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CHEMOTACTIC COLLAPSE IN A PARABOLIC–ELLIPTIC SYSTEM OF MATHEMATICAL BIOLOGY

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Abstract. We study the blowup mechanism for a simplified system of chemotaxis. First, Moser's iteration scheme is applied and the blowup point of the solution is characterized by the behavior of the local Zygmund norm. Then, Gagliardo-Nirenberg's inequality gives $\varepsilon_0 > 0$ satisfying $\limsup_{t\uparrow T_{\max}} \|u(t)\|_{L^1(B_R(x_0)\cap\Omega)} \geq \varepsilon_0$ for any blowup point $x_0 \in \overline{\Omega}$ and $R > 0$. On the other hand, from the study of the Green's function it appears that $t \mapsto \|u(t)\|_{L^1(B_R(x_0)\cap\Omega)}$ has a bounded variation. Those facts imply the finiteness of blowup points, and then, the chemotactic collapse at each blowup point and an estimate of the number of blowup points follow.

1. Introduction. The present paper is devoted to a parabolic–elliptic system describing the chemotactic feature of some organisms (cellular slime molds) sensitive to the gradient of a chemical substance secreted by themselves. Precisely, it is given as

$$\begin{array}{lll} u_t = \nabla\cdot(\nabla u - \chi u \nabla v) & \text{in} & \Omega\times(0,T) \\ 0 = \Delta v - \gamma v + \alpha u & \text{in} & \Omega\times(0,T) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on} & \partial\Omega\times(0,T) \\ u\big|_{t=0} = u_0 & \text{on} & \Omega, \end{array} \tag{1}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, χ, γ, and α are positive constants, ν denotes the outer unit normal vector, $u_0 = u_0(x)$ is a smooth nonnegative function not identically 0 on $\overline{\Omega}$.

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It is proposed by Nagai [17] as a simplified model for the Keller-Segel system [14], where $u = u(x,t) \geq 0$ and $v = v(x,t) \geq 0$ stand for the density of the organisms and the concentration of the chemical substance, respectively.

The first equation shows that $\mathcal{F} = -\nabla u + \chi u \nabla v$ is the flux of u so that the effect of diffusion $-\nabla \cdot \nabla u$ and that of chemotaxis $\chi \nabla \cdot (u \nabla v)$ are competing for u to vary. Sometimes the term τv_t is added to the left-hand side of the second equation. In this case it sets up the system of Nanjundiah [20], called the *full system* in the present paper. (Still it simplifies the original system [14], where χ and α are functions of u and v.) Then it describes that v diffuses, is produced proportionary to u, and is destroyed at a certain rate. Usually the positive constant τ is very small, and neglecting the term τv_t gives (1).

There is another approximation introduced by Jäger and Luckhaus [13] describing the limiting case of $\tau \downarrow 0$ with $\chi, \alpha \sim 1$ and $\gamma \sim \tau$. There, the second equation is replaced by

$$0 = \Delta v + \alpha(u - \overline{u}_0) \qquad \text{with} \qquad \overline{u}_0 = \frac{1}{|\Omega|} \int_\Omega u.$$

Jäger and Luckhaus [13] showed the following: if $\|u_0\|_1 \ll 1$ then $T_{\max} = +\infty$ follows, while if $\|u_0\|_1 \gg 1$ then $T_{\max} < +\infty$ can happen. Here and henceforth, $T_{\max}$ denotes the maximal time for the existence of the solution, and $\|\cdot\|_p$ the standard L^p norm for $1 \leq p \leq \infty$.

The case $T_{\max} < +\infty$ is referred to as the *blowup* in a finite time of the solution, which has attracted both mathematical and biological interests. Nagai [17] showed precise results to the radially symmetric case of (1); $\|u_0\|_1 = 8\pi/(\alpha\chi)$ is the threshold for the blowup. Namely, $\|u_0\|_1 < 8\pi/(\alpha\chi)$ implies $T_{\max} = +\infty$, while $T_{\max} < +\infty$ occurs if $\|u_0\| > 8\pi/(\alpha\chi)$, which correspond exactly to what Childress [6] and Childress and Percus [7] conjectured to the full system.

Another conjecture made by [20] concerns the behavior of blowup solutions; $u(x,t)dx$ will form a delta function singularity as $t \uparrow T_{\max}$, which is referred to as the *chemotactic collapse*. A remarkable study was made by Herrero and Velázquez [11]; there are solutions to the Jäger-Luckhaus model satisfying

$$w^* - \lim_{t \uparrow T_{\max}} u(x,t)dx = m\delta_{x_0}(dx) + f(x)\,dx \tag{2}$$

in $\mathcal{M}(\overline{\Omega})$, the space of measures on $\overline{\Omega}$, where $\Omega = \left\{x \in \mathbf{R}^2 : |x| < 1\right\}$, $u(x,t) = u(|x|, t)$, $x_0 = 0$, $m = 8\pi/(\alpha\chi)$, and $f(x) = f(|x|) \in C\left(\overline{\Omega} \setminus \{0\}\right) \cap L^1(\Omega)$ is a nonnegative function. Actually much sharper descriptions are presented there concerning the asymptotic behavior of the solution. Those results, the existence of a threshold and that of chemotactic collapse, were later extended to the full system by Nagai, Senba, and Yoshida [19] and Herrero and Velázquez [12], respectively.

The nonradial case is somewhat different and [19] gave only $\|u_0\|_1 < 4\pi/(\alpha\chi)$ as a sufficient condition for $T_{\max} = +\infty$. Biler [3] and Gajewski and Zacharias [8] obtained the same result independently. The proof is devoted to the full system but valid even for (1). The discrepancy between radial and nonradial cases suggests that the concentration toward the boundary occurs to nonradial blowup solutions. On the other hand conjecture [6], [7] concerning the threshold value was based on a heuristic observation to the structure of radially symmetric stationary solutions. Motivated by them, we studied isolated blowup points (Nagai, Senba, and Suzuki [18]) and the structure of nonradial stationary solutions (Senba and Suzuki [21]) in details. Consequently, we were led to the following conjectures.

1. It happens that $4\pi/(\alpha\chi) < \|u\|_1 \leq 8\pi/(\alpha\chi)$ and $T_{\max} < +\infty$. In this case, the mass $u(x,t)dx$ concentrates to a point on the boundary as $t \uparrow T_{\max}$, and in particular, radially symmetric solutions are unstable on $\Omega = \left\{x \in \mathbf{R}^2 : |x| < 1\right\}$.
2. At each blowup point $x_0 \in \overline{\Omega}$ the chemotactic collapse (2) occurs with $m = 8\pi/(\alpha\chi)$ and $m = 4\pi/(\alpha\chi)$ according to $x_0 \in \Omega$ and $x_0 \in \partial\Omega$, respectively. Here and henceforth, by definition the Dirac measure $\delta_{x_0}(dx) \in \mathcal{M}(\overline{\Omega})$ acts as $\langle \eta(x), \delta_{x_0}(dx)\rangle = \eta(x_0)$ $\left(x_0 \in \overline{\Omega}\right)$ for $\eta \in C(\overline{\Omega})$.

The present paper shows a partial answer, and the second conjecture is proven with $m = 8\pi/(\alpha\chi)$ and $m = 4\pi/(\alpha\chi)$ replaced by $m \geq 8\pi/(\alpha\chi)$ and $m \geq 4\pi/(\alpha\chi)$, respectively. As a consequence we have the finiteness of blowup points. More precisely, it holds that

$$\begin{aligned} 2 \times \sharp\,(\text{interior blowup points}) \quad &+ \quad \sharp\,(\text{boundary blowup points}) \\ &\leq \quad \alpha\chi \|u_0\|_1 / 4\pi. \end{aligned} \tag{3}$$

Therefore, if $4\pi/(\alpha\chi) \leq \|u\|_1 < 8\pi/(\alpha\chi)$ and $T_{\max} < +\infty$ occurs then $u(x,t)dx$ concentrates to a point on the boundary as $t \uparrow T_{\max}$. Inequality

(3) is also regarded as a natural refinement of the results of [17] and [19], [3], [8] concerning the continuation of the solution globally in time; in the radially symmetric case $\|u_0\|_1 < 8\pi/(\chi\alpha)$ implies $T_{\max} = +\infty$ and generally, $\|u_1\|_1 < 4\pi/(\alpha\chi)$ does. We expect that (3) is sharp. An interesting question is whether one can prescribe the numbers of interior and boundary blowup points independently.

So far, most results proven for (1) have been verified to hold in the full system; as is described, $\|u_0\|_1 < 4\pi/(\alpha\chi)$ implies $T_{\max} = +\infty$, and there are chemotactic collapses (2) for radially symmetric cases. Our method does not apply directly, but the results obtained in the present paper are expected to hold in the full system. Actually, our results hold in the Jäger-Luckhaus model with minor changes of the proof.

2. Summary. Henceforth, we put $\chi = \gamma = \alpha = 1$ for simplicity. We also suppose $u_0 \in C^2(\overline{\Omega})$, $u_0(x) \geq 0$, and $u_0(x) \not\equiv 0$. Let $-\mathcal{L}$ be the differential operator $-\Delta$ with $(\partial/\partial\nu)\cdot|_{\partial\Omega} = 0$. It generates a holomophic semigroup on $L^p(\Omega)$ denoted by $\left\{e^{-t\mathcal{L}} : t \geq 0\right\}$ for $1 \leq p \leq \infty$ (see Tanabe [23], e.g.). System (1) is reduced to the abstract equation

$$u(t) = e^{-t\mathcal{L}}u_0 + \int_0^t e^{-(t-s)\mathcal{L}}\nabla\cdot\Big(u(s)\nabla\left(\mathcal{L}+1\right)^{-1}u(s)\Big)\,ds, \tag{4}$$

and the method of Yagi [24] or Biler [3] applies. Existence, uniqueness, regularity, and positivity hold for the time-local solution.

The first theorem justifies the terminology blowup.

Theorem 1. *If $T_{\max} < +\infty$, then*

$$\lim_{t\uparrow T_{\max}} \|u(t)\|_\infty = +\infty. \tag{5}$$

Regarding this, we define the *blowup set* $\mathcal{B}$ of u usually as

$$\mathcal{B} = \big\{x_0 \in \overline{\Omega} : \text{there exist} \quad t_k \uparrow T_{\max} \quad \text{and} \quad x_k \to x_0 \\ \text{such that} \quad u(x_k, t_k) \to +\infty \quad \text{as} \quad k \to \infty\big\}$$

and call each $x_0 \in \mathcal{B}$ a *blowup point*. Condition $T_{\max} < +\infty$ implies $\mathcal{B} \neq \emptyset$, but more importantly, the finiteness of blowup points follows.

Theorem 2. *If $T_{\max} < +\infty$, then*

$$2\sharp\left(\mathcal{B}\cap\Omega\right) + \sharp\left(\mathcal{B}\cap\partial\Omega\right) \leq \|u_0\|_1/(4\pi). \tag{6}$$

Also chemotactic collapse occurs at each blowup point.

Theorem 3. *If $T_{\max} < +\infty$, there is a mapping $m : \mathcal{B} \to [4\pi, +\infty)$ with $m|_{\mathcal{B}\cap\Omega} \geq 8\pi$ and a nonnegative function $f = f(x)$ in*

$$f \in C(\overline{\Omega} \setminus \mathcal{B}) \cap L^1(\Omega) \tag{7}$$

satisfying in $\mathcal{M}(\overline{\Omega})$,

$$w^* - \lim_{t\uparrow T_{\max}} u(x,t)dx = \sum_{x_0\in\mathcal{B}} m(x_0)\delta_{x_0}(dx) + f(x)\,dx. \tag{8}$$

Theorems 2 and 3 are proven in the following way. First, we show the finiteness of blowup points. This implies that any blowup point x_0 is isolated, and by the method of [18], the estimates [19], [3], [8] can be localized around x_0. Consequently, chemotactic collapse (8) is proven, and then the estimate of the number of blowup points, inequality (6), follows because the L^1 norm of u is preserved:

$$\|u(t)\|_1 = \|u_0\|_1 \qquad (0 \leq t < T_{\max})\,. \tag{9}$$

In such arguments, the crucial part is showing $\sharp\mathcal{B} < +\infty$. Fortunately, system (1) admits for local L^1 norms of u to have bounded variations as $t \uparrow T_{\max}$. Combining this fact with the Gagliardo-Nirenberg-type inequalities implies the finiteness of blowup points.

The present paper is divided into eight sections. Taking preliminaries in Section 3, we characterize the blowup point in terms of the localized Zygmund norm in Section 4. Then, Theorem 1 is proven in Section 5. Section 6 is a remark on the Green's function. The finiteness of blowup points is proven in Section 7 and the proof of Theorem 3 is completed in Section 8.

3. Preliminaries. A form of the Gagliardo-Nirenberg inequality in two space dimensions is indicated as

$$\|w\|_2^2 \leq K^2 \left(\|\nabla w\|_1^2 + \|w\|_1^2\right) \qquad w \in W^{1,1}(\Omega), \tag{10}$$

where $K > 0$ is a constant determined by Ω (see Adams [1]). In this section we shall show some inequalities derived from (10) for later use. Henceforth we set $B_R(x_0) = \left\{x \in \mathbf{R}^2 : |x - x_0| < R\right\}$.

First, we introduce the cutoff function φ satisfying

$$0 \leq \varphi \leq 1 \quad \text{in} \quad \mathbf{R}^2, \qquad \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega. \tag{11}$$

Actually, it is taken in the following way.

Given $x_0 \in \Omega$, we have $0 < R' < R$ with $B_{2R}(x_0) \subset \Omega$. Then we take φ satisfying

$$\varphi(x) = \begin{cases} 1 & (x \in B_{R'}(x_0)) \\ 0 & \left(x \in \mathbf{R}^2 \setminus B_R(x_0)\right). \end{cases} \tag{12}$$

Next we prepare $\zeta \in C_0^\infty(\mathbf{R}^2)$ satisfying $\zeta = \zeta(|y|)$, $0 \leq \zeta \leq 1$ in $\mathbf{R}^2$, and

$$\zeta(y) = \begin{cases} 1 & \left(y \in B_{1/2}(0)\right) \\ 0 & \left(y \in \mathbf{R}^2 \setminus B_1(0)\right). \end{cases}$$

Given $x_0 \in \partial\Omega$, we take a smooth conformal mapping $X : B_{2R}(x_0) \cap \overline{\Omega} \to \mathbf{R}^2$ satisfying $x_0 \mapsto 0$ and

$$\begin{aligned} X\left(B_{2R}(x_0) \cap \Omega\right) &\subset \{(x_1, x_2) : x_2 > 0\} \\ X\left(B_{2R}(x_0) \cap \partial\Omega\right) &\subset \{(x_1, x_2) : x_2 = 0\} \\ X\left(B_{R'}(x_0) \cap \Omega\right) \subset B_{1/2}(0), \quad X\left(\Omega \setminus B_R(x_0)\right) &\subset \mathbf{R}^2 \setminus B_1(0) \end{aligned}$$

for $0 < R' < R \ll 1$. Then we set $\varphi(x) = \zeta\left(X(x)\right)$. It holds that

$$\frac{\partial}{\partial \nu} \zeta \circ X = \frac{\partial X}{\partial \nu} \cdot (\nabla \zeta \circ X) = 0 \qquad \text{on} \qquad \partial\Omega$$

because $(\partial X)/(\partial \nu)$ is proportional to $(0, -1)$ on $\partial\Omega$, and such a φ satisfies (11) and (12).

The above φ is sometimes written as $\varphi_{x_0,R',R}$. Then, $\psi = \left(\varphi_{x_0,R',R}\right)^6$ satisfies

$$\begin{gathered} \psi(x) = \begin{cases} 1 & (x \in B_{R'}(x_0)) \\ 0 & \left(x \in \mathbf{R}^2 \setminus B_R(x_0)\right) \end{cases} \\ 0 \leq \psi \leq 1 \quad \text{in} \quad \mathbf{R}^2, \qquad \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \\ |\nabla \psi| \leq A\psi^{5/6}, \quad |\triangle \psi| \leq B\psi^{2/3} \quad \text{in} \quad \mathbf{R}^2, \end{gathered}$$

where $A > 0$ and $B > 0$ are constants determined by $0 < R' < R \ll 1$.

Lemma 4. *The following inequalities hold for any $s > 1$, where $C > 0$ is a constant:*

$$\int_\Omega u^2\psi \le 2K^2 \int_{B_R(x_0)\cap\Omega} u \cdot \int_\Omega u^{-1} |\nabla u|^2 \psi + K^2\big(\frac{A^2}{2}+1\big) \|u\|_1^2 \tag{13}$$

$$\int_\Omega u^2 \le \frac{2K^2}{\log s}\int_\Omega \big(u\log u + e^{-1}\big)\cdot \int_\Omega u^{-1}|\nabla u|^2 + 2K^2\|u\|_1^2 + 3s^2|\Omega| \tag{14}$$

$$\int_\Omega u^3\psi \le \frac{72K^2}{\log s}\int_{B_R(x_0)\cap\Omega}\big(u\log u + e^{-1}\big)\cdot\int_\Omega |\nabla u|^2\psi + C\|u\|^3_{L^1(B(x_0,R)\cap\Omega)} + 10|\Omega|s^3 \tag{15}$$

Proof. Putting $w = u\psi^{1/2}$, we have

$$\Big\{\int_\Omega |\nabla w|\Big\}^2 \le 2\Big\{\int_\Omega |\nabla u|\,\psi^{1/2}\Big\}^2 + 2\Big\{\int_\Omega u|\nabla\psi^{1/2}|\Big\}^2$$
$$\le 2\Big\{\int_\Omega |\nabla u|\,\psi^{1/2}\Big\}^2 + \frac{A^2}{2}\|u\|_1^2 \le 2\int_{B_R(x_0)\cap\Omega} u\cdot\int_\Omega u^{-1}|\nabla u|^2\psi + \frac{A^2}{2}\|u\|_1^2.$$

Hence (13) follows from (10) and $\|w\|_1 \le \|u\|_1$.

We turn to (14). Take $w = (u-s)_+$ with $a_+ = \max\{a, 0\}$. We have

$$\|w\|_2^2 = \int_{\{u>s\}}(u-s)^2 \ge \int_{\{u>s\}}\Big(\frac12 u^2 - s^2\Big)$$
$$= \int_\Omega \frac12 u^2 - \int_{\{u\le s\}}\frac12 u^2 - \int_\Omega s^2 \ge \frac12\int_\Omega u^2 - \frac32 s^2|\Omega|.$$

On the other hand we have $\|w\|_1^2 \le \|u\|_1^2$ and

$$\|\nabla w\|_1^2 \le \Big\{\int_{\{u>s\}}|\nabla u|\Big\}^2 \le \int_{\{u>s\}} u\cdot\int_{\{u>s\}} u^{-1}|\nabla u|^2$$
$$\le \frac{1}{\log s}\int_\Omega\big(u\log u + e^{-1}\big)\cdot\int_\Omega u^{-1}|\nabla u|^2$$

because $s\log s \ge -e^{-1}$ for any $s > 0$. This implies (14).

Finally, take $w = (u-s)_+^{3/2}\psi^{1/2}$. We have

$$\|w\|_2^2 = \int_{\{u>s\}}(u-s)_+^3\psi \ge \int_{\{u>s\}}\big(\frac14 u^3 - s^3\big)\psi \ge \frac14\int_\Omega u^3\psi - \frac54 s^3|\Omega|.$$

Because $|\nabla w| \le \frac{3}{2}(u-s)_+^{1/2}|\nabla u|\psi^{1/2} + \frac{1}{2}A(u-s)_+^{3/2}\psi^{1/3}$, we have

$$\|\nabla w\|_1^2 \le \frac{9}{2}\Big\{\int_{\{u>s\}}(u-s)^{1/2}|\nabla u|\psi^{1/2}\Big\}^2 + \frac{A^2}{2}\Big\{\int_{\{u>s\}}(u-s)^{3/2}\psi^{1/3}\Big\}^2.$$

Here, it holds that

$$\Big\{\int_{\{u>s\}}(u-s)^{1/2}|\nabla u|\psi^{1/2}\Big\}^2 \le \Big\{\int_{\{u>s\}} u^{1/2}|\nabla u|\psi^{1/2}\Big\}^2$$
$$\le \int_{B_R(x_0)\cap\{u>s\}} u \int_{\{u>s\}}|\nabla u|^2\psi \le \frac{1}{\log s}\int_{B_R(x_0)\cap\Omega}\big(u\log u + e^{-1}\big)\cdot\int_\Omega |\nabla u|^2\psi$$

and

$$\begin{aligned}
&\Big\{\int_{\{u>s\}}(u-s)^{3/2}\psi^{1/3}\Big\}^2 \le \Big\{\int_{\{u>s\}} u\psi^{1/3}u^{1/2}\Big\}^2 \\
&\le \Big\{\int_\Omega u^3\psi\Big\}^{2/3}\|u\|_{L^1(B_R(x_0)\cap\Omega)}\,|\Omega|^{1/3} \\
&\le \varepsilon\int_\Omega u^3\psi + \frac{1}{3}(\frac{3}{2}\varepsilon)^{-2}|\Omega|\,\|u\|^3_{L^1(B_R(x_0)\cap\Omega)}, \qquad (16)
\end{aligned}$$

where $\varepsilon > 0$. Writing $C_\varepsilon = \frac{4}{27}\varepsilon^{-2}$, we have

$$\begin{aligned}
\|\nabla w\|_1^2 \;\le\; & \frac{9}{2\log s}\int_{B_R(x_0)\cap\Omega}\big(u\log u + e^{-1}\big)\cdot\int_\Omega|\nabla u|^2\psi \\
& + \frac{A^2}{2}\varepsilon\int_\Omega u^3\psi + \frac{A^2}{2}C_\varepsilon|\Omega|\,\|u\|^3_{L^1(B_R(x_0)\cap\Omega)}.
\end{aligned}$$

Since $\psi^{1/2} \le \psi^{1/3}$, it follows from (16) that

$$\|w\|_1^2 \le \varepsilon\int_\Omega u^3\psi + C_\varepsilon|\Omega|\,\|u\|^3_{L^1(B_R(x_0)\cap\Omega)}.$$

We get

$$\begin{aligned}
&\big(\frac{1}{4} - K^2\big(\frac{A^2}{2}+1\big)\varepsilon\big)\int_\Omega u^3\psi \\
&\le \frac{9K^2}{\log s}\int_{B_R(x_0)\cap\Omega}\big(u\log u + e^{-1}\big)\cdot\int_\Omega|\nabla u|^2\,\psi \\
&\qquad + K^2C_\varepsilon\,|\Omega|\,\big(\frac{A^2}{2}+1\big)\,\|u\|^3_{L^1(B_R(x_0)\cap\Omega)} + \frac{5}{4}s^3|\Omega|
\end{aligned}$$

by (10). Taking $\varepsilon > 0$ as $\frac{1}{4} - K^2(\frac{A^2}{2}+1)\varepsilon = \frac{1}{8}$, we obtain (15).

4. Characterization of blowup points. Henceforth, we always assume $T_{\max} < +\infty$ and $\mathcal{B}$ denotes the blowup set of u. Generic positive constants are denoted as $C_1, C_2, \cdots$, successively. In the case that their dependence on the parameters, say $\alpha, \beta, \cdots$, have to be indicated precisely, we write them as $C_\alpha, C_{\alpha,\beta}, \cdots$, and so forth.

The first equation of (1), provided with the boundary conditions, gives $\frac{d}{dt}\int_\Omega u = 0$, or (9) by $u > 0$. Then the L^1 estimate (see [4]) to the second equation of (1) gives

$$\sup_{0 \le t < T_{\max}} \left\{ \|v(t)\|_{W^{1,q}(\Omega)} + \|v(t)\|_p \right\} < +\infty \tag{17}$$

for $q \in [1, 2)$ and $p \in [1, \infty)$. Here and henceforth, $W^{m,q}(\Omega)$ denotes the usual Sobolev space: the set of q-integrable functions up to m-th order of differentiation. (Sometimes m becomes fractional. See Henry [10].)

We prove the following lemma.

Lemma 5. *$x_0 \in \overline{\Omega}$ is a blowup point of u if and only if*

$$\limsup_{t \uparrow T_{\max}} \int_{B_R(x_0)\cap\Omega} (u \log u)(\cdot, t) = +\infty$$

for $R > 0$ sufficiently small.

Proof. The "only if" part is obvious, because $x_0 \notin \mathcal{B}$ implies

$$\sup_{0 \le t < T_{\max}} \|u(t)\|_{L^\infty(B_R(x_0)\cap\Omega)} < +\infty$$

for $0 < R \ll 1$ by the definition. To prove the "if" part, we take $0 < R \ll 1$ and suppose

$$\sup_{0 \le t < T_{\max}} \int_{B_R(x_0)\cap\Omega} (u \log u)(\cdot, t) < +\infty. \tag{18}$$

Localizing the estimates of [19], [3], [8], we shall show that then $x_0 \notin \mathcal{B}$ follows.

Step 1. Let $0 < R' < R$ and $\psi = \left(\varphi_{x_0,R',R}\right)^6$. We multiply the first equation of (1) by $u\psi$ to get the identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\psi + \int_\Omega |\nabla u|^2\psi + \int_\Omega u\nabla u \cdot \nabla\psi = \int_\Omega u\left(\nabla v \cdot \nabla u\right)\psi + \int_\Omega u^2 \nabla v \cdot \nabla\psi. \tag{19}$$

The first term of the right-hand side of (19) is treated by the second equation of (1) to get

$$\begin{aligned}&\int_\Omega u\left(\nabla v\cdot\nabla u\right)\psi=\frac{1}{2}\int_\Omega\left(\nabla u^2\cdot\nabla v\right)\psi=-\frac{1}{2}\int_\Omega\left(u^2\Delta v\right)\psi-\frac{1}{2}\int_\Omega u^2\nabla v\cdot\nabla\psi\\&=\frac{1}{2}\int_\Omega u^3\psi-\frac{1}{2}\int_\Omega u^2v\psi-\frac{1}{2}\int_\Omega u^2\nabla v\cdot\nabla\psi\le\frac{1}{2}\int_\Omega u^3\psi-\frac{1}{2}\int_\Omega u^2\nabla v\cdot\nabla\psi\\&=\frac{1}{2}\int_\Omega u^3\psi+\frac{1}{2}\int_\Omega v\nabla u^2\cdot\nabla\psi+\frac{1}{2}\int_\Omega u^2v\Delta\psi.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\psi+\int_\Omega|\nabla u|^2\psi+\int_\Omega u\nabla u\cdot\nabla\psi\\&\qquad\le\frac{1}{2}\int_\Omega u^3\psi-\frac{1}{2}\int_\Omega v\nabla u^2\cdot\nabla\psi-\frac{1}{2}\int_\Omega u^2v\Delta\psi.\end{aligned}\tag{20}$$

Here, Young's inequality is applied to each term as

$$\begin{aligned}&\Big|\int_\Omega u\nabla u\cdot\nabla\psi\Big|\le A\int_\Omega u\psi^{1/3}\cdot|\nabla u|\,\psi^{1/2}\\&\le A|\Omega|^{\frac{1}{6}}\Big\{\int_\Omega u^3\psi\Big\}^{\frac{1}{3}}\Big\{\int_\Omega|\nabla u|^2\psi\Big\}^{\frac{1}{2}}\le\frac{1}{4}\int_\Omega|\nabla u|^2\psi+\frac{1}{3}\int_\Omega u^3\psi+\frac{4A^6|\Omega|}{3},\\&\frac{1}{2}\Big|\int_\Omega v\nabla u^2\cdot\nabla\psi\Big|\le\frac{A}{2}\int_\Omega v\cdot u\psi^{1/3}\cdot|\nabla u|\,\psi^{1/2}\\&\le\frac{A}{2}\|v\|_6\Big\{\int_\Omega u^3\psi\Big\}^{\frac{1}{3}}\Big\{\int_\Omega|\nabla u|^2\psi\Big\}^{\frac{1}{2}}\le\frac{1}{4}\int_\Omega|\nabla u|^2\psi+\frac{1}{3}\int_\Omega u^3\psi+\frac{A^6\|v\|_6^6}{48},\\&\frac{1}{2}\Big|\int_\Omega u^2v\Delta\psi\Big|\le\frac{B}{2}\int_\Omega v\cdot u^2\psi^{2/3}\le\frac{B}{2}\|v\|_3\Big\{\int_\Omega u^3\psi\Big\}^{\frac{2}{3}}\le\frac{1}{3}\int_\Omega u^3\psi+\frac{B^3\|v\|_3^3}{6}.\end{aligned}$$

We recall (17), and deduce from (20) that

$$\frac{d}{dt}\int_\Omega u^2\psi+\int_\Omega|\nabla u|^2\psi\le 2\int_\Omega u^3\psi+C_1.\tag{21}$$

Now combine (15) with (21), (18), and (9). Making $s\gg 1$, we have

$$\frac{d}{dt}\int_\Omega u^2\psi+\frac{1}{2}\int_\Omega|\nabla u|^2\psi\le C_2.$$

This implies

$$\sup_{0\le t<T_{\max}} \int_\Omega u^2(\cdot,t)\psi < +\infty. \tag{22}$$

Step 2. Multiplying the first equation of (1) by $u^2\psi$ gives

$$\begin{aligned}\frac{1}{3}\frac{d}{dt}\int_\Omega u^3\psi + 2\int_\Omega u\left|\nabla u\right|^2\psi + \int_\Omega u^2\nabla u\cdot\nabla\psi \\ = 2\int_\Omega u^2\left(\nabla v\cdot\nabla u\right)\psi + \int_\Omega u^3\nabla v\cdot\nabla\psi.\end{aligned}$$

For $w = u^{3/2}$ this means that

$$\begin{aligned}\frac{1}{3}\frac{d}{dt}\int_\Omega w^2\psi + \frac{8}{9}\int_\Omega\left|\nabla w\right|^2\psi + \frac{2}{3}\int_\Omega w\nabla w\cdot\nabla\psi \\ = \frac{4}{3}\int_\Omega w\left(\nabla v\cdot\nabla w\right)\psi + \int_\Omega w^2\nabla v\cdot\nabla\psi.\end{aligned}$$

Here, by using the second equation of (1) we have

$$\begin{aligned}&\int_\Omega w\left(\nabla v\cdot\nabla w\right)\psi = \frac{1}{2}\int_\Omega\left(\nabla v\cdot\nabla w^2\right)\psi \le \frac{1}{2}\int_\Omega uw^2\psi - \frac{1}{2}\int_\Omega w^2\nabla v\cdot\nabla\psi \\ &\le \frac{1}{2}\left(\int_\Omega w^3\psi\right)^{8/9}|\Omega|^{1/9} + \frac{1}{2}\int_\Omega v\nabla w^2\cdot\nabla\psi + \frac{1}{2}\int_\Omega w^2 v\Delta\psi \\ &\le \frac{1}{2}\int_\Omega w^3\psi + \frac{1}{2}\int_\Omega v\nabla w^2\cdot\nabla\psi + \frac{1}{2}\int_\Omega w^2 v\Delta\psi + \frac{1}{2}\cdot\left(\frac{8}{9}\right)^8\cdot\frac{|\Omega|}{9}.\end{aligned}$$

We obtain

$$\begin{aligned}&\frac{1}{3}\frac{d}{dt}\int_\Omega w^2\psi + \frac{8}{9}\int_\Omega\left|\nabla w\right|^2\psi + \frac{2}{3}\int_\Omega w\nabla w\cdot\nabla\psi \\ &\quad\le \frac{2}{3}\int_\Omega w^3\psi - \frac{1}{3}\int_\Omega v\nabla w^2\cdot\nabla\psi - \frac{1}{3}\int_\Omega w^2 v\Delta\psi + \frac{2}{3}\cdot\left(\frac{8}{9}\right)^8\cdot\frac{|\Omega|}{9},\end{aligned}$$

which obeys a similar form of (20). Inequality (22) implies

$$\sup_{0\le t<T_{\max}} \int_\Omega w^{4/3}(\cdot,t)\psi < +\infty.$$

In particular, we have

$$\sup_{0\le t<T_{\max}} \int_{B_{R'}(x_0)\cap\Omega} (w\log w)(\cdot,t) < +\infty$$
$$\sup_{0\le t<T_{\max}} \|w(\cdot,t)\|_{L^1(B_{R'}(x_0)\cap\Omega)} < +\infty.$$

Therefore, taking $R''\in(0,R')$, we can apply the arguments of step 1 with u, R, and $\psi=\left(\varphi_{x_0,R',R}\right)^6$, replacing by w, R', and $\psi_1=\left(\varphi_{x_0,R'',R'}\right)^6$, respectively. Similarly to (22) it follows that

$$\sup_{0\le t<T_{\max}} \|w(t)\|^{2/3}_{L^2(B_r(x_0)\cap\Omega)} = \sup_{0\le t<T_{\max}} \|u(t)\|_{L^3(B_r(x_0)\cap\Omega)} < +\infty$$

for any $r\in(0,R)$, because $R'\in(0,R)$ and $R''\in(0,R')$ are arbitrary. From the second equation of (1) this implies $\sup_{0\le t<T_{\max}}\|v(t)\|_{W^{2,3}(B_{r'}(x_0)\cap\Omega)} < +\infty$ for $r'\in(0,r)$. Therefore,

$$\sup_{0\le t<T_{\max}} \|v(t)\|_{C^1(B_r(x_0)\cap\Omega)} < +\infty \tag{23}$$

holds for any $r\in(0,R)$. Repeating the arguments once more, we have

$$\sup_{0\le t<T_{\max}} \|u(t)\|_{L^4(B_r(x_0)\cap\Omega)} < +\infty. \tag{24}$$

Step 3. Take $r'\in(0,r)$ and put $\psi_1=\left(\varphi_{x_0,r',r}\right)^6$. For $p\ge1$ we multiply the first equation of (1) by $u^p\psi_1^{p+1}$ and get

$$\frac{d}{dt}\frac{1}{p+1}\int_\Omega (u\psi_1)^{p+1} = -\int_\Omega \nabla(u^p\psi_1^{p+1})\cdot\nabla u + \int_\Omega u\nabla(u^p\psi_1^{p+1})\cdot\nabla v = -I+II.$$

Here we have

$$\begin{aligned} I &= \int_\Omega \left(pu^{p-1}\psi_1^{p+1}\nabla u + u^p\nabla\psi_1^{p+1}\right)\cdot\nabla u \\ &= \frac{4p}{(p+1)^2}\int_\Omega \left|\nabla u^{\frac{p+1}{2}}\right|^2\psi_1^{p+1} + \frac{1}{p+1}\int_\Omega \nabla\psi_1^{p+1}\cdot\nabla u^{p+1} \\ &= \frac{4p}{(p+1)^2}\int_\Omega \left|\nabla u^{\frac{p+1}{2}}\right|^2\psi_1^{p+1} + \frac{4}{p+1}\int_\Omega \psi_1^{\frac{p+1}{2}}\nabla u^{\frac{p+1}{2}}\cdot u^{\frac{p+1}{2}}\nabla\psi_1^{\frac{p+1}{2}} \end{aligned}$$

$$
\begin{aligned}
&= \left\{ \frac{4p}{(p+1)^2} - \frac{2}{p+1} \right\} \int_\Omega \left| \nabla u^{\frac{p+1}{2}} \right|^2 \psi_1^{p+1} \\
&+ \frac{2}{p+1} \int_\Omega \left| \nabla (u\psi_1)^{\frac{p+1}{2}} \right|^2 - \frac{2}{p+1} \int_\Omega u^{p+1} \left| \nabla \psi_1^{\frac{p+1}{2}} \right|^2 \\
&\geq \frac{2}{p+1} \int_\Omega \left| \nabla (u\psi_1)^{\frac{p+1}{2}} \right|^2 - \frac{p+1}{2} \int_\Omega u^{p+1} \psi_1^{p-1} \left| \nabla \psi_1 \right|^2 \\
&\geq \frac{2}{p+1} \int_\Omega \left| \nabla (u\psi_1)^{\frac{p+1}{2}} \right|^2 - \frac{A^2(p+1)}{2} \int_\Omega (u\psi_1)^{p+(2/3)} \cdot u^{1/3} \\
&\geq \frac{2}{p+1} \int_\Omega \left| \nabla (u\psi_1)^{\frac{p+1}{2}} \right|^2 - \frac{A^2(p+1)}{2} \left\| u_0 \right\|_{L^1(\Omega)}^{1/3} \left\{ \int_\Omega (u\psi_1)^{1+(3/2)p} \right\}^{2/3}.
\end{aligned}
$$

On the other hand, estimate (23) means that

$$
L \equiv \sup_{0 \leq t < T_{\max}} \| \nabla v \|_{L^\infty(B_r(x_0) \cap \Omega)} < +\infty.
$$

We obtain

$$
\begin{aligned}
II &\leq L \int_\Omega |u \nabla (u^p \psi_1^{p+1})| = L \int_\Omega |\frac{p}{p+1} \nabla (u\psi_1)^{p+1} + u^{p+1} \psi_1^p \nabla \psi_1| \\
&\leq L \Big\{ \frac{p}{p+1} \int_\Omega |\nabla (u\psi_1)^{p+1}| + (p+1) \int_\Omega u^{p+1} \psi_1^p |\nabla \psi_1| \Big\} \\
&\leq \frac{2pL}{p+1} \int_\Omega (u\psi_1)^{\frac{p+1}{2}} \left| \nabla (u\psi_1)^{\frac{p+1}{2}} \right| + LA(p+1) \int_\Omega (u\psi_1)^{p+(5/6)} u^{1/6} \\
&\leq \frac{1}{p+1} \int_\Omega \left| \nabla (u\psi_1)^{\frac{p+1}{2}} \right|^2 + 4L^2(p+1) \int_\Omega (u\psi_1)^{p+1} \\
&\qquad + LA(p+1) \left\| u_0 \right\|_{L^1(\Omega)}^{1/6} \left\{ \int_\Omega (u\psi_1)^{1+(6/5)p} \right\}^{5/6}.
\end{aligned}
$$

It holds that

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega u_1^{p+1} &\leq - \int_\Omega |\nabla u_1^{\frac{p+1}{2}}|^2 + C_3(p+1)^2 \int_\Omega u_1^{p+1} \\
&+ C_3(p+1)^2 \Big(\Big\{ \int_\Omega u_1^{1+(3/2)p} \Big\}^{2/3} + \Big\{ \int_\Omega u_1^{1+(6/5)p} \Big\}^{5/6} \Big), \qquad (25)
\end{aligned}
$$

where $u_1 = u\psi_1$. Here, $C_3 > 0$ is independent of $p \geq 1$ and we can apply an iteration scheme of Moser's type (see Alikakos [2]). To this end we make use of Gagliardo-Nirenberg's inequality in the form of

$$\|w\|_{L^q(\Omega)} \leq K\left(\|\nabla w\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)}\right)^{\frac{1-(1/q)}{2}} \|w\|^{1/q}_{L^1(\Omega)}, \tag{26}$$

where $K > 0$ is independent of $q \in [1, q_0]$ for given $q_0 > 1$.

First, we apply (26) for $w = u_1^{(p+1)/2}$ and $q = \frac{3p+2}{p+1} \in [\frac{5}{2}, 3)$. We have

$$\begin{aligned} &C_3(p+1)^2\Big\{\int_\Omega u_1^{1+(3/2)p}\Big\}^{\frac{2}{3}} \\ &\leq C_3(p+1)^2\Big\{\int_\Omega |\nabla u_1^{\frac{p+1}{2}}|^2 + \int_\Omega u_1^{p+1}\Big\}^{\frac{2p+1}{3p+3}}\Big\{\int_\Omega u_1^{\frac{p+1}{2}}\Big\}^{\frac{2}{3}}. \end{aligned}$$

Because $\frac{2p+1}{3p+3} < \frac{2}{3}$, the right-hand side is dominated from above by

$$\begin{aligned} &C_3(p+1)^2\Big\{\int_\Omega \left(|\nabla u_1^{\frac{p+1}{2}}|^2 + u_1^{p+1}\right) + 1\Big\}^{2/3} \cdot \Big\{\int_\Omega u_1^{\frac{p+1}{2}}\Big\}^{2/3} \\ &\leq \frac{1}{6}\Big\{\int_\Omega \left(|\nabla u_1^{\frac{p+1}{2}}|^2 + u_1^{p+1}\right) + 1\Big\} + 16C_3^3(p+1)^6\Big\{\int_\Omega u_1^{\frac{p+1}{2}} + 1\Big\}^2. \end{aligned}$$

Second, we apply (26) for $w = u_1^{(p+1)/2}$ and $q = \frac{12p+10}{5p+5} \in [\frac{22}{10}, \frac{12}{5})$. We have

$$\begin{aligned} &C_3(p+1)^2\Big\{\int_\Omega u_1^{1+(6/5)p}\Big\}^{5/6} \\ &\leq C_3(p+1)^2\Big\{\int_\Omega |\nabla u_1^{\frac{p+1}{2}}|^2 + \int_\Omega u_1^{p+1}\Big\}^{\frac{7p+5}{12p+12}} \cdot \Big\{\int_\Omega u_1^{\frac{p+1}{2}}\Big\}^{5/6} \\ &\leq C_3(p+1)^2\Big\{\int_\Omega \left(|\nabla u_1^{\frac{p+1}{2}}|^2 + u_1^{p+1}\right) + 1\Big\}^{7/12} \cdot \Big\{\int_\Omega u_1^{\frac{p+1}{2}}\Big\}^{5/6} \\ &\leq \frac{1}{6}\Big\{\int_\Omega \left(|\nabla u_1^{\frac{p+1}{2}}|^2 + u_1^{p+1}\right) + 1\Big\} + (\frac{7}{2})^{7/5}C_3^{12/5}(p+1)^{24/5}\Big\{\int_\Omega u_1^{\frac{p+1}{2}}\Big\}^2 \\ &\leq \frac{1}{6}\Big\{\int_\Omega \left(|\nabla u_1^{\frac{p+1}{2}}|^2 + u_1^{p+1}\right) + 1\Big\} + (\frac{7}{2})^{7/5}C_3^{12/5}(p+1)^6\Big\{\int_\Omega u_1^{\frac{p+1}{2}} + 1\Big\}^2. \end{aligned}$$

Finally, we apply (26) for $w = u_1^{(p+1)/2}$ and $q = 2$. We have

$$\begin{aligned} &C_3(p+1)^2\int_\Omega u_1^{p+1} \leq C_3(p+1)^2\Big\{\int_\Omega \left(|\nabla u_1^{\frac{p+1}{2}}|^2 + u_1^{p+1}\right)\Big\}^{1/2}\Big\{\int_\Omega u_1^{\frac{p+1}{2}}\Big\} \\ &\leq \frac{1}{6}\Big\{\int_\Omega \left(|\nabla u_1^{\frac{p+1}{2}}|^2 + u_1^{p+1}\right) + 1\Big\} + 3C_3^2(p+1)^4\Big\{\int_\Omega u_1^{\frac{p+1}{2}} + 1\Big\}^2. \end{aligned}$$

Thus, inequality (25) gives that

$$\frac{d}{dt}\int_\Omega u_1^{p+1} + \frac{1}{2}\int_\Omega |\nabla u_1^{\frac{p+1}{2}}|^2 \le \frac{1}{2}\int_\Omega u_1^{p+1} + C_4(p+1)^6\Big\{\int_\Omega u_1^{\frac{p+1}{2}} + 1\Big\}^2.$$

However, again (26) for $q = 2$ implies

$$\begin{aligned} \|w\|_{L^2(\Omega)}^2 &\le K^2\left(\|\nabla w\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2\right)^{1/2}\|w\|_{L^1(\Omega)} \\ &\le \frac{1}{4}\left(\|\nabla w\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2\right) + K^4\|w\|_{L^1(\Omega)}^2, \end{aligned}$$

and hence

$$\Big\|u_1^{\frac{p+1}{2}}\Big\|_{L^2(\Omega)}^2 \le \frac{1}{3}\Big\|\nabla u_1^{\frac{p+1}{2}}\Big\|_{L^2(\Omega)}^2 + \frac{4K^4}{3}\Big\|u_1^{\frac{p+1}{2}}\Big\|_{L^1(\Omega)}^2.$$

We obtain

$$\frac{d}{dt}\int_\Omega u_1^{p+1} + \frac{1}{3}\int_\Omega |\nabla u^{\frac{p+1}{2}}|^2 \le C_4(p+1)^6\Big\{\int_\Omega u_1^{\frac{p+1}{2}} + 1\Big\}^2 + \frac{2}{3}K^2\Big\{\int_\Omega u_1^{\frac{p+1}{2}}\Big\}^2,$$

and therefore

$$\frac{d}{dt}\int_\Omega u_1^{p+1} + \int_\Omega u_1^{p+1} \le C_5(p+1)^6\Big\{\int_\Omega u_1^{\frac{p+1}{2}} + 1\Big\}^2.$$

This gives that

$$\begin{aligned} &\sup_{0\le t<T_{\max}}\Big\{\int_\Omega u_1^{p+1} + 1\Big\} \\ &\quad \le C_5\max\Big\{(p+1)^6\sup_{0\le t<T_{\max}}\Big\{\int_\Omega u_1^{\frac{p+1}{2}} + 1\Big\}^2, \|u_0\|_{L^\infty(\Omega)}^{p+1}|\Omega| + 1\Big\}. \end{aligned}$$

Therefore, $\Phi_k = \sup_{0\le t<T_{\max}}\int_\Omega u_1^{2^k} + 1$ satisfies

$$\begin{aligned} \Phi_{k+1} &\le C_5\max\Big\{2^{6(k+1)}\Phi_k^2,\ \Big(|\Omega|+1\Big)\Big(\|u_0\|_{L^\infty(\Omega)} + 1\Big)^{2^{k+1}}\Big\} \\ &\le C_5 2^{6(k+1)}\max\Big\{\Phi_k^2,\ \Big(\|u_0\|_{L^\infty(\Omega)} + 1\Big)^{2^{k+1}}\Big\} \end{aligned} \tag{27}$$

for $k = 1, 2, \cdots$. Let $d = \|u_0\|_{L^\infty(\Omega)} + 1$. Then, (27) is reduced to

$$\Phi_{k+1} \le C_5^{2^{k-1}-1}\cdot 2^{\sum_{\ell=2}^k 6(\ell+1)2^{k-\ell}}\cdot\max\Big\{\Phi_2^{2^{k-1}},\ d^{2^{k+1}}\Big\}$$

for $k = 2, 3, \cdots$. We have

$$\sup_{0\le t<T_{\max}} \left\{ \int_\Omega u_1^{2^{k+1}} \right\}^{\frac{1}{2^{k+1}}} \le \Phi_{k+1}^{\frac{1}{2^{k+1}}} \le C_5^{\frac{2^{k-1}-1}{2^{k+1}}} \cdot 2^{6\sum_{j=1}^{\infty} j2^{-j}} \cdot \max\left\{ \Phi_2^{1/4},\ d \right\},$$

and, letting $k \to +\infty$,

$$\sup_{0\le t<T_{\max}} \|u_1(\cdot,t)\|_{L^\infty(\Omega)} \le C_5 \max\left\{ \Big(\sup_{0\le t<T_{\max}} \|u_1(\cdot,t)\|_{L^4(\Omega)}^4 + 1 \Big)^{1/4},\ d \right\}$$

follows. By using (24), we obtain

$$\sup_{0\le t<T_{\max}} \|u_1(\cdot,t)\|_{L^\infty(\Omega)} = \sup_{0\le t<T_{max}} \|u(\cdot,t)\psi_1\|_{L^\infty(\Omega)} < +\infty,$$

or $\limsup_{t\uparrow T_{\max}} \|u(t)\|_{L^\infty(B_{r'}(x_0)\cap\Omega)} < +\infty$. This means $x_0 \notin \mathcal{B}$ and the proof is complete.

5. Proof of Theorem 1. The global version of Lemma 5 is expressed as follows:

$$\limsup_{t\uparrow T} \int_\Omega u \log u < +\infty \tag{28}$$

implies

$$\limsup_{t\uparrow T} \|u(t)\|_\infty < +\infty. \tag{29}$$

In fact, this is proven just by replacing the cutoff function φ with the constant function 1. If (29) follows, then equation (4) assures for the solution u to be continued after $t = T$. We shall show that (28) follows from

$$\liminf_{t\uparrow T} \int_\Omega u \log u < +\infty. \tag{30}$$

Then, $T_{\max} < +\infty$ holds only if

$$\liminf_{t\uparrow T_{\max}} \int_\Omega u \log u = +\infty,$$

and in particular relation (5) follows.

To this end, we multiply $\log u$ by the first equation of (1). By using the second equation of (1) we have

$$\frac{d}{dt}\int_\Omega u\log u + \int_\Omega u^{-1}\left|\nabla u\right|^2 + \int_\Omega uv = \int_\Omega u^2. \tag{31}$$

The right-hand side is dominated by (14). It follows that

$$\frac{d}{dt}\int_\Omega u\log u + \Big(1-\frac{2K^2}{\log s}\int_\Omega\left(u\log u+e^{-1}\right)\Big)\cdot\int_\Omega u^{-1}|\nabla u|^2 \le C\left\|u_0\right\|_1^2+3s^2|\Omega|.$$

Taking $s = s(t) = \exp\left(2K^2\int_\Omega\left(u\log u+e^{-1}\right)\right) > 1$, we have

$$\frac{dJ}{dt} \le C\left\|u_0\right\|_1^2 + 3|\Omega|\exp\left(4K^2 J\right), \tag{32}$$

where $J = \int_\Omega\left(u\log u+e^{-1}\right)$. Inequality (32) and $\liminf_{t\uparrow T} J(t) < +\infty$ imply $\limsup_{t\uparrow T} J(t) < +\infty$ by the comparison theorem for ordinary differential equations. In particular, (30) implies (28). The proof is complete.

6. Estimate on the Green's function. Given $x_0 \in \overline{\Omega}$, we take $0 < R' < R \ll 1$ and set $\psi = \left(\varphi_{x_0,R',R}\right)^6$. Let $G = G(x,y)$ be the Green's function of the operator $\mathcal{L}+1$, so that it solves

$$\left(-\Delta_y+1\right)G = \delta(y-x) \qquad (y\in\Omega)$$

with $\frac{\partial}{\partial\nu_y}G = 0 \ (y\in\partial\Omega)$ for $x\in\Omega$. From the elliptic regularity, it is extended to a smooth function on $\overline{\Omega}\times\overline{\Omega}\setminus\left\{(x,x) : x\in\overline{\Omega}\right\}$. Also the symmetry $G(x,y) = G(y,x)$ follows.

In this section we show the following.

Lemma 6. *The function $\rho(x,y) = \nabla\psi(x)\cdot\nabla_x G(x,y)+\nabla\psi(y)\cdot\nabla_y G(x,y)$ belongs to $L^\infty(\Omega\times\Omega)$.*

Proof. Case 1. $x_0\in\Omega$**.** Let $K_0(r)$ be the modified Bessel function of the second kind:

$$K_0(r) = -\{r+\log\frac{r}{2}\}I_0(r) + \sum_{k=1}^\infty \frac{1}{(k!)^2}\Big(\sum_{\ell=1}^k\frac{1}{\ell}\Big)\left(\frac{r}{2}\right)^{2k},$$

where $I_0(r) = \sum_{k=0}^\infty \frac{1}{(k!)^2}\left(\frac{r}{2}\right)^{2k}$ denotes the modified Bessel function of the first kind. Then the classical theory guarantees that $e_0(r) = \frac{1}{2\pi}K_0(r)$ provides a fundamental solution for $-\Delta+1$, so that $(-\Delta+1)e_0(|x|) = \delta(0)$

holds in $\mathbf{R}^2$. Furthermore, the above expression of K_0 gives that

$$e_0(|x|) = \frac{1}{2\pi}\big(1 + \frac{|x|^2}{4}\big) \log \frac{1}{|x|} + (C^{2,\theta} \text{ function}) \tag{33}$$

with $\theta \in (0,1)$. Given $x \in \Omega$, let $G(x,y) = e_0\left(|x-y|\right) + K_0(x,y)$. Then it holds that $\left(-\Delta_y + 1\right) K_0 = 0$ $(y \in \Omega)$ with

$$\frac{\partial}{\partial \nu_y} K_0 = -\frac{\partial}{\partial \nu_y} e_0\left(|x-y|\right) \qquad (y \in \partial\Omega).$$

Therefore, the elliptic regularity gives $K_0 \in C^{2,\theta}_{loc}\left(\Omega \times \overline{\Omega}\right)$. Combining this with (33), we obtain $G(x,y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} + K(x,y)$ with $K \in C^{1,\theta}_{loc}\left(\Omega \times \overline{\Omega}\right)$. Because $G(x,y)$ is symmetric, we also have $K \in C^{1,\theta}_{loc}\left(\overline{\Omega} \times \Omega\right)$. Therefore,

$$\begin{aligned}\rho(x,y) = &-(x-y)\cdot\left(\nabla\psi(x) - \nabla\psi(y)\right)/(2\pi\,|x-y|^2) \\ &+\nabla\psi(x)\cdot\nabla_x K(x,y) + \nabla\psi(y)\cdot\nabla_y K(x,y)\end{aligned}$$

belongs to $L^\infty\left(\Omega \times \Omega\right)$ by supp $\psi \subset \Omega$.

Case 2: $x_0 \in \partial\Omega$**.** We take the conformal mapping $X : B_{2R}(x_0) \cap \overline{\Omega} \to \mathbf{R}^2$ of Section 3. The function $c = |X'|^2 \circ X^{-1}$ has a bounded, smooth extension on $\overline{\mathbf{R}^2_+}$, where $\mathbf{R}^2_+ = \{(x_1,x_2) : x_2 > 0\}$. Let $\hat{c} = c(\xi) \in C^{0,1}_{loc}(\mathbf{R}^2)$ be its even extension. A fundamental solution $e(\xi,\eta)$ of $-\Delta + \hat{c}$ exists so that $\left(-\Delta_\eta + \hat{c}(\eta)\right) e(\xi,\eta) = \delta(\eta - \xi)$ holds for $\eta, \xi \in \mathbf{R}^2$.

On the other hand, in the same way as in *Case 1* we can take $\alpha = \alpha(\xi) \in C^{0,1}_{loc}\left(\mathbf{R}^2\right)$ and $\tilde{K} = \tilde{K}(\xi,\eta) \in C^{0,1}_{loc}\left(\mathbf{R}^2 \times \mathbf{R}^2\right)$ with $\xi \in \overline{\mathbf{R}^2_+} \mapsto \tilde{K}(\xi,\cdot) \in C^{2,\theta}_{loc}(\mathbf{R}^2)$ locally θ-continuous and

$$\tilde{e}(\xi,\eta) = \frac{1}{2\pi}\left(1 + \alpha(\xi)\,|\xi-\eta|^2\right) \log \frac{1}{|\xi-\eta|} + \tilde{K}(\xi,\eta)$$

satisfying $\left(-\Delta_\eta + \hat{c}(\xi)\right) \tilde{e}(\xi,\eta) = \delta(\eta-\xi)$. In particular, we have

$$\left(-\Delta_\eta + \hat{c}(\eta)\right)\left(\tilde{e}(\xi,\eta) - e(\xi,\eta)\right) = \left(\hat{c}(\eta) - \hat{c}(\xi)\right)\tilde{e}(\xi,\eta),$$

of which the right-hand side belongs to $C^\theta_{loc}\left(\mathbf{R}^2 \times \mathbf{R}^2\right)$. The elliptic regularity gives the local θ-continuity of $\xi \in \mathbf{R}^2 \mapsto \tilde{e}(\xi,\cdot) - e(\xi,\cdot) \in C^{2,\theta}_{loc}(\mathbf{R}^2)$, and we obtain

$$e(\xi,\eta) = \frac{1}{2\pi} \log \frac{1}{|\xi-\eta|} + K_1(\xi,\eta) \tag{34}$$

with $\xi \in \overline{\mathbf{R}^2_+} \mapsto K_1(\xi,\cdot) \in C^{1,\theta}_{loc}(\overline{\mathbf{R}^2_+})$ locally θ-continuous. In particular, $\nabla_\eta K_1 \in C^{\theta}_{loc}(\overline{\mathbf{R}^2_+} \times \overline{\mathbf{R}^2_+})$ follows. Let $E(\xi,\eta) = e(\xi,\eta) + e(\xi,\eta^*)$ with $\eta^* = (\eta_1, -\eta_2)$ for $\eta = (\eta_1,\eta_2)$. Then, for $\xi \in \mathbf{R}^2_+$ it holds that

$$(-\Delta_\eta + c(\eta))\, E(\xi,\eta) = \delta(\eta-\xi) \qquad \left(\eta \in \mathbf{R}^2_+\right)$$

and

$$\frac{\partial}{\partial \nu_\eta} E(\xi,\eta) = 0 \qquad \left(\eta \in \partial\mathbf{R}^2_+\right).$$

Because X is conformal, this implies for $x \in B_{2R}(x_0) \cap \Omega$ that

$$(-\Delta_y + 1)\, E\left(X(x), X(y)\right) = \delta(y-x) \qquad (y \in B_{2R}(x_0)\cap\Omega)$$

and

$$\frac{\partial}{\partial \nu_y} E\left(X(x), X(y)\right) = 0 \qquad (y \in B_{2R}(x_0)\cap\partial\Omega).$$

The Green's function $G(x,y)$ satisfies the same relation, and the elliptic regularity assures the θ-continuity of

$$x \in \overline{B_R(x_0)\cap\Omega} \quad\mapsto\quad G(x,\cdot) - E\left(X(x), X(\cdot)\right) \in C^{1,\theta}\left(\overline{B_R(x_0)\cap\Omega}\right).$$

We have by (34) that

$$G(x,y) = \frac{1}{2\pi}\log\frac{1}{|X(x)-X(y)|} + \frac{1}{2\pi}\log\frac{1}{|X(x)-X(y)^*|} + K_2(x,y)$$

with $\nabla_y K_2 \in C^{\theta}(\overline{B_R(x_0)\cap\Omega} \times \overline{B_R(x_0)\cap\Omega})$. Because $G(x,y)$ is symmetric, so is $K_2(x,y)$. We have $K_2 \in C^{1,\theta}(\overline{B_R(x_0)\cap\Omega} \times \overline{B_R(x_0)\cap\Omega})$.

Here, the term $G_1(x,y) = \frac{1}{2\pi}\log\frac{1}{|X(x)-X(y)|}$ is treated as before; writing $\Psi = \psi \circ X^{-1}$ and $e_1(\xi,\eta) = \frac{1}{2\pi}\log\frac{1}{|\xi-\eta|}$, we have

$$\begin{aligned}
&\nabla\psi(x)\cdot\nabla_x G_1(x,y) + \nabla\psi(y)\cdot\nabla_y G_1(x,y)\\
&\quad = c(\xi)\nabla\Psi(\xi)\cdot\nabla_\xi e_1(\xi,\eta) + c(\eta)\nabla\Psi(\eta)\cdot\nabla_\eta e_1(\xi,\eta)\\
&\quad = -\frac{(\xi-\eta)\cdot\left(c(\xi)\nabla\Psi(\xi) - c(\eta)\nabla\Psi(\eta)\right)}{2\pi\,|\xi-\eta|^2}\\
&\quad \in L^\infty\left((B_R(x_0)\cap\Omega)\times(B_R(x_0)\cap\Omega)\right).
\end{aligned}$$

Concerning the term involving $G_2(x,y) = \frac{1}{2\pi}\log\frac{1}{|X(x)-X(y)^*|}$, we make use of $\left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0$. This gives that $\left.\frac{\partial\Psi}{\partial\xi_2}\right|_{\xi_2=0} = 0$, and hence

$$\begin{aligned}
&\nabla\psi(x)\cdot\nabla_x G_2(x,y) + \nabla\psi(y)\cdot\nabla_y G_2(x,y)\\
&\quad = c(\xi)\nabla\Psi(\xi)\cdot\nabla_\xi e_1(\xi,\eta^*) + c(\eta)\nabla\Psi(\eta)\cdot\nabla_\eta e_1(\xi,\eta^*)\\
&\quad = \{c(\xi)\Psi_{\xi_1}(\xi) - c(\eta)\Psi_{\eta_1}(\eta)\}(\xi_1-\eta_1)/(2\pi|\xi-\eta^*|^2)\\
&\qquad + \{c(\xi)\Psi_{\xi_2}(\xi) + c(\eta)\Psi_{\eta_2}(\eta)\}(\xi_1+\eta_2)/(2\pi|\xi-\eta^*|^2)\\
&\quad \in L^\infty\left((B_R(x_0)\cap\Omega)\times(B_R(x_0)\cap\Omega)\right).
\end{aligned}$$

We obtain $\rho \in L^\infty\left((B_R(x_0)\cap\Omega)\times(B_R(x_0)\cap\Omega)\right)$ and the conclusion $\rho \in L^\infty(\Omega\times\Omega)$ follows because supp $\psi \subset B_R(x_0)\cap\overline{\Omega}$ and $G(x,y)$ is smooth on $\overline{\Omega}\times\overline{\Omega}\setminus\left\{(x,x) : x\in\overline{\Omega}\right\}$. The proof is complete.

7. Finiteness of blowup points. In this section we show the finiteness of blowup points. We first show the following.

Lemma 7. *It holds that*

$$\frac{d}{dt}\int_\Omega (u\log u)\psi + \frac{1}{4}\int_\Omega u^{-1}|\nabla u|^2\psi \le 2\int_\Omega u^2\psi + C_6. \tag{35}$$

Proof. The first equation of (1) gives

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u\log u)\,\psi = \int_\Omega u_t(\log u + 1)\psi\\
&= -\int_\Omega \nabla u\cdot\nabla\left((\log u+1)\,\psi\right) + \int_\Omega u\nabla v\cdot\nabla\left((\log u+1)\,\psi\right) = -I + II.
\end{aligned}$$

Here, the second equation of (1) applies as

$$\begin{aligned}
II &= \int_\Omega \psi\nabla v\cdot\nabla u + \int_\Omega u\,(\log u+1)\,\nabla v\cdot\nabla\psi\\
&= -\int_\Omega u\nabla\cdot(\psi\nabla v) + \int_\Omega u\,(\log u+1)\,\nabla v\cdot\nabla\psi\\
&= \int_\Omega u\psi(u-v) + \int_\Omega u\log u\nabla v\cdot\nabla\psi.
\end{aligned}$$

We also note that

$$I = \int_\Omega u^{-1}|\nabla u|^2\psi + \int_\Omega (\log u+1)\,\nabla u\cdot\nabla\psi.$$

Then we obtain the localized version of (31):

$$\frac{d}{dt}\int_\Omega (u\log u)\psi + \int_\Omega u^{-1}\left|\nabla u\right|^2\psi + \int_\Omega uv\psi = \int_\Omega u^2\psi - \int_\Omega \left(\log u + 1\right)\nabla u\cdot\nabla\psi + \int_\Omega (u\log u)\nabla v\cdot\nabla\psi. \tag{36}$$

Recall the elementary inequality: Let $\alpha > 0$ and $0 < \beta < 2$. Then, it holds that $\left(|\log u| + 1\right)^\alpha u^\beta \le u^2 + C_{\alpha,\beta}$ $(u > 0)$. The second term of the right-hand side of (36) is dominated as

$$\begin{aligned}
&\left|\int_\Omega (\log u + 1)\nabla u\cdot\nabla\psi\right| \le A\int_\Omega \left(|\log u| + 1\right)u^{1/2}\psi^{1/3}\cdot u^{-1/2}|\nabla u|\psi^{1/2} \\
&\le A|\Omega|^{1/6}\left\{\int_\Omega \left(|\log u| + 1\right)^3 u^{3/2}\psi\right\}^{1/3}\left\{\int_\Omega u^{-1}|\nabla u|^2\psi\right\}^{1/2} \\
&\le A\left|\Omega\right|^{1/6}\left\{\int_\Omega u^2\psi + C_{3,3/2}\left|\Omega\right|\right\}^{1/3}\left\{\int_\Omega u^{-1}\left|\nabla u\right|^2\psi\right\}^{1/2} \\
&\le \frac{1}{4}\int_\Omega u^{-1}|\nabla u|^2\psi + \frac{1}{3}\int_\Omega u^2\psi + \frac{4A^6\left|\Omega\right|}{3} + \frac{C_{3,3/2}\left|\Omega\right|}{3}.
\end{aligned}$$

The third term of the right-hand side of (36) is equal to

$$-\int_\Omega v\nabla\cdot(u\log u\nabla\psi) = -\int_\Omega v(\log u + 1)\nabla u\cdot\nabla\psi - \int_\Omega (vu\log u)\Delta\psi. \tag{37}$$

Each term of the right-hand side of equality (37) is dominated as follows:

$$\begin{aligned}
&\left|\int_\Omega v(\log u + 1)\nabla u\cdot\nabla\psi\right| \le A\int_\Omega v\cdot u^{1/2}\left(|\log u| + 1\right)\psi^{1/3}\cdot u^{-1/2}|\nabla u|\psi^{1/2} \\
&\le A\left\|v\right\|_6\left\{\int_\Omega u^{3/2}\left(|\log u| + 1\right)^3\psi\right\}^{1/3}\left\{\int_\Omega u^{-1}|\nabla u|^2\psi\right\}^{1/2} \\
&\le \frac{1}{4}\int_\Omega u^{-1}|\nabla u|^2\psi + \frac{1}{3}\int_\Omega u^2\psi + \frac{4A^6\left\|v\right\|_6^6}{3} + \frac{C_{3,3/2}\left|\Omega\right|}{3} \\
&\left|\int_\Omega (vu\log u)\Delta\psi\right| \le B\int_\Omega v\left|u\log u\right|\psi^{2/3} \le B\left\|v\right\|_3\left\{\int_\Omega \left|u\log u\right|^{3/2}\psi\right\}^{2/3} \\
&\le \frac{1}{3}\int_\Omega u^2\psi + \frac{4B^3\left\|v\right\|_3^3}{3} + \frac{C_{3/2,3/2}\left|\Omega\right|}{3}.
\end{aligned}$$

Inequality (35) follows from (17), and the proof is complete.

We are ready to prove the following.

Lemma 8. *The blowup set $\mathcal{B}$ of u is finite.*

Proof. There is $\varepsilon_0 > 0$ such that any $x_0 \in \mathcal{B}$ and $0 < R \ll 1$ admit the estimate

$$\limsup_{t \uparrow T_{\max}} \int_{B_R(x_0)\cap\Omega} u \geq \varepsilon_0. \tag{38}$$

In fact, take $R' \in (0, R)$ and set $\psi = \left(\varphi_{x_0,R',R}\right)^6$. Combining (35) and (13), we have

$$\frac{d}{dt}\int_\Omega (u\log u)\psi + \frac{1}{4}\Big(1 - 16K^2\int_{B_R(x_0)\cap\Omega} u\Big)\cdot\int_\Omega u^{-1}|\nabla u|^2\psi \leq C_7.$$

Therefore, if $\limsup_{t\uparrow T_{\max}} \int_{B_R(x_0)\cap\Omega} u < \varepsilon_0 \equiv \frac{1}{16K^2}$, then

$$\limsup_{t\uparrow T_{\max}} \int_{B_{R'}(x_0)\cap\Omega} u\log u \leq \limsup_{t\uparrow T_{\max}} \int_\Omega (u\log u)\psi < +\infty.$$

This implies $x_0 \notin \mathcal{B}$ by Lemma 5, a contradiction.

Next we show that

$$\left|\frac{d}{dt}\int_\Omega u\psi\right| \leq B\left\|u_0\right\|_1 + \frac{1}{2}\left\|\rho\right\|_{L^\infty(\Omega\times\Omega)}\left\|u_0\right\|_1^2. \tag{39}$$

In fact, the first equation of (1) gives

$$\frac{d}{dt}\int_\Omega u\psi = \int_\Omega u_t\psi = \int_\Omega u\Delta\psi + \int_\Omega u\nabla v\cdot\nabla\psi.$$

Here, it is obvious that $\left|\int_\Omega u\Delta\psi\right| \leq B\left\|u_0\right\|_1$. By using the notations of the previous section we have

$$\begin{aligned}\int_\Omega u\nabla v\cdot\nabla\psi &= \int_\Omega\int_\Omega u(x,t)\nabla\psi(x)\cdot\nabla_x G(x,y)u(y,t)\,dy\,dx \\ &= \frac{1}{2}\int_\Omega\int_\Omega \rho(x,y)u(x,t)u(y,t)\,dx\,dy.\end{aligned}$$

Because $\rho(x,y) \in L^\infty(\Omega\times\Omega)$ we have

$$\left|\int_\Omega\int_\Omega \rho(x,y)u(x,t)u(y,t)\,dx\,dy\right| \leq \left\|\rho\right\|_{L^\infty(\Omega\times\Omega)}\left\|u_0\right\|_1^2,$$

and inequality (39) has been proven.

This implies that the value

$$\lim_{t\uparrow T_{\max}} \int_\Omega u\psi = \int_\Omega u_0(x)\psi + \int_0^{T_{\max}} \left(\frac{d}{dt} \int_\Omega u(\cdot,t)\psi \right) dt$$

exists. Because $0 < R \ll 1$ is arbitrary, inequality (38) is improved as

$$\begin{aligned} \liminf_{t\uparrow T_{\max}} \int_{B_R(x_0)\cap\Omega} u(x,t)\,dx &\geq \lim_{t\uparrow T_{\max}} \int_\Omega u(x,t)\psi(x)\,dx \\ &\geq \limsup_{t\uparrow T_{\max}} \int_{B_{R'}(x_0)\cap\Omega} u(x,t)\,dx \geq \varepsilon_0. \end{aligned}$$

Therefore, by using (9) we conclude $\sharp\mathcal{B} \leq \|u_0\|_1/\varepsilon_0 < +\infty$. The proof is complete.

8. Proof of Theorem 3. Once the finiteness of blowup points is proven, chemotactic collapse (8) follows from localizing the estimates of [19], [3], [8]. Then, inequality (6) is a consequence of (9). The final section shows (8), simplifying the arguments of our previous work [18].

Given $0 < \varepsilon \ll 1$, we set $\mathcal{B}_\varepsilon = \bigcup_{x_0\in\mathcal{B}} B_\varepsilon(x_0)$. We have

$$\sup_{0\leq t<t_{\max}} \|u(t)\|_{L^\infty(\Omega\setminus\mathcal{B}_\varepsilon)} < +\infty,$$

and the relation

$$\sup_{0\leq t<T_{\max}} \|\nabla v(t)\|_{L^\infty(\Omega\setminus\mathcal{B}_{2\varepsilon})} < +\infty$$

follows from the second equation of (1). Then the first and the second equations of (1) assure

$$\|u\|_{C^{2+\theta,1+\theta/2}(\Omega\setminus\mathcal{B}_{3\varepsilon}\times[0,T_{\max}))} < +\infty \tag{40}$$

and

$$\|v\|_{C^{2+\theta,1+\theta/2}\left(\overline{\Omega}\setminus\mathcal{B}_{4\varepsilon}\times[0,T_{\max})\right)} < +\infty \tag{41}$$

in turn with $\theta \in (0,1)$ (see Ladyzhenskaya, Solonnikov, and Uralt'seva [15] and Gilbarg and Trudinger [9]). In particular,

$$\sup_{0\leq t<T_{\max}} \|u_t(t)\|_{C\left(\overline{\Omega}\setminus\mathcal{B}_{3\varepsilon}\right)} < +\infty$$

holds and

$$f(x) = u(x,0) + \int_0^{T_{\max}} u_t(x,t)\,dt = \lim_{t \uparrow T_{\max}} u(x,t) \geq 0 \tag{42}$$

exists for any $x \in \overline{\Omega} \setminus \mathcal{B}$. Convergence (42) is locally uniform on $\overline{\Omega} \setminus \mathcal{B}$ and relation (7) follows from (9). The family $\{u(x,t)dx : 0 \leq t < T_{\max}\} \subset \mathcal{M}(\overline{\Omega})$ is bounded so that it is sequentially weak star pre-compact as $t \uparrow T_{\max}$. We shall show the following.

Lemma 9. *For any $x_0 \in \mathcal{B}$ and $0 < R \ll 1$, it holds that*

$$\liminf_{t \uparrow T_{\max}} \int_{B_R(x_0) \cap \Omega} u(x,t)\,dx \geq m_* = \begin{cases} 8\pi & (x_0 \in \Omega) \\ 4\pi & (x_0 \in \partial\Omega)\,. \end{cases} \tag{43}$$

Lemma 9 implies Theorem 3 as follows. First, any sequence $t_k \uparrow T_{\max}$ admits a subsequence $\{t'_k\}$ such that $w^* - \lim_{k \to \infty} u(x,t'_k)dx = \mu(dx)$ exists. Because $\mu(dx) - f(x)dx \in \mathcal{M}(\overline{\Omega})$ has the support on the finite set $\mathcal{B}$ and (43) holds, we have $m' : \mathcal{B} \to [4\pi, +\infty)$ satisfying $m'|_{\mathcal{B} \cap \Omega} \geq 8\pi$ and

$$\mu(dx) = \sum_{x_0 \in \mathcal{B}} m'(x_0)\delta_{x_0}(dx) + f(x)\,dx.$$

However, from the proof of Lemma 8 we have the existence of

$$\lim_{t \uparrow T_{\max}} \int_\Omega u(x,t)\varphi(x)\,dx$$

for any smooth function φ on $\overline{\Omega}$, and the value $m'(x_0)$ is independent of the choice of $\{t_k\}$ or $\{t'_k\}$. This shows (8).

To prove Lemma 9 we take $x_0 \in \mathcal{B}$ and $0 < R' < R \ll 1$. Letting $\varphi = \varphi_{x_0,R',R}$, we introduce the localized Lyapunov function

$$W_\varphi(t) = \int_\Omega \left\{ u \log u - uv + \frac{1}{2}\left(|\nabla v|^2 + v^2\right) \right\} \varphi.$$

We have the following.

Lemma 10. *It holds that*

$$\frac{d}{dt} W_\varphi(t) + \int_\Omega u\,|\nabla(\log u - v)|^2\,\varphi = \frac{d}{dt}\int_\Omega u\varphi + R_1(u,v,\varphi), \tag{44}$$

where

$$R_1(u,v,\varphi) = \int_\Omega [(1-v)\nabla u - (u \log u - uv + v_t)\nabla v] \cdot \nabla\varphi + \int_\Omega (u \log u) \triangle \varphi.$$

Proof. Multiplying $(\log u - v)\varphi$ by the first equation of (1), we have

$$\int_\Omega u_t(\log u - v)\varphi = \int_\Omega \nabla\cdot(\nabla u - u\nabla v)(\log u - v)\varphi$$
$$= -\int_\Omega u|\nabla(\log u - v)|^2\varphi - \int_\Omega (\log u - v)(\nabla u - u\nabla v)\cdot\nabla\varphi. \tag{45}$$

Here, it holds that

$$\int_\Omega u_t(\log u - v)\varphi = \frac{d}{dt}\int_\Omega (u\log u - uv)\varphi - \frac{d}{dt}\int_\Omega u\varphi + \int_\Omega uv_t\varphi \tag{46}$$

and

$$\begin{aligned}\int_\Omega (\log u)\nabla u\cdot\nabla\varphi &= -\int_\Omega u\nabla\cdot(\log u\nabla\varphi) + \int_{\partial\Omega}(u\log u)\frac{\partial\varphi}{\partial\nu}\\ &= -\int_\Omega \{(u\log u)\triangle\varphi + \nabla u\cdot\nabla\varphi\}.\end{aligned}\tag{47}$$

By using the second equation of (1), we have

$$\int_\Omega uv_t\varphi = \int_\Omega(-\triangle v + v)v_t\varphi = \frac{1}{2}\frac{d}{dt}\int_\Omega \left(|\nabla v|^2 + v^2\right)\varphi + \int_\Omega v_t\nabla v\cdot\nabla\varphi.$$

This, together with (45), (46) and (47), leads to

$$\frac{d}{dt}W_\varphi + \int_\Omega u|\nabla(\log u - v)|^2\varphi = \frac{d}{dt}\int_\Omega u\varphi + \int_\Omega (u\log u)\triangle\varphi$$
$$+ \int_\Omega [(1-v)\nabla u - (u\log u - uv + v_t)\nabla v]\cdot\nabla\varphi.$$

The proof is complete.

The above lemma implies the following.

Lemma 11. *Let $x_0 \in \mathcal{B}$ and $\varphi = \varphi_{x_0,R',R}$ for $0 < R' < R \ll 1$. Then we have*

$$W^* \equiv \sup_{0\le t<T_{\max}} W_\varphi(t) < +\infty \tag{48}$$

and

$$\limsup_{t\uparrow T_{\max}} \int_\Omega |\nabla v|^2\varphi = +\infty. \tag{49}$$

Proof. Recall (44) and put

$$F(t) = W_\varphi(t) - \int_0^t R_1(u,v,\varphi)\,ds - \int_\Omega u\varphi.$$

Relations (9), (40), and (41) imply

$$\left|\int_\Omega u\varphi\right| \le \|u_0\|_{L^1(\Omega)} \qquad \text{and} \qquad \sup_{0\le t<T_{max}} |R_1(u,v,\varphi)| < +\infty.$$

By Lemma 10, F is monotone decreasing in $[0, T_{max})$ and (48) follows. Then we have

$$\int_\Omega (u\log u)\varphi \le W^* + \int_\Omega uv\varphi,$$

and Lemma 5 gives

$$\limsup_{t\uparrow T_{\max}} \int_\Omega uv\varphi = +\infty.$$

By using Young's inequality we have

$$\begin{aligned} a\int_\Omega uv\varphi &\le \int_\Omega (u\log u)\,\varphi + \frac{1}{e}\int_\Omega e^{av}\varphi \\ &\le W_\varphi + \int_\Omega uv\varphi + \frac{1}{e}\int_\Omega e^{av}\varphi \le W^* + \int_\Omega uv\varphi + \frac{1}{e}\int_\Omega e^{av}\varphi, \end{aligned}$$

and hence

$$(a-1)\int_\Omega uv\varphi \le \frac{1}{e}\int_\Omega e^{av}\varphi + W^*.$$

Therefore, we have

$$\limsup_{t\uparrow T_{\max}} \int_\Omega e^{av}\varphi = +\infty$$

for $a > 1$, which implies (49) by the following lemma.

Lemma 12. *Let $a > 0$, $x_0 \in \mathcal{B}$, and $\varphi = \varphi_{x_0,R',R}$ for $0 < R' < R \ll 1$. Then, the inequality*

$$\int_\Omega e^{av}\varphi \leq C_8 \exp\left(\frac{a^2}{8\pi}\int_\Omega |\nabla v|^2 \varphi\right) \tag{50}$$

holds on $[0, T_{\max})$. If $x_0 \in \Omega$, then we have

$$\int_\Omega e^{av}\varphi \leq C_9 \exp\left(\frac{a^2}{16\pi}\int_\Omega |\nabla v|^2 \varphi\right). \tag{51}$$

Proof. We recall the following inequalities by Moser [16] and Chang and Yang [5]: *There exists a constant K determined by Ω such that*

$$\log\left(\int_\Omega e^w\right) \leq \frac{1}{2\pi^*}\|\nabla w\|^2_{L^2(\Omega)} + \frac{1}{|\Omega|}\int_\Omega w + K$$

for $w \in X$, where

$$\pi^* = \begin{cases} 4\pi & \text{if} \quad X = H^1(\Omega) \\ 8\pi & \text{if} \quad X = H^1_0(\Omega). \end{cases}$$

First, we take $x_0 \in \mathcal{B} \cap \partial\Omega$. It holds that

$$\sup_{0 \leq t < T_{max}} \|v(\cdot,t)\|_{C^1\left(\overline{B_R(x_0)\cap\Omega}\setminus B_{R'}(x_0)\right)} < +\infty$$

by (41). Therefore, we have

$$\begin{aligned}\int_\Omega e^{av}\varphi &\leq \int_{B(x_0,R')\cap\Omega} e^{av} + \int_{B_R(x_0)\cap\Omega\setminus B_{R'}(x_0)} e^{av} \leq \int_\Omega e^{av\varphi} + C_{10} \\ &\leq e^K \exp\left(\frac{a^2}{8\pi}\|\nabla(v\varphi)\|^2_{L^2(\Omega)} + \frac{a\|v\|_1}{|\Omega|}\right) + C_{10} \\ &\leq \left(e^{K+aC_{11}} + C_{10}\right)\exp\left(\frac{a^2}{8\pi}\int_\Omega |\nabla v|^2\varphi\right)\end{aligned}$$

by (17). This shows (50). A similar calculation gives (51) for $x_0 \in \mathcal{B} \cap \Omega$. The proof is complete.

The following lemma is a modification of [19], [3], [8].

Lemma 13. *We have*

$$\int_\Omega uv\varphi \le \int_\Omega (u\log u)\varphi + M_\varphi \log\left(\int_\Omega e^v\varphi\right) - M_\varphi \log M_\varphi, \tag{52}$$

where $M_\varphi = \int_\Omega u\varphi$.

Proof. Since $-\log s$ is convex, Jensen's inequality applies as

$$\begin{aligned} -\log\left(\frac{1}{M_\varphi}\int_\Omega e^v\varphi\right) &= -\log\left(\int_\Omega \frac{e^v}{u}\frac{u}{M_\varphi}\varphi\right) \\ \le \int_\Omega \left\{-\log\left(\frac{1}{u}e^v\right)\frac{u}{M_\varphi}\varphi\right\} &= -\frac{1}{M_\varphi}\int_\Omega \left\{u\log\left(\frac{e^v}{u}\right)\varphi\right\}. \end{aligned}$$

This means (52).

We are ready to give the following.

Proof of Lemma 9. We have proven that $\lim_{t\uparrow T_{\max}} \|u\varphi\|_{L^1(\Omega)}$ exists. Suppose

$$\lim_{t\uparrow T_{\max}} M_\varphi(t) = \lim_{t\uparrow T_{\max}} \|u\varphi\|_{L^1(\Omega)} < m_*. \tag{53}$$

In the case that $x_0 \in \Omega$ we have (51). Inequality (52) implies

$$\begin{aligned} &\frac{1}{2}\int_\Omega (|\nabla v|^2 + v^2)\varphi = W_\varphi - \int_\Omega (u\log u - uv)\varphi \\ &\le W_\varphi + M_\varphi \log\left(\int_\Omega e^v\varphi\right) - M_\varphi \log M_\varphi \le W^* + \frac{M_\varphi}{16\pi}\int_\Omega |\nabla v|^2\varphi + M_\varphi \log\frac{C_9}{M_\varphi} \end{aligned}$$

by (48). It follows that

$$\frac{1}{2}\big(1 - \frac{M_\varphi}{8\pi}\big)\int_\Omega |\nabla v|^2\varphi \le W^* + M_\varphi \log\frac{C_9}{M_\varphi} \le C_{12}.$$

Therefore, (53) with $m_* = 8\pi$ gives

$$\limsup_{t\uparrow T_{\max}} \int_\Omega |\nabla v|^2\varphi < +\infty.$$

This contradicts (49). We have

$$\liminf_{t\uparrow T_{\max}} \int_{B_R(x_0)\cap\Omega} u(x,t)\,dx \ge \lim_{t\uparrow T_{\max}} \int_\Omega u(x,t)\varphi(x)\,dx \ge m_*.$$

The case $x_0 \in \partial\Omega$ can be treated similarly and the proof is complete.

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