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ON THE ASYMPTOTIC ANALYSIS OF H-SYSTEMS, I: ASYMPTOTIC BEHAVIOR OF LARGE SOLUTIONS

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Abstract. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $\gamma \in C^{3,\alpha}(\partial\Omega; \mathbb{R}^3)$ $(0 < \alpha < 1)$ and $H > 0$. We consider the asymptotic behavior of large solutions of an H-system

$$\Delta u = 2H u_{x_1} \wedge u_{x_2} \quad \text{in } \Omega, \qquad u = \gamma \quad \text{on } \partial\Omega$$

as $H \to 0$ or as $\gamma \to 0$. We show that large solutions blow up at exactly one point in Ω. The exact blow-up rate and location of blow-up point are studied.

1. Introduction. Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded domain and $\gamma \in C^{3,\alpha}(\partial\Omega; \mathbb{R}^3)$ $(0 < \alpha < 1)$ $(\gamma \not\equiv \text{const.})$. For $H > 0$, we consider the following equation, called an H-system:

$$\begin{cases} \Delta u = 2H u_{x_1} \wedge u_{x_2} & \text{in } \Omega \\ u = \gamma & \text{on } \partial\Omega. \end{cases} \tag{H1}$$

Here "$\wedge$" is the exterior product in $\mathbb{R}^3$.

This equation arises when we seek a surface in $\mathbb{R}^3$ with mean curvature H: If a solution u of (H1) satisfies $|u_{x_1}|^2 - |u_{x_2}|^2 = u_{x_1} \cdot u_{x_2} = 0$, $u(\overline{\Omega})$ represents a surface with mean curvature H (at all points $x \in \Omega$ where $\nabla u(x) \neq 0$).

(H1) is the Euler-Lagrange equation of the functional $\mathcal{E}_H$ defined in $H^1_\gamma(\Omega; \mathbb{R}^3) := \{u \in H^1(\Omega; \mathbb{R}^3) : u = \gamma \text{ on } \partial\Omega\}$:

$$\mathcal{E}_H(u) = \int_\Omega |\nabla u|^2 \, dx + \frac{4}{3} H \int_\Omega u \cdot (u_{x_1} \wedge u_{x_2}) \, dx.$$

Remark. In the definition of $\mathcal{E}_H$, it is not clear whether the cubic term $Q(u) := \int_\Omega u \cdot (u_{x_1} \wedge u_{x_2}) \, dx$ has a meaning for $u \in H^1_\gamma(\Omega; \mathbb{R}^3)$. However, one

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can uniquely extend the functional $\int_\Omega u \cdot (u_{x_1} \wedge u_{x_2})\, dx$ defined in $C^1(\overline{\Omega}; \mathbb{R}^3)$ to $H^1_\gamma(\Omega; \mathbb{R}^3)$ as a smooth functional; see [4] and [15] for details.

Let us assume that $\gamma(\partial\Omega)$ is contained in a closed ball of radius R. In [7], Hildebrandt proved that if $HR \le 1$, (H1) has at least one solution. His solution $\underline{u}_H$ is obtained by minimizing $\mathcal{E}_H$ in the set $S_R := \{u \in H^1_\gamma(\Omega; \mathbb{R}^3) : \|u\|_{L^\infty(\Omega)} \le R\}$: $\mathcal{E}_H(\underline{u}_H) = \inf_{u \in S_R} \mathcal{E}_H(u)$. $\underline{u}_H$ is usually called a small solution. In [4] and [14], Brezis-Coron and Struwe independently proved that (H1) admits more solutions if $HR \le 1$. These solutions are called large solutions. Brezis-Coron's solution $\overline{u}_H$ is obtained as $\overline{u}_H = \underline{u}_H - \frac{J_H}{2H} v^0_H$, where v^0_H is a solution of the minimization problem:

$$J_H := \inf_{\substack{v \in H^1_0(\Omega; \mathbb{R}^3) \\ |Q(v)|=1}} \left\{ \int_\Omega |\nabla v|^2\, dx + 4H \int_\Omega \underline{u}_H \cdot (v_{x_1} \wedge v_{x_2})\, dx \right\}. \tag{1.1}$$

Since the functional Q is not continuous under weak convergence in $H^1_0(\Omega)$, it is not obvious that J_H is achieved. By using the information $J_H < S := (32\pi)^{1/3}$ when $\gamma \not\equiv$ const., Brezis-Coron proved that any minimizing sequence for (1.1) is compact in $H^1(\Omega)$ and J_H is achieved.

In [4], Brezis-Coron proved that the small solution is unique. On the other hand, for some cases one can expect that there are more large solutions; see the example given in [15].

One of the effective ways to study the structure of the solution space for (H1) is to study the asymptotic behavior of solutions to (H1) as $H \to 0$ or $\gamma \to 0$. In this direction, in [5] Brezis-Coron studied the asymptotic behavior of general solutions to (H1) as $H \to 0$ or $\gamma \to 0$ (in some sense). In [12] and [13] Sasahara studied the more detailed asymptotic behavior of large solutions $\overline{u}_H$ when $\Omega = \mathbb{B} := \{x \in \mathbb{R}^2 : |x| < 1\}$ as $H \to 0$. By the work of Brezis-Coron [5], large solutions $\overline{u}_H$ blow up at exactly one point in $\overline{\Omega}$. That is, there exist $\lambda_H > 0$ with $\lambda_H \to 0$ $(H \to 0)$, $a_H \in \Omega$ with $\lambda_H / d(a_H, \partial\Omega) \to 0$ $(H \to 0)$ and a solution ω of the equation $\Delta\omega = 2\omega_{x_1} \wedge \omega_{x_2}$ in $\mathbb{R}^2$ with $\int_{\mathbb{R}^2} |\nabla\omega|^2\, dx < \infty$ such that

$$\left\| \nabla \Big(H\overline{u}_H - \omega\Big(\frac{\cdot - a_H}{\lambda_H}\Big)\Big) \right\|_{L^2(\Omega)} \to 0 \quad \text{as } H \to 0. \tag{1.2}$$

In this paper, we call a cluster point of $\{a_H\}$ a blow-up point of $\overline{u}_H$. In [12] (see also the argument in [13]), when $\Omega = \mathbb{B}$, Sasahara determined a location of the blow-up point: He introduced the function

$$K(a) := (1 - |a|^2)^2 (|\nabla h_\gamma(a)|^2 + 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|) \tag{1.3}$$

defined in $\mathbb{B}$ and showed that large solutions blow up at a maximum point of K in $\mathbb{B}$. Here h_γ is the harmonic extension of γ to Ω: $\Delta h_\gamma = 0$ in Ω, $h_\gamma = \gamma$ on $\partial\Omega$. Also in [13], Sasahara proved that the converse of the above result is true; that is, he proved that if $a_0 \in \mathbb{B}$ is a strict local maximum point of K in $\mathbb{B}$ (see Theorem C below for the definition of strict local maximum), there exists a large solution $\overline{u}_H$ of (H1) for small H and $\overline{u}_H$ blows up at a_0 as $H \to 0$.

Though the arguments of the proofs of the above results (which rely on the Schwartz lemma in complex analysis, which gives the explicit form of all conformal maps from $\mathbb{B}$ to $\mathbb{B}$) given in [12] and [13] are interesting, these are not applied to multiply connected domain. (Sasahara's arguments are also applied to a simply connected domain Ω by pulling back Ω to $\mathbb{B}$ by a conformal map). Also one may ask whether there are large solutions to (H1) corresponding to general critical points of K.

Our aim in this paper and the second part of this paper [8] is the following: (1) to extend the results of Sasahara for a general domain Ω, (2) to treat the problems in a more systematic way, and (3) to give the construction of large solutions corresponding to general critical points of K. As a by product of our approach, we can also give the exact blow-up rate of large solutions.

Before stating our main results, we introduce notation: For $a \in \Omega$, define h_a^i $(i = 1, 2)$ as the solution of the equation

$$\begin{cases} \Delta h_a^i = 0 & \text{in } \Omega \\ h_a^i = \frac{2(x_i - a_i)}{|x-a|^2} & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In another term, $h_a^i(x) = -4\pi \frac{\partial}{\partial a_i} H(a, x)$, where $H(a, x)$ is the regular part of the Green's function of $-\Delta$ under a Dirichlet boundary condition: $H(a, x) = G(a, x) + \frac{1}{2\pi} \log |a - x|$, where $-\Delta_x G(a, x) = \delta(x - a)$, $G|_{\partial\Omega} = 0$.

Define $K(a, \Omega)$ $(a \in \Omega)$ as

$$K(a, \Omega) = \frac{|\nabla h_\gamma(a)|^2 + 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|}{\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)}. \tag{1.5}$$

We now state the main results of this paper and the second part of this paper [8]. The first result characterizes the location of blow-up points:

Theorem A. *Let a_∞ be a blow-up point of $\{\overline{u}_H\}$. Let $\{H_n\}$ be a sequence with $H_n \downarrow 0$ and $a_{H_n} \to a_\infty$, and $\overline{u}_n := \overline{u}_{H_n}$ a large solution to* (H1) *obtained by the method of Brezis-Coron. Then $a_\infty \in \Omega$ (interior point) and a_∞ maximizes $K(\cdot, \Omega)$ in Ω: $K(a_\infty, \Omega) = \sup_{a \in \Omega} K(a, \Omega)$.*

The next theorem gives the exact blow-up rate for $\{\overline{u}_n\}$.

Theorem B. *Let* $\{\overline{u}_n\}$ *and* a_∞ *be as in Theorem A. We have*

$$\lim_{n\to\infty}\frac{\lambda_{H_n}}{H_n}=\frac{\left(|\nabla h_\gamma(a_\infty)|^2+2|(h_\gamma)_{x_1}(a_\infty)\wedge(h_\gamma)_{x_2}(a_\infty)|\right)^{1/2}}{\frac{\partial h^1_{a_\infty}}{\partial x_1}(a_\infty)+\frac{\partial h^2_{a_\infty}}{\partial x_2}(a_\infty)},\tag{1.6}$$

$$\lim_{n\to\infty}H_n^2\|\nabla\overline{u}_n\|_{L^\infty(\Omega)}=\frac{2\sqrt{2}\left(\frac{\partial h^1_{a_\infty}}{\partial x_1}(a_\infty)+\frac{\partial h^2_{a_\infty}}{\partial x_2}(a_\infty)\right)}{\left(|\nabla h_\gamma(a_\infty)|^2+2|(h_\gamma)_{x_1}(a_\infty)\wedge(h_\gamma)_{x_2}(a_\infty)|\right)^{1/2}}.\tag{1.7}$$

The above two theorems are proved in this paper. In the second part of this paper [8], we prove the following theorems:

Theorem C. *Let* a_0 *be a strict local maximum of* $K(\cdot,\Omega)$; *that is, there exists* $R>0$ *with* $\mathbb{B}_R(a_0):=\{x\in\mathbb{R}^2:|x-a_0|<R\}\Subset\Omega$ *such that* $K(a_0,\Omega)>K(a,\Omega)$ *for all* $a\in\mathbb{B}_R(a_0)$ *with* $a\neq a_0$. *Then there exists* $H_0>0$ *such that for* $0<H\le H_0$, (H1) *has a solution* $\overline{u}_H$ *which is different from* $\underline{u}_H$. *Moreover,* $\overline{u}_H$ *blows up* (*in the sense of* (1.2)) *at exactly one point* a_0 *as* $H\to 0$.

In fact, we can construct large solutions corresponding to more general critical points of $K(\cdot,\Omega)$:

Theorem D. *Let* a_0 *be a regular point of* h_γ. *Assume also that* a_0 *is a nondegenerate critical point of* $K(\cdot,\Omega)$ *in* Ω. *Then there exists* $H_0>0$ *such that for* $0<H\le H_0$, (H1) *has a solution* $\overline{u}_H$ *which is different from* $\underline{u}_H$. *Moreover,* $\overline{u}_H$ *blows up at exactly one point* a_0 *as* $H\to 0$.

Problem (H1) is closely related to the problem

$$\begin{cases}\Delta u=2u_{x_1}\wedge u_{x_2} & \text{in }\Omega\\ u=\epsilon\gamma & \text{on }\partial\Omega.\end{cases}\tag{H2}$$

Here $\epsilon>0$. In fact (H2) is equivalent to (H1): If u_ϵ is a solution of (H2), then $v_\epsilon:=\epsilon^{-1}u_\epsilon$ is a solution of (H1) with $H=\epsilon$. The converse is also true. Thus we obtain similar results for (H2). The only difference is in the $L^\infty(\Omega)$-estimate of ∇u. (1.7) is replaced by the following (here we denote by $\overline{u}_\epsilon$ the large solution obtained by the method of Brezis-Coron and a a blow-up point of $\{\overline{u}_\epsilon\}$):

$$\lim_{\epsilon\downarrow 0}\epsilon\|\nabla\overline{u}_\epsilon\|_{L^\infty(\Omega)}=\frac{2\sqrt{2}\left(\frac{\partial h^1_a}{\partial x_1}(a)+\frac{\partial h^2_a}{\partial x_2}(a)\right)}{\left(|\nabla h_\gamma(a)|^2+2|(h_\gamma)_{x_1}(a)\wedge(h_\gamma)_{x_2}(a)|\right)^{1/2}}.\tag{1.7 bis}$$

Remark 1.1. (a) When $\Omega = \mathbb{B}$, Theorem A and Theorem C have been obtained by Sasahara ([12], [13]) by a different method. Note that when $\Omega = \mathbb{B}$, $\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a) = \frac{4}{(|a|^2-1)^2}$ (see Example 4). Theorem B and Theorem D are new even for the special case $\Omega = \mathbb{B}$.

(b) $K(\cdot, \Omega)$ is a continuous function in Ω with $K(a, \Omega) > 0$ and $K(a, \Omega) \to 0$ as $a \to \partial\Omega$; see Lemma 5.2 and Lemma 5.7. Thus $K(\cdot, \Omega)$ takes its maximum in Ω. On the other hand, $K(\cdot, \Omega)$ is not in general differentiable. For example, if we take $\gamma(x_1, x_2) = ((x_1 - a_1)(x_2 - a_2), x_1 - a_1, 0)$ for some $a \in \Omega$, then $h_\gamma = ((x_1 - a_1)(x_2 - a_2), x_1 - a_1, 0)$ and $|(h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2}| = |x_1 - a_1|$, and $K(\cdot, \Omega)$ is not differentiable at a. In this example h_γ degenerates at a. If a is a regular point of h_γ (that is, the rank of the Hessian of h_γ at a is 2 or equivalently, $(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a) \neq 0$), then $K(\cdot, \Omega)$ is C^∞ near a.

(c) Related asymptotic problems are studied by various authors. In [10] and [11], O. Rey studied the asymptotic behavior of solutions to the equations $-\Delta u = u^{n+2/n-2} + \epsilon u$ or $-\Delta u = u^{n+2/n-2} + \epsilon f$ as $\epsilon \to 0$. Since our method developed in this paper is so general, our approach gives a rigorous proof of the conjecture given in [11, Remarks]. See also [1, Remark 3]. This problem will be treated in a future work. See also [2] and [9] for related problems concerning Ginzburg-Landau equations.

This paper is written as follows: In Section 2, we study the asymptotic behavior of the small solution which is necessary to study large solutions. In Section 3, we study the asymptotic behavior of the value J_H (see (1.1)) as $H \to 0$. More precisely, first we explicitly construct a map $\varphi_{H,a} \in H_0^1(\Omega; \mathbb{R}^3)$ with $|Q(\varphi_{H,a})| = 1$ ($a \in \Omega$) such that

$$J_H(\varphi_{H,a}) := \int_\Omega |\nabla \varphi_{H,a}|^2 \, dx + 4H \int_\Omega \underline{u}_H \cdot ((\varphi_{H,a})_{x_1} \wedge (\varphi_{H,a})_{x_2}) \, dx$$
$$= S - \frac{S}{2} K(a, \Omega) H^2 + o(H^2) \quad \text{as } H \to 0. \tag{1.8}$$

Next we prove

$$J_H \geq S - \frac{S}{2} K(a_\infty, \Omega) H^2 + o(H^2) \quad \text{as } H \to 0. \tag{1.9}$$

In Section 4, using (1.8) and (1.9) we complete the proof of Theorem A. Theorem B is a by product of the proof of Theorem A. The proof of Theorem B is also given in Section 4. Theorem C and Theorem D are proved in part II [8]. Finally in Section 5, we collect technical results.

Throughout this paper, we use the following notations:

$SO(3) = \{R : R \text{ is a } 3\times 3\text{-matrix with } {}^tRR = I, \det R = 1\}$. $\mathfrak{so}(3)$ is the Lie algebra of $SO(3)$; that is, $\mathfrak{so}(3) = \{\xi : \xi \text{ is a } 3\times 3\text{-matrix with } {}^t\xi + \xi = 0\}$. Denote

$$\xi_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then $\langle \xi_1, \xi_2, \xi_3 \rangle$ is a basis of $\mathfrak{so}(3)$. exp is the exponential map defined by $\exp(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{k!}$ for $\xi \in \mathfrak{so}(3)$. $\langle e_1, e_2, e_3 \rangle$ is the standard basis of $\mathbb{R}^3$: $e_1 = {}^t(1,0,0)$, $e_2 = {}^t(0,1,0)$, $e_3 = {}^t(0,0,1)$. "$\cdot$" denotes the standard inner product in $\mathbb{R}^3$ or in $\mathbb{R}^2$. $S = (32\pi)^{1/3}$ is the best constant of the isoperimetric inequality: $\int_\Omega |\nabla v|^2\,dx \geq S|Q(v)|^{2/3}$ for all $v \in H_0^1(\Omega;\mathbb{R}^3)$. $\mathbb{B}_r(a) := \{x \in \mathbb{R}^2 : |x-a| < r\}$. $o(X)$ will denote various quantities such that $o(X)/|X| \to 0$ as $|X| \to 0$. $O(X)$ will denote various quantities such that $|O(X)| \leq C|X|$ for some $C > 0$ independent of X. $C > 0$ will denote various constants independent of H or n.

2. Asymptotic behavior of the small solution. Let $\gamma \in C^{3,\alpha}(\partial\Omega;\mathbb{R}^3)$ and $\|\gamma\|_{L^\infty(\Omega)} \leq R$. In Section 1, we have already mentioned that the small solution $\underline{u}_H$ is obtained as the solution of the minimization problem

$$\inf_{u \in S_R} \mathcal{E}_H(u),$$

where S_R is defined in Section 1.

To study the asymptotic behavior of large solutions, it turns out that we also need to know the asymptotic behavior of the small solution $\underline{u}_H$ as $H \to 0$. The following lemma gives such a result:

Lemma 2.1. *There exists $C > 0$ (independent of H) such that*

$$\|\underline{u}_H - h_\gamma\|_{C^3(\overline{\Omega})} \leq CH \quad \text{as } H \to 0. \tag{2.1}$$

Proof. By regularity theory (see [7]), there exists $C > 0$ (independent of small H) such that $\|\underline{u}_H\|_{C^{3,\alpha}(\overline{\Omega})} \leq C$. Thus $\underline{u}_H \to h_\gamma$ in $C^3(\overline{\Omega})$ as $H \to 0$. Set $r_H := \underline{u}_H - h_\gamma$. Since $\underline{u}_H$ satisfies (H1), r_H satisfies

$$\begin{aligned} \Delta r_H = 2H((r_H)_{x_1} \wedge (r_H)_{x_2} + (r_H)_{x_1} \wedge (h_\gamma)_{x_2} \\ + (h_\gamma)_{x_1} \wedge (r_H)_{x_2} + (h_\gamma)_{x_1} \wedge (h_\gamma)_{x_2}) \end{aligned} \tag{2.2}$$

with $r_H|_{\partial\Omega} = 0$. The elliptic regularity theory and the Sobolev imbedding $W^{2,4/3} \hookrightarrow W^{1,4}$ imply

$$\begin{aligned}
&\|r_H\|_{W^{2,4/3}(\Omega)} \\
&\le CH(\||\nabla r_H|^2\|_{L^{4/3}(\Omega)} + \|\nabla h_\gamma\|_{L^2(\Omega)}\|\nabla r_H\|_{L^4(\Omega)} + \||\nabla h_\gamma|^2\|_{L^{4/3}(\Omega)}) \\
&\le CH(\|\nabla r_H\|^2_{L^{8/3}(\Omega)} + \|\nabla h_\gamma\|_{L^2(\Omega)}\|\nabla r_H\|_{L^4(\Omega)} + \|\nabla h_\gamma\|^2_{L^{8/3}(\Omega)}) \\
&\le CH(\|\nabla r_H\|_{L^{8/3}(\Omega)}\|r_H\|_{W^{2,4/3}(\Omega)} + \|\nabla h_\gamma\|_{L^2(\Omega)}\|r_H\|_{W^{2,4/3}(\Omega)} \\
&\quad + \|\nabla h_\gamma\|^2_{L^{8/3}(\Omega)}).
\end{aligned}$$

Since $r_H \to 0$ in $C^2(\overline{\Omega})$ as $H \to 0$, the above inequality implies

$$\|r_H\|_{W^{2,4/3}(\Omega)} \le CH \quad \text{as } H \to 0. \tag{2.3}$$

Iterating a similar argument (using (2.3) and Sobolev imbedding) implies the assertion of Lemma 2.1. □

In the following sections, we study the asymptotic behavior of large solutions.

3. Asymptotic behavior of J_H. In this section, we prove (1.8) and (1.9). Obviously, we have

$$J_H = S + o(1) \tag{3.1}$$

as $H \to 0$. Let v_H^0 be a solution to the problem (1.1). (1.2) and (3.1) imply

$$\Big\| -\frac{S}{2}v_H^0 - \omega\Big(\frac{\cdot - a_H}{\lambda_H}\Big)\Big\|_{H^1(\Omega)} \to 0 \quad \text{as } H \to 0. \tag{3.2}$$

Define

$$v_H := -\frac{S}{2}v_H^0. \tag{3.3}$$

Then

$$\Big\|v_H - \omega\Big(\frac{\cdot - a_H}{\lambda_H}\Big)\Big\|_{H^1(\Omega)} \to 0 \tag{3.4}$$

$$Q(v_H) = -4\pi. \tag{3.5}$$

In (3.2) and (3.4), ω is a solution of

$$\Delta\omega = 2\omega_{x_1} \wedge \omega_{x_2} \quad \text{in } \mathbb{R}^2 \tag{3.6}$$

with

$$\int_{\mathbb{R}^2} |\nabla\omega|^2\,dx < \infty, \quad \int_{\mathbb{R}^2} \omega\cdot(\omega_{x_1}\wedge\omega_{x_2})\,dx = -4\pi$$

and $\omega(\infty)=0$.

All solutions to (3.6) are known (see [5, Lemma A.1]) and given by $\omega(z) = \Pi^{-1}\big(\frac{P(z)}{Q(z)}\big)+C$, $z=(x_1,x_2)=x_1+ix_2$, where $\Pi:\mathbb{S}^2\to\mathbb{C}\cup\{\infty\}$ denotes the stereographic projection from the north pole, P and Q are polynomials with $\max\{\deg P,\deg Q\}=1$ and C is a constant. In another term, it is written as $\omega(x_1,x_2)=R\widehat{U}_{\lambda,a}$ for some $\lambda>0$, $a\in\mathbb{R}^2$ and $R\in SO(3)$, where

$$U_{\lambda,a}(x_1,x_2)=\frac{2\lambda}{\lambda^2+|x-a|^2}\begin{pmatrix}x_1-a_1\\ x_2-a_2\\ -\lambda\end{pmatrix}+\begin{pmatrix}0\\0\\1\end{pmatrix},\quad \widehat{U}_{\lambda,a}=U_{\lambda,a}-\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{3.7}$$

Let $h_{\lambda,a}$ be the harmonic extension of $\widehat{U}_{\lambda,a}|_{\partial\Omega}$ to Ω: $\Delta h_{\lambda,a}=0$ in Ω and $h_{\lambda,a}=\widehat{U}_{\lambda,a}$ on $\partial\Omega$. Define $PU_{\lambda,a}:=\widehat{U}_{\lambda,a}-h_{\lambda,a}$. For $\epsilon>0$, we set

$$M(\epsilon):=\Big\{v\in H_0^1(\Omega;\mathbb{R}^3): \exists R\in SO(3),\exists a\in\Omega,\exists\lambda>0 \text{ with } \lambda/d(a,\partial\Omega)<\epsilon \\ \text{such that } \|\nabla(v-RPU_{\lambda,a})\|_{L^2(\Omega)}<\epsilon\Big\}.$$

For $v\in M(\epsilon)$, we consider the following problem:

$$\inf\Big\{\|\nabla(v-\alpha RPU_{\lambda,a})\|_{L^2(\Omega)}:\tfrac12<\alpha<2,\ R\in SO(3),\\ \lambda>0,\ a\in\Omega,\ \lambda/d(a,\partial\Omega)<2\epsilon\Big\}. \tag{3.8}$$

By Lemma 5.1, for small $\epsilon>0$, (3.8) has a unique solution.

Let $a_\infty\in\overline{\Omega}$ be a blow-up point of large solutions $\{\overline{u}_H\}$. Then there exist $H_n\to0$, $R_n\in SO(3)$, $\lambda_n\downarrow0$, $\Omega\ni a_n\to a_\infty$ such that (see (3.4)) ($v_n:=v_{H_n}$, $d_n:=d(a_n,\partial\Omega)$)

$$\|\nabla(v_n-R_nPU_{\lambda_n,a_n})\|_{L^2(\Omega)}\to0,\quad \lambda_n/d_n\to0. \tag{3.9}$$

(3.9) implies that there exists $\epsilon_n\downarrow0$ such that $v_n\in M(\epsilon_n)$. Apply Lemma 5.1 to v_n and also denote the solution to (3.8) (for $v=v_n$, $\epsilon=\epsilon_n$) by

$(\alpha_n, R_n, \lambda_n, a_n)$. Define $w_n := v_n - \alpha_n R_n PU_{\lambda_n,a_n} \in H_0^1(\Omega;\mathbb{R}^3)$. By our choice of $(\alpha_n, R_n, \lambda_n, a_n)$, we have

$$\begin{aligned}0 &= \int_\Omega \nabla w_n \cdot \nabla (R_n PU_{\lambda_n,a_n})\,dx \\ &= \int_\Omega \nabla w_n \cdot \nabla \big(R_n \frac{\partial}{\partial a_i} PU_{\lambda_n,a_n}\big)\,dx \quad (i=1,2) \\ &= \int_\Omega \nabla w_n \cdot \nabla \big(R_n \frac{\partial}{\partial \lambda} PU_{\lambda_n,a_n}\big)\,dx \\ &= \int_\Omega \nabla w_n \cdot \nabla (R_n \xi_i PU_{\lambda_n,a_n})\,dx \quad (i=1,2,3). \end{aligned} \tag{3.10}$$

In the following, we estimate

$$J_n := \int_\Omega |\nabla v_n|^2\,dx + 4H_n \int_\Omega \underline{u}_n \cdot (v_n)_{x_1} \wedge (v_n)_{x_2}\,dx,$$

where $\underline{u}_n := \underline{u}_{H_n}$.

Lemma 3.1. *We have the following (as $n \to \infty$):*

$$\begin{aligned}J_n &= \alpha_n^2\big(8\pi - 4\pi\big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\big)\lambda_n^2\big) \\ &\quad - 8\pi\alpha_n^2((\underline{u}_n)_{x_1}(a_n)\cdot R_n e_1 + (\underline{u}_n)_{x_2}(a_n)\cdot R_n e_2)\lambda_n H_n \\ &\quad + \|\nabla w_n\|^2_{L^2(\Omega)} + O\big(\frac{\lambda_n^3}{d_n^3}\big) + O\big(\frac{\lambda_n^2 H_n}{d_n^2}\big) + O\big(\frac{\lambda_n^2 H_n}{d_n}|\log\lambda_n|\big) \\ &\quad + O(\lambda_n H_n |\log\lambda_n|^{1/2}\|\nabla w_n\|_{L^2(\Omega)}) + O(H_n\|\nabla w_n\|^2_{L^2(\Omega)}).\end{aligned}$$

Proof. By (3.10) and Lemma 5.2, we have

$$\begin{aligned}\int_\Omega |\nabla v_n|^2\,dx &= \alpha_n^2 \int_\Omega |\nabla PU_{\lambda_n,a_n}|^2\,dx + \int_\Omega |\nabla w_n|^2\,dx \\ &= \alpha_n^2\big(8\pi - 4\pi\big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\big)\lambda_n^2\big) + \|\nabla w_n\|^2_{L^2(\Omega)} + O\big(\frac{\lambda_n^3}{d_n^3}\big).\end{aligned} \tag{3.11}$$

Next we consider

$$\begin{aligned}\int_\Omega \underline{u}_n \cdot (v_n)_{x_1} \wedge (v_n)_{x_2}\,dx &= \alpha_n^2 \int_\Omega \underline{u}_n \cdot (R_n PU_{\lambda_n,a_n})_{x_1} \wedge (R_n PU_{\lambda_n,a_n})_{x_2}\,dx \\ &\quad + \alpha_n \int_\Omega \underline{u}_n \cdot (R_n PU_{\lambda_n,a_n})_{x_1} \wedge (w_n)_{x_2}\,dx\end{aligned}$$

$$+ \alpha_n \int_\Omega \underline{u}_n \cdot (w_n)_{x_1} \wedge (R_n PU_{\lambda_n,a_n})_{x_2}\, dx + \int_\Omega \underline{u}_n \cdot (w_n)_{x_1} \wedge (w_n)_{x_2}\, dx$$
$$= I_1 + I_2 + I_3 + I_4.$$

Here by Lemma A.3 and Lemma A.4 in [4],

$$\begin{aligned} I_2 + I_3 &= \alpha_n \int_\Omega R_n PU_{\lambda_n,a_n} \cdot ((\underline{u}_n)_{x_1} \wedge (w_n)_{x_2} + (w_n)_{x_1} \wedge (\underline{u}_n)_{x_2})\, dx \\ &= O\Big(\int_\Omega |PU_{\lambda_n,a_n}||\nabla w_n|\, dx\Big) = O(\|PU_{\lambda_n,a_n}\|_{L^2(\Omega)}\|\nabla w_n\|_{L^2(\Omega)}) \\ &= O(\lambda_n|\log\lambda_n|^{1/2}\|\nabla w_n\|_{L^2(\Omega)}) + O\Big(\frac{\lambda_n}{d_n}\|\nabla w_n\|_{L^2(\Omega)}\Big), \end{aligned} \tag{3.12}$$

$$I_4 = O(\|\nabla w_n\|^2_{L^2(\Omega)}). \tag{3.13}$$

To estimate I_1, extend γ to a bounded domain $\tilde\Omega \Supset \Omega$ (we also denote it by γ) such that $\mathrm{supp}(\gamma) \subset \tilde\Omega$. Also extend $\underline{u}_n$ to $\tilde\Omega$ by $\underline{u}_n = \gamma$ in $\tilde\Omega \setminus \Omega$ and h_{λ_n,a_n} to $\tilde\Omega$ by $\widehat{U}_{\lambda_n,a_n}$ in $\tilde\Omega \setminus \Omega$. Then ($h_n := h_{\lambda_n,a_n}$)

$$\begin{aligned} I_1/\alpha_n^2 &= \int_{\tilde\Omega} \underline{u}_n \cdot (R_n\widehat{U}_{\lambda_n,a_n} - R_n h_n)_{x_1} \wedge (R_n\widehat{U}_{\lambda_n,a_n} - R_n h_n)_{x_2}\, dx \\ &= \int_{\tilde\Omega} \underline{u}_n \cdot (R_n\widehat{U}_{\lambda_n,a_n})_{x_1} \wedge (R_n\widehat{U}_{\lambda_n,a_n})_{x_2}\, dx \\ &- \int_{\tilde\Omega} \underline{u}_n \cdot (R_n\widehat{U}_{\lambda_n,a_n})_{x_1} \wedge (R_n h_n)_{x_2}\, dx - \int_{\tilde\Omega} \underline{u}_n \cdot (R_n h_n)_{x_1} \wedge (R_n\widehat{U}_{\lambda_n,a_n})_{x_2}\, dx \\ &+ \int_{\tilde\Omega} \underline{u}_n \cdot (R_n h_n)_{x_1} \wedge (R_n h_n)_{x_2}\, dx = I_1^1 + I_1^2 + I_1^3 + I_1^4. \end{aligned}$$

Here by [4, Lemma A.4]

$$\begin{aligned} I_1^2 + I_1^3 &= -\int_{\tilde\Omega} R_n h_n \cdot ((R_n\widehat{U}_{\lambda_n,a_n})_{x_1} \wedge (\underline{u}_n)_{x_2} + (\underline{u}_n)_{x_1} \wedge (R_n\widehat{U}_{\lambda_n,a_n})_{x_2})\, dx \\ &= O(\|h_n\|_{L^\infty(\tilde\Omega)}\|\nabla\widehat{U}_{\lambda_n,a_n}\|_{L^1(\tilde\Omega)}) = O\Big(\frac{\lambda_n^2}{d_n}|\log\lambda_n|\Big), \end{aligned} \tag{3.14}$$

since $\|h_n\|_{L^\infty(\tilde\Omega)} = O\big(\frac{\lambda_n}{d_n}\big)$ (by the maximum principle and the fact that $\|\widehat{U}_{\lambda_n,a_n}\|_{L^\infty(\mathbb{R}^2\setminus\Omega)} = O\big(\frac{\lambda_n}{d_n}\big)$) and $\|\nabla\widehat{U}_{\lambda_n,a_n}\|_{L^1(\tilde\Omega)} = O(\lambda_n|\log\lambda_n|)$. By the calculation in the proof of Lemma 5.2, we have

$$I_1^4 = O\Big(\int_{\tilde\Omega} |\nabla h_n|^2\, dx\Big) = O\big(\frac{\lambda_n^2}{d_n^2}\big). \tag{3.15}$$

Next we estimate I_1^1.

$$
\begin{aligned}
I_1^1 &= \int_{\tilde\Omega} \underline{u}_n \cdot (R_n \widehat{U}_{\lambda_n,a_n})_{x_1} \wedge (R_n \widehat{U}_{\lambda_n,a_n})_{x_2}\, dx \\
&= -\frac{1}{2} \int_{\tilde\Omega} |\nabla U_{\lambda_n,a_n}|^2 \underline{u}_n \cdot R_n U_{\lambda_n,a_n}\, dx \text{ (by (5.14))} \\
&= -4 \int_{\mathbb{B}_{d_n}(a_n)} \frac{\lambda_n^2}{(\lambda_n^2+r^2)^2} \underline{u}_n \cdot R_n U_{\lambda_n,a_n}\, dx \\
&\quad - 4 \int_{\tilde\Omega\setminus \mathbb{B}_{d_n}(a_n)} \frac{\lambda_n^2}{(\lambda_n^2+r^2)^2} \underline{u}_n \cdot R_n U_{\lambda_n,a_n}\, dx. \qquad (3.16)
\end{aligned}
$$

Here the second term in (3.16) is estimated as

$$
\begin{aligned}
\int_{\tilde\Omega\setminus \mathbb{B}_{d_n}(a_n)} \frac{\lambda_n^2}{(\lambda_n^2+r^2)^2} \underline{u}_n \cdot R_n U_{\lambda_n,a_n}\, dx &= O\Big(\int_{\tilde\Omega\setminus \mathbb{B}_{d_n}(a_n)} \frac{\lambda_n^2}{(\lambda_n^2+r^2)^2}\, dx\Big) \\
&= O\Big(\frac{\lambda_n^2}{d_n^2}\Big). \qquad (3.17)
\end{aligned}
$$

To estimate the first term, by Taylor's formula we have

$$
\underline{u}_n = \underline{u}_n(a_n) + \nabla \underline{u}_n(a_n)\cdot(x-a_n) + \frac{1}{2}\nabla^2 \underline{u}_n(a_n)(x-a_n)\cdot(x-a_n) + O(|x-a_n|^3).
$$

Then by using the oddness of the integral, we have

$$
\begin{aligned}
&\int_{\mathbb{B}_{d_n}(a_n)} \frac{\lambda_n^2}{(\lambda_n^2+r^2)^2} \underline{u}_n \cdot R_n U_{\lambda_n,a_n}\, dx \\
&= \int_{\mathbb{B}_{d_n}(a_n)} \frac{\lambda_n^2}{(\lambda_n^2+r^2)^3} \underline{u}_n \cdot (2\lambda_n(x_1-a_n^1) R_n e_1 \\
&\quad + 2\lambda_n(x_2-a_n^2)R_n e_2 + (r^2-\lambda_n^2) R_n e_3)\, dx \\
&= \int_{\mathbb{B}_{d_n}(a_n)} \frac{2\lambda_n^3}{(\lambda_n^2+r^2)^3} (x_1-a_n^1)^2 (\underline{u}_n)_{x_1}(a_n)\cdot R_n e_1\, dx \\
&\quad + \int_{\mathbb{B}_{d_n}(a_n)} \frac{2\lambda_n^3}{(\lambda_n^2+r^2)^3} (x_2-a_n^2)^2 (\underline{u}_n)_{x_2}(a_n)\cdot R_n e_2\, dx \\
&\quad + \int_{\mathbb{B}_{d_n}(a_n)} \frac{\lambda_n^2(r^2-\lambda_n^2)}{(\lambda_n^2+r^2)^3} \underline{u}_n(a_n)\cdot R_n e_3\, dx + O\Big(\int_{\mathbb{B}_{d_n}(a_n)} \frac{\lambda_n^3 r^4}{(\lambda_n^2+r^2)^3}\, dx\Big) \\
&\quad + O\Big(\int_{\mathbb{B}_{d_n}(a_n)} \frac{\lambda_n^2 |r^2-\lambda_n^2| r^2}{(\lambda_n^2+r^2)^3}\, dx\Big). \qquad (3.18)
\end{aligned}
$$

Here the first term in (3.18) is calculated as

$$\begin{aligned}
&\int_{\mathbb{B}_{d_n}(a_n)} \frac{2\lambda_n^3}{(\lambda_n^2+r^2)^3}(x_1-a_n^1)^2(\underline{u}_n)_{x_1}(a_n)\cdot R_n e_1\,dx\\
&= 2(\underline{u}_n)_{x_1}(a_n)\cdot R_n e_1 \int_0^{d_n}\int_0^{2\pi} \frac{\lambda_n^3 r^3\cos^2\theta}{(\lambda_n^2+r^2)^3}\,d\theta\,dr\\
&= 2\pi(\underline{u}_n)_{x_1}(a_n)\cdot R_n e_1\lambda_n \int_0^{d_n/\lambda_n} \frac{s^3}{(1+s^2)^3}\,ds\\
&= \frac{\pi}{2}(\underline{u}_n)_{x_1}(a_n)\cdot R_n e_1\lambda_n + O\Big(\frac{\lambda_n^3}{d_n^2}\Big). \qquad (3.19)
\end{aligned}$$

The second term is calculated similarly. The third term is calculated as

$$\begin{aligned}
&\int_{\mathbb{B}_{d_n}(a_n)} \frac{\lambda_n^2(r^2-\lambda_n^2)}{(\lambda_n^2+r^2)^3}\underline{u}_n(a_n)\cdot R_n e_3\,dx\\
&= 2\pi\underline{u}_n(a_n)\cdot R_n e_3 \int_0^{d_n/\lambda_n} \frac{s(s^2-1)}{(1+s^2)^3}\,ds \qquad (3.20)\\
&= -2\pi\underline{u}_n(a_n)\cdot R_n e_3 \int_{d_n/\lambda_n}^{\infty} \frac{s(s^2-1)}{(1+s^2)^3}\,ds = O\Big(\frac{\lambda_n^2}{d_n^2}\Big),
\end{aligned}$$

where we have used $\int_0^\infty \frac{s(s^2-1)}{(1+s^2)^3}\,ds = 0$. Since

$$\int_{\mathbb{B}_{d_n}(a_n)} \frac{\lambda_n^3 r^4}{(\lambda_n^2+r^2)^3}\,dx = O\Big(\lambda_n^3\int_0^{d_n/\lambda_n} \frac{s^5}{(1+s^2)^3}\,ds\Big) = O(\lambda_n^3|\log\lambda_n|)$$

and

$$\int_{\mathbb{B}_{d_n}(a_n)} \frac{\lambda_n^2|r^2-\lambda_n^2|r^2}{(\lambda_n^2+r^2)^3}\,dx = O\Big(\lambda_n^2\int_0^{d_n/\lambda_n} \frac{s^3|s^2-1|}{(1+s^2)^3}\,ds\Big) = O(\lambda_n^2|\log\lambda_n|),$$

from (3.12)–(3.20) we have

$$\begin{aligned}
&\int_\Omega \underline{u}_n\cdot(v_n)_{x_1}\wedge(v_n)_{x_2}\,dx = -2\pi\alpha_n^2((\underline{u}_n)_{x_1}(a_n)\cdot R_n e_1\\
&+ (\underline{u}_n)_{x_2}(a_n)\cdot R_n e_2)\lambda_n + O\Big(\frac{\lambda_n^2}{d_n^2}\Big) + O\Big(\frac{\lambda_n^2}{d_n}|\log\lambda_n|\Big)\\
&+ O(\lambda_n|\log\lambda_n|^{1/2}\|\nabla w_n\|_{L^2(\Omega)}) + O(\|\nabla w_n\|^2_{L^2(\Omega)}). \qquad (3.21)
\end{aligned}$$

(3.11) and (3.21) imply the conclusion of Lemma 3.1. □

The asymptotic behavior of α_n as $n\to\infty$ is given by the next lemma:

Lemma 3.2. *We have, as $n \to \infty$,*

$$\alpha_n = 1 + \frac{1}{4\pi}\int_\Omega R_n \widehat{U}_{\lambda_n,a_n} \cdot (w_n)_{x_1} \wedge (w_n)_{x_2}\, dx + \frac{1}{2}\Big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\Big)\lambda_n^2 + o\Big(\frac{\lambda_n^2}{d_n^2}\Big) + o(\|\nabla w_n\|^2_{L^2(\Omega)}).$$

Proof. By (3.5), we have (extend v_n, PU_{λ_n,a_n} and w_n to $\mathbb{R}^2$ by 0)

$$-4\pi = \int_{\mathbb{R}^2} v_n \cdot (v_n)_{x_1} \wedge (v_n)_{x_2}\, dx \tag{3.22}$$
$$= \int_{\mathbb{R}^2} (\alpha_n PU_{\lambda_n,a_n} + w_n) \cdot (\alpha_n PU_{\lambda_n,a_n} + w_n)_{x_1} \wedge (\alpha_n PU_{\lambda_n,a_n} + w_n)_{x_2}\, dx.$$

We set $W_n := -\alpha_n R_n h_n + w_n$ ($h_n = h_{\lambda_n,a_n}$ and h_n is extended to $\mathbb{R}^2$ by $\widehat{U}_{\lambda_n,a_n}$ in $\mathbb{R}^2 \setminus \Omega$). Then (5.19) implies

$$\begin{aligned}(3.22) &= \alpha_n^3 \int_{\mathbb{R}^2} R_n\widehat{U}_{\lambda_n,a_n} \cdot (R_n\widehat{U}_{\lambda_n,a_n})_{x_1} \wedge (R_n\widehat{U}_{\lambda_n,a_n})_{x_2}\, dx \\ &\quad + 3\alpha_n^2 \int_{\mathbb{R}^2} W_n \cdot (R_n\widehat{U}_{\lambda_n,a_n})_{x_1} \wedge (R_n\widehat{U}_{\lambda_n,a_n})_{x_2}\, dx \\ &\quad + 3\alpha_n \int_{\mathbb{R}^2} R_n\widehat{U}_{\lambda_n,a_n} \cdot (W_n)_{x_1} \wedge (W_n)_{x_2}\, dx + \int_{\mathbb{R}^2} W_n \cdot (W_n)_{x_1} \wedge (W_n)_{x_2}\, dx \\ &= -4\pi\alpha_n^3 + Q_1 + Q_2 + Q_3.\end{aligned} \tag{3.23}$$

Here, by (5.20),

$$Q_3 = O(\|\nabla W_n\|^3_{L^2(\Omega)}). \tag{3.24}$$

Since $\Delta\widehat{U}_{\lambda_n,a_n} = 2(\widehat{U}_{\lambda_n,a_n})_{x_1} \wedge (\widehat{U}_{\lambda_n,a_n})_{x_2}$ and h_n is harmonic in Ω, we have by (3.10) and the calculation in the proof of Lemma 5.2

$$\begin{aligned}&\int_{\mathbb{R}^2} W_n \cdot (R_n\widehat{U}_{\lambda_n,a_n})_{x_1} \wedge (R_n\widehat{U}_{\lambda_n,a_n})_{x_2}\, dx = \frac{1}{2}\int_{\mathbb{R}^2} W_n \cdot \Delta\widehat{U}_{\lambda_n,a_n}\, dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^2} \nabla W_n \cdot \nabla(R_n\widehat{U}_{\lambda_n,a_n})\, dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^2} \nabla(w_n - \alpha_n R_n h_n) \cdot \nabla(R_n\widehat{U}_{\lambda_n,a_n})\, dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^2} \nabla(w_n - \alpha_n R_n h_n) \cdot \nabla(R_n PU_{\lambda_n,a_n})\, dx \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^2} \nabla(w_n - \alpha_n R_n h_n) \cdot \nabla(R_n h_n)\, dx\end{aligned} \tag{3.25}$$

$$= \frac{\alpha_n}{2}\int_{\mathbb{R}^2}|\nabla h_n|^2\,dx = 2\pi\alpha_n\Big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n)+\frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\Big)\lambda_n^2 + O\Big(\frac{\lambda_n^3}{d_n^3}\Big).$$

Next we estimate Q_2. For simplicity, we set $\tilde{h}_n := \alpha_n R_n h_n$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^2} R_n\widehat{U}_{\lambda_n,a_n}\cdot(W_n)_{x_1}\wedge(W_n)_{x_2}\,dx\\
&= \int_{\mathbb{R}^2} R_n\widehat{U}_{\lambda_n,a_n}\cdot(w_n-\tilde{h}_n)_{x_1}\wedge(w_n-\tilde{h}_n)_{x_2}\,dx\\
&= \int_{\mathbb{R}^2} R_n\widehat{U}_{\lambda_n,a_n}\cdot(w_n)_{x_1}\wedge(w_n)_{x_2}\,dx\\
&\quad -\int_{\mathbb{R}^2} R_n\widehat{U}_{\lambda_n,a_n}\cdot((w_n)_{x_1}\wedge(\tilde{h}_n)_{x_2}+(\tilde{h}_n)_{x_1}\wedge(w_n)_{x_2})\,dx\\
&\quad +\int_{\mathbb{R}^2} R_n\widehat{U}_{\lambda_n,a_n}\cdot(\tilde{h}_n)_{x_1}\wedge(\tilde{h}_n)_{x_2}\,dx = Q_2^1+Q_2^2+Q_2^3.
\end{aligned}$$

Claim 1. $Q_2^2 = o\big(\frac{\lambda_n}{d_n}\|\nabla w_n\|_{L^2(\Omega)}\big)$.
Proof of Claim 1. By [4, Lemma A.4],

$$Q_2^2 = \int_\Omega w_n\cdot((R_n\widehat{U}_{\lambda_n,a_n})_{x_1}\wedge(\tilde{h}_n)_{x_2}+(\tilde{h}_n)_{x_1}\wedge(R_n\widehat{U}_{\lambda_n,a_n})_{x_2})\,dx. \tag{3.26}$$

Let $\varphi_n \in H^1_0(\Omega;\mathbb{R}^3)$ be the solution of

$$\begin{cases} -\Delta\varphi_n = (R_n\widehat{U}_{\lambda_n,a_n})_{x_1}\wedge(\tilde{h}_n)_{x_2}+(\tilde{h}_n)_{x_1}\wedge(R_n\widehat{U}_{\lambda_n,a_n})_{x_2} & \text{in } \Omega\\ \varphi_n = 0 & \text{on } \partial\Omega.\end{cases}$$

Then by the estimate in [4, Lemma A.1] (see also [3] and [6])

$$\|\varphi_n\|_{L^\infty(\Omega)}+\|\nabla\varphi_n\|_{L^2(\Omega)} \le C\|\nabla\widehat{U}_{\lambda_n,a_n}\|_{L^2(\Omega)}\|\nabla\tilde{h}_n\|_{L^2(\Omega)} = O\big(\frac{\lambda_n}{d_n}\big). \tag{3.27}$$

Then we have

$$Q_2^2 = \int_\Omega \nabla w_n\cdot\nabla\varphi_n\,dx = O(\|\nabla w_n\|_{L^2(\Omega)}\|\nabla\varphi_n\|_{L^2(\Omega)}). \tag{3.28}$$

To estimate $\|\nabla\varphi_n\|_{L^2(\Omega)}$, first we have (by (3.27))

$$\begin{aligned}
\|\nabla\varphi_n\|^2_{L^2(\Omega)} &= -\int_\Omega \varphi_n\cdot\Delta\varphi_n\,dx \qquad (3.29)\\
&\le \|\varphi_n\|_{L^\infty(\Omega)}\int_\Omega|\Delta\varphi_n|\,dx = O\Big(\frac{\lambda_n}{d_n}\int_\Omega|\nabla U_{\lambda_n,a_n}||\nabla h_n|\,dx\Big).
\end{aligned}$$

Let $\delta > 0$ be arbitrary. Define $\epsilon_n = \delta^{-1}\lambda_n$. Since $\lambda_n/d_n \to 0$, for large n we have $\mathbb{B}_{\epsilon_n}(a_n) \Subset \Omega$ and

$$\begin{aligned}\int_\Omega |\nabla U_{\lambda_n,a_n}||\nabla h_n|\,dx =& \int_{\mathbb{B}_{\epsilon_n}(a_n)}|\nabla U_{\lambda_n,a_n}||\nabla h_n|\,dx + \int_{\Omega\setminus\mathbb{B}_{\epsilon_n}(a_n)}|\nabla U_{\lambda_n,a_n}||\nabla h_n|\,dx\\
\le& \|\nabla U_{\lambda_n,a_n}\|_{L^2(\mathbb{B}_{\epsilon_n}(a_n))}\|\nabla h_n\|_{L^2(\mathbb{B}_{\epsilon_n}(a_n))}\\
&+ \|\nabla U_{\lambda_n,a_n}\|_{L^2(\Omega\setminus\mathbb{B}_{\epsilon_n}(a_n))}\|\nabla h_n\|_{L^2(\Omega\setminus\mathbb{B}_{\epsilon_n}(a_n))}\\
\le& C\|\nabla h_n\|_{L^\infty(\mathbb{B}_{\epsilon_n}(a_n))}\epsilon_n + C\frac{\lambda_n}{d_n}\|\nabla U_{\lambda_n,a_n}\|_{L^2(\Omega\setminus\mathbb{B}_{\epsilon_n}(a_n))}\\
\le& C\Big(\frac{\lambda_n\epsilon_n}{d_n^2} + \frac{\lambda_n^2}{d_n\epsilon_n}\Big) = C\frac{\lambda_n}{d_n}\Big(\delta^{-1}\frac{\lambda_n}{d_n} + \delta\Big).\end{aligned} \tag{3.30}$$

In (3.30), we have used the fact that

$$\|\nabla h_n\|_{L^\infty(\mathbb{B}_{\epsilon_n}(a_n))} \le C\frac{\lambda_n}{d_n^2} \tag{3.31}$$

and

$$\int_{\Omega\setminus\mathbb{B}_{\epsilon_n}(a_n)}|\nabla U_{\lambda_n,a_n}|^2\,dx = O\Big(\int_{\mathbb{R}^2\setminus\mathbb{B}_{\epsilon_n}(a_n)}\frac{\lambda_n^2}{(\lambda_n^2+r^2)^2}\,dx\Big) = O\Big(\frac{\lambda_n^2}{\epsilon_n^2}\Big).$$

(3.31) is a consequence of the elliptic estimate (since h_n is harmonic)

$$d_x|\nabla h_n(x)| \le C\|h_n\|_{L^\infty(\Omega)} = O\Big(\frac{\lambda_n}{d_n}\Big)$$

for $x \in \Omega$, and $d_x \ge \frac{1}{2}d_n$ for $x \in \mathbb{B}_{\epsilon_n}(a_n)$ and large n, where $d_x = d(x, \partial\Omega)$.

Since $\lambda_n/d_n \to 0$ and $\delta > 0$ is arbitrary, (3.29) and (3.30) imply

$$\|\nabla\varphi_n\|_{L^2(\Omega)} = o\Big(\frac{\lambda_n}{d_n}\Big). \tag{3.32}$$

(3.28) and (3.32) prove Claim 1.

Claim 2. $Q_2^3 = o\big(\frac{\lambda_n^2}{d_n^2}\big)$.

Proof of Claim 2. By (5.19), we have

$$Q_2^3 = \frac{1}{2}\int_{\mathbb{R}^2}\tilde{h}_n\cdot((R_nU_{\lambda_n,a_n})_{x_1}\wedge(\tilde{h}_n)_{x_2} + (\tilde{h}_n)_{x_1}\wedge(R_nU_{\lambda_n,a_n})_{x_2})\,dx. \tag{3.33}$$

Let $G(\cdot,\cdot)$ be a Green's function in $\mathbb{R}^2$ and set

$$\phi_n(x) = \int_{\mathbb{R}^2} G(x,y)((R_nU_{\lambda_n,a_n})_{x_1} \wedge (\tilde{h}_n)_{x_2} + (\tilde{h}_n)_{x_1} \wedge (R_nU_{\lambda_n,a_n})_{x_2})\,dx.$$

By [3] or [6], we have

$$\|\phi_n\|_{L^\infty(\mathbb{R}^2)} + \|\nabla\phi_n\|_{L^2(\mathbb{R}^2)} \le C\|\nabla U_{\lambda_n,a_n}\|_{L^2(\mathbb{R}^2)}\|\nabla h_n\|_{L^2(\mathbb{R}^2)} = O\Big(\frac{\lambda_n}{d_n}\Big).$$

Then we proceed as in the proof of Claim 1 and we get $Q_2^3 = o\Big(\frac{\lambda_n^2}{d_n^2}\Big)$.

¿From Claim 1 and Claim 2, we have

$$\int_{\mathbb{R}^2} R_n\widehat{U}_{\lambda_n,a_n}\cdot(W_n)_{x_1}\wedge(W_n)_{x_2}\,dx = \int_\Omega R_n\widehat{U}_{\lambda_n,a_n}\cdot(w_n)_{x_1}\wedge(w_n)_{x_2}\,dx + o\Big(\frac{\lambda_n}{d_n}\|\nabla w_n\|_{L^2(\Omega)}\Big) + o\Big(\frac{\lambda_n^2}{d_n^2}\Big). \tag{3.34}$$

(3.22), (3.23), (3.24), (3.25), (3.34) and the fact that

$$\|W_n\|_{L^2(\mathbb{R}^2)} = O(\|\nabla w_n\|_{L^2(\Omega)}) + O\big(\frac{\lambda_n}{d_n}\big)$$

imply

$$\begin{aligned}-4\pi = -4\pi\alpha_n^3 &+ 3\alpha_n\int_\Omega R_n\widehat{U}_{\lambda_n,a_n}\cdot(w_n)_{x_1}\wedge(w_n)_{x_2}\,dx\\ &+ 6\pi\alpha_n^3\Big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\Big)\lambda_n^2 + o\Big(\frac{\lambda_n}{d_n}\|\nabla w_n\|_{L^2(\Omega)}\Big)\\ &+ o\Big(\frac{\lambda_n^2}{d_n^2}\Big) + O(\|\nabla w_n\|^3_{L^2(\Omega)}).\end{aligned} \tag{3.35}$$

By (3.35), $\alpha_n \to 1$ $(n\to\infty)$ and

$$\begin{aligned}\alpha_n &= 1 + \frac{3\alpha_n}{4\pi(\alpha_n^2+\alpha_n+1)}\int_\Omega R_n\widehat{U}_{\lambda_n,a_n}\cdot(w_n)_{x_1}\wedge(w_n)_{x_2}\,dx\\ &\quad+ \frac{3\alpha_n^3}{2(\alpha_n^2+\alpha_n+1)}\Big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\Big)\lambda_n^2\\ &\quad+ o\Big(\frac{\lambda_n}{d_n}\|\nabla w_n\|_{L^2(\Omega)}\Big) + o\Big(\frac{\lambda_n^2}{d_n^2}\Big) + O(\|\nabla w_n\|^3_{L^2(\Omega)})\\ &= 1 + \frac{1}{4\pi}\int_\Omega R_n\widehat{U}_{\lambda_n,a_n}\cdot(w_n)_{x_1}\wedge(w_n)_{x_2}\,dx \qquad (3.36)\\ &\quad+ \frac{1}{2}\Big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\Big)\lambda_n^2 + o\Big(\frac{\lambda_n^2}{d_n^2}\Big) + o(\|\nabla w_n\|^2_{L^2(\Omega)}).\end{aligned}$$

In the calculation of (3.36), we have used the estimate $d_n|\nabla h_{a_n}(a_n)| \leq C\|h_{a_n}\|_{L^\infty(\Omega)} = O(d_n^{-1})$. This completes the proof. $\square$

Combining Lemma 3.1 and Lemma 3.2, we have

Corollary 3.3. *We have (as $n \to \infty$)*

$$\begin{aligned}
J_{H_n} :&= \inf_{\substack{v \in H_0^1(\Omega;\mathbb{R}^3) \\ |Q(v)|=1}} \left\{ \int_\Omega |\nabla v|^2\,dx + 4H_n \int_\Omega \underline{u}_n \cdot (v_{x_1} \wedge v_{x_2})\,dx \right\} \\
&= S + \frac{S}{2}\Big(\frac{\partial h_{a_n}^1}{\partial x_1}(a_n) + \frac{\partial h_{a_n}^2}{\partial x_2}(a_n)\Big)\lambda_n^2 \\
&\quad - S((\underline{u}_n)_{x_1}(a_n) \cdot R_n e_1 + (\underline{u}_n)_{x_2}(a_n) \cdot R_n e_2)\lambda_n H_n \\
&\quad + \frac{4}{S^2}\bigg(\int_\Omega |\nabla w_n|^2\,dx + 4\int_\Omega R_n \widehat{U}_{\lambda_n,a_n} \cdot (w_n)_{x_1} \wedge (w_n)_{x_2}\,dx\bigg) \\
&\quad + o(\|\nabla w_n\|_{L^2(\Omega)}^2) + o\Big(\frac{\lambda_n^2}{d_n^2}\Big) + O\Big(\frac{\lambda_n^2 H_n}{d_n}|\log\lambda_n|\Big) \\
&\quad + O(H_n\lambda_n|\log\lambda_n|^{1/2}\|\nabla w_n\|_{L^2(\Omega)}).
\end{aligned}$$

To proceed further, we need the following:

Lemma 3.4. *For any $a \in \Omega$ and $H > 0$, there exists $\varphi_{H,a} \in H_0^1(\Omega;\mathbb{R}^3)$ such that*

$$\begin{aligned}
J_H(\varphi_{H,a}) :&= \frac{\int_\Omega |\nabla\varphi_{H,a}|^2\,dx + 4H\int_\Omega \underline{u}_H \cdot (\varphi_{H,a})_{x_1} \wedge (\varphi_{H,a})_{x_2}\,dx}{|Q(\varphi_{H,a})|^{2/3}} \\
&= S - \frac{S}{2}K(a,\Omega)H^2 + o(H^2) \quad \text{as } H \to 0.
\end{aligned}$$

Proof. For $a \in \Omega$ and $H > 0$, define $\varphi_{H,a} \in H_0^1(\Omega;\mathbb{R}^3)$ by

$$\varphi_{H,a} := -\frac{2}{S}R_H PU_{\lambda_H,a}, \tag{3.37}$$

where

$$\lambda_H := \frac{(|\nabla h_\gamma(a)|^2 + 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|)^{1/2}}{\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)}H \tag{3.38}$$

and $R_H \in SO(3)$ satisfies (see Lemma 5.4)

$$(h_\gamma)_{x_1}(a)\cdot R_H e_1 + (h_\gamma)_{x_2}(a)\cdot R_H e_2 = (|\nabla h_\gamma(a)|^2 + 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|)^{1/2}. \tag{3.39}$$

By Lemma 5.2, we have

$$\int_\Omega |\varphi_{H,a}|^2\,dx = \frac{4}{S^2}\Big(8\pi - 4\pi\Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda_H^2\Big) + o(H^2). \tag{3.40}$$

By an argument similar to the proofs of (3.21) and Lemma 2.1, we have

$$\begin{aligned}
&\int_\Omega \underline{u}_H \cdot (\varphi_{H,a})_{x_1} \wedge (\varphi_{H,a})_{x_2}\,dx \qquad (3.41)\\
&= -\frac{8\pi}{S^2}((\underline{u}_H)_{x_1}(a)\cdot R_H e_1 + (\underline{u}_H)_{x_2}(a)\cdot R_H e_2)\lambda_H + o(H)\\
&= -\frac{8\pi}{S^2}((h_\gamma)_{x_1}(a)\cdot R_H e_1 + (h_\gamma)_{x_2}(a)\cdot R_H e_2)\lambda_H + o(H).\\
&Q(\varphi_{H,a}) = -\frac{8}{S^3}Q(PU_{\lambda_H,a}),
\end{aligned}$$

where (as before, $h_{\lambda_H,a}$ is extended to $\mathbb{R}^2$ by $\widehat{U}_{\lambda_H,a}$)

$$\begin{aligned}
&Q(PU_{\lambda_H,a})\\
&= \int_{\mathbb{R}^2} (\widehat{U}_{\lambda_H,a} - h_{\lambda_H,a})\cdot(\widehat{U}_{\lambda_H,a} - h_{\lambda_H,a})_{x_1} \wedge (\widehat{U}_{\lambda_H,a} - h_{\lambda_H,a})_{x_2}\,dx\\
&= \int_{\mathbb{R}^2} \widehat{U}_{\lambda_H,a}\cdot(\widehat{U}_{\lambda_H,a})_{x_1}\wedge(\widehat{U}_{\lambda_H,a})_{x_2}\,dx\\
&\quad - 3\int_{\mathbb{R}^2} h_{\lambda_H,a}\cdot(\widehat{U}_{\lambda_H,a})_{x_1}\wedge(\widehat{U}_{\lambda_H,a})_{x_2}\,dx\\
&\quad + 3\int_{\mathbb{R}^2} \widehat{U}_{\lambda_H,a}\cdot(h_{\lambda_H,a})_{x_1}\wedge(h_{\lambda_H,a})_{x_2}\,dx\\
&\quad - \int_{\mathbb{R}^2} h_{\lambda_H,a}\cdot(h_{\lambda_H,a})_{x_1}\wedge(h_{\lambda_H,a})_{x_2}\,dx.
\end{aligned} \tag{3.42}$$

Here

$$\int_{\mathbb{R}^2} h_{\lambda_H,a}\cdot(\widehat{U}_{\lambda_H,a})_{x_1}\wedge(\widehat{U}_{\lambda_H,a})_{x_2}\,dx = -\frac{1}{2}\int_{\mathbb{R}^2} |\nabla\widehat{U}_{\lambda_H,a}|^2 U_{\lambda_H,a}\cdot h_{\lambda_H,a}\,dx$$

(by (5.14))

$$= -2\pi\Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda_H^2 + o(H^2) \quad \text{(by (5.17))}, \tag{3.43}$$

$$\int_{\mathbb{R}^2} \widehat{U}_{\lambda_H,a}\cdot(h_{\lambda_H,a})_{x_1}\wedge(h_{\lambda_H,a})_{x_2}\,dx = o(H^2), \tag{3.44}$$

$$\int_{\mathbb{R}^2} h_{\lambda_H,a}\cdot(h_{\lambda_H,a})_{x_1}\wedge(h_{\lambda_H,a})_{x_2}\,dx = O(\|\nabla h_{\lambda_H,a}\|^3_{L^2(\mathbb{R}^2)}) = O(H^3). \tag{3.45}$$

(3.44) is proved by an argument similar to the proofs of Claim 1 and Claim 2 in the proof of Lemma 3.2. (3.45) is a consequence of the fact that $\|\nabla h_{\lambda_H,a}\|_{L^2(\Omega)} = O(\lambda_H) = O(H)$ (see the proof of Lemma 5.2) and

$$\begin{aligned}\|h_{\lambda_H,a}\|_{L^2(\mathbb{R}^2\setminus\Omega)} &= \left(\int_{\mathbb{R}^2\setminus\Omega} |\nabla \widehat{U}_{\lambda_H,a}|^2\,dx\right)^{1/2}\\ &= O\left(\left(\int_{\mathbb{R}^2\setminus\mathbb{B}_d(a)} \frac{\lambda_H^2}{(\lambda_H^2+|x-a|^2)^2}\,dx\right)^{1/2}\right) \quad (d=\operatorname{dist}(a,\partial\Omega))\\ &= O\left(\left(\int_d^\infty \frac{\lambda_H^2 r}{(\lambda_H^2+r^2)^2}\,dr\right)^{1/2}\right) = O(\lambda_H) = O(H).\end{aligned}$$

From (3.42)–(3.45), we have

$$\begin{aligned}Q(\varphi_{H,a}) &= -\frac{8}{S^3}\Big\{-4\pi + 6\pi\Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda_H^2\Big\} + o(H^2)\\ &= 1 - \frac{3}{2}\Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda_H^2 + o(H^2), \qquad (3.46)\end{aligned}$$

and from (3.40), (3.41) and (3.46), we have

$$\begin{aligned}J_H(\varphi_{H,a}) &= \Big[S\Big\{1 - \frac{1}{2}\Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda_H^2\Big\}\\ &\quad - S((h_\gamma)_{x_1}(a)\cdot R_H e_1 + (h_\gamma)_{x_2}(a)\cdot R_H e_2)\lambda_H H\Big]\\ &\quad \times\Big\{1 + \Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda_H^2\Big\} + o(H^2)\\ &= S + \frac{S}{2}\Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda_H^2\\ &\quad - S((h_\gamma)_{x_1}(a)\cdot R_H e_1 + (h_\gamma)_{x_2}(a)\cdot R_H e_2)\lambda_H H + o(H^2)\\ &= S - \frac{S}{2}\frac{|\nabla h_\gamma(a)|^2 + 2|(h_\gamma)_{x_1}(a)\wedge(h_\gamma)_{x_2}(a)|}{\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)}H^2 + o(H^2)\\ &= S - \frac{S}{2}K(a,\Omega)H^2 + o(H^2),\end{aligned}$$

where the third equality follows from our choices of λ_H and R_H (see (3.38) and (3.39)). This completes the proof of Lemma 3.4. □

Lemma 3.4 proves (1.8). Next we prove (1.9). For this purpose, we introduce the following spaces:

$$W := \Big\{\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^3) : \int_{\mathbb{R}^2} \Big(|\nabla\phi|^2 + \frac{|\phi|^2}{(1+|x|^2)^2}\Big)\,dx < \infty\Big\},$$

$$W(R\widehat{U}_{\lambda,a}) = \operatorname{span}\Big\langle R\widehat{U}_{\lambda,a}, R\frac{\partial \widehat{U}_{\lambda,a}}{\partial a_1}, R\frac{\partial \widehat{U}_{\lambda,a}}{\partial a_2}, R\frac{\partial \widehat{U}_{\lambda,a}}{\partial \lambda}, R\xi_i\widehat{U}_{\lambda,a\,(i=1,2,3)}\Big\rangle,$$

where $R \in SO(3)$, $\lambda > 0$ and $a \in \mathbb{R}^2$,

$$W(R\widehat{U}_{\lambda,a})^{\perp} = \Big\{\phi \in W : \int_{\mathbb{R}^2} \nabla\phi\cdot\nabla\varphi\,dx = 0 \quad \forall\varphi \in W(R\widehat{U}_{\lambda,a})\Big\}.$$

Note that $\phi \in W$ is equivalent to the condition $\phi\circ\Pi \in H^1(\mathbb{S}^2;\mathbb{R}^3)$, where $\Pi : \mathbb{S}^2 \to \mathbb{R}^2 \cup \{\infty\}$ is the stereographic projection from the north pole.

In [12] (see also [13]), Sasahara proved that the kernel of the linearized operator of $u \mapsto -\Delta u + 2u_{x_1}\wedge u_{x_2}$ at $R\widehat{U}_{\lambda,a}$ in W is $W(R\widehat{U}_{\lambda,a}) \oplus \mathbb{R}^3$.

Lemma 3.5. *The blow-up point a_∞ is in the interior of Ω, and there exists $C > 0$ (independent of n) such that $C^{-1} \le \frac{\lambda_n}{H_n} \le C$.*

Proof. Assume that $a_\infty \in \partial\Omega$; i.e., $d_n(= d(a_n,\partial\Omega)) \to 0$ as $n \to \infty$. Then by Corollary 3.3, Lemma 5.5 and Lemma 5.7, there exist constants $C_1, C_2, C_3 > 0$ (independent of n) such that

$$\begin{aligned} J_{H_n} &= S + \frac{S}{2}\Big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\Big)\lambda_n^2 \\ &\quad - S((\underline{u}_n)_{x_1}(a_n)\cdot R_n e_1 + (\underline{u}_n)_{x_2}(a_n)\cdot R_n e_2)\lambda_n H_n \\ &\quad + \frac{4}{S^2}\Big(\int_\Omega |\nabla w_n|^2\,dx + 4\int_\Omega R_n\widehat{U}_{\lambda_n,a_n}\cdot(w_n)_{x_1}\wedge(w_n)_{x_2}\,dx\Big) \\ &\quad + o(\|\nabla w_n\|^2_{L^2(\Omega)}) + o\Big(\frac{\lambda_n^2}{d_n^2}\Big) + O\Big(\frac{\lambda_n^2 H_n}{d_n}|\log\lambda_n|\Big) \\ &\quad + O(H_n\lambda_n|\log\lambda_n|^{1/2}\|\nabla w_n\|_{L^2(\Omega)}) \\ &\ge S + C_1\frac{\lambda_n^2}{d_n^2} - C_2\lambda_n H_n + C_3\|\nabla w_n\|^2_{L^2(\Omega)} + o(H_n^2) \\ &\ge S - Cd_n^2H_n^2 + o(H_n^2) = S + o(H_n^2). \end{aligned} \tag{3.47}$$

Here we have used the fact that $C_1\frac{\lambda_n^2}{d_n^2} - C_2\lambda_n H_n \ge -Cd_n^2H_n^2$ for some $C > 0$.

On the other hand, Lemma 3.4 implies $J_{H_n} \leq S - CH_n^2$ for some $C > 0$. This contradicts (3.47). Therefore a_∞ is in the interior of Ω.

Since $d_n \geq C$ for some $C > 0$ independent of n, we may drop d_n in the expansion of J_{H_n} and obtain

$$\begin{aligned} J_{H_n} \geq S &+ \frac{S}{2}\Big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\Big)\lambda_n^2 \\ &- S((\underline{u}_n)_{x_1}(a_n) \cdot R_n e_1 + (\underline{u}_n)_{x_2}(a_n) \cdot R_n e_2)\lambda_n H_n \\ &+ C\|\nabla w_n\|^2_{L^2(\Omega)} + o(\lambda_n^2) + o(H_n^2) \\ &\geq S + C_4\lambda_n^2 - C_5\lambda_n H_n + o(H_n^2). \end{aligned} \tag{3.48}$$

Since $J_{H_n} \leq S - CH_n^2$, (3.48) gives us $C_4\lambda_n^2 - C_5\lambda_n H_n \leq -CH_n^2$. The second assertion of the lemma follows from this. $\square$

We now give the proof of (1.9). By Lemma 2.1, Corollary 3.3, Lemma 3.5 and Lemma 5.4, we have

$$\begin{aligned} J_{H_n} \geq S &+ \frac{S}{2}\Big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\Big)\lambda_n^2 \\ &- S(|\nabla h_\gamma(a_n)|^2 + 2|(h_\gamma)_{x_1}(a_n) \wedge (h_\gamma)_{x_2}(a_n)|)^{1/2}\lambda_n H_n + o(H_n^2). \end{aligned}$$

Since $a\lambda^2 + b\lambda H \geq -\frac{b^2}{4a}H^2$ if $a > 0$, we have

$$\begin{aligned} &\frac{S}{2}\Big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\Big)\lambda_n^2 \\ &- S(|\nabla h_\gamma(a_n)|^2 + 2|(h_\gamma)_{x_1}(a_n) \wedge (h_\gamma)_{x_2}(a_n)|)^{1/2}\lambda_n H_n \\ &\geq -\frac{S}{2}\frac{|\nabla h_\gamma(a_n)|^2 + 2|(h_\gamma)_{x_1}(a_n) \wedge (h_\gamma)_{x_2}(a_n)|}{\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)}H_n^2 = -\frac{S}{2}K(a_n, \Omega)H_n^2. \end{aligned}$$

Thus (1.9) is proved since $a_n \to a_\infty$. $\square$

In the next section, we prove Theorem A and Theorem B.

4. Proofs of Theorem A and Theorem B.

Proof of Theorem A. Theorem A is a direct consequence of (1.8) and (1.9). In fact, for any $a \in \Omega$ we have (by (1.8) and (1.9))

$$S - \frac{S}{2}K(a_\infty, \Omega)H_n^2 + o(H_n^2) \leq J_{H_n} \leq S - \frac{S}{2}K(a, \Omega)H_n^2 + o(H_n^2) \tag{4.1}$$

as $n \to \infty$. From (4.1), we have

$$K(a,\Omega) \leq K(a_\infty,\Omega). \tag{4.2}$$

In (4.2), $a \in \Omega$ is arbitrary. Thus we have $\sup_{a\in\Omega} K(a,\Omega) = K(a_\infty,\Omega)$. This proves Theorem A. $\square$

Proof of Theorem B. By Lemma 3.5, there exists $C > 0$ such that $C^{-1} \leq \frac{\lambda_n}{H_n} \leq C$ for all $n \in \mathbb{N}$. Thus there exist a subsequence of $\{\frac{\lambda_n}{H_n}\}$ (still denoted by $\{\frac{\lambda_n}{H_n}\}$) and $\beta > 0$ such that $\frac{\lambda_n}{H_n} \to \beta$. Then arguing as in the proof of (1.9), we have

$$\begin{aligned} J_{H_n} \geq S &+ \frac{S}{2}\Big(\frac{\partial h^1_{a_\infty}}{\partial x_1}(a_\infty) + \frac{\partial h^2_{a_\infty}}{\partial x_2}(a_\infty)\Big)\beta^2 H_n^2 \\ &- S(|\nabla h_\gamma(a_\infty)|^2 + 2|(h_\gamma)_{x_1}(a_\infty) \wedge (h_\gamma)_{x_2}(a_\infty)|)^{1/2}\beta H_n^2 + o(H_n^2). \end{aligned} \tag{4.3}$$

Combining (4.3) with Lemma 3.4 (applied to $a = a_\infty$), we have

$$\begin{aligned} &\frac{S}{2}\Big(\frac{\partial h^1_{a_\infty}}{\partial x_1}(a_\infty) + \frac{\partial h^2_{a_\infty}}{\partial x_2}(a_\infty)\Big)\beta^2 \\ &- S(|\nabla h_\gamma(a_\infty)|^2 + 2|(h_\gamma)_{x_1}(a_\infty) \wedge (h_\gamma)_{x_2}(a_\infty)|)^{1/2}\beta \\ \leq\ & -\frac{S}{2}\frac{|\nabla h_\gamma(a_\infty)|^2 + 2|(h_\gamma)_{x_1}(a_\infty) \wedge (h_\gamma)_{x_2}(a_\infty)|}{\frac{\partial h^1_{a_\infty}}{\partial x_1}(a_\infty) + \frac{\partial h^2_{a_\infty}}{\partial x_2}(a_\infty)}. \end{aligned} \tag{4.4}$$

(4.4) implies

$$\beta = \frac{(|\nabla h_\gamma(a_\infty)|^2 + 2|(h_\gamma)_{x_1}(a_\infty) \wedge (h_\gamma)_{x_2}(a_\infty)|)^{1/2}}{\frac{\partial h^1_{a_\infty}}{\partial x_1}(a_\infty) + \frac{\partial h^2_{a_\infty}}{\partial x_2}(a_\infty)}. \tag{4.5}$$

Thus (1.6) is proved for some subsequence of $\{\lambda_n\}$ and $\{H_n\}$. However, the uniqueness of the possible limit implies that (1.6) holds for the full sequence.

Next we prove (1.7). Set $v_n := H_n\overline{u}_n$. Then v_n satisfies

$$\begin{cases} \Delta v_n = 2(v_n)_{x_1} \wedge (v_n)_{x_2} & \text{in } \Omega \\ v_n = H_n\gamma & \text{on } \partial\Omega. \end{cases}$$

Define $(\overline{\lambda}_n)^{-1} := \|\nabla v_n\|_{L^\infty(\Omega)} = H_n\|\nabla\overline{u}_n\|_{L^\infty(\Omega)}$. Then $\overline{\lambda}_n \to 0$ (as $n \to \infty$). As in [5], we have

$$\Big\|\nabla\Big(v_n - R\widehat{U}_{\lambda,a}\Big(\frac{\cdot - \overline{a}_n}{\overline{\lambda}_n}\Big)\Big)\Big\|_{L^2(\Omega)} \to 0$$

for some $R \in SO(3)$, $\lambda > 0$, $a \in \mathbb{R}^2$, where $\overline{a}_n \in \overline{\Omega}$ is such that $|\nabla v_n(\overline{a}_n)| = (\overline{\lambda}_n)^{-1}$. Moreover, $\overline{\lambda}_n/d(\overline{a}_n, \partial\Omega) \to 0$ and $(v_n)_{\overline{\lambda}_n,\overline{a}_n} := v_n(\overline{\lambda}_n \cdot + \overline{a}_n) \to R\widehat{U}_{\lambda,a}$ in $C^k_{\rm loc}(\mathbb{R}^2)$ for any $k \geq 1$. Since $|\nabla((v_n)_{\overline{\lambda}_n.\overline{a}_n})(0)| = \overline{\lambda}_n|\nabla v_n(\overline{a}_n)| = 1$ and $\|\nabla(v_n)_{\overline{\lambda}_n,\overline{a}_n}\|_{L^\infty(\overline{\lambda}_n^{-1}(\Omega-\overline{a}_n))} \leq 1$, we have

$$|\nabla(R\widehat{U}_{\lambda,a})(0)| = 1, \quad \|\nabla(R\widehat{U}_{\lambda,a})\|_{L^\infty(\mathbb{R}^2)} \leq 1. \tag{4.6}$$

Since $|\nabla(R\widehat{U}_{\lambda,a})(x)| = \frac{2\sqrt{2}\lambda}{\lambda^2+|x-a|^2}$, (4.6) implies $a = 0$ and $\lambda = 2\sqrt{2}$.

Let α_n, R_n, λ_n and a_n be as in the proof of Theorem A. Then

$$\left\|\nabla\Big(RU_{2\sqrt{2},0}\Big(\frac{\cdot - \overline{a}_n}{\overline{\lambda}_n}\Big) - \alpha_n R_n U_{\lambda_n,a_n}\Big)\right\|_{L^2(\Omega)} \to 0. \tag{4.7}$$

¿From (4.7) (as in the proof of Lemma 5.1),

$$\frac{\lambda_n}{2\sqrt{2}\overline{\lambda}_n} = 1 + o(1). \tag{4.8}$$

Combining (4.8) with (1.6), we obtain (1.7). □

Remark 4.1. Arguing as above, if $R_n \to R_\infty$, then R_∞ satisfies

$$\begin{aligned} &(h_\gamma)_{x_1}(a_\infty)\cdot R_\infty e_1 + (h_\gamma)_{x_2}(a_\infty)\cdot R_\infty e_2 \\ &= (|\nabla h_\gamma|^2(a_\infty) + 2|(h_\gamma)_{x_1}(a_\infty)\wedge(h_\gamma)_{x_2}(a_\infty)|)^{1/2} \\ &= \max_{R\in SO(3)}\{(h_\gamma)_{x_1}(a_\infty)\cdot Re_1 + (h_\gamma)_{x_2}(a_\infty)\cdot Re_2\}. \end{aligned}$$

We end this section by considering the special case $\Omega = \mathbb{B}$.

Example 4.2. We explicitly calculate h^i_a $(i = 1, 2)$. First consider the problem

$$\begin{cases} \Delta_x H(a,x) = 0 & \text{in } \Omega, \\ H(a,x) = \log|x-a|^2 & \text{on } \partial\Omega. \end{cases} \tag{4.9}$$

Then $h^i_a(x) = -\frac{\partial H}{\partial a_i}(a,x)$. The solution of (4.9) is given by the Poisson integral formula:

$$H(a,x) = \frac{(1-|x|^2)}{2\pi}\int_{\partial\Omega}\frac{\log|z-a|^2}{|z-a|^2}\,dz.$$

An explicit calculation shows that

$$H(a,x) = 2\log\left|x - \frac{a}{|a|^2}\right| + 2\log|a|, \tag{4.10}$$

$$h_a^1(x) = \frac{2(|a|^2x_1 + a_1 - 2(a\cdot x)a_1)}{||a|^2x - a|^2} - \frac{2a_1}{|a|^2} \tag{4.11}$$

$$h_a^2(x) = \frac{2(|a|^2x_2 + a_2 - 2(a\cdot x)a_2)}{||a|^2x - a|^2} - \frac{2a_2}{|a|^2}. \tag{4.12}$$

Thus we have

$$\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a) = \frac{4}{(|a|^2-1)^2}. \tag{4.13}$$

In (4.10)–(4.13), we have assumed $a \neq 0$; however, (4.13) also holds for $a = 0$ since $h_0^i(x) = 2x_i$. Thus when $\Omega = \mathbb{B}$,

$$K(a,\mathbb{B}) = \frac{1}{4}(1-|a|^2)^2(|\nabla h_\gamma(a)|^2 + 2|(h_\gamma)_{x_1}(a)\wedge(h_\gamma)_{x_2}(a)|),$$

and Theorem A recovers the result of Sasahara [12]. Moreover, in this case

$$\lim_{n\to\infty}\frac{\lambda_n}{H_n} = \frac{1}{4}(1-|a_\infty|^2)^2(|\nabla h_\gamma(a_\infty)|^2 + 2|(h_\gamma)_{x_1}(a_\infty)\wedge(h_\gamma)_{x_2}(a_\infty)|)^{1/2},$$

$$\lim_{n\to\infty} H_n^2\|\nabla\overline{u}_n\|_{L^\infty(\Omega)}$$

$$= \frac{8\sqrt{2}}{(1-|a_\infty|^2)^2(|\nabla h_\gamma(a_\infty)|^2 + 2|(h_\gamma)_{x_1}(a_\infty)\wedge(h_\gamma)_{x_2}(a_\infty)|)^{1/2}}.$$

5. Technical lemmas.

Lemma 5.1. *There exists* $\epsilon_0 > 0$ *such that* (3.8) *has a unique solution for all* $0 < \epsilon \le \epsilon_0$.

Proof. The proof of uniqueness needs some calculations given in part II [8], so it is proved in part II. Here we prove only the existence of a solution to (3.8). Let $(\alpha_k, R_k, \lambda_k, a_k) \in (\frac{1}{2}, 2)\times SO(3)\times(0,\infty)\times\Omega$ be a minimizing sequence. Since $v \in M(\epsilon)$, there exist $R \in SO(3)$, $a \in \Omega$ and $\lambda > 0$ such that

$$\|\nabla(v - RPU_{\lambda,a})\|_{L^2(\Omega)} < \epsilon, \quad \lambda/d(a,\partial\Omega) < \epsilon. \tag{5.1}$$

So we have

$$\begin{aligned}\|\nabla(v - \alpha_k R_k PU_{\lambda_k,a_k})\|_{L^2(\Omega)} &\le \|\nabla(v - RPU_{\lambda,a})\|_{L^2(\Omega)} + o_k(1)\\ &\le \epsilon + o_k(1)\end{aligned} \tag{5.2}$$

as $k \to \infty$. Here $o_k(1) \to 0$ as $k \to \infty$. (5.2) implies

$$\begin{aligned}&\|\nabla v\|^2_{L^2(\Omega)} - 2\alpha_k(\nabla v, R_k \nabla(PU_{\lambda_k,a_k}))_{L^2(\Omega)} + \alpha_k^2 \|\nabla(PU_{\lambda_k,a_k})\|^2_{L^2(\Omega)} \\ &\le \epsilon^2 + o_k(1).\end{aligned} \tag{5.3}$$

¿From Lemma 5.2 and Lemma 5.7, we have

$$\|\nabla(PU_{\lambda,a})\|^2_{L^2(\Omega)} = 8\pi + O\Big(\frac{\lambda^2}{d^2}\Big) \quad (d = d(a, \partial\Omega)). \tag{5.4}$$

(5.1), (5.4), and the fact that $\lambda_k d_k^{-1} \le 2\epsilon$ ($d_k = d(a_k, \partial\Omega)$) and the fact that $\lambda d^{-1} < \epsilon$ imply

$$\|\nabla(PU_{\lambda_k,a_k})\|^2_{L^2(\Omega)} = 8\pi + O(\epsilon^2), \quad \|\nabla v\|^2_{L^2(\Omega)} = 8\pi + O(\epsilon). \tag{5.5}$$

¿From (5.3) and (5.5), we have

$$\alpha_k = 1 + O(\epsilon) + o_k(1). \tag{5.6}$$

Since $\{\alpha_k\}$ is bounded, we may assume (passing to a subsequence if necessary) $\alpha_k \to \alpha_\infty$. Then by (5.6), we have

$$\alpha_\infty = 1 + O(\epsilon). \tag{5.7}$$

In particular, if $\epsilon > 0$ is small, $\alpha_\infty \in (\frac{1}{2}, 2)$.

On the other hand, assume that $\lambda_k \to \infty$ or $\lambda_k \to 0$. Then $R_k \nabla U_{\lambda_k,a_k} \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^2)$ and

$$(\nabla v, R_k \nabla(PU_{\lambda_k,a_k}))_{L^2(\Omega)} = (\nabla v, R_k \nabla U_{\lambda_k,a_k})_{L^2(\Omega)} \to 0.$$

This contradicts (5.3) if $\epsilon > 0$ is small. Thus there exists $\lambda_\infty > 0$ such that (for some subsequence of $\{\lambda_k\}$) $\lambda_k \to \lambda_\infty$.

Since $SO(3)$ and $\overline{\Omega}$ are compact, there exist $R_\infty \in SO(3)$ and $a_\infty \in \overline{\Omega}$ such that (for some subsequence) $R_n \to R_\infty$ and $a_n \to a_\infty$.

It only remains to show that $\lambda_\infty / d(a_\infty, \partial\Omega) < 2\epsilon$. If this is shown, $(\alpha_\infty, R_\infty, \lambda_\infty, a_\infty)$ is a minimizer for (3.8).

We already know that $\lambda_\infty / d(a_\infty, \partial\Omega) \le 2\epsilon$. Assume that there exist a sequence $\{\epsilon_k\} \downarrow 0$, $v_k \in M(\epsilon_k)$ and $(\alpha_\infty^k, R_\infty^k, \lambda_\infty^k, a_\infty^k) \in (\frac{1}{2}, 2) \times SO(3) \times (0, \infty) \times \Omega$ obtained by the above argument for $v = v_k$ such that $\lambda_\infty^k/d(a_\infty^k, \partial\Omega) = 2\epsilon_k$. Since $v_k \in M(\epsilon_k)$, there exists $(R'_k, \lambda'_k, a'_k) \in SO(3) \times (0,\infty) \times \Omega$ such that

$$\|\nabla(v_k - R'_k PU_{\lambda'_k,a'_k})\|_{L^2(\Omega)} < \epsilon_k, \quad \lambda'_k/d(a'_k, \partial\Omega) < \epsilon_k. \tag{5.8}$$

Since $\|\nabla(v_k - \alpha_\infty^k R_\infty^k PU_{\lambda_\infty^k,a_\infty^k})\|_{L^2(\Omega)} \le \epsilon_k$, (5.8) implies

$$\|\nabla(\alpha_\infty^k R_\infty^k PU_{\lambda_\infty^k,a_\infty^k} - R'_k PU_{\lambda'_k,a'_k})\|_{L^2(\Omega)} < 2\epsilon_k. \tag{5.9}$$

Since $\alpha_\infty^k = 1 + O(\epsilon_k^2) = 1 + o_k(1)$ (see (5.7)), (5.9) implies

$$\|\nabla(R_\infty^k PU_{\lambda_\infty^k,a_\infty^k} - R'_k PU_{\lambda'_k,a'_k})\|_{L^2(\Omega)} = o_k(1). \tag{5.10}$$

(5.10) implies

$$\|\nabla(R_\infty^k U_{\lambda_\infty^k,a_\infty^k} - R'_k U_{\lambda'_k,a'_k})\|_{L^2(\mathbb{R}^2)} = o_k(1). \tag{5.11}$$

¿From (5.11), we have

$$(R_\infty^k)^{-1} R'_k = I + o_k(1), \quad \frac{\lambda'_k}{\lambda_\infty^k} = 1 + o_k(1), \quad \frac{a_\infty^k - a'_k}{\lambda_\infty^k} = o_k(1). \tag{5.12}$$

(5.12) implies

$$\begin{aligned}\frac{1}{2\epsilon_k} &= (\lambda_\infty^k)^{-1} d(a_\infty^k, \partial\Omega) \ge (\lambda_\infty^k)^{-1} d(a'_k, \partial\Omega) - (\lambda_\infty^k)^{-1}|a_\infty^k - a'_k| \\ &\ge (1 + o_k(1))(\lambda'_k)^{-1} d(a'_k, \partial\Omega) + o_k(1) \ge (1 + o_k(1))\frac{1}{\epsilon_k} + o_k(1).\end{aligned}$$

This is a contradiction if k is large enough. Thus we have $\lambda_\infty/d(a_\infty, \partial\Omega) < 2\epsilon$. □

Lemma 5.2. *As $n \to \infty$, we have*

$$\int_\Omega |\nabla PU_{\lambda_n,a_n}|^2\, dx = 8\pi - 4\pi\Big(\frac{\partial h_{a_n}^1}{\partial x_1}(a_n) + \frac{\partial h_{a_n}^2}{\partial x_2}(a_n)\Big)\lambda_n^2 + O\Big(\frac{\lambda_n^3}{d_n^3}\Big).$$

Proof. We have ($h_n := h_{\lambda_n,a_n}$)

$$\begin{aligned}\int_\Omega |\nabla PU_{\lambda_n,a_n}|^2 dx &= \int_\Omega |\nabla \widehat{U}_{\lambda_n,a_n} - \nabla h_n|^2 dx = -\int_\Omega \Delta \widehat{U}_{\lambda_n,a_n}(\widehat{U}_{\lambda_n,a_n} - h_n)\, dx \\ &= -2\int_\Omega (\widehat{U}_{\lambda_n,a_n})_{x_1} \wedge (\widehat{U}_{\lambda_n,a_n})_{x_2} \cdot (\widehat{U}_{\lambda_n,a_n} - h_n)\, dx \qquad (5.13) \\ &= \int_\Omega |\nabla U_{\lambda_n,a_n}|^2 U_{\lambda_n,a_n} \cdot (U_{\lambda_n,a_n} - e_3 - h_n)\, dx \\ &= \int_\Omega |\nabla U_{\lambda_n,a_n}|^2 dx - \int_\Omega |\nabla U_{\lambda_n,a_n}|^2 U_{\lambda_n,a_n} e_3\, dx - \int_\Omega |\nabla U_{\lambda_n,a_n}|^2 U_{\lambda_n,a_n} h_n\, dx.\end{aligned}$$

Here we have used the fact that

$$-2(U_{\lambda_n,a_n})_{x_1} \wedge (U_{\lambda_n,a_n})_{x_2} = |\nabla U_{\lambda_n,a_n}|^2 U_{\lambda_n,a_n}. \tag{5.14}$$

Here

$$\int_\Omega |\nabla U_{\lambda_n,a_n}|^2\,dx = \int_\Omega \frac{8\lambda_n^2}{(\lambda_n^2+r^2)^2}\,dx\ (r=|x-a_n|) \tag{5.15}$$

$$= 8\pi - \int_{\mathbb{R}^2\setminus\Omega} \frac{8\lambda_n^2}{(\lambda_n^2+r^2)^2}\,dx = 8\pi - 8\lambda_n^2 \int_{\mathbb{R}^2\setminus\Omega} \frac{1}{r^4}\,dx + O\Big(\frac{\lambda_n^4}{d_n^4}\Big),$$

$$\int_\Omega |\nabla U_{\lambda_n,a_n}|^2 U_{\lambda_n,a_n}\cdot e_3 = \int_\Omega \frac{8\lambda_n^2(r^2-\lambda_n^2)}{(\lambda_n^2+r^2)^3}\,dx$$

$$= -\int_{\mathbb{R}^2\setminus\Omega} \frac{8\lambda_n^2(r^2-\lambda_n^2)}{(\lambda_n^2+r^2)^3}\,dx \quad \Big(\text{since} \int_{\mathbb{R}^2} \frac{8\lambda_n^2(r^2-\lambda_n^2)}{(\lambda_n^2+r^2)^3}\,dx = 0\Big)$$

$$= -8\lambda_n^2 \int_{\mathbb{R}^2\setminus\Omega} \frac{1}{r^4}\,dx + O\Big(\frac{\lambda_n^4}{d_n^4}\Big) \tag{5.16}$$

and

$$\int_\Omega |\nabla U_{\lambda_n,a_n}|^2 U_{\lambda_n,a_n}\cdot h_n\,dx = \int_\Omega \frac{16\lambda_n^3}{(\lambda_n^2+r^2)^3}(x_1-a_n^1)h_n^1$$

$$+ \int_\Omega \frac{16\lambda_n^3}{(\lambda_n^2+r^2)^3}(x_2-a_n^2)h_n^2\,dx + \int_\Omega \frac{8\lambda_n^2(r^2-\lambda_n^2)}{(\lambda_n^2+r^2)^3}h_n^3\,dx$$

$$= 4\pi\Big(\frac{\partial h_n^1}{\partial x_1}(a_n) + \frac{\partial h_n^2}{\partial x_2}(a_n)\Big)\lambda_n + O\Big(\frac{\lambda_n^3}{d_n^3}\Big)$$

$$= 4\pi\Big(\frac{\partial h_{a_n}^1}{\partial x_1}(a_n) + \frac{\partial h_{a_n}^2}{\partial x_2}(a_n)\Big)\lambda_n^2 + O\Big(\frac{\lambda_n^3}{d_n^3}\Big), \tag{5.17}$$

where $a_n = (a_n^1, a_n^2)$, $h_n = (h_n^1, h_n^2, h_n^3)$.

In calculating (5.17), we have used an argument similar to that used in (3.18), (3.19), (3.20) and the estimates

$$\|\nabla^2 h_n^i\|_{L^\infty(\mathbb{B}_{d_n/2}(a_n))} = O\Big(\frac{\lambda_n}{d_n^3}\Big),$$

$$\nabla h_n^i(a_n) - \lambda_n \nabla h_{a_n}^i(a_n) = O\Big(\frac{\lambda_n^2}{d_n^3}\Big)\ (i=1,2), \tag{5.18}$$

$$\|h_n^3\|_{L^\infty(\Omega)} = O\Big(\frac{\lambda_n^2}{d_n^2}\Big), \quad \|\nabla^2 h_n^3\|_{L^\infty(\mathbb{B}_{d_n/2}(a_n))} = O\Big(\frac{\lambda_n^2}{d_n^4}\Big).$$

(5.18) is a consequence of $\|h_n^i\|_{L^\infty(\Omega)} = O\left(\frac{\lambda_n}{d_n}\right)$, $\|h_n^i - \lambda_n h_{a_n}^i\|_{L^\infty(\Omega)} = O\left(\frac{\lambda_n^2}{d_n^2}\right)$ and the elliptic estimates $d_n^2\|\nabla^2 h_n^i\|_{L^\infty(\mathbb{B}_{d_n/2}(a_n))} \le C\|h_n^i\|_{L^\infty(\Omega)}$, $d_n|\nabla h_n^i(a_n) - \lambda_n \nabla h_{a_n}^i(a_n)| \le C\|h_n^i - \lambda_n h_{a_n}^i\|_{L^\infty(\Omega)}$ and

$$d_n^2\|\nabla^2 h_n^3\|_{L^\infty(\mathbb{B}_{d_n/2}(a_n))} \le C\|h_n^3\|_{L^\infty(\Omega)}$$

since h_n^i, h_{a_n} and h_n^3 are harmonic in Ω. From (5.15)–(5.17), we obtain the conclusion of Lemma 5.2. Note that the above calculation also shows that

$$\int_\Omega |\nabla h_n|^2\,dx + \int_{\mathbb{R}^2\setminus\Omega} |\nabla U_{\lambda_n,a_n}|^2\,dx = 4\pi\Big(\frac{\partial h_{a_n}^1}{\partial x_1}(a_n) + \frac{\partial h_{a_n}^2}{\partial x_2}(a_n)\Big)\lambda_n^2 + o\Big(\frac{\lambda_n^2}{d_n^2}\Big).$$

¿From this, we have, in particular,

$$\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a) > 0 \quad \text{for } a \in \Omega.$$

Lemma 5.3. *Assume $u, v, w \in W$ (for the definition of W, see Section 3). We then have*

$$\int_{\mathbb{R}^2} u\cdot(v_{x_1}\wedge w_{x_2} + w_{x_1}\wedge v_{x_2})\,dx = \int_{\mathbb{R}^2} v\cdot(u_{x_1}\wedge w_{x_2} + w_{x_1}\wedge u_{x_2})\,dx. \quad (5.19)$$

$$\Big|\int_{\mathbb{R}^2} w\cdot(u_{x_1}\wedge u_{x_2})\,dx\Big| \le C\|\nabla w\|_{L^2(\mathbb{R}^2)}\|\nabla u\|_{L^2(\mathbb{R}^2)}^2. \quad (5.20)$$

Proof. When one of u, v, and w has compact support, these are deduced from the results of Brezis-Coron [4]. The general case follows from a density argument. □

Lemma 5.4. *Let $a, b \in \mathbb{R}^3$. Then* $\max_{R\in SO(3)}\{a\cdot Re_1 + b\cdot Re_2\} = (|a|^2 + |b|^2 + 2|a\wedge b|)^{1/2}$.

Proof. Since this result (more precisely, the explicit form of maximizer) also plays an important role in part II, we give a proof in detail. We may assume (by rotating a and b if necessary) that $a = {}^t(a_1, 0, 0)$ and $b = {}^t(b_1, b_2, 0)$ with $a_1 b_2 \ge 0$. For $p, q \in \mathbb{R}^3$, denote by $\langle p, q\rangle$ the linear subspace of $\mathbb{R}^3$ spanned by p and q.

Step 1. If $R \in SO(3)$ satisfies $\langle a, b\rangle \subset \langle Re_1, Re_2\rangle$, then

$$a\cdot Re_1 + a\cdot Re_2 \le (|a|^2 + |b|^2 + 2|a\wedge b|)^{1/2}. \quad (5.21)$$

Proof of Step 1. If $b_2 = 0$, then (5.21) is trivially satisfied. So we assume $b_2 \neq 0$. Then $\langle Re_1, Re_2\rangle = \langle a, b\rangle = \langle e_1, e_2\rangle$ and $Re_3 = \pm e_3$. Thus R has the following form: $R = \begin{pmatrix} R' & 0 \\ 0 & \pm 1 \end{pmatrix}$, $R' = \begin{pmatrix} \pm\cos\alpha & \mp\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$. Then

$$\begin{aligned} a \cdot Re_1 + b \cdot Re_2 &= \pm a_1 \cos\alpha \mp b_1 \sin\alpha + b_2 \cos\alpha \le ((b_2 \pm a_1)^2 + b_1^2)^{1/2} \\ &= (|a|^2 + |b|^2 \pm 2|a \wedge b|)^{1/2} \le (|a|^2 + |b|^2 + 2|a \wedge b|)^{1/2}. \end{aligned}$$

Step 2. Completion of the proof: Let $R \in SO(3)$ be arbitrary. Let a_R be the orthogonal projection of a to $\langle Re_1, Re_2\rangle$: $a_R = (a \cdot Re_1)Re_1 + (a \cdot Re_2)Re_2$. Define b_R similarly. Then by Step 1

$$\begin{aligned} a \cdot Re_1 + b \cdot Re_2 = a_R \cdot Re_1 + b_R \cdot Re_2 &\le (|a_R|^2 + |b_R|^2 + 2|a_R \wedge b_R|)^{1/2} \\ &\le (|a|^2 + |b|^2 + 2|a \wedge b|)^{1/2}. \end{aligned}$$

Thus $\max_{R\in SO(3)}\{a \cdot Re_1 + b \cdot Re_2\} \le (|a|^2 + |b|^2 + 2|a \wedge b|)^{1/2}$. In fact, the argument in Step 1 shows that $\max_{R\in SO(3)}\{a \cdot Re_1 + b \cdot Re_2\} = (|a|^2 + |b|^2 + 2|a \wedge b|)^{1/2}$. Moreover, the maximum is attained by $R \in SO(3)$ with $\langle a, b\rangle = \langle Re_1, Re_2\rangle$. $\square$

The equivalent statement of the next lemma can be found in [13, Lemma 9], however, the proof given there seems to be incomplete, so we give a complete proof.

Lemma 5.5. *There exists a constant $C > 0$ (independent of R, λ and a) such that*

$$\int_{\mathbb{R}^2} |\nabla\phi|^2\, dx + 4\int_{\mathbb{R}^2} R\widehat{U}_{\lambda,a} \cdot \phi_{x_1} \wedge \phi_{x_2}\, dx \ge C\|\nabla\phi\|^2_{L^2(\mathbb{R}^2)}$$

for all $\phi \in W(R\widehat{U}_{\lambda,a})^\perp$.

Proof. Step 1. For all $\phi \in W$ with $\int_{\mathbb{R}^2} \nabla(R\widehat{U}_{\lambda,a}) \cdot \nabla\phi\, dx = 0$, we prove

$$\int_{\mathbb{R}^2} |\nabla\phi|^2\, dx + 4\int_{\mathbb{R}^2} R\widehat{U}_{\lambda,a} \cdot \phi_{x_1} \wedge \phi_{x_2}\, dx \ge 0.$$

In fact, by assumption we have, for any $t \in \mathbb{R}$,

$$\int_{\mathbb{R}^2} |\nabla(t\phi) + \nabla(R\widehat{U}_{\lambda,a})|^2\, dx = t^2 \int_{\mathbb{R}^2} |\nabla\phi|^2\, dx + \int_{\mathbb{R}^2} |\nabla(R\widehat{U}_{\lambda,a})|^2\, dx, \quad (5.22)$$

$$
\begin{aligned}
Q(t\phi + R\widehat{U}_{\lambda,a}) = &\int_{\mathbb{R}^2} (R\widehat{U}_{\lambda,a}) \cdot (R\widehat{U}_{\lambda,a})_{x_1} \wedge (R\widehat{U}_{\lambda,a})_{x_2}\, dx \\
&+ 3t \int_{\mathbb{R}^2} \phi \cdot (R\widehat{U}_{\lambda,a})_{x_1} \wedge (R\widehat{U}_{\lambda,a})_{x_2}\, dx \qquad (5.23)\\
&+ 3t^2 \int_{\mathbb{R}^2} R\widehat{U}_{\lambda,a} \cdot \phi_{x_1} \wedge \phi_{x_2}\, dx + t^3 \int_{\mathbb{R}^2} \phi \cdot \phi_{x_1} \wedge \phi_{x_2}\, dx \\
= &\, Q(R\widehat{U}_{\lambda,a}) + 3t^2 \int_{\mathbb{R}^2} R\widehat{U}_{\lambda,a} \cdot \phi_{x_1} \wedge \phi_{x_2}\, dx + O(t^3),
\end{aligned}
$$

where we have used the fact that

$$
\Delta(R\widehat{U}_{\lambda,a}) = 2(R\widehat{U}_{\lambda,a})_{x_1} \wedge (R\widehat{U}_{\lambda,a})_{x_2} \qquad (5.24)
$$

and the assumption $\int_{\mathbb{R}^2} \nabla(R\widehat{U}_{\lambda,a}) \cdot \nabla\phi\, dx = 0$. By the isoperimetric inequality

$$
\frac{\int_{\mathbb{R}^2} |\nabla(t\phi) + \nabla(R\widehat{U}_{\lambda,a})|^2\, dx}{|Q(t\phi + R\widehat{U}_{\lambda,a})|^{2/3}} \geq S, \qquad \frac{\int_{\mathbb{R}^2} |\nabla(R\widehat{U}_{\lambda,a})|^2\, dx}{|Q(R\widehat{U}_{\lambda,a})|^{2/3}} = S,
$$

(5.22) and (5.23), we have

$$
t^2 \int_{\mathbb{R}^2} |\nabla\phi|^2\, dx + 4t^2 \int_{\mathbb{R}^2} R\widehat{U}_{\lambda,a} \cdot \phi_{x_1} \wedge \phi_{x_2}\, dx + O(t^3) \geq 0. \qquad (5.25)
$$

Since (5.25) holds for all $t \in \mathbb{R}$, we have

$$
\int_{\mathbb{R}^2} |\nabla\phi|^2\, dx + 4 \int_{\mathbb{R}^2} R\widehat{U}_{\lambda,a} \cdot \phi_{x_1} \wedge \phi_{x_2}\, dx \geq 0.
$$

Step 2. For $\phi \in W(R\widehat{U}_{\lambda,a})^{\perp}$ with $\phi \not\equiv$ const.,

$$
\int_{\mathbb{R}^2} |\nabla\phi|^2\, dx + 4 \int_{\mathbb{R}^2} R\widehat{U}_{\lambda,a} \cdot \phi_{x_1} \wedge \phi_{x_2}\, dx > 0.
$$

($\because$) Assume that there exists $\phi \in W(R\widehat{U}_{\lambda,a})^{\perp}$ with $\phi \not\equiv$ const. such that

$$
\int_{\mathbb{R}^2} |\nabla\phi|^2\, dx + 4 \int_{\mathbb{R}^2} R\widehat{U}_{\lambda,a} \cdot \phi_{x_1} \wedge \phi_{x_2}\, dx = 0. \qquad (5.26)
$$

We may assume without loss of generality $\int_{\mathbb{R}^2} \frac{\phi}{(1+|x|^2)^2}\, dx = 0$ (otherwise, we consider $\phi - \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\phi}{(1+|x|^2)^2}\, dx$).

By Step 1, for any $\epsilon \in \mathbb{R}$ and any $\varphi \in W$ with $\int_{\mathbb{R}^2} \nabla\varphi \cdot \nabla(R\widehat{U}_{\lambda,a})\,dx = 0$ and $\int_{\mathbb{R}^2} \frac{\varphi}{(1+|x|^2)^2}\,dx = 0$, we have

$$\int_{\mathbb{R}^2} |\nabla\phi + \epsilon\nabla\varphi|^2\,dx + 4\int_{\mathbb{R}^2} R\widehat{U}_{\lambda,a} \cdot (\phi+\epsilon\varphi)_{x_1} \wedge (\phi+\epsilon\varphi)_{x_2}\,dx \geq 0. \quad (5.27)$$

(5.26) and (5.27) imply

$$\epsilon\biggl(\int_{\mathbb{R}^2} \nabla\phi\cdot\nabla\varphi\,dx + 2\int_{\mathbb{R}^2} R\widehat{U}_{\lambda,a}\cdot(\phi_{x_1}\wedge\varphi_{x_2} + \varphi_{x_1}\wedge\phi_{x_2})\,dx\biggr) + O(\epsilon^2) \geq 0.$$

Since $\epsilon \in \mathbb{R}$ is arbitrary, we have (using (5.19))

$$\begin{aligned}&\int_{\mathbb{R}^2} \nabla\phi\cdot\nabla\varphi\,dx + 2\int_{\mathbb{R}^2} R\widehat{U}_{\lambda,a}\cdot(\phi_{x_1}\wedge\varphi_{x_2} + \varphi_{x_1}\wedge\phi_{x_2})\,dx \quad (5.28)\\ &= \int_{\mathbb{R}^2} \nabla\phi\cdot\nabla\varphi\,dx + 2\int_{\mathbb{R}^2} \varphi\cdot(\phi_{x_1}\wedge(R\widehat{U}_{\lambda,a})_{x_2} + (R\widehat{U}_{\lambda,a})_{x_1}\wedge\phi_{x_2})\,dx = 0.\end{aligned}$$

Define $\overline{W} := W \cap \bigl\{\phi : \int_{\mathbb{R}^2} \frac{\phi}{(1+|x|^2)^2}\,dx = 0\bigr\}$. Since $\int_{\mathbb{R}^2} \frac{\phi}{(1+|x|^2)^2}\,dx = 0$ is equivalent to $\int_{\mathbb{S}^2} \phi\circ\Pi\,dx = 0$, where $\Pi : \mathbb{S}^2 \to \mathbb{R}\cup\{\infty\}$ is the stereographic projection from the north pole, the Poincaré inequality implies that

$$\begin{aligned}\int_{\mathbb{R}^2} \Bigl(|\nabla\phi|^2 + \frac{|\phi|^2}{(1+|x|^2)^2}\Bigr)\,dx &\leq C\int_{\mathbb{S}^2} (|\nabla(\phi\circ\Pi)|^2 + |\phi\circ\Phi|^2)\,dx\\ &\leq C\int_{\mathbb{S}^2} |\nabla(\phi\circ\Pi)|^2\,dx \leq C\int_{\mathbb{S}^2} |\nabla\phi|^2\,dx. \quad (5.29)\end{aligned}$$

(5.29) shows that $(\cdot,\cdot) : \overline{W}\times\overline{W} \ni (\phi,\varphi) \mapsto \langle\phi,\varphi\rangle := \int_{\mathbb{R}^2} \nabla\phi\cdot\nabla\varphi\,dx$ defines a scalar product in $\overline{W}$. Since

$$(\phi,\varphi) \mapsto \int_{\mathbb{R}^2} \nabla\phi\cdot\nabla\varphi\,dx + 2\int_{\mathbb{R}^2} \varphi\cdot(\phi_{x_1}\wedge(R\widehat{U}_{\lambda,a})_{x_2} + (R\widehat{U}_{\lambda,a})_{x_1}\wedge\phi_{x_2})\,dx$$

is a continuous symmetric bilinear form on $\overline{W}\times\overline{W}$, there exists a bounded self-adjoint operator $L : \overline{W} \to \overline{W}$ $(= -\Delta\cdot + 2(\cdot)_{x_1}\wedge(R\widehat{U}_{\lambda,a})_{x_2} + 2(R\widehat{U}_{\lambda,a})_{x_1}\wedge(\cdot)_{x_2})$ such that

$$\langle L\phi,\varphi\rangle = \int_{\mathbb{R}^2} \nabla\phi\cdot\nabla\varphi\,dx + 2\int_{\mathbb{R}^2} \varphi\cdot(\phi_{x_1}\wedge(R\widehat{U}_{\lambda,a})_{x_2} + (R\widehat{U}_{\lambda,a})_{x_1}\wedge\phi_{x_2})\,dx.$$

Then by (5.28) and the fact that

$$\begin{aligned}\langle L\phi, R\widehat{U}_{\lambda,a}\rangle &= \int_{\mathbb{R}^2} \nabla\phi \cdot \nabla(R\widehat{U}_{\lambda,a})\, dx \\ &\quad + 2\int_{\mathbb{R}^2} R\widehat{U}_{\lambda,a} \cdot ((R\widehat{U}_{\lambda,a})_{x_1} \wedge \phi_{x_2} + \phi_{x_1} \wedge (R\widehat{U}_{\lambda,a})_{x_2})\, dx \\ &= -\int_{\mathbb{R}^2} \nabla\phi \cdot \nabla(R\widehat{U}_{\lambda,a})\, dx = 0,\end{aligned}$$

$L\phi = 0$ and $\phi \in W(R\widehat{U}_{\lambda,a})$. Since $\phi \in W(RU_{\lambda,a})^{\perp}$, we get $\phi \equiv 0$. This is a contradiction.

Step 3. We complete the proof of Lemma 5.5. Note that it is sufficient to prove that there exists at least one $C > 0$ which satisfies the condition of Lemma 5.5. Then the assertion of Lemma 5.5 also holds for other R', λ' and a' by the same constant C.

We prove the lemma by a contradiction argument. So assume that there exists a sequence $\{\phi_n\} \subset W(R\widehat{U}_{\lambda,a})^{\perp}$ such that

$$\int_{\mathbb{R}^2} |\nabla\phi_n|^2\, dx = 1, \tag{5.30}$$

$$\int_{\mathbb{R}^2} |\nabla\phi_n|^2\, dx + 4\int_{\mathbb{R}^2} R\widehat{U}_{\lambda,a} \cdot (\phi_n)_{x_1} \wedge (\phi_n)_{x_2}\, dx \to 0. \tag{5.31}$$

Since (5.30) and (5.31) are invariant by the translation $\phi_n \mapsto \phi_n - c$ ($c \in \mathbb{R}^3$ is a constant), we may assume $\int_{\mathbb{R}^2} \frac{\phi_n}{(1+|x|^2)^2}\, dx = 0$ for all n. Then as in (5.29), $\{\phi_n\}$ is $H^1(\mathbb{R}^2)$-bounded and there exist a subsequence of $\{\phi_n\}$ (we also denote it by $\{\phi_n\}$) and $\phi \in H^1(\mathbb{R}^2; \mathbb{R}^3)$ such that $\phi_n \rightharpoonup \phi$ weakly in $H^1(\mathbb{R}^2)$. Then by the weak lower semicontinuity of $\phi \mapsto \langle L\phi, \phi\rangle$ and (5.31), we have

$$0 \le \langle L\phi, \phi\rangle \le \liminf_{n\to\infty} \langle L\phi_n, \phi_n\rangle = 0.$$

Then by Step 2, $\phi \equiv 0$. Thus by Lemma 5.6 below, we have

$$\begin{aligned}o(1) &= \int_{\mathbb{R}^2} |\nabla\phi_n|^2\, dx + 4\int_{\mathbb{R}^2} R\widehat{U}_{\lambda,a} \cdot (\phi_n)_{x_1} \wedge (\phi_n)_{x_2}\, dx \\ &= \int_{\mathbb{R}^2} |\nabla\phi_n|^2\, dx + o(1) = 1 + o(1).\end{aligned}$$

This is a contradiction. This completes the proof. □

The following lemma is essentially proved in [4, Lemma A.9].

Lemma 5.6. *Assume $u \in W \cap L^\infty(\mathbb{R}^2)$ and let $\phi_n \in H^1(\mathbb{R}^2)$ be a sequence such that $\phi_n \rightharpoonup \phi$ weakly in $H^1(\mathbb{R}^2)$. Then*

$$\int_{\mathbb{R}^2} u \cdot (\phi_n)_{x_1} \wedge (\phi_n)_{x_2}\, dx \to \int_{\mathbb{R}^2} u \cdot \phi_{x_1} \wedge \phi_{x_2}\, dx.$$

Proof. It suffices to consider the case $\phi = 0$. For any $\epsilon > 0$, there exists $\tilde{u} \in C_0^\infty(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} |\nabla(u-\tilde{u})|^2\, dx + \int_{\mathbb{R}^2} \frac{|u-\tilde{u}|^2}{(1+|x|^2)^2}\, dx < \epsilon^2$. Then by (5.20), we have

$$\Big| \int_{\mathbb{R}^2} u \cdot (\phi_n)_{x_1} \wedge (\phi_n)_{x_2}\, dx - \int_{\mathbb{R}^2} \tilde{u} \cdot (\phi_n)_{x_1} \wedge (\phi_n)_{x_2}\, dx \Big| \le C\epsilon. \tag{5.32}$$

On the other hand, by (5.19)

$$\int_{\mathbb{R}^2} \tilde{u} \cdot (\phi_n)_{x_1} \wedge (\phi_n)_{x_2}\, dx = \frac{1}{2} \int_{\mathbb{R}^2} \phi_n \cdot (\tilde{u}_{x_1} \wedge (\phi_n)_{x_1} + (\phi_n)_{x_1} \wedge \tilde{u}_{x_2})\, dx \to 0. \tag{5.33}$$

Since $\epsilon > 0$ is arbitrary, (5.32) and (5.33) imply the assertion of Lemma 5.6. □

Lemma 5.7. *Set $d = dist(a, \partial\Omega)$. Then we have*

$$\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a) = \frac{1}{d^2} + o\Big(\frac{1}{d^2}\Big) \quad \text{as } d \to 0. \tag{5.34}$$

Proof. Assume d is small enough. Define $\Omega_d := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > d\}$. By definition, $a \in \partial\Omega_d$. Let ν be the outer normal of $\partial\Omega_d$ at a. Set $\overline{a} = a + 2d\nu$. Define $g^i(x) := \frac{2(x_i - \overline{a}_i - 2\nu_i(x-\overline{a})\cdot\nu)}{|x-\overline{a}|^2}$ $(i = 1, 2)$. g^i $(i = 1, 2)$ is harmonic in Ω. Moreover, $h_a^i(x) - g^i(x) = o\big(\frac{1}{d}\big)$ on $\partial\Omega$ (as $d \to 0$). Thus by the maximum principle, we have

$$\|h_a^i - g^i\|_{L^\infty(\Omega)} = o\Big(\frac{1}{d}\Big) \quad (\text{as } d \to 0). \tag{5.35}$$

Then by the elliptic estimate, we have (using (5.35))

$$d|\nabla h_a^i(a) - \nabla g^i(a)| \le C\|h_a^i - g^i\|_{L^\infty(\Omega)} = o\Big(\frac{1}{d}\Big). \tag{5.36}$$

(5.36) implies

$$\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a) = \frac{\partial g^1}{\partial x_1}(a) + \frac{\partial g^2}{\partial x_2}(a) + o\Big(\frac{1}{d^2}\Big) = \frac{1}{d^2} + o\Big(\frac{1}{d^2}\Big) \quad \text{as } d \to 0.$$

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