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GLOBAL EXISTENCE AND UNIFORM DECAY RATES FOR THE KIRCHHOFF-CARRIER EQUATION WITH NONLINEAR DISSIPATION

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Abstract. This paper is devoted to the existence of global solutions of the Kirchhoff-Carrier equation

$$u_{tt} - M(t, \int_\Omega |\nabla u|^2\, dx)\Delta u = 0$$

subject to nonlinear boundary dissipation. Assuming that $M(t,\lambda) \geq m_0 > 0$, we prove the existence and uniqueness of regular solutions without any smallness on the initial data. Moreover, uniform decay rates are obtained by assuming a nonlinear feedback acting on the boundary.

1. Introduction. This paper is concerned with the existence and uniform decay of solutions of the quasilinear wave equation with nonlinear boundary damping

$$(*) \qquad \begin{cases} u_{tt} - M\big(t, \int_\Omega |\nabla u|^2\, dx\big)\Delta u = 0 & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty) \\ \frac{\partial u}{\partial \nu} + g(u_t) = 0 & \text{on } \Gamma_1 \times (0,\infty) \\ u(x,0) = u^0(x); \quad u_t(x,0) = u^1(x) & \text{in } \Omega \end{cases}$$

where Ω is a bounded, star-shaped domain of $\mathbf{R}^n$, $n \geq 1$, with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint and ν represents the unit outward normal to Γ.

Problem $(*)$ has its origin in the canonical model of Kirchhoff and Carrier, which describes small vibrations of an elastic stretched string. More precisely, the mathematical model related to it is given by

$$\rho \frac{\partial^2 u}{\partial t^2} = \Big(P_0 + \frac{Eh}{2L}\int_0^L \big(\frac{\partial u}{\partial x}\big)^2 dx\Big)\frac{\partial^2 u}{\partial x^2},$$

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$0 \le x \le L$; $t \ge 0$, where u is the lateral deflection, ρ is the mass density, h is the cross-sectional area, L is the length, E is the Young's modulus and P_0 is the initial axial tension.

The bibliography related to this subject is extremely long; see, for instance, [1, 7, 12, 20] and a long list of references therein.

It is interesting to observe that up to now and as far as we are concerned, there is no general existence result for weak solutions to problem $(*)$, that is, when $(u(t), u'(t)) \in H^1(\Omega) \times L^2(\Omega)$. Now, for strong solutions, that is, when $(u(t), u'(t)) \in H^2(\Omega) \times H^1(\Omega)$, the majority of results in the literature presents local-in-time existence or global existence for small initial data; see for example, [4, 14, 15, 16, 17]. Now, when problem $(*)$ presents a strong damping we refer the reader to the following works: [3, 9, 13].

The main goal of this paper is exactly to give an answer to a question which has remained open in the literature, namely, to obtain global existence for strong solutions without any smallness on the initial data even if $g(s) = 0$. It is important to have in mind that, in general, a linear or nonlinear feedback plays an essential role to obtain decay rates which allow one to extend the solution to the whole interval $(0, \infty)$.

We claim that our results improve some aspects of the most recent results in this direction; see [6, 10, 18]. In [18] only the unidimensional case was considered. In [10] the authors also considered $M = M(t, \lambda)$ but making use of a linear feedback ($g(s) = s$) as in [18]. Now, in [6] the authors considered the particular case when $M(\lambda) = 1 + \lambda$ and a nonlinear damping satisfying similar conditions to those ones considered in the present work. However, in order to obtain the existence of global solutions and uniform decay rates the three works above mentioned considered small initial data.

Inspired by [11] we are able to construct a special basis to the nonlinear case, and combining this fact with techniques developed in our recent work [2] we obtain a priori estimates to the *linearized problem*:

$$\begin{cases} w_{tt} - \mu(t)\Delta w = 0 & \text{in } \ \Omega \times (0,\infty) \\ w = 0 & \text{on } \ \Gamma_0 \times (0,\infty) \\ \frac{\partial w}{\partial \nu} + g(w_t) = 0 & \text{on } \ \Gamma_1 \times (0,\infty) \\ w(0) = u^0, \ \ w_t(0) = u^1 \end{cases} \tag{1.1}$$

where $\mu(t) \ge \mu_0 > 0$.

So, making use of the estimates to the linearized problem, the fixed-point theorem and continuity arguments we obtain a solution u to $(*)$ on the maximal interval $[0, T_{max})$. Furthermore, by uniqueness of solutions of problems $(*)$ and (1.1) when $\mu(t) = M(|\nabla u(t)|^2)$ we are able to extend this solution to the whole interval $[0,\infty)$ by using the a priori estimates mentioned above. Moreover, uniform decay rates (exponential and algebraic) are also obtained by applying the integral inequalities due to Komornik.

Our paper is organized as follows: In Section 2 we state the notation and main results; in Section 3 we prove existence and uniqueness of strong solutions to the linearized problem (1.1). In Section 4 we obtain local and global solutions to problem $(*)$, and finally in Section 5 we give the proofs of the uniform decay.

2. Notation and main result. Consider the Hilbert spaces $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\}$ endowed with the inner product

$$(u,v)_V = \int_\Omega \nabla u \cdot \nabla v \, dx$$

and $H = \{v \in V : \Delta v \in L^2(\Omega)\}$ equipped with the natural inner product

$$(u,v)_H = (u,v)_V + (\Delta u, \Delta v)_{L^2(\Omega)} \, .$$

We define the following:

$$(u,v) = \int_\Omega u(x)v(x)\, dx; \quad (u,v)_{\Gamma_1} = \int_{\Gamma_1} u(x)v(x)\, d\Gamma,$$

$$|u|^2 = \int_\Omega |u(x)|^2 \, dx; \;\; |u|^2_{\Gamma_1} = \int_{\Gamma_1} |u(x)|^2 \, d\Gamma; \;\; ||u||^p_{\Gamma_1,p} = \int_{\Gamma_1} |u(x)|^p \, d\Gamma.$$

Now, we state the general hypotheses.

(A.1) Assumptions on $M(t,\lambda)$. Consider $M = M(t,\lambda)$, a function such that

$$M \in W^{1,\infty}_{loc}\left((0,\infty)\times(0,\infty)\right) \;\text{ and }\; M(t,\lambda) \geq m_0 > 0 \tag{H.1}$$

for some $m_0 > 0$,

$$\frac{\partial M}{\partial t}(t,\lambda) \leq 0, \quad \frac{\partial M}{\partial \lambda}(t,\lambda) \geq 0 \tag{H.2}$$

and given $a > 0$ there exists $K > 0$ such that

$$\Big|\frac{\partial M}{\partial \lambda}(t,\lambda)\Big| \leq K \quad \text{for all } \; \lambda \in [0,a] \; \text{ and } \; t \geq 0. \tag{H.3}$$

(A.2) Assumptions on g. Assume that $g : \mathbf{R} \to \mathbf{R}$ is a nondecreasing Lipschitzian function such that

$$g(s)s > 0 \quad \text{for all } s \neq 0, \tag{H.4}$$

and suppose that there exist $C_i > 0; i = 1, 2, 3, 4$ such that

$$C_1 |s|^p \leq |g(s)| \leq C_2 |s|^{1/p} \quad \text{if } |s| \leq 1 \tag{H.5}$$

$$C_3 |s| \leq |g(s)| \leq C_4 |s| \quad \text{if } |s| > 1, \tag{H.6}$$

where $p \geq 1$.

(A.3) Assumptions on the initial data. Assume that

$$\left\{u^0, u^1\right\} \in H \times V \tag{H.7}$$

satisfying the compatibility conditions

$$\frac{\partial u^0}{\partial \nu} + g(u^1) = 0 \quad \text{on } \Gamma_1. \tag{H.8}$$

The energy related to problem $(*)$ is given by

$$E(t) = \frac{1}{2} |u'(t)|^2 + \frac{1}{2} \hat{M}\left(t, |\nabla u(t)|^2\right), \tag{2.1}$$

where

$$\hat{M}(t, \lambda) = \int_0^\lambda M(t, s)\, ds. \tag{2.2}$$

Now, we are in a position to state our main result.

Theorem 2.1. *Under assumptions* (A.1), (A.2), and (A.3), *problem* $(*)$ *possesses a unique solution* $u : (0, \infty) \to \mathbf{R}$ *in the class*

$$u \in L^\infty(0, \infty; H), \;\; u' \in L^\infty(0, \infty; V), \;\; u'' \in L^\infty(0, \infty; L^2(\Omega)).$$

Moreover, the energy determined by u *has the following decay rates*:

$$E(t) \leq CE(0)e^{-\omega t} \quad \text{if} \;\; p = 1, \;\; t \geq 0,$$

$$E(t) \leq \frac{CE(0)}{(1+t)^{2/(p-1)}} \quad \text{if} \;\; p > 1, \;\; t \geq 0$$

where C and ω are positive constants.

To end this section, we recall the following useful lemma:

Lemma 2.2 [5, Lemma 9.1]. *Let $E : \mathbf{R}_+ \to \mathbf{R}_+$ be a nonincreasing function, and assume that there exist two constants $p > 0$ and $C > 0$ such that*

$$\int_S^{+\infty} E^{\frac{p+1}{2}}(t)\, dt \leq CE(S), \quad 0 \leq S < +\infty.$$

Then, we have

$$E(t) \leq CE(0)(1+t)^{\frac{-2}{p-1}} \quad \text{for all } t \geq 0 \text{ if } p > 1,$$
$$E(t) \leq CE(0)e^{1-\omega t} \quad \text{for all } t \geq 0 \text{ if } p = 1,$$

where C and ω are positive constants.

3. The linearized equation with nonlinear boundary damping. This section is concerned with the existence and uniqueness of global solutions of the linearized equation with nonlinear boundary feedback,

$$\begin{cases} u_{tt} - \mu(t)\Delta u = 0 & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty) \\ \frac{\partial u}{\partial \nu} + g(u_t) = 0 & \text{on } \Gamma_1 \times (0,\infty) \\ u(0) = u^0, \ \ u_t(0) = u^1, \end{cases} \tag{3.1}$$

where u^0, u^1 and g satisfy the assumptions introduced before and

$$\mu \in W^{1,1}_{loc}(0,\infty); \quad \mu(t) \geq m_0 > 0 \tag{3.2}$$

for some $m_0 > 0$.

In what follows we obtain, by using Galerkin procedure, two a priori estimates that will play an essential role later, in order to obtain a global solution to problem $(*)$. However, we need some technical lemmas that will be used to construct a special basis, following similar ideas first introduced by Milla Miranda and Medeiros in [11] to the linear case, that is, when $g(s) = s$.

Lemma 3.1. *Given $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\Gamma_1)$ the unique solution of the boundary value problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial \nu} = g & \text{on } \Gamma_1 \end{cases} \tag{3.3}$$

belongs to H. Moreover, $||u||_H \leq C[|f| + ||g||_{H^{-1/2}(\Gamma_1)}]$.

Proof. Let $\{0,\ \hat{g}\} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ defined by

$$\hat{g} = \begin{cases} g & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_0. \end{cases}$$

By a trace theorem, there exists $h \in H$ such that $\gamma_0 h = 0$, $\gamma_1 h = \hat{g}$ and $||h||_H \leq C\, ||g||_{H^{-1/2}(\Gamma_1)}$. Let w be the solution of the following boundary value problem:

$$\begin{cases} -\Delta w = f + \Delta h & \text{in } \Omega \\ w = 0 & \text{on } \Gamma_0 \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \Gamma_1. \end{cases}$$

By a weak solution of (3.3) we mean to say that $w \in V$ and

$$\int_\Omega \nabla w \cdot \nabla v\, dx = \int_\Omega f v\, dx - \int_\Omega \Delta h v\, dx$$

for all $v \in V$. Thus, it follows that $w \in H^2(\Omega)$, and by using results of elliptic regularity one has

$$||w||_{H^2(\Omega)} \leq C\, [|f| + |\Delta h|]\, .$$

Then, $u = w + h$ belongs to H, is a solution of (3.3) and

$$\begin{aligned} ||u||_H = ||w+h||_H &\leq C(||w||_{H^2(\Omega)} + ||h||_H) \leq C\, [|f| + |\Delta h| + ||h||_H] \\ &\leq C\, [|f| + ||h||_H] \leq C[|f| + ||g||_{H^{-1/2}(\Gamma_1)}]. \end{aligned}$$

Lemma 3.2. *In H the norm of H and the one given by*

$$u \mapsto \left[|\Delta u|^2 + \left\| \frac{\partial u}{\partial \nu} \right\|^2_{H^{-1/2}(\Gamma_1)} \right]^{\frac{1}{2}}$$

are equivalent.

Proof. Let $u \in H$. Then, by Lemma 3.1, we have

$$\|u\|_H^2 \leq C \left[|\Delta u|^2 + \left\| \frac{\partial u}{\partial \nu} \right\|^2_{H^{-1/2}(\Gamma_1)} \right].$$

We also have $|\Delta u| \leq ||u||_H$ and $\left|\left| \frac{\partial u}{\partial \nu} \right|\right|_{H^{-1/2}(\Gamma_1)} \leq C\, ||u||_H$ where the second inequality is obtained by a trace theorem. □

Lemma 3.3. *Suppose $u^0 \in H$, $u^1 \in V$ and $\frac{\partial u^0}{\partial \nu} + g(u^1) = 0$ on Γ_1. Then, for each $\varepsilon > 0$ there exist w and z in H such that*

$$\left|\left|w - u^0\right|\right|_H < \varepsilon \quad \text{and} \quad \left|\left|z - u^1\right|\right|_V < \varepsilon$$

with $\frac{\partial w}{\partial \nu} + g(z) = 0$ on Γ_1.

Proof. Since H is dense in V, for each $\varepsilon > 0$ there exists $z \in H$ such that $\|z - u^1\|_V < \varepsilon$. Let us consider $w \in H$ a solution of

$$\begin{cases} -\Delta w = -\Delta u^0 & \text{in } \Omega \\ w = 0 & \text{on } \Gamma_0 \\ \frac{\partial w}{\partial \nu} = -g(z) & \text{on } \Gamma_1. \end{cases}$$

Notice that from assumptions $(A.2)$ and $(A.3)$ we have $g(z), g(u^1) \in L^2(\Gamma_1) \subset H^{-1/2}(\Gamma_1)$.

Considering Lemma 3.1 it follows that $w \in H$, and from Lemma 3.2 and since g is Lipschitzian we deduce

$$\begin{aligned} \|w - u^0\|_H^2 &= \left|\Delta w - \Delta u^0\right|^2 + \left\|\frac{\partial w}{\partial \nu} - \frac{\partial u^0}{\partial \nu}\right\|_{H^{-1/2}(\Gamma_1)}^2 \\ &\le C\left|\frac{\partial w}{\partial \nu} - \frac{\partial u^0}{\partial \nu}\right|_{\Gamma_1}^2 = C\left|g(z) - g(u^1)\right|_{\Gamma_1}^2 \\ &\le C\left|z - u^1\right|_{\Gamma_1}^2 \le C\left|\left|z - u^1\right|\right|_{H^{1/2}(\Gamma_1)} \le C\left|\left|z - u^1\right|\right|_V^2 < \varepsilon^2. \end{aligned}$$

Theorem 3.4. *Under assumptions* (A.2), (A.3) *and* (3.2), *problem* (3.1) *admits a unique solution u in the class*

$$u \in C_s^0(0, T; H) \cap C_s^1(0, T; V) \cap C_s^2(0, T; L^2(\Omega)) \quad \text{for all } T > 0.$$

Proof. From Lemma 3.3 we obtain two sequences (u_k^0) and (u_k^1) of vectors belonging to H satisfying the following conditions:

$$\lim_{k \to +\infty} u_k^0 = u^0 \quad \text{in} \quad H, \quad \lim_{k \to +\infty} u_k^1 = u^1 \quad \text{in} \quad V \quad \text{and} \tag{3.4}$$

$$\frac{\partial u_k^0}{\partial \nu} + g(u_k^1) = 0 \quad \text{on} \quad \Gamma_1, \quad \text{for all} \quad k \in \mathbf{N}.$$

We fix $k \in \mathbf{N}$. For u_k^0, u_k^1 linearly independent, we define the first two vectors $w_1^k = u_k^0$; $w_2^k = u_k^1$. By an orthonormalization process we construct a basis in H represented by

$$\{w_1^k, w_2^k, \dots, w_j^k, \dots\} \quad \text{for each} \quad k \in \mathbf{N}. \tag{3.5}$$

If for fixed $k \in \mathbf{N}$ the vectors u_k^0, u_k^1 are linearly dependent, we choose $w_k^1 = u_k^0$ and for w_2^k any vector outside of the line λu_k^0 and continue the above process. In fact we are interested that u_k^0 and u_k^1 belong to the basis. If $u_k^1 = \lambda u_k^0$ for some $\lambda \in \mathbf{R}$ then we already have u_k^1 belonging to the space generated by the above basis, and therefore we can choose as the second element any vector in the H that is linearly independent with u_k^0.

For $m \in \mathbf{N}$ we consider the subspace $V_m^k = [w_1^k, w_2^k, \dots, w_m^k]$ generated by the first m vectors w_j^k of (3.5). Let

$$u_{km}(t) = \sum_{j=1}^{m} \gamma_{kjm}(t) w_j^k$$

be the solution to the Cauchy problem

$$(u''_{km}(t), w) + \mu(t)\,(\nabla u_{km}(t), \nabla w) + \mu(t)\,(g(u'_{km}(t)), w)_{\Gamma_1} = 0 \tag{3.6}$$
$$u_{km}(0) = u_k^0;\;\; u'_{km}(0) = u_k^1.$$

By standard methods in differential equations, we can prove the existence of a solution to (3.6) on some interval $[0, t_{km})$. Then, this solution can be extended to the closed interval $[0, T]$, $T > 0$, by use of the first estimate below.

The First Estimate: Taking $w = u'_{km}(t)$ in (3.6) we obtain

$$\begin{aligned}\frac{d}{dt}\Big\{\frac{1}{2}|u'_{km}(t)|^2 + \frac{\mu(t)}{2}|\nabla u_{km}(t)|^2\Big\} + \mu(t)(g(u'_{km}(t)), u'_{km}(t))_{\Gamma_1} \\ = \frac{\mu'(t)}{2}|\nabla u_{km}(t)|^2.\end{aligned} \tag{3.7}$$

Integrating (3.7) over (0,t), it holds that

$$\begin{aligned}&|u'_{km}(t)|^2 + \mu(t)\,|\nabla u_{km}(t)|^2 + 2\int_0^t \mu(s)\,(g(u'_{km}(s)), u'_{km}(s))_{\Gamma_1}\,ds \\ &\le 2\int_0^t \frac{|\mu'(s)|}{\mu(s)}\Big[\,|u'_{km}(s)|^2 + \mu(s)\,|\nabla u_{km}(s)|^2 \\ &+ 2\int_0^s \mu(\xi)\,(g(u'_{km}(\xi)), u'_{km}(\xi))_{\Gamma_1}\,d\xi\Big]ds + \Big(\big|u_k^1\big|^2 + \mu(0)\,\big|\nabla u_k^0\big|^2\Big).\end{aligned}$$

Employing Gronwall's lemma, from the last inequality we obtain the first estimate

$$\begin{aligned}&|u'_{km}(t)|^2 + \mu(t)\,|\nabla u_{km}(t)|^2 + 2\int_0^t \mu(s)\,(g(u'_{km}(s)), u'_{km}(s))_{\Gamma_1}\;ds\\&\leq \big(|u^1_k|^2 + \mu(0)|\nabla u^0_k|^2\big)\exp\Big(2\int_0^t \frac{|\mu'(s)|}{\mu(s)}ds\Big);\;\; 0\leq t\leq T. \qquad (3.8)\end{aligned}$$

The Second Estimate: First of all we are going to estimate $u''_{km}(0)$ in the L^2 norm. Considering $t=0$ in (3.6) and $w = u''_{km}(0)$ we deduce

$$|u''_{km}(0)|^2 = -\mu(0)\,(\nabla u_{km}(0), \nabla u''_{km}(0)) - \mu(0)\,(g(u'_{km}(0)), u''_{km}(0))_{\Gamma_1}$$

for all $w \in V^k_m$. The use of the generalized Green's theorem yields

$$|u''_{km}(0)|^2 = \mu(0)\big(\Delta u_{km}(0), u''_{km}(0)\big) - \mu(0)\big(\frac{\partial u_{km}(0)}{\partial \nu} + g(u'_{km}(0)), u'_{km}(0)\big)_{\Gamma_1}.$$

At this point the importance of the special basis becomes clear. As $\frac{\partial u^0_k}{\partial\nu} + g(u^1_k) = 0$ on Γ_1 we conclude that

$$|u''_{km}(0)| \leq \mu(0)\left|\Delta u^0_k\right|. \qquad (3.9)$$

Now, getting the derivative of (3.6) with respect to t and considering $w = u''_{km}(t)$ it follows that

$$\begin{aligned}&\frac{d}{dt}\Big\{\frac{1}{2}|u''_{km}(t)|^2 + \frac{\mu(t)}{2}|\nabla u'_{km}(t)|^2\Big\} + \mu'(t)(\nabla u_{km}(t), \nabla u''_{km}(t))\\&+\mu(t)\int_{\Gamma_1} g'(u'_{km}(t))\,(u''_{km}(t))^2\,d\Gamma + \mu'(t)\int_{\Gamma_1} g(u'_{km}(t))u''_{km}(t)\,d\Gamma\\&= \frac{\mu'(t)}{2}\,|\nabla u'_{km}(t)|^2. \qquad (3.10)\end{aligned}$$

On the other hand, taking $w = \frac{\mu'(t)}{\mu(t)}u''_{km}(t)$ in (3.6) we infer

$$\begin{aligned}&\frac{\mu'(t)}{\mu(t)}\,|u''_{km}(t)|^2 + \mu'(t)\,(\nabla u_{km}(t), \nabla u''_{km}(t))\\&+\mu'(t)\int_{\Gamma_1} g(u'_{km}(t))u''_{km}(t)\,d\Gamma = 0. \qquad (3.11)\end{aligned}$$

Combining (3.10) and (3.11) we deduce

$$\frac{d}{dt}\Big\{\frac{1}{2}|u''_{km}(t)|^2+\frac{\mu(t)}{2}|\nabla u'_{km}(t)|^2\Big\}+\mu(t)\int_{\Gamma_1}g'(u'_{km}(t))(u''_{km}(t))^2d\Gamma$$
$$\leq\frac{|\mu'(t)|}{\mu(t)}\Big[\,|u''_{km}(t)|^2+\frac{\mu(t)}{2}\,|\nabla u'_{km}(t)|^2\Big].\tag{3.12}$$

Integrating (3.12) over $(0,t)$ and taking (3.9) into account, we obtain

$$|u''_{km}(t)|^2+\mu(t)\,|\nabla u'_{km}(t)|^2+2\int_0^t\mu(s)\int_{\Gamma_1}g'(u'_{km}(s))\,(u''_{km}(s))^2\,d\Gamma\,ds$$
$$\leq\mu^2(0)\,|\Delta u_k^0|^2+\mu(0)\,|\nabla u_k^1|^2+2\int_0^t\frac{|\mu'(s)|}{\mu(s)}\Big[\,|u''_{km}(s)|^2+\mu(s)\,|\nabla u'_{km}(s)|^2$$
$$+2\int_0^s\mu(\xi)\int_{\Gamma_1}g'(u'_{km}(\xi))\,(u''_{km}(\xi))^2\,d\Gamma\,d\xi\Big]\,ds.$$

The last inequality combined with Gronwall's lemma leads us to the second estimate

$$|u''_{km}(t)|^2+\mu(t)\,|\nabla u'_{km}(t)|^2+2\int_0^t\mu(s)\int_{\Gamma_1}g'(u'_{km}(s))\,(u''_{km}(s))^2\,d\Gamma\,ds$$
$$\leq\big(\mu^2(0)|\Delta u_k^0|^2+\mu(0)|\nabla u_k^1|^2\big)\exp\Big(2\int_0^t\frac{|\mu'(s)|}{\mu(s)}ds\Big).\tag{3.13}$$

We note that from the first estimate given by (3.8) and taking the assumptions (H.4), (H.5), (H.6) and (3.2) into account we also obtain the following estimates:

$$\int_0^t|u'_{km}(s)|^2_{\Gamma_1}\,ds\leq L;\quad\int_0^t|g(u'_{km}(s)|^2_{\Gamma_1}\,ds\leq L\tag{3.14}$$

where L is a positive constant independent of k,m and t.

For a fixed k, the a priori estimates (3.8), (3.13) and (3.14) permit, by using induction and diagonal processes, to obtain a subsequence $(u_{km}^{(p)})$ of (u_{km}) which from now on will also be denoted by (u_{km}) and a function $u_k:\Omega\times(0,\infty)\to\mathbf{R}$ satisfying:

$$u_{km}\rightharpoonup u_k\quad\text{weak star in }L^\infty_{loc}(0,\infty;V),\tag{3.15}$$
$$u'_{km}\rightharpoonup u'_k\quad\text{weak star in }L^\infty_{loc}(0,\infty;V),\tag{3.16}$$
$$u''_{km}\rightharpoonup u''_k\quad\text{weak star in }L^\infty_{loc}(0,\infty;L^2(\Omega)),\tag{3.17}$$
$$u'_{km}\rightharpoonup u'_k\quad\text{weakly in }L^2_{loc}(0,\infty;L^2(\Gamma_1)),\tag{3.18}$$
$$g(u'_{km})\rightharpoonup\chi_k\quad\text{weakly in }L^2_{loc}(0,\infty;L^2(\Gamma_1)).\tag{3.19}$$

Indeed, for the first time interval $[0,1]$ there exists a subsequence $(u_{km}^{(1)})$ of (u_{km}) and a function $u_k^{(1)} : \Omega \times (0,1) \to \mathbf{R}$ such that

$$u_{km}^{(1)} \rightharpoonup u_k^{(1)} \quad \text{weak star in } L^\infty(0,1;V).$$

For the interval $[0,p]$, for all $p \in \mathbf{N}$, $p \geq 2$, there exists a subsequence $(u_{km}^{(p)})$ of $(u_{km}^{(p-1)})$ and a function $u_k^{(p)} : \Omega \times (0,p) \to \mathbf{R}$ with $u_k^{(p)} = u_k^{(p-1)}$ on $(0,p-1)$ such that

$$u_{km}^p \rightharpoonup u_k^p \quad \text{weak star in } L^\infty(0,p;V).$$

Applying the induction with respect to p and the diagonal process, we obtain the existence of the function u_k and convergence (3.15). The other convergences (3.16)–(3.19) are obtained by analogous arguments.

The convergences (3.15)–(3.19) permit us to pass to the limit in (3.6). Since (w_j^k) is a basis of H, then for all $\theta \in D(0,\infty)$ and for all $v \in H$ after passing to the limit we obtain

$$\int_0^\infty (u_k''(t),v)\theta\,dt + \int_0^\infty \mu(t)(\nabla u_k(t),\nabla v)\theta\,dt + \int_0^\infty \mu(t)\int_{\Gamma_1} \chi_k v\,d\Gamma\,\theta\,dt = 0. \tag{3.20}$$

From (3.20) we obtain

$$u_k'' - \mu(t)\Delta u_k = 0 \quad \text{in } D'(\Omega \times (0,\infty)).$$

Now, since $u_k'' \in L_{loc}^\infty(0,\infty;L^2(\Omega))$ we have $\mu\Delta_k \in L_{loc}^\infty(0,\infty;L^2(\Omega))$, and therefore

$$u_k'' - \mu(t)\Delta u_k = 0 \quad \text{in } L^\infty(0,\infty;L^2(\Omega)). \tag{3.21}$$

Taking (3.21) into account and making use of the generalized Green's formula we deduce

$$\frac{\partial u_k}{\partial \nu} + \chi_k = 0 \quad \text{in } D'(0,\infty;H^{-1/2}(\Gamma_1)),$$

and since $\chi_k \in L_{loc}^2(0,\infty;L^2(\Gamma_1))$ we infer

$$\frac{\partial u_k}{\partial \nu} + \chi_k = 0 \quad \text{in } L_{loc}^2(0,\infty;L^2(\Gamma_1)). \tag{3.22}$$

Our goal is to show that $\chi_k = g(u_k')$. Indeed, considering $w = u_{km}$ in (3.6) and integrating the obtained expression over $(0,T)$, it holds that

$$\int_0^T (u_{km}''(t), u_{km}(t))\, dt + \int_0^T \mu(t)\, |\nabla u_{km}(t)|^2\, dt + \int_0^T \mu(t)\, (g(u_{km}'(t)), u_{km}(t))\, dt = 0. \tag{3.23}$$

From the first and second estimates and thanks to the Aubin-Lions theorem there exists a subsequence of (u_{km}) still denoted by the same notation such that

$$u_{km} \to u_k \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \tag{3.24}$$

$$u_{km}' \to u_k' \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{3.25}$$

Now, since $H^{1/2}(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$ is compact and noting that

$$||u_{km}(t)||_{H^{1/2}(\Gamma_1)} \le C\, |\nabla u_{km}(t)| \quad \text{and} \quad |u_{km}'(t)|_{\Gamma_1} \le C\, |\nabla u_{km}'(t)|$$

from the first and second estimates and again making use of the Aubin-Lions theorem, we deduce

$$u_{km} \to u_k \quad \text{strongly in } L^2(0,T;L^2(\Gamma_1)). \tag{3.26}$$

Then, from the convergences (3.17), (3.19), (3.24) and (3.26) we can pass to the limit in (3.23) in order to obtain

$$\lim_{m\to\infty} \int_0^T \mu(t)|\nabla u_{km}(t)|^2 dt = -\int_0^T (u_k''(t), u_k(t)) dt - \int_0^T \mu(t)(\chi_k(t), u_k(t))_{\Gamma_1} dt. \tag{3.27}$$

Combining (3.21), (3.22) and (3.27) and taking the generalized Green's formula into account, we deduce

$$\lim_{m\to\infty} \int_0^T \mu(t)\, |\nabla u_{km}(t)|^2\, dt = \int_0^T \mu(t)\, |\nabla u_k(t)|^2\, dt,$$

which implies, since $\mu(t) \ge \mu_0 > 0$, that

$$\nabla u_{mk} \to \nabla u_k \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{3.28}$$

Now, considering $w = u'_{mk}$ in (3.6) and integrating the obtained result over (0,T), it follows that

$$\int_0^T (u''_{km}(t), u'_{km}(t))\,dt + \int_0^T \mu(t)\,(\nabla u_{km}(t), \nabla u'_{km}(t))\,dt + \int_0^T \mu(t)\,(g(u'_{km}(t)), u'_{km}(t))_{\Gamma_1}\,dt = 0. \tag{3.29}$$

The convergences (3.16), (3.17), (3.25) and (3.28) together with (3.29) yield

$$\lim_{m\to\infty} \int_0^T \mu(t)\,(g(u'_{km}(t)), u'_{km}(t))_{\Gamma_1}\,dt = -\int_0^T (u''_k(t), u'_k(t))\,dt - \int_0^T \mu(t)\,(\nabla u_k(t), \nabla u'_k(t))\,dt,$$

and from (3.21) and (3.22), by applying Green's formula, we obtain

$$\lim_{m\to\infty} \int_0^T \mu(t)\,(g(u'_{km}(t)), u'_{km}(t))_{\Gamma_1}\,dt = \int_0^T \mu(t)\,(\chi_k(t), u'_k(t))_{\Gamma_1}\,dt. \tag{3.30}$$

On the other hand, since g is a nondecreasing monotone function, one has

$$\int_0^T \mu(t)\,(g(u'_{km}(t)) - g(\psi), u'_{km}(t) - \psi)_{\Gamma_1}\,dt \geq 0$$

for all $\psi \in L^2(\Gamma_1)$. The last inequality yields

$$\int_0^T \mu(t)\,(g(u'_{km}(t)), \psi)_{\Gamma_1}\,dt + \int_0^T \mu(t)\,(g(\psi), u'_{km}(t) - \psi)_{\Gamma_1}\,dt \leq \int_0^T \mu(t)\,(g(u'_{km}(t)), u'_{km}(t))_{\Gamma_1}\,dt. \tag{3.31}$$

From (3.31) we deduce

$$\liminf_{m\to\infty} \int_0^T \mu(t)\,(g(u'_{km}(t)), \psi)_{\Gamma_1}\,dt + \liminf_{m\to\infty} \int_0^T \mu(t)\,(g(\psi), u'_{km}(t) - \psi)_{\Gamma_1}\,dt \leq \liminf_{m\to\infty} \int_0^T \mu(t)\,(g(u'_{km}(t)), u'_{km}(t))_{\Gamma_1}\,dt.$$

Considering the convergences (3.18), (3.19) and (3.30) we infer

$$\int_0^T \mu(t)\left(\chi_k(t) - g(\psi), u_k'(t) - \psi\right)_{\Gamma_1} dt \geq 0. \tag{3.32}$$

Getting $\psi = u_k' + \lambda\xi$ where ξ is an arbitrary element of $L^2(\Gamma_1)$ and $\lambda > 0$, we can write

$$\int_0^T \mu(t)\left(\chi_k(t) - g(u_k'(t) + \lambda\xi), (-\lambda\xi)\right)_{\Gamma_1} dt \geq 0.$$

Consequently,

$$\int_0^T \mu(t)\left(\chi_k(t) - g(u_k'(t) + \lambda\xi), \xi\right) dt \leq 0 \quad \text{ for all } \xi \in L^2(\Gamma_1).$$

As the operator $g : L^2(\Gamma_1) \to \left(L^2(\Gamma_1)\right)' = L^2(\Gamma_1)$; $v \mapsto g(v)$ is hemicontinuous, one has

$$\int_0^T \mu(t)\left(\chi_k(t) - g(u_k'(t)), \xi\right)_{\Gamma_1} dt \leq 0 \quad \text{ for all } \xi \in L^2(\Gamma_1).$$

Hence,

$$\int_0^T \mu(t)\left(\chi_k(t) - g(u_k'(t)), \xi\right)_{\Gamma_1} dt = 0$$

for all $\xi \in L^2(\Gamma_1)$, which implies that $\chi_k = g(u_k')$.

We observe that the estimates (3.8), (3.13) and (3.14) also hold for all $k \in \mathbf{N}$. Thus, by the same process used to obtain the convergences (3.15)–(3.19) we get a diagonal sequence $(u_k^{(k)})$ still denoted by (u_k) and a function $u : \Omega \times (0, \infty) \to \mathbf{R}$ such that

$$u_k \rightharpoonup u \quad \text{weak star in } L^\infty_{loc}(0, \infty; V), \tag{3.33}$$

$$u_k' \rightharpoonup u' \quad \text{weak star in } L^\infty_{loc}(0, \infty; V), \tag{3.34}$$

$$u_k'' \rightharpoonup u'' \quad \text{weak star in } L^\infty_{loc}(0, \infty; L^2(\Omega)), \tag{3.35}$$

$$u_k' \rightharpoonup u' \quad \text{weakly } L^2_{loc}(0, \infty; L^2(\Gamma_1)), \tag{3.36}$$

$$g(u_k) \rightharpoonup \chi \quad \text{weakly } L^2_{loc}(0, \infty; L^2(\Gamma_1)). \tag{3.37}$$

By similar arguments to those used above, we can pass to the limit in (3.20) when $k \to +\infty$ to obtain

$$u'' - \mu(t)\Delta u = 0 \quad \text{in } L^\infty_{loc}(0,\infty; L^2(\Omega)), \tag{3.38}$$

$$\frac{\partial u}{\partial \nu} + g(u') = 0 \quad \text{in } L^2_{loc}(0,\infty; L^2(\Gamma_1)). \tag{3.39}$$

3.5. Uniqueness. Let u_1 and u_2 be two solutions of problem (3.1). Then, $z = u_1 - u_2$ satisfies

$$(z''(t), w) + \mu(t)\,(\nabla z(t), \nabla w) + \mu(t)\,(g(u_1') - g(u_2'), w)_{\Gamma_1} = 0, \tag{3.40}$$

for all $w \in V$. Substituting $w = z'(t)$ in (3.40) and observing that g is a monotone function it holds that

$$\frac{d}{dt}\Big\{\frac{1}{2}\,|z'(t)|^2 + \frac{\mu(t)}{2}\,|\nabla z(t)|^2\Big\} \le \frac{1}{2}\,|\mu'(t)|\,|\nabla z(t)|^2 .$$

Integrating the last inequality over $(0,t)$ we deduce

$$|z'(t)|^2 + \mu(t)\,|\nabla z(t)|^2 \le \int_0^t \frac{|\mu'(s)|}{\mu(s)} \Big(\mu(s)\,|\nabla z(t)|^2 + |z'(s)|^2\Big)\,ds.$$

Employing Gronwall's lemma the last inequality yields $|z'(t)| = |\nabla z(t)| = 0$. □

To finish this section it remains to prove that the unique solution of problem (3.1) belongs to the class

$$u \in C^0_s(0,T;H) \cap C^1_s(0,T;V) \cap C^2_s(0,T;L^2(\Omega)), \quad \text{for all } T > 0. \tag{3.41}$$

Indeed, we observe that from the above estimates and for all $T > 0$ one has

$$u \in C^0([0,T];V) \cap C^1([0,T];L^2(\Omega)). \tag{3.42}$$

Since $u \in L^\infty(0,T;H)$ and $u' \in L^\infty(0,T;V)$ from J.L. Lions and E. Magenes [Chapter 3, Lemma 8.1] we deduce

$$u \in C^0_s(0,T;H) \cap C^1_s(0,T;V). \tag{3.43}$$

On the other hand, since $u'' = \mu(t)\Delta u$ and $\mu(t) \ge \mu_0 > 0$, we also have $u'', \Delta u \in C^0([0,T];V') \cap L^\infty(0,T;L^2(\Omega))$. Therefore, $u'', \Delta u \in C^0_s(0,T;L^2(\Omega))$ which together (3.43) proves (3.41). Then Theorem 3.4 is proved.

4. Local and global solutions. In this section we first prove that problem $(*)$ possesses a local solution on some interval $[0, T_0]$ making use of a fixed-point method and moreover that this solution is unique. Then, considering the uniqueness of both problems $(*)$ and (3.1) and continuity arguments, we can extend this solution to the whole interval $(0, \infty)$ considering the a priori estimates in Section 3 to the linearized case.

Let us consider a real number $0 < T_0 < 1$ to be determined later and $R > 0$ such that $R > m_0^{-1/2}(R_1 + R_2)$, where

$$R_1 = (|u^1| + 1) + M^{1/2}(0, |\nabla u^0|^2)(|\nabla u^0| + 1), \tag{4.1}$$

$$R_2 = M(0, |\nabla u^0|^2)(|\Delta u^0| + 1) + M^{1/2}(0, |\nabla u^0|^2)(|\nabla u^1| + 1). \tag{4.2}$$

We denote by $\mathcal{H}$ the set of solutions u in $[0, T_0]$ of problem (3.1) in the class $u \in L^\infty(0, T_0; V)$; $u' \in L^\infty(0, T_0; V) \cap C^0([0, T_0]; L^2(\Omega))$.

So, let us induce in $\mathcal{H}$ the metric given by

$$d(u, v) = ||u - v||_{L^\infty(0,T_0;V)} + ||u' - v'||_{C^0([0,T_0];L^2(\Omega))},$$

and let B_{R,T_0} be the subset of $\mathcal{H}$ defined by

$$B_{R,T_0} = \Big\{ u \in L^\infty(0, T_0; V) : u' \in L^\infty(0, T_0; V) \cap C^0([0, T_0]; L^2(\Omega)),$$
$$||u||_{L^\infty(0,T_0;V)} + ||u'||_{L^\infty(0,T_0;V)} \le R, \ u(0) = u^0; u'(0) = u^1 \Big\}.$$

Let us notice that $\big(B_{R,T_0}, d(u, v)\big)$ is a complete metric space. Indeed, let (u_μ) be a Cauchy sequence in B_{R,T_0}. Then

$$u_\mu \to \xi \quad \text{strongly in} \quad L^\infty(0, T_0; V), \tag{4.3}$$

$$u'_\mu \to \xi' \quad \text{strongly in} \ C^0([0, T_0]; L^2(\Omega)). \tag{4.4}$$

As (u_μ) and (u'_μ) are bounded in $L^\infty(0, T_0; V)$ there exists a subsequence of (u_μ), still denoted by the same notation, such that

$$u_\mu \rightharpoonup u \quad \text{weak star in} \ L^\infty(0, T_0; V), \tag{4.5}$$

$$u'_\mu \rightharpoonup u' \quad \text{weak star in} \quad L^\infty(0, T_0; V). \tag{4.6}$$

Therefore, $u = \xi$ and

$$||u||_{L^\infty(0,T_0;V)} \le \liminf_{\mu\to\infty} ||u_\mu||_{L^\infty(0,T_0;V)},$$
$$||u'||_{L^\infty(0,T_0;V)} \le \liminf_{\mu\to\infty} ||u'_\mu||_{L^\infty(0,T_0;V)}.$$

Those inequalities combined with convergences (4.3)–(4.6) yield

$$||u||_{L^\infty(0,T_0;V)} + ||u'||_{L^\infty(0,T_0;V)} \leq R, \quad u(0) = u^0;\ u'(0) = u^1;$$

that is, $u \in B_{R,T_0}$. Moreover, by (4.3) and (4.4), $d(u_\mu, u)$ converges to zero. This concludes the desired argument.

Let S be the nonlinear mapping defined in the following way: $S : B_{R,T_0} \to \mathcal{H}$, $v \mapsto Sv = z$, where z is the unique solution of the problem

$$\begin{cases} z'' - M\big(t, |\nabla v(t)|^2\big)\Delta z = 0 & \text{in } \Omega \times (0, T_0) \\ z = 0 & \text{in } \Gamma_0 \times (0, T_0) \\ \frac{\partial z}{\partial \nu} + g(z') = 0 & \text{on } \Gamma_1 \times (0, T_0) \\ z(0) = u^0;\ z'(0) = u^1. \end{cases} \tag{4.7}$$

We observe that $\mu(t) = M\big(t, |\nabla v(t)|^2\big)$ belongs to $W^{1,1}(0, T_0)$ since

$$\mu'(t) = \frac{\partial M}{\partial t}\big(t, |\nabla v(t)|^2\big) + 2\frac{\partial M}{\partial \lambda}\big(t, |\nabla v(t)|^2\big)\big(\nabla v(t), \nabla v'(t)\big), \tag{4.8}$$

and, moreover, from assumptions (H.1)–(H.3) and the definition of B_{R,T_0} we have μ and μ' bounded in $[0, T_0]$. Then, by Theorem 3.4 there exists a unique solution z of problem (4.7), and this solution has the regularity of vectors in B_{R,T_0}. Therefore, the map S is well defined.

Our goal is to show that $S\big(B_{R,T_0}\big) \subset B_{R,T_0}$ and, moreover, that S is a strict contraction. Let

$$K^* = \max\Big\{\Big|\frac{\partial M}{\partial t}(t, \lambda)\Big|, \Big|\frac{\partial M}{\partial \lambda}(t, \lambda)\Big|; 0 \leq t \leq 1,\ \ 0 \leq \lambda \leq R^2\Big\}$$

and $\mu(t) = M\big(t, |\nabla v(t)|^2\big)$ with $v \in B_{R,T_0}$. Then, from (4.8) it holds that

$$|\mu'(t)| \leq K^*(1 + 2R^2);\ \ 0 \leq t \leq T_0. \tag{4.9}$$

Let $z \in S\big(B_{R,T_0}\big)$. Then, $z = S(v)$ for some $v \in B_{R,T_0}$. We will show that $z \in B_{R,T_0}$ for some T_0 sufficiently small. Indeed, since z is the solution of (4.7), it follows immediately that z belongs to the class of vectors of B_{R,T_0}, $z(0) = u^0$; $z'(0) = u^1$. It remains to show that

$$||z||_{L^\infty(0,T_0;V)} + ||z'||_{L^\infty(0,T_0;V)} \leq R. \tag{4.10}$$

In fact, from assumption (H.1), (3.8), (3.13) , (4.1) and (4.2) we deduce

$$m_0^{1/2}\big|\nabla z_{km}(t)\big| \leq R_1 \exp\Big(\int_0^{T_0} \frac{|\mu'(t)|}{\mu(t)}\,dt\Big), \tag{4.11}$$

$$m_0^{1/2}\big|\nabla z'_{km}(t)\big| \leq R_2 \exp\Big(\int_0^{T_0} \frac{|\mu'(t)|}{\mu(t)}\,dt\Big), \tag{4.12}$$

for all $t \in [0, T_0]$, $k \geq k_1$ and $m \geq 1$. The inequalities (4.11) and (4.12) combined with (4.9) give us

$$||z||_{L^\infty(0,T_0;V)} + ||z'||_{L^\infty(0,T_0;V)} \leq m_0^{-1/2}(R_1 + R_2)\exp(K_1 T_0)$$

where $K_1 = m_0^{-1} K^*(1 + 2R^2)$. Since $R > m_0^{-1/2}(R_1 + R_2)$ there exists $0 < T_0 < 1$ sufficiently small such that (4.10) holds.

It remains to prove that S is a strict contraction. To this end we consider $v_1, v_2 \in B_{R,T_0}$, $z_1 = S(v_1)$, $z_2 = S(v_2)$ and $w = z_1 - z_2$. Then, the first equation of (4.7) implies that

$$w'' - M\big(t, \big|\nabla v_1(t)\big|^2\big)\Delta w = \big[M\big(t, \big|\nabla v_1(t)\big|^2\big) - M\big(t, \big|\nabla v_2(t)\big|^2\big)\big]\Delta z_2.$$

Taking the inner product of $L^2(\Omega)$ on both sides of the above identity with w' and noting that

$$\frac{\partial w}{\partial \nu} = g(z'_2) - g(z'_1) \quad \text{on } \Gamma_1$$

we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big|w'(t)\big|^2 + M\big(t, \big|\nabla v_1(t)\big|^2\big)\big(\nabla w(t), \nabla w'(t)\big) \\ &+ M\big(t, \big|\nabla v_1(t)\big|^2\big)\big(g(z'_1(t)) - g(z'_2(t)), w'(t)\big)_{\Gamma_1} \\ &= \big[M\big(t, \big|\nabla v_1(t)\big|^2\big) - M\big(t, \big|\nabla v_2(t)\big|^2\big)\big]\big(\Delta z_2(t), w'(t)\big). \end{aligned} \tag{4.13}$$

Since g is a monotonic increasing function and

$$\begin{aligned} &\frac{d}{dt}[M(t, |\nabla v_1(t)|^2)|\nabla w(t)|^2] \\ &= \mu'_1(t)|\nabla w(t)|^2 + 2M(t, |\nabla v_1(t)|^2)(\nabla w(t), \nabla w'(t)), \end{aligned}$$

where

$$\mu_1'(t) = \frac{\partial M}{\partial t}\big(t, \big|\nabla v_1(t)\big|^2\big) + 2\frac{\partial M}{\partial \lambda}\big(t, \big|\nabla v_1(t)\big|^2\big)\big(\nabla v_1(t), \nabla v_1'(t)\big), \quad (4.14)$$

from (4.13) we obtain

$$\frac{d}{dt}\Big\{\frac{1}{2}\big|w'(t)\big|^2 + \frac{1}{2}M\big(t, \big|\nabla v_1(t)\big|^2\big)\big|\nabla w(t)\big|^2\Big\} \quad (4.15)$$
$$\leq \frac{1}{2}\big|\mu_1'(t)\big|\big|\nabla w(t)\big|^2 + \big[M\big(t, \big|\nabla v_1(t)\big|^2\big) - M\big(t, \big|\nabla v_2(t)\big|^2\big)\big]\big(\Delta z_2(t), w'(t)\big).$$

But, by using the same arguments used in (4.9) we infer

$$\big|\mu_1'(t)\big| \leq K^*(1 + 2R^2); \quad 0 \leq t \leq T_0. \quad (4.16)$$

On the other hand, for a fixed t, there exists λ^* such that

$$M\big(t, \big|\nabla v_1(t)\big|^2\big) - M\big(t, \big|\nabla v_2(t)\big|^2\big) = \frac{\partial M}{\partial \lambda}(t, \lambda^*)\big(\big|\nabla v_1(t)\big|^2 - \big|\nabla v_2(t)\big|^2\big).$$

The identity above and the fact that

$$\Big|\frac{\partial M}{\partial \lambda}(t, \lambda^*)\Big|\Big|\big|\nabla v_1(t)\big|^2 - \big|\nabla v_2(t)\big|^2\Big| \leq K^*\big(\big|\nabla v_1(t)\big| + \big|\nabla v_2(t)\big|\big)d(v_1, v_2)$$

imply

$$\Big|M\big(t, \big|\nabla v_1(t)\big|^2\big) - M\big(t, \big|\nabla v_2(t)\big|^2\big)\Big| \leq 2K^* R d(v_1, v_2). \quad (4.17)$$

Also, the second estimate (3.13) yields

$$\big|z_2''(t)\big| \leq R_2 \exp(K_1 T_0); \quad 0 \leq t \leq T_0.$$

The last inequality and the equation

$$z_2'' = M\big(t, \big|\nabla v_2(t)\big|^2\big)\Delta z_2$$

imply that

$$\big|\Delta z_2(t)\big| \leq m_0^{-1} R_2 \exp(K_1 T_0); \quad 0 \leq t \leq T_0. \quad (4.18)$$

Integrating (4.15) over $(0,t)$, $t \in [0,T_0]$ and taking (4.16), (4.17) and (4.18) into account we deduce

$$\begin{aligned} &\frac{1}{2}\left|w'(t)\right|^2 + \frac{m_0}{2}\left|\nabla w(t)\right|^2 \qquad (4.19) \\ &\le K^* R m_0^{-2} R_2^2 \exp(2K_1 T_0) T_0 d(v_1,v_2) + \frac{L}{2}\int_0^t \left(m_0\left|\nabla w(s)\right|^2 + \left|w'(s)\right|^2\right) ds \end{aligned}$$

where $L = \max\left\{K^* m_0^{-1}(1+2R^2), 2K^* R(d(v_1,v_2)\right\}$. From (4.19) and applying Gronwall's inequality, it holds that

$$\left|w'(t)\right|^2 + m_0\left|\nabla w(t)\right|^2 \le N d(v_1,v_2) T_0 \exp(LT_0)$$

where $N = 2K^* R m_0^{-2} R_2^2 \exp(2K_1 T_0)$. The last inequality implies that we can choose T_0 sufficiently small such that

$$d(Sv_1, Sv_2) \le \alpha d(v_1,v_2); \quad 0 < \alpha < 1. \qquad (4.20)$$

Thus, S is a strict contraction. Taking T_0 that satisfies (4.10) and (4.20) simultaneously we deduce that S has a unique fixed point u. This function u is the required solution.

Our purpose, from now on, is to extend the solution u to the whole interval $[0,\infty)$. To this end, let us consider the *linearized problem*:

$$\begin{cases} w'' - \mu(t)\Delta w = 0 & \text{in } \Omega \times (0,T_0) \\ w = 0 & \text{on } \Gamma_0 \times (0,T_0) \\ \frac{\partial w}{\partial \nu} + g(w') = 0 & \text{on } \Gamma_1 \times (0,T_0) \\ w(0) = u^0;\ \ w'(0) = u^1 & \text{in } \Omega \end{cases} \qquad (4.21)$$

with $\mu(t) = M\left(t, \left|\nabla u(t)\right|^2\right)$; $t \in [0,T_0]$ where u is the local solution obtained by the fixed-point method. As $u \in B_{R,T_0}$ it follows that $\mu \in W^{1,1}(0,T_0)$. Moreover, $\mu(t) \ge m_0 > 0$ for all $t \in [0,T_0]$. According to Theorem 3.4 the solution w of (4.21) belongs to

$$w \in C_s^0(0,T_0;H) \cap C_s^1(0,T_0;V) \cap C_s^2(0,T_0;L^2(\Omega)). \qquad (4.22)$$

As u is a solution of problem (4.21) and this problem has uniqueness of solutions, we have $u = w$; therefore u possesses the regularity given by (4.22).

Notice that $u(T_0)$ and $u'(T_0)$ have properties analogous to $u^0$and u^1. Then, we can obtain a solution v on $[0, T_1]$, $T_1 > 0$ of problem $(*)$ with $M_1(t, \lambda) = M(t + T_0, \lambda)$ and initial data $u(T_0)$, $u'(T_0)$. The function

$$w(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq T_0 \\ v(t - T_0) & \text{if } T_0 \leq t \leq T_0 + T_1 \end{cases}$$

is a solution on $[0, T_0 + T_1]$ of problem $(*)$ with initial data u^0 and u^1.

We observe that uniqueness of the solution of problem $(*)$ on $[0, T]$, where $[0, T]$ is any interval of existence, follows exactly making use of similar arguments introduced before to prove that S is a strict contraction. In fact it is only possible because we are dealing with strong solutions.

Let $\big(u_i(t)\big)_{t \in I}$ be the family of solutions on $[0, T_i]$ of problem $(*)$ with initial data $\{u^0, u^1\}$ satisfying the assumptions $(A.3)$, fixed in the space $H \times V$. The uniqueness of solutions implies that if $T_i < T_j$ then u_i and u_j coincide on $[0, T_i]$. Then, from Zorn's lemma we find a maximal interval $[0, T_{max})$ given by $[0, T_{max}) = \cup_{i \in I}[0, T_i)$ where the solution of problem $(*)$ is given by $u(t) = u_i(t), t \in [0, T_i]$. We observe that

$$u \in C^0_s(0, T; H) \cap C^1_s(0, T; V) \cap C^2_s(0, T; L^2(\Omega)), \tag{4.23}$$

for all $0 < T < T_{\max}$.

In order to prove that $T_{\max} = +\infty$ let us consider firstly the following:

Lemma 4.1. *Let u be the solution of $(*)$ in $[0, T_{\max})$. We have the following:*

$$\textit{If } \limsup_{t \to T_{\max}} \left[\left|\Delta u(t)\right|^2 + \left|\nabla u'(t)\right|^2\right] < +\infty, \quad \textit{then } T_{\max} = +\infty.$$

Proof. We argue by contradiction. So, let us suppose that $T_{max} < \infty$. Then, since $\left|\Delta u(t)\right|^2 + \left|\nabla u'(t)\right|^2 \leq L$ for some $L > 0$ and for all $0 \leq t \leq T_{\max}$ and considering that $\frac{\partial u}{\partial \nu} + g(u') = 0$ on Γ_1 it follows that there exists $C > 0$ such that

$$\left|\Delta u(t)\right| + \left|\left|\frac{\partial u}{\partial \nu}(t)\right|\right|_{H^{-1/2}(\Gamma_1)} + \left|\nabla u'(t)\right| \leq C; \quad 0 \leq t \leq T_{\max}. \tag{4.24}$$

Now, from the fact

$$\left|\nabla u(t_2) - \nabla u(t_1)\right| \leq \int_{t_1}^{t_2} \left|\nabla u'(s)\right| ds \leq C\left|t_1 - t_2\right|$$

it holds that

$$u(t) \to \xi \quad \text{strongly in } V \quad \text{as } t \to T_{\max}. \tag{4.25}$$

Let (t_μ) be a sequence in $[0, T_{\max})$ with $t_\mu \to T_{\max}$. Then, by (4.24) there exists a subsequence $(t_{\mu'})$ of (t_μ) such that

$$u(t_{\mu'}) \rightharpoonup \alpha \quad \text{weakly in } H. \tag{4.26}$$

From (4.25) and (4.26) it follows that $\alpha = \xi$; that is, there exists the H-weak limit of $(u(t_\mu))$, and this limit is independent of the sequence (t_μ). Therefore

$$u(t) \rightharpoonup \xi \quad \text{weakly in } H \quad \text{as } t \to T_{\max}. \tag{4.27}$$

As $u'' = M\big(t, \left|\nabla u(t)\right|^2\big)\Delta u(t)$, inequality (4.24) implies that there exists a constant C_1 such that $\left|u''(t)\right| \le C_1$, for all $0 \le t < T_{\max}$. Then, by analogous arguments to those used to obtain (4.27) we deduce

$$u'(t) \rightharpoonup \psi \quad \text{weakly in } V \quad \text{as } t \to T_{\max}. \tag{4.28}$$

On the other hand, we observe that from (4.28) and taking the assumptions (H.5) and (H.6) into account, we infer

$$g(u'(t)) \rightharpoonup \chi \quad \text{weakly in } L^2(\Gamma_1). \tag{4.29}$$

However, from (4.27) we have

$$g(u'(t)) = \frac{\partial u(t)}{\partial \nu} \rightharpoonup \frac{\partial \xi}{\partial \nu} \quad \text{weakly in } H^{-1/2}(\Gamma_1),$$

and consequently $\frac{\partial \xi}{\partial \nu} = \chi$. Our purpose is to show that

$$\chi = g(\psi). \tag{4.30}$$

Indeed, in order to prove (4.30) and since g is a monotonic increasing function it is sufficient to prove that

$$\big(g(u'(t)), u'(t)\big)_{\Gamma_1} \to \big(\chi, \psi\big)_{\Gamma_1} \quad \text{as } t \to T_{\max}. \tag{4.31}$$

But, to prove (4.31) let us note that $u''(t) = M\big(t, \left|\nabla u(t)\right|^2\big)\Delta u(t)$ for all $t \in [0, T_{\max})$; then getting the inner product of $L^2(\Omega)$ in both sides of the last identity with $u'(t)$, taking the above weak and strong convergences into account and making use of the generalized Green's formula we deduce (4.31) and consequently (4.30). Then, from (4.27), (4.29) and (4.30) we have

$$\frac{\partial u(t)}{\partial \nu} + g(u'(t)) \rightharpoonup \frac{\partial \xi}{\partial \nu} + g(\psi) \quad \text{weakly in } H^{-\frac{1}{2}}(\Gamma_2) \text{ as } t \to T_{\max},$$

and therefore $\frac{\partial \xi}{\partial \nu} + g(\psi) = 0$ on Γ_1.

Let us consider v the solution of problem $(*)$ with $M_1(t, \lambda) = M(t + T_{\max}, \lambda)$ and initial conditions ξ, ψ. Then, $v(t)$ is defined on $[0, T_1)$ for some $T_1 > 0$. Let

$$w(t) = \begin{cases} u(t) & \text{if } 0 \le t < T_{\max} \\ v(t - T_{\max}) & \text{if } T_{\max} \le t < T_{max} + T_1. \end{cases}$$

Thus, w extends u which is a contradiction. Consequently $T_{\max} = +\infty$. □

Based on Lemma 4.1 we are able to extend the solution of problem $(*)$ to the whole interval $[0, \infty)$. Indeed, for this purpose let us consider the problem

$$\begin{cases} w'' - \mu(t)\Delta w = 0 & \text{in } \Omega \times (0, T_{\max}) \\ w = 0 & \text{on } \Gamma_0 \times (0, T_{\max}) \\ \frac{\partial w}{\partial \nu} + g(w') = 0 & \text{on } \Gamma_1 \times (0, T_{\max}) \\ w(0) = u^0; \;\; w'(0) = u^1 \end{cases} \tag{4.32}$$

with $\mu(t) = M\big(t, \left|\nabla u(t)\right|^2\big)$; $t \in [0, T_{\max})$, where u is the maximal solution of problem $(*)$. We recall that $u \in C_s^0(0, T; H) \cap C_s^1(0, T; V) \cap C_s^2(0, T; L^2(\Omega))$, for all $0 < T < T_{\max}$. Therefore, $\mu \in W_{loc}^{1,1}(0, T_{\max})$. As u is also a solution of problem (4.32) it coincides with the unique solution w of the same problem. If $T_{\max} < +\infty$, then, from the second a priori estimate (3.13) we deduce

$$\left|w''(t)\right|^2 + m_0\left|\nabla w'(t)\right|^2 \le C(T_{\max}) \tag{4.33}$$

for all $t \in [0, T_{\max})$ where $C(T_{\max})$ is a constant which depends on $T_{\max}$. Now, since $w'' = \mu(t)\Delta w$ and $\mu(t) \ge m_0 > 0$ the estimate (4.33) holds for $\left|\Delta w(t)\right|^2$. In other words, we also have

$$\left|\Delta w(t)\right|^2 + \left|\nabla w'(t)\right|^2 \le C(T_{max}) \tag{4.34}$$

for all $t \in [0, T_{\max})$.

Next, we are going to prove that $T_{\max} = +\infty$. Let us argue by contradiction. So, let us suppose that $T_{\max} < +\infty$. According to Lemma 4.1

$$\text{If } T_{\max} < +\infty, \quad \text{then} \quad \limsup_{t \to T_{\max}} \left[\left| \Delta u(t) \right|^2 + \left| \nabla u'(t) \right|^2 \right] = +\infty. \tag{4.35}$$

But, as $u = w$ in $[0, T_{\max})$ from (4.34) we deduce that

$$\limsup_{t \to T_{\max}} \left[\left| \Delta u(t) \right|^2 + \left| \nabla u'(t) \right|^2 \right] \le C(T_{\max}),$$

which, in view of (4.35), is a contradiction. Therefore, $T_{\max} = +\infty$ as we desired to prove.

Remark. As we realize, the comparison, by uniqueness, of the solutions of the linearized problem (4.30) and the nonlinear one $(*)$ plays an essential role in obtaining the extension to the whole interval $[0, \infty)$. Unfortunately this can be applied only when we have strong solutions. But, at least, now it is clear that we don't need smallness of the initial data to obtain the desired extended solution.

5. Uniform decay. In this section we prove the exponential and algebraic decay of solutions of $(*)$. For the rest of this section let x^0 be a fixed point in $\mathbf{R}^n$. Then, consider $m(x) = x - x^0$; $R = \max_{x \in \overline{\Omega}} \|m(x)\|$ and a partition of the boundary Γ into two pieces $\Gamma_0 = \{x \in \Gamma : m(x) \cdot \nu(x) \le 0\}$ and $\Gamma_1 = \{x \in \Gamma : m(x) \cdot \nu(x) \ge \delta > 0\}$. From $(*)$ and (H.1) we deduce that

$$E'(t) \le -m_0 \int_{\Gamma_1} g(u')u' \, d\Gamma \le 0. \tag{5.1}$$

Set

$$Nu := 2\big(m \cdot \nabla u\big) + (n-1)u. \tag{5.2}$$

from $(*)$ after integration by parts, we obtain

$$\begin{aligned} 0 &= \int_S^T E^{\frac{p-1}{2}} \Big(u''(t) - M\big(t, \left|\nabla u(t)\right|^2\big)\Delta u(t), (Nu)(t)\Big)\, dt \qquad (5.3) \\ &= \Big[E^{\frac{p-1}{2}} \big(u'(t), (Nu)(t)\big)\Big]_S^T - \frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E' \big(u'(t), (Nu)(t)\big)\, dt \\ &\quad - \int_S^T E^{\frac{p-1}{2}} \big(u'(t), (Nu')(t)\big) dt - \int_S^T E^{\frac{p-1}{2}} M\big(t, \left|\nabla u(t)\right|^2\big) \big(\Delta u(t), (Nu)(t)\big) dt, \end{aligned}$$

where $0 \le S < T < +\infty$. Using Green's and Gauss's formula, we infer

$$\big(u'(t),(Nu')(t)\big) = -\big|u'(t)\big|^2 + \int_{\Gamma_1}(m\cdot\nu)\big|u'\big|^2 d\Gamma \tag{5.4}$$

and

$$\begin{aligned}
&M\big(t,\big|\nabla u(t)\big|^2\big)\big(\Delta u(t),(Nu)(t)\big) = -M\big(t,\big|\nabla u(t)\big|^2\big)\big|\nabla u(t)\big|^2\\
&+2M\big(t,\big|\nabla u(t)\big|^2\big)\int_\Gamma \frac{\partial u}{\partial\nu}\big(m\cdot\nabla u\big)d\Gamma - M\big(t,\big|\nabla u(t)\big|^2\big)\int_\Gamma\big(m\cdot\nu\big)\big|\nabla u\big|^2 d\Gamma\\
&+(n-1)M\big(t,\big|\nabla u(t)\big|^2\big)\int_{\Gamma_1} g(u')u\,d\Gamma.
\end{aligned}\tag{5.5}$$

Thus, (5.3)–(5.5) yield

$$\begin{aligned}
&\int_S^T E^{\frac{p-1}{2}}\big[\big|u'(t)\big|^2 + M\big(t,\big|\nabla u(t)\big|^2\big)\big|\nabla u(t)\big|^2\big]\,dt = -\big[E^{\frac{p-1}{2}}\big(u'(t),(Nu)(t)\big)\big]_S^T\\
&+\frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\big(u'(t),(Nu)(t)\big)\,dt\\
&+\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(m\cdot\nu)\big|u'\big|^2 d\Gamma dt + 2\int_S^T E^{\frac{p-1}{2}}M\big(t,\big|\nabla u(t)\big|^2\big)\int_\Gamma\frac{\partial u}{\partial\nu}\big(m\cdot\nabla u\big)d\Gamma dt\\
&-\int_S^T E^{\frac{p-1}{2}}M\big(t,\big|\nabla u(t)\big|^2\big)\int_\Gamma\big(m\cdot\nu\big)\big|\nabla u\big|^2 d\Gamma dt\\
&-(n-1)\int_S^T E^{\frac{p-1}{2}}M\big(t,\big|\nabla u(t)\big|^2\big)\int_{\Gamma_1} g(u')u\,d\Gamma dt.
\end{aligned}\tag{5.6}$$

On the other hand, observe that from assumptions (H.2) and (H.3) we have

$$M\big(t,\big|\nabla u(t)\big|^2\big) \le M\big(0,\big|\nabla u^0\big|^2\big) + K\big[\big|\nabla u(t)\big|^2 + \big|\nabla u^0\big|^2\big],$$

and since $u \in L^\infty(0,\infty;V)$ it follows that

$$m_0 \le M\big(t,\big|\nabla u(t)\big|^2\big) \le m_1 \quad \text{for all } t \ge 0 \tag{5.7}$$

where m_1 is a positive constant. Then, from (H.2) and (5.7) we deduce

$$\hat{M}\big(t,\big|\nabla u(t)\big|^2\big) \le \frac{m_1}{m_0}M\big(t,\big|\nabla u(t)\big|^2\big)\big|\nabla u(t)\big|^2 \quad \text{for all } t \ge 0. \tag{5.8}$$

Furthermore, on Γ_0 we have $\frac{\partial u}{\partial x_k} = \frac{\partial u}{\partial \nu}\nu_k$, which implies that

$$m \cdot \nabla u = \big(m \cdot \nu\big)\big(\frac{\partial u}{\partial \nu}\big) \ \text{ and } \ \big|\nabla u\big|^2 = \big(\frac{\partial u}{\partial \nu}\big)^2 \ \text{ on } \ \Gamma_0.$$

Consequently,

$$-\int_{\Gamma}\big(m \cdot \nu\big)\big|\nabla u\big|^2 d\Gamma = -\int_{\Gamma_1}\big(m \cdot \nu\big)\big|\nabla u\big|^2 d\Gamma - \int_{\Gamma_0}\big(m \cdot \nu\big)\big(\frac{\partial u}{\partial \nu}\big)^2 d\Gamma, \quad (5.9)$$

$$2\int_{\Gamma}\frac{\partial u}{\partial \nu}\big(m \cdot \nabla u\big)d\Gamma = 2\int_{\Gamma_1}\frac{\partial u}{\partial \nu}\big(m \cdot \nabla u\big)d\Gamma + 2\int_{\Gamma_0}\big(m \cdot \nu\big)\big(\frac{\partial u}{\partial \nu}\big)^2 d\Gamma. \quad (5.10)$$

Observing that $\int_{\Gamma_0}\big(m \cdot \nu\big)\big(\frac{\partial u}{\partial \nu}\big)^2 d\Gamma \leq 0$ and $\frac{\partial u}{\partial \nu} = -g(u')$ on Γ_1, from (5.6), (5.8), (5.9) and (5.10) it holds that

$$\begin{aligned}
&L^{-1}\int_S^T E^{\frac{p-1}{2}}\big[\big|u'(t)\big|^2 + \hat{M}\big(t, \big|\nabla u(t)\big|^2\big)\big]dt \leq -\big[E^{\frac{p-1}{2}}\big(u'(t),(Nu)(t)\big)\big]_S^T \\
&+ \frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\big(u'(t),(Nu)(t)\big)dt + \int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(m \cdot \nu)\big|u'\big|^2 d\Gamma dt \\
&- 2\int_S^T E^{\frac{p-1}{2}}M\big(t, \big|\nabla u(t)\big|^2\big)\int_{\Gamma_1} g(u')\big(m \cdot \nabla u\big)d\Gamma dt \\
&- \int_S^T E^{\frac{p-1}{2}}M\big(t, \big|\nabla u(t)\big|^2\big)\int_{\Gamma_1}\big(m \cdot \nu\big)\big|\nabla u\big|^2 d\Gamma dt \\
&+ (n-1)\int_S^T E^{\frac{p-1}{2}}M\big(t, \big|\nabla u(t)\big|^2\big)\int_{\Gamma_1} g(u')u\, d\Gamma dt, \qquad (5.11)
\end{aligned}$$

where $L = \max\big\{1, \frac{m_1}{m_0}\big\}$. Then we conclude

$$-2\int_{\Gamma_1} g(u')\big(m \cdot \nabla u\big)d\Gamma \leq \delta\int_{\Gamma_1}\big|\nabla u\big|^2 d\Gamma + \frac{R^2}{\delta}\int_{\Gamma_1}\big|g(u')\big|^2 d\Gamma. \quad (5.12)$$

Combining (5.11) and (5.12), and since $m \cdot \nu \geq \delta > 0$ on Γ_1, we obtain

$$\begin{aligned}
&L^{-1}\int_S^T E^{\frac{p-1}{2}}\big[\big|u'(t)\big|^2 + \hat{M}\big(t, \big|\nabla u(t)\big|^2\big)\big]dt \qquad (5.13) \\
&\leq -\big[E^{\frac{p-1}{2}}\big(u'(t),(Nu)(t)\big)\big]_S^T + \frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\big(u'(t),(Nu)(t)\big)dt
\end{aligned}$$

$$+\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(m\cdot\nu)\big|u'\big|^2 d\Gamma dt+\frac{R^2 m_1}{\delta}\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}\big|g(u')\big|^2 d\Gamma dt$$
$$+(n-1)m_1\int_S^T E^{\frac{p-1}{2}}\Big|\int_{\Gamma_1} g(u')u\,d\Gamma\Big|\,dt.$$

Next, we are going to analyze terms on the right-hand side of (5.13). Although it is a well-known procedure we are going to repeat it so that it will become clear to the reader.

Estimate for $I_1 := -\big[E^{\frac{p-1}{2}}\big(u'(t),(Nu)(t)\big)\big]_S^T$. Here and in the sequel C will denote various positive constants which may be different at different occurrences. We observe initially that

$$\big|\big(u'(t),(Nu)(t)\big)\big|\le CE(t).$$

Consequently,

$$\big|I_1\big|=\big|\big[E^{\frac{p-1}{2}}\big(u'(t),(Nu)(t)\big)\big]_S^T\big|\le CE(S). \tag{5.14}$$

Estimate for $I_2 := \frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\big(u'(t),(Nu)(t)\big)dt$. We have

$$\big|E^{\frac{p-3}{2}}E'\big(u'(t),(Nu)(t)\big)\big|\le CE^{\frac{p-3}{2}}\big|E'\big|E\le -C\big(E^{\frac{p+1}{2}}\big)'.$$

Then

$$\big|I_2\big|\le -C\int_S^T\big(E^{\frac{p+1}{2}}\big)'dt\le CE(S). \tag{5.15}$$

Estimate for $I_3 := \int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(m\cdot\nu)\big|u'\big|^2 d\Gamma dt$. From the assumptions (H.5) and (H.6) and making use of Hölder's inequality, we deduce

$$\int_{|u'|\le 1}(m\cdot\nu)\big|u'\big|^2 d\Gamma\le C\int_{|u'|\le 1}(m\cdot\nu)\big(u'g(u')\big)^{\frac{2}{p+1}}d\Gamma$$
$$\le C\Big(\int_{\Gamma_1}u'g(u')d\Gamma\Big)^{\frac{2}{p+1}}\le C\big(-E'(t)\big)^{\frac{2}{p+1}}$$

and

$$\int_{|u'|>1}(m\cdot\nu)\big|u'\big|^2 d\Gamma\le C\int_{|u'|>1}(m\cdot\nu)u'g(u')\,d\Gamma\le -CE'(t).$$

Hence,

$$
\begin{aligned}
\big|I_3\big| &\le C\int_S^T E^{\frac{p-1}{2}}(t)\big(-E'(t)\big)^{\frac{2}{p+1}}dt - C\int_S^T E^{\frac{p-1}{2}}(t)E'(t)\,dt\\
&\le \int_S^T \big(\varepsilon E^{\frac{p+1}{2}}(t) - C(\varepsilon)E'(t)\big)dt - C\int_S^T \big(E^{\frac{p+1}{2}}\big)'dt\\
&\le \varepsilon\int_S^T E^{\frac{p+1}{2}}(t)\,dt + C(\varepsilon)E(S) + CE(S). \qquad (5.16)
\end{aligned}
$$

Estimate for $I_4 := \frac{R^2m_1}{\delta}\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}\big|g(u')\big|^2 d\Gamma dt$. From assumptions (H.5) and (H.6) we can write

$$
\begin{aligned}
\frac{R^2m_1}{\delta}\int_{|u'|\le 1}\big|g(u')\big|^2 d\Gamma &\le C\int_{|u'|\le 1}\big(u'g(u')\big)^{\frac{2}{p+1}}d\Gamma\\
&\le C\Big(\int_{\Gamma_1} u'g(u')d\Gamma\Big)^{\frac{2}{p+1}} \le C\big(-E'(t)\big)^{\frac{2}{p+1}}
\end{aligned}
$$

and

$$
\frac{R^2m_1}{\delta}\int_{|u'|>1}\big|g(u')\big|^2 d\Gamma \le C\int_{|u'|>1} u'g(u')\,d\Gamma \le -CE'(t).
$$

Analogously we deduce

$$
\big|I_4\big| \le \varepsilon\int_S^T E^{\frac{p+1}{2}}(t)\,dt + C(\varepsilon)E(S) + CE(S). \qquad (5.17)
$$

Estimate for $I_5 := (n-1)m_1\int_S^T E^{\frac{p-1}{2}}\big|\int_{\Gamma_1} g(u')u\,d\Gamma\big|\,dt$. Using trace theory, we obtain

$$
(n-1)m_1\Big|\int_{\Gamma_1} g(u')u\,d\Gamma\Big| \le \frac{C^2}{4\varepsilon}\big|g(u'(t))\big|_{\Gamma_1}^2 + \varepsilon\big|\nabla u(t)\big|^2.
$$

Then

$$
\big|I_5\big| \le \varepsilon\int_S^T E^{\frac{p+1}{2}}(t)\,dt + C(\varepsilon)E(S) + CE(S) + \varepsilon C\int_S^T E^{\frac{p+1}{2}}(t)\,dt. \qquad (5.18)
$$

Thus, combining (5.13)–(5.18) we arrive at

$$
\big(2L^{-1} - (3+C)\varepsilon\big)\int_S^T E^{\frac{p+1}{2}}(t)\,dt \le CE(S) + C(\varepsilon)E(S).
$$

Considering ε sufficiently small and by using Lemma 2.2 we obtain the above-mentioned decay rates. The proof is now completed.

Further remarks. All the considerations presented in this paper can be applied when one has an internal damping, that is, when we consider the problem

$$\begin{cases} u'' - M\big(t, \big|\nabla u(t)\big|^2\big)\Delta u + g(u') = 0 & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \Gamma \times (0,\infty) \\ u(0) = u^0; \ \ u'(0) = u^1 \end{cases}$$

where $\{u^0, u^1\} \in H_0^1 \cap H^2(\Omega) \times H_0^1(\Omega)$ and $g(s) = \big|s\big|^\gamma s$, $0 < \gamma \leq \frac{2}{n-2}$, $n \geq 3$ and $\gamma > 0$ if $n = 1, 2$, for instance. In this case we have an easier situation because we do not need to construct a special basis in order to obtain the a priori estimates as in Section 3. Also, the uniform decay is easily obtained by use of a standard Liapunov functional. We also observe that if we change $u = 0$ by $\frac{\partial u}{\partial \nu} = 0$ on Γ the procedure is also similar.

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