\{(x_2, 0)\}$, and Γ is not locally Lipschitz at $(x_2, 0)$. It follows that $\mathcal{H}^1(\Gamma \cap \mathcal{S}) \geq \mathcal{H}^1(\tilde{\Gamma}) = \mathcal{L}^1(K) > 0$. □

6. Appendix

We state here a comparison result for unbounded, continuous viscosity solutions. This result was proved by Ishii [19]. We give a simplified proof which avoids the classical step-by-step argument.

Theorem 6.1. (*Comparison result for unbounded solutions*) *Assume* **(H1-**β**)** *and* **(H2)** *and let* $f, g \in C(\mathbb{R}^n \times [0, T])$. *Let* $x_0 \in \mathbb{R}^n$ *and* $r > 0$, *and define* $L = A_2 + B_2 + A_2|x_0|$. *If* $u \in C(\bar{B}(x_0, r) \times [0, T])$ *is a viscosity subsolution of*

$$\frac{\partial \omega}{\partial t} + H(x, t, D_x\omega) = f \text{ in } B(x_0, r) \times (0, T) \text{ and } \omega(x, 0) = u_0(x) \text{ in } B(x_0, r) \quad (6.1)$$

and v *is a viscosity supersolution of*

$$\frac{\partial \omega}{\partial t} + H(x, t, D_x\omega) = g \text{ in } B(x_0, r) \times (0, T) \text{ and } \omega(x, 0) = v_0(x) \text{ in } B(x_0, r), \quad (6.2)$$

then

$$u(x, t) - v(x, t) \leq \sup_{y\in\bar{B}(x_0,r)} \{u_0(y) - v_0(y)\} + \int_0^t \sup_{y\in\bar{B}(x_0,r)} \{f(y, s) - g(y, s)\}\, ds \quad (6.3)$$

for every $(x, t) \in \bar{\mathcal{D}}(x_0, r)$ *where*

$$\mathcal{D}(x_0, r) = \{(x, t) \in B(x_0, r) \times (0, T) : \mathrm{e}^{LT}(1 + |x - x_0|) - 1 \leq r\}.$$

In particular, if $r \geq 2\mathrm{e}^{LT}$, *then* (6.3) *holds in* $\bar{B}(x_0, \mathrm{e}^{-LT}r/2) \times [0, T]$.

Remark 6.1. The main point is that assumption **(H2)** implies a property of "finite speed of propagation," which can be described from two points of view:

- the values of the solution u at the point (x, t) do not depend on the initial data u_0 in the whole space $\mathbb{R}^n$ but only in a ball $\bar{B}(x, r(x, t))$ where $r(x, t) = \mathrm{e}^{(A_2+B_2+A_2|x_0|)t} - 1$;

- the values of the initial data in some ball $B(x_0, r)$ completely determine the values of the solution u of the equation in the domain of dependence $\bar{\mathcal{D}}(x_0, r)$ (cf. Figure 2).

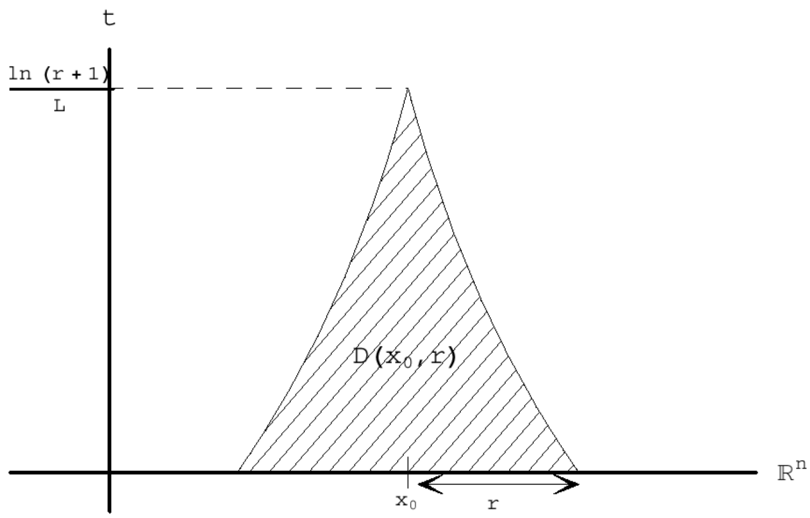

Figure 2. $\mathcal{D}(x_0, r) = \{(x, t) \in \bar{B}(x_0, r) \times [0, T] : e^{LT}(1 + |x - x_0|) - 1 \le r\}$.

We start by a lemma, proving it and using it to prove the theorem.

Lemma 6.1. *Under the assumptions of Theorem 6.1, the function $w = u - v$ is a viscosity subsolution of*

$$\frac{\partial \omega}{\partial t} - (A_2|x| + B_2)\,|D_x\omega| \le f - g \quad \text{in } B(x_0, r) \times (0, T). \tag{6.4}$$

Proof of Lemma 6.1. Let $\varphi \in C^1(B(x_0, r) \times (0, T))$, and let $(\bar{x}, \bar{t}) \in B(x_0, r) \times (0, T)$ be a local maximum of $w - \varphi$. Without loss of generality, we can suppose that $(\bar{x}, \bar{t})$ is a strict local maximum in $\bar{B}(\bar{x}, \sigma) \times [\bar{t} - \sigma, \bar{t} + \sigma]$ for $\sigma > 0$ small enough. We define

$$\Psi_{\varepsilon,\alpha}(x, y, t, s) = u(x, t) - v(y, s) - \frac{|x - y|^2}{\varepsilon^2} - \frac{|t - s|^2}{\alpha^2} - \varphi(x, t)$$

and

$$M_{\varepsilon,\alpha} \;=\; \max_{\bar{B}(\bar{x},\sigma)^2 \times [\bar{t}-\sigma, \bar{t}+\sigma]^2} \Psi_{\varepsilon,\alpha} \;=\; \Psi_{\varepsilon,\alpha}(x_{\varepsilon,\alpha}, y_{\varepsilon,\alpha}, t_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}).$$

Since $(\bar{x}, \bar{t})$ is a strict maximum, we know (see Barles [7]) that

$$M_{\varepsilon,\alpha} \to \max_{\bar{B}(\bar{x},\sigma) \times [\bar{t}-\sigma, \bar{t}+\sigma]} w - \varphi = (w - \varphi)(\bar{x}, \bar{t}),$$

$x_{\varepsilon,\alpha}, y_{\varepsilon,\alpha} \to \bar{x}$ and $t_{\varepsilon,\alpha}, s_{\varepsilon,\alpha} \to \bar{t}$ when $\varepsilon, \alpha \to 0$. As u is a subsolution of (6.1) and v is a supersolution of (6.2), by a straightforward calculation, we obtain

$$2\frac{t_{\varepsilon,\alpha} - s_{\varepsilon,\alpha}}{\alpha^2} + \frac{\partial \varphi}{\partial t}(x_{\varepsilon,\alpha}, t_{\varepsilon,\alpha}) + H\Big(x_{\varepsilon,\alpha}, t_{\varepsilon,\alpha}, 2\frac{x_{\varepsilon,\alpha} - y_{\varepsilon,\alpha}}{\varepsilon^2} + D_x\varphi(x_{\varepsilon,\alpha}, t_{\varepsilon,\alpha})\Big)$$

$$\leq f(x_{\varepsilon,\alpha}, t_{\varepsilon,\alpha}) \tag{6.5}$$

and

$$2\frac{t_{\varepsilon,\alpha} - s_{\varepsilon,\alpha}}{\alpha^2} + H\Big(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}, 2\frac{x_{\varepsilon,\alpha} - y_{\varepsilon,\alpha}}{\varepsilon^2}\Big) \geq g(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}). \tag{6.6}$$

Subtracting (6.6) from (6.5) and letting α go to 0, we get that $x_{\varepsilon,\alpha} \to x_\varepsilon$, $y_{\varepsilon,\alpha} \to y_\varepsilon$, $t_{\varepsilon,\alpha} \to t_\varepsilon$ and

$$\frac{\partial \varphi}{\partial t}(x_\varepsilon, t_\varepsilon) + H\big(x_\varepsilon, t_\varepsilon, 2\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} + D_x\varphi(x_\varepsilon, t_\varepsilon)\big) - H\big(y_\varepsilon, t_\varepsilon, 2\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2}\big) \leq (f-g)(x_\varepsilon, t_\varepsilon).$$

From **(H1-β)** and **(H2)**, we have

$$\begin{aligned}
&H\big(x_\varepsilon, t_\varepsilon, 2\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} + D_x\varphi(x_\varepsilon, t_\varepsilon)\big) - H\big(y_\varepsilon, t_\varepsilon, 2\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2}\big)\\
&\geq -\big(A_2|x_\varepsilon| + B_2\big)|D_x\varphi(x_\varepsilon, t_\varepsilon)| - C_1\big(\beta + 2\frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon^2}\big)|x_\varepsilon - y_\varepsilon|\\
&\geq -\big(A_2|x_\varepsilon| + B_2\big)|D_x\varphi(x_\varepsilon, t_\varepsilon)| - \frac{\beta C_1 \varepsilon^2}{2} - \big(\frac{\beta C_1}{2} + 2C_1\big)\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2}.
\end{aligned}$$

It follows that

$$\frac{\partial \varphi}{\partial t}(x_\varepsilon, t_\varepsilon) - (A_2|x_\varepsilon| + B_2)\,|D_x\varphi(x_\varepsilon, t_\varepsilon)| \leq \frac{C_1\varepsilon^2}{2} + \frac{5C_1}{2}\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} + (f-g)(x_\varepsilon, t_\varepsilon).$$

Letting ε go to 0, we end the proof since $|x_\varepsilon - y_\varepsilon|^2/\varepsilon^2 \to 0$, x_ε, $y_\varepsilon \to \bar{x}$ and $t_\varepsilon \to \bar{t}$. □

Proof of Theorem 6.1. Let $0 < \varepsilon < r$ and $\chi_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ be a C^1 function in $[0, r)$ such that $\chi_\varepsilon \equiv 0$ in $[0, r-\varepsilon]$, χ_ε is increasing in $[0, r)$ with limit $+\infty$ at r and $\chi_\varepsilon \equiv +\infty$ in $[r, +\infty)$. Let $\eta > 0$. The function $w(x,t) - \eta t - \frac{\eta}{T-t} - \int_0^t \sup_{y \in \bar{B}(x_0,r)}\{f(y,s) - g(y,s)\}\,ds - \chi_\varepsilon(\mathrm{e}^{Lt}(1 + (|x - x_0|^2 + \varepsilon^2)^{\frac{1}{2}}) - 1)$ achieves its maximum at a point $(\bar{x}, \bar{t}) \in B(x_0, r) \times [0, T)$. If $\bar{t} > 0$, then we have a local maximum and, using that w is a subsolution of (6.4) in $B(x_0, r) \times (0, T)$, we obtain from Lemma 6.1,

$$\begin{aligned}
&\eta + \frac{\eta}{(T-\bar{t})^2} + \sup_{y \in \bar{B}(x_0,r)}\{f(y,t) - g(y,t)\} - (f(\bar{x}, \bar{t}) - g(\bar{x}, \bar{t})) \\
&+ \chi'_\varepsilon \cdot \mathrm{e}^{L\bar{t}}\left(L(1 + (|\bar{x} - x_0|^2 + \varepsilon^2)^{1/2}) - (A_2|\bar{x}| + B_2)\,\frac{|\bar{x} - x_0|}{(|\bar{x} - x_0|^2 + \varepsilon^2)^{1/2}}\right) \leq 0.
\end{aligned} \tag{6.7}$$

But

$$(A_2|\bar{x}| + B_2)\frac{|\bar{x} - x_0|}{(|\bar{x} - x_0|^2 + \varepsilon^2)^{1/2}} \leq A_2|\bar{x} - x_0| + A_2|x_0| + B_2,$$

which implies that (6.7) leads to a contradiction since $L = A_2 + B_2 + |x_0|$. Thus $\bar{t} = 0$ and we obtain that, for every $(x,t) \in \mathbb{R}^n \times [0, T]$,

$$w(x,t) - \eta t - \frac{\eta}{T-t} - \int_0^t \sup_{y \in \bar{B}(x_0,r)}\{f(y,s) - g(y,s)\}\,ds$$

$$-\chi_\varepsilon\left(\mathrm{e}^{Lt}(1+(|x-x_0|^2+\varepsilon^2)^{\frac{1}{2}})-1\right)\le w(\bar{x},0)\le \sup_{y\in\bar{B}(x_0,r)} w(y,0).$$

Letting η go to 0, we have the desired comparison (6.3) for every (x,t) such that

$$\chi_\varepsilon\big(\mathrm{e}^{Lt}(1+(|x-x_0|^2+\varepsilon^2)^{\frac{1}{2}})-1\big)=0 \iff \mathrm{e}^{Lt}(1+(|x-x_0|^2+\varepsilon^2)^{\frac{1}{2}})-1\le r-\varepsilon.$$

Letting ε go to 0, we get the comparison (6.3) for every $(x,t)\in\bar{\mathcal{D}}(x_0,r)$. Moreover, if $r\ge 2\mathrm{e}^{LT}$, a straightforward computation (taking $\varepsilon=1$) shows that (6.3) is fulfilled for every $(x,t)\in\bar{B}(x_0,\mathrm{e}^{-LT}r/2)\times[0,T]$, which completes the proof. □

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