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LOCAL EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS FOR NONLINEAR PARABOLIC EQUATIONS WITH SUPERLINEAR GROWTH AND UNBOUNDED INITIAL DATA

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1. Introduction. A wide, and nowadays classical, literature has dealt with the Cauchy–Dirichlet problem for the superlinear heat equation:

$$\begin{cases} u_t - \Delta u = |u|^{p-1}u & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is an open, bounded subset of $\mathbf{R}^N$, $p > 1$ and u_0 belongs to a Lebesgue space $L^q(\Omega)$ for some $q \geq 1$. Well–known examples show that a global (in time) solution of (1.1) can not in general be expected, so problem (1.1) needs to be formulated inside a maximal interval $(0, T_{\max})$, where $T_{\max}$ depends on u_0. Since the works by F. B. Weissler ([10] and [11]), several authors have investigated different features of this problem under the assumption that u_0 only belongs to $L^q(\Omega)$, with $q < +\infty$ (see [6], [9], [2], [4]), obtaining existence, uniqueness and continuous-dependence results for (1.1) always in the framework of linear semigroup theory and for the so–called mild, or integral, solutions, namely if u satisfies

$$u(t) = T_t u_0 + \int_0^t T_{t-s}|u(s)|^{p-1}u(s)\, ds\,, \tag{1.2}$$

where T_t denotes the heat semigroup. A condition is required in the link between q and p in order to have existence, that is either $q > \frac{N(p-1)}{2}$ and

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$q \geq 1$, or $q = \frac{N(p-1)}{2}$ and $q > 1$; see [2] and Section 7.6 in [4] for examples showing that there exists at least a positive function u_0 belonging to $L^q(\Omega)$ for every $q < \frac{N(p-1)}{2}$ such that (1.1) admits no positive solution.

The recent paper [4] by H. Brezis and T. Cazenave, which motivated our work, provides a detailed treatment of problem (1.1) together with a new, stronger uniqueness result, proving that there exist a time $T(u_0)$ and a unique classical solution of (1.1) in $[0, T(u_0)]$ (i.e. u is C^1 in $t \in (0, T(u_0))$ and C^2 in $x \in \bar{\Omega}$) belonging to $C^0([0, T(u_0)]; L^q(\Omega))$. Furthermore, this solution has the following properties:

(i) if $u_0 \geq 0$, then $u(t) \geq 0$, for every $t \in [0, T(u_0)]$,

(ii) $\lim_{t \to 0^+} t^{\frac{N}{2q}} \|u(t)\|_{L^\infty(\Omega)} = 0$

(iii) $\|u(t) - v(t)\|_{L^q(\Omega)} + t^{\frac{N}{2q}} \|u(t) - v(t)\|_{L^\infty(\Omega)} \leq C \|u_0 - v_0\|_{L^q(\Omega)}$,

for every solution v having v_0 as initial datum and for all $t \leq \min(T(u_0), T(v_0))$, with C denoting a positive constant possibly depending on u_0 and v_0. The main novelty of this last result was to establish uniqueness without assuming that (ii) holds true, while (ii) is proved as an extra property of the solution.

The aim of this paper is to extend some of these results to a nonlinear setting, replacing the context of linear semigroup theory and mild solutions with strongly monotone divergence form operators and weak solutions (for this kind of approach see [8], [7], [5]). By means of nonlinear techniques of a priori estimates, we are able both to obtain an analogous theorem of existence, uniqueness and continuous dependence of local (in time) weak solutions for a more general class of operators, and to recover some of the results already contained in the literature from another point of view which also makes use of a slightly different functional setting. Let us stress that existence results in a similar framework can also be found in [1].

Let us now state more precisely our assumptions and results. We deal with the Cauchy-Dirichlet problem for the following class of nonlinear parabolic equations:

$$\begin{cases} u_t - \operatorname{div}(a(x,t,\nabla u)) = \rho(x,t)|u|^{p-1}u & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.3}$$

where Ω is an open, bounded subset of $\mathbf{R}^N$ $(N > 2)$, $p > 1$, ρ is a bounded function in $\Omega \times (0,T)$ and $a(x,t,\xi) : \Omega \times (0,T) \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function such that, for all ξ, η in $\mathbf{R}^N$, for almost every (x,t) in $\Omega \times (0,T)$,

(a_1) $(a(x,t,\xi)-a(x,t,\eta))(\xi-\eta)\geq\alpha|\xi-\eta|^2\,,\quad \alpha>0,$
(a_2) $a(x,t,0)=0\,,$
(a_3) $|a(x,t,\xi)|\leq\alpha'(k(x,t)+|\xi|)\,,\quad \alpha'>0,\quad k(x,t)\in L^2(\Omega\times(0,T))\,.$

The initial datum u_0 will be taken in $L^q(\Omega)$, with $q\geq\frac{N(p-1)}{2}$ and $q>1$. We will not deal with the case $q=1$ and $q>\frac{N(p-1)}{2}$.

Definition 1.1. A function u will be said to be a weak solution of (1.3) if u belongs to $C([0,T];L^q(\Omega))\cap L^2_{\rm loc}(0,T;H^1_0(\Omega))$ and satisfies

$$\int_\Omega[(u\varphi)(t_2)-(u\varphi)(t_1)]\,dx+\int_{t_1}^{t_2}\int_\Omega\{-u\varphi_\tau+a(x,\tau,\nabla u)\nabla\varphi\}\,dx\,d\tau$$
$$=\int_{t_1}^{t_2}\int_\Omega\rho(x,\tau)|u|^{p-1}u\varphi\,dx\,d\tau\,,$$

for every $0<t_1<t_2\leq T$ and all functions φ in $C^\infty([0,T]\times\bar\Omega)$ such that $\varphi=0$ on $\partial\Omega\times(0,T)$.

Remark 1.2. Note that since a weak solution u belongs to $L^\infty(0,T;L^q(\Omega))$, then $|u|^{p-1}$ belongs to $L^\infty(0,T;L^{\frac{q}{p-1}}(\Omega))$. Thus, using that $\frac{q}{p-1}\geq\frac{N}{2}$ and u is in $L^2_{\rm loc}(0,T;L^{\frac{2N}{N-2}}(\Omega))$, we easily get that $\rho|u|^{p-1}u$ belongs to $L^2_{\rm loc}(0,T;H^{-1}(\Omega))$. This implies that every weak solution u is such that u_t belongs to $L^2_{\rm loc}(0,T;H^{-1}(\Omega))$ and u also satisfies

$$\int_{t_1}^{t_2}{}_{H^{-1}(\Omega)}\langle u_\tau,\varphi\rangle_{H^1_0(\Omega)}\,d\tau+\int_{t_1}^{t_2}\int_\Omega a(x,\tau,\nabla u)\nabla\varphi\,dx\,d\tau$$
$$=\int_{t_1}^{t_2}\int_\Omega\rho(x,\tau)\,|u|^{p-1}u\varphi\,dx\,d\tau\,,\tag{1.4}$$

for every $0<t_1<t_2\leq T$ and all functions φ in $L^2_{\rm loc}(0,T;H^1_0(\Omega))$.

Our main result is to prove that (1.3) is locally (in time) well posed in the class of weak solutions. Let us stress that to our knowledge some of our results (especially as far as uniqueness is concerned) are new even in the simple case of linear operators of the type $-{\rm div}(A(x,t)\nabla u)$, where $A(x,t)$ is a bounded coercive matrix, since our techniques allow us to have a dependence on t in the operator.

Theorem 1.3. *Assume that $\rho(x,t)$ is a bounded function on $\Omega\times\mathbf{R}$, $N>2$, $p>1$ and u_0 belongs to $L^q(\Omega)$ with $q>\max\{1,\frac{N(p-1)}{2}\}$ (or $q=\frac{N(p-1)}{2}$ and $q>1$). Then there exists a time $T=T(u_0)$, depending on u_0 and $\|\rho\|_\infty$,* *such that* (1.3) *has a unique weak solution* u *in* $[0, T(u_0)]$. *This solution* u *has the following further properties:*

(i) *if* $u_0 \geq 0$, *then* $u(t) \geq 0$, *for every* $t \in [0, T(u_0)]$,

(ii) $\lim_{t\to 0^+} t^{\frac{N}{2q}} \|u(t)\|_{L^\infty(\Omega)} = 0$

(iii) $\|u(t) - v(t)\|_{L^q(\Omega)} + t^{\frac{N}{2q}} \|u(t) - v(t)\|_{L^\infty(\Omega)} \leq C\|u_0 - v_0\|_{L^q(\Omega)}$,

for any other solution v *having* v_0 *as initial datum and for all* $t \leq \min(T(u_0), T(v_0))$, *where* C *is a positive constant depending on* α, $|\Omega|$, N, p, q, *and also on* u_0 *or* v_0 *if* $q = \frac{N(p-1)}{2}$. *Finally, a uniform time* $T(B)$ *can be chosen for any initial datum* u_0 *belonging to a bounded (compact, if* $q = \frac{N(p-1)}{2}$ *and* $q > 1$*) subset* B *of* $L^q(\Omega)$.

Remark 1.4. The time $T(u_0)$ found in Theorem 1.3 should not be confused with the maximal time $T_{\max}$ of existence of the solution. Indeed, Theorem 1.3 provides the existence of a time interval $[0, T(u_0)]$ where (1.3) has a unique weak solution and states that $u(T(u_0))$ belongs to $L^q(\Omega)$, so that u can be extended beyond $T(u_0)$.

Remark 1.5. The assumption of boundedness of $\rho(x, t)$ for $t \in \mathbf{R}$ is not the sharpest one, but it allows a simpler statement; actually, $T = T(u_0)$ depends on ρ only through $\|\rho\|_{L^\infty(\Omega\times(0,T))}$. For instance, if $q > \max\{1, \frac{N(p-1)}{2}\}$ (1.3) admits a unique weak solution in $[0, T]$ if $T < \frac{C}{\|\rho\|_{L^\infty(\Omega\times(0,T))}\|u_0\|^{\gamma}_{L^q(\Omega)}}$, where $\gamma = \frac{2q(p-1)}{2q-N(p-1)}$, and C is a positive constant depending on α, q, p, N and $|\Omega|$.

Our method of proof, although it largely differs from the one in [4], owes to that paper the fundamental hint of turning to problem (1.3) after a careful study of the initial boundary value problem with linear growth:

$$\begin{cases} u_t - \operatorname{div}(a(x, t, \nabla u)) = \psi(x, t)u & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.5}$$

where ψ belongs to $L^\infty(0, T; L^\sigma(\Omega))$, $\sigma \geq \frac{N}{2}$. Most of our results will in fact be proved reasoning on (1.5), then applying the same methods to equation (1.3) where the term $|u|^{p-1}$ plays a role similar to the function ψ in (1.5). In some sense, this also offers a motivation for the bound appearing in the link between q and p, since a solution of (1.3) should belong to $L^\infty(0, T; L^q(\Omega))$;

hence $|u|^{p-1}$ belongs to $L^\infty(0,T;L^{\frac{q}{p-1}}(\Omega))$, and we have $\frac{q}{p-1} \geq \frac{N}{2}$. Thus we collect in the next theorem the results obtained on (1.5) for weak solutions defined as follows.

Definition 1.6. Let ψ be in $L^\infty(0,T;L^\sigma(\Omega))$, with $\sigma \geq \frac{N}{2}$, $T > 0$. A weak solution of (1.5) is a function u in $C([0,T];L^q(\Omega)) \cap L^2_{\mathrm{loc}}(0,T;H^1_0(\Omega))$ which satisfies

$$\int_\Omega [(u\varphi)(t_2) - (u\varphi)(t_1)]\,dx + \int_{t_1}^{t_2}\int_\Omega \{-u\varphi_\tau + a(x,\tau,\nabla u)\nabla\varphi\}\,dx\,d\tau$$
$$= \int_{t_1}^{t_2}\int_\Omega \psi(x,\tau)u\varphi\,dx\,d\tau\,,$$

for every $0 < t_1 < t_2 \leq T$ and all functions φ in $C^\infty([0,T]\times\bar{\Omega})$ such that $\varphi = 0$ on $\partial\Omega \times (0,T)$.

Let us note that the same arguments used in Remark 1.2 show that if u is a weak solution of (1.5) then u_t belongs to $L^2_{\mathrm{loc}}(0,T;H^{-1}(\Omega))$ and u satisfies

$$\int_{t_1}^{t_2} {}_{H^{-1}(\Omega)}\langle u_\tau, \varphi\rangle_{H^1_0(\Omega)}\,d\tau + \int_{t_1}^{t_2}\int_\Omega a(x,\tau,\nabla u)\nabla\varphi\,dx\,d\tau$$
$$= \int_{t_1}^{t_2}\int_\Omega \psi(x,\tau)\,u\varphi\,dx\,d\tau\,, \tag{1.6}$$

for every $0 < t_1 < t_2 < T$ and all functions φ in $L^2_{\mathrm{loc}}(0,T;H^1_0(\Omega))$.

Now we state the existence and uniqueness theorem for problem (1.5), in the case $\sigma > \frac{N}{2}$.

Theorem 1.7. *Assume that $N > 2$ and that ψ belongs to $L^\infty(0,T;L^\sigma(\Omega))$, with $\sigma > \frac{N}{2}$. Then, for every $T > 0$ and every u_0 in $L^q(\Omega)$ with $q > 1$, there exists a unique weak solution u of* (1.5). *Moreover this solution u satisfies, for a positive constant $C = C(\alpha,q,N,|\Omega|,\sigma)$, and for every t in $[0,T]$,*

$$\|u(t)\|_{L^q(\Omega)} \leq Ce^{Ct\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}}\|u_0\|_{L^q(\Omega)}\,,$$
$$\|u(t)\|_{L^r(\Omega)} \leq Ce^{Ct\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}}t^{-\frac{N(r-q)}{2qr}}\|u_0\|_{L^q(\Omega)} \qquad \forall r > q\,, \tag{1.7}$$
$$\|u(t)\|_{L^\infty(\Omega)} \leq Ce^{Ct\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}}\,t^{-\frac{N}{2q}}\,\|u_0\|_{L^q(\Omega)}\,,$$

with $\beta = \frac{2\sigma}{2\sigma - N}$.

Let us explain the plan of the paper. The proof of Theorem 1.3 (as the one of Theorem 1.7) will be achieved in several steps, since it needs many technical devices. The main idea is to construct the unique weak solution of (1.3) via a fixed-point argument which is carried on, differently from the classical approach, in the space $X = C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, which plays a natural role for equation (1.3). A difficulty arises in the definition of the contraction mapping, since we can not use the integral formulation (1.2); so we will define an operator $\mathcal{F}$ such that, for all v in X, $\mathcal{F}(v)$ is the function u (which will be proved to be unique) obtained as limit, in the strong topology of X, of the sequence $\{u_n\}$ of solutions of the problems

$$\begin{cases} (u_n)_t - \operatorname{div}(a(x,t,\nabla u_n)) = \rho(x,t)|v_n|^{p-1}v_n & \text{in } \Omega \times (0,T), \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(x,0) = \Theta_n(u_0) & \text{in } \Omega, \end{cases} \tag{1.8}$$

where $\{v_n\}$ is a sequence of bounded functions strongly convergent to v in X, and $\Theta_n(s) = \max(-n, \min(s,n))$ is the truncation function at levels $\pm n$. Indeed, we are able to get from (1.8) both a priori estimates and stability results in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, to show first that u_n converges in X and its limit is independent of the choice of the sequence v_n approximating v in X, and then that $\mathcal{F}$ is a contraction on a suitable closed, convex subset of X. Next we prove that the unique fixed point found in X is a weak solution of (1.3) (and in particular, it belongs to $L^\infty_{\mathrm{loc}}(0,T;L^\infty(\Omega))$) and that uniqueness holds in this class. This approach allows us to deal with weak solutions (which are usually obtained as limit of solutions of more regular problems) and at the same time points out the importance of the space $X = C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$.

In order to provide the tools needed for this program, in Section 2 we prove the basic *a priori* estimates for (1.3) and (1.5), while Section 3 is devoted to stability and compactness properties of the equation, and finally the proofs of our results are completed in Section 4.

2. A priori estimates. In the following, for $T > 0$, we set $Q \equiv \Omega \times (0,T)$, and recall that given functions $b \in L^\infty(Q)$, $\psi \in L^\infty(0,T;L^\sigma(\Omega))$ with $\sigma > \frac{N}{2}$, and u_0 in $L^\infty(\Omega)$, there exists a unique bounded weak solution of the Cauchy–Dirichlet problem

$$\begin{cases} u_t - \operatorname{div}(a(x,t,\nabla u)) = \psi(x,t)b(x,t) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega; \end{cases}$$

that is, u is in $L^\infty(Q) \cap L^2(0,T;H_0^1(\Omega))$ and satisfies

$$\int_{t_1}^{t_2} {}_{H^{-1}(\Omega)}\langle u_\tau, \varphi\rangle_{H_0^1(\Omega)}\, d\tau + \int_{t_1}^{t_2}\int_\Omega a(x,\tau,\nabla u)\nabla\varphi\, dx d\tau$$
$$= \int_{t_1}^{t_2}\int_\Omega \psi(x,\tau)\, b(x,\tau)\varphi\, dx d\tau\,, \tag{2.1}$$

for every $0 \le t_1 < t_2 \le T$ and all functions φ in $L^2(0,T;H_0^1(\Omega))$ (see [7], [8] and [5]).

Proposition 2.1. *Let ψ belong to $L^\infty(0,T;L^\sigma(\Omega))$, $\sigma > \frac{N}{2}$, $N > 2$, $1 < q < \infty$, and let u_0, z_0 belong to $L^\infty(\Omega)$, v and w belong to $L^\infty(Q)$. Let u and z be the weak solutions of the following problems:*

$$\begin{cases} u_t - \operatorname{div}(a(x,t,\nabla u)) = \psi(x,t)v & \textit{in } \Omega\times(0,T), \\ u = 0 & \textit{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) & \textit{in } \Omega, \end{cases} \tag{2.2}$$

$$\begin{cases} z_t - \operatorname{div}(a(x,t,\nabla z)) = \psi(x,t)w & \textit{in } \Omega\times(0,T), \\ z = 0 & \textit{on } \partial\Omega\times(0,T), \\ z(x,0) = z_0(x) & \textit{in } \Omega. \end{cases} \tag{2.3}$$

Then there exists a positive constant C_0, depending on N, $|\Omega|$, α, σ and q, such that, setting $\beta = \frac{2\sigma}{2\sigma - N}$, we have, for every t in $[0,T]$,

$$\Big[1 - t\, C_0\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}\Big] \sup_{s\in[0,t]}\int_\Omega |u(s)-z(s)|^q\, dx$$
$$+ \frac{1}{2}\int_0^t \left(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\, dx\right)^{\frac{N-2}{N}} d\tau$$
$$\le C_0\int_\Omega |u_0 - z_0|^q\, dx + tC_0\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}\sup_{s\in[0,t]}\int_\Omega |v(s)-w(s)|^q\, dx$$
$$+ \frac{1}{4}\int_0^t \left(\int_\Omega |v-w|^{\frac{Nq}{N-2}}\, dx\right)^{\frac{N-2}{N}} d\tau\,. \tag{2.4}$$

Proof. For the first part of the proof we distinguish two cases:

a) $q \ge 2$. We take $|u-z|^{q-2}(u-z)$ as test function in formulation (2.1) written for problems (2.2) and (2.3), in the time interval $(0,s)$, with $s \le t$.

Subtracting both equations and integrating by parts we get

$$\begin{aligned}&\frac{1}{q}\int_\Omega |u(s)-z(s)|^q\,dx\\&+(q-1)\int_0^s\int_\Omega [a(x,\tau,\nabla u)-a(x,\tau,\nabla z)]\cdot\nabla(u-z)\,|u-z|^{q-2}\,dx\,d\tau\\&\le \frac{1}{q}\int_\Omega |u_0-z_0|^q\,dx+\int_0^s\int_\Omega |\psi|\,|v-w|\,|u-z|^{q-1}\,dx\,d\tau\,;\end{aligned}$$

hence, applying assumption (a_1),

$$\begin{aligned}&\frac{1}{q}\int_\Omega |u(s)-z(s)|^q\,dx+(q-1)\alpha\int_0^s\int_\Omega |\nabla(u-z)|^2\,|u-z|^{q-2}\,dxd\tau\\&\le \frac{1}{q}\int_\Omega |u_0-z_0|^q\,dx+\int_0^s\int_\Omega |\psi|\,|v-w|\,|u-z|^{q-1}\,dxd\tau\,.\end{aligned}$$

Since $|u-z|^{q-2}|\nabla(u-z)|^2=\frac{4}{q^2}|\nabla(|u-z|^{\frac{q}{2}-1}(u-z))|^2$, the Sobolev embedding theorem allows us to obtain (we denote by S the constant given by Sobolev inequality)

$$\begin{aligned}&\frac{1}{q}\int_\Omega |u(s)-z(s)|^q\,dx+(q-1)S\frac{4}{q^2}\alpha\int_0^s\left(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\,dx\right)^{\frac{N-2}{N}}d\tau\\&\le \frac{1}{q}\int_\Omega |u_0-z_0|^q\,dx+\int_0^s\int_\Omega |\psi|\,|v-w|\,|u-z|^{q-1}\,dxd\tau\,.\end{aligned}\tag{2.5}$$

b) $1<q<2$. In this case, for $\varepsilon>0$, we choose $\frac{u-z}{(\varepsilon+|u-z|)^{2-q}}$ as test function in both equations (2.2) and (2.3) and we subtract. Setting $\Phi_\varepsilon(s)=\int_0^s\frac{r}{(\varepsilon+|r|)^{2-q}}\,dr$ we have,

$$\begin{aligned}&\int_\Omega \Phi_\varepsilon(u-z)(s)\,dx\\&+\int_0^s\int_\Omega \Phi_\varepsilon''(u-z)[a(x,\tau,\nabla u)-a(x,\tau,\nabla z)]\cdot\nabla(u-z)\,dxd\tau\le\\&\le\int_\Omega \Phi_\varepsilon(u_0-z_0)\,dx+\int_0^s\int_\Omega \psi\,(v-w)\,\frac{u-z}{(\varepsilon+|u-z|)^{2-q}}\,dxd\tau\,.\end{aligned}$$

Since $\Phi''(s) \geq \frac{q-1}{(\varepsilon+|s|)^{2-q}}$, $\frac{1}{8}\,|s|^q\,\chi_{\{|s|>\varepsilon\}} \leq \Phi_\varepsilon(s)\chi_{\{|s|>\varepsilon\}} \leq \Phi_\varepsilon(s) \leq \frac{1}{q}\,|s|^q$, we obtain, using also (a_1),

$$\frac{1}{8}\int\limits_{\{x\,:\,|(u-z)(x,s)|>\varepsilon\}} |(u-z)(s)|^q\,dx + \alpha\,(q-1)\int_0^s\int_\Omega \frac{|\nabla(u-z)|^2}{(\varepsilon+|u-z|)^{2-q}}\,dxd\tau$$
$$\leq \frac{1}{q}\int_\Omega |u_0-z_0|^q\,dx + \int_0^s\int_\Omega |\psi|\,|v-w|\,|u-z|^{q-1}\,dxd\tau\,.$$

Now, observing that $\frac{|\nabla(u-z)|^2}{(\varepsilon+|u-z|)^{2-q}} = \frac{4}{q^2}|\nabla[(\varepsilon+|u-z|)^{\frac{q}{2}}-\varepsilon]|^2$, by the Sobolev inequality we get

$$\frac{1}{8}\int\limits_{\{x:|(u-z)(x,s)|>\varepsilon\}} |(u-z)(s)|^q dx$$
$$+\,\alpha\,(q-1)\frac{4}{q^2}S\int_0^s\Big(\int_\Omega |(\varepsilon+|u-z|)^{\frac{q}{2}}-\varepsilon|^{\frac{2N}{N-2}}dx\Big)^{\frac{N-2}{N}}d\tau$$
$$\leq \frac{1}{q}\int_\Omega |u_0-z_0|^q\,dx + \int_0^s\int_\Omega |\psi||v-w||u-z|^{q-1}\,dxd\tau\,.$$

Letting ε go to zero we find, for every $1<q<2$,

$$\begin{aligned}\frac{1}{8}\int_\Omega |u(s)-z(s)|^q\,dx + (q-1)S\frac{4}{q^2}\alpha\int_0^s\left(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\,dx\right)^{\frac{N-2}{N}}d\tau\\ \leq \frac{1}{q}\int_\Omega |u_0-z_0|^q\,dx + \int_0^s\int_\Omega |\psi|\,|v-w|\,|u-z|^{q-1}\,dxd\tau\,.\end{aligned}\tag{2.6}$$

Henceforth, we proceed with $1<q<\infty$. Then, Young's inequality applied to (2.5) and (2.6) gives, for every $1<q<\infty$,

$$\begin{aligned}\min(\frac{1}{8},\frac{1}{q})\int_\Omega |u(s)-z(s)|^q\,dx + 4S\alpha\frac{q-1}{q^2}\int_0^s\left(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\,dx\right)^{\frac{N-2}{N}}d\tau\\ \leq \frac{1}{q}\int_\Omega |u_0-z_0|^q\,dx + \int_0^s\int_\Omega |\psi|\,|v-w|^q\,dxd\tau + \int_0^s\int_\Omega |\psi|\,|u-z|^q\,dxd\tau\,.\end{aligned}\tag{2.7}$$

Using Hölder's inequality with exponents $(\sigma;\sigma')$ and setting

$$c_0 = [\min(\frac{1}{8},\frac{1}{q},4S\alpha\frac{q-1}{q^2})]^{-1}$$

we obtain

$$\begin{aligned}\int_\Omega |u(s)-z(s)|^q\,dx + \int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau \\ \le \frac{c_0}{q}\int_\Omega |u_0-z_0|^q\,dx + c_0\int_0^s \|\psi\|_{L^\sigma(\Omega)}\Big(\int_\Omega |v-w|^{q\sigma'}\,dx\Big)^{\frac{1}{\sigma'}}\,d\tau \\ + c_0\int_0^s \|\psi\|_{L^\sigma(\Omega)}\Big(\int_\Omega |u-z|^{q\sigma'}\,dx\Big)^{\frac{1}{\sigma'}}\,d\tau\,.\end{aligned} \tag{2.8}$$

Since $\frac{N}{2}<\sigma<\infty$, the interpolation inequality implies

$$\Big(\int_\Omega |v-w|^{q\sigma'}\,dx\Big)^{\frac{1}{\sigma'}} \le \Big(\int_\Omega |v-w|^q\,dx\Big)^{\frac{2\sigma-N}{2\sigma}}\Big(\int_\Omega |v-w|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{2\sigma}}\,,$$

so (2.8) yields

$$\begin{aligned}\int_\Omega |u(s)-z(s)|^q\,dx + \int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau \le \frac{c_0}{q}\int_\Omega |u_0-z_0|^q\,dx \\ + c_0\|\psi\|_{L^\infty(0,T;L^\sigma(\Omega))}\int_0^s \Big(\int_\Omega |v-w|^q\,dx\Big)^{\frac{2\sigma-N}{2\sigma}}\Big(\int_\Omega |v-w|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{2\sigma}}\,d\tau \\ + c_0\|\psi\|_{L^\infty(0,T;L^\sigma(\Omega))}\int_0^s \Big(\int_\Omega |u-z|^q\,dx\Big)^{\frac{2\sigma-N}{2\sigma}}\Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{2\sigma}}\,d\tau.\end{aligned}$$

Thus, by Young's inequality, we have, setting $\beta=\frac{2\sigma}{2\sigma-N}$ and

$$C_0 = 2\max\{c_0, (\frac{8}{\beta'})^{\beta-1}\frac{1}{\beta}c_0^\beta\},$$

$$\begin{aligned}\int_\Omega |u(s)-z(s)|^q\,dx + \int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau \\ \le \frac{C_0}{2}\int_\Omega |u_0-z_0|^q\,dx + \frac{1}{8}\int_0^s \Big(\int_\Omega |v-w|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau \\ + \frac{C_0}{2}\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}\int_0^s\int_\Omega |v-w|^q\,dx\,d\tau \\ + \frac{1}{2}\int_0^s \Big(\int_\Omega |u-z|^{q\frac{N}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau \\ + \frac{C_0}{2}\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}\int_0^s\int_\Omega |u-z|^q\,dx\,d\tau\,,\end{aligned}$$

from which we deduce

$$
\begin{aligned}
&\int_\Omega |u(s)-z(s)|^q\,dx + \frac{1}{2}\int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau \\
&\quad\le \frac{C_0}{2}\int_\Omega |u_0-z_0|^q\,dx + \frac{1}{8}\int_0^t \Big(\int_\Omega |v-w|^{q\frac{N}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau \\
&\quad + t\,\frac{C_0}{2}\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}\sup_{[0,t]}\int_\Omega |v-w|^q\,dx \\
&\qquad + t\,\frac{C_0}{2}\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}\sup_{[0,t]}\int_\Omega |u-z|^q\,dx\,, \qquad \forall s\le t\,.
\end{aligned}
$$

Since the two terms in the left-hand side are both positive, taking the supremum over s in $[0,t]$ for both we obtain (2.4). □

The technique used in the previous proposition will be now applied to the superlinear Dirichlet problem (1.3).

Proposition 2.2. *Let $\rho(x,t)$ belong to $L^\infty(\Omega\times\mathbf{R})$, $q>\frac{N(p-1)}{2}$, $N>2$, and let u_0, z_0 belong to $L^\infty(\Omega)$, v and w belong to $L^\infty(Q)$. Let u and z be the weak solutions of the problems*

$$
\begin{cases}
u_t - \operatorname{div}(a(x,t,\nabla u)) = \rho(x,t)|v|^{p-1}v & \text{in } \Omega\times(0,T),\\
u=0 & \text{on } \partial\Omega\times(0,T),\\
u(x,0)=u_0(x) & \text{in } \Omega,
\end{cases}
\tag{2.9}
$$

$$
\begin{cases}
z_t - \operatorname{div}(a(x,t,\nabla z)) = \rho(x,t)|w|^{p-1}w & \text{in } \Omega\times(0,T),\\
z=0 & \text{on } \partial\Omega\times(0,T),\\
z(x,0)=z_0(x) & \text{in } \Omega.
\end{cases}
\tag{2.10}
$$

Then there exists a positive constant C_1, depending on N, $|\Omega|$, α, q and p, such that, setting $\gamma=\frac{2q(p-1)}{2q-N(p-1)}$, we have, for all t in $[0,T]$,

$$
\begin{aligned}
&\Big[1 - tC_1\|\rho\|_\infty^{\frac{\gamma}{p-1}}\big(\|v\|^\gamma_{L^\infty(0,T;L^q(\Omega))} + \|w\|^\gamma_{L^\infty(0,T;L^q(\Omega))}\big)\Big] \\
&\times \sup_{s\in[0,t]}\int_\Omega |u(s)-z(s)|^q\,dx + \frac{1}{2}\int_0^t \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau \\
&\le C_1\int_\Omega |u_0-z_0|^q\,dx + tC_1\|\rho\|_\infty^{\frac{\gamma}{p-1}}\Big[\|v\|^\gamma_{L^\infty(0,T;L^q(\Omega))} + \|w\|^\gamma_{L^\infty(0,T;L^q(\Omega))}\Big] \\
&\times \sup_{s\in[0,t]}\int_\Omega |v(s)-w(s)|^q\,dx + \frac{1}{4}\int_0^t \Big(\int_\Omega |v-w|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\,.
\end{aligned}
\tag{2.11}
$$

Proof. As in Proposition 2.1, we take as test function, in formulation (2.1) of both equations (2.9) and (2.10), either $|u-z|^{q-2}(u-z)$, if $q \geq 2$, or $\frac{u-z}{(\varepsilon+|u-z|)^{2-q}}$, if $1 < q < 2$, and then we let ε tend to zero. Then we have the following inequality, for every $s \leq t$,

$$\int_\Omega |u(s)-z(s)|^q \, dx + \int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{N-2}{N}} \, d\tau$$
$$\leq \frac{c_0}{q} \int_\Omega |u_0 - z_0|^q \, dx + c_0 \int_0^s \int_\Omega |\rho| \, \Big| |v|^{p-1} v - |w|^{p-1} w \Big| \, |u-z|^{q-1} \, dx d\tau \,,$$

with $c_0 = [\min(\frac{1}{8}, \frac{1}{q}, 4S\alpha \frac{q-1}{q^2})]^{-1}$. Using also Young's inequality we easily get

$$\int_\Omega |u(s)-z(s)|^q \, dx + \int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{N-2}{N}} \, d\tau \tag{2.12}$$
$$\leq \frac{c_0}{q} \int_\Omega |u_0 - z_0|^q \, dx + pc_0 \|\rho\|_\infty \int_0^s \int_\Omega \Big(|v|^{p-1} + |w|^{p-1} \Big) |v-w|^q \, dx \, d\tau$$
$$+ pc_0 \|\rho\|_\infty \int_0^s \int_\Omega (|v|^{p-1} + |w|^{p-1}) \, |u-z|^q \, dx \, d\tau \,.$$

Setting $\sigma = \frac{q}{p-1}$, since $\frac{q}{p-1} > \frac{N}{2}$, we are in the same situation as in the previous proposition, so using first Hölder's inequality with exponents (σ, σ'), then the interpolation inequality for $1 < \sigma' < \frac{N}{N-2}$, we have

$$\int_0^s \int_\Omega \Big(|v|^{p-1} + |w|^{p-1} \Big) |v-w|^q \, dx \, d\tau$$
$$\leq \int_0^s \Big(\|v\|_{L^q(\Omega)}^{p-1} + \|w\|_{L^q(\Omega)}^{p-1} \Big) \Big(\int_\Omega |v-w|^{q\sigma'} \, dx \Big)^{\frac{1}{\sigma'}}$$
$$\leq \Big(\|v\|_{L^\infty(0,T;L^q(\Omega))}^{p-1} + \|w\|_{L^\infty(0,T;L^q(\Omega))}^{p-1} \Big) \times$$
$$\times \int_0^s \Big(\int_\Omega |v-w|^q \, dx \Big)^{\frac{2q-N(p-1)}{2q}} \Big(\int_\Omega |v-w|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{(N-2)(p-1)}{2q}} \, d\tau \,,$$

which together with (2.12) implies

$$\int_\Omega |u(s)-z(s)|^q \, dx + \int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{N-2}{N}} \, d\tau$$
$$\leq \frac{c_0}{q} \int_\Omega |u_0 - z_0|^q \, dx + pc_0 \|\rho\|_\infty \Big(\|v\|_{L^\infty(0,T;L^q(\Omega))}^{p-1} + \|w\|_{L^\infty(0,T;L^q(\Omega))}^{p-1} \Big)$$

$$\times \int_0^s \Big(\int_\Omega |v-w|^q \, dx \Big)^{\frac{2q-N(p-1)}{2q}} \Big(\int_\Omega |v-w|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{(N-2)(p-1)}{2q}} \, d\tau$$
$$+ pc_0 \|\rho\|_\infty \Big(\|v\|^{p-1}_{L^\infty(0,T;L^q(\Omega))} + \|w\|^{p-1}_{L^\infty(0,T;L^q(\Omega))} \Big)$$
$$\times \int_0^s \Big(\int_\Omega |u-z|^q \, dx \Big)^{\frac{2q-N(p-1)}{2q}} \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{(N-2)(p-1)}{2q}} \, d\tau \, .$$

By Young's inequality, setting $\gamma = \frac{2q(p-1)}{2q-N(p-1)}$ and

$$C_1 = 2 \max\{ \frac{c_0}{q}, \Big(\frac{4N(p-1)}{q} \Big)^{\frac{N\gamma}{2q}} \frac{p-1}{\gamma} (pc_0)^{\frac{\gamma}{p-1}} \},$$

we obtain

$$\int_\Omega |u(s)-z(s)|^q \, dx + \int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{N-2}{N}} \, d\tau \le \frac{C_1}{2} \int_\Omega |u_0 - z_0|^q \, dx$$
$$+ \frac{1}{8} \int_0^s \Big(\int_\Omega |v-w|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{N-2}{N}} \, d\tau + \frac{1}{2} \int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{N-2}{N}} \, d\tau$$
$$+ \frac{C_1}{2} \|\rho\|_\infty^{\frac{\gamma}{p-1}} \Big(\|v\|^\gamma_{L^\infty(0,T;L^q(\Omega))} + \|w\|^\gamma_{L^\infty(0,T;L^q(\Omega))} \Big) \int_0^s \int_\Omega |v-w|^q \, dx \, d\tau$$
$$+ \frac{C_1}{2} \|\rho\|_\infty^{\frac{\gamma}{p-1}} \Big(\|v\|^\gamma_{L^\infty(0,T;L^q(\Omega))} + \|w\|^\gamma_{L^\infty(0,T;L^q(\Omega))} \Big) \int_0^s \int_\Omega |u-z|^q \, dx \, d\tau \, .$$

Thus we finally get, for every $s \le t$,

$$\int_\Omega |u(s)-z(s)|^q \, dx + \frac{1}{2} \int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{N-2}{N}} \, d\tau$$
$$\le \frac{C_1}{2} \int_\Omega |u_0 - z_0|^q \, dx + \frac{1}{8} \int_0^t \Big(\int_\Omega |v-w|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{N-2}{N}} \, d\tau$$
$$+ t \, \frac{C_1}{2} \|\rho\|_\infty^{\frac{\gamma}{p-1}} \Big(\|v\|^\gamma_{L^\infty(0,T;L^q(\Omega))} + \|w\|^\gamma_{L^\infty(0,T;L^q(\Omega))} \Big) \sup_{[0,t]} \int_\Omega |v-w|^q \, dx$$
$$+ t \, \frac{C_1}{2} \|\rho\|_\infty^{\frac{\gamma}{p-1}} \Big(\|v\|^\gamma_{L^\infty(0,T;L^q(\Omega))} + \|w\|^\gamma_{L^\infty(0,T;L^q(\Omega))} \Big) \sup_{[0,t]} \int_\Omega |u-z|^q \, dx \, ,$$

which yields (2.11). □

For our purposes, the case $\sigma = \frac{N}{2}$ in Proposition 2.1, or similarly the case $q = \frac{N(p-1)}{2}$ in Proposition 2.2, are more delicate to deal with, and the content of next propositions, which essentially makes use of what has already been proved and may appear redundant, will become clear at the moment of applications in Section 4.

Proposition 2.3. *Let $N > 2$, $q > 1$, and let ψ belong to $L^\infty(0,T;L^{\frac{N}{2}}(\Omega))$. Assume that u_0, z_0 are in $L^\infty(\Omega)$ and v, w belong to $L^\infty(Q)$, and that u and z are bounded weak solutions of the following problems:*

$$\begin{cases} u_t - \mathrm{div}(a(x,t,\nabla u)) = \psi(x,t)v & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

$$\begin{cases} z_\tau - \mathrm{div}(a(x,t,\nabla z)) = \psi(x,t)w & \text{in } \Omega \times (0,T), \\ z = 0 & \text{on } \partial\Omega \times (0,T), \\ z(x,0) = z_0(x) & \text{in } \Omega. \end{cases}$$

Then for any $\bar\sigma > \frac{N}{2}$, and for any bounded function $\bar\psi$, there exists a positive constant $C_2 = C_2(\alpha, |\Omega|, N, \bar\sigma, q)$ such that, setting $\beta = \frac{2\bar\sigma}{2\bar\sigma - N}$, we have, for every t in $[0,T]$,

$$\begin{aligned} &\Big(1 - t\, C_2 \|\bar\psi\|^\beta_{L^\infty(0,T;L^{\bar\sigma}(\Omega))}\Big) \sup_{s\in[0,t]} \int_\Omega |u(s) - z(s)|^q \, dx \qquad (2.13) \\ &+ \Big[\frac{1}{2} - C_2 \|\psi - \bar\psi\|_{L^\infty(0,T;L^{\frac{N}{2}}(\Omega))}\Big] \int_0^t \Big(\int_\Omega |u - z|^{\frac{Nq}{N-2}} \, dx\Big)^{\frac{N-2}{N}} d\tau \\ &\le C_2 \int_\Omega |u_0 - z_0|^q \, dx + t\ C_2 \|\bar\psi\|^\beta_{L^\infty(0,T;L^{\bar\sigma}(\Omega))} \sup_{s\in[0,t]} \int_\Omega |v(s) - w(s)|^q \, dx \\ &+ \Big[\frac{1}{4} + C_2 \|\psi - \bar\psi\|_{L^\infty(0,T;L^{\frac{N}{2}}(\Omega))}\Big] \int_0^t \Big(\int_\Omega |v - w|^{\frac{Nq}{N-2}} \, dx\Big)^{\frac{N-2}{N}} d\tau \,. \end{aligned}$$

Proof. Let us start from inequality (2.7) obtained in the proof of Proposition 2.1:

$$\begin{aligned} &\int_\Omega |u(s) - z(s)|^q dx + \int_0^s \Big(\int_\Omega |u - z|^{\frac{Nq}{N-2}} \, dx\Big)^{\frac{N-2}{N}} d\tau \\ &\le \frac{c_0}{q} \int_\Omega |u_0 - z_0|^q dx + c_0 \int_0^s \int_\Omega |\psi||v - w|^q dx \, d\tau + c_0 \int_0^s \int_\Omega |\psi||u - z|^q dx \, d\tau, \end{aligned}$$

for every s in $(0,t)$ and with $c_0 = [\min(\frac{1}{8}, \frac{1}{q}, 4S\alpha\frac{q-1}{q^2})]^{-1}$, which implies, for any bounded function $\bar{\psi}$,

$$\int_\Omega |u(s)-z(s)|^q\,dx + \int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau \le \frac{c_0}{q}\int_\Omega |u_0-z_0|^q\,dx$$
$$+ c_0\int_0^s\int_\Omega |\psi-\bar{\psi}|\,|v-w|^q\,dx\,d\tau + c_0\int_0^s\int_\Omega |\bar{\psi}|\,|v-w|^q\,dx\,d\tau$$
$$+ c_0\int_0^s\int_\Omega |\psi-\bar{\psi}|\,|u-z|^q\,dx\,d\tau + c_0\int_0^s\int_\Omega |\bar{\psi}|\,|u-z|^q\,dx\,d\tau\,.$$

By Hölder's inequality with exponents $(\frac{N}{2}, \frac{N}{N-2})$ we get

$$\int_\Omega |u(s)-z(s)|^q\,dx + \int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau \le \frac{c_0}{q}\int_\Omega |u_0-z_0|^q\,dx$$
$$+ c_0\|\psi-\bar{\psi}\|_{L^\infty(0,T;L^{\frac{N}{2}}(\Omega))}\int_0^s \Big(\int_\Omega |v-w|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau \tag{2.14}$$
$$+ c_0\int_0^s\int_\Omega |\bar{\psi}|\,|v-w|^q\,dx\,d\tau + c_0\|\psi-\bar{\psi}\|_{L^\infty(0,T;L^{\frac{N}{2}}(\Omega))}$$
$$\times\int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau + c_0\int_0^s\int_\Omega |\bar{\psi}|\,|u-z|^q\,dx\,d\tau\,.$$

The last term, and the analog with $v-w$, can now be dealt with as in Proposition 2.1. For every $\bar{\sigma} > \frac{N}{2}$, using first Hölder's inequality with exponents $(\bar{\sigma}, \bar{\sigma}')$, then the interpolation inequality with $1 < \bar{\sigma}' < \frac{N}{N-2}$, and finally Young's inequality we have

$$c_0\int_0^s\int_\Omega |\bar{\psi}|\,|u-z|^q\,dx\,d\tau \le \frac{1}{2}\int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau$$
$$+ \frac{C_2}{2}\,\|\bar{\psi}\|^\beta_{L^\infty(0,T;L^{\bar{\sigma}}(\Omega))}\;t\sup_{[0,t]}\int_\Omega |u-z|^q\,dx\,,$$

with $\beta = \frac{2\bar{\sigma}}{2\bar{\sigma}-N}$ and $C_2 = 2\max(c_0, \big(\frac{8}{\beta'}\big)^{\beta-1}\frac{1}{\beta}c_0^\beta)$. Therefore (2.14) implies, for every s in $(0,t)$,

$$\int_\Omega |u(s)-z(s)|^q\,dx$$
$$+ \big[\frac{1}{2} - \frac{C_2}{2}\|\psi-\bar{\psi}\|_{L^\infty(0,T;L^{\frac{N}{2}}(\Omega))}\big]\int_0^s \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau$$
$$\le \frac{C_2}{2}\int_\Omega |u_0-z_0|^q\,dx$$

$$+ \left[\frac{1}{8} + \frac{C_2}{2}\|\psi - \bar\psi\|_{L^\infty(0,T;L^{\frac{N}{2}}(\Omega))}\right] \int_0^s \Big(\int_\Omega |v-w|^{\frac{Nq}{N-2}}\, dx\Big)^{\frac{N-2}{N}} d\tau$$
$$+ t\, \frac{C_2}{2}\|\bar\psi\|^\beta_{L^\infty(0,T;L^{\bar\sigma}(\Omega))} \sup_{[0,t]} \int_\Omega |v-w|^q\, dx$$
$$+ t\frac{C_2}{2}\|\bar\psi\|^\beta_{L^\infty(0,T;L^{\bar\sigma}(\Omega))} \sup_{[0,t]} \int_\Omega |u-z|^q\, dx\,,$$

and taking the supremum over s we obtain (2.13). □

It is now easy to see that, starting from (2.12) and reasoning as we have done in the previous proposition, we can deduce the following a priori estimate.

Proposition 2.4. *Let ρ be in $L^\infty(\Omega \times \mathbf{R})$, $q = \frac{N(p-1)}{2}$, $N > 2$, and let u_0, z_0 be in $L^\infty(\Omega)$, v and w belong to $L^\infty(Q)$. Let u and z be the weak solutions of*

$$\begin{cases} u_t - \operatorname{div}(a(x,t,\nabla u)) = \rho(x,t)|v|^{p-1}v & \text{in } \Omega\times(0,T),\\ u = 0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0) = u_0(x) & \text{in } \Omega,\end{cases}$$

$$\begin{cases} z_t - \operatorname{div}(a(x,t,\nabla z)) = \rho(x,t)|w|^{p-1}w & \text{in } \Omega\times(0,T),\\ z = 0 & \text{on } \partial\Omega\times(0,T),\\ z(x,0) = z_0(x) & \text{in } \Omega.\end{cases}$$

Then for any $r > q$ there exists a positive constant C_3, depending on N, $|\Omega|$, α, q, p and r, such that we have, for every couple of bounded functions $\bar\varphi$ and $\bar\zeta$, with $\gamma = \frac{2r(p-1)}{2r-N(p-1)}$,

$$\begin{aligned} &\Big[1 - tC_3\|\rho\|_\infty^{\frac{\gamma}{p-1}} \Big(\|\bar\varphi\|^\gamma_{L^\infty(0,T;L^r(\Omega))} + \|\bar\zeta\|^\gamma_{L^\infty(0,T;L^r(\Omega))}\Big)\Big]\\ &\times \sup_{s\in[0,t]} \int_\Omega |u(s)-z(s)|^q\, dx\\ &+ \Big[\frac{1}{2} - C_3\Big(\|v-\bar\varphi\|^{p-1}_{L^\infty(0,T;L^q(\Omega))} + \|w-\bar\zeta\|^{p-1}_{L^\infty(0,T;L^q(\Omega))}\Big)\Big]\\ &\times \int_0^t \Big(\int_\Omega |u-z|^{\frac{Nq}{N-2}}\, dx\Big)^{\frac{N-2}{N}} d\tau\\ &\le C_3 \int_\Omega |u_0 - z_0|^q\, dx + tC_3\|\rho\|_\infty^{\frac{\gamma}{p-1}} \Big[\|\bar\varphi\|^\gamma_{L^\infty(0,T;L^r(\Omega))} + \|\bar\zeta\|^\gamma_{L^\infty(0,T;L^r(\Omega))}\Big] \end{aligned} \tag{2.15}$$

$$\times \sup_{s\in[0,t]} \int_\Omega |v(s)-w(s)|^q\,dx + \Big[\frac{1}{4} + C_3\Big(\|v-\bar\varphi\|^{p-1}_{L^\infty(0,T;L^q(\Omega))} + $$
$$+ \|w-\bar\zeta\|^{p-1}_{L^\infty(0,T;L^q(\Omega))}\Big)\Big] \int_0^t \Big(\int_\Omega |v-w|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\,. \qquad \square$$

The a priori estimates we have previously derived will play the fundamental role in the construction of the contraction mapping on the space $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$. Our next goal is to find further estimates concerning the solution at a fixed time $t>0$, which will be used to prove regularity for the solution of (1.3). In this way we can recover some classical properties of the heat equation, which are contained in the literature, by means of nonlinear techniques; to this purpose, we will follow some ideas of the paper by H. Brezis and T. Cazenave.

Let us start with the following lemma.

Lemma 2.5. *Let ψ be in $L^\infty(0,T;L^\sigma(\Omega))$ with $\sigma > \frac{N}{2}$, and let u be a weak solution of*

$$\begin{cases} u_t - \operatorname{div}(a(x,t,\nabla u)) = \psi(x,t)u & \text{in } \Omega\times(0,T),\\ u=0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x) & \text{in } \Omega, \end{cases} \tag{1.5}$$

with $u_0 \in L^q(\Omega)$, $q>1$. Let $r>1$ and suppose that u belongs to $L^r_{\rm loc}(0,T;L^{\frac{Nr}{N-2}}(\Omega))$. Then the function $\int_\Omega |u(t)|^r\,dx$ belongs to $W^{1,1}_{\rm loc}(0,T)$ (hence it is continuous on $(0,T)$) and satisfies

$$\frac{1}{r}\frac{d}{dt}\Big(\int_\Omega |u(t)|^r\,dx\Big) + (r-1)\int_\Omega a(x,t,\nabla u)\nabla u\,|u|^{r-2}\,dx = \int_\Omega \psi(x,t)|u|^r\,dx\,,$$

for almost every t in $(0,T)$. Moreover, there exists a positive constant C_4 depending on α, N, $|\Omega|$ such that the following differential inequality holds true almost everywhere in $(0,T)$.

$$\frac{d}{dt}\Big(\int_\Omega |u|^r\,dx\Big) + \frac{r-1}{r}\Big(\int_\Omega |u|^{\frac{Nr}{N-2}}\,dx\Big)^{\frac{N-2}{N}} \le C_4\, r\int_\Omega \psi(x,t)|u|^r\,dx\,. \tag{2.16}$$

Proof. We start first assuming that $r\ge 2$. Let us denote by $\Theta_k(s) = \max(-k,\min(s,k))$ the truncation function at levels $\pm k$. First of all, since

u is in $L^2_{\text{loc}}(0,T;H^1_0(\Omega))$, we have that $\varphi = |\Theta_k(u)|^{r-2}\Theta_k(u)$ belongs to $L^2_{\text{loc}}(0,T;H^1_0(\Omega))$, so it can be taken as test function in (1.6) to have

$$\begin{aligned}&\int_{t_1}^{t_2} {}_{H^{-1}(\Omega)}\langle u_\tau, |\Theta_k(u)|^{r-2}\Theta_k(u)\rangle_{H^1_0(\Omega)}\, d\tau\\&+(r-1)\int_{t_1}^{t_2}\int_\Omega a(x,\tau,\nabla\Theta_k(u))\cdot\nabla\Theta_k(u)|\Theta_k(u)|^{r-2}dx\,d\tau\\&\le\int_{t_1}^{t_2}\int_\Omega|\psi(x,\tau)|\,|u|\,|\Theta_k(u)|^{r-1}dx\,d\tau\,.\end{aligned}$$

Setting $S_k(t)=\int_0^t|\Theta_k(s)|^{r-2}\Theta_k(s)\,ds$ and integrating by parts we obtain, using also (a_1),

$$\begin{aligned}&\int_\Omega S_k(u)(t_2)dx+(r-1)\int_{t_1}^{t_2}\int_\Omega a(x,\tau,\nabla\Theta_k(u))\cdot\nabla\Theta_k(u)|\Theta_k(u)|^{r-2}\,dx\,d\tau\\&\le\int_\Omega S_k(u)(t_1)dx+\int_{t_1}^{t_2}\int_\Omega|\psi(x,\tau)|\,|u|\,|\Theta_k(u)|^{r-1}\,dx\,d\tau\,.\end{aligned}\tag{2.17}$$

Since u belongs to $L^r_{\text{loc}}(0,T;L^{\frac{Nr}{N-2}}(\Omega))$, $u(t)$ is in $L^r(\Omega)$ for almost every t in $(0,T)$, so without loss of generality we can assume that $u(t_1)$ belongs to $L^r(\Omega)$. Moreover, since ψ is in $L^\infty(0,T;L^\sigma(\Omega))$ with $\sigma>\frac{N}{2}$, the product $|\psi(x,t)|\,|u|^r$ is certainly in $L^1_{\text{loc}}(0,T;L^1(\Omega))$, so the right-hand side of (2.17) is bounded uniformly on k, and letting k tend to infinity we deduce by Fatou's lemma that $a(x,t,\nabla u)\cdot\nabla u\,|u|^{r-2}$ belongs to $L^1_{\text{loc}}(0,T;L^1(\Omega))$.

Now, take a function ζ in $C^\infty_c(0,T)$, and let us choose as test function in (1.6) $\varphi=|\Theta_k(u)|^{r-2}\Theta_k(u)\zeta(t)$, with $t_2=T$, t_1 small enough to have $\zeta\equiv 0$ in $[0,t_1]$. Then we obtain

$$\begin{aligned}&-\int_0^T\zeta'(\tau)\int_\Omega S_k(u(\tau))\,dx\,d\tau\\&+(r-1)\int_0^T\zeta\int_\Omega a(x,\tau,\nabla\Theta_k(u))\nabla\Theta_k(u)|\Theta_k(u)|^{r-2}\,dx\,d\tau\\&\quad=\int_0^T\zeta\int_\Omega\psi(x,\tau)u\,|\Theta_k(u)|^{r-2}\Theta_k(u)\,dx\,d\tau\,,\end{aligned}$$

for every $\zeta\in C^\infty_c(0,T)$. Therefore we have, in the sense of distributions,

$$\begin{aligned}\frac{d}{dt}\Big(\int_\Omega S_k(u(t))\,dx\Big)&=-(r-1)\int_\Omega a(x,t,\nabla\Theta_k(u))\nabla\Theta_k(u)\,|\Theta_k(u)|^{r-2}dx\\&+\int_\Omega\psi(x,t)u\,|\Theta_k(u)|^{r-2}\Theta_k(u)\,dx\,.\end{aligned}\tag{2.18}$$

Recalling that u is in $L^r_{\rm loc}(0,T;L^{\frac{Nr}{N-2}}(\Omega))$, and $a(x,t,\nabla u)\cdot\nabla u\,|u|^{r-2}$ belongs to $L^1_{\rm loc}(0,T;L^1(\Omega))$, by the Lebesgue theorem we deduce from (2.18) that $\frac{d}{dt}\Big(\int_\Omega S_k(u(t))\,dx\Big)$ is strongly convergent in $L^1_{\rm loc}(0,T)$ as k tends to infinity. Thus, since $S_k(s)$ converges to $\frac{|s|^r}{r}$, we get that $\int_\Omega |u(t)|^r\,dx$ belongs to $W^{1,1}_{\rm loc}(0,T)$, and for almost every t in $(0,T)$

$$\frac{1}{r}\frac{d}{dt}\Big(\int_\Omega |u(t)|^r\,dx\Big)+(r-1)\int_\Omega a(x,t,\nabla u)\nabla u\,|u|^{r-2}\,dx=\int_\Omega \psi(x,t)|u|^r\,dx\,. \tag{2.19}$$

Let us show that the same equality holds true if $1<r<2$. Indeed, choosing, for $\varepsilon>0$, $\frac{u}{(\varepsilon+|u|)^{2-r}}$ as test function in (1.6) and setting $\Phi_\varepsilon(s)=\int_0^s\frac{\xi}{(\varepsilon+|\xi|)^{2-r}}\,d\xi$ we have

$$\begin{aligned}&\int_\Omega \Phi_\varepsilon(u(t_2))\,dx+\int_{t_1}^{t_2}\int_\Omega \Phi''_\varepsilon(u)a(x,\tau,\nabla u)\nabla u\,dx\,d\tau\\ &\le\int_\Omega \Phi_\varepsilon(u(t_1))\,dx+\int_{t_1}^{t_2}\int_\Omega \psi(x,t)\;\frac{u^2}{(\varepsilon+|u|)^{2-r}}\,dx\,d\tau\,.\end{aligned} \tag{2.20}$$

Since $0\le\Phi_\varepsilon(s)\le\frac{1}{r}\,|s|^r$, $\Phi''_\varepsilon(s)\ge0$ and $\Phi''_\varepsilon(s)$ converges to $(r-1)|s|^{r-2}$ as ε tends to zero, by Fatou's lemma we deduce from (2.20) (recall that u is in $L^2_{\rm loc}(0,T;H^1_0(\Omega))$) that $a(x,t,\nabla u)\nabla u\,|u|^{r-2}$ belongs to $L^1_{\rm loc}(0,T;L^1(\Omega))$. Now, if we take as test function in (1.6) $\frac{u}{(\varepsilon+|u|)^{2-r}}\zeta$, with $\zeta\in C^\infty_c(0,T)$, we get as above that

$$\begin{aligned}&\frac{d}{dt}\Big(\int_\Omega \Phi_\varepsilon(u(t))\,dx\Big)\\ &=-\int_\Omega a(x,t,\nabla u)\nabla u\,\Phi''_\varepsilon(u(t))\,dx+\int_\Omega \psi(x,t)\frac{u^2}{(\varepsilon+|u|)^{2-r}}\,dx\,,\end{aligned}$$

in the sense of distributions. Since $a(x,t,\nabla u)\nabla u\,|u|^{r-2}$ belongs to $L^1_{\rm loc}(0,T;L^1(\Omega))$, the Lebesgue theorem implies that $\frac{d}{dt}\big(\int_\Omega \Phi_\varepsilon(u(t))\,dx\big)$ is strongly convergent in $L^1_{\rm loc}(0,T)$; as $\Phi_\varepsilon(s)$ converges to $\frac{|s|^r}{r}$ this means that the function $\int_\Omega |u(t)|^r\,dx$ is in $W^{1,1}_{\rm loc}(0,T)$ and (2.19) holds true almost everywhere in $(0,T)$ for $1<r<2$ as well.

Moreover, for any positive ζ in $C^\infty_c(0,T)$, we have, using (a_1), (a_2) and the Sobolev embedding theorem

$$\begin{aligned}&-\int_0^T \zeta'(\tau)\int_\Omega \Phi_\varepsilon(u(\tau))\,dx\,d\tau\\&+4\alpha S\frac{(r-1)}{r^2}\int_0^T \zeta\Big(\int_\Omega |(\varepsilon+|u|)^{\frac{r}{2}}-\varepsilon|^{\frac{2N}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\\&\le \int_0^T \zeta\int_\Omega \psi(x,\tau)\frac{u^2}{(\varepsilon+|u|)^{2-r}}\,dx\,d\tau\,,\end{aligned}$$

which yields, as ε tends to zero, the following differential inequality:

$$\frac{1}{r}\frac{d}{dt}\Big(\int_\Omega |u|^r\,dx\Big)+4\alpha S\frac{(r-1)}{r^2}\Big(\int_\Omega |u|^{\frac{Nr}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\le \int_\Omega \psi(x,t)|u|^r\,dx\,.$$

Of course this inequality can still be obtained from (2.19) if $r \ge 2$, and this proves (2.16). □

From (2.16) we obtain important estimates at a fixed time t; to this purpose, we will also use the following simple lemma concerning ordinary differential inequalities.

Lemma 2.6. *Let $f(t)$ be a strictly positive function belonging to $W^{1,1}_{\rm loc}(0,T)$ such that*

$$f'(t)+Af(t)^{1+\theta}\le Bf(t)\,,\qquad \textit{almost everywhere in } (0,T)\,,$$

for some positive constants A, B and θ. Then the function $t^{\frac{1}{\theta}}f(t)$ is bounded in $(0,T)$, and in particular we have

$$t^{\frac{1}{\theta}}f(t)\le \Big(\frac{1}{A\theta}\Big)^{\frac{1}{\theta}}e^{Bt}\qquad \forall t\in(0,T)\,. \tag{2.21}$$

Proof. Setting $g(t)=f(t)^{-\theta}$, thanks to the fact that $f(t)>0$ we have that g belongs to $W^{1,1}_{\rm loc}(0,T)$ and satisfies, for almost every t in $(0,T)$

$$g'+\theta Bg\ge\theta A,\quad g(t)>0\quad \forall t\in(0,T)\,.$$

Hence, by a well-known comparison principle, g is bigger than the solution of the corresponding linear ordinary differential equation with zero initial datum; that is, $g(t) \geq \frac{A}{B}(1 - e^{-\theta Bt})$ for all t in $(0,T)$ (recall that g is continuous in $(0,T)$), which implies

$$tf(t)^\theta \leq \frac{B}{A} e^{\theta Bt} \frac{t}{e^{\theta Bt} - 1} \qquad \forall t \in (0,T)\,,$$

from which we easily deduce (2.21). $\square$

We are ready to prove the following regularity result.

Proposition 2.7. *Let $\psi(x,t)$ belong to $L^\infty(0,T;L^\sigma(\Omega))$, with $\sigma > \frac{N}{2}$, and let u_0 be in $L^q(\Omega)$, $q > 1$. Let $r \geq q$, and assume that u is a weak solution of (1.5) belonging to $L^r_{\rm loc}(0,T;L^{\frac{Nr}{N-2}}(\Omega))$. Then there exist a positive constant C, depending on α, $|\Omega|$, N, q and σ, and a positive constant C_r, depending also on r, such that*

$$\|u(t)\|_{L^q(\Omega)} \leq e^{Ct\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}} \|u_0\|_{L^q(\Omega)} \tag{2.22}$$

$$\|u(t)\|_{L^r(\Omega)} \leq C_r e^{C_r\, t\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}} t^{-\frac{N(r-q)}{2qr}} \|u_0\|_{L^q(\Omega)} \quad \text{if } r > q, \tag{2.23}$$

for every t in $(0,T]$, with $\beta = \frac{2\sigma}{2\sigma - N}$. Moreover, if u belongs to $L^r_{\rm loc}(0,T;L^r(\Omega))$ for every $r > 1$, then we also have (for a possibly different constant $C = C(\alpha, |\Omega|, N, q, \sigma)$)

$$\|u(t)\|_{L^\infty(\Omega)} \leq C e^{Ct\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}}\, t^{-\frac{N}{2q}}\, \|u_0\|_{L^q(\Omega)}\,, \tag{2.24}$$

for every t in $(0,T]$.

Proof. For $r \geq q$, consider (2.16). Using Hölder's inequality with exponents (σ, σ'), and the interpolation for $1 < \sigma' < \frac{N}{N-2}$, we obtain (henceforth we will denote by c_i, $i = 1, 2, \ldots$, positive constants depending only on α, q, σ, N and $|\Omega|$, while we will write c_r to denote possibly different constants depending on r beyond α, q, σ, N and $|\Omega|$)

$$\begin{aligned}\frac{d}{dt}\Big(\int_\Omega |u|^r\,dx\Big) + \Big(\int_\Omega |u|^{\frac{Nr}{N-2}}\,dx\Big)^{\frac{N-2}{N}} &\leq c_0\, r\|\psi(t)\|_{L^\sigma(\Omega)} \Big(\int_\Omega |u|^{r\sigma'}\,dx\Big)^{\frac{1}{\sigma'}} \\ &\leq c_0 r\, \|\psi\|_{L^\infty(0,T;L^\sigma(\Omega))} \Big(\int_\Omega |u|^r\,dx\Big)^{\frac{2\sigma-N}{2\sigma}} \Big(\int_\Omega |u|^{\frac{Nr}{N-2}}\Big)^{\frac{N-2}{2\sigma}}\,,\end{aligned}$$

which yields, thanks to Young's inequality,

$$\frac{d}{dt}\Big(\int_\Omega |u|^r\,dx\Big) + \frac{1}{2}\Big(\int_\Omega |u|^{\frac{Nr}{N-2}}\,dx\Big)^{\frac{N-2}{N}} \leq c_r\, \|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))} \int_\Omega |u|^r\,dx\,. \tag{2.25}$$

If $r = q$, then we obtain

$$\begin{cases} \frac{d}{dt}\Big(\int_\Omega |u|^q\,dx\Big) \le c_1\,\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}\int_\Omega |u|^q\,dx\,, \qquad \text{a.e.} \quad \text{in} \quad (0,T) \\ \int_\Omega |u|^q(0)\,dx = \|u_0\|^q_{L^q(\Omega)}\,, \end{cases}$$

which implies (2.22).

On the other hand, if $q < r < r\frac{N}{N-2}$, we have, using also (2.22),

$$\begin{aligned} \int_\Omega |u|^r\,dx &\le \Big(\int_\Omega |u|^q\,dx\Big)^{\frac{2r}{Nr-q(N-2)}}\Big(\int_\Omega |u|^{\frac{Nr}{N-2}}\,dx\Big)^{\frac{(r-q)(N-2)}{Nr-q(N-2)}} \\ &\le \Big(e^{Ct\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}}\|u_0\|_{L^q(\Omega)}\Big)^{\frac{2rq}{Nr-q(N-2)}}\Big(\int_\Omega |u|^{\frac{Nr}{N-2}}\,dx\Big)^{\frac{(r-q)(N-2)}{Nr-q(N-2)}}. \end{aligned}$$

Therefore, by (2.25) we deduce, setting $M = e^{Ct\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}}\|u_0\|_{L^q(\Omega)}$ and $\theta = \frac{2q}{N(r-q)}$,

$$\frac{d}{dt}\Big(\int_\Omega |u|^r\,dx\Big) + \frac{1}{2M^{\theta r}}\Big(\int_\Omega |u|^r\,dx\Big)^{1+\theta} \le c_r\,\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}\int_\Omega |u|^r\,dx\,. \tag{2.26}$$

Thanks to (2.26) we can apply Lemma 2.6 to the function $\int_\Omega |u(t)|^r\,dx$, so it follows that there exists a constant c_r such that

$$\|u(t)\|^r_{L^r(\Omega)}\,t^{\frac{N(r-q)}{2q}} \le c_r\|u_0\|^r_{L^q(\Omega)}\,e^{c_r\,t\,\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}} \qquad \forall t \le T\,,$$

which implies estimate (2.23).

Now assume that u belongs to $L^r_{\mathrm{loc}}(0,T;L^r(\Omega))$ for every $r > 1$; to obtain an $L^\infty(\Omega)$ estimate, we use a Moser-type iteration method. First of all, using (2.16) we have the following inequality for every $s \ge \frac{q}{2}$:

$$\frac{d}{dt}\Big(\int_\Omega |u|^{2s}\,dx\Big) + \Big(\int_\Omega |u|^{s\frac{2N}{N-2}}\,dx\Big)^{\frac{N-2}{N}} \le c_2\,s\int_\Omega |\psi|\,|u|^{2s}\,dx\,. \tag{2.27}$$

Setting $v \equiv |u|^s$, from (2.27) using Hölder's inequality we have

$$\begin{aligned} \frac{d}{dt}\Big(\int_\Omega |v|^2\,dx\Big) + \Big(\int_\Omega |v|^{\frac{2N}{N-2}}\,dx\Big)^{\frac{N-2}{N}} &\le c_2\,s\int_\Omega |\psi|\,|v|^2\,dx \\ &\le c_2\,s\,\|\psi\|_{L^\infty(0,T;L^\sigma(\Omega))}\Big(\int_\Omega |v|^{2\sigma'}\,dx\Big)^{\frac{1}{\sigma'}}. \end{aligned}$$

The interpolation inequality with $2 < 2\sigma' < \frac{2N}{N-2}$ implies

$$\frac{d}{dt}\Big(\int_\Omega |v|^2\,dx\Big) + \Big(\int_\Omega |v|^{\frac{2N}{N-2}}\,dx\Big)^{\frac{N-2}{N}}$$
$$\le c_2\, s\,\|\psi\|_{L^\infty(0,T;L^\sigma(\Omega))}\Big(\int_\Omega |v|^2\,dx\Big)^{\frac{N-\sigma'(N-2)}{2\sigma'}}\Big(\int_\Omega |v|^{\frac{2N}{N-2}}\,dx\Big)^{\frac{(\sigma'-1)(N-2)}{2\sigma'}},$$

so that Young's inequality yields, with $\beta = \frac{2\sigma}{2\sigma-N}$,

$$\frac{d}{dt}\Big(\int_\Omega |v|^2\,dx\Big) + \frac{1}{2}\Big(\int_\Omega |v|^{\frac{2N}{N-2}}\,dx\Big)^{\frac{N-2}{N}} \le c_3(s\|\psi\|_{L^\infty(0,T;L^\sigma(\Omega))})^\beta \int_\Omega |v|^2\,dx. \tag{2.28}$$

We also have, by interpolation between 1 and $\frac{2N}{N-2}$, for every $\delta > 0$ (δ will be chosen later),

$$\int_\Omega |v|^2\,dx \le \Big(\int_\Omega |v|\,dx\Big)^{\frac{4}{2+N}}\Big(\int_\Omega |v|^{\frac{2N}{N-2}}\,dx\Big)^{\frac{N-2}{N+2}}$$
$$\le \delta\Big(\int_\Omega |v|^{\frac{2N}{N-2}}\,dx\Big)^{\frac{N-2}{N}} + \frac{c_4}{\delta^{\frac{N}{2}}}\Big(\int_\Omega |v|\,dx\Big)^2.$$

Then (2.28) becomes

$$\frac{d}{dt}\Big(\int_\Omega |v|^2\,dx\Big) + \frac{1}{2\delta}\int_\Omega |v|^2\,dx \tag{2.29}$$
$$\le c_3\,(s\|\psi\|_{L^\infty(0,T;L^\sigma(\Omega))})^\beta \int_\Omega |v|^2\,dx + \frac{c_4}{2\delta^{1+\frac{N}{2}}}\Big(\int_\Omega |v|\,dx\Big)^2.$$

Let us now define the positive function $z(t) \equiv t^\mu \int_\Omega |v(t)|^2\,dx$, with μ to be chosen. By Lemma 2.5, $z(t)$ belongs to $W^{1,1}_{\mathrm{loc}}(0,T)$, and thanks to (2.29) $z(t)$ satisfies the following differential inequality:

$$z'(t) \le \Big[\frac{\mu}{t} + c_3(s\|\psi\|_{L^\infty(0,T;L^\sigma(\Omega))})^\beta - \frac{1}{2\delta}\Big]z(t) + \frac{c_4}{2}t^\mu\Big[\frac{1}{\delta^{1+\frac{N}{2}}}\Big]\Big(\int_\Omega |v|\,dx\Big)^2.$$

Choosing $\delta = (2[\frac{\mu}{t} + c_3(s\|\psi\|_{L^\infty(0,T;L^\sigma(\Omega))})^\beta])^{-1}$ we get

$$z'(t) \le c_5\,t^\mu\,\Big[(s\|\psi\|_{L^\infty(0,T;L^\sigma(\Omega))})^\beta + \frac{\mu}{t}\Big]^{1+\frac{N}{2}}\Big(\int_\Omega |v|\,dx\Big)^2;$$

hence, for $t \leq \frac{1}{\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}$ we have, setting $\nu = 1 + \frac{N}{2}$,

$$z'(t) \leq c_5\, t^{\mu-\nu}[\mu + s^{\beta}]^{\nu} \left(\int_\Omega |v|\,dx \right)^2 . \tag{2.30}$$

Now we start with the iterating process, setting $s = s_n = q2^{n-1}$, so $z(t) = t^{\mu}\int_\Omega |u(t)|^{2s_n}\,dx = t^{\mu}\int_\Omega |u(t)|^{q2^n}\,dx$. We will choose $\mu = \mu_n$ such that $\mu_n > \frac{N(2s_n-q)}{2q}$, so as to have, by (2.23) that $z(t) = t^{\mu_n}\int_\Omega |u|^{2s_n}\,dx$ can be continuously extended to 0 by putting $z(0) = 0$. Therefore, the differential inequality (2.30) satisfied by $z(t)$, together with the fact that $z(0) = 0$, allow us to deduce

$$t^{\mu_n}\int_\Omega |u|^{q2^n}\,dx \leq c_5\,[\mu_n + s_n^{\beta}]^{\nu} \int_0^t \xi^{\mu_n-\nu}\left[\int_\Omega |u|^{q2^{n-1}}(\xi)\,dx\right]^2 d\xi\,.$$

Defining $\Phi_n(t) = \frac{z(t)}{t^{\mu_n}} = \int_\Omega |u(t)|^{q2^n}\,dx$, we get, for all $t \leq \frac{1}{\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}$,

$$\Phi_n(t) \leq c_5 \frac{[\mu_n + s_n^{\beta}]^{\nu}}{t^{\mu_n}} \int_0^t \xi^{\mu_n-\nu}\Phi_{n-1}(\xi)^2\,d\xi\,, \tag{2.31}$$

where $\nu = 1 + \frac{N}{2}$, $s_n = q2^{n-1}$, and provided that $\mu_n > \frac{N(2s_n-q)}{2q} = \frac{N(2^n-1)}{2}$. Since by (2.22)

$$\Phi_0(t) = \int_\Omega |u|^q\,dx \leq e^{C\,q\,t\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}\ \|u_0\|^q_{L^q(\Omega)}\,,$$

setting $L = e^{C\,q\,t\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}\ \|u_0\|^q_{L^q(\Omega)}$ we get, at the first step of recurrence,

$$\Phi_1(t) \leq c_5 \frac{(\mu_1 + q^{\beta})^{\nu}}{t^{\mu_1}}\, L^2 \int_0^t \xi^{\mu_1-\nu}\,d\xi\,.$$

Choosing $\mu_1 = \nu$, we obtain $\Phi_1(t) \leq c_5 \frac{(\mu_1+q^{\beta})^{\nu}}{t^{\nu-1}}\, L^2$; hence, by (2.31), we get

$$\Phi_2(t) \leq c_5 \frac{(\mu_2 + (2q)^{\beta})^{\nu}}{t^{\mu_2}}\,(c_5(\mu_1 + q^{\beta})^{\nu})^2 L^4 \int_0^t \xi^{\mu_2-\nu-2(\nu-1)}\,d\xi\,.$$

We will choose $\mu_2 = \nu + 2(\nu-1)$, and then at every step $\mu_n = \nu + 2(\mu_{n-1}-1)$, so going on we prove by induction

$$\Phi_n(t) \leq c_5^{\sum_{j=0}^{n-1} 2^j}\ L^{2^n}\, b_1^{2^{n-1}}\, b_2^{2^{n-2}} \cdots b_n\, t^{-(\mu_n-1)}\,,$$

where $\mu_n = \nu + (\nu-1)\sum_{j=1}^{n-1} 2^j$, $L = e^{Cqt\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}\|u_0\|^q_{L^q(\Omega)}$, $b_n = (\mu_n + q^\beta 2^{\beta(n-1)})^\nu$. We need to remark that with this choice $\mu_n = 1 + \frac{N}{2}(2^n-1)$, so the condition $\mu_n > \frac{N(2^n-1)}{2}$ required for (2.31) to hold is always satisfied. Thus we finally proved that, for every $t \le \frac{1}{\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}$,

$$\Phi_n^{\frac{1}{q2^n}} = \Big(\int_\Omega |u(t)|^{q2^n}\,dx\Big)^{\frac{1}{q2^n}} \le c_5^{\sum_{j=1}^{n}\frac{1}{q2^j}}\, L^{\frac{1}{q}}\Big(b_1^{\frac12} b_2^{\frac14}\cdots b_n^{\frac{1}{2^n}}\Big)^{\frac1q}\, t^{-\frac{\mu_n-1}{q2^n}}\,. \quad (2.32)$$

Our goal is now to let n tend to infinity, since the left-hand side of (2.32) will converge to $\|u(t)\|_{L^\infty(\Omega)}$; first of all, we can estimate the b_n's as follows: $b_n \le c_6 2^{n\beta\nu}$; hence,

$$b_1^{\frac12} b_2^{\frac14}\cdots b_n^{\frac{1}{2^n}} \le c_6^{\sum_{j=1}^{n}\frac{1}{2^j}}\, 2^{\sum_{j=1}^{n}\frac{\beta\nu j}{2^j}} \le c_7\,,$$

since the series $\sum_{j=1}^{+\infty}\frac{j}{2^j}$ is convergent. Moreover, we have $\lim_{n\to+\infty}\frac{\mu_n-1}{q2^n} = \frac{N}{2q}$, so that, passing to the limit in (2.32), we obtain, with $\beta = \frac{2\sigma}{2\sigma-N}$,

$$\|u(t)\|_{L^\infty(\Omega)} \le \frac{c_8 e^{c_8 t\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}\|u_0\|_{L^q(\Omega)}}{t^{\frac{N}{2q}}}\,, \qquad \forall t \le \frac{1}{\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}\,;$$

hence, $\|u(t)\|_{L^\infty(\Omega)} \le c_9\, t^{-\frac{N}{2q}}\,\|u_0\|_{L^q(\Omega)}$, $\forall t \le \frac{1}{\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}$. Now, this estimate implies (2.24) for all t in $[0,T]$. Indeed, set $\tau_0 = \frac{1}{2\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}$; then for t in $[2\tau_0, 3\tau_0] = [\frac{1}{\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}, \frac{3}{2\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}]$ we can repeat the same estimate to obtain

$$\begin{aligned}\|u(t)\|_{L^\infty(\Omega)} &\le c_9\frac{1}{(t-\tau_0)^{\frac{N}{2q}}}\|u(\tau_0)\|_{L^q(\Omega)}\\ &\le c_9\frac{2^{\frac{N}{2q}}}{t^{\frac{N}{2q}}}\, e^{C\,\tau_0\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}\,\|u_0\|_{L^q(\Omega)}\,.\end{aligned}$$

More generally, for t in $[k\tau_0, (k+1)\tau_0]$ we find

$$\begin{aligned}\|u(t)\|_{L^\infty(\Omega)} &\le c_9\frac{k^{\frac{N}{2q}}}{t^{\frac{N}{2q}}}\, e^{Ck\tau_0\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}\,\|u_0\|_{L^q(\Omega)}\\ &\le c_{10}\frac{1}{t^{\frac{N}{2q}}}\, e^{c_{10}t\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}\,\|u_0\|_{L^q(\Omega)}\,,\end{aligned}$$

so (2.24) is proved.

Remark 2.8. If u belongs to $L^r_{\rm loc}(0,T;L^r(\Omega))$ for every $1<r<+\infty$, then (2.23) can be slightly improved, in the sense that the same estimate holds true for every r with a constant not depending on r, as it can be easily obtained by interpolation between (2.22) and (2.24).

Remark 2.9. It is worth pointing out that in the proofs of Lemma 2.5 and Proposition 2.7 we used only the coercivity of the operator, i.e., the assumption that $a(x,t,\xi)\cdot\xi \geq \alpha|\xi|^2$ for every ξ in $\mathbf{R}^N$, so the result could also be applied to nonmonotone operators.

On the other hand, using the strong monotonicity assumption (a_1) an identical proof can be performed to obtain the following result.

Corollary 2.9. *Let $\psi(x,t)$ belong to $L^\infty(0,T;L^\sigma(\Omega))$, with $\sigma > \frac{N}{2}$, and assume that u_0, v_0 are in $L^q(\Omega)$, $q>1$. Let $r\geq q$, and let u, v be weak solutions of*

$$\begin{cases} u_t - \operatorname{div}(a(x,t,\nabla u)) = \psi(x,t)u & \text{in } \Omega\times(0,T),\\ u=0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x) & \text{in } \Omega, \end{cases}$$

$$\begin{cases} v_t - \operatorname{div}(a(x,t,\nabla v)) = \psi(x,t)v & \text{in } \Omega\times(0,T),\\ v=0 & \text{on } \partial\Omega\times(0,T),\\ v(x,0)=v_0(x) & \text{in } \Omega. \end{cases}$$

Assume that u and v belong to $L^r_{\rm loc}(0,T;L^{\frac{Nr}{N-2}}(\Omega))$, then there exist a positive constant C, depending on α, $|\Omega|$, N, q and σ, and a positive constant C_r, depending also on r, such that

$$\|u(t)-v(t)\|_{L^q(\Omega)} \leq e^{Ct\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}}\|u_0-v_0\|_{L^q(\Omega)}\,, \tag{2.33}$$

$$\|u(t)-v(t)\|_{L^r(\Omega)} \leq C_r e^{C_r\,t\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}} t^{-\frac{N(r-q)}{2qr}}\|u_0-v_0\|_{L^q(\Omega)} \tag{2.34}$$

if $r>q$, for every t in $(0,T]$, with $\beta=\frac{2\sigma}{2\sigma-N}$. If in addition u and v belong to $L^r_{\rm loc}(0,T;L^r(\Omega))$ for every $r>1$, then we also have (for a possibly different constant $C=C(\alpha,|\Omega|,N,q,\sigma)$)

$$\|u(t)-v(t)\|_{L^\infty(\Omega)} \leq Ce^{Ct\|\psi\|^\beta_{L^\infty(0,T;L^\sigma(\Omega))}}\,t^{-\frac{N}{2q}}\,\|u_0-v_0\|_{L^q(\Omega)}\,, \tag{2.35}$$

for every t in $(0,T]$.

3. Compactness and stability properties. In this section we study the properties of continuity and compactness for equation (1.5), which will have important consequences in the study of (1.3). We still work in the framework of the space $C([0,T];L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, we recall that $q>1$, $N>2$.

Proposition 3.1. *Let $\{\psi_n(x,t)\}$ be a sequence bounded in $L^\infty(0,T;L^\sigma(\Omega))$, for $\sigma>\frac{N}{2}$. Assume also that $\{u_{0n}\}\subset L^\infty(\Omega)$ strongly converges in $L^q(\Omega)$, and that $\{v_n\}\subset L^\infty(Q)$ is a bounded sequence in*

$$C([0,T];L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega)),$$

and let u_n be the weak solution of

$$\begin{cases} (u_n)_t-\operatorname{div}(a(x,t,\nabla u_n))=\psi_n(x,t)v_n & \text{in } \Omega\times(0,T),\\ u_n=0 & \text{on } \partial\Omega\times(0,T),\\ u_n(x,0)=u_{0n}(x) & \text{in } \Omega.\end{cases} \tag{3.1}$$

Then u_n is relatively compact in $C([0,T];L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$.

Proof. First of all, the a priori estimates in Proposition 2.1 imply that the sequence $\{u_n\}$ is bounded in $C([0,T];L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$; hence there exists a function u in $L^\infty(0,T;L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$ such that, up to a subsequence, u_n converges to u $*$-weakly in $L^\infty(0,T;L^q(\Omega))$ and weakly in $L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$. On the other hand, the sequence $\{\psi_n\, v_n\}$ is clearly bounded in $L^1(Q)$; hence, for instance applying the results in [3], we also have, again for a subsequence, not relabeled, that $u_n\to u$ almost everywhere in Q. Denoting by $\Theta_k(u)=\max(-k,\min(u,k))$ the truncation function at levels $\pm k$ already introduced above, we can deduce, by the almost-everywhere convergence, that

$$\Theta_k(u_n)\to\Theta_k(u)\ \text{ strongly in } L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega)),\text{ for every } k>0. \tag{3.2}$$

Then setting $G_k(s)=s-\Theta_k(s)$, we turn to the study of $G_k(u_n)$. As in the proof of Proposition 2.1, we need to distinguish between the cases $1<q<2$ and $q\ge 2$. If $1<q<2$, we choose as test function in (3.1) $\frac{G_k(u_n)}{(\varepsilon+|G_k(u_n)|)^{2-q}}$, with $\varepsilon>0$, and we set $\Phi_\varepsilon^k(s)=\int_0^s\frac{G_k(r)}{(\varepsilon+|G_k(r)|)^{2-q}}\,dr$. Then we have, by (a_1) and (a_2),

$$\begin{aligned}&\int_\Omega\Phi_\varepsilon^k(u_n)(T)\,dx+\alpha(q-1)\int_0^T\int_\Omega\frac{|\nabla G_k(u_n)|^2}{(\varepsilon+|G_k(u_n)|)^{2-q}}\,dx\,d\tau\\ &\le\int_\Omega\Phi_\varepsilon^k(u_{0n})\,dx+\int_0^T\int_\Omega|\psi_n|\,|v_n||G_k(u_n)|^{q-1}\,dx\,d\tau\,.\end{aligned}$$

Note that $0 \le \Phi_\varepsilon^k(s) \le \frac{1}{q}\,|G_k(s)|^q$ and that $G_k(s) \equiv 0$ if $|s| \le k$; thus, using Sobolev's embedding theorem and Young's inequality we obtain (henceforth we denote by c_i positive constants depending on q, α, $|\Omega|$, N and σ)

$$c_1 \int_0^T \Big(\int_\Omega |(\varepsilon + |G_k(u_n)|)^{\frac{q}{2}} - \varepsilon|^{\frac{2N}{N-2}}\, dx\Big)^{\frac{N-2}{N}} d\tau \le \int_\Omega |G_k(u_{0n})|^q\, dx$$
$$+ \int_0^T \int_\Omega |\psi_n|\, |v_n|^q\, \chi_{\{|u_n|>k\}}\, dx\, d\tau + \int_0^T \int_\Omega |\psi_n|\, |G_k(u_n)|^q\, dx\, d\tau\,,$$

which yields, as ε tends to zero, for $1 < q < 2$,

$$\begin{aligned} &c_1 \int_0^T \Big(\int_\Omega |G_k(u_n)|^{\frac{Nq}{N-2}}\, dx\Big)^{\frac{N-2}{N}} d\tau \le \int_\Omega |G_k(u_{0n})|^q\, dx \\ &+ \int_0^T \int_\Omega |\psi_n|\, |v_n|^q\, \chi_{\{|u_n|>k\}}\, dx\, d\tau + \int_0^T \int_\Omega |\psi_n|\, |G_k(u_n)|^q\, dx\, d\tau\,. \end{aligned} \tag{3.3}$$

It is easier to get (3.3) if $q \ge 2$, simply by taking $|G_k(u_n)|^{q-2}G_k(u_n)$ as test function in (3.1), so we can proceed with $1 < q < \infty$. Our next goal is to prove that the last two terms of (3.3), which play a similar role, go to zero as k tends to infinity uniformly with respect to n. Indeed, using several times Hölder's inequality, we have, for every $\lambda > 0$

$$\begin{aligned} &\int_0^T\!\!\int_\Omega |\psi_n|\, |v_n|^q \chi_{\{|u_n|>k\}} dx\, d\tau \le \int_0^T \|\psi_n\|_{L^\sigma(\Omega)} \Big(\int_\Omega |v_n|^{q\sigma'} \chi_{\{|u_n|>k\}}\, dx\Big)^{\frac{1}{\sigma'}} d\tau \\ &\le \|\psi_n\|_{L^\infty(0,T;L^\sigma(\Omega))} \int_0^T \Big(\int_\Omega |v_n|^{q\sigma'(1+\lambda)} dx\Big)^{\frac{1}{\sigma'(1+\lambda)}} \Big(\int_\Omega \chi_{\{|u_n|>k\}}\, dx\Big)^{\frac{\lambda}{\sigma'(1+\lambda)}} d\tau \\ &\le c_2 \Big(\int_0^T \Big(\int_\Omega |v_n|^{q\sigma'(1+\lambda)}\, dx\Big)^{\frac{1}{\sigma'}} d\tau\Big)^{\frac{1}{1+\lambda}} \Big(\int_0^T \Big(\int_\Omega \chi_{\{|u_n|>k\}} dx\Big)^{\frac{1}{\sigma'}} d\tau\Big)^{\frac{\lambda}{(1+\lambda)}} \\ &\le c_3 \Big(\int_0^T \Big(\int_\Omega (|v_n|^{\frac{q}{2}})^{2\sigma'(1+\lambda)} dx\Big)^{\frac{2(1+\lambda)}{2(1+\lambda)\sigma'}} d\tau\Big)^{\frac{1}{1+\lambda}} \operatorname{meas}\{(x,t) : |u_n| > k\}^{\frac{\lambda}{(1+\lambda)\sigma'}}. \end{aligned} \tag{3.4}$$

We recall that by interpolation the space $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;L^{\frac{2N}{N-2}}(\Omega))$ injects into the space $L^r(0,T;L^p(\Omega))$ for every couple (r,p) such that $2 \le r$, $2 \le p \le \frac{2N}{N-2}$ and $\frac{1}{r} + \frac{N}{2p} = \frac{N}{4}$ (see [8]). Choosing $\lambda = \frac{2}{N} - \frac{1}{\sigma}$ (recall that $\sigma > \frac{N}{2}$) we have

$$\frac{1}{2(1+\lambda)} + \frac{N}{(2\sigma'(1+\lambda))2} = \frac{N}{4}\,; \tag{3.5}$$

thus, since $|v_n|^{\frac{q}{2}}$ is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;L^{\frac{2N}{N-2}}(\Omega))$, by (3.5) and the interpolation theorem we have that $|v_n|^{\frac{q}{2}}$ is also bounded in $L^{2(1+\lambda)}(0,T;L^{2\sigma'(1+\lambda)}(\Omega))$, so that (3.4) implies

$$\int_0^T \int_\Omega |\psi_n|\,|v_n|^q\,\chi_{\{|u_n|>k\}}\,dx\,d\tau \le c_4\,\text{meas}\{(x,t)\,:\,|u_n|>k\}^{\frac{\lambda}{(1+\lambda)\sigma'}}\,;$$

hence, $\{u_n\}$ being bounded in $L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$,

$$\lim_{k\to+\infty}\int_0^T \int_\Omega |\psi_n|\,|v_n|^q\,\chi_{\{|u_n|>k\}}\,dx\,d\tau = 0 \quad \text{uniformly with respect to } n.$$

The term $\int_0^T\int_\Omega |\psi_n|\,|G_k(u_n)|^q\,dx$ in (3.3) can be dealt with in the same way; hence it follows from (3.3)

$G_k(u_n) \overset{k\to+\infty}{\to} 0$ strongly in $L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, uniformly with respect to n.

Together with (3.2) this implies

$$u_n \to u \qquad \text{strongly in } L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega)). \tag{3.6}$$

Moreover, if we take $|u_n-u_m|^{q-2}(u_n-u_m)$—if $q\ge 2$, otherwise we need to use the auxiliary test function $\frac{r}{(\varepsilon+|r|)^{2-q}}$, and then let ε tend to zero, as at the beginning of the proof—as test function both in the equation relative to u_n and in that of u_m, subtracting we get, for all $s\le T$,

$$\begin{aligned}
&\int_\Omega |u_n-u_m|^q(s)\,dx \qquad\qquad (3.7)\\
&+\int_0^s\int_\Omega (a(x,\tau,\nabla u_n)-a(x,\tau,\nabla u_m))\nabla(u_n-u_m)|u_n-u_m|^{q-2}\,dx\,d\tau\\
&\le c_5\int_0^s\int_\Omega (|\psi_n|\,|v_n-v_m|+|v_m|\,|\psi_n-\psi_m|)\,|u_n-u_m|^{q-1}\,dx\,d\tau\\
&\quad + c_5\int_\Omega |u_{0n}-u_{0m}|^q dx.
\end{aligned}$$

Using (a_1) and Hölder's inequality, and the fact that ψ_n is bounded in $L^\infty(0,T;L^\sigma(\Omega))$, we have:

$$\begin{aligned}
&\int_0^s\int_\Omega (|\psi_n|\,|v_n-v_m|+|v_m|\,|\psi_n-\psi_m|)\,|u_n-u_m|^{q-1}\,dx\,d\tau\\
&\le c_6\int_0^T\left(\int_\Omega |v_n-v_m|^{\sigma'}\,|u_n-u_m|^{(q-1)\sigma'}\,dx\right)^{\frac{1}{\sigma'}}\,d\tau\\
&+\|\psi_n-\psi_m\|_{L^\infty(0,T;L^\sigma(\Omega))}\int_0^T\left(\int_\Omega |v_m|^{\sigma'}\,|u_n-u_m|^{(q-1)\sigma'}\,dx\right)^{\frac{1}{\sigma'}}\,d\tau\,,
\end{aligned}$$

so that we obtain:

$$\int_\Omega |u_n - u_m|^q(s)\,dx \le c_7 \|v_n - v_m\|_{L^q(0,T;L^{q\sigma'}(\Omega))} \|u_n - u_m\|^{q-1}_{L^q(0,T;L^{q\sigma'}(\Omega))}$$
$$+ \|v_m\|_{L^q(0,T;L^{q\sigma'}(\Omega))} \|u_n - u_m\|^{q-1}_{L^q(0,T;L^{q\sigma'}(\Omega))} + \int_\Omega |u_{0n} - u_{0m}|^q dx, \qquad (3.8)$$

$\forall s \le T$. Since $\sigma' < \frac{N}{N-2}$, and v_n is bounded in $L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, and thanks to (3.6), taking the supremum on s in (3.8) we deduce that u_n is a Cauchy sequence in $C([0,T];L^q(\Omega))$; hence u_n is relatively compact in

$$C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega)).$$

Remark 3.1. At the beginning of the proof of Proposition 3.1 we used the results in [3] to deduce that, up to a subsequence, the solutions u_n of (3.1) almost everywhere converge in Q. The same results also imply that the sequence $\{\nabla u_n\}$ is almost-everywhere convergent in Q. If $q = 2$, this can be also deduced from (3.7) and (3.8), which in this case yield, using assumption (a_1), that u_n is strongly convergent in $L^2(0,T;H^1_0(\Omega))$. Of course this still holds true if $\{u_{0n}\}$ is strongly convergent in $L^q(\Omega)$ with $q > 2$.

Let us also point out that, in order to obtain the compactness result in Proposition 3.1, we only used that $a(x,t,\xi)\cdot\xi \ge \alpha|\xi|^2$ and $(a(x,t,\xi) - a(x,t,\eta))\cdot(\xi-\eta) > 0$ for every ξ and η in $\mathbf{R}^N$ $(\xi \ne \eta)$.

Remark 3.2. In the case $\sigma = \frac{N}{2}$, the result of Proposition 3.1 still holds true if we assume that v_n strongly converges in $L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$ and ψ_n strongly converges in $L^\infty(0,T;L^{\frac{N}{2}}(\Omega))$. Indeed, this follows easily from the following inequality obtained subtracting the equations solved by u_n and u_m:

$$\int_\Omega |u_n - u_m|^q(s)\,dx + \int_0^s \Big(\int_\Omega |u_n - u_m|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}} d\tau$$
$$\le c_7 \int_0^s \int_\Omega |\psi_n v_n - \psi_m v_m||v_n - v_m||u_n - u_m|^{q-1} dx\,d\tau + c_7 \int_\Omega |u_{0n} - u_{0m}|^q\,dx.$$

Using Hölder's inequality with exponents $(\frac{N}{2}, \frac{N}{N-2})$ and Young's inequality with (q, q'), we get

$$\int_\Omega |u_n - u_m|^q(s)\,dx + \frac{1}{2}\int_0^s \Big(\int_\Omega |u_n - u_m|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau$$
$$\le c_8 \int_0^s \Big(\int_\Omega |v_n - v_m|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau + c_8\|\psi_n - \psi_m\|^q_{L^\infty(0,T;L^{\frac{N}{2}}(\Omega))}$$
$$+ c_8 \int_\Omega |u_{0n} - u_{0m}|^q\,dx\,,$$

which yields the convergence of u_n in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$.

The following result adds to Proposition 3.1 some simple information we will often use later.

Proposition 3.2. *Let $\{\psi_n(x,t)\}$ be strongly convergent in $L^\infty(0,T;L^\sigma(\Omega))$ to ψ, with $\sigma \ge \frac{N}{2}$, and let $\{u_{0n}\}, \{\bar u_{0n}\} \subset L^\infty(\Omega)$ be strongly convergent in $L^q(\Omega)$ to the same function u_0. Assume that $\{v_n\} \subset L^\infty(Q)$ and $\{\bar v_n\} \subset L^\infty(Q)$ converge to the same function v strongly in $L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$ and $*$-weakly in $L^\infty(0,T;L^q(\Omega))$. If u_n and $\bar u_n$ are the solutions of the following problems,*

$$\begin{cases} (u_n)_t - \operatorname{div}(a(x,t,\nabla u_n)) = \psi_n(x,t)v_n & \text{in } \Omega\times(0,T),\\ u_n = 0 & \text{on } \partial\Omega\times(0,T),\\ u_n(x,0) = u_{0n}(x) & \text{in } \Omega,\end{cases}$$

$$\begin{cases} (\bar u_n)_t - \operatorname{div}(a(x,t,\nabla \bar u_n)) = \psi_n(x,t)\bar v_n & \text{in } \Omega\times(0,T),\\ \bar u_n = 0 & \text{on } \partial\Omega\times(0,T),\\ \bar u_n(x,0) = \bar u_{0n}(x) & \text{in } \Omega,\end{cases}$$

then u_n and $\bar u_n$ strongly converge to the same limit u in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$.

Proof. By Proposition 3.1 and Remark 3.2 there are subsequences, still denoted u_n and $\bar u_n$, strongly convergent in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, say, respectively to functions u and $\bar u$. Writing (3.8) for u_n and $\bar u_n$ we get, for all $s \le T$,

$$\int_\Omega |u_n - \bar u_n|^q(s)\,dx \le c_6\,\|v_n - \bar v_n\|_{L^q(0,T;L^{q\sigma'}(\Omega))}\,\|u_n - \bar u_n\|^{q-1}_{L^q(0,T;L^{q\sigma'}(\Omega))}$$
$$+ c_6\|u_{0n} - \bar u_{0n}\|^q_{L^q(\Omega)}$$
$$\le c_7\|v_n - \bar v_n\|_{L^q(0,T;L^{q\sigma'}(\Omega))} + c_6\|u_{0n} - \bar u_{0n}\|^q_{L^q(\Omega)}\,.$$

Since $q\sigma' \le \frac{Nq}{N-2}$ (note that here we admit the value $\sigma = \frac{N}{2}$) passing to the limit we deduce that $u = \bar{u}$; this also implies that the whole sequences u_n and $\bar{u}_n$ converge, as well. $\square$

It will be useful to clearly state a similar result in the case of problem (1.3) as well.

Proposition 3.3. *Let $\rho(x,t)$ be a bounded function, and let $\{u_{0n}\}, \{\bar{u}_{0n}\} \subset L^\infty(\Omega)$ be strongly convergent to u_0 in $L^q(\Omega)$, with $q \ge \frac{N(p-1)}{2}$. Assume that $\{v_n\} \subset L^\infty(Q)$ and $\{\bar{v}_n\} \subset L^\infty(Q)$ converge to a same function v strongly in $L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$ and $*$-weakly in $L^\infty(0,T;L^q(\Omega))$. If u_n and $\bar{u}_n$ are the solutions of the following problems,*

$$\begin{cases} (u_n)_t - \operatorname{div}(a(x,t,\nabla u_n)) = \rho(x,t)|v_n|^{p-1}v_n & \text{in } \Omega \times (0,T), \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(x,0) = u_{0n}(x) & \text{in } \Omega, \end{cases}$$

$$\begin{cases} (\bar{u}_n)_t - \operatorname{div}(a(x,t,\nabla \bar{u}_n)) = \rho(x,t)|\bar{v}_n|^{p-1}\bar{v}_n & \text{in } \Omega \times (0,T), \\ \bar{u}_n = 0 & \text{on } \partial\Omega \times (0,T), \\ \bar{u}_n(x,0) = \bar{u}_{0n}(x) & \text{in } \Omega, \end{cases}$$

then u_n and $\bar{u}_n$ strongly converge to the same limit u in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$.

Proof. It is enough to observe that in this case (3.7) with u_n and $\bar{u}_n$ becomes

$$\begin{aligned} &\int_\Omega |u_n - \bar{u}_n|^q(s)\,dx \\ &+ \int_0^s \int_\Omega (a(x,\tau,\nabla u_n) - a(x,\tau,\nabla \bar{u}_n))\nabla(u_n - \bar{u}_n)\,|u_n - \bar{u}_n|^{q-2}\,dx\,d\tau \\ &\le c_5 \int_0^s \int_\Omega |\rho| \left||v_n|^{p-1}v_n - |\bar{v}_n|^{p-1}\bar{v}_n\right| |u_n - \bar{u}_n|^{q-1}\,dx\,d\tau \\ &+ c_5 \int_\Omega |u_{0n} - \bar{u}_{0n}|^q\,dx \\ &\quad \le c_5\|\rho\|_\infty \int_0^T \int_\Omega \Big(|v_n|^{p-1} + |\bar{v}_n|^{p-1}\Big)|v_n - \bar{v}_n|\,|u_n - \bar{u}_n|^{q-1}\,dx\,d\tau \\ &+ c_5 \int_\Omega |u_{0n} - \bar{u}_{0n}|^q\,dx\,, \end{aligned}$$

which yields, by Hölder's inequality, with $\sigma = \frac{q}{p-1}$,

$$\int_\Omega |u_n - \bar{u}_n|^q(s)\,dx + \int_0^s \Big(\int_\Omega |u_n - \bar{u}_n|^{\frac{Nq}{N-2}}\,dx \Big)^{\frac{N-2}{N}} d\tau$$
$$\le c_6\, \|v_n - \bar{v}_n\|_{L^q(0,T;L^{q\sigma'}(\Omega))}\, \|u_n - \bar{u}_n\|^{q-1}_{L^q(0,T;L^{q\sigma'}(\Omega))}$$
$$+\, c_6 \|u_{0n} - \bar{u}_{0n}\|^q_{L^q(\Omega)}\,,$$

and again the conclusion follows from the fact that $q\sigma' \le \frac{Nq}{N-2}$ and using the strong convergence of v_n and $\bar{v}_n$. $\square$

4. Existence and uniqueness results via a contraction mapping. We have now all the tools needed to prove our main result. Let us first concentrate on problem (1.3), although most ideas come from dealing with the problem with linear growth (1.5), which we will treat later. Let us also stress that the method used, that is, the construction of a contraction mapping in the space $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, is new even for the heat equation.

Proof of Theorem 1.1, if $q > \max\{1, \frac{N(p-1)}{2}\}$**.**

Step 1: Construction of the mapping. Let us define the following operator in the space $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$. Denoting by $\Theta_k(s)$ the truncation function as before, take v in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, and let $\{v_n\}$ be a sequence of bounded functions strongly convergent to v in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$. Then we consider the solutions u_n of the following Cauchy-Dirichlet problems:

$$\begin{cases} (u_n)_t - \operatorname{div}(a(x,t,\nabla u_n)) = \rho(x,t)|v_n|^{p-1}v_n & \text{in } \Omega \times (0,T), \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(x,0) = \Theta_n(u_0)(x) & \text{in } \Omega, \end{cases} \tag{4.1}$$

and we define $\mathcal{F}(v) = \lim\limits_{n\to\infty} u_n$, the limit being intended in the topology of $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$. The definition is well posed because of Proposition 3.1 (recall that $\frac{q}{p-1} > \frac{N}{2}$), which says that u_n is compact in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, and thanks to Proposition 3.3, which states that the whole sequence u_n converges and moreover $\mathcal{F}(v)$ does not depend on the choice of the sequence v_n approximating v in $C([0,T];L^q(\Omega))\cap$

$L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$. The possibility of obtaining $\mathcal{F}(v)$ by choosing possibly different approximations of v will be often used in the sequel. It is also worth noting that, if $q \geq 2$, by Remark 3.1 we know that u_n strongly converges in $L^2(0,T;H_0^1(\Omega))$; hence, $\mathcal{F}(v)$ is clearly a weak solution of

$$\begin{cases} u_t - \operatorname{div}(a(x,t,\nabla u)) = \rho(x,t)|v|^{p-1}v & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Step 2: The contraction mapping principle. By virtue of Proposition 2.2, we can apply twice (2.11) to the approximating problems (4.1), then passing to the limit as n tends to infinity we get, for a positive constant $C_1 = C_1(\alpha, |\Omega|, N, p, q)$, and with $\gamma = \frac{2q(p-1)}{2q-N(p-1)}$,

$$\begin{aligned} &\Big[1 - T\,C_1 \|\rho\|_\infty^{\frac{\gamma}{p-1}} \, \|v\|^\gamma_{L^\infty(0,T;L^q(\Omega))}\Big] \, \|\mathcal{F}(v)\|^q_{L^\infty(0,T;L^q(\Omega))} \\ &+ \frac{1}{2}\int_0^T \Big(\int_\Omega |\mathcal{F}(v)|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}} d\tau \\ &\leq C_1 \int_\Omega |u_0|^q\,dx + \Big[T\,C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}} \, \|v\|^\gamma_{L^\infty(0,T;L^q(\Omega))}\Big] \, \|v\|^q_{L^\infty(0,T;L^q(\Omega))} \\ &+ \frac{1}{4}\int_0^T \Big(\int_\Omega |v|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}} d\tau\,, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\Big[1 - T\,C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}} \Big(\|v\|^\gamma_{L^\infty(0,T;L^q(\Omega))} + \|w\|^\gamma_{L^\infty(0,T;L^q(\Omega))}\Big)\Big] \\ &\times \|\mathcal{F}(v) - \mathcal{F}(w)\|^q_{L^\infty(0,T;L^q(\Omega))} + \frac{1}{2}\int_0^T \Big(\int_\Omega |\mathcal{F}(v)-\mathcal{F}(w)|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}} d\tau \\ &\leq T\,C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}} \Big(\|v\|^\gamma_{L^\infty(0,T;L^q(\Omega))} + \|w\|^\gamma_{L^\infty(0,T;L^q(\Omega))}\Big)\|v-w\|^q_{L^\infty(0,T;L^q(\Omega))} \\ &+ \frac{1}{4}\int_0^T \Big(\int_\Omega |v-w|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}} d\tau\,. \end{aligned} \tag{4.3}$$

Let us now define

$$\begin{aligned} K = \Big\{\varphi \in C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega)) :& \\ \|\varphi\|^q_{C([0,T];L^q(\Omega))} \leq 4C_1\|u_0\|^q_{L^q(\Omega)}\,,& \\ \int_0^T \Big(\int_\Omega |\varphi|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}} d\tau \leq 8C_1\,\|u_0\|^q_{L^q(\Omega)}\Big\}.& \end{aligned}$$

Since K depends on T and u_0 we will alternatively write $K(T,u_0)$ instead of simply K. Note that it is a closed, convex subset of $C([0,T];L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$; hence, it is a complete metric space with the inherited distance; we will show that there exists a small T such that $\mathcal{F}$ is a contraction mapping from K to K. Indeed, from (4.2) we deduce that if v belongs to K, then

$$\begin{aligned}&\Big[1-C_1\,T\|\rho\|_\infty^{\frac{\gamma}{p-1}}(4C_1)^{\frac{\gamma}{q}}\|u_0\|^{\gamma}_{L^q(\Omega)}\Big]\,\|\mathcal{F}(v)\|^q_{C([0,T];L^q(\Omega))}\\&+\frac12\int_0^T\Big(\int_\Omega|\mathcal{F}(v)|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\\&\le 3C_1\|u_0\|^q_{L^q(\Omega)}+C_1\,T\,\|\rho\|_\infty^{\frac{\gamma}{p-1}}(4C_1)^{1+\frac{\gamma}{q}}\|u_0\|^{\gamma}_{L^q(\Omega)}\,\|u_0\|^q_{L^q(\Omega)}\,.\end{aligned}$$

It is now easy to see that there exists T such that $\mathcal{F}(v)$ belongs to K for all v in K; for instance we can take, to fix our ideas, $T\le\frac{1}{8C_1(4C_1)^{\frac{\gamma}{q}}\|\rho\|_\infty^{\frac{\gamma}{p-1}}\|u_0\|^{\gamma}_{L^q(\Omega)}}$, to have

$$\frac78\|\mathcal{F}(v)\|^q_{C([0,T];L^q(\Omega))}+\frac12\int_0^T\Big(\int_\Omega|\mathcal{F}(v)|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\le\frac72C_1\|u_0\|^q_{L^q(\Omega)}\,,$$

so that we get that $\mathcal{F}(v)$ belongs to K as well.

Moreover, for v and w in K, we have from (4.3)

$$\begin{aligned}&\Big[1-2\,T\,C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}}\,(4C_1)^{\frac{\gamma}{q}}\|u_0\|^{\gamma}_{L^q(\Omega)}\Big]\,\|\mathcal{F}(v)-\mathcal{F}(w)\|^q_{C([0,T];L^q(\Omega))}\\&+\frac12\int_0^T\Big(\int_\Omega|\mathcal{F}(v)-\mathcal{F}(w)|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\\&\le 2\,T\,C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}}\,(4C_1)^{\frac{\gamma}{q}}\|u_0\|^{\gamma}_{L^q(\Omega)}\|v-w\|^q_{C([0,T];L^q(\Omega))}\\&+\frac14\int_0^T\Big(\int_\Omega|v-w|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\,;\end{aligned}$$

hence, for $T\le\frac{1}{8C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}}\,(4C_1)^{\frac{\gamma}{q}}\|u_0\|^{\gamma}_{L^q(\Omega)}}$, denoting by dist_K the distance inherited by $K(T,u_0)$ as a subset of $C([0,T];L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, we have that

$$\mathrm{dist}_K(\mathcal{F}(v),\mathcal{F}(w))\le\Big(\frac12\Big)^{\frac1q}\mathrm{dist}_K(v,w)\,.$$

Thus we have proved that there exists a time $T = T(u_0)$ such that $\mathcal{F}$ is a contraction from $K(T, u_0)$ to $K(T, u_0)$; hence, it admits a unique fixed point $u \in K$. It is just to fix our ideas that we can set, henceforth, $T(u_0) = \frac{1}{8C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}} (4C_1)^{\frac{\gamma}{q}} \|u_0\|^\gamma_{L^q(\Omega)}}$, and, since $T(u_0)$ depends only on $\|u_0\|_{L^q(\Omega)}$, a uniform time can be chosen for all initial data belonging to a bounded subset of $L^q(\Omega)$ (see the statement of the theorem).

Step 3: Characterization of the fixed point. Here we show that the fixed point u in K is actually a weak solution of (1.3) in the sense of Definition 1.1. Let us start with the case u_0 in $L^\infty(\Omega)$.

Lemma 4.1. *Let u_0 be in $L^\infty(\Omega)$, and let us call u_q the unique fixed point found in $C([0,T]; L^q(\Omega)) \cap L^q(0,T; L^{\frac{Nq}{N-2}}(\Omega))$ by Step 2, for a time $T(u_0) = \frac{1}{8C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}} (4C_1)^{\frac{\gamma}{q}} \|u_0\|^\gamma_{L^q(\Omega)}}$. Then u_q belongs to $L^\infty(0, T(u_0); L^\infty(\Omega))$, and it is a weak solution of* (1.3).

Proof. Let $r > \max(2, q)$. Considering u_0 as an element of $L^r(\Omega)$, by Step 2 we can find a time $T_r = T_r(u_0)$ and a fixed point u_r for an operator $\mathcal{F}_r$ defined in $C([0,T]; L^r(\Omega)) \cap L^r(0,T; L^{\frac{Nr}{N-2}}(\Omega))$. Our first remark is that $u_r = u_q$ in $[0, \min(T(u_0), T_r(u_0))]$. Indeed, using Proposition 2.2 for the approximating problems yielding u_q and u_r, and passing to the limit, we have, for all $t \le \min(T(u_0), T_r(u_0))$, with $\gamma = \frac{2q(p-1)}{2q-N(p-1)}$,

$$\Big[1 - 2\,t\,C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}} \Big(\|u_r\|^\gamma_{L^\infty(0,t;L^q(\Omega))} + \|u_q\|^\gamma_{L^\infty(0,t;L^q(\Omega))}\Big)\Big]$$
$$\times \|u_r - u_q\|^q_{L^\infty(0,t;L^q(\Omega))} + \frac{1}{4}\int_0^t \Big(\int_\Omega |u_r - u_q|^{\frac{Nq}{N-2}}\, dx\Big)^{\frac{N-2}{N}} d\tau \le 0\,.$$

For $t \le \min(T(u_0), T_r(u_0))$ we have that $2\,t\,C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}} \|u_q\|^\gamma_{L^\infty(0,t;L^q(\Omega))} \le \frac{1}{4}$; moreover, setting $M_r = \|u_r\|^\gamma_{L^\infty(0,T_r;L^q(\Omega))}$, for $t \le \frac{1}{8C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}} M_r}$ we have $2\,t\,C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}} \|u_r\|^\gamma_{L^\infty(0,t;L^q(\Omega))} \le \frac{1}{4}$; hence, $u_r = u_q$. Repeating the same estimate for time intervals of the same length, in a finite number of steps (depending on u_r) we can then prove that $u_r = u_q$ in $[0, \min(T_r(u_0), T(u_0)]$.

Next we claim that u_q belongs to $L^r(0, T(u_0); L^{\frac{Nr}{N-2}}(\Omega))$ for every $r > q$. To this purpose, let $r > \max(2, q)$ be fixed, and let us define the set

$$C_r = \{t \in (0, T(u_0)] \,:\, u_q \in C^0([0,t]; L^r(\Omega)) \cap L^r(0,t; L^{\frac{Nr}{N-2}}(\Omega))\}\,.$$

Since $u_r = u_q$ in $[0, \min(T_r(u_0), T(u_0)]$, we have that $C_r \neq \emptyset$. Let us set $\tau_r = \sup C_r$, and argue by contradiction, assuming that $\tau_r < T(u_0)$. For every $t < \tau_r$, we have that u_q belongs to $C^0([0,t]; L^r(\Omega)) \cap L^r(0,t; L^{\frac{Nr}{N-2}}(\Omega))$, so in the approximation procedure used to find u_q in Step 2 we can choose to approximate u_q with a sequence of bounded functions strongly convergent in $C^0([0,t]; L^r(\Omega)) \cap L^r(0,t; L^{\frac{Nr}{N-2}}(\Omega))$. Using Remark 3.1 (recall that $r > 2$) and since $u_q = \mathcal{F}(u_q)$ we get that u_q belongs to $L^2(0,t; H_0^1(\Omega))$ and it is a weak solution of

$$\begin{cases} (u_q)_\tau - \operatorname{div}(a(x,\tau,\nabla u_q)) = \rho(x,\tau)|u_q|^{p-1}u_q & \text{in } \Omega \times (0,t), \\ u_q = 0 & \text{on } \partial\Omega \times (0,t), \\ u_q(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{4.5}$$

Then we can apply, to equation (4.5), Proposition 2.5 with $\psi = \rho\,|u_q|^{p-1}$ and $\sigma = \frac{q}{p-1}$, to obtain, for all fixed $t < \tau_r$,

$$\|u_q(t)\|_{L^r(\Omega)} \leq c_r e^{c_r\, t\, \|\rho\|_\infty^{\frac{\gamma}{p-1}} \|u_q\|^\gamma_{L^\infty(0,t;L^q(\Omega))}}\; t^{-\frac{N(r-q)}{2qr}} \|u_0\|_{L^q(\Omega)}\,,$$

where c_r is a positive constant depending on r, α, $|\Omega|$, N, q and p. Since

$$t\|\rho\|_\infty^{\frac{\gamma}{p-1}} \|u_q\|^\gamma_{L^\infty(0,t;L^q(\Omega))} \leq T(u_0)\|\rho\|_\infty^{\frac{\gamma}{p-1}} \|u_q\|^\gamma_{L^\infty(0,T(u_0);L^q(\Omega))}\,,$$

recalling the value of $T(u_0)$ we deduce that $\|u_q(t)\|_{L^r(\Omega)}$ is bounded for all $t < \tau_r$, so that as t approaches τ_r we get that $u_q(\tau_r)$ belongs to $L^r(\Omega)$. Then we can apply again the fixed-point argument in Step 2 in an interval $[\tau_r, t)$, and in the space $C^0([\tau_r,t]; L^r(\Omega)) \cap L^r(\tau_r,t; L^{\frac{Nr}{N-2}}(\Omega))$ in order to find a unique fixed point relative to the problem

$$\begin{cases} z_\tau - \operatorname{div}(a(x,\tau,\nabla z)) = \rho|z|^{p-1}z\,, & \Omega \times (\tau_r,t), \\ z = 0 & \text{on } \partial\Omega \times (\tau_r,t), \\ z(\tau_r) = u_q(\tau_r)\,, & \text{in } \Omega, \end{cases}$$

obtaining, as above, that $z = u_q$, so u_q belongs to $C^0([0,t]; L^r(\Omega)) \cap L^r(0,t; L^{\frac{Nr}{N-2}}(\Omega))$ for $t > \tau_r$, thus contradicting the definition of τ_r. Thus we have proved that, for every fixed $r > q$, u_q belongs to $L^r(0,T(u_0); L^{\frac{Nr}{N-2}}(\Omega))$, and is a weak solution of (1.3), and the conclusion follows by standard regularity results. $\square$

Now, let us turn to the general case of u_0 in $L^q(\Omega)$, and consider the Cauchy-Dirichlet problem

$$\begin{cases} (u_n)_t - \operatorname{div}(a(x,t,\nabla u_n)) = \rho(x,t)|u_n|^{p-1}u_n & \text{in } \Omega\times(0,T), \\ u_n = 0 & \text{on } \partial\Omega\times(0,T), \\ u_n(x,0) = \Theta_n(u_0)(x) & \text{in } \Omega. \end{cases} \tag{4.6}$$

By Step 2 and Lemma 4.1, there exists a unique weak solution u_n of (4.6) which is obtained through the contraction mapping principle in the convex set K_n defined starting from $\Theta_n(u_0)$ and for the time $[0,\mathcal{T}_n]$, with $\mathcal{T}_n = \frac{1}{8C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}}(4C_1)^{\frac{\gamma}{q}}\|\Theta_n(u_0)\|^\gamma_{L^q(\Omega)}}$. Moreover, for fixed n, u_n is bounded in $[0,\mathcal{T}_n]$. But since $\|\Theta_n(u_0)\|_{L^q(\Omega)} \le \|u_0\|_{L^q(\Omega)}$, we have both that $\mathcal{T}_n > T(u_0)$ and that u_n belongs to $K(T(u_0),u_0)$ for all n. Setting $T = T(u_0)$ for shortness, this implies that u_n is bounded in $L^\infty(0,T;L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$; hence, $|u_n|^{p-1}$ is bounded in $L^\infty(0,T;L^{\frac{q}{p-1}}(\Omega))$, and since $\frac{q}{p-1} > \frac{N}{2}$ by Proposition 3.1 u_n itself is strongly convergent to a function u in $C([0,T];L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$. By definition of $\mathcal{F}$, we find, passing to the limit as n tends to infinity, that $u = \mathcal{F}(u)$, and since u_n belongs to K, then u is the fixed point previously found. On the other hand, for fixed n, u_n is a bounded weak solution, and we can apply to (4.6) the estimates in Proposition 2.5 with $\psi = |u_n|^{p-1}\rho$ and $\sigma = \frac{q}{p-1}$; using that $\|\rho|u_n|^{p-1}\|^{\frac{2\sigma}{2\sigma-N}}_{L^\infty(0,T;L^\sigma(\Omega))} \le \|\rho\|_\infty^{\frac{\gamma}{p-1}}\|u_n\|^\gamma_{L^\infty(0,T;L^q(\Omega))}$, with $\gamma = \frac{2q(p-1)}{2q-N(p-1)}$, and thanks to the fact that $\|\Theta_n(u_0)\|_{L^q(\Omega)} \le \|u_0\|_{L^q(\Omega)}$, we obtain (we will denote henceforth by c_i positive constants depending on q, p, N, α and $|\Omega|$)

$$\|u_n(t)\|_{L^r(\Omega)} \le c_1\, t^{-\frac{N(r-q)}{2qr}}\,\|u_0\|_{L^q(\Omega)}\, e^{c_1\, t\,\|\rho\|_\infty^{\frac{\gamma}{p-1}}\|u_n\|^\gamma_{L^\infty(0,T;L^q(\Omega))}} \qquad \forall r > q\,,$$
$$\|u_n(t)\|_{L^\infty(\Omega)} \le c_1\, t^{-\frac{N}{2q}}\,\|u_0\|_{L^q(\Omega)}\, e^{c_1\, t\,\|\rho\|_\infty^{\frac{\gamma}{p-1}}\|u_n\|^\gamma_{L^\infty(0,T;L^q(\Omega))}}\,.$$

By simply using Fatou's lemma, these estimates hold true for u as well, and observing that $t\,\|\rho\|_\infty^{\frac{\gamma}{p-1}}\|u_n\|^\gamma_{L^\infty(0,T;L^q(\Omega))} \le T(u_0)\|\rho\|_\infty^{\frac{\gamma}{p-1}}(4C_1)^{\frac{\gamma}{q}}\,\|u_0\|^\gamma_{L^q(\Omega)} \le c_2$, we can deduce that u belongs to $L^\infty_{\rm loc}(0,T;L^\infty(\Omega))$ and satisfies

$$\|u(t)\|_{L^r(\Omega)} \le C\, t^{-\frac{N(r-q)}{2qr}}\,\|u_0\|_{L^q(\Omega)} \qquad \forall r > q\,,$$
$$\|u(t)\|_{L^\infty(\Omega)} \le C\, t^{-\frac{N}{2q}}\,\|u_0\|_{L^q(\Omega)}\,,$$

for a constant C which does not depend on u_0. Moreover, applying Corollary 2.9 to u_n and u_m we also deduce that u_n strongly converges to u in $L^\infty_{\mathrm{loc}}(0,T;L^\infty(\Omega))$, which implies that in fact u_n strongly converges to u also in $L^2_{\mathrm{loc}}(0,T;H^1_0(\Omega))$, so u is a weak solution of (1.3) in the sense of Definition 1.1.

Step 4. Uniqueness of the weak solution. Here we prove uniqueness in the class of weak solutions. We have just proved that u is a weak solution of (1.2); suppose that there exists another weak solution w. Recalling Remark 1.2, subtracting the equations we deduce that $u-w$ belongs to $L^2_{\mathrm{loc}}(0,T;H^1_0(\Omega))$ and satisfies

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega {}_{H^{-1}(\Omega)}\langle (u-w)_t,\varphi(t)\rangle_{H^1_0(\Omega)}\,dx\,d\tau \\ + \int_{t_1}^{t_2}\int_\Omega [a(x,\tau,\nabla u)-a(x,\tau,\nabla w)]\cdot\nabla\varphi\,dx\,d\tau \\ = \int_{t_1}^{t_2}\int_\Omega \rho(|u|^{p-1}u-|w|^{p-1}w)\varphi\,dx\,d\tau\,, \end{aligned} \tag{4.7}$$

for every $0<t_1<t_2\le T$ and all functions φ in $L^2_{\mathrm{loc}}(0,T;H^1_0(\Omega))$.

Let us deal with the case $1<q<2$. Then, for $\varepsilon>0$, we choose $\varphi = \frac{u-w}{(\varepsilon+|u-w|)^{2-q}}$ in (4.7) (it is easy to check that φ is in $L^2_{\mathrm{loc}}(0,T;H^1_0(\Omega))$) and we proceed as in Proposition 2.1. Setting $\Phi_\varepsilon(s)=\int_0^s \frac{r}{(\varepsilon+|r|)^{2-q}}\,dr$ we have, integrating by parts,

$$\begin{aligned}&\int_\Omega \Phi_\varepsilon(u-w)(t_2)\,dx \\ &+\alpha\int_{t_1}^{t_2}\int_\Omega \Phi''_\varepsilon(u-w)[a(x,\tau,\nabla u)-a(x,\tau,\nabla w)]\cdot\nabla(u-w)\,dx\,d\tau \\ &\le \int_\Omega \Phi_\varepsilon(u-w)(t_1)\,dx + \|\rho\|_\infty \int_{t_1}^{t_2}\int_\Omega (|u|^{p-1}+|w|^{p-1})\,|u-w|^q\;dx\,d\tau\,.\end{aligned}$$

Since $\Phi''(s)\ge \frac{q-1}{(\varepsilon+|s|)^{2-q}}$, while $\frac{1}{8}\,|s|^q\,\chi_{\{|s|>\varepsilon\}} \le \Phi_\varepsilon(s)\chi_{\{|s|>\varepsilon\}}\le \Phi_\varepsilon(s)\le \frac{1}{q}\,|s|^q$, we obtain

$$\begin{aligned}&\frac{1}{8}\int_{\{x\,:\,|(u-w)(x,t_2)|>\varepsilon\}} |(u-w)(t_2)|^q dx + \alpha\,(q-1)\int_{t_1}^{t_2}\int_\Omega \frac{|\nabla(u-w)|^2}{(\varepsilon+|u-w|)^{2-q}}dx\,d\tau \\ &\le \frac{1}{q}\int_\Omega |(u-w)(t_1)|^q\,dx + \|\rho\|_\infty\int_0^{t_2}\int_\Omega (|u|^{p-1}+|w|^{p-1})|u-w|^q\,dx\,d\tau\,,\end{aligned}$$

which yields, by Sobolev's inequality,

$$\begin{aligned}&\frac{1}{8}\int_{\{x:|(u-w)(x,t_2)|>\varepsilon\}}|(u-w)(t_2)|^q dx\\&+\alpha(q-1)\frac{4}{q^2}S\int_{t_1}^{t_2}\Big(\int_\Omega |(\varepsilon+|u-w|)^{\frac{q}{2}}-\varepsilon|^{\frac{2N}{N-2}}dx\Big)^{\frac{N-2}{N}}d\tau\\&\le \frac{1}{q}\int_\Omega |(u-w)(t_1)|^q dx+\|\rho\|_\infty\int_0^{t_2}\int_\Omega(|u|^{p-1}+|w|^{p-1})|u-w|^q dx\, d\tau\,.\end{aligned}$$

Letting first ε and then t_1 tend to zero, using that $u-w$ belongs to $C([0,T];L^q(\Omega))$ we get

$$\begin{aligned}&\frac{1}{8}\int_\Omega |(u-w)(t_2)|^q dx+\alpha\,(q-1)\frac{4}{q^2}S\int_0^{t_2}\Big(\int_\Omega |u-w|^{\frac{Nq}{N-2}}dx\Big)^{\frac{N-2}{N}}d\tau\\&\qquad\le \|\rho\|_\infty\int_0^{t_2}\int_\Omega(|u|^{p-1}+|w|^{p-1})\,|u-w|^q\,dx\,d\tau\,,\qquad \forall\ 0<t_2\le T\,.\end{aligned}\tag{4.8}$$

Note that (4.8) is exactly the estimate (2.12) (with $v=u$ and $z=w$) we found in Proposition 2.2. Therefore, we can obtain (2.11) for u and w, that is (recall that $\gamma=\frac{2q(p-1)}{2q-N(p-1)}$)

$$\begin{aligned}&\Big[1-2t\ C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}}\Big(\|u\|^\gamma_{L^\infty(0,T;L^q(\Omega))}+\|w\|^\gamma_{L^\infty(0,T;L^q(\Omega))}\Big)\Big]\\&\times\sup_{s\in[0,t]}\int_\Omega |u(s)-w(s)|^q\,dx\\&+\frac{1}{4}\int_0^t\Big(\int_\Omega |u-w|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\le 0\,,\qquad \forall t\le T=T(u_0)\,.\end{aligned}$$

Since $2t\ C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}}\ \|u\|^\gamma_{L^\infty(0,T;L^q(\Omega))}\le\frac{1}{4}$, if we set

$$T(w)=\frac{1}{8C_1\|\rho\|_\infty^{\frac{\gamma}{p-1}}\ \|w\|^\gamma_{L^\infty(0,T;L^q(\Omega))}}\,,$$

for $t\le\min(T(u_0),T(w))$ we easily get that $u=w$. The estimate can then be iterated for time intervals of the same length, and in a finite number of steps (depending on $\|w\|_{C([0,T];L^q(\Omega))}$) we find that $u=w$ in $[0,T]$.

The case $q \geq 2$ is simpler, since $C([0,T];L^q(\Omega)) \subset C([0,T];L^2(\Omega))$ it is enough to take $u-w$ in (4.7) to prove that $u = w$ reasoning exactly in the same way as above.

Step 5: Continuous dependence. Let now u and v be the solutions with initial data respectively u_0 and v_0. Since u and v are locally bounded, we can apply Corollary 2.9 with $|\psi| \leq |\rho| \, [|u|^{p-1} + |v|^{p-1}]$, in a time $[0,T_0]$, with $T_0 = \min(T(u_0), T(v_0))$, and $\sigma = \frac{q}{p-1}$. Since $\|\psi\|_{L^\infty(0,T;L^\sigma(\Omega))}^{\frac{2\sigma}{2\sigma-N}} \leq \|\rho\|_\infty^{\frac{\gamma}{p-1}} (\|u\|^\gamma_{L^\infty(0,T;L^q(\Omega))} + \|v\|^\gamma_{L^\infty(0,T;L^q(\Omega))})$ we get, from (2.33) and (2.35),

$$\begin{aligned}\|u(t)-v(t)\|_{L^q(\Omega)} &+ t^{\frac{N}{2q}} \|u(t)-v(t)\|_{L^\infty(\Omega)} \\ &\leq c_3 e^{c_3\, t\, \|\rho\|_\infty^{\frac{\gamma}{p-1}} (\|u\|^\gamma_{L^\infty(0,t;L^q(\Omega))} + \|v\|^\gamma_{L^\infty(0,t;L^q(\Omega))})} \|u_0 - v_0\|_{L^q(\Omega)} \,.\end{aligned}$$

We observe now that u belongs to the convex set $K(T(u_0))$ related to u_0 and so does v with $K(T(v_0))$, so the estimates on $T(u_0)$ and $T(v_0)$ imply

$$t \, \|\rho\|_\infty^{\frac{\gamma}{p-1}} (\|u\|^\gamma_{L^\infty(0,t;L^q(\Omega))} + \|v\|^\gamma_{L^\infty(0,t;L^q(\Omega))}) \leq \frac{1}{8C_1(4C_1)^{\frac{\gamma}{q}}} \,, \qquad \forall t \leq T_0 \,,$$

and so we obtain the following continuous-dependence estimate:

$$\|u(t)-v(t)\|_{L^q(\Omega)} + t^{\frac{N}{2q}} \|u(t)-v(t)\|_{L^\infty(\Omega)} \leq C \, \|u_0 - v_0\|_{L^q(\Omega)} \,, \tag{4.9}$$

where C is a positive constant depending on α, $|\Omega|$, p, q and N but not on u_0 and v_0. In order to obtain (ii) in Theorem 1.3, consider the solution u_n of (4.6), which belongs to $L^\infty(Q)$ for fixed n. From (4.9), written for u and u_n, we get

$$t^{\frac{N}{2q}} \|u(t)\|_{L^\infty(\Omega)} \leq t^{\frac{N}{2q}} \|u_n\|_{L^\infty(\Omega)} + C \, \|u_0 - \Theta_n(u_0)\|_{L^q(\Omega)} \,, \qquad \forall t \in [0, T(u_0)] \,.$$

Letting first t tend to zero then n go to infinity in the previous inequality we find

$$\lim_{t\to 0^+} t^{\frac{N}{2q}} \|u(t)\|_{L^\infty(\Omega)} = 0 \,.$$

Finally, (i) is easily obtained since if $u_0 \geq 0$, all the u_n's are positive and so $u \geq 0$. ☐

The case when $\frac{q}{p-1} = \frac{N}{2}$ is a little bit more delicate, since the construction of the metric space for a fixed-point argument is less evident. Again we will divide the proof in various steps in the attempt to make it clearer.

Proof of Theorem 1.3; the case $q = \frac{N(p-1)}{2}$ **and** $q > 1$.

Step 1: Construction of the mapping. The first step is identical to the one in Theorem 1.3, the case $q > \frac{N(p-1)}{2}$; that is, for v in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, we take a sequence of bounded functions $\{v_n\}$ strongly converging to v in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$ and we consider the solutions u_n of the following Cauchy-Dirichlet problems:

$$\begin{cases} (u_n)_t - \operatorname{div}(a(x,t,\nabla u_n)) = \rho(x,t)|v_n|^{p-1}v_n & \text{in } \Omega \times (0,T), \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(x,0) = \Theta_n(u_0)(x) & \text{in } \Omega. \end{cases} \tag{4.10}$$

Then we define $\mathcal{F}(v) = \lim_{n\to\infty} u_n$ in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, the definition being well posed thanks to Remark 3.2 and Proposition 3.3.

Step 2: The contraction mapping principle. Let us take a small positive constant $\varepsilon > 0$ that will be fixed later, and let us define n_ε as follows:

$$n_\varepsilon = \min\{k \in \mathbf{N} \,:\, \|\Theta_m(u_0) - \Theta_k(u_0)\|_{L^q(\Omega)} \le \varepsilon\,, \quad \forall m \ge k\}\,. \tag{4.11}$$

Let us also fix $\bar r$ such that $\bar r > q = \frac{N(p-1)}{2}$ and set $\bar\gamma = \frac{2\bar r(p-1)}{2\bar r - N(p-1)}$. We will call u_{n_ε} the unique weak solution, previously found, of the Dirichlet problem

$$\begin{cases} (u_{n_\varepsilon})_t - \operatorname{div}(a(x,t,\nabla u_{n_\varepsilon})) = \rho(x,t)|u_{n_\varepsilon}|^{p-1}u_{n_\varepsilon} & \text{in } \Omega \times (0,\bar T), \\ u_{n_\varepsilon} = 0 & \text{on } \partial\Omega \times (0,\bar T), \\ u_{n_\varepsilon}(x,0) = \Theta_{n_\varepsilon}(u_0)(x) & \text{in } \Omega, \end{cases} \tag{4.12}$$

with $\bar T = \frac{1}{8C_1\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}(4C_1)^{\frac{\bar\gamma}{q}}\|\Theta_{n_\varepsilon}(u_0)\|_{L^{\bar r}(\Omega)}^{\bar\gamma}}$, which is $T(\Theta_{n_\varepsilon}(u_0))$ previously found if considering $\Theta_{n_\varepsilon}(u_0)$ in $L^{\bar r}(\Omega)$. We also have that there exists a positive constant $\bar C = \bar C(\bar r, p, N, \alpha, |\Omega|)$ such that

$$\|u_{n_\varepsilon}\|_{L^\infty(0,T;L^{\bar r}(\Omega))}^{\bar\gamma} \le \bar C \|\Theta_{n_\varepsilon}(u_0)\|_{L^{\bar r}(\Omega)}^{\bar\gamma}\,. \tag{4.13}$$

Let now $T \le \bar T$. Applying twice to the approximating problems (4.10) Proposition 2.4, first with $z = w = z_0 = \bar\zeta = 0$ and $\bar\varphi = u_{n_\varepsilon}$, then with $\bar\zeta = \bar\varphi = u_{n_\varepsilon}$, and then passing to the limit, we get, for a constant $C_3 =$

$C_3(\alpha, N, |\Omega|, p, q, \bar r)$,

$$\begin{aligned}&\big[1 - T\,C_3\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\,\|u_{n_\varepsilon}\|^{\bar\gamma}_{L^\infty(0,T;L^{\bar r}(\Omega))}\big]\,\|\mathcal F(v)\|^q_{L^\infty(0,T;L^q(\Omega))}\\&+\big[\frac12 - C_3\|v-u_{n_\varepsilon}\|^{p-1}_{L^\infty(0,T;L^q(\Omega))}\big]\int_0^T\Big(\int_\Omega|\mathcal F(v)|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\\&\quad\le C_3\int_\Omega|u_0|^q\,dx + T\,C_3\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\,\|u_{n_\varepsilon}\|^{\bar\gamma}_{L^\infty(0,T;L^{\bar r}(\Omega))}\|v\|^q_{L^\infty(0,T;L^q(\Omega))}\\&+\big[\frac14 + C_3\|v-u_{n_\varepsilon}\|^{p-1}_{L^\infty(0,T;L^q(\Omega))}\big]\int_0^T\Big(\int_\Omega|v|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\,,\end{aligned}\tag{4.14}$$

and

$$\begin{aligned}&\big[1 - 2\,T\,C_3\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\,\|u_{n_\varepsilon}\|^{\bar\gamma}_{L^\infty(0,T;L^{\bar r}(\Omega))}\big]\|\mathcal F(v)-\mathcal F(w)\|^q_{L^\infty(0,T;L^q(\Omega))}\\&\quad+\big[\frac12 - C_3\Big(\|v-u_{n_\varepsilon}\|^{p-1}_{L^\infty(0,T;L^q(\Omega))}+\|w-u_{n_\varepsilon}\|^{p-1}_{L^\infty(0,T;L^q(\Omega))}\Big)\big]\\&\quad\times\int_0^T\Big(\int_\Omega|\mathcal F(v)-\mathcal F(w)|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\le\\&\quad\le 2\,T\,C_3\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\,\|u_{n_\varepsilon}\|^{\bar\gamma}_{L^\infty(0,T;L^{\bar r}(\Omega))}\|v-w\|^q_{L^\infty(0,T;L^q(\Omega))}\\&+\big[\frac14 + C_3\big(\|v-u_{n_\varepsilon}\|^{p-1}_{L^\infty(0,T;L^q(\Omega))}+\|w-u_{n_\varepsilon}\|^{p-1}_{L^\infty(0,T;L^q(\Omega))}\big)\big]\\&\times\int_0^T\Big(\int_\Omega|v-w|^{\frac{Nq}{N-2}}dx\Big)^{\frac{N-2}{N}}\,d\tau.\end{aligned}\tag{4.15}$$

For positive constants δ, δ', L and L' that will be fixed later (at any rate, they all, and ε as well, will depend only on C_3 and u_0), let us define the following closed-convex set of $C([0,T];L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$:

$$\begin{aligned}K\equiv\Big\{&v\in C([0,T];L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))\;:\;\|v\|^q_{C([0,T];L^q(\Omega))}\le L'\,,\\&\int_0^T\Big(\int_\Omega|v|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\le L\,,\\&\|v-u_{n_\varepsilon}\|^{p-1}_{L^\infty(0,T;L^q(\Omega))}\le\delta\,,\quad\int_0^T\Big(\int_\Omega|v-u_{n_\varepsilon}|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\le\delta'\Big\}.\end{aligned}$$

Once δ, δ', L and L' have been chosen, it will be easily seen that K is nonempty since it contains u_{n_ε}; let us skip this point for a while, in order to show that an accurate choice of the constants lets $\mathcal{F}$ be a contraction on K, which is a complete metric space endowed with the distance inherited by $C([0,T];L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$.

Firstly, if v belongs to K, (4.14) implies, taking into account (4.13), and setting $C_4=2C_3\,\bar C$,

$$\begin{aligned}&\big[1-T\,C_4\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\,\|\Theta_{n_\varepsilon}(u_0)\|^{\bar\gamma}_{L^{\bar r}(\Omega)}\big]\,\|\mathcal{F}(v)\|^q_{C([0,T];L^q(\Omega))}\\&+[\frac12-C_3\delta]\int_0^T\Big(\int_\Omega|\mathcal{F}(v)|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\\&\quad\le C_3\int_\Omega|u_0|^q\,dx+T\;C_4\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\,\|\Theta_{n_\varepsilon}(u_0)\|^{\bar\gamma}_{L^{\bar r}(\Omega)}L'+[\frac14+C_3\delta]L\,.\end{aligned}\tag{4.16}$$

Moreover, by (4.12), we can use (2.15) with $w=z=u_{n_\varepsilon}$ and $z_0=\Theta_{n_\varepsilon}(u_0)$, and $\bar\varphi=\bar\zeta=u_{n_\varepsilon}$, to obtain, for all v in K,

$$\begin{aligned}&\Big[1-TC_4\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\|\Theta_{n_\varepsilon}(u_0)\|^{\bar\gamma}_{L^{\bar r}(\Omega)}\Big]\,\|\mathcal{F}(v)-u_{n_\varepsilon}\|^q_{C([0,T];L^q(\Omega))}\\&+[\tfrac12-C_3\delta]\int_0^T\Big(\int_\Omega|\mathcal{F}(v)-u_{n_\varepsilon}|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\\&\le C_3\int_\Omega|u_0-\Theta_{n_\varepsilon}(u_0)|^q\,dx+TC_4\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\|\Theta_{n_\varepsilon}(u_0)\|^{\bar\gamma}_{L^{\bar r}(\Omega)}\delta^{\frac N2}+[\tfrac14+C_3\delta]\delta'.\end{aligned}\tag{4.17}$$

Recalling (4.11), in order to have, by (4.16) and (4.17), that $\mathcal{F}(v)$ belongs to K, it would be enough to find ε, δ, δ', L and L' such that the following conditions are satisfied:

$$\begin{aligned}&C_3\int_\Omega|u_0|^q\,dx+T\;C_4\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\,\|\Theta_{n_\varepsilon}(u_0)\|^{\bar\gamma}_{L^{\bar r}(\Omega)}L'+[\frac14+C_3\delta]L\\&\le\min\left\{\Big(1-T\;C_4\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\,\|\Theta_{n_\varepsilon}(u_0)\|^{\bar\gamma}_{L^{\bar r}(\Omega)}\Big)L',\Big(\frac12-C_3\delta\Big)L\right\}\,,\end{aligned}\tag{4.18}$$

$$\begin{aligned}&C_3\varepsilon+T\;C_4\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\,\|\Theta_{n_\varepsilon}(u_0)\|^{\bar\gamma}_{L^{\bar r}(\Omega)}\delta^{\frac N2}\;+[\frac14+C_3\delta]\delta'\\&\le\min\left\{\Big(1-T\;C_4\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\,\|\Theta_{n_\varepsilon}(u_0)\|^{\bar\gamma}_{L^{\bar r}(\Omega)}\Big)\delta^{\frac N2},\Big(\frac12-C_3\delta\Big)\delta'\right\}\,.\end{aligned}\tag{4.19}$$

It is easy to see that, for a time T sufficiently small, a similar choice of the constants can be done. Indeed, choose δ such that

$$\tfrac14-2C_3\delta>\tfrac18\,.\tag{4.20}$$

Then, (4.18) and (4.19) are implied by

$$
\begin{aligned}
&C_3\int_\Omega |u_0|^q\,dx + T\ C_4\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\ \|\Theta_{n_\varepsilon}(u_0)\|_{L^{\bar r}(\Omega)}^{\bar\gamma} L' + \frac{5}{16}L \\
&\le \min\left\{\left(1 - T\ C_4\|\rho\|_{L^\infty(Q)}^{\frac{\bar\gamma}{p-1}}\ \|\Theta_{n_\varepsilon}(u_0)\|_{L^{\bar r}(\Omega)}^{\bar\gamma}\right)L',\, \frac{7}{16}L\right\}\,,
\end{aligned}
\tag{4.21}
$$

and

$$
\begin{aligned}
&C_3\varepsilon + T\,C_4\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\ \|\Theta_{n_\varepsilon}(u_0)\|_{L^{\bar r}(\Omega)}^{\bar\gamma}\delta^{\frac{N}{2}}\ + \frac{5}{16}\delta' \\
&\le \min\left\{\left(1 - T\ C_4\|\rho\|_{L^\infty(Q)}^{\frac{\bar\gamma}{p-1}}\ \|\Theta_{n_\varepsilon}(u_0)\|_{L^{\bar r}(\Omega)}^{\bar\gamma}\right)\delta^{\frac{N}{2}},\, \frac{7}{16}\delta'\right\}\,.
\end{aligned}
\tag{4.22}
$$

For instance, we can take $L = 16C_3\|u_0\|_{L^q(\Omega)}^q$, $L' = 7C_3\|u_0\|_{L^q(\Omega)}^q$, $\varepsilon = \frac{1}{12C_3}\delta^{\frac{N}{2}}$, $\delta' = \frac{4}{3}\delta^{\frac{N}{2}}$, so that (4.17) and (4.18) are satisfied if, for example, $T \le \frac{1}{16C_4\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\|\Theta_{n_\varepsilon}(u_0)\|_{L^{\bar r}(\Omega)}^{\bar\gamma}}$.

Therefore, we will consider from now on to have chosen a fixed δ satisfying (4.20), and then to have also fixed the other constants ε, δ', L and L'—satisfying (4.18) and (4.19)—which simply depend on C_3 and $\|u_0\|_{L^q(\Omega)}^q$. Then by (4.11) we have found u_{n_ε} and we have defined the closed convex set $K(T(u_0))$, where we set, to fix our ideas,

$$
T = T(u_0) = \frac{1}{16C_4\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\|\Theta_{n_\varepsilon}(u_0)\|_{L^{\bar r}(\Omega)}^{\bar\gamma}}.
$$

Thus, using (4.18) and (4.19) in (4.16) and (4.17) we have proved that $\mathcal{F}$ maps K into K. Moreover, applying (2.13) with $w = z = z_0 = \bar\zeta = 0$ and with $u = v = \bar\varphi = u_{n_\varepsilon}$ and $u_0 = \Theta_{n_\varepsilon}(u_0)$, we get

$$
\begin{aligned}
&\left[1 - 2\ T\ C_4\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\ \|\Theta_{n_\varepsilon}(u_0)\|_{L^{\bar r}(\Omega)}^{\bar\gamma}\right]\|u_{n_\varepsilon}\|_{L^\infty(0,T;L^q(\Omega))}^q \\
&\quad + \frac{1}{4}\int_0^T\left(\int_\Omega |u_{n_\varepsilon}|^{\frac{Nq}{N-2}}\,dx\right)^{\frac{N-2}{N}}d\tau \le C_3\int_\Omega |u_0|^q\,dx\,,
\end{aligned}
$$

so, with the choice of L, L' and $T(u_0)$ we have made, it follows that u_{n_ε} belongs to K; hence, K is a nonempty complete metric space. Finally, for v

and w in K we have, from (4.15),

$$\begin{aligned}&\big[1 - T\ C_4 \|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\, \|\Theta_{n_\varepsilon}(u_0)\|_{L^{\bar r}(\Omega)}^{\bar\gamma}\big] \|\mathcal{F}(v) - \mathcal{F}(w)\|^q_{C([0,T];L^q(\Omega))}\\ &+ [\frac{1}{2} - 2C_3\delta] \int_0^T \Big(\int_\Omega |\mathcal{F}(v) - \mathcal{F}(w)|^{\frac{Nq}{N-2}}\, dx\Big)^{\frac{N-2}{N}}\, d\tau \le\\ &\quad \le T\ C_4 \|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\, \|\Theta_{n_\varepsilon}(u_0)\|_{L^{\bar r}(\Omega)}^{\bar\gamma} \|v - w\|^q_{C([0,T];L^q(\Omega))}\\ &+ [\frac{1}{4} + 2C_3\delta] \int_0^T \Big(\int_\Omega |v - w|^{\frac{Nq}{N-2}}\, dx\Big)^{\frac{N-2}{N}}\, d\tau\,;\end{aligned}$$

hence, for $T = T(u_0)$ we have

$$\mathrm{dist}_K(\mathcal{F}(v), \mathcal{F}(w)) \le \Big(\frac{\frac14 + 2C_3\delta}{\frac12 - 2C_3\delta}\Big)^{\frac1q} \mathrm{dist}_K(v, w)\,,$$

so that $\mathcal{F}$ is a contraction on K as a consequence of (4.20).

Note also that the time $T(u_0)$ depends on u_0 through the norm of $\Theta_{n_\varepsilon}(u_0)$ in $L^{\bar r}(\Omega)$, $\bar r > q$; in particular, we can write $n_\varepsilon = n_\varepsilon(u_0)$, since the value of n_ε depends itself on u_0, while ε is fixed and depends only on the constants of the problem through δ given by (4.20). Thus, if K is a compact subset of $L^q(\Omega)$, there exist l functions u_i, $i = 1, \dots, l$ such that u_i belongs to K for every i and $K \subset \bigcup_{i=1}^{l} B(u_i, \frac{\varepsilon}{4})$ ($B(u_i, \frac{\varepsilon}{4})$ denotes the ball centered in u_i with radius $\frac{\varepsilon}{4}$). Let us set

$$n_\varepsilon^i = \min\{k \in \mathbf{N} \,:\, \|\Theta_m(u_i) - \Theta_k(u_i)\|_{L^q(\Omega)} \le \frac{\varepsilon}{2} \quad \forall\, m \ge k\}\,.$$

Then for every u in K belonging to $B(u_i, \frac{\varepsilon}{4})$ we have, for every $m \ge n_\varepsilon^i$,

$$\begin{aligned}&\|\Theta_m(u) - \Theta_{n_\varepsilon^i}(u)\|_{L^q(\Omega)} \le \|\Theta_m(u) - \Theta_m(u_i)\|_{L^q(\Omega)}\\ &\quad + \|\Theta_{n_\varepsilon^i}(u_i) - \Theta_{n_\varepsilon^i}(u)\|_{L^q(\Omega)} + \|\Theta_m(u_i) - \Theta_{n_\varepsilon^i}(u_i)\|_{L^q(\Omega)}\\ &\quad \le 2\|u - u_i\|_{L^q(\Omega)} + \frac{\varepsilon}{2} \le \varepsilon\,,\end{aligned}$$

which implies, by the definition of $n_\varepsilon(u)$ in (4.11), that $n_\varepsilon(u) \le n_\varepsilon^i$. Thus, setting $L = \max\limits_{1\le i\le l}\{n_\varepsilon^i\}$, we have that

$$\|\Theta_{n_\varepsilon}(u)\|_{L^{\bar r}(\Omega)} \le L\,|\Omega|^{\frac{1}{\bar r}} \qquad \forall\, u \in K\,;$$

hence, the time $T(u)$ is uniformly bounded from below for u in K, so that a uniform time can be chosen for compact subsets of $L^q(\Omega)$ (see the statement of the theorem).

Step 3: Regularity of the fixed point. Again we start with the case u_0 in $L^\infty(\Omega)$.

Lemma 4.2. *Let u_0 be in $L^\infty(\Omega)$, and let us call u_q the unique fixed point found in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$ by Step 2, for a time $T(u_0) = \frac{1}{16C_4\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\|\Theta_{n_\varepsilon}(u_0)\|^{\bar\gamma}_{L^{\bar r}(\Omega)}}$. Then u_q belongs to $L^\infty(0,T(u_0);L^\infty(\Omega))$, and it is a weak solution of* (1.3).

Proof. By Theorem 1.3 if $q > \frac{N(p-1)}{2}$, (see Lemma 4.1), there exists a unique bounded weak solution U of

$$\begin{cases} U_t - \operatorname{div}(a(x,t,\nabla U)) = \rho(x,t)|U|^{p-1}U & \text{in } \Omega\times(0,T_U),\\ U = 0 & \text{on } \partial\Omega\times(0,T_U),\\ U(x,0) = u_0(x) & \text{in } \Omega,\end{cases}$$

for $T_U = \frac{1}{8C_1\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}(4C_1)^{\frac{\bar\gamma}{q}}\|u_0\|^{\bar\gamma}_{L^{\bar r}(\Omega)}}$. Let us first prove that $U = u_q$ in $[0,\min(T_U,T(u_0))]$, and, to this purpose, let u_{n_ε} be the auxiliary function used in the construction of u_q. Applying Proposition 2.4 to U and to the approximating problems for u_q, we get

$$\begin{aligned}&\big[1 - 2\,t\,C_3\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\big(\|U\|^{\bar\gamma}_{L^\infty(0,t;L^{\bar r}(\Omega))} + \|u_{n_\varepsilon}\|^{\bar\gamma}_{L^\infty(0,t;L^{\bar r}(\Omega))}\big)\big]\\ &\quad\times \sup_{s\in[0,t]}\int_\Omega |u_q(s)-U(s)|^q\,dx\\ &+ \big[\frac14 - 2C_3\delta\big]\int_0^t\Big(\int_\Omega |u_q - U|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau \le 0\,,\end{aligned}$$

for all $t \le \min(T_U,T(u_0))$. But the estimate on $T(u_0)$ and (4.13) imply

$$2\,t\,C_3\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\|u_{n_\varepsilon}\|^{\bar\gamma}_{L^\infty(0,t;L^{\bar r}(\Omega))} \le \frac18\,,$$

so for $t = \frac{1}{4C_3\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\|U\|^{\bar\gamma}_{L^\infty(0,T_U;L^{\bar r}(\Omega))}}$ we deduce, using also (4.20), that $U = u_q$ in $[0,t]$. The estimate can then be iterated in intervals of the same length, to obtain, in a finite number of steps, that $U = u_q$ in $[0,\min(T_U,T(u_0))]$.

Then we define $C_\infty = \{t \in (0, T(u_0)] \,:\, u_q \in L^\infty(0,t; L^\infty(\Omega))\}$. We have just proved that $C_\infty \neq \emptyset$, since $u_q = U$ in $[0, \min(T_U, T(u_0)]$. Let us set $\tau_\infty = \sup C_\infty$, and argue by contradiction, assuming that $\tau_\infty < T(u_0)$. Since u_q is a locally bounded weak solution in $[0, \tau_\infty)$, from Lemma 2.5 we have, for all t in $[0, \tau_\infty)$ and for every $r > q$ (we will denote by c_i positive constants depending only on α, q, σ, N, $\bar r$ and $|\Omega|$, while we will write c_r to denote possibly different constants depending on r beyond α, q, σ, N, $\bar r$ and $|\Omega|$,)

$$\frac{d}{dt}\Big(\int_\Omega |u_q(t)|^r\,dx\Big) + \Big(\int_\Omega |u_q(t)|^{\frac{Nr}{N-2}}\,dx\Big)^{\frac{N-2}{N}} \le c_r \int_\Omega |\rho||u_q|^{p-1}|u_q|^r\,dx\,;$$

hence, we get

$$\begin{aligned}&\frac{d}{dt}\Big(\int_\Omega |u_q(t)|^r\,dx\Big) + \Big(\int_\Omega |u_q(t)|^{\frac{Nr}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\\ &\le c_r \int_\Omega |\rho|\,|u_q - u_{n_\varepsilon}|^{p-1}|u_q|^r\,dx + c_r \int_\Omega |\rho||u_{n_\varepsilon}|^{p-1}|u_q|^r\,dx\,.\end{aligned}$$

Using Hölder's inequality with exponents $(\frac{N}{2}, \frac{N}{N-2})$ and since

$$\|u_q - u_{n_\varepsilon}\|^{p-1}_{L^\infty(0,T;L^q(\Omega))} \le \delta,$$

we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\int_\Omega |u_q(t)|^r\,dx\Big) + \Big(\int_\Omega |u_q(t)|^{\frac{Nr}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\\ &\le c_r\|\rho\|_\infty\delta\Big(\int_\Omega |u_q(t)|^{\frac{Nr}{N-2}}\,dx\Big)^{\frac{N-2}{N}} + c_r \int_\Omega |\rho||u_{n_\varepsilon}|^{p-1}|u_q|^r\,dx\,.\end{aligned}$$

We can choose $r = \bar r$, which was fixed above, so as to have that $\frac{\bar r}{p-1} > \frac{N}{2}$; choosing δ even smaller, if necessary (note that in (4.20) above we only asked that $\delta < \frac{1}{16C_3}$), we arrive at

$$\frac{d}{dt}\Big(\int_\Omega |u_q(t)|^{\bar r}\,dx\Big) + \Big(\int_\Omega |u_q(t)|^{\frac{N\bar r}{N-2}}\,dx\Big)^{\frac{N-2}{N}} \le c_1 \int_\Omega |\rho|\,|u_{n_\varepsilon}|^{p-1}|u_q|^{\bar r}\,dx\,.$$

¿From now on we can follow the lines of Proposition 2.7, replacing $\psi(x,t)$ with $|\rho||u_{n_\varepsilon}(x,t)|^{p-1}$, and fixing $\sigma = \frac{\bar r}{p-1}$. Thus we obtain (2.23), which in this case becomes

$$\|u_q(t)\|_{L^{\bar r}(\Omega)}\, t^{\frac{N(\bar r - q)}{2q\bar r}} \le c_2 e^{c_2\ t\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\ \|u_{n_\varepsilon}\|^{\bar\gamma}_{L^\infty(0,T;L^{\bar r}(\Omega))}} \|u_0\|_{L^q(\Omega)}, \quad \forall t \in [0, \tau_\infty),$$

which yields, recalling that $t < \tau_\infty < T(u_0)$ and using (4.13),

$$\|u_q(t)\|_{L^{\bar r}(\Omega)} \le c_3\, t^{-\frac{N(\bar r - q)}{2q\bar r}}\, \|u_0\|_{L^q(\Omega)} \qquad \forall t < \tau_\infty\,. \tag{4.23}$$

Since

$$\|\rho|u_q|^{p-1}\|^{\frac{2\sigma}{2\sigma-N}}_{L^\infty(\frac{t}{2},t;L^\sigma(\Omega))} \le c_4\, \|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\, \|u_q\|^{\bar\gamma}_{L^\infty(\frac{t}{2},t;L^{\bar r}(\Omega))},$$

again from Proposition 2.7 applied to u_q in the interval $(\frac{t}{2}, t)$ with $t < \tau_\infty$, we get

$$\|u_q(t)\|_{L^\infty(\Omega)} \le c_5 e^{c_5\, \|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\; t\; \|u_q\|^{\bar\gamma}_{L^\infty(\frac{t}{2},t;L^{\bar r}(\Omega))}}\; t^{-\frac{N}{2q}} \|u_q(\frac{t}{2})\|_{L^q(\Omega)}\,,$$

so, by (4.23),

$$\|u_q(t)\|_{L^\infty(\Omega)} \le c_6 e^{c_6 \|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\; t^{1-\frac{N(\bar r-q)\bar\gamma}{2q\bar r}}\; \|u_0\|^{\bar\gamma}_{L^q(\Omega)}}\; t^{-\frac{N}{2q}} \|u_q(\frac{t}{2})\|_{L^q(\Omega)}\,.$$

Since $q = \frac{N(p-1)}{2}$, we have $1 - \frac{N(\bar r - q)\bar\gamma}{2q\bar r} = 0$, and so we finally find

$$\|u(t)\|_{L^\infty(\Omega)} \le C(\|u_0\|_{L^q(\Omega)}, \|\rho\|_\infty) t^{-\frac{N}{2q}}\,, \qquad \forall t < \tau_\infty\,, \tag{4.24}$$

where $C(\|u_0\|_{L^q(\Omega)}, \|\rho\|_\infty)$ denotes a constant depending on the norm of u_0 in $L^q(\Omega)$ and on $\|\rho\|_\infty$. Then (4.24) implies that $u_q(\tau_\infty) \in L^\infty(\Omega)$, so we can solve problem (1.3) for $t > \tau_\infty$ with initial datum $u_q(\tau_\infty)$. As we have seen above, in this way we find that u_q is bounded for $t > \tau_\infty$ as well, finding a contradiction with the definition of τ_∞. This proves that $\tau_\infty = T(u_0)$, and of course u_q is a weak solution in $[0, T(u_0)]$. □

Now, for general u_0 in $L^q(\Omega)$, let us consider the initial–boundary value problem (1.3) with $\Theta_m(u_0)$ as initial datum. Since $\Theta_m(u_0)$ belongs to $L^\infty(\Omega)$, by Step 2 and Lemma 4.2 there exists a unique bounded function u_m, obtained as fixed-point of a mapping $\mathcal{F}_m$ in an interval $[0, \mathcal{T}_m]$, with $\mathcal{T}_m$ given by Step 2, which is a solution of

$$\begin{cases} (u_m)_t - \operatorname{div}(a(x,t,\nabla u_m)) = \rho|u_m|^{p-1}u_m & \text{in } \Omega \times (0, \mathcal{T}_m), \\ u_m = 0 & \text{on } \partial\Omega \times (0, \mathcal{T}_m), \\ u_m(x,0) = \Theta_m(u_0)(x)\,, & \text{in } \Omega, \end{cases}$$

in the sense of Definition 1.1. But if u_{n_ε} are the functions used to obtain the fixed point u relative to u_0, for $m \ge n_\varepsilon$ we have $\Theta_{n_\varepsilon}(\Theta_m(u_0)) = \Theta_{n_\varepsilon}(u_0)$, so by (4.11) and by the fact that $\|\Theta_m(u_0)\|_{L^q(\Omega)} \leq \|u_0\|_{L^q(\Omega)}$ we easily see that $\mathcal{T}_m \geq T(u_0)$ for all $m \geq n_\varepsilon$ and the the solutions u_m of (4.25) are all bounded weak solutions belonging to $K(T(u_0), T(u_0))$, hence satisfying

$$\|u_m - u_{n_\varepsilon}\|^{p-1}_{L^q(\Omega)} \leq \delta \qquad \forall m \geq n_\varepsilon \,. \tag{4.26}$$

Thus, applying now (2.15) with $v = u$, $w = z = u_m$ and $\bar{\varphi} = \bar{\zeta} = u_{n_\varepsilon}$ we obtain

$$\begin{aligned}&\left[1 - 4\, t\, C_3 \|\rho\|_\infty^{\frac{\bar{\gamma}}{p-1}} \|u_{n_\varepsilon}\|^{\bar{\gamma}}_{L^\infty(0,T;L^{\bar{r}}(\Omega))}\right] \|u - u_m\|^q_{L^\infty(0,T;L^q(\Omega))} \\ &+ [\frac{1}{4} - 4C_3\delta] \int_0^t \Big(\int_\Omega |u - u_m|^{\frac{Nq}{N-2}}\, dx\Big)^{\frac{N-2}{N}} d\tau \leq C_3 \int_\Omega |u_0 - \Theta_m(u_0)|^q\, dx\,,\end{aligned}$$

which implies, for δ as above and $t = T(u_0)$, that u is in fact the limit of u_m in the strong topology of $C([0,T]; L^q(\Omega)) \cap L^q(0,T; L^{\frac{Nq}{N-2}}(\Omega))$. On the other hand, since u_m is locally bounded, thanks to (4.26) we can perform the same steps as in Lemma 4.2, replacing u_q with u_m, in order to arrive at the following estimate:

$$\|u_m(t)\|_{L^\infty(\Omega)} \leq C(\|u_0\|_{L^q(\Omega)}, \|\rho\|_\infty) t^{-\frac{N}{2q}}\,, \qquad \forall t \leq T(u_0)\,, \tag{4.27}$$

where $C(\|u_0\|_{L^q(\Omega)}, \|\rho\|_\infty)$ denotes a constant depending on the norm of u_0 in $L^q(\Omega)$ and on $\|\rho\|_\infty$. Passing to the limit in (4.27) as m tends to infinity, we deduce that u is locally bounded and satisfies (4.27) as well, and finally that it is a weak solution of (1.3).

Step 4: Uniqueness of the weak solution. Let w be another weak solution of (1.2), hence $w \in C([0,T]; L^q(\Omega))$, given $\eta > 0$ such that

$$\frac{1}{4} - 2(2+\eta) C_3 \delta > 0\,,$$

since $u, w : [0,T] \mapsto L^q(\Omega)$ are uniformly continuous, there exists $C(\eta, u, w)$ such that

$$\begin{aligned}\|u(t) - u(s)\|_{L^q(\Omega)} + \|w(t) - w(s)\|_{L^q(\Omega)} &\leq \big((1+\eta)^{\frac{1}{p-1}} - 1\big)\delta^{\frac{1}{p-1}} \\ \forall t, s \in [0,T] : |t - s| &\leq C(\eta, u, w)\,.\end{aligned} \tag{4.28}$$

Let $1 < q < 2$; as in the proof of Theorem 1.3 in the case $q > \frac{N(p-1)}{2}$, we can obtain the following inequality,

$$\frac{1}{8}\int_\Omega |(u-w)(t)|^q\,dx + 4\alpha S\frac{q-1}{q^2}\int_0^t \Big(\int_\Omega |u-w|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau$$
$$\leq \int_0^t\int_\Omega \rho(|u|^{p-1}u-|w|^{p-1}w)|u-w|^{q-1}\,dx\,d\tau\,, \qquad \forall t\in[0,T]\,,$$

which yields the following estimate, as in Proposition 2.4:

$$\begin{aligned}&\big[1-4\;t\;C_3\|\rho\|_\infty^{\frac{\bar\gamma}{p-1}}\,\|u_{n_\varepsilon}\|^{\bar\gamma}_{L^\infty(0,T;L^{\bar r}(\Omega))}\big]\sup_{s\in[0,t]}\int_\Omega |u(s)-w(s)|^q\,dx\\ &+\big[\frac{1}{4}-2C_3\big(\|u-u_{n_\varepsilon}\|^{p-1}_{L^\infty(0,T;L^q(\Omega))}+\|w-u_{n_\varepsilon}\|^{p-1}_{L^\infty(0,T;L^q(\Omega))}\big)\big]\\ &\qquad\times\int_0^t\Big(\int_\Omega |u-w|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\leq 0\,.\end{aligned}\tag{4.29}$$

Since u belongs to K, thanks to (4.28) we have, for $t\leq C(\eta,u,w)$,

$$\begin{aligned}\|w(t)-u_{n_\varepsilon}(t)\|_{L^q(\Omega)}&\leq \|w(t)-u_0\|_{L^q(\Omega)}+\|u(t)-u_0\|_{L^q(\Omega)}\\ &+\|u-u_{n_\varepsilon}\|_{L^\infty(0,T;L^q(\Omega))}\leq (1+\eta)\delta)^{\frac{1}{p-1}}\ ,\end{aligned}$$

so that the choice of η and (4.29) imply that $u=w$ in $[0,t]$, for $t\leq \min(C(\eta,u,w),T(u_0))$. Iterating the estimate in a finite number of steps (less than $[\frac{T(u_0)}{C(\eta,u,w)}]+1$) we get that $u=w$ in $[0,T]$. The case $q\geq 2$ is easier and uses the same arguments.

Step 5: Continuous dependence. Let u and v be the weak solutions relative to u_0 and v_0 respectively, and u_{n_ε}, $v_{\bar n_\varepsilon}$ the functions used in the previous steps for the fixed-point argument, which are solutions of (1.3) with $\Theta_{n_\varepsilon}(u_0)$ and $\Theta_{\bar n_\varepsilon}(v_0)$, respectively. Let $\bar r>q$, as above. By Lemma 2.5 we have

$$\frac{d}{dt}\Big(\int_\Omega |u(t)-v(t)|^{\bar r}dx\Big)+\Big(\int_\Omega |u(t)-v(t)|^{\frac{N\bar r}{N-2}}dx\Big)^{\frac{N-2}{N}}$$
$$\leq c_{\bar r}\|\rho\|_\infty\int_\Omega (|u|^{p-1}+|v|^{p-1})|u-v|^{\bar r}dx,$$

which yields

$$\begin{aligned}&\frac{d}{dt}\Big(\int_\Omega |u-v|^{\bar r}\,dx\Big)\\ &+\big[1-c_{\bar r}\|\rho\|_\infty\big(\|u-u_{n_\varepsilon}\|^{p-1}_{L^q(\Omega)}+\|v-v_{\bar n_\varepsilon}\|^{p-1}_{L^q(\Omega)}\big)\big]\Big(\int_\Omega |u-v|^{\frac{N\bar r}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\\ &\leq c_{\bar r}\|\rho\|_\infty\int_\Omega (|u_{n_\varepsilon}|^{p-1}+|v_{\bar n_\varepsilon}|^{p-1})|u-v|^{\bar r}\,dx\,.\end{aligned}$$

By Step 2, we have $(\|u-u_{n_\varepsilon}\|^{p-1}_{L^q(\Omega)}+\|v-v_{\bar n_\varepsilon}\|^{p-1}_{L^q(\Omega)})\le 2\delta$, and choosing δ smaller if necessary (note that in (4.20) above we only asked that $\delta<\frac{1}{16C_3}$) we get

$$\begin{aligned}&\frac{d}{dt}\Big(\int_\Omega |u-v|^{\bar r}\,dx\Big)+\Big(\int_\Omega |u-v|^{\frac{N\bar r}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\\ &\le C_{\bar r}\|\rho\|_\infty\int_\Omega(|u_{n_\varepsilon}|^{p-1}+|v_{\bar n_\varepsilon}|^{p-1})|u-v|^{\bar r}\,dx\,.\end{aligned}$$

Reasoning as in Step 3, with the help of Corollary 2.9, we prove, for $t<\min(T(u_0),T(v_0))$,

$$\|u(t)-v(t)\|_{L^q(\Omega)}+t^{\frac{N}{2q}}\|u(t)-v(t)\|_{L^\infty(\Omega)}\le C(\|\rho\|_\infty,u_0,v_0)\ \|u_0-v_0\|_{L^q(\Omega)}\,,$$

with $C(\|\rho\|_\infty,u_0,v_0)$ that depends on $\|\rho\|_\infty$, u_0 and v_0. Finally, (ii) and (iii) are obtained as in the case when $q>\frac{N(p-1)}{2}$.

Remark 4.1. It is clear from the proof of Theorem 1.3, for both cases, that uniqueness also holds in the class of approximated solutions, that is to say in the class of functions u belonging to $C([0,T];L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$ such that there exist two sequences $\{w_n\}$ and $\{z_n\}$ of bounded functions strongly converging to u in $C([0,T];L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$ and satisfying

$$\begin{cases}(w_n)_t-\operatorname{div}(a(x,t,\nabla w_n))=\rho(x,t)|z_n|^{p-1}z_n & \text{in } \Omega\times(0,T),\\ w_n=0 & \text{on } \partial\Omega\times(0,T),\\ w_n(x,0)=u_{0n} & \text{in } \Omega,\end{cases}$$

with u_{0n} strongly converging to u_0 in $L^q(\Omega)$.

Finally, we are left with the proof of Theorem 1.7, which essentially provided the tools for obtaining the previous results. Since the method is very close to the one used above, we will only sketch some arguments, mainly pointing out the difference between the superlinear and the linear growth of the right-hand side.

Proof of Theorem 1.7. We define a mapping as above: for v in $C([0,T];L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, $\mathcal{F}(v)$ is the unique approximated solution of

$$\begin{cases}(u)_t-\operatorname{div}(a(x,t,\nabla u))=\psi(x,t)v & \text{in } \Omega\times(0,T),\\ u=0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0 & \text{in } \Omega,\end{cases}$$

in the sense that it is obtained by approximating v in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$ with whatever sequence of bounded functions and u_0 with the sequence of its truncations, and then passing to the limit (as already said, the limit is independent of the approximation,; see Proposition 3.2). The estimates in Proposition 2.1 yield the following inequalities, with $\beta = \frac{2\sigma}{2\sigma - N}$ and $C_0 = C_0(\alpha, |\Omega|, N, q, \sigma)$, for all $t \le T$:

$$
\begin{aligned}
&(1 - t\, C_0 \|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}) \sup_{s\in[0,t]} \int_\Omega |\mathcal{F}(v)|^q \, dx \\
&\quad + \frac{1}{2}\int_0^t \Big(\int_\Omega |\mathcal{F}(v)|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{N-2}{N}} \, d\tau \le \\
&\le C_0 \int_\Omega |u_0|^q \, dx + t\, C_0 \|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))} \sup_{s\in[0,t]} \int_\Omega |v(s)|^q \, dx \\
&+ \frac{1}{4}\int_0^t \Big(\int_\Omega |v|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{N-2}{N}} \, d\tau \,,
\end{aligned}
\tag{4.30}
$$

and

$$
\begin{aligned}
&(1 - t\, C_0 \|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}) \sup_{s\in[0,t]} \int_\Omega |\mathcal{F}(v) - \mathcal{F}(w)|^q \, dx \\
&+ \frac{1}{2}\int_0^t \Big(\int_\Omega |\mathcal{F}(v) - \mathcal{F}(w)|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{N-2}{N}} \, d\tau \\
&\le t\, C_0 \|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))} \sup_{s\in[0,t]} \int_\Omega |v(s) - w(s)|^q \, dx \\
&+ \frac{1}{4}\int_0^t \Big(\int_\Omega |v - w|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{N-2}{N}} \, d\tau \,.
\end{aligned}
\tag{4.31}
$$

Let us fix k points $\tau_1 < \tau_2 < \cdots < \tau_k$ in $[0,T]$, such that the interval $[0,T]$ is divided into $k+1$ subintervals (τ_j, τ_{j+1}) (with $\tau_0 = 0$, $\tau_{k+1} = T$), each of length $\delta = \frac{1}{8C_0\|\psi\|^{\beta}_{L^\infty(0,T;L^\sigma(\Omega))}}$ but for the last one, which may be smaller. Setting I_i the interval $[\tau_i, \tau_{i+1}]$ for $i = 0, \ldots, k$, we define the following closed convex sets:

$$
\begin{aligned}
K_i = \Big\{ &\varphi \in C(I_i; L^q(\Omega)) \cap L^q(I_i; L^{\frac{Nq}{N-2}}(\Omega) \, : \\
&\max_{I_i} \int_\Omega |\varphi|^q \, dx \le 4^{i+1} C_0 \|u_0\|^q_{L^q(\Omega)} \,, \\
&\int_{I_i} \Big(\int_\Omega |\varphi|^{\frac{Nq}{N-2}} \, dx \Big)^{\frac{N-2}{N}} \, d\tau \le 8 C_0 4^i \, \|u_0\|^q_{L^q(\Omega)} \Big\} \,.
\end{aligned}
$$

Note, incidentally, that an estimate like (4.30) can be obtained in each subinterval I_i, so that, if v belongs to K_i for every $i = 0, \dots, k$, reasoning inductively we deduce

$$\frac{7}{8}\sup_{s\in I_i}\int_\Omega |\mathcal{F}(v)|^q\,dx + \frac{1}{2}\int_{I_i}\Big(\int_\Omega |\mathcal{F}(v)|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau \tag{4.32}$$
$$\leq 4^i C_0\|u_0\|^q_{L^q(\Omega)} + \frac{1}{8}4^{i+1}C_0\|u_0\|^q_{L^q(\Omega)} + 2C_0 4^i\|u_0\|^q_{L^q(\Omega)} = \frac{7}{2}C_0 4^i\|u_0\|^q_{L^q(\Omega)},$$

which implies that $\mathcal{F}(v)$ belongs to K_i for every $i = 0, \dots, k$. This means that the convex set $K = \bigcap_{i=0}^{k} K_i$ is an invariant closed convex set for $\mathcal{F}$.

In fact, in order to apply the contraction mapping principle, we need to restrict ourselves to each subinterval I_i. Thus we begin by setting $\mathcal{F}_0 = \mathcal{F}_{|I_0}$, the restriction of $\mathcal{F}$ to I_0. We have already seen that $\mathcal{F}_0$ maps K_0 into itself; moreover, (4.31) allows us to obtain

$$\frac{7}{8}\sup_{s\in I_0}\int_\Omega |\mathcal{F}_0(v)-\mathcal{F}_0(w)|^q\,dx + \frac{1}{2}\int_{I_0}\Big(\int_\Omega |\mathcal{F}_0(v)-\mathcal{F}_0(w)|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau$$
$$\leq \frac{1}{8}\sup_{s\in I_0}\int_\Omega |v(s)-w(s)|^q\,dx + \frac{1}{4}\int_{I_0}\Big(\int_\Omega |v-w|^{\frac{Nq}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\,d\tau\,,$$

so that we deduce

$$\mathrm{dist}_{K_0}(\mathcal{F}_0(v),\mathcal{F}_0(w)) \leq \big(\tfrac{1}{2}\big)^{\frac{1}{q}}\,\mathrm{dist}_{K_0}(v,w)\,.$$

Therefore, $\mathcal{F}_0$ admits a unique fixed point z_0 in K_0. It is clear the the same argument can be repeated in the interval I_1, defining an operator $\mathcal{F}_1$ on the convex set K_1 from the differential problem

$$\begin{cases} u_t - \mathrm{div}(a(x,t,\nabla u)) = \psi(x,t)u & \text{in } \Omega\times I_1,\\ u = 0 & \text{on } \partial\Omega\times I_1,\\ u(x,\tau_1) = z_0(x,\tau_1) & \text{in } \Omega. \end{cases}$$

We can find a fixed point z_1 in K_1 which is evidently an extension of z_0. By induction we can define similar operators $\mathcal{F}_i$, each on the convex set K_i and related to the equation (1.5) in the interval I_i where the initial datum for $t = \tau_i$ is given by the fixed point found for $\mathcal{F}_{i-1}$. Then each $\mathcal{F}_i$ yields a fixed point z_i in I_i, and clearly the function $u = \sum_{i=0}^{k} z_i\chi_{I_i}$ is a fixed point for the operator $\mathcal{F}$ defined in $[0,T]$. It is worth noting that it is the only fixed point of $\mathcal{F}$ in K as a consequence of (4.32); indeed, every fixed point w of $\mathcal{F}$ in K is in particular a fixed point for $\mathcal{F}_0$ in K_0, so it must be equal to z_0 on K_0. But this implies that w is also a fixed point for $\mathcal{F}_1$ in K_1, and so on it is proved that w must be equal to u.

The rest of the proof is identical to Theorem 1.3. Indeed, u is the limit of $\{u_n\}$, solutions of

$$\begin{cases} (u_n)_t - \operatorname{div}(a(x,t,\nabla u_n)) = \psi(x,t)u_n & \text{in } \Omega \times (0,T), \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(x,0) = \Theta_n(u_0)(x) & \text{in } \Omega, \end{cases}$$

which is a sequence of bounded weak solutions. Thus, applying Proposition 2.7 we get local estimates for u_n which imply that u belongs to $L^\infty_{\text{loc}}(0,T;L^\infty(\Omega)) \cap L^2_{\text{loc}}(0,T;H^1_0(\Omega))$ and satisfies (1.7). The uniqueness of the weak solution can be proved reasoning, as in Step 4 of the proof of Theorem 1.3 when $q > \max(1, \frac{N(p-1)}{2})$, for each time subinterval I_i, hence in all $[0,T]$. $\square$

Remark 4.2. The existence of a fixed point u for the operator $\mathcal{F}$ defined in the previous proof can also be obtained by using Schauder's fixed-point theorem. Indeed, we have already found a closed convex set K which is invariant for $\mathcal{F}$, so it only remains to prove that $\mathcal{F}$ is a continuous and compact operator. To this purpose, we can use Proposition 3.1; let $\{v_j\}$ be a sequence bounded in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$; then by definition of $\mathcal{F}$ there exist sequences $\{v_j^n\}$ and $\{w_j^n\}$ of bounded functions such that

$$\begin{cases} (w_j^n)_t - \operatorname{div}(a(x,t,\nabla w_j^n)) = \psi(x,t)v_j^n & \text{in } \Omega \times (0,T), \\ w_j^n = 0 & \text{on } \partial\Omega \times (0,T), \\ w_j^n(x,0) = \Theta_n(u_0) & \text{in } \Omega, \end{cases} \tag{4.33}$$

and v_j^n, w_j^n strongly converge in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$ to v_j and to $\mathcal{F}(v_j)$, respectively. Then we can take a subsequence $\{n_j\}$ such that

$$\|w_j^{n_j} - \mathcal{F}(v_j)\|_{L^\infty(0,T;L^q(\Omega))} + \|w_j^{n_j} - \mathcal{F}(v_j)\|_{L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))} \le \frac{1}{j} \qquad \forall j \in \mathbf{N}\,. \tag{4.34}$$

Since the sequence $\{v_j^{n_j}\}$ is bounded in $C([0,T];L^q(\Omega)) \cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, applying Proposition 3.1 to the problems (4.33) with $n = n_j$, we deduce that the sequence $\{w_j^{n_j}\}$ is relatively compact in $C([0,T];L^q(\Omega))\cap L^q(0,T;L^{\frac{Nq}{N-2}}(\Omega))$, which implies, thanks to (4.34), that $\{\mathcal{F}(v_j)\}$ is relatively compact. In a similar way we can prove the continuity of the operator, so $\mathcal{F}$ admits a fixed point u in K by Schauder's theorem. This method has the advantage that it works for monotone (not necessarily strongly monotone) coercive operators since it only makes use of Proposition 3.1 (see also Remark 3.1), but on the contrary it is then more difficult to prove that u is the unique fixed point for $\mathcal{F}$.

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