NONLINEAR GEOMETRICAL OPTICS FOR OSCILLATORY WAVE TRAINS WITH A CONTINUOUS OSCILLATORY SPECTRUM

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Abstract. The frequency and the direction of propagation of an oscillatory wave train may be read on its oscillatory spectrum. Many works in geometrical optics allow the study of at most countable oscillatory spectra. In these works, the number of directions of propagation is therefore at most countable, while many physical effects would require a continuous infinity of directions of propagation. The goal of this paper is to make the nonlinear geometrical optics for wave trains with such a continuous oscillatory spectrum. This requires the introduction of new spaces, which are Wiener algebras associated to spaces of vector-valued measures with bounded total variation. We also make qualitative studies on the properties of wave trains with continuous oscillatory spectrum, and on the incidence of the nonlinearity on such oscillations. We finally suggest an application of the results of this paper to the study of both the spontaneous and the stimulated Raman scatterings.

1. Introduction

1.1. **Need for continuous oscillating spectra.** Geometrical optics are used in the study of the propagation of oscillations by semilinear hyperbolic systems. Until now, the general frame in the modeling of the oscillations was furnished by periodic or almost-periodic functions (see [11, 12]), whose oscillating spectrum plays an important role in such a study.

Let u be an almost-periodic function of $\mathbb{R}^D$ with values in $\mathbb{C}$,

$$a(x) = \sum_{\beta \in \mathbb{R}^D} a_\beta e^{i\beta \cdot x},$$

satisfying $\sum_{\beta \in \mathbb{R}^D} |a_\beta| < \infty$. Its oscillating spectrum is given by $\sigma(a) := \{\beta \in \mathbb{R}^D, a_\beta \neq 0\}$, and is therefore at most countable. That is why the

almost-periodic frame allows us only to study oscillations whose oscillating spectrum is at most countable, when various physical phenomena would require "oscillations" whose spectrum is a continuous subset of the characteristic variety associated to the problem. This is the case for instance of many scattering effects due to parametric instabilities of electromagnetic waves in plasmas (see [7]): Brillouin and Raman scattering, Compton scattering, etc. In fact, light scattering is likely to furnish many examples which require such "oscillations."

In this paper, we describe two examples of conical emission, the spontaneous and the stimulated Raman scattering.
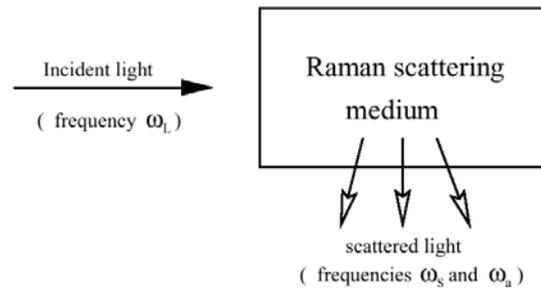
1.2. **Spontaneous and stimulated Raman scattering.** When an incident laser beam of given frequency ω_L meets a Raman-scattering medium, one assists to the creation of scattered light (see [4, 17]); the emission is then nearly isotropic.

The scattered light may have two frequencies, denoted by ω_S and ω_a, which correspond to Raman-Stokes and Raman-anti-Stokes scattering. One has $\omega_S < \omega_L < \omega_a$. Normally, Stokes lines are more intense than anti-Stokes lines, but remain quite weak. However, when the laser beam becomes very intense, the scattering may grow very efficient and is then called stimulated Raman-scattering. The light is then emitted in a narrow cone in the forward and in the backward directions.

In order to explain both effects, we use the three-level Maxwell-Bloch model. Then any beam of frequency ω and wave number k must satisfy $(\omega, k) \in \mathcal{C}$, where $\mathcal{C}$ is the characteristic variety associated to this model.

Take now a scattered Stokes light beam for instance; its frequency is ω_S, and it can be emitted in any direction k such that $(\omega_S, k) \in \mathcal{C}$. The set $\{(\omega_S, k), (\omega_S, k) \in \mathcal{C}\}$ should therefore be the oscillating spectrum of the scattered beam, and is usually a continuous (and hence not countable) subset of $\mathcal{C}$. Using almost-periodic functions to describe this phenomenon

would give priority to a countable (though possibly dense; see [15]) number of directions of propagation, without any physical reason. We will also see how the amplification in the stimulated Raman effect may be seen as a nonlinear instability. We will detail these examples at the end of this paper.

1.3. **Modeling.** We briefly introduce here the framework we use in our study. This framework should allow us to take into account both discrete and continuous spectra. Take again an almost-periodic function u of $\mathbb{R}^D$ with values in $\mathbb{C}$,

$$a(x) = \sum_{\beta \in \mathbb{R}^D} a_\beta e^{i\beta \cdot x} \quad \text{with} \quad \sum_{\beta \in \mathbb{R}^D} |a_\beta| < \infty.$$

Its Fourier transform $\lambda := \mathcal{F}_x a$ is a measure of bounded variation,

$$\lambda = \mathcal{F}_x a = \sum_{\beta \in \mathbb{R}^D} a_\beta \delta_\beta,$$

where δ_β is the Dirac delta function with mass at β. Its oscillating spectrum is given by $\sigma(u) = \text{Supp}\,\lambda$; this fact will be used to introduce continuous oscillating spectra.

Almost-periodic functions are inverse Fourier transforms of bounded variation measures, and their oscillating spectrum is the support of this Fourier transform. That is why we will consider the Wiener algebra of functions whose Fourier transform is a measure of bounded variation. The support of such measures can be either countable or continuous, thus providing a general frame for the study of oscillations.

2. Preliminaries in vector-measure theory

2.1. **Total variation.** In this section and in the following, B will denote a Banach space. A B-valued Borel measure on $\mathbb{R}^D$ is a countably additive set function λ defined on $\mathcal{B}(\mathbb{R}^D)$ (the Borel sets of $\mathbb{R}^D$), with values in B. To any B-valued Borel measure λ, we can associate a positive Borel measure $v(\lambda)$, called the *total variation* of λ, and defined for all Borel sets E as

$$v(\lambda)(E) = \sup \sum_{i=1}^n \|\lambda(E_i)\|_B,$$

where the supremum is taken over all finite sequences $\{E_i\}$ of disjoint sets in $\mathcal{B}(\mathbb{R}^D)$ with $E_i \subset E$.

The set function λ is of *bounded variation* if $v(\lambda)(\mathbb{R}^D) < \infty$. We denote by $\mathcal{BV}(\mathbb{R}^D, B)$ the set of all B-valued Borel measures of bounded variation

on $\mathbb{R}^D$. The total variation induces a norm on the space $\mathcal{BV}(\mathbb{R}^D, B)$, defined as $|\lambda|_{\mathcal{BV}} := v(\lambda)(\mathbb{R}^D)$, for all $\lambda \in \mathcal{BV}(\mathbb{R}^D, B)$.

The normed vector space $(\mathcal{BV}(\mathbb{R}^D, B), |\cdot|_{\mathcal{BV}})$ is then a Banach space.

2.2. **Integration in Banach spaces.** The Bochner integral (see [8]) is usually used when integrating vector functions with respect to scalar measures. When integrating scalar functions with respect to vector measures, a Dunford integral ([3, 9]) is generally used. In this paper we have to deal with those two cases. We even need to integrate vector functions with respect to vector measures, and that is why we will use the general Bartle integral (see [1]), which generalizes the Bochner and Dunford integrals. Unless otherwise specified, the results stated in this section may be found in [1]. In this section, B_1, B_2 and B_3 will denote three Banach spaces, and we assume that there is a bilinear mapping $(x, y) \mapsto x \cdot y$, defined on $B_1 \times B_2$ with values in B_3, such that $\|x \cdot y\|_{B_3} \leq k\|x\|_{B_1}\|y\|_{B_2}$ for some fixed, positive number k. The Bartle integral provides an integration of B_1-valued functions with respect to B_2-valued measures.

Let λ be an element of $\mathcal{BV}(\mathbb{R}^D, B_2)$. We first define the integration of λ-simple functions. Let $f : \mathbb{R}^D \to B_1$ be a λ-simple function

$$f = \sum_{i=1}^{n} x_i \chi_{E_i},$$

where the x_i are elements of B_1 and the E_i are disjoint Borel sets of $\mathbb{R}^D$. For any Borel set E of $\mathbb{R}^D$, we define the integral of f over E by

$$\int_E f(z)\lambda(dz) := \sum_{i=1}^{n} x_i \cdot \lambda(E_i \cap E).$$

This is an element of B_3 which does not depend on the representation we have chosen for f. This integral is extended to a broader class of functions than the λ-simple functions.

Proposition 1. *f is said to be λ-integrable if and only if there exists a sequence $\{f_n\}$ of λ-simple functions such that*

- f_n converges to f λ-almost-everywhere,

- the sequences $\lambda_n(E) := \int_E f_n(z)\lambda(dz)$ converges in B_3 for all $E \in \mathcal{B}(\mathbb{R}^D)$.

The integral of f over E is then defined as

$$\int_E f(z)\lambda(dz) := \lim_{n \to \infty} \int_E f_n(z)\lambda(dz).$$

Remark. i) If B_2 is scalar and $B_1 = B_3$, then the Bartle integral is a generalization of the usual Bochner integral, since if f is Bochner integrable, it is also Bartle integrable to the same value.

ii) If B_1 is scalar and $B_2 = B_3$, then the Bartle integral also generalizes the results of the integral defined in this framework in [3].

We now give a few results we will use in further sections. The first proposition (see [8] for the proof) gives a commutativity property between the integral and bounded linear operators, and a convenient way to compute the total variation of vector measures defined by integrals.

Proposition 2. *Suppose B_2 is scalar and $B_1 = B_3$. Let f be a λ-integrable function. Then*
i) *if T is a bounded linear operator on B_1 to another Banach space B, then Tf is λ-integrable, and*

$$\int_E Tf(z)\lambda(dz) = T\int_E f(z)\lambda(dz), \qquad \forall E \in \mathcal{B}(\mathbb{R}^D);$$

ii) *if in addition the scalar measure λ is positive, then the vector measure F defined for all Borel sets E as $F(E) := \int_E f(z)\lambda(dz)$ is in $\mathcal{BV}(\mathbb{R}^D, B_2)$ and one has*

$$v(F)(E) = \int \|f(z)\|_{B_1}\lambda(dz), \qquad \forall E \in \mathcal{B}(\mathbb{R}^D).$$

Thanks to the general Bartle integral (integration of vector functions with respect to vector measures), we can easily define the product $T\lambda$ of an operator T and a vector measure.

Proposition 3. *Let λ belong to $\mathcal{BV}(\mathbb{R}^D, B_2)$, and T be a λ-integrable function defined on $\mathbb{R}^D$ with values in $B_1 := \mathcal{L}_C(B_2, B_3)$ (continuous linear operators from B_2 to B_3). Then the product $T\lambda$ defined as*

$$T\lambda(E) = \int_E T(z)\lambda(dz), \quad \forall E \in \mathcal{B}(\mathbb{R}^D),$$

is an element of $\mathcal{BV}(\mathbb{R}^D, B_3)$.

One cannot generalize the Lebesgue dominated convergence theorem for the integration of vector-valued functions with respect to vector-valued measures (it is false for instance that the integrability of $\|f\|$ implies the integrability of f), but we have the following generalization of Vitali theorem.

Theorem 1 (Vitali)**.** *Let $\lambda \in \mathcal{BV}(\mathbb{R}^D, B)$. Let $\{f_n\}$ be a sequence of integrable functions which are such that*

i) *the sequence $\{f_n\}$ converges $v(\lambda)$-almost-everywhere to f;*

ii) *given $\varepsilon > 0$ there is a $\delta > 0$ such that $E \in \mathcal{B}(\mathbb{R}^D)$ and $v(\lambda)(E) < \delta$*
imply

$$\left| \int_E f_n(z)\lambda(dz) \right| < \varepsilon, \quad n = 1, 2, \ldots.$$

Then we may conclude that f is λ-integrable and

$$\int_E f(z)\lambda(dz) = \lim_{n \to \infty} \int_E f_n(z)\lambda(dz),$$

uniformly for $E \in \mathcal{B}(\mathbb{R}^D)$.

However, when either B_1 or B_2 is scalar, the Lebesgue dominated convergence theorem and its consequences remain true.

Theorem 2 (Lebesgue)**.** *If either B_1 or B_2 is scalar, then*

i) *Lebesgue's dominated convergence theorem is valid;*

ii) *Lebesgue's theorem on the continuity of functions defined by integrals*
is true.

2.3. **Convolutions of vector-valued measures.** Before studying convolutions of vector-valued measures, we have to define the product of such measures. In this section, we still denote by B_1, B_2 and B_3 three Banach spaces and by $\cdot$ a bilinear mapping with the properties stated in the above section. Let λ be in $\mathcal{BV}(\mathbb{R}^D, B_1)$ and μ be in $\mathcal{BV}(\mathbb{R}^D, B_2)$. There exists an unique measure $\nu \in \mathcal{BV}(\mathbb{R}^{2D}, B_3)$ such that for any measurable rectangle $E \times F \in \mathcal{B}(\mathbb{R}^D) \times \mathcal{B}(\mathbb{R}^D)$, one has $\nu(E \times F) := \lambda(E) \cdot \mu(F)$.

Definition 1. This measure ν is called the *product measure* of λ and μ, and satisfies the following inequality:

$$|\nu|_{\mathcal{BV}} \leq k|\lambda|_{\mathcal{BV}}|\mu|_{\mathcal{BV}}.$$

We denote the product measure ν by $\lambda \cdot \mu$.

We now define the convolution of a B_1-valued measure and a B_2-valued measure, as done in [10].

Definition 2. Let $\lambda \in \mathcal{BV}(\mathbb{R}^D, B_1)$ and $\mu \in \mathcal{BV}(\mathbb{R}^D, B_2)$. For all $E \in \mathcal{B}(\mathbb{R}^D)$, let us define $\lambda \overset{\cdot}{*} \mu(E)$ as $\lambda \overset{\cdot}{*} \mu(E) := (\lambda \cdot \mu)(E_2)$, where $E_2 = \{(x, y) : x + y \in E\}$.

Remark. The convolution so defined depends on the bilinear mapping $(x, y) \mapsto x \cdot y$, and in particular, if this mapping is not symmetric (respectively associative), convolution is not a commutative (respectively associative) law.

The following property (see [10]) is a Hölder inequality with respect to the norm $|\cdot|_{\mathcal{BV}}$, and will be used when studying Wiener algebras.

Proposition 4. *For all $\lambda \in \mathcal{BV}(\mathbb{R}^D, B_1)$ and $\mu \in \mathcal{BV}(\mathbb{R}^D, B_2)$, one has $\lambda \overset{.}{*} \mu \in \mathcal{BV}(\mathbb{R}^D, B_3)$ and $v(\lambda \overset{.}{*} \mu) \leq k v(\lambda) * v(\mu)$, and therefore*

$$|\lambda \overset{.}{*} \mu|_{\mathcal{BV}} \leq k |\lambda|_{\mathcal{BV}} |\mu|_{\mathcal{BV}}.$$

We now introduce the Fourier transform of bounded-variation measures.

Definition 3. Let $\lambda \in \mathcal{BV}(\mathbb{R}^D, B_1)$. Its Fourier transform $\mathcal{F}\lambda$ is the element of $C(\mathbb{R}^D, B_1)$ defined as

$$\mathcal{F}\lambda(X) := \int e^{iX\cdot\xi} \lambda(d\xi), \quad \text{for all } X \in \mathbb{R}^D.$$

Fourier transforms satisfy the following property, which is characteristic of Wiener algebras.

Proposition 5. *Let λ and μ be in $\mathcal{BV}(\mathbb{R}^D, B_1)$. One has*

$$\mathcal{F}(\lambda \overset{.}{*} \mu) = \mathcal{F}\lambda \cdot \mathcal{F}\mu,$$

2.4. **The Radon-Nikodym property.** The Radon-Nikodym property plays an important role in this paper; this is not surprising since it is closely linked to representation problems (it is in a sense equivalent to the Riesz representation theorem) which naturally occur in the present study.

This property is used for instance to provide a representation of a vector measure under the form of the integral of a vector function with respect to a scalar measure. Such a representation proves easier to handle, especially when estimating the total variation of vector-valued measures.

The Radon-Nikodym property is stated like this:

Definition 4 (Radon-Nikodym property)**.** A Banach space B satisfies the Radon-Nikodym property if for all finite positive Borel measures μ on $\mathbb{R}^D$ and all $\lambda \in \mathcal{BV}(\mathbb{R}^D, B)$ which are μ-continuous, i.e., such that

$$\lim_{\mu(E)\to 0} \lambda(E) = 0 \quad \text{in } B,$$

there exists a B-valued integrable function r_λ such that

$$\lambda(E) = \int_E r_\lambda(z) \mu(dz), \quad \text{for all } E \in \mathcal{B}(\mathbb{R}^D).$$

This property is not true in general: it fails, for instance, for c_0-valued measures. However, we need it only for Sobolev spaces H^s, in which case it is satisfied thanks to the following theorem (see [8] for instance).

Theorem 3 (Phillips). *Reflexive   Banach   spaces   satisfy   the   Radon-Nikodym property.*

As a consequence of the Radon-Nikodym property, we give the following properties.

Proposition 6. *Let B be a reflexive Banach space and $\lambda \in \mathcal{BV}(\mathbb{R}^D, B)$; then*
   i) *there exists a B-valued function r_λ, which is $v(\lambda)$-integrable and satisfies $\|r_\lambda\| = 1$ $v(\lambda)$-almost-everywhere, such that*

$$\lambda(E) = \int_E r_\lambda(z) v(\lambda)(dz), \qquad \forall E \in \mathcal{B}(\mathbb{R}^D);$$

ii) *when $B = H^s(\mathbb{R}^d_y)$, then the Fourier transform $\widehat{\lambda(E)}$ with respect to the variable y is in $B_1 := L^2_s(\mathbb{R}^d)$, the L^2-space with weight $w(\eta) := (1 + |\eta|^2)^s$. The set function $\widehat{\lambda}$ defined as $\widehat{\lambda}(E) := \widehat{\lambda(E)}$ is in $\mathcal{BV}(\mathbb{R}^D, B_1)$ and is written*

$$\widehat{\lambda}(E) = \int_E \widehat{r_\lambda(z)} v(\lambda)(dz).$$

Moreover, one has $v(\widehat{\lambda}) = v(\lambda)$.

Proof. i) We just have to notice that $v(\lambda)$ is a finite, positive Borel measure and that λ is $v(\lambda)$-continuous. The Radon-Nikodym property then gives the desired representation of $\lambda(E)$ since B is a reflexive Banach space.
The total variation of λ is given by

$$v(\lambda)(E) = \int_E \|r_\lambda(z)\| v(\lambda)(dz), \qquad \forall E \in \mathcal{B}(\mathbb{R}^D),$$

and therefore $\|r_\lambda\| = 1$, $v(\lambda)$-almost-everywhere.
   ii) Since the Fourier transform defines a bounded operator on B to B_1, we deduce from Proposition 2 that

$$\widehat{\lambda}(E) = \int_E \widehat{r_\lambda(z)} v(\lambda)(dz).$$

One then has

$$v(\widehat{\lambda})(E) = \int_E \|\widehat{r_\lambda(z)}\|_{L^2_s} v(\lambda)(dz) = \int_E \|r_\lambda(z)\|_{H^s} v(\widehat{\lambda})(dz) = v(\lambda)(E).$$

3. Semilinear hyperbolic systems for oscillations with continuous oscillating spectrum

We consider semilinear hyperbolic systems of the type

$$L^\varepsilon \mathfrak{u}^\varepsilon + F(\mathfrak{u}^\varepsilon, \mathfrak{u}^\varepsilon) = 0, \quad \mathfrak{u}^\varepsilon|_{t=0} = \mathfrak{u}_0^\varepsilon(y), \tag{1}$$

where $L^\varepsilon(\partial_x) := \partial_t + \sum_{i=1}^d A_i \partial_{y_i} + L_0/\varepsilon$, and the $N \times N$ matrices A_i are symmetric, while L_0 is skew-symmetric. F denotes a bilinear mapping defined on $\mathbb{C}^N \times \mathbb{C}^N$ with values in $\mathbb{C}^N$.

We consider the case of initial data oscillating with a possibly continuous oscillating spectrum.

3.1. **The spaces.** From now on, we write $x = (t, y)$, where $t \in \mathbb{R}$ and $y \in \mathbb{R}^d$, and similarly, $X = (T, Y) \in \mathbb{R}^{1+d}$. We consider Fourier transforms with respect to y and to X. We will always denote by η the dual variable of y, and by ξ the dual variable of X. The letter s will always denote a positive real number such that $s > d/2$.

We introduce some spaces.

Definition 5. i) The space A_0^s denotes the set of functions defined on $\mathbb{R}_y^d \times \mathbb{R}_X^{1+d}$ with values in $\mathbb{C}^N$ whose Fourier transform with respect to the X variable is in $\mathcal{BV}(\mathbb{R}_\xi^{1+d}, H^s(\mathbb{R}_y^d)^N)$.

ii) The space $A_{\underline{t}}^s$ denotes the set of functions defined on $\mathbb{R}_x^{1+d} \times \mathbb{R}_X^{1+d}$ with values in $\mathbb{C}^N$ whose Fourier transform with respect to the X variable is in $\mathcal{C}([0, \underline{t}], \mathcal{BV}(\mathbb{R}_\xi^{1+d}, H^s(\mathbb{R}_y^d)^N))$, where $\underline{t} > 0$.

The spaces A_0^s and $A_{\underline{t}}^s$ are equipped with the norms

$$\begin{aligned}
\|f\|_{A_0^s} &:= |\mathcal{F}_X f|_{\mathcal{BV}}, & f \in A_0^s, \\
\|f\|_{A_{\underline{t}}^s} &:= \sup_{0 \le t \le \underline{t}} |\mathcal{F}_X f(t, \cdot)|_{\mathcal{BV}}, & f \in A_{\underline{t}}^s.
\end{aligned}$$

Proposition 7. i) *The bilinear mapping F defined on $\mathbb{C}^N \times \mathbb{C}^N$ extends to a continuous bilinear mapping on $H^s(\mathbb{R}^d)^N \times H^s(\mathbb{R}^d)^N$ with values in $H^s(\mathbb{R}^d)^N$.*

ii) *The normed spaces $(A_0^s, \|\cdot\|_{A_0^s})$ and $(A_{\underline{t}}^s, \|\cdot\|_{A_{\underline{t}}^s})$ are complete, and there exists a constant $k > 0$ (independent of $\underline{t}$) such that*

$$\begin{aligned}
\|F(f, g)\|_{A_0^s} &\le k\|f\|_{A_0^s}\|g\|_{A_0^s}, & \forall f, g \in A_0^s, \tag{2} \\
\|F(f, g)\|_{A_{\underline{t}}^s} &\le k\|f\|_{A_{\underline{t}}^s}\|g\|_{A_{\underline{t}}^s}, & \forall f, g \in A_{\underline{t}}^s. \tag{3}
\end{aligned}$$

Proof. i) Since we have chosen $s > d/2$, $H^s(\mathbb{R}^d)$ is an algebra and there exists a constant $k' > 0$ such that for all $u, v \in H^s(\mathbb{R}^d)$, one has $\|uv\|_s \leq k'\|u\|_s\|v\|_s$. It is then straightforward to see that F is indeed continuous on $H^s(\mathbb{R}^d)^N \times H^s(\mathbb{R}^d)^N$ with values in $H^s(\mathbb{R}^d)^N$; i.e., there exists $k > 0$ such that for all $u, v \in H^s(\mathbb{R}^d)^N$, one has

$$\|F(u,v)\|_{H^s} \leq k\|u\|_{H^s}\|v\|_{H^s}.$$

ii) We can therefore use the results of Section 2.3 on product vector measures and convolution. We will take $B_1 = B_2 = B_3 = H^s(\mathbb{R}^d)^N$ and $x{\cdot}y := F(x,y)$. For all $f, g \in A_{\underline{t}}^s$, one then has $\mathcal{F}_X F(f,g)(t) = \mathcal{F}_X f(t) \overset{F}{*} \mathcal{F}_X g(t)$, and thanks to Proposition 4,

$$|\mathcal{F}_X F(f,g)(t)|_{\mathcal{BV}} \leq k|\mathcal{F}_X f(t)|_{\mathcal{BV}}|\mathcal{F}_X g(t)|_{\mathcal{BV}}.$$

The desired inequality (3) is then straightforward.

   We now prove that $(A_{\underline{t}}^s, \|\cdot\|_T)$ is complete. Let $\{f_n\}$ be a Cauchy sequence in $A_{\underline{t}}^s$. Then $\{\mathcal{F}_X f_n\}$ is a Cauchy sequence in $C([0,\underline{t}], \mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N))$, which is complete. Therefore, there exists $\lambda \in C([0,\underline{t}], \mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N))$ such that $\mathcal{F}_X f_n \to \lambda$ uniformly in t for the norm of the total variation. If we define $f(t, y, X) := \int_{\mathbb{R}^{1+d}} e^{iX\cdot\xi}\lambda(t, d\xi)$, then $f_n \to f$ in $A_{\underline{t}}^s$, which proves that $A_{\underline{t}}^s$ is complete. $\square$

3.2. **Almost-periodic functions and density functions.** In this section, we give two important classes of functions which are in the spaces introduced above. The first of them has already been discussed in the Introduction and is the space of almost-eriodic functions; the framework defined in this paper thus generalizes the usual frame of [11] and [12].

Definition 6. (*Almost-periodic functions*) A Wiener almost-periodic function with coefficients in B (where B denotes $H^s(\mathbb{R}^d)^N$ or $C([0,\underline{t}], H^s(\mathbb{R}^d)^N)$), is a function defined by an absolutely convergent series

$$a(X) := \sum_{\beta \in \mathbb{R}^{1+d}} a_\beta e^{i\beta\cdot X}, \qquad \sum_{\beta \in \mathbb{R}^{1+d}} \|a_\beta\|_B < \infty.$$

   While almost-periodic functions have necessarily a countable oscillating spectrum, this is not the case of the density functions we introduce now.

Definition 7. (*Density functions*) Let $\mathcal{M}$ be a submanifold of $\mathbb{R}^{1+d}$ of dimension n and α in $L^1(\mathcal{M}, B)$, where B denotes either $H^s(\mathbb{R}^d)^N$ or $C([0,\underline{t}],$

$H^s(\mathbb{R}^d)^N$). A density function of support $\mathcal{M}$ and density α is a function f defined as

$$f(X) = \int_{\mathcal{M}} e^{iX\xi} \alpha(\xi) \sigma(d\xi),$$

where σ denotes the Lebesgue measure of $\mathcal{M}$.

We can now formulate the following proposition.

Proposition 8. i) *Almost periodic functions with coefficients in $H^s(\mathbb{R}^d)^N$ (respectively in $C([0,\underline{t}], H^s(\mathbb{R}^d)^N)$) belong to A_0^s (respectively to $A_{\underline{t}}^s$).*

ii) *Density functions with support a submanifold $\mathcal{M}$ and with density $\alpha \in L^1(\mathcal{M}, B)$, where $B = H^s(\mathbb{R}^d)^N$ (respectively $C([0,\underline{t}], H^s(\mathbb{R}^d)^N)$), belong to A_0^s (respectively to $A_{\underline{t}}^s$).*

Proof. Let a be an almost-periodic function with coefficients in $H^s(\mathbb{R}^d)^N$ (resp. in $C([0,\underline{t}], H^s(\mathbb{R}^d)^N)$).

$$a = \sum_{\beta \in \mathbb{R}^{1+d}} a_\beta e^{i\beta \cdot X} \quad \text{with} \quad \sum_{\beta \in \mathbb{R}^{1+d}} \|a_\beta\| < \infty,$$

where the norm is taken in $H^s(\mathbb{R}^d)^N$ (respectively in $C([0,\underline{t}], H^s(\mathbb{R}^d)^N)$). The Fourier transform of a is thus given by $\mathcal{F}_X a = \sum_\beta a_\beta \delta_\beta$, which is in $\mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N)$ (respectively in $C([0,\underline{t}], \mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N))$) and a is therefore in A_0^s (respectively $A_{\underline{t}}^s$).

Let us now consider a density function f of support $\mathcal{M}$ and density $\alpha \in L^1(\mathcal{M}, B)$, where $B = H^s(\mathbb{R}^d)^N$ (respectively $C([0,\underline{t}], H^s(\mathbb{R}^d)^N)$).

We have to prove that the set function $\lambda = \mathcal{F}_X f$ defined in $\mathcal{B}(\mathbb{R}^{1+d})$ as

$$\lambda(E) := \int_{\mathcal{M}} \chi_E(\xi) \alpha(\xi) \sigma(d\xi),$$

where χ_E is the characteristic function of the Borel set E, is in the space $\mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N)$ (resp. in $C([0,\underline{t}], \mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N))$). This is the case thanks to the results of Section 2. $\qquad \square$

Remark. i) Let f be a density function with support $\mathcal{M}$ and density $\alpha \in L^1(\mathcal{M}, (C([0,T], H^s(\mathbb{R}^d)^N)))$; then one has

$$\|f\|_{A_{\underline{t}}^s} = \int_{\mathcal{M}} \|\alpha(\xi)\|_{C([0,\underline{t}], H^s)} \sigma(d\xi).$$

ii) An almost-periodic function can also be seen as a density function whose support is a submanifold of dimension 0.

Example. i) Fourier expansions of the form $\sum_{n \in \mathbb{Z}} a_n e^{inX}$ where the Fourier coefficients a_n belong to $H^s(\mathbb{R}^d)^N$ and satisfy $\sum_n \|a_n\|_{H^s} < \infty$, are in A_0^s since they are (almost-) periodic functions.

ii) The space A_0^s contains Bessel expansions. We assume here that $d = 2$. Let the coefficients a_n be as above, and consider now the density function f whose support is the circle $C((1,0,0);1) \subset \mathbb{R}^{1+d}$, defined as

$$f(y, X) = \frac{1}{2\pi} \int_0^{2\pi} e^{iX \cdot (1, \cos\theta, \sin\theta)} \sum_{n \in \mathbb{Z}} a_n e^{in\theta} d\theta.$$

Denoting $X = (T, Y) = (T, Y_1, Y_2)$, one then has

$$f(y, X) = e^{iT} \sum_{n \in \mathbb{Z}} a_n J_n(|Y|) e^{in\phi(Y)},$$

where J_n denotes the nth Bessel function, and $\tan\phi(Y) = Y_1/Y_2$.

iii) When $d \geq 3$ we find spherical harmonics.

3.3. **Regularity properties.** We now give two regularity properties satisfied by elements of A_0^s and $A_{\underline{t}}^s$.

Proposition 9. *Let f be in A_0^s (respectively $A_{\underline{t}}^s$). Then*

i) *the function f belongs to $C(\mathbb{R}^d \times \mathbb{R}^{d+1})$ (respectively $C([0, \underline{t}] \times \mathbb{R}^d \times \mathbb{R}^{d+1})$). Moreover, f is bounded and there exists a positive number k' such that*

$$\|f\|_\infty \leq k' \|f\|_{A_0^s}, \quad (\text{respectively } \|f\|_\infty \leq k' \|f\|_{A_{\underline{t}}^s});$$

ii) *the function f^ε defined on $\mathbb{R}^d$ (respectively $\mathbb{R}^{d+1}$) as $f^\varepsilon(y) := f(y, 0, y/\varepsilon)$ (respectively $f^\varepsilon(x) := f(x, x/\varepsilon)$) belongs to $L^2(\mathbb{R}^d)^N$ (respectively $C([0, \underline{t}], L^2(\mathbb{R}^d)^N)$). Moreover, one has*

$$\|f^\varepsilon\|_{L^2} \leq \|f\|_{A_0^s} \quad (\text{respectively } \|f^\varepsilon\|_{C([0, \underline{t}]; L^2(\mathbb{R}^d))^N} \leq \|f\|_{A_{\underline{t}}^s}).$$

Proof. i) We will prove the result for $f \in A_{\underline{t}}^s$; the Fourier transform $\lambda := \mathcal{F}_X f$ then belongs to $C([0, \underline{t}], \mathcal{BV}(\mathbb{R}^{d+1}, H^s(\mathbb{R}^d)^N))$. Let $(x_0, X_0) \in \mathbb{R}^{2d+2}$; we want to prove that

$$\|f(x, X) - f(x_0, X_0)\|_{\mathbb{C}^N} \to 0, \quad \text{when } (x, X) \to (x_0, X_0).$$

One has

$$\begin{aligned} \|f(x, X) - f(x_0, X_0)\|_{\mathbb{C}^N} &\leq \|f(x, X) - f(t_0, y, X)\|_{\mathbb{C}^N} \\ &+ \|f(t_0, y, X) - f(t_0, y_0, X_0)\|_{\mathbb{C}^N} + \|f(t_0, y_0, X) - f(x_0, X_0)\|_{\mathbb{C}^N} \\ &= A + B + C. \end{aligned}$$

Let us prove that each of these terms converges to 0 when $(x, X) \to (x_0, X_0)$.

Term A: One can bound

$$A \leq |\lambda(t) - \lambda(t_0)|_{\mathcal{BV}}.$$

Since $\lambda \in C([0, \underline{t}], \mathcal{BV}(\mathbb{R}^{d+1}, H^s(\mathbb{R}^d)^N))$, one has $|\lambda(t) - \lambda(t_0)|_{\mathcal{BV}} \to 0$ when $t \to t_0$, and therefore $A \to 0$ when $(x, X) \to (x_0, X_0)$.

Term B: Thanks to Proposition 6, $\lambda(t_0)$ reads

$$\lambda(t_0)(E) = \int_E r_0(\xi) v(\lambda(t_0))(d\xi), \quad \forall E \in \mathcal{B}(\mathbb{R}^{d+1}),$$

where r_0 is an $H^s(\mathbb{R}^d)^N$-valued $v(\lambda(t_0))$-integrable function such that $\|r_0(\xi)\|_{H^s(\mathbb{R}^d)^N} = 1$ almost everywhere.

We can then write $f(t_0, y, X)$ in the form

$$f(t_0, y, X) = \int e^{iX \cdot \xi} r_0(\xi)(y) v(\lambda(t_0))(d\xi),$$

and therefore

$$\|f(t_0, y, X) - f(t_0, y_0, X)\|_{\mathbb{C}^N} = \int e^{iX \cdot \xi} (r_0(\xi)(y) - r_0(\xi)(y_0)) v(\lambda(t_0))(d\xi).$$

It is a consequence of Lebesgue's dominated convergence theorem that this expression converges to 0 when $(x, X) \to (x_0, X_0)$. We just have to check that we can use this theorem.

· For all $\xi \in \mathbb{R}^{1+d}$, the mapping $y \mapsto e^{iX \cdot \xi} r_0(\xi)(y)$ is continuous at y_0 since $H^s(\mathbb{R}^d) \subset C(\mathbb{R}^d)$ (we recall that $s > d/2$).

· For all $\xi \in \mathbb{R}^{1+d}$, one has $\|e^{iX \cdot \xi} r_0(\xi)(y)\|_{\mathbb{C}^N} \leq \|r_0(\xi)(\cdot)\|_\infty \leq k' \|r_0(\xi)\|_{H^s} \leq k'$. Since $v(\lambda(t_0)) \in \mathcal{BV}(\mathbb{R}^{d+1}, \mathbb{R})$, the constants are $v(\lambda(t_0))$-integrable, and the domination hypothesis is thus fulfilled.

We may then conclude that $B \to 0$.

Term C: One has $C \to 0$ with the same proof as for the B term.

We have then the convergence to 0 of A, B and C, and the continuity of f follows.

In order to prove the boundedness property, we use Proposition 6 to write f in the form

$$f(t, y, X) = \int e^{iX \cdot \xi} r_t(\xi) v(\lambda(t))(d\xi),$$

which yields

$$\|f(t,y,X)\|_{\mathbb{C}^N} \leq \int \|r_t(\xi)\|_\infty v(\lambda(t))(d\xi) \leq k' \int \|r_t(\xi)\|_{H^s} v(\lambda(t))(d\xi)$$
$$= k'|\lambda(t)|_{\mathcal{BV}},$$

and the desired inequality follows.

ii) Here again, we will prove the result for $f \in A_{\underline{t}}^s$. Thanks to Proposition 6, we can write

$$f(t,y,X) = \int e^{iX\cdot\xi} r_t(\xi)(y)v(\lambda(t))(d\xi),$$

where for all $\xi \in \mathbb{R}^{d+1}$, one has $r_t(\xi) \in H^s(\mathbb{R}^d)^N$. In particular, one has

$$f(t,y,x/\varepsilon) = \int e^{i\frac{x\cdot\xi}{\varepsilon}} r_t(\xi)(y)v(\lambda(t))(d\xi),$$

and thus

$$\|f(t,\cdot,t/\varepsilon,\cdot/\varepsilon)\|_{L^2} = \|\int e^{i\frac{(t,\cdot)\cdot\xi}{\varepsilon}} r_t(\xi)(\cdot)v(\lambda(t))(d\xi)\|_{L^2}$$
$$\leq \int \|r_t(\xi)\|_{L^2} v(\lambda(t))(d\xi) \leq \int \|r_t(\xi)\|_{H^s} v(\lambda(t))(d\xi) = |\lambda(t)|_{\mathcal{BV}}.$$

Point ii) of the lemma then follows. $\qquad\square$

3.4. **Solving the Cauchy problem (1).** As in [13], we look for solutions to (1) of the form

$$\mathfrak{u}^\varepsilon(x) := u^\varepsilon(x,x/\varepsilon), \tag{4}$$

with $u^\varepsilon \in A_{\underline{t}}^s$. Plugging $\mathfrak{u}^\varepsilon$ defined by (4) into equation (1) formally yields

$$[L_1(\partial_x)u^\varepsilon(x,X) + \varepsilon^{-1}L(\partial_X)u^\varepsilon(x,X) + F(u^\varepsilon,u^\varepsilon)]_{X=x/\varepsilon} = 0, \tag{5}$$

where $L_1(\partial_x) := \partial_t + \sum_i A_i \partial_{x_i}$ and $L(\partial_X) := \partial_T + \sum_i A_i \partial_{X_i} + L_0$. We will therefore look for $u^\varepsilon \in A_{\underline{t}}^s$ such that

$$L_1(\partial_x)u^\varepsilon(x,X) + \varepsilon^{-1}L(\partial_X)u^\varepsilon(x,X) + F(u^\varepsilon,u^\varepsilon) = 0. \tag{6}$$

If u^ε were differentiable with respect to the X variable, then the derivation of (5) obtained by plugging (4) into equation (1) would be rigorous. But the only thing we know about u^ε is that it is continuous, thanks to Proposition 9. The fact that equation (6) implies that (4) defines a solution to problem (1) is stated by the following lemma.

Lemma 1. *Suppose that u^ε is in $A_{\underline{t}}^s$ and satisfies*

$$L_1(\partial_x)u^\varepsilon(x, X) + \varepsilon^{-1}L(\partial_X)u^\varepsilon(x, X) + F(u^\varepsilon, u^\varepsilon) = 0.$$

Then $\mathfrak{u}^\varepsilon := u^\varepsilon(x, x/\varepsilon)$ is a solution in $\mathcal{D}'([0, \underline{t}] \times \mathbb{R}^{1+d+dN})$ of

$$L^\varepsilon(\partial_x)\mathfrak{u}^\varepsilon + F(\mathfrak{u}^\varepsilon, \mathfrak{u}^\varepsilon) = 0.$$

Proof. Let $\{\rho_n\}$ defined on $\mathbb{R}_x^{1+d} \times \mathbb{R}_X^{1+d}$ be a regularizing sequence. We have

$$([L_1(\partial_x) + \varepsilon^{-1}L(\partial_X)]u^\varepsilon) * \rho_n(x, X) = -F(u^\varepsilon, u^\varepsilon) * \rho_n(x, X).$$

Thanks to Propositions 7 and 9, $F(u^\varepsilon, u^\varepsilon)$ is continuous, and therefore, $F(u^\varepsilon, u^\varepsilon) * \rho_n(x, X)$ converges uniformly in (x, X) to $F(u^\varepsilon, u^\varepsilon)(x, X)$. Therefore, $F(u^\varepsilon, u^\varepsilon) * \rho_n(x, x/\varepsilon)$ converges uniformly in x to $F(u^\varepsilon, u^\varepsilon)(x, x/\varepsilon)$. On the other hand, we have

$$([L_1(\partial_x) + \varepsilon^{-1}L(\partial_X)]u^\varepsilon) * \rho_n(x, X) = [L_1(\partial_x) + \varepsilon^{-1}L(\partial_X)](u^\varepsilon * \rho_n)(x, X),$$

and therefore

$$\begin{aligned}
([L_1(\partial_x) &+ \varepsilon^{-1}L(\partial_X)]u^\varepsilon) * \rho_n(x, x/\varepsilon) \\
&= [L_1(\partial_x) + \varepsilon^{-1}L(\partial_X)](u^\varepsilon * \rho_n)(x, x/\varepsilon) = L^\varepsilon(\partial_x)(u^\varepsilon * \rho_n(x, x/\varepsilon)).
\end{aligned}$$

We then have the convergence of $L(\partial_x)(u^\varepsilon * \rho_n(x, x/\varepsilon))$ to $-F(u^\varepsilon, u^\varepsilon)(x, x/\varepsilon)$, uniformly in x. Since $u^\varepsilon * \rho_n(x, x/\varepsilon)$ converges uniformly in x to $\mathfrak{u}^\varepsilon(x) = u^\varepsilon(x, x/\varepsilon)$, we can deduce that $L^\varepsilon(\partial_x)\mathfrak{u}^\varepsilon = -F(\mathfrak{u}^\varepsilon, \mathfrak{u}^\varepsilon)$ in $\mathcal{D}'$, and the lemma is thus proved. □

We can now formulate the following theorem.

Theorem 4. *Let u^0 be in A_0^s. There exists a positive real number $\underline{t} > 0$ such that for all $\varepsilon > 0$, the Cauchy problem*

$$L^\varepsilon\mathfrak{u}^\varepsilon + F(\mathfrak{u}^\varepsilon, \mathfrak{u}^\varepsilon) = 0, \quad \mathfrak{u}^\varepsilon|_{t=0} = u^0(y, 0, y/\varepsilon),$$

has a solution $\mathfrak{u}^\varepsilon$ in $C([0, \underline{t}] \times \mathbb{R}^d) \cap C([0, \underline{t}], L^2(\mathbb{R}^d)^N)$ which is unique. Moreover, $\mathfrak{u}^\varepsilon$ can be written in the form $\mathfrak{u}^\varepsilon = u^\varepsilon(x, x/\varepsilon)$ where $u^\varepsilon \in A_{\underline{t}}^s$ is uniquely determined by

$$L_1(\partial_x)u^\varepsilon + \varepsilon^{-1}L(\partial_X)u^\varepsilon + F(u^\varepsilon, u^\varepsilon) = 0, \quad u^\varepsilon|_{t=0} = u^0, \tag{7}$$

and its norm in $A_{\underline{t}}^s$ is bounded uniformly in ε.

Proof. Let us prove first that there exist a $\underline{t} > 0$ and a unique $u^\varepsilon \in A_{\underline{t}}^s$ such that (7) is satisfied. The proof will be by Picard iteration. The first iterate

u_1^ε solves the linear problem, which one gets by setting $F \equiv 0$ in (6). The other iterates are determined solving

$$L_1(\partial_x)u_{\nu+1}^\varepsilon + \varepsilon^{-1}L(\partial_X)u_{\nu+1}^\varepsilon = -F(u_\nu^\varepsilon, u_\nu^\varepsilon), \quad u|_{t=0} = u^0, \qquad (8)$$

for $\nu \geq 1$.

In order to solve these equations, we will need the following lemma:

Lemma 2. *Let $g \in A_{\underline{t}}^s$ and $f^0 \in A_0^s$. The linear problem*

$$L_1(\partial_x)f + \varepsilon^{-1}L(\partial_X)f = g, \quad f|_{t=0} = f^0, \qquad (9)$$

has a unique solution in $A_{\underline{t}}^s$. Moreover, one has

$$\|f\|_{A_{\underline{t}}^s} \leq \|f^0\|_{A_0^s} + \underline{t}\|g\|_{A_{\underline{t}}^s}. \qquad (10)$$

Proof. Let us denote by λ and μ the elements of $C([0,\underline{t}], \mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N))$ defined as $\lambda := \mathcal{F}_X f$ and $\mu := \mathcal{F}_X g$, and $\lambda_0 := \mathcal{F}_X f^0 \in \mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N)$. We also introduce $A(\eta) := \sum_{j=1}^d A_j \eta_j$ and $L(\xi) := \xi_0 I + \sum_{j=1}^d a_j \xi_j + \frac{1}{i} L_0$. Taking the Fourier transform of equation (9) with respect to X and y yields

$$(\partial_t + iA(\eta) + \varepsilon^{-1}iL(\xi))\widehat{\lambda} = \widehat{\mu},$$

where the notation $\widehat{\lambda}(E) := \widehat{\lambda(E)}$ has been introduced in Proposition 6. In the space $C([0,\underline{t}], \mathcal{S}'(\mathbb{R}^{1+d+dN}))$, we know that we have existence and uniqueness of a solution to this equation with initial conditions in $\mathcal{S}'(\mathbb{R}^{1+d+dN})$. This solution is given by the formula

$$\widehat{\lambda} = e^{-it(A(\eta)+\varepsilon^{-1}L(\xi))}\widehat{\lambda}_0 + \int_0^t e^{-i(t-u)(A(\eta)+\varepsilon^{-1}L(\xi))}\widehat{\mu(u)}\,du. \qquad (11)$$

We thus have to prove that the element of $C([0,\underline{t}], \mathcal{S}'(\mathbb{R}^{1+d+dN}))$ given by (11) is in fact in $C([0,\underline{t}], \mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N))$. Let $T_{t,u}$ be a function on $\mathbb{R}_\xi^{d+1}$ with values in $\mathcal{L}_C(L_s^2(\mathbb{R}^d), L_s^2(\mathbb{R}^d))$ defined as

$$T_{t,u}(\xi)h(\eta) := e^{-i(t-u)(A(\eta)+\varepsilon^{-1}L(\xi))}h(\eta), \quad \forall h \in L_s^2(\mathbb{R}^d).$$

Since $\|T_{t,u}(\xi)\| = 1$ for all ξ, $T_{t,u}$ is integrable. Moreover, we know that for all $t \in \mathbb{R}$, $\widehat{\lambda(t)} \in \mathcal{BV}(\mathbb{R}^{1+d}, L_s^2(\mathbb{R}^d)^N)$; Proposition 3 then allows us to conclude that $T_{t,u}\widehat{\mu(u)}$ defined as

$$T_{t,u}\widehat{\mu(u)}(E) := \int_E e^{-i(t-u)(A(\eta)+\varepsilon^{-1}L(\xi))}\widehat{\mu(u)}(d\xi), \quad \forall E \in \mathcal{B}(\mathbb{R}^{1+d}),$$

is in $\mathcal{BV}(\mathbb{R}^{1+d}, L_s^2(\mathbb{R}^d)^N)$; moreover, one has $|T_{t,u}\widehat{\mu}(u)|_{\mathcal{BV}} \leq |\widehat{\mu(u)}|_{\mathcal{BV}}$, and the set function $\widehat{\lambda}_1$ defined for all $E \in \mathcal{B}(\mathbb{R}^{1+d})$ as

$$\widehat{\lambda_1(t)}(E) := \int_0^t T_{t,u}\widehat{\mu(u)}(E)\, du$$

is in $\mathcal{BV}(\mathbb{R}^{1+d}, L_s^2(\mathbb{R}^d)^N)$ and satisfies

$$|\widehat{\lambda_1(t)}|_{\mathcal{BV}} \leq \int_0^t |T_{t,u}\widehat{\mu(u)}|_{\mathcal{BV}}\, du \leq \int_0^t |\widehat{\mu(u)}|_{\mathcal{BV}}\, du.$$

We also define the set function $\widehat{\lambda_2(t)} := T_{t,0}\widehat{\lambda}_0$; one then has

$$|\widehat{\lambda_2(t)}|_{\mathcal{BV}} \leq |\widehat{\lambda}_0|_{\mathcal{BV}}.$$

Since the solution $\widehat{\lambda}$ given by (11) is written $\widehat{\lambda} = \widehat{\lambda_1} + \widehat{\lambda_2}$, we have thus proved that it is in $C([0, \underline{t}], \mathcal{BV}(\mathbb{R}^{1+d}, L_s^2(\mathbb{R}^d)^N))$. We also have the estimate

$$|\widehat{\lambda(t)}|_{\mathcal{BV}} \leq |\widehat{\lambda}_0|_{\mathcal{BV}} + \int_0^t |\widehat{\mu(u)}|_{\mathcal{BV}}\, du. \tag{12}$$

We have seen in Proposition 6 that one has $v(\widehat{\alpha}) = v(\alpha)$ for any α belonging to $\mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N)$. Therefore, $|\widehat{\alpha}|_{\mathcal{BV}} = |\alpha|_{\mathcal{BV}}$, and it is thus easy to deduce from the above inequality that

$$\|f\|_{A_{\underline{t}}^s} \leq \|f^0\|_{A_0^s} + \underline{t}\|g\|_{A_{\underline{t}}^s}.$$

The proof of the lemma is then complete. $\qquad\square$

We now return to the Picard iteration. Thanks to Proposition 7, we know that there exists a constant $k > 0$ such that

$$\|F(f, f)\|_{A_{\underline{t}}^s} \leq k\|f\|_{A_{\underline{t}}^s}^2, \quad \forall f \in A_{\underline{t}}^s.$$

Applying Lemma 2 to (8) and using the above inequality yields

$$\|u_{\nu+1}^\varepsilon\|_{A_{\underline{t}}^s} \leq \|u^0\|_{A_0^s} + k\underline{t}\|u_\nu^\varepsilon\|_{A_{\underline{t}}^s}^2.$$

Taking $\underline{t}$ small enough, we can assume that for all ν we have

$$\|u_\nu\|_{A_{\underline{t}}^s} \leq 2\|u^0\|_{A_0^s}. \tag{13}$$

Let us now bound the difference $w_\nu^\varepsilon := u_{\nu+1}^\varepsilon - u_\nu^\varepsilon$. The function w_ν^ε solves

$$L(\partial_x)w_{\nu+1}^\varepsilon + \varepsilon^{-1}L(\partial_X)w_{\nu+1}^\varepsilon = F(w_\nu^\varepsilon, u_{\nu-1}^\varepsilon) - F(u_\nu^\varepsilon, w_{\nu-1}^\varepsilon), \quad w_\nu^\varepsilon|_{t=0} = 0.$$

Applying Lemma 2 to this system and using inequality (13) yields

$$\|w_\nu^\varepsilon\|_{A_{\underline{t}}^s} \leq 4k\underline{t}\|u^0\|_{A_0^s}\|w_{\nu-1}^\varepsilon\|_{A_{\underline{t}}^s},$$

and thus

$$\|u_{\nu+1} - u_\nu\|_{A_{\underline{t}}^s} \le 2\|u^0\|_{A_0^s}(4k\underline{t})^\nu.$$

If $\underline{t}$ is small enough so that $4k\underline{t} < 1$, then (u_ν^ε) is a Cauchy sequence in $A_{\underline{t}}^s$ which, thanks to Proposition 7, converges to an element u^ε of $A_{\underline{t}}^s$. This function u^ε solves system (7). Thanks to Lemma 1, the corresponding function $\mathfrak{u}^\varepsilon$ solves the Cauchy problem (1). Thanks to Proposition 9, we know that $\mathfrak{u}^\varepsilon \in C([0,\underline{t}] \times \mathbb{R}^d) \cap C([0,\underline{t}], L^2(\mathbb{R}^d)^N)$ and the existence part of the theorem is thus proved. Uniqueness of the solution $\mathfrak{u}^\varepsilon$ is proved using the classical L^2 uniqueness argument. $\qquad\square$

4. Geometrical optics for problem (1)

4.1. **Ansatz for the approximate solution.** Our goal is to determine an approximate solution v^ε to (1) in the form

$$\mathbf{v}^\varepsilon(x) := v_0(x, x/\varepsilon) + \varepsilon v_1(x, x/\varepsilon), \tag{14}$$

where v_0 and v_1 belong to $A_{\underline{t}}^s$. This means that we want to find an approximate solution which preserves the structure of the initial condition, i.e., which is an oscillation with possibly continuous oscillating spectrum. Plugging (14) into (1) yields the following formal expansion in powers of ε:

$$\begin{aligned}
L^\varepsilon \mathbf{v}^\varepsilon + F(\mathbf{v}^\varepsilon, \mathbf{v}^\varepsilon) &= \frac{1}{\varepsilon}[L(\partial_X)v_0(x, X)]_{X=x/\varepsilon} \\
&\quad + [L(\partial_X)v_1(x, X) + L_1(\partial_x)v_0(x, X) + F(v_0, v_0)]_{X=x/\varepsilon} \\
&\quad + \varepsilon[L_1(\partial_x)v_1(x, X) + F(v_0, v_1) + F(v_1, v_0)]_{X=x/\varepsilon}, \tag{15}
\end{aligned}$$

where $L(\partial_X) := \partial_T + \sum_{i=1}^d \partial_{Y_i} + L_0$, and $L_1(\partial_x) := \partial_t + \sum_i A_i \partial_{y_i}$. The strategy consists in searching for v_0 and v_1 in $A_{\underline{t}}^s$ such that the first two terms of the above expansion vanish. We will then prove that the associated function v^ε of (14) is an approximate solution to Problem (1) and that it satisfies a stability property.

4.2. **A few tools.** As always in geometrical optics, the characteristic variety associated to problem (1) and polarization conditions play an important role. We now introduce these objects.

Definition 8. i) We denote by $L(\xi)$ the symbol

$$L(\xi) := \xi_0 I + \sum_{i=1}^d \xi_i A_i + L_0/i = \xi_0 I + A(\xi_I) + L_0/i,$$

where $\xi_I := (\xi_1, \ldots, \xi_d)$ and $\xi := (\xi_0, \xi_I)$.

ii) The characteristic variety associated to problem (1) is the set $\mathcal{C}$ defined as $\mathcal{C} := \{\xi \in \mathbb{R}^{d+1} : \det L(\xi) = 0\}$. We will also denote by *Sing* $\mathcal{C}$ the set of the singular points of $\mathcal{C}$.

iii) Since $L(\xi)$ is a symmetric matrix for all ξ, denote by $\pi(\xi)$ the orthogonal projector on its kernel and by $Q(\xi)$ the partial inverse of $L(\xi)$, defined as $Q(\xi)\pi(\xi) = 0$, and $Q(\xi)L(\xi) = I - \pi(\xi)$.

If v is a monochromatic wave $v := v_\beta e^{i\beta \cdot X}$, then annihilating the first term in expansion (15) reads $L(\beta)v_\beta = 0$; that is, $\pi(\beta)v_\beta = v_\beta$. If v is an almost-periodic function, $v := \sum v_\beta e^{i\beta \cdot X}$, then this condition reads $\Pi v = v$, where the operator Π is defined on almost-periodic functions as

$$\Pi(\sum a_\beta e^{i\beta \cdot X}) := \sum \pi(\beta)a_\beta e^{i\beta \cdot X}.$$

This operator Π defined on almost-periodic functions is in fact equal to the Fourier multiplier $\pi(D_X)$, since for any almost-periodic function a, one has

$$\lambda := \mathcal{F}_X a = \sum_{\beta \in \mathbb{R}^{d+1}} a_\beta \delta_\beta,$$

and therefore,

$$\pi\lambda = \sum_{\beta \in \mathbb{R}^{d+1}} \pi(\beta)a_\beta \delta_\beta,$$

whose inverse Fourier transform is $\sum \pi(\beta)a_\beta e^{i\beta \cdot X}$. This yields $\pi(D_X)a = \Pi a$. This relation is used to define the operator Π on every function of $A_{\underline{t}}^s$. The only thing we need is to prove that the Fourier multiplier $\pi(D_X)$ is well defined on $A_{\underline{t}}^s$.

We similarly want to define the Fourier multiplier $Q(D_X)$ on $A_{\underline{t}}^s$. In order to do this, we first introduce the notion of π- and Q-regularity.

Definition 9. We will say that $f \in A_{\underline{t}}^s$ is π-regular (respectively Q-regular) if π (respectively Q) is λ-integrable, where $\lambda := \mathcal{F}_X f$.

We can now formulate the following proposition, which says when the Fourier multiplier $\pi(D_X)$ and $Q(D_X)$ are well defined.

Proposition 10. i) *Every element of $A_{\underline{t}}^s$ is π-regular and the Fourier multiplier $\pi(D_X)$ is well defined on $A_{\underline{t}}^s$.* ii) *The Fourier multiplier $Q(D_X)$ is well defined on Q-regular elements of $A_{\underline{t}}^s$.*

Proof. Let f belong to $A_{\underline{t}}^s$ and $\lambda := \mathcal{F}_X f$.

i) Since $\|\pi(\xi)\| \leq 1$ for all $\xi \in \mathbb{R}^{1+d}$, the function π is λ-integrable thanks to Proposition 3; i.e., f is π-regular. The product $\pi\lambda$ is therefore well defined in $C([0,\underline{t}], \mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N))$, and the Fourier multiplier is thus well defined on $A_{\underline{t}}^s$.

ii) The partial inverse $Q(\xi)$ is not bounded in norm, and thus not necessarily λ-integrable. That is why we have to assume that f is Q-regular. The proof is then similar to the proof of point i). $\square$

Remark. A measure λ such that $\pi\lambda = \lambda$ has its support in $\mathcal{C}$, since $\pi(\xi) = 0$ for all $\xi \notin \mathcal{C}$.

The following proposition gives a class of Q-regular functions.

Proposition 11. *Let $f \in A_{\underline{t}}^s$ and $\lambda := \mathcal{F}_X f$. Let us assume that* Supp λ *is such that $d(\mathrm{Supp}\ \lambda, \mathrm{Sing}\ \mathcal{C}) > 0$. Then f is Q-regular.*

Proof. For any $\xi_I \in \mathbb{R}^d$, there exist $m(\xi_I)$ functions τ_j such that

$$A(\xi_I) + L_0/i = -\sum_{j=1}^{m(\xi_I)} \tau_j(\xi_I)\pi(\tau_j(\xi_I), \xi_I).$$

Therefore one has

$$L(\xi) = L(\xi_0, \xi_I) = \sum_{j=1}^{m(\xi_I)} (\xi_0 - \tau_j(\xi_I))\pi(\tau_j(\xi_I), \xi_I)$$

and

$$Q(\xi) = \sum_{\tau_j(\xi_s) \neq \xi_0} \frac{1}{\xi_0 - \tau_j(\xi_I)} \pi(\tau_j(\xi_I), \xi_I).$$

Since $m(\xi_I)$ is constant and the functions τ_j are smooth on each connected component of $\{\xi_I \in \mathbb{R}^d : (\tau_j(\xi_I), \xi_I) \text{ is not singular}, j = 1, \ldots, m(\xi_I)\}$, one has $\xi_0 - \tau_j(\xi_I) \to 0$ if and only if $d(\xi = (\xi_0, \xi_I), \mathrm{Sing}\ \mathcal{C}) \to 0$. Thanks to the assumptions made in the proposition, we thus know that Q is bounded on Supp λ, and thus integrable. $\square$

Remark. This proposition proves very useful to ensure that an almost-periodic function or a density function is Q-regular.

4.3. **Annihilating $\mathbf{L}(\partial_{\mathbf{X}})\mathbf{v_0}$.** The equation $L(\partial_X)v_0 = 0$ is the equation we obtain when annihilating the ε^{-1} term in expansion (15).

The following lemma says that $\pi(D_X)$ is a good generalization of operator Π in the sense that the equivalence $L(\partial_X)a = 0 \iff \pi(D_X)a = a$, which is true for almost-periodic functions, remains true for every function in $A_{\underline{t}}^s$.

Lemma 3. *Let f be in $A_{\underline{t}}^s$. Then one has*

$$L(\partial_X)f = 0 \quad \text{if and only if} \quad \pi(D_X)f = f.$$

Proof. Since $f \in A_{\underline{t}}^s$, one has $\lambda \in C([0,\underline{t}], \mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N))$. We will prove the desired result for all $t \in [0,\underline{t}]$ and will use throughout this proof the notation λ instead of $\lambda(t)$. We recall that thanks to Proposition 6, λ can read

$$\lambda(E) = \int_E r_\lambda(\xi) v(\lambda)(d\xi), \quad \forall E \in \mathcal{B}(\mathbb{R}^{1+d}).$$

Since the condition $\pi(D_X)f = f$ can also read $\pi\lambda = \lambda$, it is therefore equivalent to

$$\int_E r_\lambda(\xi) v(\lambda)(d\xi) = \int_E \pi r_\lambda(\xi) v(\lambda)(d\xi), \quad \forall E \in \mathcal{B}(\mathbb{R}^{1+d}),$$

which means that $\pi(\xi) r_\lambda(\xi) = r_\lambda(\xi)$, $v(\lambda)$-almost everywhere, i.e., that one has $L(\xi) r_\lambda(\xi) = 0$, $v(\lambda)$-almost everywhere. This last condition also reads

$$\int_E L(\xi) r_\lambda(\xi) v(\lambda)(d\xi) = 0, \quad \forall E \in \mathcal{B}(\mathbb{R}^{1+d}).$$

It is then a consequence of Theorem 1 that this is also equivalent to

$$L \text{ is } \lambda\text{-integrable, and } \int_E L(\xi)\lambda(d\xi) = 0, \quad \forall E \in \mathcal{B}(\mathbb{R}^{1+d}),$$

which, thanks to Proposition 3, means that the product $L\lambda$ is well defined and is equal to 0, i.e., that $L(\partial_X)f$ is well defined and equal to 0. $\qquad\square$

The first term v_0 of our ansatz must therefore satisfy

$$\pi(D_X)v_0 = v_0 \tag{16}$$

in order to annihilate the ε^{-1} term in expansion (15). This condition is usually called the *polarization condition*.

Remark. From now on, we will assume that the initial condition u^0 satisfies the polarization condition

$$\pi(D_X)u^0 = u^0.$$

This condition is necessary for two reasons. The first one is because we want to have $v_0|_{t=0} = u^0$ together with (16). The second one is more subtle. We take in this paper initial conditions for problem (1) of the form $\mathfrak{u}^\varepsilon|_{t=0}(y) = u^0(y, 0, y/\varepsilon)$, with $u^0 \in A_0^s$. For instance, we can take $u^0(y, X) = g(y)e^{i(w,k)\cdot(T,Y)}$, with $g \in H^s(\mathbb{R}^d)^N$. One then has $\mathfrak{u}^\varepsilon|_{t=0}(y) = g(y)e^{ik\cdot y/\varepsilon}$. In the usual frame, such an initial condition gives birth to more than one oscillation (as many as the number of sheets of the characteristic variety $\mathcal{C}$ above

k), while in this paper we have implicitly chosen the oscillation associated to (ω, k). Such a choice corresponds to the above polarization condition of the initial condition.

4.4. **Annihilating the ε^0 term in expansion (15).** Annihilating the ε^0 term in expansion (15) yields the following equation:

$$L(\partial_X)v_1 + L_1(\partial_x)v_0 + F(v_0, v_0) = 0. \tag{17}$$

In order to study this equation, we use the following lemma, which generalizes Lemma 3.

Lemma 4. *Let f and g be in $A_{\underline{t}}^s$. Then one has*

$$L(\partial_X)f = g \quad \text{if and only if} \quad \begin{cases} \pi(D_X)g = 0, \text{ and } g \text{ is } Q\text{-regular,} \\ (I - \pi(D_X))f = Q(D_X)g. \end{cases}$$

Proof. Let us prove first the direct implication. Assuming that $L(\partial_X)f = g$, denote by λ and μ the Fourier transforms $\lambda := \mathcal{F}_X f$ and $\mu := \mathcal{F}_X g$. Since the function $\xi \mapsto iL(\xi)$ is λ-integrable, one has $iL\lambda = \mu$. It is easy to see that $\pi(D_X)g = 0$; we prove that Q is μ-integrable and that $(I - \pi)\lambda = Q\mu$. In order to do this, we will construct a sequence $\{Q_n\}$ of integrable functions which converges to Q, and use Theorem 1. We can decompose the characteristic variety $\mathcal{C}$ (see [2]) as follows: $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_m$, where $\mathcal{C}_j$ is the set of roots of multiplicity j of $\det L$. We define $\mathcal{C}_0$ as $\mathbb{R}^{1+d} \backslash \mathcal{C}$. We also define the sets A_i^n as $A_i^n := \{x \in \mathcal{C}_i : \mathrm{d}(x, \mathcal{C}_{i+1}) \geq 1/n\}$, for $i = 1, \ldots, m-1$, and $A_m^n := \mathcal{C}_m$.

We now introduce the sets $A_n := A_1^n \cup \cdots \cup A_m^n$ and χ_n, the characteristic function of $A_n \cap B(0, n)$. Then χ_n has compact support and $\chi_n \to 1$ everywhere when $n \to \infty$. The same arguments as in the proof of Proposition 11 yield that the functions $Q_n := \chi_n Q$ are μ-integrable. They also converge everywhere to Q and satisfy

$$\int_E Q_n(\xi)\mu(d\xi) = \int_E (I - \pi(\xi))\lambda(d\xi), \quad \forall E \in \mathcal{B}(\mathbb{R}^{1+d}).$$

Thanks to Theorem 1 we therefore conclude that Q is μ integrable and that for all $E \in \mathcal{B}(\mathbb{R}^{1+d})$,

$$\int_E Q(\xi)\mu(d\xi) = \lim_{n \to \infty} \int_E Q_n(\xi)\mu(d\xi) = \int_E (I - \pi(\xi))\lambda(d\xi).$$

We have thus shown that $Q\mu = (1 - \pi)\lambda$, and the proof of the direct implication is then complete.

We omit the proof of the reverse implication, since it is very similar to the proof of Lemma 3. $\qquad\square$

Equation (17) is therefore equivalent to

$$\pi(D_X)L_1(\partial_x)\pi(D_X)v_0 + \pi(D_X)F(v_0, v_0) = 0, \tag{18}$$

and

$$\begin{cases} L_1(\partial_x)v_0 + F(v_0, v_0) \text{ is } Q\text{-regular} \\ (I - \pi(D_X))v_1 = -Q(D_X)[L_1(\partial_x)v_0 + F(v_0, v_0)]. \end{cases} \tag{19}$$

The following proposition says that equation (18) is well-posed in A_T^{s+2} with initial data A_0^{s+2}; the first term $v_0 = \pi(D_X)v_0$ of our ansatz will then be fully determined.

Proposition 12. *Let $\sigma > d/2$ and $u^0 = \pi(D_X)u^0$ be in A_0^σ. Then there exists a positive real number $\underline{t} > 0$ such that the system*

$$\pi(D_X)L_1(\partial_x)\pi(D_X)f + \pi(D_X)F(f, f) = 0, \quad f|_{t=0} = u^0,$$

has a unique solution $f \in A_{\underline{t}}^\sigma$ such that $f = \pi(D_X)f$.

Proof. The proof of this proposition is by Picard iteration, just as was done to solve equation (6) in the proof of Theorem 4. We will not detail it and will just give the proof of the following lemma, which is the key step in Picard iteration. $\qquad\square$

Lemma 5. *Let $g \in A_{\underline{t}}^\sigma$ and $f^0 = \pi(D_X)f^0 \in A_0^\sigma$. The linear problem*

$$\pi(D_X)L_1(\partial_x)\pi(D_X)f = g, \quad f|_{t=0} = f^0,$$

has a unique solution $f \in A_{\underline{t}}^\sigma$ such that $f = \pi(D_X)f$. Moreover, one has

$$\|f\|_{A_{\underline{t}}^\sigma} \le \|f^0\|_{A_0^\sigma} + \underline{t}\|g\|_{A_{\underline{t}}^\sigma}.$$

Proof. The proof of this lemma is very similar to that of Lemma 2; one just has to replace the functions $T_{t,u}$ by

$$S_{t,u}(\xi)h(\eta) := e^{-i(t-u)\pi(\xi)A(\eta)\pi(\xi)}h(\eta), \quad \forall h \in L_\sigma^2(\mathbb{R}^d).$$

Remark. We will apply this proposition with $\sigma = s + 2$ since the computations of v_1 and of the residual have a regularity cost. For instance, the term $L_1(\partial_x)v_0 + F(v_0, v_0)$ which appears in the right hand side of equation (19) is only in $A_{\underline{t}}^{s+1}$.

Thanks to this proposition, we can solve equation (18). We recall that the goal of the present section is to solve equation (17). This will be done if we can solve (19). In order to do this, we make the following assumption.

Assumption 1. *The profile v_0 solution to* (18) *is such that $L_1(\partial_x)v_0 + F(v_0, v_0)$ is Q-regular.*

Remark. We will comment on this assumption in the last section, and we will see in an example that it is easily checked. Under this assumption, solving (19) is the same as taking $(I - \pi(D_X))v_1 = -Q(D_X)[L_1(\partial_x)v_0 + F(v_0, v_0)]$, which is possible, since v_1 does not need to satisfy any other condition. We have therefore solved Equations (18) and (19), and thus annihilated the ε^0 term (17).

In order to do so, we have at this stage determined v_0 and $(I - \pi(D_X))v_1$. We now choose to take $\pi(D_X)v_1 = 0$, so that the ansatz $v^\varepsilon = v_0 + \varepsilon v_1$ we are looking for is now fully determined.

4.5. **Smallness of the residual.** We have seen in Section 3.2 that the profile u^ε of the exact solution $\mathfrak{u}^\varepsilon$ to problem (1) satisfies

$$L_1(\partial_x)u^\varepsilon + \varepsilon^{-1}L(\partial_X)u^\varepsilon + F(u^\varepsilon, u^\varepsilon) = 0.$$

Thanks to the above sections, we know that the profile $v^\varepsilon := v_0 + \varepsilon v_1$ of the approximate solution $\mathfrak{v}^\varepsilon$ is almost a solution of this equation in the sense that

$$L_1(\partial_x)v^\varepsilon + \varepsilon^{-1}L(\partial_X)v^\varepsilon + F(v^\varepsilon, v^\varepsilon) = \varepsilon r^\varepsilon,$$

where $r^\varepsilon := L_1(\partial_x)v_1 + F(v_0, v_1) + F(v_1, v_0) + \varepsilon F(v_1, v_1)$ is in $A_{\underline{t}}^s$. One then has

$$L_1(\partial_x)v^\varepsilon + \varepsilon^{-1}L(\partial_X)v^\varepsilon + F(v^\varepsilon, v^\varepsilon) = O(\varepsilon) \quad \text{in } A_{\underline{t}}^s.$$

A straightforward adaptation of the proof of Lemma 1 then yields

$$L^\varepsilon(\partial_x)\mathfrak{v}^\varepsilon = \varepsilon \mathfrak{r}^\varepsilon, \quad \text{in } \mathcal{D}',$$

where $\mathfrak{r}^\varepsilon(x) := r^\varepsilon(x, x/\varepsilon)$. Since r^ε is in $A_{\underline{t}}^s$, Proposition 9 yields the following result:

Proposition 13. *Under Assumption* 1, *the approximate solution of geometrical optics $\mathfrak{v}^\varepsilon(x) = v^\varepsilon(x, x/\varepsilon)$ almost solves Problem* (1) *in the sense that $L^\varepsilon(\partial_x)\mathfrak{v}^\varepsilon = O(\varepsilon)$, both in $C([0, \underline{t}] \times \mathbb{R}^d)$ and in $C([0, \underline{t}], L^2(\mathbb{R}^d)^N)$.*

4.6. **Stability of geometrical optics.** We have proved in the above section that $\mathfrak{v}^\varepsilon$ almost solves (1), but we have not yet proved that the difference $\mathfrak{u}^\varepsilon - \mathfrak{v}^\varepsilon$ remains small, i.e., that $\mathfrak{v}^\varepsilon$ is a good approximation of $\mathfrak{u}^\varepsilon$. This is what the following theorem says.

Theorem 5 (Stability). *Suppose that the initial condition $u^0 \in A_0^{s+2}$ is polarized so that $\pi(D_X)u^0 = u^0$, and assume that Assumption* 1 *is fulfilled.*

Then the exact solution u^ε is in $A_T^{s+2} \subset A_{\underline{t}}^s$ and the approximate solution $\mathbf{v}^\varepsilon(x) = v^\varepsilon(x, x/\varepsilon)$ is stable in the sense that $u^\varepsilon = v_0 + O(\varepsilon)$ in $A_{\underline{t}}^s$, and $\mathbf{u}^\varepsilon = \mathbf{v}_0 + O(\varepsilon)$, both in $C([0, \underline{t}] \times \mathbb{R}^d)$ and in $C([0, \underline{t}], L^2(\mathbb{R}^d)^N)$.

Proof. One has

$$L_1(\partial_x)u^\varepsilon + \varepsilon^{-1}L(\partial_X)u^\varepsilon = -F(u^\varepsilon, u^\varepsilon),$$
$$L_1(\partial_x)v^\varepsilon + \varepsilon^{-1}L(\partial_X)v^\varepsilon = -F(v^\varepsilon, v^\varepsilon) + \varepsilon r^\varepsilon.$$

Taking the difference of these two equalities yields

$$L_1(\partial_x)w^\varepsilon + \varepsilon^{-1}L(\partial_X)w^\varepsilon = -F(u^\varepsilon, w^\varepsilon) + F(w^\varepsilon, v^\varepsilon) + \varepsilon r^\varepsilon := R^\varepsilon,$$

where $w^\varepsilon := u^\varepsilon - v^\varepsilon$ also satisfies $w^\varepsilon|_{t=0} = 0$.

The result stated in Lemma 9 is not sharp enough to give an interesting bound for $\|w^\varepsilon\|_{A_{\underline{t}}^s}$, but in the proof of this lemma, we proved the sharper estimation (12), which now yields

$$|\mathcal{F}_X w^\varepsilon(t)|_{\mathcal{BV}} \le \int_0^t |\mathcal{F}_X R^\varepsilon(u)|_{\mathcal{BV}}\, du.$$

Moreover, $|\mathcal{F}_X R^\varepsilon|_{\mathcal{BV}}$ is bounded by

$$|\mathcal{F}_X R^\varepsilon|_{\mathcal{BV}} \le C_1|\mathcal{F}_X w^\varepsilon|_{\mathcal{BV}} + \varepsilon C_2,$$

where C_1 depends on F and on an ε-independent upper bound of $\|u^\varepsilon\|_{A_{\underline{t}}^s}$ and $\|v^\varepsilon\|_{A_{\underline{t}}^s}$, while C_2 is an ε-independent upper bound of $\|r^\varepsilon\|_{A_{\underline{t}}^s}$. We then have

$$|\mathcal{F}_X w^\varepsilon(t)|_{\mathcal{BV}} \le \varepsilon \underline{t} C_2 + \int_0^t C_1|\mathcal{F}_X v^\varepsilon(w)|_{\mathcal{BV}}\, du,$$

which yields, thanks to Gronwall's inequality,

$$|\mathcal{F}_X w^\varepsilon(t)|_{\mathcal{BV}} \le \varepsilon \underline{t} C_2 e^{C_1 \underline{t}}.$$

One then has

$$u^\varepsilon = v^\varepsilon + O(\varepsilon) \quad \text{in } A_{\underline{t}}^s.$$

Since we also have $v^\varepsilon = v_0 + O(\varepsilon)$ in $A_{\underline{t}}^s$, we can deduce that $u^\varepsilon = v_0 + O(\varepsilon)$ in $A_{\underline{t}}^s$. The last two asymptotic relations stated in the theorem are a consequence of Proposition 9. $\qquad\square$

5. Qualitative properties and resonances

5.1. **Qualitative properties of the linear problem.** In this section we consider again the linear system

$$\pi(D_X)L_1(\partial_x)\pi(D_X)f = g, \quad f|_{t=0} = f^0, \tag{20}$$

where $\pi(D_X)g = g \in A_{\underline{t}}^s$ and $\pi(D_X)f^0 = f^0 \in A_0^s$.

We have proved in Section 4.4 that there is a unique solution $f = \pi(D_X)f$ to this problem. The following proposition gives a property satisfied by the oscillating spectrum of f.

Proposition 14. *Denote* $\mathcal{S}_0 := \operatorname{Supp} \mathcal{F}_X f^0$, *and assume that there exists a subset* $\mathcal{S}$ *of* $\mathcal{C}$ *such that for all* $t \in [0, \underline{t}]$ *one has* $\operatorname{Supp} \mathcal{F}_X g(t, \cdot) \subset \mathcal{S}$. *Then one has* $\operatorname{Supp} \mathcal{F}_X f(t, \cdot) \subset \mathcal{S}_0 \cup \mathcal{S}$, $\forall t \in [0, \underline{t}]$.

Proof. As usual, denote $\lambda := \mathcal{F}_X f$, $\mu := \mathcal{F}_X g$ and $\lambda_0 := \mathcal{F}_X f^0$. We have seen in Section 4.4 that $\widehat{\lambda}$ is given by

$$\widehat{\lambda}(t) = e^{-it\pi(\xi)A(\eta)\pi(\xi)}\widehat{\lambda}_0 + \int_0^t e^{-i(t-u)\pi(\xi)A(\eta)\pi(\xi)}\widehat{\mu}(u)\,du,$$

which yields the desired result. $\square$

We will now prove that the operator $\pi(D_X)L_1(\partial_x)\pi(D_X)$ may often be seen as a scalar operator, which simplifies the computations a lot. If ξ is in the characteristic variety $\mathcal{C}$, we can write $\xi = (\xi_0, \xi_I) := (\tau(\xi_I), \xi_I)$, where the function τ is a local parametrization of $\mathcal{C}$ in a neighborhood of ξ. If ξ is a smooth point of $\mathcal{C}$, then one has (see [6])

$$\pi(\xi)L_1(\partial_x)\pi(\xi) = \pi(\xi)(\partial_t - \tau'(\xi_I) \cdot \partial_y), \tag{21}$$

and $\pi(\xi)L_1(\partial_x)\pi(\xi)$ is thus a transport vector field at the group velocity $\tau'(\xi_I)$ on the range of $\pi(\xi)$.

For any $\xi_I \in \mathbb{R}^d$, there is normally more than one point of $\mathcal{C}$ with $\mathbb{R}^d$ coordinate ξ_I; they are written $\xi^1 := (\tau_1(\xi_I), \xi_I), \ldots, \xi^r := (\tau_r(\xi_I), \xi_I)$, where r may depend on ξ_I. Therefore, any $\xi \in \mathcal{C}\backslash\operatorname{Sing} \mathcal{C}$ may read $\xi = (\tau_i(\xi_I), \xi_I)$ for a unique i. We will use the notation $\tau'(\xi)$ instead of $\tau_i'(\xi_I)$. Equation (21) thus reads

$$\pi(\xi)L_1(\partial_x)\pi(\xi) = \pi(\xi)(\partial_t - \tau'(\xi) \cdot \partial_y);$$

we can now formulate a proposition which generalizes this property.

Proposition 15. *Suppose that* $\mathcal{S}_0$ *and* $\mathcal{S}$ *are as in the above proposition. If in addition,* $(\mathcal{S}_0 \cup \mathcal{S}) \cap \operatorname{Sing} \mathcal{C} = \emptyset$, *then* f *solves the scalar system*

$$(\partial_t - \tau'(D_X) \cdot \partial_y)\pi(D_X)f = g, \quad f|_{t=0} = f^0.$$

The following corollary gives the explicit solution of the linear problem without source term and for initial conditions being almost-periodic functions or density functions.

Corollary 1. *Let us consider the linear problem* (20) *in the case where g=0.*
 i) *If $f^0 = \pi(D_X)f^0$ is an almost-periodic function given by*

$$f^0(y, X) := \sum_{\beta \in \mathcal{S}_0} a_\beta(y)e^{i\beta \cdot X},$$

where $\mathcal{S}_0$ contains no singular point of $\mathcal{C}$, then the solution $f = \pi(D_X)f$ to (20) *is given by*

$$f(x, X) := \sum_{\beta \in \mathcal{S}_0} a_\beta(y + \tau'(\beta)t)e^{i\beta \cdot X}.$$

ii) *If $f^0 = \pi(D_X)f^0$ is a density function of support $\mathcal{M}$ and density α,*

$$f^0(y, X) = \int_{\mathcal{M}} e^{iX \cdot \xi}\alpha(\xi)(y)\sigma(d\xi),$$

and if $\mathcal{M}$ contains no singular point of $\mathcal{C}$, then the solution $f = \pi(D_X)f$ to (20) *is given by*

$$f(x, X) = \int_{\mathcal{M}} e^{iX \cdot \xi}\alpha(\xi)(y + t\tau'(\xi))\sigma(d\xi).$$

Proof. The case of almost-periodic functions is well known (see [12] for instance); let us prove part ii) of the corollary. It is easy to see using the above proposition, the assumptions made on $\mathcal{M}$, and the fact that $\pi(\xi)L_1(\partial_x)\pi(\xi)$ is scalar when ξ is a nonsingular point of $\mathcal{C}$, that the Fourier transform $\lambda := \mathcal{F}_X f$ is given by

$$\widehat{\lambda} = e^{it\tau'(\xi) \cdot \eta}\widehat{\lambda_0}.$$

The Fourier transform $\lambda_0 := \mathcal{F}_X f^0$ is given for all $E \in \mathcal{B}(\mathbb{R}^{1+d})$ by

$$\lambda_0(E) = \int_{\mathcal{M}} \chi_E(\xi)\alpha(\xi)\sigma(d\xi),$$

where χ_E is the characteristic function of E. Thanks to Proposition 2, $\widehat{\lambda_0}$ is given by

$$\widehat{\lambda_0}(E) = \int_{\mathcal{M}} \chi_E(\xi)\widehat{\alpha(\xi)}\sigma(d\xi),$$

and thus $\widehat{\lambda}$ takes the form

$$\widehat{\lambda}(E)(t, \eta) = \int_{\mathcal{M}} \chi_E(\xi)e^{it\tau'(\xi) \cdot \eta}\widehat{\alpha(\xi)}(\eta)\sigma(d\xi),$$

and Proposition 2 yields this time

$$\lambda(E)(t,y) = \int_{\mathcal{M}} \chi_E(\xi)\alpha(\xi)(y + t\tau'(\xi))\sigma(d\xi).$$

Taking the inverse Fourier transform of λ with respect to ξ yields the desired result. $\qquad\square$

5.2. **Resonances.** We have seen that the profile $v_0 = \pi(D_X)v_0$ given by geometrical optics for Problem (1) is determined by

$$\pi(D_X)L_1(\partial_x)\pi(D_X)v_0 + \pi(D_X)F(v_0, v_0) = 0, \quad v_0|_{t=0} = u^0.$$

We will see that in many cases, we can omit the nonlinearity $\pi(D_X)F(v_0, v_0)$; the only cases where it will be of importance will be when resonances occur.

Definition 10. Let $\lambda = \pi\lambda$ and $\mu = \pi\mu$ be in $\mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N)$. We will say that λ and μ resonate if and only if one has $v(\mu) * v(\lambda)(\mathcal{C}) \neq 0$. We will say that two functions of A_0^s or $A_{\underline{t}}^s$ resonate if their Fourier transforms resonate.

Remark. If λ and μ are written $\lambda = a_{\beta_1}\delta_{\beta_1}$ and $\mu = a_{\beta_2}\delta_{\beta_2}$ with $\pi(\beta_1)a_{\beta_1} = a_{\beta_1}$ and $\pi(\beta_2)a_{\beta_2} = a_{\beta_2}$, then λ and μ resonate if and only if $\beta_1 + \beta_2 \in \mathcal{C}$, which is the usual resonance condition.

Generally, $\mathcal{C}$ admits 0 as a center of symmetry, and 0 is also in $\mathcal{C}$, so that if $\beta \in \mathcal{C}$, one has $-\beta \in \mathcal{C}$ and $\beta + (-\beta) = 0 \in \mathcal{C}$; i.e., $a_\beta\delta_\beta$ and $a_{-\beta}\delta_{-\beta}$ resonate. Therefore, any measure whose support is a point usually resonates with another measure; but for general measures, this is no longer true, and that is why we give the following definition.

Definition 11. Let $\lambda = \pi\lambda \in \mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N)$. λ is absolutely nonresonant if for all $\mu = \pi\mu \in \mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N)$ such that $\mu(\{0\}) = 0$, λ and μ do not resonate. In the opposite case, we say that λ is potentially resonant. We also say that an element of A_0^s or $A_{\underline{t}}^s$ is absolutely nonresonant if its Fourier transform is absolutely nonresonant.

Example. i) Consider the wave equation with $d = 2$. The associated characteristic variety $\mathcal{C}$ is the cone $\{(\xi_0, \xi_1, \xi_2) : \xi_0 = \pm\sqrt{\xi_1^2 + \xi_2^2}\}$. Let $\mathcal{M}$ be a circular section of $\mathcal{C}$ given by $\mathcal{M} = \{(\omega, \xi_1, \xi_2) \in \mathcal{C}\}$, where $\omega > 0$. Then any density function f of support $\mathcal{M}$ and density $\alpha \in C(\mathcal{M}, H^s(\mathbb{R}^d)^N)$ is absolutely nonresonant. Indeed, its Fourier transform λ is given by

$$\lambda(E) = \int_{\mathcal{M}} \chi_E(\xi)\alpha(\xi)\sigma(d\xi), \qquad \forall E \in \mathcal{B}(\mathbb{R}^{1+d}),$$

so that

$$v(\lambda)(E) = \int_{\mathcal{M}} \chi_E(\xi) \|\alpha(\xi)\|_{H^s} \sigma(d\xi), \qquad \forall E \in \mathcal{B}(\mathbb{R}^{1+d}).$$

Let $\mu = \pi\mu \in \mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N)$ such that $\mu(\{0\}) = 0$. One then has

$$v(\lambda) * v(\mu)(\mathcal{C}) = \int_{\mathcal{C}} \int_{\mathcal{M}} \chi(\xi_1 + \xi_2) \|\alpha(\xi_1)\|_{H^s} \sigma(d\xi_1) v(\mu)(d\xi_2)$$

$$\leq M \int_{\mathcal{C}} \int_{\mathcal{M}} \chi(\xi_1 + \xi_2) \sigma(d\xi_1) v(\mu)(d\xi_2),$$

where for all ξ, one has $\|\alpha(\xi)\|_{H^s} \leq M$. For all nonzero $\xi_2 \in \mathcal{C}$ one has $\mathcal{C} \cap (\mathcal{M} + \xi_2) = \emptyset$, and therefore $v(\lambda) * v(\mu)(\mathcal{C}) \leq M\sigma(\mathcal{C})\mu(\{0\}) = 0$. Therefore, λ and μ do not resonate, and we have thus proved that f is absolutely nonresonant.

ii) On the other hand, any almost-periodic function whose spectrum is on $\mathcal{C}$ is potentially resonant, as it is easy to see using the fact that $\mathcal{C}$ is homogeneous.

We will consider initial conditions of the form $u^0 = \sum_{i=1}^{m} u^i$, with $m \geq 1$ and $u^i = \pi(D_X)u^i \in A_0^s$, for $i = 1, \dots, m$.
One can decompose $[1, m]$ under the form $[1, m] = \mathcal{P} + \mathcal{N}$, where $\mathcal{P}$ and $\mathcal{N}$ are defined as

$$i \in \mathcal{P} \iff \mathcal{F}_X u^i \text{ is potentially resonant}$$
$$i \in \mathcal{N} \iff \mathcal{F}_X u^i \text{ is absolutely nonresonant.}$$

The following theorem says that the nonlinearity $\pi(D_X)F(v_0, v_0)$ does not see the absolutely nonresonant part of u^0.

Theorem 6. *Let $v_0 = \pi(D_X)v_0$ be the solution given by geometrical optics. Let $v^p = \pi(D_X)v^p$ and $v^n = \pi(D_X)v^n$ be the solutions of*

$$\begin{cases} \pi(D_X)L_1(\partial_x)\pi(D_X)v^p + \pi(D_X)F(v^p, v^p) = 0 \\ v^p|_{t=0} = \sum_{i \in \mathcal{P}} u^i := u^{0,p}, \end{cases}$$

and

$$\begin{cases} \pi(D_X)L_1(\partial_x)\pi(D_X)v^n = 0 \\ v^n|_{t=0} = \sum_{i \in \mathcal{N}} u^i := u^{0,n}. \end{cases}$$

Then one has $v_0 = v^p + v^n$.

Proof. Let us show that $\pi(D_X)F(v^p + v^n, v^p + v^n) = \pi(D_X)F(v^p, v^p)$. It is sufficient to prove that $g := \pi(D_X)F(v^n, f)$ is equal to 0 for all $f \in A_{\underline{t}}^s$. Denoting $\lambda := \mathcal{F}_X f$ and $\mu^n := \mathcal{F}_X v^n$, one has $\mathcal{F}_X g = \pi\mu^n * \lambda$. Since $\pi(\xi) = 0$ for all $\xi \notin \mathcal{C}$, we also have $\mathcal{F}_X g = \pi\chi_{\mathcal{C}}\mu^n * \lambda$, where $\chi_{\mathcal{C}}$ denotes the

characterisitic function of $\mathcal{C}$. We also know that $|\chi_{\mathcal{C}}\mu^n * \lambda|_{\mathcal{BV}} = v(\mu^n * \lambda)(\mathcal{C})$, which yields thanks to Proposition 4

$$|\chi_{\mathcal{C}}\mu^n * \lambda|_{\mathcal{BV}} \le k\, v(\mu^n) * v(\lambda)(\mathcal{C}). \tag{22}$$

But we also have seen that we can find an explicit expression for μ^n which yields $v(\mu^n) = v(\mathcal{F}_X u^{0,n})$. Equation (22) thus reads

$$|\chi_{\mathcal{C}}\mu^n * \lambda|_{\mathcal{BV}} \le k\, v(\mathcal{F}_X u^{0,n}) * v(\lambda)(\mathcal{C}) = 0,$$

the last equality being a consequence of the fact that $\mathcal{F}_X u^{0,n}$ is absolutely nonresonant. It is then straightforward to conclude that $g = 0$ and therefore that we have $\pi(D_X)F(v^p + v^n, v^p + v^n) = \pi(D_X)F(v^p, v^p)$. Thanks to this property, one has

$$\pi(D_X)L_1(\partial_x)\pi(D_X)(v^p + v^n) + \pi(D_X)F(v^p + v^n, v^p + v^n) = 0.$$

Since we also have $(v^p + v^n)|_{t=0} = u^0$, $v^p + v^n$ and v_0 satisfy the same Cauchy problem and are thus equal, thanks to Proposition 12. $\qquad\square$

Remark: i)We have seen that measures whose support is discrete are usually potentially resonant, but we will see that many density functions are absolutely nonresonant. This simplifies the computations since density functions are far more difficult to handle than almost-periodic functions. The above theorem says that absolutely nonresonant density functions solve a linear problem, and Corollary 1 gives a simple form of the solution in many cases. ii) This theorem may easily be sharpened when dealing with concrete examples, as we will see when studying the stimulated Raman scattering.

The following corollary should be used when checking Assumption 1.

Corollary 2. *If $u^{0,n}$ is Q-regular, then so is v^n.*

Proof. It is a straightforward consequence of Theorem 6, Propositions 11 and Proposition 14. $\qquad\square$

5.3. **Conditions for resonances between almost-periodic functions and density functions; transparency.** In order to study possible resonances between almost-periodic functions or density functions, we will have to study the convolutions of their Fourier transforms. We will consider this for elements $\lambda_1, \lambda_2, \mu_1$ and μ_2 of $C([0, \underline{t}], \mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N))$ given by

$$\lambda_1 = a_{\beta_1}\delta_{\beta_1} \quad \text{and} \quad \lambda_2 = a_{\beta_2}\delta_{\beta_2},$$

$$\mu_1(E) = \int_{\mathcal{M}_1} \chi_E(\xi)\alpha_1(\xi)\sigma_1(d\xi), \quad \forall E \in \mathcal{B}(\mathbb{R}^{1+d}),$$

$$\mu_2(E) = \int_{\mathcal{M}_2} \chi_E(\xi)\alpha_2(\xi)\sigma_2(d\xi), \quad \forall E \in \mathcal{B}(\mathbb{R}^{1+d}),$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are submanifolds of dimension n_1 and n_2 respectively. One then has

$$i) \quad v(\lambda_1) * v(\lambda_2) = \|a_{\beta_1}\|\|a_{\beta_2}\|\delta_{\beta_1+\beta_2}$$

$$ii) \quad v(\lambda_1) * v(\mu_1)(E) = \|a_{\beta_1}\| \int_{\mathcal{M}_1} \chi_E(\beta_1 + \xi)\|\alpha_1(\xi)\|\sigma_1(d\xi)$$

$$iii) \quad v(\mu_1) * v(\mu_2)(E) =$$
$$\int_{\mathcal{M}_1} \int_{\mathcal{M}_2} \chi_E(\xi_1 + \xi_2)\|\alpha_1(\xi_1)\|\|\alpha_2(\xi_2)\|\sigma_1(d\xi_1)\sigma_2(d\xi_2).$$

We have therefore the following necessary conditions for resonances.

Proposition 16. *With the notation used above, and assuming that $\lambda_i = \pi\lambda_i$, and $\mu_i = \pi\mu_i$, for $i = 1, 2$, then*
i) λ_1 *and* λ_2 *resonate if and only if* $\beta_1 + \beta_2 \in \mathcal{C}$;
ii) λ_1 *and* μ_1 *resonate if and only if* $\mathcal{R} := \{\xi \in \mathcal{M} : \beta_1 + \xi \in \mathcal{C}$ *and* $\alpha_1(\xi) \neq 0\}$ *is a subset of* $\mathcal{M}_1$ *whose Lebesgue measure* $\sigma_1(\mathcal{R})$ *is nonzero,*
iii) μ_1 *and* μ_2 *resonate if and only if* $\mathcal{R} := \{(\xi_1, \xi_2) \in \mathcal{M}_1 \times \mathcal{M}_2 : \xi_1 + \xi_2 \in \mathcal{C}, \alpha_1(\xi_1) \neq 0$ *and* $\alpha_2(\xi_2) \neq 0\}$ *is a subset of* $\mathcal{M}_1 \times \mathcal{M}_2$ *whose Lebesgue measure* $(\sigma_1 \times \sigma_2)(\mathcal{R})$ *is nonzero.*

The above proposition gives resonance conditions, However, two measures λ and μ may resonate in the sense of Proposition 10, i.e., $v(\lambda) * v(\mu)(\mathcal{C}) \neq 0$, while $\pi(\lambda * \mu) = 0$. In the derivation of geometrical optics, the nonlinear factor takes the form of this last term. If it is equal to 0, the nonlinearity then vanishes, as if there were no resonances. Such a phenomenon is called *transparency*; when it occurs, equations that should be nonlinear are in fact linear. In the case of discrete oscillating spectra, this phenomenon is well known (see [5, 16] for examples and [14] for a general study). With the above notation, the transparency conditions read

$$i) \quad \pi(\beta_1 + \beta_2)F(a_{\beta_1}, a_{\beta_2}) = 0$$

$$ii) \quad \int_{\mathcal{M}_1} \pi(\beta_1 + \xi)F(a_{\beta_1}, \alpha_1(\xi))\sigma_1(d\xi) = 0$$

$$iii) \quad \int_{\mathcal{M}_1} \int_{\mathcal{M}_2} \pi(\xi_1 + \xi_2)F(\alpha_1(\xi_1), \alpha_2(\xi_2))\sigma_1(d\xi_1)\sigma_2(d\xi_2) = 0.$$

5.4. **Spontaneous Raman scattering.** The three-level Maxwell-Bloch model is used to explain this phenomenon (see [4, 17], as well as [14] for a mathematical point of view). This system consists in the Maxwell equations

$$\begin{cases} \partial_t E - \mathrm{curl}B = -\partial_t P \\ \partial_t B + \mathrm{curl}E = 0 \\ \mathrm{div}E = -\mathrm{div}P \\ \mathrm{div}B = 0, \end{cases}$$

where E and B denote respectively the electric and magnetic fields, while P is the polarization. The link between E and P is given by the Bloch equations

$$i\varepsilon\partial_t\rho = [\Omega, \rho] - [E \cdot \Gamma, \rho], \quad P = tr(\Gamma\rho),$$

where Ω is the electronic Hamiltonian in absence of external field, and $-\Gamma$ is the dipole moment operator. The relevant part of the characteristic variety in space dimension $d = 2$ has the following form:

Characteristic variety C
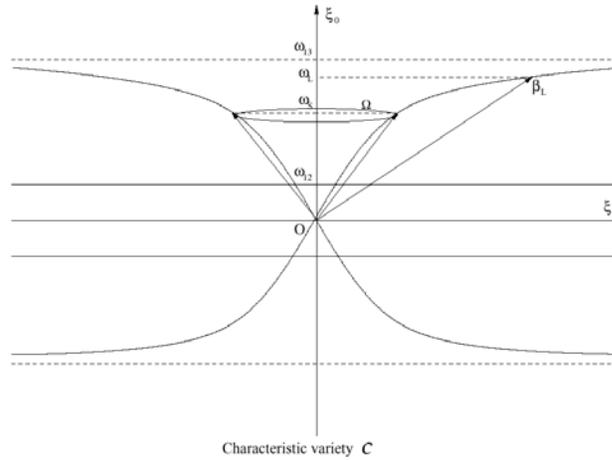
where $(O\xi_0)$ is an axis of revolution.

We consider the two-dimensional case $d = 2$ and suppose in this section that the incident light is weak enough to neglect the nonlinearities. In our case, this means that the bilinear mapping F of (1) is taken equal to 0.

As in the introduction, assume that the incident light has frequency ω_L. Energy-level diagrams show that light is emitted at the Stokes frequency $\omega_S = \omega_L - \omega_{12}$ and at the anti-Stokes frequency $\omega_a = \omega_L + \omega_{12}$. Boltzmann's law explains why Stokes lines are stronger than anti-Stokes lines.

The intersection of $\mathcal{C}$ with the plane $\xi_0 = \omega_S$ is a circle Ω, and as said in the Introduction, the oscillating spectrum of the emitted oscillation should be this circle Ω. We assume here that we can describe it with a density function u^0 of support Ω and density $\alpha \in C(\Omega, H^s(\mathbb{R}^d)^N)$,

$$u^0(y, X) = \int_\Omega e^{iX \cdot \xi} \alpha(\xi)(y)\sigma(d\xi).$$

Since Ω does not contain any singular point of $\mathcal{C}$, we know thanks to Corollary 1 and since the nonlinearity F is equal to 0, that the profile v_0 of geometrical optics reads

$$v_0(t, y, X) = \int_\Omega e^{iX \cdot \xi} \alpha(\xi)(y + t\tau'(\xi))\sigma(\xi)d\xi, \tag{23}$$

where $\xi \mapsto \tau(\xi)$ is a parametrization of $\mathcal{C}$ in a neighborhood of Ω.

Thanks to Proposition 11 it is straightforward to check that Assumption 1 is satisfied, so that we can conclude thanks to Theorem 5 that (23) gives a good approximation to the exact solution.

The set of all the directions of propagation of the approximate solution (23) is $\{\tau'(\xi) : \xi \in \Omega\}$, which is of revolution, so that as observed experimentally, the emission is nearly (because α may depend slightly on ξ) isotropic.

5.5. **Stimulated Raman scattering.** In this section, we still consider the two-dimensional case $d = 2$, and use the three-level Maxwell-Bloch model whose characteristic variety has been given in the above section.

We assume here that the incident light is a very strong laser beam, so that nonlinear phenomena occur. We show in this section that the stimulated Raman scattering may be seen as an instability effect, in the sense that the nonlinear effects may only amplify the Stokes frequency, among all the frequencies initially present.

We can assume that the incident light gives a contribution to the initial condition to Problem (1) of the form $\mathbf{u}_L^0(y) := 2a_{\beta_L} \cos(k_L \cdot y/\varepsilon)$, where $\beta_L := (\omega_L, k_L)$ is in $\mathcal{C}$ and $\pi(\beta_L)a_{\beta_L} = a_{\beta_L}$. The associated profile $u_L^0 \in A_0^s$ is thus given by $u_L^0 = a_{\beta_L}(\delta_{\beta_L} + \delta_{-\beta_L})$.

We will also assume that there are other terms in the initial condition, whose oscillating spectrum is continuous, since, as said in Section 1, there is no physical reason to give priority to a countable number of directions of propagation. We thus consider an initial condition whose profile u^0 is given by

$$u^0 = u_L^0 + \sum_{i=1}^n u_i^0,$$

where the $u_i^0 = \pi(D_X)u_i^0$ are density functions of one- or two-dimensional support.

Our goal is to look for the resonant part of this initial condition, i.e., the part that will be affected by the nonlinearity present in the equations of geometrical optics.

In order to apply Theorem 6, we give the following lemma.

Lemma 6. i) *Density functions with two-dimensional support are absolutely nonresonant.* ii) *Density functions with a one-dimensional support $\mathcal{M}$ which is nowhere horizontal (i.e., such that $\mathcal{M} \cap P$ is discrete for all horizontal planes P) are absolutely nonresonant.*

Proof. i) Let us consider a density function f of two-dimensional support $\mathcal{M}$ and density α; its Fourier transform λ is given for all Borel sets E by $\lambda(E) = \int_{\mathcal{M}} \chi_E(\xi)\alpha(\xi)\,d\xi$. Let $\mu = \pi\mu \in \mathcal{BV}(\mathbb{R}^{1+d}, H^s(\mathbb{R}^d)^N)$ such that $\mu(\{0\}) = 0$. One has

$$v(\lambda) * v(\mu)(\mathcal{C}) \leq M \int_{\mathcal{C}} (\int_{\mathcal{M}} \chi_{\mathcal{C}}(\xi_1 + \xi_2)\sigma(d\xi_1))v(\mu)(d\xi_2),$$

where $\|\alpha(\xi)\| \leq M$, for all $\xi \in \mathcal{M}$.

In the particular case we are dealing with, it appears clearly on the graph of $\mathcal{C}$ that the set $\mathcal{C} \cap (\xi_2 + \mathcal{C})$, and therefore $\mathcal{C} \cap (\xi_2 + \mathcal{M})$, has a Lebesgue measure equal to 0, for all $\xi_2 \in \mathcal{C}\backslash\{0\}$; i.e., $\int_{\mathcal{M}} \chi_{\mathcal{C}}(\xi_1 + \xi_2)\sigma(d\xi_1) = 0$, unless $\xi_2 = 0$. Since $v(\mu)(\{0\}) = 0$, we then have $v(\lambda) * v(\mu)(\mathcal{C}) = 0$, and therefore λ and μ do not resonate, which yields the first point of the lemma. The proof of the second point is quite similar, and we omit it. $\qquad\square$

Thanks to Theorem 6, we know that the terms u_i^0 of the initial condition u^0 which are as those given in this lemma are not affected by the nonlinearity. This is not the case of the density function whose one-dimensional support $\mathcal{M}$ is horizontal (i.e., there exists a horizontal plane P such that $\mathcal{M} \cap P = \mathcal{M}$). The vertical coordinate of such a support will be called the *frequency* of the density function.

We introduce the following assumption on the components of the initial condition u^0.

Assumption 2. *The density functions u_i^0 have one-dimensional horizontal support $\mathcal{M}_i$ of frequency $\omega^i \neq \omega_{12}$, for $i = 1, \ldots, m_1$, and have either a two-dimensional or nowhere plane support for $i = m_1 + 1, \ldots, m$.*

Moreover, for all $1 \leq i, j \leq m_1$, one has $\omega^i + \omega^j \neq \pm\omega_{12}$.

Up to a change of indices we can suppose that $\omega^1 = \omega_L - \omega_{12}$ and $\omega^2 = -\omega^1$. If these frequencies are not present in the components of the initial

condition, just assume $u_1^0 = u_2^0 = 0$. We can now formulate the following theorem, which is sharper than Theorem 6, but uses the particular structure of the characteristic variety of the present example.

Theorem 7. *Suppose the initial condition is given by*

$$u^0 = a_{\beta_L}(\delta_{\beta_L} + \delta_{-\beta_L}) + \sum_{i=1}^{m} u_i^0,$$

with $\pi(\beta_L)a_{\beta_L} = a_{\beta_L}$ and the density functions $u_i^0 = \pi(D_X)u_i^0$ are as in Assumption 2. Let $v_0 = \pi(D_X)v_0$ be the profile given by geometrical optics, and let $v_{lin} = \pi(D_X)v_{lin}$ and $v_{n.lin} = \pi(D_X)v_{n.lin}$ be the solutions of

$$\begin{cases} \pi(D_X)L_1(\partial_x)\pi(D_X)v_{n.lin} + \pi(D_X)F(v_{n.lin}, v_{n.lin}) = 0, \\ v_{n.lin}|_{t=0} = a_{\beta_L}(\delta_{\beta_L} + \delta_{-\beta_L}) + u_1^0 + u_2^0, \end{cases} \tag{24}$$

and

$$\begin{cases} \pi(D_X)L_1(\partial_x)\pi(D_X)v_{lin} = 0, \\ v_{lin}|_{t=0} = \sum_{i \geq 3} u_i^0. \end{cases}$$

Then one has $v_0 = v_{lin} + v_{n.lin}$. Moreover, the spectrum $\mathcal{S} := \mathrm{Supp}\, \mathcal{F}_X v_{n.lin}$ of $v_{n.lin}$ satisfies $\mathcal{S} \subset \{0, \pm\beta\} \cup \mathcal{M}_1 \cup \mathcal{M}_2 \cup (\mathcal{M}_1 - \beta_L) \cup (\mathcal{M}_2 + \beta_L) \subset \mathcal{C}$. In particular, if all the components of u^0 are Q-regular, then so is v_0.

Proof. Thanks to Lemma 6 and Theorem 6, we can suppose all the density functions u^i have a horizontal one-dimensional support.

One can see that each iterate $v_{n.lin}^{\nu}$ of the Picard iterates used to solve (24) has a spectrum $\mathcal{S}_\nu := \mathrm{Supp}\, \mathcal{F}_X v_{n.lin}^{\nu}$ which satisfies $\mathcal{S}_\nu \subset \{0, \pm\beta\} \subset \mathcal{M}_1 \cup \mathcal{M}_2 \cup (\mathcal{M}_1 - \beta_L) \cup (\mathcal{M}_2 + \beta_L)$. Therefore, $\mathcal{S}$ satisfies the same inclusion. This fact, together with Assumption 2, yields that

$$\pi(D_X)F(v_0, v_0) = \pi(D_X)F(v_{n.lin}, v_{n.lin}),$$

and we can then conclude as in the proof of Theorem 6 that

$$v_0 = v_{n.lin} + v_{lin}.$$

Commentary. We have proved that among all the potentially resonant components of the initial condition, only those which have frequency $\pm(\omega_L - \omega_{12})$ are effectively resonant. The frequency $\omega_S := \omega_L - \omega_{12}$ is the Stokes frequency we introduced in Section 1. In the graph of $\mathcal{C}$, we have taken $\omega_a = \omega_L + \omega_{12} > \omega_{13}$, so that the anti-Stokes lines do not occur; the analysis would be of the same kind taking them into account. All the other potentially resonant components are not affected by the nonlinearity, and therefore are not amplified.

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The nonlinearity thus chooses only density functions of horizontal one-dimensional support at the Stokes frequency, among all the density functions possibly present in the initial condition. Whether this component will be amplified or not then depends on the coefficients of F, but if something is effectively amplified it has necessarily the Stokes frequency, and its oscillating spectrum satisfies the relation given in the theorem.

One could object that since this amplified oscillation has an oscillating spectrum of the same kind as the one observed with the spontaneous Raman-scattering, the emission should also be isotropic. However, one observes experimentally that the light is emitted in a narrow cone in the forward and in the backward direction. In order to explain this apparent discrepancy, assume that a component v_0^1 of the approximate solution v_0 may be read as a density function of support Ω and density $\alpha \in C(\Omega, (C([0,T], H^s(\mathbb{R}^d)^N)))$. That is, one has

$$v_0^1(t, y, X) = \int_\Omega e^{iX \cdot \xi} \alpha(\xi)(t, y) \sigma(d\xi).$$

In the spontaneous Raman scattering, the time dependence of $\alpha(\xi)$ is found by solving a transport equation for all ξ. In the stimulated Raman scattering, this is no longer the case and the time dependence is now nonlinear and traduces the amplification. Even in simple examples, one can notice that this amplification is nonisotropic, so that even if the light is emitted in all the directions, its intensity may strongly depend on the direction. Therefore, in the case of the stimulated Raman scattering, one can think that the amplification factor $\alpha(\xi)$ takes its strongest values in the bold part of Ω in the following graphic.

With such an α, the most intense part of the emitted light is then in a narrow cone in the forward and in the backward direction, as experimentally observed.
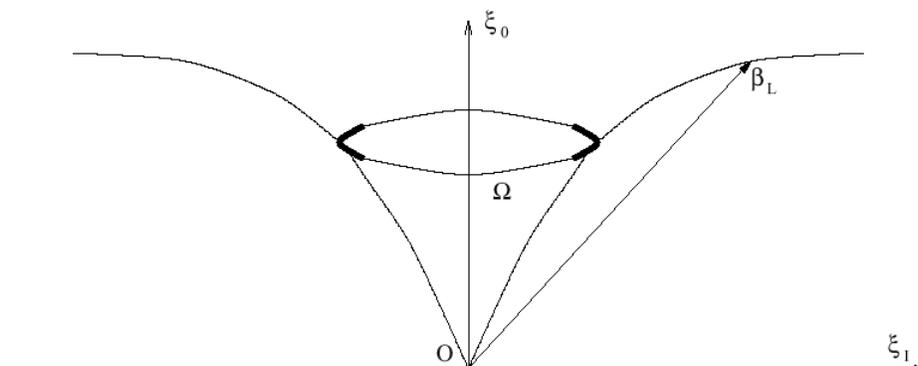
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