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SOME COMPARISON, SYMMETRY AND MONOTONICITY RESULTS FOR CARNOT-CARATHÉODORY SPACES

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Abstract. We consider here some differential operators arising from the so called Carnot-Carathéodory metric spaces associated with a family of vector fields $X = (X_1, \dots, X_k)$, which include the Hörmander type as a special case. We prove some weak and strong comparison results for solutions of the relevant differential *inequalities*. We then use these results to get some symmetry and monotonicity properties of solutions of the relevant partial differential equations.

1. Preliminaries. Let $X = (X_1, \dots, X_k)$ be a family of C^∞ vector fields in $\mathbb{R}^n$; we say that X satisfies Hörmander's condition if

$$\text{rankLie}[X_1, \dots, X_k](x) = n, \quad \text{for any } \ x \in \mathbb{R}^n. \tag{1}$$

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A piecewise C^1-curve $\gamma : [0,T] \longrightarrow \mathbb{R}^n$ is called horizontal, if whenever $\gamma'(t)$ exists,

$$\gamma'(t) = \sum_{j=1}^{k} c_j(t) X_j(\gamma(t)) \text{ satisfying } \sum_{j=1}^{k} c_j(t)^2 \le 1. \tag{2}$$

T is called the horizontal length of γ. Denote by $\mathcal{H}$ the space of all horizontal curves, and given any $x, y \in \mathbb{R}^n$, we define

$$d(x,y) = \inf_{\gamma\in\mathcal{H}} \{T, \gamma(0) = x, \gamma(T) = y\}. \tag{3}$$

Thanks to the works of Nagel, Stein and Wainger [26] and Chow [10], we get the following basic properties:

(H1) $(\mathbb{R}^n, d)$ is a metric space; i.e., $d(x,y) < \infty$ for any $x, y \in \mathbb{R}^n$.

(H2) For any U a bounded, open set of $\mathbb{R}^n$, there exist C_1 and $R_0 > 0$ such that

$$\forall\, x_0 \in U, r \le R_0, \quad |B_d(x_0, 2r)| \le C_1 |B_d(x_0, r)| \quad \text{(doubling condition)}, \tag{4}$$

where $|\cdot|$ denotes Lebesgue's measure on $\mathbb{R}^n$ and B_d denotes the ball in $(\mathbb{R}^n, d)$. For any Lipschitz function u, we denote by $|Xu| = [\sum_{1\le j\le k} (X_j u)^2]^{\frac{1}{2}}$ the length of the horizontal gradient $Xu = (X_1 u, \dots, X_k u)$; we can then introduce the corresponding Sobolev spaces. For any U an open set of $\mathbb{R}^n$ and $1 \le p < \infty$, define

$$\|u\|_{W^{1,p}} = \Big(\int_U |Xu|^p + |u|^p dx \Big)^{1/p} \tag{5}$$

and $W_0^{1,p}(U)$ (respectively $W^{1,p}(U)$) the completion of $C_0^\infty(U)$ (respectively the completion of $\{u \in C^\infty(U) : \|u\|_{W^{1,p}} < \infty\}$) under the norm $\|\cdot\|_{W^{1,p}}$. When $p = 2$, we denote $W_0^{1,2}$ (respectively $W^{1,2}$) by H_0^1 (respectively H^1).

Of course, all the definitions (2), (3) and (5) can be generalized for any real-valued, locally Lipschitz vector fields:

$$X_j = \sum_{l=1}^{n} a_{jl} \frac{\partial}{\partial x_l}, \qquad j = 1, \dots, k. \tag{6}$$

Clearly, for any $u \in W^{1,p}(U)$, $X_j u$ is given in the sense of distribution:

$$\langle X_j u, \varphi\rangle = \int_U u X_j^* \varphi \, dx, \qquad \forall\, \varphi \in C_0^\infty(U) \tag{7}$$

where $X_j^* = -\sum_{1\le l\le n} \partial_l (a_{jl} \cdot)$ denotes the formal adjoint of X_j.

Remark 1. The definitions and conditions (1)–(7) have sense also for a Riemannian manifold $(\mathcal{M}, g)$ with a subbundle of the tangent bundle $\mathcal{TM}$ (see [22]). An interesting case is homogeneous nilpotent stratified groups, the so-called Carnot groups, for example, the Heisenberg group. The definitions and conditions (2)–(7) work also for the metric generated by nonsmooth vector fields of Baouendi-Grushin type (see Example 7.2).

We assume that (H1) and (H2) always hold true; therefore $(\mathbb{R}^n, d)$ is called the Carnot-Carathéodory metric space associated to $(X_j)_{1\leq j\leq k}$ given by (6). We consider the differential operator L as follows:

$$Lu = \sum_{j=1}^{k} X_j^* A_j(x, Xu) + \sum_{j=1}^{k} b_j X_j u + \sum_{j=1}^{k} X_j^*(c_j u), \tag{8}$$

and we consider the solution or subsolution of the equation

$$\begin{cases} Lu &= (\leq)\ f(u), \quad \text{in } \Omega, \\ u &= (\leq)\ 0 \qquad \text{on } \partial\Omega \end{cases} \tag{9}$$

for Ω a bounded, open set of $\mathbb{R}^n$.

Remark 2. A typical case of L is just $L_0 = \sum_{j\leq k} X_j^* X_j$. But (8) defines a more general operator other than that in divergence form. For example, $L_1 = -\sum_{j\leq k}(X_j)^2$ can be written as $L_1 = L_0 - \sum_{j\leq k}(X_j + X_j^*)X_j$, where $(X_j + X_j^*) = -\sum_{l\leq n} \partial_l a_{jl}$ is an L^∞ function, for any $1 \leq j \leq k$; hence L_1 is an operator of the form (8).

Many works have been done to understand the density of regular functions, Sobolev's embedding properties, the isoperimetric inequalities or the Hölder regularities of solutions for differential operators of type L, in general Carnot-Carathéodory spaces (see [19], [23] and references therein). A major difficulty lies essentially in the degeneracy of vector fields X, that is, $\{X_1(x), \dots, X_k(x)\}$ can not span $\mathbb{R}^n$ at some points x. Thus, this fact leads to the degeneracy of L, which is no longer uniformly elliptic.

In this paper, our aim is to establish some comparison or Harnack-type results under rather weak conditions for L, u and f in some general Carnot-Carathéodory spaces. Then we will use these results to get some symmetry and monotonicity results, by a generalization of the "moving plane method."

The moving plane method goes back to Alexandrov's famous paper [1] on constant mean curvature hypersurfaces. Serrin [28] reintroduced it for studying elliptic equations. This method has known a great development since the symmetry results of Gidas, Ni and Nirenberg [20], where they used essentially the maximum principle and the reflection invariance of the Laplacian. Later on, Berestycki and Nirenberg [3] improved this method by taking advantage of "small" domains; we can find also many works for unbounded domains, for special symmetric domains in $\mathbb{S}^n$ [27] or $\mathbb{H}^n$ [25], and recently for some general manifolds in [2]. But all these works treat uniformly elliptic operators.

For a degenerate elliptic operator, many interesting results have been obtained recently in the p-Laplacian case by Brock [7]; Damascelli, Pacella and Ramaswamy [13], [14], [15]; and Serrin and Zou [29]. Birindelli and Prajapat proved in [4] some symmetry and monotonicity results for C^2 solutions in the nilpotent stratified groups case.

Our paper is organized as follows: in Section 2, we state the general assumptions and recall some basic properties of Sobolev spaces associated to X. In Section 3, we prove our comparison results. Symmetry properties for bounded and unbounded domains are established in Sections 4 and 5. Section 6 is devoted to monotonicity result, and some examples are given in the final section. In all this paper, C and C' denote always positive constants independent of u; its value could even be changed from one line to another.

2. General assumptions and preliminary results. We now introduce our general assumptions:

(H) $X = (X_1, \dots, X_k)$ is a system of vector fields given by (6) satisfying (H1), (H2) and there exist $C_2, R_0 > 0$, $\alpha \geq 1$ such that for any $x_0 \in U$, $R \leq R_0$ and any Lipschitz function u defined on $B_d(x_0, \alpha R)$, we have

$$|\{x \in B_d(x_0, R), |u(x) - \bar{u}_B| > \lambda\}| \leq \frac{C_2 R}{\lambda}\int_{B_d(x_0,\alpha R)} |Xu|\, dx, \text{ for all } \lambda > 0, \quad (10)$$

where $\bar{u}_B$ stands for the average of u over $B_d(x_0, R)$.

(H3) $(\mathbb{R}^n, d)$ is complete and $(\mathbb{R}^n, d)$ is homeomorphic to $(\mathbb{R}^n, |\cdot|)$; i.e., $(\mathbb{R}^n, d)$ defines the usual topology in $\mathbb{R}^n$.

(H4) $A = (A_j(x, \eta)) \in C^0(\bar{\Omega} \times \mathbb{R}^k)$, A is Γ-Lipschitz with respect to η on $\bar{\Omega} \times \mathbb{R}^k$, and $A(x, 0) = 0$ for any $x \in \Omega$.

(H5) There exists $C_3 > 0$ such that

$$\sum_{i,j=1}^{k} \frac{\partial A_i}{\partial \eta_j}(x, \eta)\xi_i\xi_j \geq C_3\|\xi\|^2, \text{ for any } x \in \Omega, \xi \in \mathbb{R}^k \text{ and a.e. } \eta \in \mathbb{R}^k. \quad (11)$$

(H6) $b_j, c_j, X_j^* c_j \in L^\infty(\Omega)$ for $j = 1, \dots, k$.

(H7) f is a locally Lipschitz function from $\mathbb{R}$ to $\mathbb{R}$.

We know that (H) is satisfied for a very large class of Carnot-Carathéodory spaces and a lot of work has been done to understand Sobolev spaces relative to X; we just refer to [23], [19] and the references therein for interested readers.

Note also that $Q = \log_2 C_1$ (C_1 is the constant in the doubling condition (H2)) is called the local homogeneous dimension relative to U. We suppose that $Q \geq 2$. We have

Lemma 1. (See [19].) *Suppose that* (H) *and* (H3) *hold. For any bounded open set $U \subset \mathbb{R}^n$, there exist C and $R_1 > 0$ such that for any metric ball $B = B_d(x_0, R)$ with $x_0 \in U$ and $R \leq R_1$, one has for any $1 \leq p < Q$, $1 \leq k \leq Q/(Q-p)$ and $u \in W^{1,p}(B)$*

$$\Big(\frac{1}{|B|}\int_B |u - \bar{u}_B|^{kp} dx\Big)^{1/kp} \leq CR\Big(\frac{1}{|B|}\int_B |Xu|^p dx\Big)^{1/p}. \tag{12}$$

If $p \geq Q$, then for any $1 \leq q < \infty$ and $u \in W^{1,p}(B)$, one has

$$\Big(\frac{1}{|B|}\int_B |u - \bar{u}_B|^{q} dx\Big)^{1/q} \leq CR\Big(\frac{1}{|B|}\int_B |Xu|^p dx\Big)^{1/p}. \tag{13}$$

Lemma 2. *Under the same assumptions as in Lemma* 1, *let $B = B_d(x_0, R)$ with $x_0 \in U$ and $R \leq R_1$ as above. Then for any $1 \leq p < Q$, $1 \leq k < Q/(Q-p)$ and $u \in W_0^{1,p}(B)$*

$$\Big(\frac{1}{|B|}\int_B |u|^{kp} dx\Big)^{1/kp} \leq C\Big(\frac{1}{|B|}\int_B |Xu|^p dx\Big)^{1/p}. \tag{14}$$

Lemma 2 follows easily from Lemma 1 and Sobolev's compact embedding theorem (see [19], Theorem 1.28 and Theorem 1.15).

Remark 3. Clearly, Lemmas 1 and 2 work also if we replace the Lebesgue measure dx by $\mu = \nu(x)\,dx$ with $0 < \alpha \leq \nu(x) \leq \beta < \infty$. That's why we can work on manifolds. In [18], Franchi, Serapioni and Serra Cassano have treated more general weighted Sobolev spaces associated to Lipschitz-continuous vector fields.

3. Comparison results. We begin with some maximum principles for a subsolution. The proof is similar to the classical Laplacian case, but the principles seem to be new.

Theorem 1. (Weak maximum principle) *Let $\Omega \subset U$ be bounded, open sets in $\mathbb{R}^n$. Suppose that u is in $H^1 \cap L^\infty(\Omega)$ satisfying $Lu + g(x,u) - \Lambda u \leq 0$ in Ω with $\Lambda \geq 0$, $g(x,u) \in C^0(\bar{\Omega} \times \mathbb{R})$ satisfying $g(x,s) \geq 0$ for $s \geq 0$. Then there exists a constant $C_4 > 0$ such that for any open subset Ω' in Ω with $|\Omega'| \leq C_4$, if $u \leq 0$ on $\partial\Omega'$, then $u \leq 0$ in Ω'.*

To prove this theorem, we need the following result (see for instance [18]).

Lemma 3. *Let Ω be a bounded, open set in $\mathbb{R}^n$. For any $x_0 \in \Omega$, $B_d(x_0, r) \subset\subset \Omega$ and $s < r$, we have a cutoff function $\xi \in W^{1,\infty}(\Omega)$ such that $\chi_{B_d(x_0,s)} \leq \xi \leq \chi_{B_d(x_0,r)}$ and $\|X\xi\|_\infty \leq C(\Omega)/|r-s|$.*

Proof of Theorem 1. First, we see that since $\overline{\Omega}$ is compact, there exists a finite family of metric balls $B_i = B_d(x_i, R_1/2)$ with $x_i \in U$ $(1 \leq i \leq l)$, satisfying $\Omega \subset \cup_{1\leq i\leq l} B_i$ (R_1 is the constant in Lemma 1). By hypothesis, $u^+ = \max(u,0) \in H_0^1(\Omega')$, so we can use u^+ as a test function; hence

$$\int_{\Omega'} \langle A(x,Xu), Xu^+\rangle dx + \int_{\Omega'} g(x,u)u^+ dx + \int_{\Omega'} \langle bu^+, Xu\rangle dx + \int_{\Omega'} \langle cu, Xu^+\rangle dx - \Lambda \int_{\Omega'} uu^+ dx \quad \leq 0.$$

By (H5), it is easy to see

$$\langle A(x,\eta) - A(x,\eta'), \eta - \eta'\rangle \geq C_3|\eta - \eta'|^2, \forall\, x \in \Omega, \eta, \eta' \in \mathbb{R}^k. \tag{15}$$

Using (15) with $\eta' = 0$, (H4) and $g(x,u)u^+ \geq 0$, we have

$$C_3 \int_{\Omega'} |Xu^+|^2 dx \leq \Lambda \int_{\Omega'} |u^+|^2 dx + C \int_{\Omega'} |u^+||Xu^+| dx.$$

Thus, by the fact that $C|u^+||Xu^+| \leq C_3|Xu^+|^2/2 + C^2|u^+|^2/(2C_3)$, we get

$$(C_3/2) \int_{\Omega'} |Xu^+|^2 dx \leq C' \int_{\Omega'} |u^+|^2 dx. \tag{16}$$

Now we construct ξ_i a family of cutoff functions such that $\mathrm{supp}(\xi_i)$ is included in $B_d(x_i, R_1)$ and $\xi_i \equiv 1$ in B_i with $\|X\xi_i\|_\infty \leq C/R_1$. Since $\xi_i u^+ \in H_0^1(B_d(x_i, R_1))$ for $x_i \in U$, then Lemma 2 works, $\xi_i u^+ \in L^q(B_d(x_i, R_1))$ for some $q > 2$,

$$\begin{aligned}\int_{B_i} |u^+|^2 dx &\le \int_{B_d(x_i,R_1)} |\xi_i u^+|^2 dx \le |\Omega'|^{1-2/q} \Big(\int_{B_d(x_i,R_1)} |\xi_i u^+|^q dx\Big)^{2/q} \\ &\le C|\Omega'|^{1-2/q} \int_{B_d(x_i,R_1)} |X(\xi_i u^+)|^2 dx \qquad (17) \\ &\le C|\Omega'|^{1-2/q} \Big(\int_{B_d(x_i,R_1)} \xi_i^2 |Xu^+|^2 dx + \int_{B_d(x_i,R_1)} |u^+|^2 dx\Big) \\ &\le C|\Omega'|^{1-2/q} \Big(\int_{\Omega'} |Xu^+|^2 dx + \int_{\Omega'} |u^+|^2 dx\Big).\end{aligned}$$

So we obtain

$$\int_{\Omega'} |u^+|^2 dx \le \sum_{1\le i\le l} \int_{B_i} |u^+|^2 dx \le lC|\Omega'|^{1-2/q} \Big(\int_{\Omega'} |Xu^+|^2 dx + \int_{\Omega'} |u^+|^2 dx\Big). \qquad (18)$$

If $lC|\Omega'|^{1-2/q} < 1/2$,

$$\int_{\Omega'} |u^+|^2 dx \le 2lC|\Omega'|^{1-2/q} \int_{\Omega'} |Xu^+|^2 dx.$$

Moreover, if $2lCC'|\Omega'|^{1-2/q} < C_3/2$, i.e.,

$$|\Omega'|^{1-2/q} < C_4 = \min\{1/(2lC), C_3/(4lCC')\},$$

we have $\int_{\Omega'} |Xu^+|^2 dx = 0$ by (16), which means $u^+ \equiv 0$ in Ω'. □

By the same idea, we have

Theorem 2. (Weak comparison principle) *Let $\Omega \subset U$ be bounded, open sets in $\mathbb{R}^n$ and u, v be in $H^1 \cap L^\infty(\Omega)$ satisfying*

$$Lu + g(x,u) - \Lambda u \le Lv + g(x,v) - \Lambda v \quad in \quad \Omega$$

with $\Lambda \ge 0$ and $g(x,u) \in C^0(\bar{\Omega} \times \mathbb{R})$ satisfying for any $x \in \Omega$, $g(x,s)$ is nondecreasing in s for $|s| \le \max(\|u\|_\infty, \|v\|_\infty)$. There exists a constant $C_5 > 0$ such that for any open subset Ω' in Ω with $|\Omega'| \le C_5$, if $u \le v$ on $\partial\Omega'$, then $u \le v$ in Ω'.

Proof. Using $(u-v)^+ \in H_0^1(\Omega')$ as a test function, we get

$$\begin{aligned}
&\int_{\Omega'} \langle A(x,Xu) - A(x,Xv), X(u-v)^+\rangle dx + \int_{\Omega'} [g(x,u) - g(x,v)](u-v)^+ dx \\
&\leq \Lambda \int_{\Omega'} \left|(u-v)^+\right|^2 dx - \int_{\Omega'} \langle b(u-v)^+, X(u-v)\rangle dx \\
&\qquad - \int_{\Omega'} \langle c(u-v), X(u-v)^+\rangle dx \\
&\leq \Lambda \int_{\Omega'} \left|(u-v)^+\right|^2 dx + C\int_{\Omega'} |(u-v)^+||X(u-v)^+| dx.
\end{aligned}$$

Since $[g(x,u)-g(x,v)](u-v)^+ \geq 0$ in Ω, as in the proof of Theorem 1, when $|\Omega'|$ is small enough, we get for some $q > 2$

$$\left(C_3/2 - C|\Omega'|^{1-2/q}\right) \int_{\Omega'} |X(u-v)^+|^2 dx \leq 0,$$

which completes our proof. □

We now state a Harnack-type comparison result, which generalizes the strong maximum principle obtained by Bony [5] and [6].

Theorem 3. *Let Ω be a bounded, open set in $\mathbb{R}^n$ and u, v be in $H^1 \cap L^\infty(\Omega)$ satisfying*

$$\begin{cases} Lu + \Lambda u & \leq Lv + \Lambda v \quad \text{in } \Omega \\ \qquad\quad u & \leq v \qquad\qquad\quad \text{in } \Omega \end{cases} \tag{19}$$

for a constant $\Lambda \in \mathbb{R}$. There exists $r_0 > 0$ such that for any $\delta \in (0, R_1/4)$, there exists $C_\delta > 0$ such that if $x_0 \in \Omega$ with $\overline{B_d(x_0, 4\delta)} \subset \Omega$, then

$$\|v-u\|_{L^{r_0}(B_d(x_0,2\delta))} \leq C_\delta \operatorname*{ess\,inf}_{B_d(x_0,\delta)} (v-u).$$

In addition, if u, v are continuous and if there exists $x_0 \in \Omega$ such that $u(x_0) = v(x_0)$, then $u \equiv v$ on the connected component of Ω containing x_0.

To prove the theorem, we need the following

Lemma 4. *Let Ω be a bounded, open set in $\mathbb{R}^n$ and $u \in W^{1,1}(\Omega)$. Suppose that $\exists\, K > 0$, such that for any $B_d(x,r) \subset \Omega$,*

$$\int_{B_d(x,r)} |Xu|\, dx \leq Kr^{-1}|B_d(x,r)|;$$

then there exist $C, \sigma > 0$*, such that for any* $B_d(x,s) \subset\subset \Omega$ *with* $s < R_1$*,*

$$\int_{B_d(x,s)} \exp\left(\frac{\sigma}{K}|u - \bar{u}_{B_s}|\right) \leq C|B_d(x,s)|.$$

Lemma 4 is proved by Poincaré's inequality given in Lemma 1 for $p = 1$, and the general result is proved in [8] for BMO-functions in spaces of homogeneous nature as Coifman and Weiss defined, which shows that the John-Nirenberg inequality holds true.

Proof of Theorem 3. We can always suppose that $\Lambda \geq 0$ (if not, we can take $\Lambda = 0$ since $u \leq v$). We replace v by $v + \tau$, and our result is proved by taking $\tau \to 0$. Now $v - u \geq \tau > 0$. Denoting by B_r the ball $B_d(x_0, r)$, we choose a cutoff function ξ with support in $B_{4\delta}$, so $\varphi = (v-u)^\beta \xi^2 \in H_0^1(\Omega)$ for any $\beta \leq 0$. Therefore, we can use φ as a test function, which yields

$$\begin{aligned}
&|\beta|\int_{B_{4\delta}} \xi^2 (v-u)^{\beta-1}\langle A(x,Xu) - A(x,Xv), Xu - Xv\rangle\, dx \\
&+ \int_{B_{4\delta}} 2\xi(v-u)^\beta \langle A(x,Xu) - A(x,Xv), X\xi\rangle\, dx \\
&+ \int_{B_{4\delta}} \xi^2(v-u)^\beta\langle b, X(u-v)\rangle\, dx + \int_{B_{4\delta}} (u-v)\langle c, X((v-u)^\beta \xi^2)\rangle\, dx \\
&\leq \Lambda \int_{B_{4\delta}} (v-u)^{\beta+1}\xi^2\, dx.
\end{aligned}$$

By (H4), (H5) and (H6),

$$\begin{aligned}
&|\beta| C_3 \int_{B_{4\delta}} \xi^2 (v-u)^{\beta-1}|Xu - Xv|^2\, dx \leq C\int_{B_{4\delta}} \xi|X\xi|(v-u)^\beta |X(v-u)|\, dx \\
&+ C(1+|\beta|)\int_{B_{4\delta}} \xi^2 (v-u)^\beta |X(u-v)|\, dx \\
&+ C\int_{B_{4\delta}} \xi|X\xi|(v-u)^{\beta+1}\, dx + \Lambda\int_{B_{4\delta}} (v-u)^{\beta+1}\xi^2\, dx.
\end{aligned}$$

Using Cauchy-Schwarz's inequality, we get

$$\frac{C_3|\beta|}{2}\int_{B_{4\delta}} \xi^2 (v-u)^{\beta-1}|Xu - Xv|^2 dx \leq C\left(1 + \frac{1}{|\beta|}\right)\int_{B_{4\delta}} (\xi^2 + |X\xi|^2)(v-u)^{\beta+1} dx;$$

hence

$$\int_{B_{4\delta}} \xi^2 (v-u)^{\beta-1} |Xu - Xv|^2 \, dx \le C(1 + \frac{1}{|\beta|^2}) \int_{B_{4\delta}} (\xi^2 + |X\xi|^2)(v-u)^{\beta+1} \, dx. \tag{20}$$

Take $\beta = -1$,

$$\int_{B_{4\delta}} \xi^2 \frac{|Xu - Xv|^2}{(v-u)^2} \, dx \le C \int_{B_{4\delta}} (\xi^2 + |X\xi|^2) \, dx. \tag{21}$$

Define $\Psi(t,h) = \big(\int_{B_h} (v-u)^t \, dx\big)^{1/t}$; clearly $\Psi(-\infty,h) = \mathrm{essinf}_{B_h}(v-u)$ and $\Psi(\infty,h) = \mathrm{esssup}_{B_h}(v-u)$. Now, for any $y \in B_{2\delta}$ and any $r < \delta$, we write $\tilde{B}_r = B_d(y,r)$ and take ξ given by Lemma 3 such that $\chi_{\tilde{B}_r} \le \xi \le \chi_{\tilde{B}_{2r}}$, $\|X\xi\|_\infty \le C/r$. Denote $\omega = \log(v-u)$; (21) means

$$\int_{\tilde{B}_r} |X\omega|^2 \, dx \le C(1 + 1/r^2)|\tilde{B}_{2r}| \le \frac{C}{r^2}|\tilde{B}_r|,$$

where we use also the doubling condition. Thus, it follows from Cauchy-Schwarz's inequality that

$$⨍_{\tilde{B}_r} |X\omega| \, dx \le \Big(⨍_{\tilde{B}_r} |X\omega|^2 \, dx\Big)^{1/2} \le \frac{C}{r}.$$

By Lemma 4, there exist positive numbers r_0 and C such that

$$\int_{B_{2\delta}} e^{r_0|\omega - \bar{\omega}_{B_{2\delta}}|} \, dx \le C|B_{2\delta}|. \tag{22}$$

In particular, $\big(\int_{B_{2\delta}} e^{r_0\omega} \, dx\big)\big(\int_{B_{2\delta}} e^{-r_0\omega} \, dx\big) \le C|B_{2\delta}|^2$; i.e.,

$$\Psi(r_0, 2\delta) \le C|B_{2\delta}|^{2/r_0} \Psi(-r_0, 2\delta). \tag{23}$$

Consider now $\beta < -1$, $q = (\beta+1)/2 < 0$ and $\omega = (v-u)^q$. For any $0 < h' < h'' < 2\delta$, we take a cutoff function $\chi_{B_{h'}} \le \xi \le \chi_{B_{h''}}$; by Lemma 3, (20) implies

$$\|\xi X\omega\|_{L^2(B_{h''})} \le C|q|(1 + \frac{1}{|\beta|})\frac{1}{h'' - h'}\|\omega\|_{L^2(B_{h''})} \le \frac{C|q|}{h'' - h'}\|\omega\|_{L^2(B_{h''})};$$

hence

$$\|X(\xi\omega)\|_{L^2(B_{h''})} \leq \frac{C(1+|q|)}{h''-h'}\|\omega\|_{L^2(B_{h''})}.$$

By Sobolev's embedding inequality

$$\|\omega\|_{L^{2\zeta}(B_{h'})} \leq \|\xi\omega\|_{L^{2\zeta}(B_{h''})} \leq C\|X(\xi\omega)\|_{L^2(B_{h''})} \leq \frac{C(1+|q|)}{h''-h'}\|\omega\|_{L^2(B_{h''})}$$

for some $\zeta \in (1, Q/(Q-2))$. Hence for any $\beta < -1$ (so $q < 0$), we obtain

$$\Psi(2q\zeta, h') \geq C^{1/q}(1+|q|)^{1/q}(h''-h')^{-1/q}\Psi(2q, h''). \tag{24}$$

Consider the sequence $h_n = \delta[1+(1/2)^n]$, $q_n = -r_0\zeta^n/2$ and $\beta_n = -r_0\zeta^n - 1$. By (24)

$$\begin{aligned}\Psi(2q_{n+1}, h_{n+1}) &\geq C^{\frac{1}{q_n}}(1+|q_n|)^{\frac{1}{q_n}}(1/2)^{-\frac{n+1}{q_n}}\delta^{-1/q_n}\Psi(2q_n, h_n) \qquad (25)\\ &\geq (\delta/C)^{\frac{2}{r_0\zeta^n}}(1/2)^{\frac{2(n+1)}{r_0\zeta^n}}(1+|q_n|)^{\frac{1}{q_n}}\Psi(2q_n, h_n)\\ &\;\vdots\\ &\geq (\delta/C)^{\sum_{i\leq n}\frac{2}{r_0\zeta^i}}(1/2)^{\sum_{i\leq n}\frac{2(i+1)}{r_0\zeta^i}}e^{\sum_{i\leq n}\frac{\ln(1+|q_i|)}{q_i}}\Psi(-r_0, h_0).\end{aligned}$$

Taking $n \to \infty$, $\Psi(-\infty, \delta) \geq C\delta^{\frac{2\zeta}{r_0(\zeta-1)}}\Psi(-r_0, 2\delta)$. Finally, by (23)

$$\Psi(r_0, 2\delta) \leq C|B_\delta|^{2/r_0}\Psi(-r_0, 2\delta) \leq C|B_\delta|^{2/r_0}\delta^{-\frac{2\zeta}{r_0(\zeta-1)}}\Psi(-\infty, \delta). \tag{26}$$

This completes the proof.

4. Symmetry result for bounded domains. In this section, we will apply the above results to prove some symmetry properties. For this purpose, we consider $(\mathbb{R}^n, d)$ a Carnot-Carathéodory space generated by a system X. Let $U \subset \mathbb{R}^n$ be a bounded domain. We study the second-order differential operator L defined by (8) and consider the associated "isometry" group G of L; i.e., $g \in G$ if and only if $g : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 diffeomorphism such that $g^*L = L$. Now we assume the following conditions:

i) *Isometries:* There is a family of isometries $I_t \in G$ which is C^1 in t, and such that for any $t \in (0, 2)$ there exists a family of hypersurfaces $U_t \subset \mathbb{R}^n$ such that $I_t(x) = x \Leftrightarrow x \in U_t$; i.e., U_t is the invariant hypersurface under the action of I_t.

ii) *Domain decomposition:* There exist pairwise disjoint sets V_t, $t \in (0,2)$, such that

a) $V_t \subset U_t$, for all $t \in (0,2)$.

b) For all $t_1, t_2 \in [0,2]$, $\bigcup_{t_1<t<t_2} V_t$ is an open subset of $\mathbb{R}^n$ and $\Omega = \bigcup_{0<t<2} V_t$.

iii) *Inclusion in increasing t:* Let $Q_{t_1} = \bigcup_{0<t<t_1} V_t$ and $Q^{t_1} = \bigcup_{t_1<t<2} V_t$; then

a) $I_t(Q_t) \subset Q^t$, for all $t \in (0,1)$.

b) For all $t \in (0,1)$ and for every connected component Σ of Q_t, there exists a point $x \in \partial\Sigma \cap \partial\Omega : I_t(x) \in Q^t$.

Moreover, we say that Ω is symmetric if we also have

iv) *Inclusion in decreasing t:*

a) $I_t(Q^t) \subset Q_t$, for all $t \in (1,2)$ and $Q_1 = I_1(Q^1)$.

b) For all $t \in (1,2)$ and for every connected component Σ of Q^t, there exists a point $x \in \partial\Sigma \cap \partial\Omega : I_t(x) \in Q_t$.

For $0 < t \leq 1$ and $x \in Q_t$ we define $x^t = I_t(x)$ and $u_t(x) = u \circ I_t(x) = u(x^t)$.

Theorem 4. *Assume that Ω satisfies conditions* i), ii), iii), *and* $u \in H_0^1(\Omega, \mathbb{R}) \cap C^0(\overline{\Omega}, \mathbb{R})$ *is a positive (i.e., $u > 0$ in Ω) weak solution to*

$$Lu = f(u) \quad in \quad \Omega, \quad u = 0 \quad on \quad \partial\Omega,$$

and conditions (H) *to* (H7) *are satisfied. Then*

$$u(x) < u(x^\lambda), \quad if \quad x \in Q_\lambda\,, \;\; 0 < \lambda < 1\,. \tag{27}$$

If $I_1(Q_1) \subset Q^1$, we still have the inequality

$$u(x^1) \geq u(x), \;\; \forall x \in Q_1\,. \tag{28}$$

Moreover, if Ω is symmetric, we have

$$u(x) = u(x^1), \;\; \forall x \in Q_1\,. \tag{29}$$

Proof. The major ingredient of our proof is standard (see [2]). First we see that $\partial Q_\lambda \subset V_\lambda \cup \partial\Omega$, $\forall\, \lambda \in (0,1]$, and therefore $u \leq u_\lambda$ on ∂Q_λ, $\forall\, \lambda \in (0,1]$. Begin with λ positive near 0. Since f is locally Lipschitz and u is bounded, there exists $\Lambda \geq 0$ such that $g_{\pm}(u) = \Lambda u \pm f(u)$ are nondecreasing in u. Noting $u_\lambda = u \circ I_\lambda$ and using $I_\lambda \in G$, we get

$$\begin{cases} Lu + g_-(u) - \Lambda u & = Lu_\lambda + g_-(u_\lambda) - \Lambda u_\lambda = 0 \quad \text{in } Q_\lambda \\ u & \leq u_\lambda \quad \text{on } \partial Q_\lambda. \end{cases} \tag{30}$$

By Theorem 2, then for λ small enough, we get $u \leq u_\lambda$ in Q_λ, since $|Q_\lambda| < C_5$ and $u \leq u_\lambda$ on ∂Q_λ. Now we look at $\lambda_0 = \sup\{\lambda > 0$, such that $u \leq u_\mu, \forall\, \mu \in [0, \lambda]\}$; we shall prove that $\lambda_0 = 1$. If not, $\lambda_0 < 1$ and $u \leq u_{\lambda_0}$ in Q_{λ_0} by continuity. By $Lu + \Lambda u = g_+(u) \leq g_+(u_{\lambda_0}) = Lu_{\lambda_0} + \Lambda u_{\lambda_0}$ in Q_{λ_0}, Theorem 3 implies that $u < u_{\lambda_0}$ in Q_{λ_0} since for each connected component Σ of Q_{λ_0}, we have $u_{\lambda_0} \not\equiv u$ on $\partial\Sigma$. We choose then a compact set $K \subset\subset Q_{\lambda_0}$ such that $|Q_{\lambda_0} \setminus K| < C_5/2$.

We claim that $u < u_{\lambda_0} - \epsilon$ in K for some small $\epsilon > 0$. Since u is continuous, for λ near λ_0 we have that $u < u_\lambda - \epsilon/2$ in K. On the other hand, for λ near λ_0, $|Q_\lambda \setminus K| < C_5$; then applying Theorem 2 and the fact that $u \leq u_\lambda$ on $\partial(Q_\lambda \setminus K)$, we get $u \leq u_\lambda$ in $Q_\lambda \setminus K$. Hence, $u \leq u_\lambda$ for $\lambda > \lambda_0$ and sufficiently close to λ_0; this contradicts the definition of λ_0. (27) and (28) are proved.

If Q is symmetric, we repeat our proof in the inverse direction; we get $u \geq u_1$ in Q_1, which yields $u = u_1$ in Q_1.

5. Symmetry result for unbounded domains. Let $X = \{X_1, ..., X_k\}$ be a system of C^∞ vector fields of Hörmander type. First, we remark that (H) and (H3) are locally satisfied, Lemmas 1 and 2 hold also (see [17], [26], [24] and [30]). Set $E(x) = \mathrm{Vect}\{X_j(x), 1 \leq j \leq k\}$ and $G_1 = \{g \in G : g^*(dx) = dx, d(g(x), g(y)) = d(x, y),\ \forall\ x, y \in \mathbb{R}^n,\ g^*(E) = E$ and $\exists\ c_g \geq 1$ such that $\forall\ x \in \mathbb{R}^n, Y \in E(x), \|Y\|/c_g \leq \|g^*(Y)\| \leq c_g\|Y\|\}$. We assume i), ii), iii) by replacing $t \in (0, 2)$ by $t \in \mathbb{R}$ and $I_t \in G_1$ such that $\mathbb{R}^n = \bigcup_{t\in\mathbb{R}} V_t$ with $V_t = U_t$ connected and $\lim_{t\to\infty}(\inf_{x\in V_t}|x|) = +\infty$. We study positive solutions of the following equation in $\mathbb{R}^n$, with the ground state condition at infinity, namely

$$\begin{cases} Lu & = & \displaystyle\sum_{1\leq j\leq k} X_j^* A_j(x, Xu) = f(u), & \text{in } \mathbb{R}^n, \\ u & > & 0, & \text{in } \mathbb{R}^n, \\ u(x) & \to & 0, & \text{as } |x| \to \infty. \end{cases} \tag{31}$$

Moreover, we suppose that

(H8) There exists $s_0 > 0$ such that f is nonincreasing on $(0, s_0)$.

Theorem 5. *Under the assumptions* (H4), (H5), (H7) *and* (H8), *if* $u \in H^1_{loc}(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ *is a solution of* (31), *then there exists* $t_0 \in \mathbb{R}$*, such that* $u \circ I_{t_0} = u$.

In the proof, we need the following technical lemma.

Lemma 5. *Let* $R > 0$ *and* λ *be given. Set* $P_{\gamma,\delta,R} = (\bigcup_{\gamma<t<\delta} V_t) \cap B(0,R)$. *Then there exist constants* s_1, s_2 *with* $s_1 < \lambda < s_2$ *and* $C, \delta > 0$ *such that for any* $\omega \in H^1(P_{s_1,t,2R})$ *satisfying* $\lambda \le t \le s_2$ *and* $\omega|_{(V_t \cup V_{s_1}) \cap B(0,2R)} = 0$, *we have*

$$\int_{P_{s_1,t,R}} \omega^2 \, dx \le C|supp(\omega) \cap B(0,2R)|^\delta \int_{P_{s_1,t,2R}} |X\omega|^2 \, dx, \tag{32}$$

where B *denotes the Euclidean ball in* $(\mathbb{R}^n, |\cdot|)$.

Proof. Since $V_\lambda \cap \overline{B(0,R)}$ is compact, we can find a finite number of metric balls $B_d(a_i, r)$ $(i = 1, \dots, l)$ satisfying $a_i \in V_\lambda \cap \overline{B(0,R)}$ and $r < R_1$ (R_1 is constant in Lemma 1 and 2 for $B(0,2R)$), such that $V_\lambda \cap \overline{B(0,R)} \subset \bigcup_{1\le i\le l} B_d(a_i, r/2)$.

Using the fact that I_t is continuous in t, there are s_1 and s_2 with $s_1 < \lambda < s_2$ such that $P_{s_1,s_2,R} \subset \bigcup_{1\le i\le l} B_d(a_i, r/2)$ and $V_t \cap B_d(a_i, r/2) \ne \emptyset$, $\forall\, t \in [\lambda, s_2]$. We can also assume that $B_d(x,r) \subset B(0,2R)$ for any $x \in B(0,R)$, by choosing r sufficiently small. Fix $t \in [\lambda, s_2]$ and choose $\tilde{a}_i \in V_t \cap B_d(a_i, r/2)$ $(i = 1, \dots, l)$. For any $\omega \in H^1(P_{s_1,t,2R})$ satisfying $\omega|_{V_t} = 0$, we define

$$\tilde{\omega}(x) = \begin{cases} \omega(x), & \text{if } x \in Q_t, \\ -\omega(x^t), & \text{if } x \in Q^t. \end{cases}$$

Clearly, $\tilde{\omega} \in H^1(B_i)$ for every $B_i = B_d(\tilde{a}_i, r)$ and $\fint_{B_i} \tilde{\omega}\, dx = 0$ since $I_t(B_i \cap Q_t) = B_i \cap Q^t$. Then, it follows from Lemma 1 that there exists $q > 2$ such that

$$\Big(\fint_{B_i} |\tilde{\omega}|^q \, dx\Big)^{1/q} = \Big(\fint_{B_i} |\tilde{\omega} - \tilde{\omega}_{B_i}|^q \, dx\Big)^{1/q} \le Cr\Big(\fint_{B_i} |X\tilde{\omega}|^2 \, dx\Big)^{1/2}.$$

Hence,

$$\begin{aligned} \fint_{B_i} \tilde{\omega}^2 \, dx &\le \Big(\fint_{B_i} |\tilde{\omega}|^q \, dx\Big)^{2/q}\Big(\fint_{B_i} \chi_{\text{supp}(\tilde{\omega})} \, dx\Big)^{1-2/q} \\ &\le C|\text{supp}(\tilde{\omega}) \cap B_i|^{1-2/q} \fint_{B_i} |X\tilde{\omega}|^2 \, dx \end{aligned}$$

which implies

$$\int_{B_i\cap Q_t} \omega^2\,dx \le C|\mathrm{supp}(\omega)\cap B(0,2R)|^{1-2/q}\int_{B_i\cap Q_t}|X\omega|^2\,dx.$$

Thus, we have

$$\begin{aligned}\int_{P_{s_1,t,R}}\omega^2\,dx &\le \sum_{i=1}^{l}\int_{B_i\cap Q_t}\omega^2\,dx \le C\sum_{i=1}^{l}|\mathrm{supp}(\omega)\cap B(0,2R)|^{1-2/q}\int_{B_i\cap Q_t}|X\omega|^2\,dx\\ &\le C|\mathrm{supp}(\omega)\cap B(0,2R)|^{1-2/q}\int_{P_{s_1,t,2R}}|X\omega|^2\,dx.\end{aligned}$$

The proof is completed by choosing $\delta = 1-2/q$. □

Proof of Theorem 5. The proof is divided into two steps.

Step 1. Claim : $\exists\, t_1\in\mathbb{R}$, such that $u(x)\le u_{t_1}(x)=u(x^{t_1})$, $\forall\, x\in Q_{t_1} = \bigcup_{t<t_1} V_t$. Since $I_t\in G_1$, u_t satisfies the equation $Lu_t = f(u_t)$ in Q_t for any $t\in\mathbb{R}$. It follows from assumptions that $\exists\, t_1$ such that $0\le u(x)\le s_0/2$, $\forall\, x\in Q_{t_1}$. We claim that $u\le u_{t_1}$ in Q_{t_1}. To prove this, we fix $0<\epsilon<s_0/2$ and consider the test function $(u-u_{t_1}-\epsilon)^+$. We can see that $(u-u_{t_1}-\epsilon)^+$ has compact support. Hence,

$$\int_{Q_{t_1}}(f(u)-f(u_{t_1}))(u-u_{t_1}-\epsilon)^+\,dx = \int_{Q_{t_1}}(Lu-Lu_{t_1})(u-u_{t_1}-\epsilon)^+\,dx.$$

We remark that if $x\in Q_{t_1}\cap\ \mathrm{supp}(u-u_{t_1}-\epsilon)^+$, then $(f(u)-f(u_{t_1}))(u-u_{t_1}-\epsilon)^+\le 0$, since f is nonincreasing on $(0,s_0)$. On the other hand, we have

$$\begin{aligned}&\int_{Q_{t_1}}(Lu-Lu_{t_1})(u-u_{t_1}-\epsilon)^+\,dx\\ &\ge C_3\int_{Q_{t_1}\cap\mathrm{supp}(u-u_{t_1}-\epsilon)^+}|X(u-u_{t_1}-\epsilon)^+|^2\,dx\ge 0.\end{aligned}$$

Thus, $u(x)\le u_{t_1}(x)+\epsilon$ for any $x\in Q_{t_1}$. Letting $\epsilon\to 0$, we get our claim.

Step 2. Define $\Gamma=\{\alpha\in\mathbb{R} : u_\alpha\ge u \text{ in } Q_\alpha\}$ and let $\lambda\in\Gamma$. We recall that $\exists\, R>0$ such that for any $|x|>R$, we have $0<u(x)\le s_0/2$, and there exists $\Lambda\in\mathbb{R}^+$ such that $g_\pm(x)=\Lambda x\pm f(x)$ are nondecreasing in the range of values of u. Thus,

$$Lu+\Lambda u = g_+(u)\le g_+(u_\lambda) = Lu_\lambda+\Lambda u_\lambda.$$

The strong maximum principle would yield that $u_\lambda \equiv u$ or $u < u_\lambda$ since Q_λ is connected. In the latter case, we claim that $\exists\ \delta' > 0$ such that $[\lambda, \lambda + \delta'] \subset \Gamma$. Set $Q_{\mu,R} = Q_\mu \cap \overline{B(0,R)}$ for all $\mu \in \mathbb{R}$. Choose $s \in [s_1, \lambda)$ such that $|Q_{\lambda,2R} \setminus K| < C_6$ (C_6 is a constant to be made precise), where $K = \overline{Q_{s,2R}}$. Recall that u is continuous and that $K \subset Q_\lambda$ is compact, so $\min_K(u_\lambda - u) > 0$. We see that there exists $\delta' > 0$ such that for any $\alpha \in [\lambda, \lambda + \delta']$, $u_\alpha(x) > u(x)$ in K.

On the other hand, from the continuity of I_t, it follows that for δ' sufficiently small, $|Q_{\alpha,2R} \setminus K| < 2C_6$ for any $\alpha \in [\lambda, \lambda + \delta']$. Fixing $\epsilon \in (0, s_0/2)$, $\alpha \in [\lambda, \min(s_2, \lambda + \delta')]$ and taking the test function $(u - u_\alpha - \epsilon)^+$, we obtain

$$\int_{Q_\alpha} (f(u) - f(u_\alpha))(u - u_\alpha - \epsilon)^+ \, dx = \int_{Q_\alpha} (Lu - Lu_\alpha)(u - u_\alpha - \epsilon)^+ \, dx.$$

First, using (H8) and the fact that $(f(u) - f(u_\alpha))(u - u_\alpha - \epsilon)^+ \leq 0$ in $Q_\alpha \cap (B(0,R))^c$,

$$\begin{aligned}\int_{Q_\alpha} (f(u) - f(u_\alpha))(u - u_\alpha - \epsilon)^+ \, dx &\leq \int_{Q_{\alpha,R}} (f(u) - f(u_\alpha))(u - u_\alpha - \epsilon)^+ \, dx \\ &\leq C \int_{Q_{\alpha,R}} (u - u_\alpha)^+ (u - u_\alpha - \epsilon)^+ \, dx.\end{aligned}$$

Furthermore, we have

$$\int_{Q_\alpha} (Lu - Lu_\alpha)(u - u_\alpha - \epsilon)^+ \, dx \geq C_3 \int_{Q_\alpha} |X(u - u_\alpha - \epsilon)^+|^2 \, dx.$$

Therefore,

$$\int_{Q_\alpha} |X(u - u_\alpha - \epsilon)^+|^2 \, dx \leq C \int_{Q_{\alpha,R}} (u - u_\alpha)^+ (u - u_\alpha - \epsilon)^+ \, dx.$$

Passing $\epsilon \to 0$, we obtain

$$\int_{Q_\alpha} |X(u - u_\alpha)^+|^2 \, dx \leq C' \int_{Q_{\alpha,R}} ((u - u_\alpha)^+)^2 \, dx. \tag{33}$$

By Lemma 5 (recall that $\operatorname{supp}((u - u_\alpha)^+) \cap B(0, 2R) \subset (Q_\alpha \setminus K) \cap B(0, 2R) \subset$

$P_{s_1,\alpha,2R}$ and $(u-u_\alpha)^+ = 0$ on $(V_\alpha \cup V_s) \cap B(0,2R))$,

$$\int_{Q_{\alpha,R}} ((u-u_\alpha)^+)^2\,dx \tag{34}$$
$$\leq C|\text{supp}((u-u_\alpha)^+) \cap B(0,2R)|^\delta \int_{Q_{\alpha,2R}} |X(u-u_\alpha)^+|^2\,dx.$$

Combining (33) and (34), yields that $u_\alpha \geq u$ in Q_α, when $C'C(2C_6)^\delta < 1$. Finally, $t_1 = \sup \Gamma$ exists by (H8), and $u = u_{t_1}$ follows from Step 2.

6. Monotonicity result. The purpose of this section is to obtain a monotonicity result of the same type of the one obtained by Berestycki and Nirenberg in [3] using the sliding method. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $|\partial\Omega| = 0$ and Ω can be decomposed as the union of a family of hypersurfaces as follows: there exists a hypersurface without boundary $V \subset \mathbb{R}^n$ such that $\overline{\Omega} = \cup_{t\in[0,1]} A_t(V_t)$ where the $V_t \subset V$ are compact hypersurfaces with boundary for $0 < t < 1$, C^1 depending on t, and A_t ($t \in \mathbb{R}$) is a C^1, one-parameter subgroup of the associated "isometry" group G of L, which is transversal to V at $t = 0$. More precisely, we assume the following conditions:

Foliation conditions:

$$\frac{\partial A_t(x)}{\partial t}\Big|_{t=0} \oplus T_xV = T_x\mathbb{R}^n, \quad \forall x \in V, \tag{35}$$
$$\exists\, \varepsilon > 0 \text{ such that } \forall x \in V,\ \forall t \in (0, 1+\varepsilon), \quad A_t(x) \notin V. \tag{36}$$

Directional convexity conditions:

$$\forall x \in V, \{t \in [0,1] : A_t(x) \in \overline{\Omega}\} \text{ is a closed interval denoted by } [s_1(x), s_2(x)],$$
$$\forall x \in V, \{t \in (0,1) : A_t(x) \in \Omega\} \text{ is an open interval denoted by } (\tau_1(x), \tau_2(x)).$$

The main result of this section is

Theorem 6. *Under the assumptions* (H) *to* (H7), *suppose that* $u \in H^1(\Omega) \cap C^0(\overline{\Omega})$ *is a solution of* $Lu = f(u)$ *with boundary condition* $u_{|\partial\Omega} = \phi$, *which is a strictly increasing function of* t *along the orbits of* A_t. *By this we mean that for all boundary points* $x_b, x_e \in \partial\Omega$, *which are on the same orbit of the group action, i.e.,* $\exists\, t(x_b, x_e) > 0$ *such that* $x_e = A_t(x_b)$, *we have that* $\phi(x_e) > \phi(x_b)$. *Furthermore, suppose that* u *is such that along an orbit, it takes its values in between the values of* ϕ *at the points where the orbit* *crosses the boundary; i.e., for x_b and x_e boundary points on the same orbit, as above, we have*

$$\forall\, \tau \in (0, t(x_b, x_e))\ , \quad \phi(x_b) < u(A_\tau(x_b)) < \phi(x_e). \tag{37}$$

Then, the function u is increasing along the orbits of the group action; i.e.,

$$\forall\, x \in \overline{\Omega},\ \forall\, t > 0 \text{ such that } A_t(x) \in \overline{\Omega},\ \text{we have } u(A_t(x)) > u(x). \tag{38}$$

Proof. Denote $u^\tau = u \circ A_\tau$, $\Omega_\tau = A_\tau(\Omega)$, $D_\tau = \Omega \cap \Omega_\tau$ and $\tau_1 = \sup\{\tau > 0 \text{ such that } D_\tau \neq \emptyset\}$. First, we see that $Lu^\tau = f(u^\tau)$ for all $\tau \in (0, 1)$. Then by the fact that $u < u^\tau$ on ∂D_τ, working with $g_\pm(u) = \Lambda u \pm f(u)$ and using the same argument as in the proof of Theorem 4, we have $u < u^\tau$ in D_τ for such τ since for $\tau < \tau_1$ and close to τ_1, $|D_\tau| < C_5$. Note $\tau_0 = \inf\{\tau > 0 \text{ such that } u < u^\mu, \forall\, \mu \in (\tau, \tau_1)\}$; the conclusion of the theorem is equivalent to saying that $\tau_0 = 0$. If not, by the continuity of u, $u \leq u^{\tau_0}$ in D_{τ_0}, then by Theorem 3, $u < u^{\tau_0}$ in D_{τ_0}, since $u < u^{\tau_0}$ on ∂D_{τ_0}. We choose again $K \subset\subset D_{\tau_0}$ such that $|D_{\tau_0} \setminus K| < C_5/2$; by continuity of u, we know that $u < u^\tau$ on K for τ close to τ_0. Moreover, for τ close to τ_0, $|D_\tau \setminus K| < C_5$; then by Theorem 2, $u \leq u^\tau$ on $D_\tau \setminus K$; hence $u \leq u^\tau$ in D_τ. Theorem 3 implies again that $u < u^\tau$ in D_τ for τ close to τ_0, which contradicts the choice of τ_0. This completes the proof.

7. Examples.

7.1. Carnot-Carathéodory spaces generated by vector fields of Hörmander type. We consider $\mathbb{R}^n$ for $n \geq 4$ and vector fields $X_j = \frac{\partial}{\partial x_j}$ for $j = 1, 2, \ldots, n-2$, $X_{n-1} = \frac{\partial}{\partial x_{n-1}} + x_k \frac{\partial}{\partial x_n}$ with $2 \leq k \leq n-2$. Then it is easy to see that $X = (X_1, \ldots, X_{n-1})$ satisfies (1). We take $L = \sum_{j \leq n-1} X_j^* X_j$, $\Omega = \mathbb{B}^n$ or $\Omega = [-1, 1] \times \Omega'$ with Ω' open set of $\mathbb{R}^{n-1}$, and $I_t(x_1, x') = (2(t-1) - x_1, x')$, where we denote $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. We see that all conditions required by Theorem 4 are satisfied, so we get that for any positive solution $u \in C^0 \cap H_0^1(\Omega)$ of $Lu = f(u)$ in Ω (with f locally Lipschitz), $u(x_1, x') = u(-x_1, x')$.

7.2. Carnot-Carathéodory spaces generated by vector fields of Baouendi-Grushin type. We consider $X_j = \frac{\partial}{\partial x_j}$, $j = 1, \ldots, k$ and $X_j = (x_1^2 + \cdots + x_k^2)^\gamma \frac{\partial}{\partial x_j}$, $j = k+1, \ldots, n$, where $1 \leq k < n$ and $\gamma > 1$. Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$ be a ball and $L = \sum_{j=1}^n X_j^* X_j$. We define $I_t(x) =$

$(x_1, \dots, x_k, 2(1-t)-x_{k+1}, x_{k+2}, \dots, x_n)$. We see that $I_t^* L = L$ and B satisfies the assumptions of Theorem 4, and thus we obtain the desired symmetry of u relative to the $x_{k+1} = 0$ hyperplane.

Remark 4. In this case, (H) is satisfied, but (H3) is unknown. But in [16] (see Theorems 3.1 and 3.2), Lemma 1 is proved by replacing B by $B_d(x_0, \alpha R)$ with $\alpha \geq 1$ depending only on U in the right term of (12) and (13) and by taking $k = 1$, $p = 2$. On the other hand, Lemma 2 holds true for all $u \in W_0^{1,p}(B)$ and some $k > 1$. Thus, Theorems 4 and 6 are also valid. Furthermore, we can even treat some cases for $\gamma > 0$ (see [16]).

7.3. Riemannian manifolds with constant curvature. Using Remark 3, we know that Theorem 4 is valid for manifolds. We consider $\mathbb{H}^n$ as a submanifold of the Minkowski space $\mathbb{R}^{n,1} = (\mathbb{R}^{n+1}, g)$, where g is the metric with signature $(-, +, \dots, +)$. Hyperbolic space of dimension n is the submanifold $\mathbb{H}^n = \{x = (x_0, x') \in \mathbb{R}^{n,1}$ such that $g(x,x) = -1$, and $x_0 > 0\}$. Let $L = \triangle_g$ be the Laplace-Betrami operator on $\mathbb{H}^n$. Using the stereographic coordinates $y = \Psi(x) = x'/(1 + x_0)$, we write

$$\Omega = \Psi^{-1}(\{x \in \mathbb{H}^n : x_0 < \cosh\psi\}) = \big\{y \in \mathbb{R}^n : |y| < \frac{\sinh\psi}{1+\cosh\psi}\big\} \qquad (\psi > 0)$$

and

$$L = -(1 - |y|^2)^n \sum_{j=1}^{n} \frac{\partial}{\partial y_j}\Big(\frac{1}{(1-|y|^2)^{n-2}} \frac{\partial}{\partial y_j}\Big).$$

The associated functional will be in the form

$$E(u) = \int_\Omega A(u)\, d\mu = \int_\Omega g^{ij}(y) \frac{\partial u}{\partial y_i} \frac{\partial u}{\partial y_j}\, d\mu,$$

where $d\mu = (1 - |y|^2)^{-n} dy$. We want to prove that u is symmetric relative to the $x_1 = 0$ hyperplane of $\mathbb{R}^{n,1}$. For this purpose, we write $\mathbb{R}^{n,1} = \mathbb{R}^{1,1} \times \mathbb{R}^{n-1}$, and define $A_t = \tilde{A}_t \otimes Id_{\mathbb{R}^{n-1}}$, where $\tilde{A}_t$ is the hyperbolic rotation of angle ψt in $\mathbb{R}^{1,1}$; i.e.,

$$\tilde{A}_t = \begin{pmatrix} \cosh(\psi t) & \sinh(\psi t) \\ \sinh(\psi t) & \cosh(\psi t) \end{pmatrix} \; : \; \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}.$$

We consider $I_t = \Psi \circ \tilde{A}_{t-1} \circ J \circ \tilde{A}_{1-t} \circ \Psi^{-1}$ where J is the reflection $(x_0, x_1, \dots, x_n) \mapsto (x_0, -x_1, \dots, x_n)$. It follows from Theorem 4 that u is symmetric with respect to the $y_1 = 0$ hyperplane of $\mathbb{R}^n$. Similarly, we can prove the symmetry for a convex ball in $\mathbb{S}^n$ (see [2], [27]).

7.4. Product of domains. Let $(\mathbb{R}^n, d_1)$ be a Carnot-Carathéodory space generated by vector fields of Hörmander type or Baouendi-Grushin type and suppose $\Omega_1 \subset \mathbb{R}^n$ satisfies the condition of Theorem 4. We consider L_1 to be defined by (8). Assume that (Ω_2, g) is a connected Riemannian manifold with compact closure where $\Omega_2 \subset \mathbb{R}^m$ is a coordinate card. Let

$$L_2 u = \triangle_g u = \sum_{i,j=1}^{m} -\frac{1}{\sqrt{\det g}} \frac{\partial}{\partial y_i} \big(\sqrt{\det g} g^{ij} \frac{\partial u}{\partial y_j} \big)$$

be the Laplace-Betrami operator on (Ω_2, g). We consider the space of the form $\Omega = \Omega_1 \times \Omega_2$ and $L = (L_1 + L_2)$. We can see that Remark 2 remains valid with the associated measure $\sqrt{\det g}\, dx \otimes dy$ in this case. In fact, we can set $V_t = V_t^1 \times \Omega_2$, $I_t = I_t^1 \times Id_{\Omega_2}$ where I_t^1, V_t^1 are defined as above for Ω_1. Thus, the desired directional symmetry result follows from Theorem 4.

7.5. A subgroup of the polarized Heisenberg group. The non-abelian group $\mathbb{H}^n_{pol}$ is defined by taking $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and the group operation given by $(x, y, s) * (u, v, r) = (x + u, y + v, s + r + u \cdot y)$, where $u \cdot y$ denotes the usual scalar product in $\mathbb{R}^n$. In the following, we study the subgroup of $\mathbb{H}^n_{pol}$, $G = \{(x, y, s) \in \mathbb{H}^n_{pol}$ such that $x_1 = y_1\}$ and consider the vector fields invariant on the left:

$$X_1 = \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial s}, \qquad X_l = \frac{\partial}{\partial x_l} \ \ (2 \le l \le n) \quad \text{and}$$
$$X_{n+k} = \frac{\partial}{\partial y_{k+1}} + y_{k+1} \frac{\partial}{\partial s} \ \ (1 \le k \le n-1).$$

Consider the subelliptic operator $L = \sum_{j \le 2n-1} X_j^* X_j$. Denote also $x = (x_1, x')$, $y = (x_1, y')$. Let $I(x, y, s) = ((-x_1, x'), (-x_1, y'), s)$ and $A_t(\cdot) = a_t * (\cdot) = (x^t, x^t, t^2/2) * (\cdot)$ with $x^t = (t, 0, \dots, 0) \in \mathbb{R}^n$.

Finally, we set $I_t = A_{-t} \circ I \circ A_t$. We can check that G has all the properties required by Theorem 5. Hence, we get the symmetry in the x_1-direction; namely, there exists $t_0 \in \mathbb{R}$ such that $I_{t_0}(u) = u$ for solutions of (31).

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