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ON THE ASYMPTOTIC ANALYSIS OF H-SYSTEMS, II: THE CONSTRUCTION OF LARGE SOLUTIONS

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Abstract. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $\gamma \in C^{3,\alpha}(\partial\Omega;\mathbb{R}^3)$ $(0 < \alpha < 1)$ and $H > 0$. Let h_γ be the harmonic extension of γ in Ω.

We show that if $a_0 \in \Omega$ is a regular point of h_γ and a nondegenerate critical point of $K(\cdot,\Omega)$ introduced in part I of this paper [3], then for small H, there exists a large solution $\overline{u}_H$ to the H-system

$$\Delta u = 2Hu_{x_1} \wedge u_{x_2} \quad \text{in } \Omega, \qquad u = \gamma \quad \text{on } \partial\Omega.$$

Moreover, $\overline{u}_H$ blows up (in the sense of part I) at exactly one point a_0 as $H \to 0$.

1. Introduction. This paper is a sequel to [3]. In [3], we studied the asymptotic behavior of large solutions to an H-system,

$$\begin{cases} \Delta u = 2Hu_{x_1} \wedge u_{x_2} & \text{in } \Omega \\ u = \gamma & \text{on } \partial\Omega. \end{cases} \tag{H1}$$

Here $H > 0$, $\Omega \subset \mathbb{R}^2$ is a smooth, bounded domain and $\gamma \in C^{3,\alpha}(\partial\Omega;\mathbb{R}^3)$ $(0 < \alpha < 1)$ is not constant.

Our aim in this paper is to prove the following theorems (which were announced in part I):

Theorem C. *Let $a_0 \in \Omega$ be a strict local maximum of $K(\cdot,\Omega)$; that is, there exists $R > 0$ with $\mathbb{B}_R(a_0) := \{x \in \mathbb{R}^2 : |x - a_0| < R\} \Subset \Omega$ such that $K(a_0,\Omega) > K(a,\Omega)$ for all $a \in \mathbb{B}_R(a_0)$ with $a \neq a_0$. Then there exists $H_0 > 0$ such that for $0 < H \le H_0$,* (H1) *has a solution $\overline{u}_H$ which is different from $\underline{u}_H$. Moreover, $\overline{u}_H$ blows up* (*in the sense of* (1.2) *in* [3]) *at exactly one point a_0 as $H \to 0$.*

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Theorem D. *Let $a_0 \in \Omega$ be a regular point of h_γ. Assume also that a_0 is a nondegenerate critical point of $K(\cdot,\Omega)$ in Ω. Then there exists $H_0 > 0$ such that for $0 < H \leq H_0$,* (H1) *has a solution $\overline{u}_H$ which is different from $\underline{u}_H$. Moreover, $\overline{u}_H$ blows up at exactly one point a_0 as $H \to 0$.*

Since Theorem C is proved by combining the argument in [7] (where Theorem C is proved for the special case $\Omega = \mathbb{B} := \{x \in \mathbb{R}^2 : |x| < 1\}$) and the argument in part I of this paper [3], we restrict our attention to the proof of Theorem D.

Note that, in general, Theorem C is not contained in Theorem D. (In Theorem D, we have assumed that a_0 is a regular point of h_γ; that is, the rank of the Hessian of h_γ at a_0 is 2, which is not automatically satisfied, in general.)

In Theorem D, the assumption that a_0 is a regular point of h_γ is necessary, since if this condition is not satisfied, $K(\cdot,\Omega)$ is not in general differentiable at a_0. See Remark 1.1 (b) in part I [3]. Concerning the nondegeneracy assumption of $K(\cdot,\Omega)$ at a_0, we believe that it is not necessary to hold the conclusion of Theorem D; however, in our construction of large solutions it is essential.

Here we give the outline of the proof of Theorem D. We use the same notation introduced in part I of this paper [3].

We want to find a solution $\overline{u}_H$ to (H1) as the following form:

$$\overline{u}_H = \underline{u}_H + \frac{J_H(v)}{2H|Q(v)|^{1/3}}v, \tag{1.1}$$

where $v \in H_0^1(\Omega;\mathbb{R}^3)$ and

$$J_H(v) := \frac{I_H(v)}{|Q(v)|^{2/3}} := \frac{\int_\Omega |\nabla v|^2\,dx + 4H\int_\Omega \underline{u}_H \cdot v_{x_1} \wedge v_{x_2}\,dx}{|Q(v)|^{2/3}}. \tag{1.2}$$

Then $\overline{u}_H$ is a solution to (H1) if and only if $v \in H_0^1(\Omega;\mathbb{R}^3)$ is a critical point of J_H in $H_0^1(\Omega;\mathbb{R}^3)$ (see Lemma 2.1). We find a critical point v of J_H as the following form:

$$v = RPU_{\lambda,a} + w$$

for some $R \in SO(3)$, $\lambda > 0$, $a \in \Omega$ and $w \in H_0^1(\Omega;\mathbb{R}^3)$ with $\|w\|_{H_0^1(\Omega)}$ small.

Since $RPU_{\lambda,a}$ (for $\lambda/d(a,\partial\Omega)$ small) is a good approximate solution of $dJ_H(v) = 0$ (see Lemma 4.1), one of the natural attempts to find a solution to the equation $dJ_H(v) = 0$ is to apply the implicit function theorem. However, unfortunately, there is an obstruction to using directly the implicit function theorem. It is a vector bundle $\mathcal{W}$, $H_0^1(\Omega;\mathbb{R}^3) \times M \supset \mathcal{W} \to M$, where $M := \{(R,\lambda,a) \in SO(3) \times (0,\infty) \times \Omega : \lambda/d(a,\partial\Omega) < \epsilon\}$ for some small $\epsilon > 0$. Its fiber at (R,λ,a) consists of small eigenvalues of d^2J_H at $RPU_{\lambda,a}$. Moreover it is shown that the rank of $\mathcal{W}$ is 6. Formally, $\mathcal{W}$ is given by

$$\mathcal{W} = \{RPU_{\lambda,a} : R \in SO(3),\ \lambda \in (0,\infty),\ a \in \Omega \text{ with } \lambda/d(a,\partial\Omega) \text{ small}\}.$$

Our strategy is, using this information, first to solve the equation

$$(1 - \Pi_{\mathcal{W}(R,\lambda,a)})(dJ_H(RPU_{\lambda,a} + w)) = 0 \tag{1.3}$$

for $w = w(R,\lambda,a) \in N\mathcal{W}(R,\lambda,a) \subset H_0^1(\Omega;\mathbb{R}^3)$, where $N\mathcal{W}(R,\lambda,a)$ is the fiber at (R,λ,a) of the normal bundle $N\mathcal{W} \to M$ of $\mathcal{W} \to M$ and $\Pi_{\mathcal{W}(R,\lambda,a)} : H_0^1(\Omega;\mathbb{R}^3) \to \mathcal{W}(R,\lambda,a)$ is the orthogonal projection ($\mathcal{W}(R,\lambda,a)$ is the fiber at (R,λ,a) of $\mathcal{W}$). Since the restriction of $d^2J_H(RPU_{\lambda,a})$ to $N\mathcal{W}(R,\lambda,a) \times N\mathcal{W}(R,\lambda,a)$ is invertible, (1.3) is solved by the implicit function theorem.

Next, we solve the equation for R, λ, a:

$$\Pi_{\mathcal{W}(R,\lambda,a)}(dJ_H(RPU_{\lambda,a} + w(R,\lambda,a))) = 0. \tag{1.4}$$

If (R,λ,a) solves (1.4), then $RPU_{\lambda,a} + w(R,\lambda,a)$ is a solution to $dJ_H = 0$. Thus our problem is reduced to solving the *finite-dimensional* problem (1.4). (1.4) is solved through a homotopy consideration.

The above reduction procedure is known as Lyapunov-Schmidt reduction. In [9], Taubes used a similar procedure to find solutions to the self-dual Yang-Mills equations defined over 4-manifolds. Later, in [2], [4] and [5], Floer-Weinstein and Oh applied Taubes' argument to nonlinear Schrödinger equations. In [6], O. Rey used a similar argument to construct solutions to the equations $-\Delta u = u^{n+2/n-2} + \epsilon u$ in Ω, $u > 0$ in Ω and $u|_{\partial\Omega} = 0$, where $\epsilon > 0$ is small and $\Omega \subset \mathbb{R}^n$ is a bounded domain ($n \geq 5$).

This paper is written as follows: In Section 2, we first study the equation (1.3). Next we explicitly calculate the reduced form (1.4) of the equation (H1). In Section 3, we prove the existence of a solution to (1.4) when $H > 0$ is small and complete the proof of Theorem D. In Section 4, some technical lemmas and computations are given.

See the end of Section 1 of part I [3] for the list of notations which are frequently used throughout this paper.

2. Reduction of the problem (H1). For $v \in H_0^1(\Omega;\mathbb{R}^3)$ with $Q(v) \neq 0$, define $J_H(v)$ by (1.2). Here, as in part I [3], $Q(v)$ for $v \in H_0^1(\Omega;\mathbb{R}^3)$ denotes the unique extension of $Q(u) = \int_\Omega u \cdot u_{x_1} \wedge u_{x_2}\, dx$ defined for $u \in C_0^\infty(\Omega;\mathbb{R}^3)$. The extended functional Q is C^∞ in $H_0^1(\Omega;\mathbb{R}^3)$; see [8].

Lemma 2.1. *Let $v \in H_0^1(\Omega;\mathbb{R}^3)$ be a critical point of J_H in $H_0^1(\Omega;\mathbb{R}^3)$ with $Q(v) \neq 0$. Then $\overline{u}_H := \underline{u}_H + \frac{J_H(v)}{2H|Q(v)|^{1/3}}v$ is a solution to* (H1).

Proof. Let $\varphi \in C_0^\infty(\Omega;\mathbb{R}^3)$. For $t \in \mathbb{R}$ with $|t|$ small, we have

$$I_H(v+t\varphi) = \int_\Omega |\nabla(v+t\varphi)|^2\,dx + 4H\int_\Omega \underline{u}_H \cdot (v+t\varphi)_{x_1} \wedge (v+t\varphi)_{x_2}dx \tag{2.1}$$

$$= I_H(v) + t\big(2\int_\Omega \nabla v\cdot\nabla\varphi dx + 4H\int_\Omega \underline{u}_H\cdot(v_{x_1}\wedge\varphi_{x_2} + \varphi_{x_1}\wedge v_{x_2})dx\big) + O(t^2)$$

and

$$\begin{aligned} Q(v+t\varphi) &= \int_\Omega (v+t\varphi)\cdot(v+t\varphi)_{x_1}\wedge(v+t\varphi)_{x_2}\,dx \\ &= Q(v) + 3t\int_\Omega \varphi\cdot v_{x_1}\wedge v_{x_2}\,dx + O(t^2). \end{aligned} \tag{2.2}$$

In the calculation of (2.2), we have used Lemma A.4 in [1]. We may assume without loss of generality that $Q(v) < 0$. Then for small $|t|$, $Q(v+t\varphi) < 0$ and by (2.2)

$$\begin{aligned} |Q(v+t\varphi)|^{-2/3} &= \big(-Q(v) - 3t\int_\Omega \varphi\cdot v_{x_1}\wedge v_{x_2}\,dx\big)^{-2/3} + O(t^2) \\ &= |Q(v)|^{-2/3}\big(1 - \frac{2t}{Q(v)}\int_\Omega \varphi\cdot v_{x_1}\wedge v_{x_2}\,dx\big) + O(t^2). \end{aligned} \tag{2.3}$$

From (2.1) and (2.3), we obtain

$$\begin{aligned} &J_H(v+t\varphi) = J_H(v) \\ &+ 2t|Q(v)|^{-2/3}\big(\int_\Omega \nabla v\cdot\nabla\varphi\,dx + 2H\int_\Omega \underline{u}_H\cdot(v_{x_1}\wedge\varphi_{x_2} + \varphi_{x_1}\wedge v_{x_2})\,dx \\ &- \frac{1}{Q(v)}\big(\int_\Omega |\nabla v|^2 dx + 4H\int_\Omega \underline{u}_H v_{x_1}\wedge v_{x_2}\,dx\big)\big(\int_\Omega \varphi v_{x_1}\wedge v_{x_2}\,dx\big)\big) + O(t^2). \end{aligned}$$

Thus we have

$$\begin{aligned} &\langle dJ_H(v),\varphi\rangle \\ &= 2|Q(v)|^{-2/3}\Big(\int_\Omega \nabla v\cdot\nabla\varphi\,dx + 2H\int_\Omega \underline{u}_H\cdot(v_{x_1}\wedge\varphi_{x_2} + \varphi_{x_1}\wedge v_{x_2})\,dx \\ &- \frac{1}{Q(v)}\Big(\int_\Omega |\nabla v|^2\,dx + 4H\int_\Omega \underline{u}_H\cdot v_{x_1}\wedge v_{x_2}\,dx\Big)\Big(\int_\Omega \varphi\cdot v_{x_1}\wedge v_{x_2}\,dx\Big)\Big). \end{aligned}$$

Therefore v satisfies the equation

$$-\Delta v + 2H((\underline{u}_H)_{x_1} \wedge v_{x_2} + v_{x_1} \wedge (\underline{u}_H)_{x_2}) = -\frac{J_H(v)}{|Q(v)|^{1/3}} v_{x_1} \wedge v_{x_2}. \qquad (2.4)$$

Since $\underline{u}_H$ satisfies (H1), (2.4) implies that $\underline{u}_H + \frac{J_H(v)}{2H|Q(v)|^{1/3}} v$ is a solution to (H1). $\square$

By Lemma 2.1, it is sufficient to find a critical point of J_H. We find a critical point v of J_H in the following form: $v = RPU_{\lambda,a} + w$, where $R \in SO(3)$, $\lambda > 0$, $a \in \Omega$ and $w \in W(R\widehat{U}_{\lambda,a})^\perp$ (see Section 3 of part I for the definition of $W(R\widehat{U}_{\lambda,a})^\perp$).

By Lemma 5.1 in [3], there exists $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$ and $v \in M(\epsilon)$ the problem

$$\inf\big\{\|\nabla(v - \alpha RPU_{\lambda,a})\|_{L^2(\Omega)} : \frac{1}{2} < \alpha < 2,\ R \in SO(3),$$
$$\lambda > 0,\ a \in \Omega,\ \lambda/d(a, \partial\Omega) < 2\epsilon\big\}$$

has a unique solution (see Section 3 of part I for the definition of $M(\epsilon)$). Therefore, there exists a diffeomorphism between a neighborhood of the possible "concentrated solutions" of (H1) and the set

$$\mathcal{O}(\epsilon) := \Big\{(\alpha, R, \lambda, a, w) \in \big(\frac{1}{2}, 2\big) \times SO(3) \times (0,\infty) \times \Omega \times W(RPU_{\lambda,a})^\perp :$$
$$\lambda/d(a,\partial\Omega) < 2\epsilon,\ \|\nabla w\|_{L^2(\Omega)} < 2\epsilon\Big\}$$

for some small $\epsilon > 0$. Define

$$\mathcal{O}'(\epsilon) := \Big\{(R, \lambda, a, w) \in SO(3) \times (0,\infty) \times \Omega \times W(RPU_{\lambda,a})^\perp :$$
$$\lambda/d(a,\partial\Omega) < 2\epsilon,\ \|\nabla w\|_{L^2(\Omega)} < 2\epsilon\Big\}.$$

Since the functional J_H is homogeneous of degree 0 (i.e, $J_H(\alpha v) = J_H(v)$ for $\alpha \neq 0$), we have the following:

Proposition 2.2. *Let $\epsilon_0 > 0$ be as above and $0 < \epsilon \leq \epsilon_0$. Define $\mathcal{J}_H : \mathcal{O}'(\epsilon) \to \mathbb{R}$ by $\mathcal{J}_H(R, \lambda, a, w) := J_H(RPU_{\lambda,a} + w)$. Then $(R, \lambda, a, w) \in \mathcal{O}'(\epsilon)$ is a critical point of $\mathcal{J}_H$ if and only if $RPU_{\lambda,a} + w$ is a critical point of J_H.*

By Proposition 2.2, $RPU_{\lambda,a}+w$ is a critical point of J_H if and only if there exist numbers α, β_i $(i = 1,2)$, γ, δ_i $(i = 1,2,3)$ such that the following hold (since $(R, \lambda, a, w) \in SO(3) \times (0, \infty) \times \Omega \times H^1_0(\Omega; \mathbb{R}^3)$ is a critical point of $\mathcal{J}_H$ under the constraint $\langle RPU_{\lambda,a}, w\rangle = \left\langle R\frac{\partial}{\partial a_i}PU_{\lambda,a}, w\right\rangle = \left\langle R\frac{\partial}{\partial \lambda}PU_{\lambda,a}, w\right\rangle = \langle R\xi_i PU_{\lambda,a}, w\rangle = 0$; here $\mathcal{J}_H$ is considered as a functional defined on an open subset of $SO(3) \times (0, \infty) \times \Omega \times H^1_0(\Omega; \mathbb{R}^3)$):

$$\frac{\partial \mathcal{J}_H}{\partial a_j}(R, \lambda, a, w) = \sum_{i=1}^{2} \beta_i \int_\Omega \nabla w \cdot \nabla\Big(R\frac{\partial^2}{\partial a_j \partial a_i} PU_{\lambda,a}\Big)\, dx \qquad (S1_{j\,(j=1,2)})$$

$$+ \gamma \int_\Omega \nabla w \cdot \nabla\Big(R\frac{\partial^2}{\partial a_j \partial \lambda} PU_{\lambda,a}\Big)\, dx + \sum_{i=1}^{3} \delta_i \int_\Omega \nabla w \cdot \nabla\Big(R\xi_i\frac{\partial}{\partial a_j} PU_{\lambda,a}\Big)\, dx,$$

$$\frac{\partial \mathcal{J}_H}{\partial \lambda}(R, \lambda, a, w) = \sum_{i=1}^{2} \beta_i \int_\Omega \nabla w \cdot \nabla\Big(R\frac{\partial^2}{\partial \lambda \partial a_i} PU_{\lambda,a}\Big)\, dx \qquad (S2)$$

$$+ \gamma \int_\Omega \nabla w \cdot \nabla\Big(R\frac{\partial^2}{\partial \lambda^2} PU_{\lambda,a}\Big)\, dx + \sum_{i=1}^{3} \delta_i \int_\Omega \nabla w \cdot \nabla\Big(R\xi_i\frac{\partial}{\partial \lambda} PU_{\lambda,a}\Big)\, dx,$$

$$\frac{\partial \mathcal{J}_H}{\partial \xi_j}(R, \lambda, a, w) = \sum_{i=1}^{2} \beta_i \int_\Omega \nabla w \cdot \nabla\Big(R\xi_j\frac{\partial}{\partial a_i} PU_{\lambda,a}\Big)\, dx \qquad (S3_{j\,(j=1,2,3)})$$

$$+ \gamma \int_\Omega \nabla w \cdot \nabla\Big(R\xi_j\frac{\partial}{\partial \lambda} PU_{\lambda,a}\Big)\, dx + \sum_{i=1}^{3} \delta_i \int_\Omega \nabla w \cdot \nabla(R\xi_j\xi_i PU_{\lambda,a})\, dx,$$

$$\frac{\partial \mathcal{J}_H}{\partial w}(RPU_{\lambda,a} + w) = \alpha RPU_{\lambda,a} + \sum_{i=1}^{2} \beta_i R\frac{\partial}{\partial a_i} PU_{\lambda,a} + \gamma R\frac{\partial}{\partial \lambda} PU_{\lambda,a}$$

$$+ \sum_{i=1}^{3} \delta_i R\xi_i PU_{\lambda,a}. \qquad (S4)$$

In $(S3_j)$, $\frac{\partial \mathcal{J}_H}{\partial \xi_j}(R, \lambda, a, w)$ is defined by

$$\frac{\partial \mathcal{J}_H}{\partial \xi_j}(R, \lambda, a, w) = \frac{d}{dt}\Big|_{t=0} \mathcal{J}_H(R\exp(t\xi_j), \lambda, a, w).$$

Thus the problem of finding critical points of J_H is reduced to the problem of finding $(R, \lambda, a, w) \in \mathcal{O}'(\epsilon)$, α, β_i $(i = 1, 2), \gamma, \delta_i$ $(i = 1, 2, 3) \in \mathbb{R}$ such that $(S1_j)$–$(S4)$ hold.

Since we are interested in solutions to (H1) concentrating around a given point $a_0 \in \Omega$, in $(S1)$–$(S4)$ we restrict our attention to those of (R, λ, a, w)

with $a \in U$ for some open set U with $a_0 \in U \Subset \Omega$. Thus, in particular, $d(a, \partial\Omega) \geq C$ for some $C > 0$ independent of H.

We first consider $(S4)$. We have the following:

Lemma 2.3. *There exist $H_0 > 0$ and $\epsilon_1 > 0$ such that if $R \in SO(3)$, $\lambda > 0$, $a \in \Omega$ and $H > 0$ satisfy $\lambda \leq \epsilon_1$ and $0 < H \leq H_0$, then there exist $w \in W(RPU_{\lambda,a})^\perp$, α, β_i $(i = 1, 2)$, γ and δ_i $(i = 1, 2, 3)$ such that* (S4) *holds. Moreover, $w = w(H, R, \lambda, a)$ smoothly depends on H, R, λ and a and*

$$\|\nabla w(H, R, \lambda, a)\|_{L^2(\Omega)} = O((H\lambda + \lambda^2)|\log \lambda|^{1/2})$$

as $\lambda \to 0$ and $H \to 0$.

Proof. First note that $(S4)$ is equivalent to

$$dJ_H(RPU_{\lambda,a} + w)\big|_{W(RPU_{\lambda,a})^\perp} = 0. \tag{2.5}$$

Write $\mathcal{J}_H(R, \lambda, a, w)$ as

$$\begin{aligned}\mathcal{J}_H(R, \lambda, a, w) = \mathcal{J}_H(R, \lambda, a, 0) + \frac{\partial \mathcal{J}_H}{\partial w}(R, \lambda, a, 0)w \\ + \frac{1}{2}\Big\langle \frac{\partial^2 \mathcal{J}_H}{\partial w^2}(R, \lambda, a, 0)w, w\Big\rangle + \mathcal{R}(R, \lambda, a, w),\end{aligned} \tag{2.6}$$

where, as in the proof of Lemma 2.1,

$$\begin{aligned}&\frac{\partial \mathcal{J}_H}{\partial w}(R, \lambda, a, 0)w = 2|Q(RPU_{\lambda,a})|^{-2/3} \\ &\times \Big\{2H\Big(\int_\Omega \underline{u}_H \cdot ((RPU_{\lambda,a})_{x_1} \wedge w_{x_2} + w_{x_1} \wedge (RPU_{\lambda,a})_{x_2})\,dx\Big) \\ &\quad - \frac{I_H(RPU_{\lambda,a})}{Q(RPU_{\lambda,a})}\int_\Omega w \cdot (RPU_{\lambda,a})_{x_1} \wedge (RPU_{\lambda,a})_{x_2}\,dx\Big\},\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\Big\langle \frac{\partial^2 \mathcal{J}_H}{\partial w^2}(R, \lambda, a, 0)w, w\Big\rangle \\ &= 2|Q(RPU_{\lambda,a})|^{-2/3}\Big\{\int_\Omega |\nabla w|^2\,dx + 4H\int_\Omega \underline{u}_H \cdot w_{x_1} \wedge w_{x_2}\,dx \\ &\quad - \frac{8H}{Q(RPU_{\lambda,a})}\Big(\int_\Omega \underline{u}_H \cdot ((RPU_{\lambda,a})_{x_1} \wedge w_{x_2} + w_{x_1} \wedge (RPU_{\lambda,a})_{x_2})\,dx\Big) \\ &\quad \times \Big(\int_\Omega w \cdot (RPU_{\lambda,a})_{x_1} \wedge (RPU_{\lambda,a})_{x_2}\,dx\Big) \\ &\quad - \frac{2I_H(RPU_{\lambda,a})}{Q(RPU_{\lambda,a})}\int_\Omega RPU_{\lambda,a} \cdot w_{x_1} \wedge w_{x_2}\,dx\end{aligned}$$

$$+\frac{5I_H(RPU_{\lambda,a})}{|Q(RPU_{\lambda,a})|^2}\Big(\int_\Omega w\cdot(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2}\,dx\Big)^2\Big\}, \tag{2.8}$$

$$|\mathcal{R}(R,\lambda,a,w)|\le C\|\nabla w\|^3_{L^2(\Omega)},\quad \Big\|\frac{\partial\mathcal{R}}{\partial w}(R,\lambda,a,w)\Big\|_{H^{-1}(\Omega)}\le C\|\nabla w\|^2_{L^2(\Omega)}. \tag{2.9}$$

We estimate the right side of (2.8). First note that

$$I_H(RPU_{\lambda,a})\to 8\pi,\quad Q(RPU_{\lambda,a})\to -4\pi \tag{2.10}$$

as $H\to 0$ and $\epsilon\to 0$. By (2.9), (2.10) and Lemma 5.5 of part I [3], there exist $H_0>0$ and $\epsilon_0>0$ such that if $0<H\le H_0$ and $0<\epsilon\le\epsilon_0$, we have

$$\Big\langle\frac{\partial^2\mathcal{J}_H}{\partial w^2}(R,\lambda,a,0)w,w\Big\rangle\ge C\|\nabla w\|^2_{L^2(\Omega)} \tag{2.11}$$

for $(R,\lambda,a,w)\in\mathcal{O}'(\epsilon)$. Since

$$\frac{\partial\mathcal{J}_H}{\partial w}(R,\lambda,a,w)=\frac{\partial\mathcal{J}_H}{\partial w}(R,\lambda,a,0)+\frac{\partial^2\mathcal{J}_H}{\partial w^2}(R,\lambda,a,0)w+\frac{\partial\mathcal{R}}{\partial w}(R,\lambda,a,w),$$

(2.9), (2.11) and the implicit function theorem show that if $\big\|\frac{\partial\mathcal{J}_H}{\partial w}(R,\lambda,a,0)\big\|$ is small, then there exists $w(H,R,\lambda,a)\in W(RPU_{\lambda,a})^\perp$ (smoothly depending on H, R, λ and a) such that $\frac{\partial\mathcal{J}_H}{\partial w}(R,\lambda,a,w)=0$; that is, $(S4)$ holds. Moreover, we have the estimate

$$\|\nabla w(H,R,\lambda,a)\|_{L^2(\Omega)}\le C\Big\|\frac{\partial\mathcal{J}_H}{\partial w}(R,\lambda,a,0)\Big\|. \tag{2.12}$$

Here $\big\|\frac{\partial\mathcal{J}_H}{\partial w}(R,\lambda,a,0)\big\|$ is the operator norm of $\frac{\partial\mathcal{J}_H}{\partial w}(R,\lambda,a,0):W(RPU_{\lambda,a})^\perp\to\mathbb{R}$. In Section 4 Lemma 4.1, we prove the estimate

$$\Big\|\frac{\partial\mathcal{J}_H}{\partial w}(R,\lambda,a,0)\Big\|=O((H\lambda+\lambda^2)|\log\lambda|^{1/2}). \tag{2.13}$$

Thus, for small H and λ there exists $w=w(H,R,\lambda,a)$ such that $\frac{\partial\mathcal{J}_H}{\partial w}(R,\lambda,a,w)=0$. Moreover, from (2.12) and (2.13) we have

$$\|\nabla w(H,R,\lambda,a)\|_{L^2(\Omega)}=O((H\lambda+\lambda^2)|\log\lambda|^{1/2}).$$

This completes the proof of Lemma 2.3. □

Next we estimate α, β_i $(i=1,2)$, γ and δ_i $(i=1,2,3)$. For this we use $(S4)$. Taking the inner product of $(S4)$ with $RPU_{\lambda,a}$, $R\frac{\partial}{\partial a_i}PU_{\lambda,a}$ $(i=1,2)$, $R\frac{\partial}{\partial\lambda}PU_{\lambda,a}$, and $R\xi_iPU_{\lambda,a}$ $(i=1,2,3)$ respectively, we obtain the matrix equation

$$A\begin{pmatrix}\alpha\\ \beta_1\\ \beta_2\\ \gamma\\ \delta_1\\ \delta_2\\ \delta_3\end{pmatrix}=\begin{pmatrix}\langle dJ_H(RPU_{\lambda,a}+w),RPU_{\lambda,a}\rangle\\ \langle dJ_H(RPU_{\lambda,a}+w),R\frac{\partial}{\partial a_1}PU_{\lambda,a}\rangle\\ \langle dJ_H(RPU_{\lambda,a}+w),R\frac{\partial}{\partial a_2}PU_{\lambda,a}\rangle\\ \langle dJ_H(RPU_{\lambda,a}+w),R\frac{\partial}{\partial \lambda}PU_{\lambda,a}\rangle\\ \langle dJ_H(RPU_{\lambda,a}+w),R\xi_1PU_{\lambda,a}\rangle\\ \langle dJ_H(RPU_{\lambda,a}+w),R\xi_2PU_{\lambda,a}\rangle\\ \langle dJ_H(RPU_{\lambda,a}+w),R\xi_3PU_{\lambda,a}\rangle\end{pmatrix}, \tag{2.14}$$

where $w=w(H,R,\lambda,a)$ is given by Lemma 2.3, and by Lemma 4.2 A is given as $A=A_1+A_2$, where

$$A_1=\begin{pmatrix}8\pi&0&0&0&0&0&0\\ 0&\frac{32\pi}{3}\lambda^{-2}&0&0&0&0&0\\ 0&0&\frac{32\pi}{3}\lambda^{-2}&0&0&0&0\\ 0&0&0&\frac{16\pi}{3}\lambda^{-2}&0&0&0\\ 0&0&0&0&\frac{16\pi}{3}&0&0\\ 0&\frac{16\pi}{3}\lambda^{-1}&0&0&0&\frac{16\pi}{3}&0\\ 0&0&\frac{16\pi}{3}\lambda^{-1}&0&0&0&\frac{16\pi}{3}\end{pmatrix}$$

and

$$A_2=\begin{pmatrix}O(\lambda^2)&O(\lambda^2)&O(\lambda^2)&O(\lambda)&0&0&0\\ O(\lambda^2)&O(1)&O(1)&O(\lambda^{-1})&O(\lambda)&\frac{16\pi}{3}\lambda^{-1}+O(\lambda)&O(\lambda)\\ O(\lambda^2)&O(1)&O(1)&O(\lambda^{-1})&O(\lambda)&O(\lambda)&\frac{16\pi}{3}\lambda^{-1}+O(\lambda)\\ O(\lambda)&O(\lambda^{-1})&O(\lambda^{-1})&O(\lambda^{-1})&O(\lambda^2)&O(\lambda^2|\log\lambda|)&O(\lambda^2|\log\lambda|)\\ 0&O(\lambda)&O(\lambda)&O(\lambda^2)&O(\lambda)&O(\lambda)&O(\lambda)\\ 0&O(\lambda)&O(\lambda)&O(\lambda^2|\log\lambda|)&O(\lambda)&O(\lambda)&O(\lambda)\\ 0&O(\lambda)&O(\lambda)&O(\lambda^2|\log\lambda|)&O(\lambda)&O(\lambda)&O(\lambda)\end{pmatrix}.$$

Let $\tilde{A}=\begin{pmatrix}\mathbf{a}_1\\ \lambda^2\mathbf{a}_2\\ \lambda^2\mathbf{a}_3\\ \lambda^2\mathbf{a}_4\\ \mathbf{a}_5\\ \mathbf{a}_6\\ \mathbf{a}_7\end{pmatrix}$, where $\mathbf{a}_i$ $(1\le i\le 7)$ are row vectors of A; $A=\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\\ \mathbf{a}_4\\ \mathbf{a}_5\\ \mathbf{a}_6\\ \mathbf{a}_7\end{pmatrix}$.

Define $\tilde{A}_1$ and $\tilde{A}_2$ similarly. Then by Lemma 4.3, (2.14) is reduced to the following:

$$\tilde{A}\begin{pmatrix}\alpha\\ \beta_1\\ \beta_2\\ \gamma\\ \delta_1\\ \delta_2\\ \delta_3\end{pmatrix} = \begin{pmatrix}V_\alpha(H,R,\lambda,a)\\ V_{\beta_1}(H,R,\lambda,a)\\ V_{\beta_2}(H,R,\lambda,a)\\ V_\gamma(H,R,\lambda,a)\\ V_{\delta_1}(H,R,\lambda,a)\\ V_{\delta_2}(H,R,\lambda,a)\\ V_{\delta_3}(H,R,\lambda,a)\end{pmatrix}, \tag{2.15}$$

where V_α, V_{β_i} $(i=1,2)$, V_γ, and V_{δ_i} $(i=1,2,3)$ are smooth functions of (H,R,λ,a) satisfying

$$V_\alpha(H,R,\lambda,a) = O(\lambda^2) + O(\lambda H), \tag{2.16}$$

$$\begin{aligned}V_{\beta_1}(H,R,\lambda,a) &= S\Big\{\Big(\frac{\partial^2}{\partial x_1^2}h_a^1(a) + \frac{\partial^2}{\partial x_1 \partial x_2}h_a^2(a)\Big)\lambda^4\\ &- \Big(\frac{\partial^2 h_\gamma}{\partial x_1^2}(a)\cdot Re_1 + \frac{\partial^2 h_\gamma}{\partial x_1 \partial x_2}(a)\cdot Re_2\Big)\lambda^3 H\Big\} + O(H\lambda^{2/p+2}|\log\lambda|^{1/2})\\ &+ O(H^2\lambda^{2/p+1}|\log\lambda|^{1/2}) + O(\lambda^{2/p+3}|\log\lambda|^{1/2})\ (1<\forall p<2),\end{aligned} \tag{2.17}$$

$$\begin{aligned}V_{\beta_2}(H,R,\lambda,a) &= S\Big\{\Big(\frac{\partial^2}{\partial x_1 \partial x_2}h_a^1(a) + \frac{\partial^2}{\partial x_2^2}h_a^2(a)\Big)\lambda^4\\ &- \Big(\frac{\partial^2 h_\gamma}{\partial x_1 \partial x_2}(a)\cdot Re_1 + \frac{\partial^2 h_\gamma}{\partial x_2^2}(a)\cdot Re_2\Big)\lambda^3 H\Big\} + O(H\lambda^{2/p+2}|\log\lambda|^{1/2})\\ &+ O(H^2\lambda^{2/p+1}|\log\lambda|^{1/2}) + O(\lambda^{2/p+3}|\log\lambda|^{1/2})\ (1<\forall p<2),\end{aligned} \tag{2.18}$$

$$\begin{aligned}V_\gamma(H,R,\lambda,a) &= S\Big\{\Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda^3 - ((h_\gamma)_{x_1}(a)\cdot Re_1\\ &+ (h_\gamma)_{x_2}(a)\cdot Re_2)\lambda^2 H\Big\} + O(\lambda^4|\log\lambda|) + O(H\lambda^3|\log\lambda|),\end{aligned} \tag{2.19}$$

$$\begin{aligned}V_{\delta_1}(H,R,\lambda,a) &= -S((h_\gamma)_{x_1}(a)\cdot(R\xi_1 e_1) + (h_\gamma)_{x_2}(a)\cdot(R\xi_1 e_2))\lambda H\\ &+ O(H\lambda^2|\log\lambda|^{1/2}) + O(\lambda H^2) + O(\lambda^3|\log\lambda|),\end{aligned} \tag{2.20}$$

$$\begin{aligned}V_{\delta_2}(H,R,\lambda,a) &= -S((h_\gamma)_{x_1}(a)\cdot(R\xi_2 e_1) + (h_\gamma)_{x_2}(a)\cdot(R\xi_2 e_2))\lambda H\\ &+ O(H\lambda^2|\log\lambda|^{1/2}) + O(\lambda H^2) + O(\lambda^3|\log\lambda|),\end{aligned} \tag{2.21}$$

$$\begin{aligned}V_{\delta_3}(H,R,\lambda,a) &= -S((h_\gamma)_{x_1}(a)\cdot(R\xi_3 e_1) + (h_\gamma)_{x_2}(a)\cdot(R\xi_3 e_2))\lambda H\\ &+ O(H\lambda^2|\log\lambda|^{1/2}) + O(\lambda H^2) + O(\lambda^3|\log\lambda|).\end{aligned} \tag{2.22}$$

Write $\tilde{A} = \tilde{A}_1 + \tilde{A}_2 = \tilde{A}_1(I + \tilde{A}_1^{-1}\tilde{A}_2)$, where

$$\tilde{A}_1^{-1} = \begin{pmatrix} (8\pi)^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{32\pi}{3}\right)^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(\frac{32\pi}{3}\right)^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{16\pi}{3}\right)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(\frac{16\pi}{3}\right)^{-1} & 0 & 0 \\ 0 & -\left(\frac{32\pi}{3}\right)^{-1}\lambda^{-1} & 0 & 0 & 0 & \left(\frac{16\pi}{3}\right)^{-1} & 0 \\ 0 & 0 & -\left(\frac{32\pi}{3}\right)^{-1}\lambda^{-1} & 0 & 0 & 0 & \left(\frac{16\pi}{3}\right)^{-1} \end{pmatrix}$$

and

$$\tilde{A}_1^{-1}\tilde{A}_2 = \begin{pmatrix} O(\lambda^2) & O(\lambda^2) & O(\lambda^2) & O(\lambda) & 0 & 0 & 0 \\ O(\lambda^4) & O(\lambda^2) & O(\lambda^2) & O(\lambda) & O(\lambda^3) & O(\lambda) & O(\lambda^3) \\ O(\lambda^4) & O(\lambda^2) & O(\lambda^2) & O(\lambda) & O(\lambda^3) & O(\lambda^3) & O(\lambda) \\ O(\lambda^3) & O(\lambda) & O(\lambda) & O(\lambda) & O(\lambda^4) & O(\lambda^4|\log\lambda|) & O(\lambda^4|\log\lambda|) \\ 0 & O(\lambda) & O(\lambda) & O(\lambda^2) & O(\lambda) & O(\lambda) & O(\lambda) \\ O(\lambda^3) & O(\lambda) & O(\lambda) & O(1) & O(\lambda) & -\frac{1}{2} + O(\lambda) & O(\lambda) \\ O(\lambda^3) & O(\lambda) & O(\lambda) & O(1) & O(\lambda) & O(\lambda) & -\frac{1}{2} + O(\lambda) \end{pmatrix}.$$

Thus $\tilde{A} = \tilde{A}_1(A_3 + O(\lambda)) = \tilde{A}_1 A_3(I + O(\lambda))$, where

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & O(1) & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & O(1) & 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad A_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & O(1) & 0 & 2 & 0 \\ 0 & 0 & 0 & O(1) & 0 & 0 & 2 \end{pmatrix},$$

and we have

$$\tilde{A}^{-1} = (I + O(\lambda))A_3^{-1}\tilde{A}_1^{-1} = (I + O(\lambda)) \tag{2.23}$$
$$\times \begin{pmatrix} (8\pi)^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{32\pi}{3}\right)^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(\frac{32\pi}{3}\right)^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{16\pi}{3}\right)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(\frac{16\pi}{3}\right)^{-1} & 0 & 0 \\ 0 & -2\left(\frac{32\pi}{3}\right)^{-1}\lambda^{-1} & 0 & O(1) & 0 & 2\left(\frac{16\pi}{3}\right)^{-1} & 0 \\ 0 & 0 & -2\left(\frac{32\pi}{3}\right)^{-1}\lambda^{-1} & O(1) & 0 & 0 & 2\left(\frac{16\pi}{3}\right)^{-1} \end{pmatrix}.$$

Since

$$\begin{pmatrix} V_\alpha(H,R,\lambda,a) \\ V_{\beta_1}(H,R,\lambda,a) \\ V_{\beta_2}(H,R,\lambda,a) \\ V_\gamma(H,R,\lambda,a) \\ V_{\delta_1}(H,R,\lambda,a) \\ V_{\delta_2}(H,R,\lambda,a) \\ V_{\delta_3}(H,R,\lambda,a) \end{pmatrix} = \begin{pmatrix} O(\lambda^2)+O(\lambda H) \\ O(\lambda^4)+O(\lambda^3 H)+O(H^2\lambda^{2/p+1}|\log\lambda|^{1/2}) \\ O(\lambda^4)+O(\lambda^3 H)+O(H^2\lambda^{2/p+1}|\log\lambda|^{1/2}) \\ O(\lambda^3)+O(\lambda^2 H) \\ O(\lambda H)+O(\lambda^3|\log\lambda|) \\ O(\lambda H)+O(\lambda^3|\log\lambda|) \\ O(\lambda H)+O(\lambda^3|\log\lambda|) \end{pmatrix} \quad (1<\forall p<2),$$

by (2.15) and (2.23) we obtain

$$\begin{pmatrix} \alpha \\ \beta_1 \\ \beta_2 \\ \gamma \\ \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} O(\lambda^2)+O(\lambda H) \\ O(\lambda^3)+O(\lambda^2 H) \\ O(\lambda^3)+O(\lambda^2 H) \\ O(\lambda^3)+O(\lambda^2 H) \\ O(\lambda H)+O(\lambda^3|\log\lambda|) \\ O(\lambda H)+O(\lambda^3|\log\lambda|) \\ O(\lambda H)+O(\lambda^3|\log\lambda|) \end{pmatrix}. \tag{2.24}$$

Now, we return to $(S1)$–$(S3)$. By $(S1)$–$(S3)$, (2.24), Lemma 2.3, (4.11), (4.20) and Lemma 4.4 we obtain

$$\begin{aligned} \frac{\partial \mathcal{J}_H}{\partial a_j}(R,\lambda,a,w) &= O\Big(\sum_{i=1}^{2}|\beta_i|\Big\|\nabla\Big(\frac{\partial^2}{\partial a_i\partial a_j}PU_{\lambda,a}\Big)\Big\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}\Big) \\ &\quad + O\Big(|\gamma|\Big\|\nabla\Big(\frac{\partial^2}{\partial\lambda\partial a_j}PU_{\lambda,a}\Big)\Big\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}\Big) \\ &\quad + O\Big(\sum_{i=1}^{3}|\delta_i|\Big\|\nabla\Big(\frac{\partial}{\partial a_j}PU_{\lambda,a}\Big)\Big\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}\Big) \\ &= O(H\lambda^2|\log\lambda|^{1/2})+O(\lambda^3|\log\lambda|^{1/2})+O(H^2\lambda|\log\lambda|^{1/2}), \end{aligned} \tag{2.25}$$

$$\begin{aligned} \frac{\partial \mathcal{J}_H}{\partial \lambda}(R,\lambda,a,w) &= O\Big(\sum_{i=1}^{2}|\beta_i|\Big\|\nabla\Big(\frac{\partial^2}{\partial\lambda\partial a_i}PU_{\lambda,a}\Big)\Big\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}\Big) \\ &\quad + O\Big(|\gamma|\Big\|\nabla\Big(\frac{\partial^2}{\partial\lambda^2}PU_{\lambda,a}\Big)\Big\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}\Big) \\ &\quad + O\Big(\sum_{i=1}^{3}|\delta_i|\Big\|\nabla\Big(\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big)\Big\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}\Big) \\ &= O(H\lambda^2|\log\lambda|^{1/2})+O(\lambda^3|\log\lambda|^{1/2})+O(H^2\lambda|\log\lambda|^{1/2}), \end{aligned} \tag{2.26}$$

$$\frac{\partial \mathcal{J}_H}{\partial \xi_j}(R,\lambda,a,w) = O\Big(\sum_{i=1}^{2}|\beta_i|\Big\|\nabla\Big(\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big)\Big\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}\Big)$$

$$
\begin{aligned}
&+O\Big(|\gamma|\Big\|\nabla\Big(\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big)\Big\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}\Big) \qquad (2.27)\\
&+O\Big(\sum_{i=1}^{3}|\delta_i|\|\nabla PU_{\lambda,a}\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}\Big)\\
&=O(H\lambda^3|\log\lambda|^{1/2})+O(\lambda^4|\log\lambda|^{1/2})+O(H^2\lambda^2|\log\lambda|^{1/2}).
\end{aligned}
$$

Since $\frac{\partial \mathcal{J}_H}{\partial a_j}(R,\lambda,a,w)=\big\langle dJ_H(RPU_{\lambda,a}+w),R\frac{\partial}{\partial a_j}PU_{\lambda,a}\big\rangle$, $\frac{\partial \mathcal{J}_H}{\partial \lambda}(R,\lambda,a,w)=\big\langle dJ_H(RPU_{\lambda,a}+w),R\frac{\partial}{\partial \lambda}PU_{\lambda,a}\big\rangle$ and $\frac{\partial \mathcal{J}_H}{\partial \xi_j}(R,\lambda,a,w)=\langle dJ_H(RPU_{\lambda,a}+w),$ $R\xi_jPU_{\lambda,a}\rangle$, by (2.25)–(2.27) and (4.61)–(4.63), we obtain

Lemma 2.4. *$RPU_{\lambda,a}+w(H,R,\lambda,a)$ is a critical point of J_H if and only if $(R,\lambda,a)\in SO(3)\times(0,\infty)\times\Omega$ satisfies the following $(T1)$–$(T3)$:*

$$
\begin{aligned}
&\Big(\frac{\partial^2}{\partial x_1\partial x_i}h_a^1(a)+\frac{\partial^2}{\partial x_2\partial x_i}h_a^2(a)\Big)\lambda^2-\Big(\frac{\partial^2 h_\gamma}{\partial x_1\partial x_i}(a)\cdot Re_1+\frac{\partial^2 h_\gamma}{\partial x_2\partial x_i}(a)\cdot Re_2\Big)\lambda H\\
&=V_{1i}(H,R,\lambda,a), \qquad (T1_{i(i=1,2)})\\
&\Big(\frac{\partial h_a^1}{\partial x_1}(a)+\frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda-((h_\gamma)_{x_1}(a)\cdot Re_1+(h_\gamma)_{x_2}(a)\cdot Re_2)H\\
&=V_2(H,R,\lambda,a), \qquad (T2)\\
&((h_\gamma)_{x_1}(a)\cdot(R\xi_ie_1)+(h_\gamma)_{x_1}(a)\cdot(R\xi_ie_2))\lambda H=V_{3i}(H,R,\lambda,a), \qquad (T3_{i(i=1,2,3)})
\end{aligned}
$$

where V_{1i} $(i=1,2)$, V_2, and V_{3i} $(i=1,2,3)$ are smooth functions of (H,R,λ,a) satisfying

$$
V_{1i}(H,R,\lambda,a)=O(H\lambda^{\frac{2}{p}}|\log\lambda|^{\frac12})+O(\lambda^{\frac{2}{p+1}}|\log\lambda|^{\frac12})+O(H^2\lambda^{\frac{2}{p-1}}|\log\lambda|^{\frac12}),
$$
$$
V_2(H,R,\lambda,a)=O(\lambda^2|\log\lambda|)+O(H\lambda|\log\lambda|)+O(H^2),
$$
$$
V_{3i}(H,R,\lambda,a)=O(H\lambda^2|\log\lambda|^{1/2})+O(\lambda^3|\log\lambda|)+O(\lambda H^2).
$$

Here $1<p<2$ is arbitrary.

In the next section, we prove the existence of a solution to $(T1)$–$(T3)$ under the assumptions of Theorem D in Section 1.

3. Proof of Theorem D. In this section, we prove the existence of a solution to $(T1)$–$(T3)$ under the assumptions of Theorem D.

•1 Let $a_0\in\Omega$ be a regular point of h_γ. Also assume that a_0 is a nondegenerate critical point of $K(\cdot,\Omega)$.

•2 For $a \in \Omega$, denote by $R_a \in SO(3)$ the solution of the problem

$$\begin{aligned}&(h_\gamma)_{x_1}(a)\cdot R_a e_1 + (h_\gamma)_{x_2}(a)\cdot R_a e_2\\ =& \max_{R\in SO(3)}\{(h_\gamma)_{x_1}(a)\cdot Re_1 + (h_\gamma)_{x_2}(a)\cdot Re_2\}\\ =& (|\nabla h_\gamma(a)|^2 + 2|(h_\gamma)_{x_1}(a)\wedge(h_\gamma)_{x_2}(a)|)^{1/2}\end{aligned}$$

constructed in part I, Lemma 5.4.

•3 For $a \in \Omega$, define $\lambda_a = \frac{(|\nabla h_\gamma(a)|^2+2|(h_\gamma)_{x_1}(a)\wedge(h_\gamma)_{x_2}(a)|)^{1/2}}{\frac{\partial h_a^1}{\partial x_1}(a)+\frac{\partial h_a^2}{\partial x_2}(a)}H$. We need the following lemma:

Lemma 3.1. *Let $a, b \in \mathbb{R}^3$ with $a\wedge b \neq 0$. Define $F_{a,b} : SO(3) \to \mathbb{R}$ by $F_{a,b}(R) = a\cdot Re_1 + b\cdot Re_2$. Denote by $R_{a,b} \in SO(3)$ the solution of the problem $\max_{R\in SO(3)}\{a\cdot Re_1 + b\cdot Re_2\}$ constructed in part I, Lemma 5.4. We then have the following:*

(1) *$R_{a,b}$ smoothly depends on $a, b \in \mathbb{R}^3$ with $a \wedge b \neq 0$.*

(2) *Define $\mathbb{F} : \mathfrak{so}(3) \to \mathbb{R}^3$ by $\mathbb{F} = \begin{pmatrix} f_1\\ f_2\\ f_3\end{pmatrix}$, where*

$$f_i(\xi) = a\cdot(R_{a,b}\xi\xi_i e_1) + b\cdot(R_{a,b}\xi\xi_i e_2)\ (1\le i\le 3).$$

Then $\mathbb{F}$ is an isomorphism.

Proof. First note that $R_{a,b}$ is uniquely determined by the construction given in the proof of Lemma 5.4 in part I if $a\wedge b\neq 0$. Then (1) is obvious from the construction of $R_{a,b}$. We prove (2). As in Lemma 5.4 in part I, we may assume without loss of generality that $a = (a_1, 0, 0)$, $b = (b_1, b_2, 0)$ with $a_1 b_2 > 0$. Since $|a|F_{\frac{a}{|a|},\frac{b}{|a|}} = F_{a,b}$, we may also assume that $a_1 = 1$. By the construction of $R_{a,b}$ (see the proof of Lemma 5.4 in part I [3]), $R_{a,b}$ takes the following form $R_{a,b} = \begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}$, $\cos\alpha = \frac{1+b_2}{\sqrt{|b|^2+1+2b_2}}$, $\sin\alpha = -\frac{b_1}{\sqrt{|b|^2+1+2b_2}}$. Direct computation shows the following:

$$f_1(\xi_1) = -\sqrt{|b|^2+1+2b_2}, \tag{3.1}$$

$$f_1(\xi_2) = 0, \tag{3.2}$$

$$f_1(\xi_3) = 0, \tag{3.3}$$

$$f_2(\xi_1) = 0, \tag{3.4}$$

$$f_2(\xi_2) = -\frac{1+b_2}{\sqrt{|b|^2+1+2b_2}}, \tag{3.5}$$

$$f_2(\xi_3) = -\frac{b_1}{\sqrt{|b|^2+1+2b_1}}, \tag{3.6}$$

$$f_3(\xi_1) = 0, \tag{3.7}$$

$$f_3(\xi_2) = -\frac{b_1}{\sqrt{|b|^2+1+2b_2}}, \tag{3.8}$$

$$f_3(\xi_3) = -\frac{|b|^2+b_2}{\sqrt{|b|^2+1+2b_2}}. \tag{3.9}$$

Thus $\mathbb{F}$ is represented as

$$\mathbb{F} = -\frac{1}{\sqrt{|b|^2+1+2b_2}} \begin{pmatrix} |b|^2+1+2b_2 & 0 & 0 \\ 0 & 1+b_2 & b_1 \\ 0 & b_1 & |b|^2+b_2 \end{pmatrix}$$

with respect to the basis $\langle \xi_1, \xi_2, \xi_3 \rangle$ of $\mathfrak{so}(3)$ and the basis $\langle e_1, e_2, e_3 \rangle$ of $\mathbb{R}^3$. Then since $b_2 > 0$, $\det \mathbb{F} = -b_2(|b|^2+1+2b_2)^{1/2} \neq 0$. This completes the proof. □

Remark 3.2. $\mathbb{F}$ is the Hessian of $F_{a,b}$ at $R_{a,b}$. The above proof shows that $R_{a,b}$ is a nondegenerate maximum of $F_{a,b}$ if and only if $a \wedge b \neq 0$.

We find a solution (R, λ, a) to $(T1)$–$(T3)$ of the following form:

⋆1 a is in some small neighborhood of a_0.

⋆2 R is of the form $R = R_a \exp \xi$, where $\xi \in \mathfrak{so}(3)$ and $|\xi|$ is small.

⋆3 $\lambda = \lambda_a(1+\eta)$ and $|\eta|$ is small.

We may assume without loss of generality that $a_0 = 0$. We rewrite $(T1)$–$(T3)$ in terms of a, ξ and η. For this, we first note that by Lemma 3.1 (1), R_a is differentiable with respect to a if a is in some small neighborhood of 0 since 0 is a regular point of h_γ. Next one can write $K(a, \Omega)$ as

$$K(a,\Omega) = \frac{((h_\gamma)_{x_1}(a)\cdot R_a e_1 + (h_\gamma)_{x_2}(a)\cdot R_a e_2)^2}{\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)},$$

and by Lemma 4.5

$$\tfrac{1}{2}\Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)^2 \frac{\partial}{\partial a_i} K(a,\Omega) = \{(h_\gamma)_{x_1 x_i}(a)\cdot R_a e_1 + (h_\gamma)_{x_2 x_i}(a)\cdot R_a e_2\}$$

$$\times (|\nabla h_\gamma(a)|^2 + 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|)^{1/2}\big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\big)$$
$$- (|\nabla h_\gamma(a)|^2 + 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|)\Big(\frac{\partial^2}{\partial x_1 \partial x_i} h_a^1(a) + \frac{\partial^2}{\partial x_2 \partial x_i} h_a^2(a)\Big), \tag{3.10}$$

where we have used $a \cdot \left(\frac{\partial}{\partial a_i} R_a e_1\right) + b \cdot \left(\frac{\partial}{\partial a_i} R_a e_2\right) = 0$ by our choice of R_a.

By (3.10), $(T1)$ is rewritten as follows:

$$-\frac{\left(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\right)^2}{2(|\nabla h_\gamma(a)|^2 + 2|(h_\gamma)_{x_1}(a) \wedge (h_\gamma)_{x_2}(a)|)^{1/2}} \frac{\partial}{\partial a_i} K(a) = V'_{1i}(H, \xi, \eta, a), \tag{3.11}$$

where V'_{1i} is a smooth function of (H, ξ, η, a) satisfying

$$V'_{i1}(H, \xi, \eta, a) = O(|\eta|) + O(|\xi|) + O(H^{2/p-1} |\log H|^{1/2}). \tag{3.12}$$

Since 0 is a nondegenerate critical point of K, $\nabla K(a, \Omega) = \nabla^2 K(0, \Omega) a + o(|a|)$ and $\nabla^2 K(0, \Omega)$ is invertible. Thus (3.11) is reduced to the following:

$$\begin{cases} a = W_1(H, \xi, \eta, a), \\ W_1(H, \xi, \eta, a) = O(|\eta|) + O(|\xi|) + o(|a|) + O(H^{2/p-1} |\log H|^{1/2}). \end{cases} \tag{$T1'$}$$

Simple computation shows that $(T2)$ is reduced to the following:

$$\begin{cases} \eta = W_2(H, \xi, \eta, a), \\ W_2(H, \xi, \eta, a) = O(|\xi|^2) + O(H |\log H|). \end{cases} \tag{$T2'$}$$

Since $R = R_a \exp \xi$ and by our choice of R_a,

$$\begin{aligned} &(h_\gamma)_{x_1}(a) \cdot (R\xi_i e_1) + (h_\gamma)_{x_2}(a) \cdot (R\xi_i e_2) \\ &= (h_\gamma)_{x_1}(a) \cdot (R_a \xi_i e_1) + (h_\gamma)_{x_2}(a) \cdot (R_a \xi_i e_2) \\ &+ (h_\gamma)_{x_1}(a) \cdot (R_a \xi \xi_i e_1) + (h_\gamma)_{x_2}(a) \cdot (R_a \xi \xi_i e_2) + O(|\xi|^2) \\ &= (h_\gamma)_{x_1}(a) \cdot (R_a \xi \xi_i e_1) + (h_\gamma)_{x_2}(a) \cdot (R_a \xi \xi_i e_2) + O(|\xi|^2). \end{aligned} \tag{3.13}$$

By Lemma 3.1 (2) and (3.13), $(T3)$ is reduced to the following:

$$\begin{cases} \xi = W_3(H, \xi, \eta, a), \\ W_3(H, \xi, \eta, a) = O(|\xi|^2) + O(H |\log H|). \end{cases} \tag{$T3'$}$$

Thus the problem is reduced to finding a solution to the problem $(T1')$–$(T3')$. $(T1')$–$(T3')$ is written as

$$\begin{pmatrix} a \\ \eta \\ \xi \end{pmatrix} = \begin{pmatrix} W_1(H,\xi,\eta,a) \\ W_2(H,\xi,\eta,a) \\ W_3(H,\xi,\eta,a) \end{pmatrix} =: W(H,\xi,\eta,a). \tag{3.14}$$

To solve (3.14), set

$$M_H := \Big\{(a,\eta,\xi) \in \Omega \times (-\tfrac{1}{2},\tfrac{1}{2}) \times \mathfrak{so}(3) : |a| \le H^{\frac{1}{4}},\ |\eta| \le H^{\frac{1}{2}},\ |\xi| \le H^{\frac{1}{2}}\Big\}.$$

Lemma 3.3. *Take p such that $1 < p \le \frac{4}{3}$. There exists $H_0 > 0$ such that for any H with $0 < H \le H_0$, (3.14) has a solution $(a,\eta,\xi) \in M_H$.*

Proof. For $0 \le t \le 1$, define $W_t(H,\xi,\eta,a) := \begin{pmatrix} a \\ \eta \\ \xi \end{pmatrix} - tW(H,\xi,\eta,a)$. Assume $(a,\eta,\xi) \in M_H$. If $|a| = H^{1/4}$,

$$|a| = H^{1/4} > |W_1(H,\xi,\eta,a)| \tag{3.15}$$

if H is small. If $|\eta| = H^{1/2}$,

$$|\eta| = H^{1/2} > |W_2(H,\xi,\eta,a)| \tag{3.16}$$

if H is small. If $|\xi| = H^{1/2}$,

$$|\xi| = H^{1/2} > |W_3(H,\xi,\eta,a)| \tag{3.17}$$

if H is small.

From (3.15)–(3.17), if H is small, W_t does not vanish on ∂M_H for all $0 \le t \le 1$. Assume that W_1 does not vanish in M_H. Then $\frac{W_1}{|W_1|} : M_H \to \mathbb{S}^5$ is continuous and $\frac{W_t}{|W_t|} : \partial M_H \to \mathbb{S}^5$ $(0 \le t \le 1)$ gives a homotopy between $\frac{W_0}{|W_0|}$ and $\frac{W_1}{|W_1|}$. Thus, in particular, $\frac{W_0}{|W_0|} : \partial M_H \to \mathbb{S}^5$ is null homotopic. However, the degree of $\frac{W_0}{|W_0|} : \partial M_H \to \mathbb{S}^5$ is obviously 1. This is a contradiction. The proof of Lemma 3.3 is complete. □

Completion of the proof of Theorem D. Let $(a,\eta,\xi) \in M_H$ be a solution of (3.14) obtained by Lemma 3.3. Define $R = R_a \exp \xi$ and $\lambda = \lambda_a(1+\eta)$. Set

$$u_H := \underline{u}_H + \frac{J_H(RPU_{\lambda,a} + w(R,\lambda,a))}{2H|Q(RPU_{\lambda,a} + w(R,\lambda,a))|^{1/3}}(RPU_{\lambda,a} + w(R,\lambda,a)).$$

Then by our construction, u_H is a solution of (H1) which satisfies the required condition in Theorem D. □

4. Technical lemmas and computations. Throughout this section, we assume $d(a, \partial\Omega) \geq C$ for some $C > 0$.

Lemma 4.1. *Let* $\mathcal{J}_H$ *be as in Proposition* 2.2; *then*

$$\left\| \frac{\partial \mathcal{J}_H}{\partial w}(R, \lambda, a, 0) \right\| = O((H\lambda + \lambda^2)|\log \lambda|^{1/2})$$

as $\lambda \to 0$ *and* $H \to 0$.

Proof. For $w \in W(RPU_{\lambda,a})^{\perp}$, $\frac{\partial \mathcal{J}_H}{\partial w}(R, \lambda, a, 0)$ is computed in (2.7). Here by Lemma A.4 in [1]

$$\begin{aligned}
&2H \int_\Omega \underline{u}_H \cdot ((RPU_{\lambda,a})_{x_1} \wedge w_{x_2} + w_{x_1} \wedge (RPU_{\lambda,a})_{x_2})\, dx \\
&= 2H \int_\Omega RPU_{\lambda,a} \cdot ((\underline{u}_H)_{x_1} \wedge w_{x_2} + w_{x_1} \wedge (\underline{u}_H)_{x_2})\, dx \\
&= O\Big(H \int_\Omega |PU_{\lambda,a}||\nabla w|\, dx\Big) = O(H \|\widehat{U}_{\lambda,a}\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)}) \\
&= O(H\lambda |\log \lambda|^{1/2} \|\nabla w\|_{L^2(\Omega)}),
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&\int_\Omega w \cdot (RPU_{\lambda,a})_{x_1} \wedge (RPU_{\lambda,a})_{x_2}\, dx \\
&= \int_\Omega w \cdot (R\widehat{U}_{\lambda,a})_{x_1} \wedge (R\widehat{U}_{\lambda,a})_{x_2}\, dx - \int_\Omega w \cdot (R\widehat{U}_{\lambda,a})_{x_1} \wedge (Rh_{\lambda,a})_{x_2}\, dx \\
&\quad - \int_\Omega w \cdot (Rh_{\lambda,a})_{x_1} \wedge (R\widehat{U}_{\lambda,a})_{x_2}\, dx + \int_\Omega w \cdot (Rh_{\lambda,a})_{x_1} \wedge (Rh_{\lambda,a})_{x_2}\, dx.
\end{aligned} \tag{4.2}$$

The first term of the right-hand side of (4.2) is 0 since $w \in W(RPU_{\lambda,a})^{\perp}$, $\Delta(R\widehat{U}_{\lambda,a}) = 2(R\widehat{U}_{\lambda,a})_{x_1} \wedge (R\widehat{U}_{\lambda,a})_{x_2}$ and

$$\int_\Omega w \cdot (R\widehat{U}_{\lambda,a})_{x_1} \wedge (R\widehat{U}_{\lambda,a})_{x_2}\, dx = -\frac{1}{2} \int_\Omega \nabla w \cdot \nabla (R\widehat{U}_{\lambda,a})\, dx = 0. \tag{4.3}$$

The second term added to the third term is estimated as (using Lemma A.4 in [1])

$$\begin{aligned}
&\int_\Omega w \cdot ((R\widehat{U}_{\lambda,a})_{x_1} \wedge (Rh_{\lambda,a})_{x_2} + (Rh_{\lambda,a})_{x_1} \wedge (R\widehat{U}_{\lambda,a})_{x_2})\, dx \\
&= \int_\Omega R\widehat{U}_{\lambda,a} \cdot (w_{x_1} \wedge (Rh_{\lambda,a})_{x_2} + (Rh_{\lambda,a})_{x_1} \wedge w_{x_2})\, dx \\
&= O\Big(\lambda \int_\Omega |\widehat{U}_{\lambda,a}||\nabla w|\, dx\Big) = O(\lambda^2 |\log \lambda|^{1/2} \|\nabla w\|_{L^2(\Omega)}).
\end{aligned} \tag{4.4}$$

The last term in (4.2) is estimated as (using Lemma A.3 in [1] and Lemma 5.2 in [3])

$$\int_\Omega w\cdot (Rh_{\lambda,a})_{x_1}\wedge (Rh_{\lambda,a})_{x_2}\,dx = O(\|\nabla w\|_{L^2(\Omega)}\|\nabla h_{\lambda,a}\|^2_{L^2(\Omega)}) \\ = O(\lambda^2\|\nabla w\|_{L^2(\Omega)}). \tag{4.5}$$

From (4.1)–(4.5), we have

$$\Big|\frac{\partial \mathcal{J}_H}{\partial w}(R,\lambda,a,0)w\Big| = O((H\lambda+\lambda^2)|\log\lambda|^{1/2}\|\nabla w\|_{L^2(\Omega)}). \tag{4.6}$$

(4.6) completes the proof of Lemma 4.1. $\square$

In the rest of this paper, we use the notation $h^i(a,x) := h^i_a(x)$ and $h^i_\lambda(a,x) := h^i_{\lambda,a}(x)$.

Lemma 4.2. *We have the following estimates:*

$$\int_\Omega |\nabla(RPU_{\lambda,a})|^2\,dx = 8\pi + O(\lambda^2), \tag{4.7}$$

$$\int_\Omega \nabla(PU_{\lambda,a})\cdot\nabla\Big(\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big)\,dx \quad (i=1,2) \tag{4.8}$$

$$= -4\pi\Big(\frac{\partial^2 h^1}{\partial x_1\partial a_i}(a,a) + \frac{\partial^2 h^2}{\partial x_2\partial a_i}(a,a)\Big)\lambda^2 + O(\lambda^3) = O(\lambda^2),$$

$$\int_\Omega \nabla(PU_{\lambda,a})\cdot\nabla\Big(\frac{\partial}{\partial \lambda}PU_{\lambda,a}\Big)\,dx$$

$$= -4\pi\Big(\frac{\partial h^1_a}{\partial x_1}(a) + \frac{\partial h^2_a}{\partial x_2}(a)\Big)\lambda + O(\lambda^2) = O(\lambda), \tag{4.9}$$

$$\int_\Omega \nabla(PU_{\lambda,a})\cdot\nabla(\xi_i PU_{\lambda,a})\,dx = 0 \quad (i=1,2,3), \tag{4.10}$$

$$\int_\Omega \Big|\nabla\Big(\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big)\Big|^2\,dx = \frac{32\pi}{3}\frac{1}{\lambda^2} + O(1) \quad (i=1,2), \tag{4.11}$$

$$\int_\Omega \nabla\Big(\frac{\partial}{\partial a_1}PU_{\lambda,a}\Big)\cdot\nabla\Big(\frac{\partial}{\partial a_2}PU_{\lambda,a}\Big)\,dx = O(1), \tag{4.12}$$

$$\int_\Omega \nabla\Big(\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big)\cdot\nabla\Big(\frac{\partial}{\partial \lambda}PU_{\lambda,a}\Big)\,dx = O\Big(\frac{1}{\lambda}\Big) \quad (i=1,2), \tag{4.13}$$

$$\int_\Omega \nabla(\xi_1 PU_{\lambda,a})\cdot\nabla\Big(\frac{\partial}{\partial a_1}PU_{\lambda,a}\Big)\,dx = O(\lambda), \tag{4.14}$$

$$\int_\Omega \nabla(\xi_2 PU_{\lambda,a})\cdot\nabla\Big(\frac{\partial}{\partial a_1}PU_{\lambda,a}\Big)\,dx = \frac{16\pi}{3}\frac{1}{\lambda} + O(\lambda), \tag{4.15}$$

$$\int_\Omega \nabla(\xi_3 PU_{\lambda,a}) \cdot \nabla\Big(\frac{\partial}{\partial a_1} PU_{\lambda,a}\Big)\,dx = O(\lambda), \tag{4.16}$$

$$\int_\Omega \nabla(\xi_1 PU_{\lambda,a}) \cdot \nabla\Big(\frac{\partial}{\partial a_2} PU_{\lambda,a}\Big)\,dx = O(\lambda), \tag{4.17}$$

$$\int_\Omega \nabla(\xi_2 PU_{\lambda,a}) \cdot \nabla\Big(\frac{\partial}{\partial a_2} PU_{\lambda,a}\Big)\,dx = O(\lambda), \tag{4.18}$$

$$\int_\Omega \nabla(\xi_3 PU_{\lambda,a}) \cdot \nabla\Big(\frac{\partial}{\partial a_2} PU_{\lambda,a}\Big)\,dx = \frac{16\pi}{3}\frac{1}{\lambda} + O(\lambda), \tag{4.19}$$

$$\int_\Omega \Big|\nabla\Big(\frac{\partial}{\partial \lambda} PU_{\lambda,a}\Big)\Big|^2\,dx = \frac{16\pi}{3}\frac{1}{\lambda^2} + O\Big(\frac{1}{\lambda}\Big), \tag{4.20}$$

$$\int_\Omega \nabla(\xi_1 PU_{\lambda,a}) \cdot \nabla\Big(\frac{\partial}{\partial \lambda} PU_{\lambda,a}\Big)\,dx = O(\lambda^2), \tag{4.21}$$

$$\int_\Omega \nabla(\xi_i PU_{\lambda,a}) \cdot \nabla\Big(\frac{\partial}{\partial \lambda} PU_{\lambda,a}\Big)\,dx = O(\lambda^2|\log\lambda|) \quad (i = 2,3), \tag{4.22}$$

$$\int_\Omega |\nabla(\xi_i PU_{\lambda,a})|^2\,dx = \frac{16\pi}{3} + O(\lambda) \quad (i = 1,2,3), \tag{4.23}$$

$$\int_\Omega \nabla(\xi_i PU_{\lambda,a}) \cdot \nabla(\xi_j PU_{\lambda,a})\,dx = O(\lambda) \quad (i \neq j). \tag{4.24}$$

Proof. (4.7) is derived from Lemma 5.2 in part I [3]. (4.10) holds since ξ_i is anti-Hermitian. Set $R = \operatorname{diam}\Omega$, $d = d(a, \partial\Omega)$ and $r = |x-a|$.
Proof of (4.8).

$$\begin{aligned}
&\int_\Omega \nabla(PU_{\lambda,a}) \cdot \nabla\big(\frac{\partial}{\partial a_i} PU_{\lambda,a}\big)\,dx = -\int_\Omega \Delta(PU_{\lambda,a}) \cdot \frac{\partial}{\partial a_i} PU_{\lambda,a}\,dx \\
&= \int_\Omega |\nabla U_{\lambda,a}|^2 U_{\lambda,a} \cdot \big(\frac{\partial}{\partial a_i} PU_{\lambda,a}\big)\,dx \\
&= \int_\Omega |\nabla U_{\lambda,a}|^2 U_{\lambda,a} \cdot \big(\frac{\partial}{\partial a_i} U_{\lambda,a} - \frac{\partial}{\partial a_i} h_{\lambda,a}\big)\,dx \\
&= -\int_\Omega |\nabla U_{\lambda,a}|^2 U_{\lambda,a} \cdot \frac{\partial}{\partial a_i} h_{\lambda,a}\,dx \text{ (since } |U_{\lambda,a}| = 1 \text{ and } U_{\lambda,a} \cdot \frac{\partial}{\partial a_i} U_{\lambda,a} = 0) \\
&= -4\pi\Big(\frac{\partial^2 h_\lambda^1}{\partial x_1 \partial a_i}(a,a) + \frac{\partial^2 h_\lambda^2}{\partial x_2 \partial a_i}(a,a)\Big)\lambda + O(\lambda^3) \\
&= -4\pi\Big(\frac{\partial^2 h^1}{\partial x_1 \partial a_i}(a,a) + \frac{\partial^2 h^2}{\partial x_2 \partial a_i}(a,a)\Big)\lambda^2 + O(\lambda^3).
\end{aligned}$$

Here the fourth line follows from an argument similar to the proof of (3.18), (3.19), (3.20) and (5.17) in part I [3], and we have used the estimates (see part I for similar estimates):

$$\Big\|\frac{\partial h^1_\lambda}{\partial a_i}(a,\cdot)\Big\|_{C^k(\overline{\Omega})} = O(\lambda), \quad \Big\|\frac{\partial h^2_\lambda}{\partial a_i}(a,\cdot)\Big\|_{C^k(\overline{\Omega})} = O(\lambda), \tag{4.25}$$

$$\Big\|\frac{\partial h^3_\lambda}{\partial a_i}(a,\cdot)\Big\|_{C^k(\overline{\Omega})} = O(\lambda^2), \tag{4.26}$$

$$\|h^1_{\lambda,a} - \lambda h^1_a\|_{C^k(\overline{\Omega})} = O(\lambda^2), \quad \|h^2_{\lambda,a} - \lambda h^2_a\|_{C^k(\overline{\Omega})} = O(\lambda^2) \tag{4.27}$$

for any $k \geq 1$.

Proof of (4.9). (4.9) is proved by an argument similar to the proof of (4.8). Here, instead of (4.25)–(4.27), we use the following estimates:

$$\Big\|\frac{\partial}{\partial\lambda} h^1_{\lambda,a}\Big\|_{C^k(\overline{\Omega})} = O(1), \quad \Big\|\frac{\partial}{\partial\lambda} h^2_{\lambda,a}\Big\|_{C^k(\overline{\Omega})} = O(1), \tag{4.28}$$

$$\Big\|\frac{\partial}{\partial\lambda} h^3_{\lambda,a}\Big\|_{C^k(\overline{\Omega})} = O(\lambda), \tag{4.29}$$

$$\Big\|\frac{\partial}{\partial\lambda} h^1_{\lambda,a} - h^1_a\Big\|_{C^k(\overline{\Omega})} = O(\lambda), \quad \Big\|\frac{\partial}{\partial\lambda} h^2_{\lambda,a} - h^2_a\Big\|_{C^k(\overline{\Omega})} = O(\lambda). \tag{4.30}$$

Proof of (4.11). We have

$$\begin{aligned}
&\int_\Omega \Big|\nabla\Big(\frac{\partial}{\partial a_i} PU_{\lambda,a}\Big)\Big|^2 dx = -\int_\Omega \frac{\partial}{\partial a_i}(\Delta PU_{\lambda,a}) \cdot \frac{\partial}{\partial a_i} PU_{\lambda,a}\, dx \\
= &\int_\Omega \frac{\partial}{\partial a_i}(|\nabla U_{\lambda,a}|^2 U_{\lambda,a}) \cdot \Big(\frac{\partial}{\partial a_i}\widehat{U}_{\lambda,a} - \frac{\partial}{\partial a_i} h_{\lambda,a}\Big)\, dx \\
= &-\int_\Omega \Big(\frac{\partial}{\partial a_i}|\nabla U_{\lambda,a}|^2\Big) U_{\lambda,a} \cdot \frac{\partial}{\partial a_i} h_{\lambda,a}\, dx + \int_\Omega |\nabla U_{\lambda,a}|^2 \Big|\frac{\partial}{\partial a_i} U_{\lambda,a}\Big|^2 dx \\
&-\int_\Omega |\nabla U_{\lambda,a}|^2 \frac{\partial}{\partial a_i} U_{\lambda,a} \cdot \frac{\partial}{\partial a_i} h_{\lambda,a}\, dx.
\end{aligned} \tag{4.31}$$

Since

$$\frac{\partial}{\partial a_i}(|\nabla U_{\lambda,a}|^2) = \frac{\partial}{\partial a_i}\Big(\frac{8\lambda^2}{(\lambda^2 + |x-a|^2)^2}\Big) = \frac{32\lambda^2(x_i - a_i)}{(\lambda^2 + |x-a|^2)^3}, \tag{4.32}$$

we have by (4.32), (4.25) and (4.26)

$$\int_\Omega \Big(\frac{\partial}{\partial a_i}|\nabla U_{\lambda,a}|^2\Big)U_{\lambda,a}\cdot\frac{\partial}{\partial a_i}h_{\lambda,a}\,dx = O\Big(\lambda\int_0^R \frac{\lambda^2 r^2}{(\lambda^2+r^2)^3}\,dr\Big) = O(1). \tag{4.33}$$

On the other hand, since

$$\frac{\partial}{\partial a_i}U_{\lambda,a} = -\frac{2\lambda}{\lambda^2+r^2}e_i + \frac{4\lambda(x_i-a_i)}{(\lambda^2+r^2)^2}\begin{pmatrix} x_1-a_1\\ x_2-a_2\\ -\lambda\end{pmatrix} \tag{4.34}$$

and

$$\Big|\frac{\partial}{\partial a_i}U_{\lambda,a}\Big|^2 = \frac{4\lambda^2}{(\lambda^2+r^2)^2}, \tag{4.35}$$

we have

$$\int_\Omega |\nabla U_{\lambda,a}|^2\Big|\frac{\partial}{\partial a_i}U_{\lambda,a}\Big|^2\,dx = 32\int_\Omega \frac{\lambda^4}{(\lambda^2+r^2)^4}\,dx \tag{4.36}$$

$$= 32\int_{\mathbb{B}_d(a)}\frac{\lambda^4}{(\lambda^2+r^2)^4}\,dx + 32\int_{\Omega\setminus\mathbb{B}_d(a)}\frac{\lambda^4}{(\lambda^2+r^2)^4}\,dx = \frac{32\pi}{3}\frac{1}{\lambda^2}+O(\lambda^4).$$

Finally by (4.25), (4.26) and (4.35) we have

$$\int_\Omega |\nabla U_{\lambda,a}|^2\frac{\partial}{\partial a_i}U_{\lambda,a}\cdot\frac{\partial}{\partial a_i}h_{\lambda,a}\,dx = O\Big(\int_\Omega\frac{\lambda^4}{(\lambda^2+r^2)^3}\,dx\Big) = O(1). \tag{4.37}$$

From (4.31), (4.33), (4.36) and (4.37) we obtain (4.11).

Proof of (4.12). We have

$$\int_\Omega \nabla\Big(\frac{\partial}{\partial a_1}PU_{\lambda,a}\Big)\cdot\nabla\Big(\frac{\partial}{\partial a_2}PU_{\lambda,a}\Big)\,dx = -\int_\Omega \frac{\partial}{\partial a_1}(\Delta PU_{\lambda,a})\cdot\frac{\partial}{\partial a_2}PU_{\lambda,a}\,dx$$

$$= \int_\Omega \frac{\partial}{\partial a_1}(|\nabla U_{\lambda,a}|^2U_{\lambda,a})\cdot\Big(\frac{\partial}{\partial a_2}\widehat{U}_{\lambda,a} - \frac{\partial}{\partial a_2}h_{\lambda,a}\Big)\,dx$$

$$= -\int_\Omega\big(\frac{\partial}{\partial a_1}|\nabla U_{\lambda,a}|^2\big)U_{\lambda,a}\cdot\frac{\partial}{\partial a_2}h_{\lambda,a}\,dx + \int_\Omega|\nabla U_{\lambda,a}|^2\frac{\partial}{\partial a_1}U_{\lambda,a}\cdot\frac{\partial}{\partial a_2}U_{\lambda,a}\,dx$$

$$= -\int_\Omega|\nabla U_{\lambda,a}|^2\frac{\partial}{\partial a_1}U_{\lambda,a}\cdot\frac{\partial}{\partial a_2}h_{\lambda,a}\,dx. \tag{4.38}$$

Here the first term in (4.38) is estimated as (4.33) and $O(1)$. The third term is estimated as (4.37) and $O(1)$. The second term is 0 since $\frac{\partial}{\partial a_1}U_{\lambda,a}\cdot\frac{\partial}{\partial a_2}U_{\lambda,a}=0$. Thus we obtain (4.12).

Proof of (4.13). We have

$$\begin{aligned}
&\int_\Omega \nabla\Big(\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big)\cdot\nabla\Big(\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big)\,dx = -\int_\Omega \frac{\partial}{\partial a_i}(\Delta PU_{\lambda,a})\cdot\frac{\partial}{\partial\lambda}PU_{\lambda,a}\,dx\\
&= \int_\Omega \frac{\partial}{\partial a_i}(|\nabla U_{\lambda,a}|^2U_{\lambda,a})\cdot\Big(\frac{\partial}{\partial\lambda}U_{\lambda,a}-\frac{\partial}{\partial\lambda}h_{\lambda,a}\Big)\,dx\\
&= -\int_\Omega\Big(\frac{\partial}{\partial a_i}|\nabla U_{\lambda,a}|^2\Big)U_{\lambda,a}\cdot\frac{\partial}{\partial\lambda}h_{\lambda,a}\,dx+\int_\Omega|\nabla U_{\lambda,a}|^2\frac{\partial}{\partial a_i}U_{\lambda,a}\cdot\frac{\partial}{\partial\lambda}U_{\lambda,a}\,dx\\
&\quad-\int_\Omega|\nabla U_{\lambda,a}|^2\frac{\partial}{\partial a_i}U_{\lambda,a}\cdot\frac{\partial}{\partial\lambda}h_{\lambda,a}\,dx. \qquad (4.39)
\end{aligned}$$

Here, by (4.28), (4.29) and (4.32), we have

$$\int_\Omega\Big(\frac{\partial}{\partial a_i}|\nabla U_{\lambda,a}|^2\Big)U_{\lambda,a}\cdot\frac{\partial}{\partial\lambda}h_{\lambda,a}\,dx = O\Big(\int_\Omega\frac{\lambda^2 r}{(\lambda^2+r^2)^3}\,dx\Big)=O(\lambda^{-1}). \qquad (4.40)$$

By (4.28), (4.29) and (4.35)

$$\int_\Omega|\nabla U_{\lambda,a}|^2\frac{\partial}{\partial a_i}U_{\lambda,a}\cdot\frac{\partial}{\partial\lambda}h_{\lambda,a}\,dx = O\Big(\int_\Omega\frac{\lambda^3}{(\lambda^2+r^2)^3}\,dx\Big)=O(\lambda^{-1}). \qquad (4.41)$$

By (4.34) and the fact that

$$\frac{\partial}{\partial\lambda}U_{\lambda,a} = -\frac{2\lambda}{\lambda^2+r^2}e_3+\frac{2(r^2-\lambda^2)}{(\lambda^2+r^2)^2}\begin{pmatrix}x_1-a_1\\ x_2-a_2\\ -\lambda\end{pmatrix}, \qquad (4.42)$$

$$\frac{\partial}{\partial a_i}U_{\lambda,a}\cdot\frac{\partial}{\partial\lambda}U_{\lambda,a}=\frac{4\lambda(x_i-a_i)}{(\lambda^2+r^2)^2}.$$

Thus we have

$$\begin{aligned}
&\int_\Omega|\nabla U_{\lambda,a}|^2\frac{\partial}{\partial a_i}U_{\lambda,a}\cdot\frac{\partial}{\partial\lambda}U_{\lambda,a}\,dx = O\Big(\int_\Omega\frac{\lambda^3(x_i-a_i)}{(\lambda^2+r^2)^4}\,dx\Big)\\
&= O\Big(\int_{\mathbb{B}_d(a)}\frac{\lambda^3(x_i-a_i)}{(\lambda^2+r^2)^4}\,dx\Big)+O\Big(\int_{\Omega\setminus\mathbb{B}_d(a)}\frac{\lambda^3(x_i-a_i)}{(\lambda^2+r^2)^4}\,dx\Big) \qquad (4.43)\\
&= O\Big(\int_{\Omega\setminus\mathbb{B}_d(a)}\frac{\lambda^3(x_i-a_i)}{(\lambda^2+r^2)^4}\,dx\Big)=O\Big(\int_d^R\frac{\lambda^3r^2}{(\lambda^2+r^2)^4}\,dr\Big)=O(\lambda^3).
\end{aligned}$$

From (4.39), (4.40), (4.41) and (4.43), we obtain (4.13).

Proof of (4.14), (4.15) and (4.16). For $1 \le i \le 3$

$$\begin{aligned}&\int_\Omega \nabla(\xi_i PU_{\lambda,a}) \cdot \nabla\Big(\frac{\partial}{\partial a_1} PU_{\lambda,a}\Big)\,dx = -\int_\Omega \xi_i \Delta(PU_{\lambda,a}) \cdot \frac{\partial}{\partial a_1} PU_{\lambda,a}\,dx \\ =\ & \int_\Omega |\nabla U_{\lambda,a}|^2(\xi_i U_{\lambda,a}) \cdot \Big(\frac{\partial}{\partial a_1} U_{\lambda,a} - \frac{\partial}{\partial a_1} h_{\lambda,a}\Big)\,dx \qquad (4.44) \\ =\ & \int_\Omega |\nabla U_{\lambda,a}|^2(\xi_i U_{\lambda,a}) \cdot \frac{\partial}{\partial a_1} U_{\lambda,a}\,dx + O(\lambda) \quad \text{(by (4.25), (4.26))}.\end{aligned}$$

When $i = 1$, $(\xi_1 U_{\lambda,a}) \cdot \frac{\partial}{\partial a_1} U_{\lambda,a} = -\frac{4\lambda^2(x_2-a_2)}{(\lambda^2+r^2)^2}$, and we have

$$\begin{aligned}(4.44) &= -32\int_\Omega \frac{\lambda^4(x_2-a_2)}{(\lambda^2+r^2)^4}\,dx + O(\lambda) \\ &= -32\int_{\Omega\setminus\mathbb{B}_d(a)} \frac{\lambda^4(x_2-a_2)}{(\lambda^2+r^2)^4}\,dx + O(\lambda) \quad \text{(by oddness)} \\ &= O\Big(\int_d^R \frac{\lambda^4 r^2}{(\lambda^2+r^2)^4}\,dr\Big) + O(\lambda) = O(\lambda).\end{aligned}$$

This proves (4.14).

When $i = 2$, $(\xi_2 U_{\lambda,a}) \cdot \frac{\partial}{\partial a_1} U_{\lambda,a} = \frac{2\lambda}{(\lambda^2+r^2)^2}(\lambda^2 - r^2 + 2(x_1-a_1)^2)$, and we have

$$\begin{aligned}(4.44) &= \int_{\mathbb{B}_d(a)} \frac{16\lambda^3}{(\lambda^2+r^2)^4}(\lambda^2 - r^2 + 2(x_1-a_1)^2)\,dx \\ &\quad + O\Big(\int_{\Omega\setminus\mathbb{B}_d(a)} \frac{\lambda^3}{(\lambda^2+r^2)^4}(\lambda^2 - r^2 + 2(x_1-a_1)^2)\,dx\Big) + O(\lambda) \\ &= 16\int_0^{2\pi}\int_0^d \frac{\lambda^3(\lambda^2 - r^2 + 2r^2\cos^2\theta)r}{(\lambda^2+r^2)^4}\,drd\theta \\ &\quad + O\Big(\int_0^{2\pi}\int_d^R \frac{\lambda^3(\lambda^2 - r^2 + 2r^2\cos^2\theta)r}{(\lambda^2+r^2)^4}\,drd\theta\Big) + O(\lambda) \\ &= 32\pi\int_0^d \frac{\lambda^5 r}{(\lambda^2+r^2)^4}\,dr + O(\lambda) = \frac{16\pi}{3}\frac{1}{\lambda} + O(\lambda).\end{aligned}$$

This proves (4.15).

When $i=3$, $(\xi_3 U_{\lambda,a})\cdot\frac{\partial}{\partial a_1}U_{\lambda,a}=\frac{4\lambda(x_1-a_1)(x_2-a_2)}{(\lambda^2+r^2)^2}$, and we have

$$\begin{aligned}(4.44) &= 32\int_\Omega \frac{\lambda^3(x_1-a_1)(x_2-a_2)}{(\lambda^2+r^2)^4}\,dx+O(\lambda)\\ &= 32\int_{\Omega\setminus\mathbb{B}_d(a)} \frac{\lambda^3(x_1-a_1)(x_2-a_2)}{(\lambda^2+r^2)^4}\,dx+O(\lambda)\quad\text{(by oddness)}\\ &= O\Big(\int_d^R \frac{\lambda^3 r^3}{(\lambda^2+r^2)^4}\,dr\Big)+O(\lambda)=O(\lambda).\end{aligned}$$

This proves (4.16).

(4.17), (4.18) and (4.19) are proved as (4.14), (4.15) and (4.16), and we omit the details.

Proof of (4.20). We have

$$\begin{aligned}&\int_\Omega \Big|\nabla\Big(\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big)\Big|^2 dx=-\int_\Omega \frac{\partial}{\partial\lambda}(\Delta PU_{\lambda,a})\cdot\frac{\partial}{\partial\lambda}PU_{\lambda,a}\,dx\\ =&\int_\Omega \frac{\partial}{\partial\lambda}(|\nabla U_{\lambda,a}|^2U_{\lambda,a})\cdot\Big(\frac{\partial}{\partial\lambda}U_{\lambda,a}-\frac{\partial}{\partial\lambda}h_{\lambda,a}\Big)\,dx\\ =&-\int_\Omega \frac{\partial}{\partial\lambda}(|\nabla U_{\lambda,a}|^2)U_{\lambda,a}\cdot\frac{\partial}{\partial\lambda}h_{\lambda,a}\,dx+\int_\Omega|\nabla U_{\lambda,a}|^2\Big|\frac{\partial}{\partial\lambda}U_{\lambda,a}\Big|^2dx\\ &-\int_\Omega|\nabla U_{\lambda,a}|^2\frac{\partial}{\partial\lambda}U_{\lambda,a}\cdot\frac{\partial}{\partial\lambda}h_{\lambda,a}\,dx.\end{aligned}\tag{4.45}$$

By (4.42),

$$\Big|\frac{\partial}{\partial\lambda}U_{\lambda,a}\Big|^2=\frac{4r^2}{(\lambda^2+r^2)^2}$$

and

$$\frac{\partial}{\partial\lambda}(|\nabla U_{\lambda,a}|^2)=\frac{\partial}{\partial\lambda}\Big(\frac{8\lambda^2}{(\lambda^2+r^2)^2}\Big)=\frac{16\lambda(r^2-\lambda^2)}{(\lambda^2+r^2)^3},$$

and we have

$$\int_\Omega|\nabla U_{\lambda,a}|^2\Big|\frac{\partial}{\partial\lambda}U_{\lambda,a}\Big|^2dx=32\int_\Omega\frac{\lambda^2r^2}{(\lambda^2+r^2)^4}\,dx=\frac{16\pi}{3}\frac{1}{\lambda^2}+O(\lambda^2),\tag{4.46}$$

$$\int_\Omega|\nabla U_{\lambda,a}|^2\frac{\partial}{\partial\lambda}U_{\lambda,a}\cdot\frac{\partial}{\partial\lambda}h_{\lambda,a}\,dx=O\Big(\int_\Omega\frac{\lambda^2r}{(\lambda^2+r^2)^3}\,dx\Big)=O(\lambda^{-1}),\tag{4.47}$$

$$\int_\Omega\frac{\partial}{\partial\lambda}(|\nabla U_{\lambda,a}|^2)U_{\lambda,a}\cdot\frac{\partial}{\partial\lambda}h_{\lambda,a}\,dx=O\Big(\int_\Omega\frac{\lambda|r^2-\lambda^2|}{(\lambda^2+r^2)^3}\,dx\Big)=O(\lambda^{-1}).\tag{4.48}$$

From (4.45)–(4.48), we obtain (4.20).

Proof of (4.21) and (4.22). For $1 \le i \le 3$, we have

$$\int_\Omega \nabla(\xi_i PU_{\lambda,a}) \cdot \nabla\Big(\frac{\partial}{\partial\lambda} PU_{\lambda,a}\Big)\, dx = -\int_\Omega \xi_i \Delta(PU_{\lambda,a}) \cdot \frac{\partial}{\partial\lambda} PU_{\lambda,a}\, dx \tag{4.49}$$

$$= \int_\Omega |\nabla U_{\lambda,a}|^2 (\xi_i U_{\lambda,a}) \cdot \frac{\partial}{\partial\lambda} U_{\lambda,a}\, dx - \int_\Omega |\nabla U_{\lambda,a}|^2 (\xi_i U_{\lambda,a}) \cdot \frac{\partial}{\partial\lambda} h_{\lambda,a}\, dx.$$

When $i = 1$, $(\xi_1 U_{\lambda,a}) \cdot \frac{\partial}{\partial\lambda} U_{\lambda,a} = 0$,

$$(\xi_1 U_{\lambda,a}) \cdot \frac{\partial}{\partial\lambda} h_{\lambda,a} = \frac{1}{\lambda^2 + r^2}\Big\{2\lambda(x_2 - a_2)\frac{\partial}{\partial\lambda} h^1_{\lambda,a} - 2\lambda(x_1 - a_1)\frac{\partial}{\partial\lambda} h^2_{\lambda,a}\Big\}$$

and

$$\begin{aligned}(4.49) &= -\int_\Omega |\nabla U_{\lambda,a}|^2 (\xi_1 U_{\lambda,a}) \cdot \frac{\partial}{\partial\lambda} h_{\lambda,a}\, dx \\ &= -\int_{\mathbb{B}_d(a)} |\nabla U_{\lambda,a}|^2 (\xi_1 U_{\lambda,a}) \cdot \frac{\partial}{\partial\lambda} h_{\lambda,a}\, dx \\ &\quad - \int_{\Omega\setminus\mathbb{B}_d(a)} |\nabla U_{\lambda,a}|^2 (\xi_1 U_{\lambda,a}) \cdot \frac{\partial}{\partial\lambda} h_{\lambda,a}\, dx. \end{aligned} \tag{4.50}$$

By (4.28) and (4.29), the second term in (4.50) is estimated as

$$\int_{\Omega\setminus\mathbb{B}_d(a)} |\nabla U_{\lambda,a}|^2 (\xi_1 U_{\lambda,a}) \cdot \frac{\partial}{\partial\lambda} h_{\lambda,a}\, dx = O\Big(\int_{\Omega\setminus\mathbb{B}_d(a)} |\nabla U_{\lambda,a}|^2\, dx\Big) = O(\lambda^2). \tag{4.51}$$

Thus by an argument similar to the proof of (3.18), (3.19), (3.20) and (5.17) in part I and (4.28) and (4.29), we have

$$\begin{aligned}(4.49) &= -16\int_{\mathbb{B}_d(a)} \frac{\lambda^3}{(\lambda^2 + r^2)^3}(x_2 - a_2)\frac{\partial}{\partial\lambda} h^1_{\lambda,a} \\ &\quad + 16\int_{\mathbb{B}_d(a)} \frac{\lambda^3}{(\lambda^2 + r^2)^3}(x_1 - a_1)\frac{\partial}{\partial\lambda} h^2_{\lambda,a}\, dx + O(\lambda^2) \\ &= -16\frac{\partial^2}{\partial x_2 \partial\lambda} h^1_{\lambda,a}(a) \int_{\mathbb{B}_d(a)} \frac{\lambda^3 (x_2 - a_2)^2}{(\lambda^2 + r^2)^3}\, dx \\ &\quad + 16\frac{\partial^2}{\partial x_1 \partial\lambda} h^2_{\lambda,a}(a) \int_{\mathbb{B}_d(a)} \frac{\lambda^3 (x_1 - a_1)^2}{(\lambda^2 + r^2)^3} dx + O\Big(\int_{\mathbb{B}_d(a)} \frac{\lambda^3 r^4}{(\lambda^2 + r^2)^3} dx\Big) \\ &= -4\pi\frac{\partial^2}{\partial x_2 \partial\lambda} h^1_{\lambda,a}(a)\lambda + 4\pi\frac{\partial^2}{\partial x_1 \partial\lambda} h^2_{\lambda,a}(a)\lambda + O(\lambda^2) \\ &= O(\lambda^2) \quad \Big(\text{since } \frac{\partial^2}{\partial x_2 \partial\lambda} h^1_{\lambda,a}(a) = \frac{\partial^2}{\partial x_1 \partial\lambda} h^2_{\lambda,a}(a);\ \text{cf. Lemma 4.5}\Big).\end{aligned}$$

This proves (4.21). When $i=2$, $(\xi_2 U_{\lambda,a})\cdot\frac{\partial}{\partial\lambda}U_{\lambda,a}=\frac{2(x_1-a_1)}{\lambda^2+r^2}$,

$$(\xi_2 U_{\lambda,a})\cdot\frac{\partial}{\partial\lambda}h_{\lambda,a}=\frac{r^2-\lambda^2}{\lambda^2+r^2}\frac{\partial}{\partial\lambda}h^1_{\lambda,a}-\frac{2\lambda(x_1-a_1)}{\lambda^2+r^2}\frac{\partial}{\partial\lambda}h^3_{\lambda,a},$$

and we have (using (4.28), (4.29) and a similar argument in the case $i=1$)

$$\begin{aligned}(4.49)&=16\int_{\Omega\setminus\mathbb{B}_d(a)}\frac{\lambda^2(x_1-a_1)}{(\lambda^2+r^2)^3}\,dx-8\int_{\mathbb{B}_d(a)}\frac{\lambda^2(r^2-\lambda^2)}{(\lambda^2+r^2)^3}\frac{\partial}{\partial\lambda}h^1_{\lambda,a}\\
&\quad+16\int_{\mathbb{B}_d(a)}\frac{\lambda^3(x_1-a_1)}{(\lambda^2+r^2)^3}\frac{\partial}{\partial\lambda}h^3_{\lambda,a}+O\Big(\int_{\Omega\setminus\mathbb{B}_d(a)}\frac{\lambda^2}{(\lambda^2+r^2)^2}\,dx\Big)\\
&=-16\pi\frac{\partial}{\partial\lambda}h^1_{\lambda,a}(a)\int_0^d\frac{\lambda^2(r^2-\lambda^2)r}{(\lambda^2+r^2)^3}\,dr+16\frac{\partial^2}{\partial x_1\partial\lambda}h^3_{\lambda,a}(a)\int_{\mathbb{B}_d(a)}\frac{\lambda^3(x_1-a_1)^2}{(\lambda^2+r^2)^3}\,dx\\
&\quad+O(\lambda^2)+O\Big(\int_{\mathbb{B}_d(a)}\frac{\lambda^2|r^2-\lambda^2|r^2}{(\lambda^2+r^2)^3}\,dx\Big)+O\Big(\lambda\int_{\mathbb{B}_d(a)}\frac{\lambda^3r^4}{(\lambda^2+r^2)^3}\,dx\Big)\\
&=16\pi\frac{\partial}{\partial\lambda}h^1_{\lambda,a}(a)\int_{d/\lambda}^\infty\frac{(s^2-1)s}{(1+s^2)^3}\,ds+O\Big(\lambda^2\int_0^{d/\lambda}\frac{s^3}{(1+s^2)^3}\,ds\Big)+O(\lambda^2)\\
&\quad+O\Big(\lambda^2\int_0^{d/\lambda}\frac{|s^2-1|s^3}{(1+s^2)^3}\,ds\Big)+O\Big(\lambda^4\int_0^{d/\lambda}\frac{s^5}{(1+s^2)^3}\,ds\Big)=O(\lambda^2|\log\lambda|).\end{aligned}$$

Here we have used $\int_0^\infty\frac{(s^2-1)s}{(1+s^2)^3}\,ds=0$.

The case $i=3$ follows from an argument similar to the proof of the case $i=2$. Thus we complete the proof of (4.22).

Proof of (4.23) and (4.24). For $1\le i\le 3$,

$$\begin{aligned}\int_\Omega|\nabla(\xi_i PU_{\lambda,a})|^2\,dx&=\int_\Omega|\nabla U_{\lambda,a}|^2(\xi_iU_{\lambda,a})\cdot(\xi_i\widehat{U}_{\lambda,a})\,dx+O(\lambda)\\
&=-\int_\Omega|\nabla U_{\lambda,a}|^2(\xi_i^2U_{\lambda,a})\cdot(\widehat{U}_{\lambda,a})\,dx+O(\lambda).\end{aligned}\tag{4.52}$$

When $i=1$,

$$\xi_1^2=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\quad -(\xi_1^2U_{\lambda,a})\cdot\widehat{U}_{\lambda,a}=\frac{4\lambda^2r^2}{(\lambda^2+r^2)^2}$$

and

$$(4.52)=32\int_\Omega\frac{\lambda^4r^2}{(\lambda^2+r^2)^4}\,dx+O(\lambda)=\frac{16\pi}{3}+O(\lambda).\tag{4.53}$$

When $i = 2$,

$$\xi_2^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad -(\xi_2^2 U_{\lambda,a}) \cdot \widehat{U}_{\lambda,a} = \frac{2\lambda^2}{(\lambda^2+r^2)^2}(2(x_1-a_1)^2 - r^2 + \lambda^2)$$

and

$$(4.52) = 16 \int_\Omega \frac{\lambda^4}{(\lambda^2+r^2)^4}(2(x_1-a_1)^2 - r^2 + \lambda^2)\, dx + O(\lambda) = \frac{16\pi}{3} + O(\lambda). \tag{4.54}$$

When $i = 3$,

$$\xi_3^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad -(\xi_3^2 U_{\lambda,a}) \cdot \widehat{U}_{\lambda,a} = \frac{2\lambda^2}{(\lambda^2+r^2)^2}(2(x_2-a_2)^2 - r^2 + \lambda^2),$$

and as in the case $i = 2$, we have

$$(4.52) = \frac{16\pi}{3} + O(\lambda). \tag{4.55}$$

(4.53)–(4.55) give (4.23). As in (4.52), we have

$$\int_\Omega \nabla(\xi_i P U_{\lambda,a}) \cdot \nabla(\xi_j P U_{\lambda,a})\, dx = -\int_\Omega |\nabla U_{\lambda,a}|^2 (\xi_j \xi_i U_{\lambda,a}) \cdot \widehat{U}_{\lambda,a}\, dx + O(\lambda). \tag{4.56}$$

We may assume $j < i$. When $j = 1$ and $i = 2$,

$$\xi_1\xi_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\xi_1\xi_2 U_{\lambda,a}) \cdot \widehat{U}_{\lambda,a} = \frac{2\lambda(\lambda^2 - r^2)(x_2 - a_2)}{(\lambda^2+r^2)^2},$$

and we have

$$\begin{aligned} (4.56) &= -\int_{\Omega \setminus \mathbb{B}_d(a)} |\nabla U_{\lambda,a}|^2 (\xi_1\xi_2 U_{\lambda,a}) \cdot \widehat{U}_{\lambda,a}\, dx + O(\lambda) \text{ (by oddness)} \\ &= O\Big(\int_{\Omega \setminus \mathbb{B}_d(a)} \frac{\lambda^2}{(\lambda^2+r^2)^2}\, dx\Big) + O(\lambda) = O(\lambda). \end{aligned} \tag{4.57}$$

When $j = 2$ and $i = 3$,

$$\xi_2\xi_3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\xi_2\xi_3 U_{\lambda,a}) \cdot \widehat{U}_{\lambda,a} = -\frac{4\lambda^2(x_1 - a_1)(x_2 - a_2)}{(\lambda^2 + r^2)^2},$$

and we have as before

$$(4.56) = O(\lambda). \tag{4.58}$$

When $j = 1$ and $i = 3$,

$$\xi_1\xi_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\xi_1\xi_3 U_{\lambda,a}) \cdot \widehat{U}_{\lambda,a} = \frac{2\lambda(r^2 - \lambda^2)(x_1 - a_1)}{(\lambda^2 + r^2)^2},$$

and we have as before

$$(4.56) = O(\lambda). \tag{4.59}$$

(4.57)–(4.59) imply (4.24). $\square$

Lemma 4.3. *We have the following estimates:*

$$\langle dJ_H(RPU_{\lambda,a} + w), RPU_{\lambda,a}\rangle = O(\lambda^2) + O(\lambda H), \tag{4.60}$$

$(1 < \forall p < 2)$

$$\begin{aligned} &\Big\langle dJ_H(RPU_{\lambda,a} + w), R\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big\rangle \ (i = 1, 2) \\ &= S\Big\{\Big(\frac{\partial^2}{\partial x_1 \partial x_i}h_a^1(a) + \frac{\partial^2}{\partial x_2 \partial x_i}h_a^2(a)\Big)\lambda^2 \\ &- \Big(\frac{\partial^2 h_\gamma}{\partial x_1 \partial x_i}(a) \cdot Re_1 + \frac{\partial^2 h_\gamma}{\partial x_2 \partial x_i}(a) \cdot Re_2\Big)\lambda H\Big\} \\ &+ O(H\lambda^{2/p}|\log\lambda|^{1/2}) + O(H^2\lambda^{2/p-1}|\log\lambda|^{1/2}) + O(\lambda^{2/p+1}|\log\lambda|^{1/2}), \end{aligned} \tag{4.61}$$

$$\begin{aligned} &\Big\langle dJ_H(RPU_{\lambda,a} + w), R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big\rangle \\ &= S\Big\{\Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda - ((h_\gamma)_{x_1}(a) \cdot Re_1 + (h_\gamma)_{x_2}(a) \cdot Re_2)H\Big\} \\ &\quad + O(\lambda^2|\log\lambda|) + O(H\lambda|\log\lambda|) + O(H^2), \end{aligned} \tag{4.62}$$

$$\begin{aligned} &\langle dJ_H(RPU_{\lambda,a} + w), R\xi_i PU_{\lambda,a}\rangle \ (i = 1, 2, 3) \\ &= -S((h_\gamma)_{x_1}(a) \cdot (R\xi_i e_1) + (h_\gamma)_{x_2}(a) \cdot (R\xi_i e_2))\lambda H \\ &\quad + O(H\lambda^2|\log\lambda|^{1/2}) + O(\lambda H^2) + O(\lambda^3|\log\lambda|), \end{aligned} \tag{4.63}$$

where $w = w(H, R, \lambda, a)$ *and* $S = (32\pi)^{1/3}$.

Proof. *Proof of* (4.60). By a calculation similar to the proof of Lemma 2.1, we have

$$\begin{aligned}
&\langle dJ_H(RPU_{\lambda,a} + w), RPU_{\lambda,a}\rangle = 2|Q(RPU_{\lambda,a} + w)|^{-2/3} \\
&\times \Big\{ -\frac{I_H(RPU_{\lambda,a} + w)}{Q(RPU_{\lambda,a} + w)} \int_\Omega RPU_{\lambda,a} \cdot (RPU_{\lambda,a} + w)_{x_1} \wedge (RPU_{\lambda,a} + w)_{x_2}\, dx \\
&+ \int_\Omega |\nabla PU_{\lambda,a}|^2\, dx + 2H \int_\Omega \underline{u}_H \cdot \{(RPU_{\lambda,a} + w)_{x_1} \wedge (RPU_{\lambda,a})_{x_2} \\
&+ (RPU_{\lambda,a})_{x_1} \wedge (RPU_{\lambda,a} + w)_{x_2}\}\, dx \Big\}. \qquad (4.64)
\end{aligned}$$

We first estimate $I_H(RPU_{\lambda,a} + w)$ and $Q(RPU_{\lambda,a} + w)$. We have

$$\begin{aligned}
I_H(RPU_{\lambda,a} + w) &= \int_\Omega |\nabla PU_{\lambda,a}|^2\, dx + \int_\Omega |\nabla w|^2\, dx \\
&+ 4H \int_\Omega \underline{u}_H \cdot (RPU_{\lambda,a})_{x_1} \wedge (RPU_{\lambda,a})_{x_2}\, dx \\
&+ 4H \int_\Omega \underline{u}_H \cdot ((RPU_{\lambda,a})_{x_1} \wedge w_{x_2} + w_{x_1} \wedge (RPU_{\lambda,a})_{x_2})\, dx \\
&+ 4H \int_\Omega \underline{u}_H \cdot w_{x_1} \wedge w_{x_2}\, dx. \qquad (4.65)
\end{aligned}$$

Here, by the integration by parts (see [Lemma A.4, 1]),

$$\begin{aligned}
&\int_\Omega \underline{u}_H \cdot (RPU_{\lambda,a})_{x_1} \wedge (RPU_{\lambda,a})_{x_2}\, dx \qquad (4.66) \\
&= \frac{1}{2} \int_\Omega RPU_{\lambda,a} \cdot ((\underline{u}_H)_{x_1} \wedge (RPU_{\lambda,a})_{x_2} + (RPU_{\lambda,a})_{x_1} \wedge (\underline{u}_H)_{x_2})\, dx \\
&= O\Big(\int_\Omega |\widehat{U}_{\lambda,a}||\nabla U_{\lambda,a}|\, dx\Big) + O\Big(\int_\Omega |\widehat{U}_{\lambda,a}||\nabla h_{\lambda,a}|\, dx\Big) \\
&= O\Big(\int_\Omega \frac{\lambda^2}{(\lambda^2 + r^2)^{3/2}}\, dx\Big) + O\Big(\lambda \int_\Omega \frac{\lambda}{(\lambda^2 + r^2)^{1/2}}\, dx\Big) = O(\lambda), \\
&\int_\Omega \underline{u}_H \cdot ((RPU_{\lambda,a})_{x_1} \wedge w_{x_2} + w_{x_1} \wedge (RPU_{\lambda,a})_{x_2})\, dx \\
&= \int_\Omega (RPU_{\lambda,a}) \cdot ((\underline{u}_H)_{x_1} \wedge w_{x_2} + w_{x_1} \wedge (\underline{u}_H)_{x_2})\, dx
\end{aligned}$$

$$= O\Big(\int_\Omega |\widehat{U}_{\lambda,a}||\nabla w|\,dx\Big) = O(\|\widehat{U}_{\lambda,a}\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)})$$
$$= O(H\lambda^2|\log\lambda|) + O(\lambda^3|\log\lambda|) \text{ (by Lemma 2.3).} \tag{4.67}$$

Lemma 2.3, (4.65), (4.66) and (4.67) imply

$$I_H(RPU_{\lambda,a}+w) = 8\pi + O(\lambda^2) + O(H\lambda). \tag{4.68}$$

Next, using Lemma A.4 in [1], we have

$$\begin{aligned} Q(RPU_{\lambda,a}+w) &= \int_\Omega (RPU_{\lambda,a}+w)\cdot(RPU_{\lambda,a}+w)_{x_1}\wedge(RPU_{\lambda,a}+w)_{x_2}\,dx \\ &= \int_\Omega RPU_{\lambda,a}\cdot(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2}\,dx \\ &\quad + 3\int_\Omega w\cdot(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2}\,dx \\ &\quad + 3\int_\Omega RPU_{\lambda,a}\cdot w_{x_1}\wedge w_{x_2}\,dx + \int_\Omega w\cdot w_{x_1}\wedge w_{x_2}\,dx. \end{aligned} \tag{4.69}$$

Here by (3.46) in part I [3], we have

$$\int_\Omega RPU_{\lambda,a}\cdot(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2}\,dx = -4\pi + O(\lambda^2), \tag{4.70}$$

and by Lemma 2.3 and Lemma A.3 in [1]

$$\begin{aligned} &\int_\Omega w\cdot(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2} \\ &= \int_\Omega w\cdot(R\widehat{U}_{\lambda,a})_{x_1}\wedge(R\widehat{U}_{\lambda,a})_{x_2}\,dx - \int_\Omega w\cdot(R\widehat{U}_{\lambda,a})_{x_1}\wedge(Rh_{\lambda,a})_{x_2}\,dx \\ &\quad - \int_\Omega w\cdot(Rh_{\lambda,a})_{x_1}\wedge(R\widehat{U}_{\lambda,a})_{x_1} + \int_\Omega w\cdot(Rh_{\lambda,a})_{x_1}\wedge(Rh_{\lambda,a})_{x_2}\,dx \\ &= O\Big(\int_\Omega |h_{\lambda,a}||\nabla w||\nabla U_{\lambda,a}|\,dx\Big) + O(\|\nabla w\|_{L^2(\Omega)}\|\nabla h_{\lambda,a}\|^2_{L^2(\Omega)}) \\ &= O(H\lambda^2|\log\lambda|^{1/2}) + O(\lambda^3|\log\lambda|^{1/2}). \end{aligned} \tag{4.71}$$

The last two terms in (4.69) are $O(\|\nabla w\|^2_{L^2(\Omega)})$, and we have from (4.69)–(4.71)

$$Q(RPU_{\lambda,a}+w) = -4\pi + O(\lambda^2) + O(\lambda H). \tag{4.72}$$

The calculation of $Q(RPU_{\lambda,a}+w)$ shows that

$$\int_\Omega RPU_{\lambda,a}\cdot(RPU_{\lambda,a}+w)_{x_1}\wedge(RPU_{\lambda,a}+w)_{x_2}\,dx=-4\pi+O(\lambda^2)+O(\lambda H). \tag{4.73}$$

The calculation of $I_H(RPU_{\lambda,a}+w)$ shows that

$$\int_\Omega \underline{u}_H((RPU_{\lambda,a}+w)_{x_1}\wedge(RPU_{\lambda,a})_{x_2}+(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a}+w)_{x_2})\,dx \\ =O(\lambda). \tag{4.74}$$

By (4.64), (4.68), (4.72), (4.73) and (4.74), we have (4.60).

Proof of (4.61). Note that

$$\Big\langle dJ_H(RPU_{\lambda,a}+w),R\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big\rangle=\frac{\partial}{\partial a_i}J_H(RPU_{\lambda,a}+w).$$

We first estimate $\frac{\partial}{\partial a_i}I_H(RPU_{\lambda,a}+w)$ and $\frac{\partial}{\partial a_i}Q(RPU_{\lambda,a}+w)$. By (4.8),

$$\frac{\partial}{\partial a_i}\int_\Omega|\nabla PU_{\lambda,a}|^2\,dx=2\int_\Omega\nabla(PU_{\lambda,a})\cdot\nabla\Big(\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big)\,dx \\ =\ -8\pi\Big(\frac{\partial^2h^1}{\partial x_1\partial a_i}(a,a)+\frac{\partial^2h^2}{\partial x_2\partial a_i}(a,a)\Big)\lambda^2+O(\lambda^3). \tag{4.75}$$

$$\frac{\partial}{\partial a_i}\int_\Omega\underline{u}_H\cdot(RPU_{\lambda,a}+w)_{x_1}\wedge(RPU_{\lambda,a}+w)_{x_2}\,dx \\ =\frac{\partial}{\partial a_i}\Big\{\int_\Omega\underline{u}_H\cdot(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2}+\int_\Omega\underline{u}_H\cdot((RPU_{\lambda,a})_{x_1}\wedge w_{x_2} \\ +w_{x_1}\wedge(RPU_{\lambda,a})_{x_2})\,dx+\int_\Omega\underline{u}_H\cdot w_{x_1}\wedge w_{x_2}\,dx\Big\}. \tag{4.76}$$

Here by Lemma A.4 in [1]

$$\frac{\partial}{\partial a_i}\int_\Omega\underline{u}_H\cdot((RPU_{\lambda,a})_{x_1}\wedge w_{x_2}+w_{x_1}\wedge(RPU_{\lambda,a})_{x_2})\,dx \\ =\frac{\partial}{\partial a_i}\int_\Omega w\cdot((RPU_{\lambda,a})_{x_1}\wedge(\underline{u}_H)_{x_2}+(\underline{u}_H)_{x_1}\wedge(RPU_{\lambda,a})_{x_2})\,dx \\ =\int_\Omega w\cdot\Big(\Big(R\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big)_{x_1}\wedge(\underline{u}_H)_{x_2}+(\underline{u}_H)_{x_1}\wedge\Big(R\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big)_{x_2}\Big)\,dx \\ =O\Big(\int_\Omega|w|\Big|\nabla\Big(\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big)\Big|\,dx\Big) \\ =O\Big(\|w\|_{L^{p/(p-1)}(\Omega)}\Big\|\nabla\Big(\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big)\Big\|_{L^p(\Omega)}\Big)\ (1<\forall p<2). \tag{4.77}$$

Since $\left|\nabla\left(\frac{\partial}{\partial a_i}\widehat{U}_{\lambda,a}\right)\right| \le C\frac{\lambda}{(\lambda^2+r^2)^{3/2}}$, we have using (4.25) and (4.26)

$$\left\|\nabla\Big(\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big)\right\|_{L^p(\Omega)} = \left\|\nabla\Big(\frac{\partial}{\partial a_i}\widehat{U}_{\lambda,a}\Big)\right\|_{L^p(\Omega)} + O(\lambda) = O(\lambda^{2/p-2}). \quad (4.78)$$

By the Sobolev imbedding theorem and Lemma 2.3, we have

$$\|w\|_{L^{p/(p-1)}(\Omega)} = O(\|\nabla w\|_{L^2(\Omega)}) = O((H\lambda+\lambda^2)|\log\lambda|^{1/2}). \quad (4.79)$$

By (4.78) and (4.79), we have

$$(4.77) = O(H\lambda^{2/p-1}|\log\lambda|^{1/2}) + O(\lambda^{2/p}|\log\lambda|^{1/2}). \quad (4.80)$$

To estimate the first term in (4.76), we smoothly extend $\underline{u}_H$ to some bounded, open set $\tilde{\Omega} \Supset \Omega$ (we also denote the extended map as $\underline{u}_H$) such that $\mathrm{supp}(\underline{u}_H) \subset \tilde{\Omega}$ and

$$\|\underline{u}_H\|_{C^2(\tilde{\Omega})} \le C \quad \text{(independent of } H). \quad (4.81)$$

This is possible in view of Lemma 2.1 in part I [3]. Then the first term in (4.76) is

$$\begin{aligned}
&\frac{\partial}{\partial a_i}\int_{\tilde{\Omega}} \underline{u}_H\cdot(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2}\,dx\\
&= \frac{\partial}{\partial a_i}\Big\{\int_{\tilde{\Omega}} \underline{u}_H\cdot(RU_{\lambda,a})x_1\wedge(RU_{\lambda,a})_{x_2}\,dx\\
&\qquad -\int_{\tilde{\Omega}} \underline{u}_H\cdot((RU_{\lambda,a})_{x_1}\wedge(Rh_{\lambda,a})_{x_2} + (Rh_{\lambda,a})_{x_1}\wedge(RU_{\lambda,a})_{x_2})\,dx\\
&\qquad +\int_{\tilde{\Omega}} \underline{u}_H\cdot(Rh_{\lambda,a})_{x_1}\wedge(Rh_{\lambda,a})_{x_2}\,dx\Big\}. \qquad (4.82)
\end{aligned}$$

Here the second term added to the third term in (4.82) is estimated as (using (4.25) and (4.26))

$$\begin{aligned}
&\frac{\partial}{\partial a_i}\int_{\tilde{\Omega}} \underline{u}_H\cdot((RU_{\lambda,a})_{x_1}\wedge(Rh_{\lambda,a})_{x_2} + (Rh_{\lambda,a})_{x_1}\wedge(RU_{\lambda,a})_{x_2})\,dx\\
&= -\int_{\tilde{\Omega}} \underline{u}_H\cdot\Big(\Big(R\frac{\partial}{\partial x_i}U_{\lambda,a}\Big)_{x_1}\wedge(Rh_{\lambda,a})_{x_2} + (Rh_{\lambda,a})_{x_1}\wedge\Big(R\frac{\partial}{\partial x_i}U_{\lambda,a}\Big)_{x_2}\Big)\,dx\\
&\quad +\int_{\tilde{\Omega}} \underline{u}_H\cdot\Big((RU_{\lambda,a})_{x_1}\wedge\Big(R\frac{\partial}{\partial a_i}h_{\lambda,a}\Big)_{x_2} + \Big(R\frac{\partial}{\partial a_i}h_{\lambda,a}\Big)_{x_1}\wedge(RU_{\lambda,a})_{x_2}\Big)\,dx\\
&\qquad \Big(\text{since } \frac{\partial}{\partial x_i}U_{\lambda,a} = -\frac{\partial}{\partial a_i}U_{\lambda,a}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\tilde\Omega} (\underline{u}_H)_{x_i} \cdot ((RU_{\lambda,a})_{x_1} \wedge (Rh_{\lambda,a})_{x_2} + (Rh_{\lambda,a})_{x_1} \wedge (RU_{\lambda,a})_{x_2})\, dx \\
&\quad + \int_{\tilde\Omega} \underline{u}_H \cdot ((R\widehat{U}_{\lambda,a})_{x_1} \wedge (Rh_{\lambda,a})_{x_2 x_i} + (Rh_{\lambda,a})_{x_1 x_i} \wedge (R\widehat{U}_{\lambda,a})_{x_2})\, dx \\
&\quad + \int_{\tilde\Omega} \underline{u}_H \cdot \Big((R\widehat{U}_{\lambda,a})_{x_1} \wedge \Big(R\frac{\partial}{\partial a_i} h_{\lambda,a}\Big)_{x_2} + \Big(R\frac{\partial}{\partial a_i} h_{\lambda,a}\Big)_{x_1} \wedge (R\widehat{U}_{\lambda,a})_{x_2}\Big)\, dx \\
&= O\Big(\lambda \int_\Omega |\nabla U_{\lambda,a}|\, dx\Big) = O(\lambda^2 |\log\lambda|). \qquad (4.83)
\end{aligned}
$$

The fourth term in (4.82) is estimated as

$$
\begin{aligned}
&\frac{\partial}{\partial a_i} \int_{\tilde\Omega} \underline{u}_H \cdot (Rh_{\lambda,a})_{x_1} \wedge (Rh_{\lambda,a})_{x_2}\, dx \\
&= \int_\Omega \underline{u}_H \cdot \Big(\Big(R\frac{\partial}{\partial a_i} h_{\lambda,a}\Big)_{x_1} \wedge (Rh_{\lambda,a})_{x_2} + (Rh_{\lambda,a})_{x_1} \wedge \Big(R\frac{\partial}{\partial a_i} h_{\lambda,a}\Big)_{x_2}\Big)\, dx \\
&= O(\lambda^2). \qquad (4.84)
\end{aligned}
$$

The first term in (4.82) is estimated as (using an argument similar to the proof of (3.16)–(3.20) in part I [3] and Lemma 2.1 in [3])

$$
\begin{aligned}
&\frac{\partial}{\partial a_i} \int_{\tilde\Omega} \underline{u}_H \cdot (RU_{\lambda,a})_{x_1} \wedge (RU_{\lambda,a})_{x_2} \qquad (4.85) \\
&\quad = - \int_{\tilde\Omega} \underline{u}_H \cdot \frac{\partial}{\partial x_i} \{(RU_{\lambda,a})_{x_1} \wedge (RU_{\lambda,a})_{x_2}\}\, dx \\
&= \int_{\tilde\Omega} (\underline{u}_H)_{x_i} \cdot (RU_{\lambda,a})_{x_1} \wedge (RU_{\lambda,a})_{x_2}\, dx \\
&= -2\pi \Big(\frac{\partial^2 \underline{u}_H}{\partial x_1 \partial x_i}(a) \cdot Re_1 + \frac{\partial^2 \underline{u}_H}{\partial x_2 \partial x_i}(a) \cdot Re_2\Big)\lambda + O(\lambda^2) \\
&= -2\pi \Big(\frac{\partial^2 h_\gamma}{\partial x_1 \partial x_i}(a) \cdot Re_1 + \frac{\partial^2 h_\gamma}{\partial x_2 \partial x_i}(a) \cdot Re_2\Big)\lambda + O(\lambda^2) + O(\lambda H).
\end{aligned}
$$

From (4.83)–(4.85), we have

$$
(4.82) = -2\pi \Big(\frac{\partial^2 h_\gamma}{\partial x_1 \partial x_i}(a) \cdot Re_1 + \frac{\partial^2 h_\gamma}{\partial x_2 \partial x_i}(a) \cdot Re_2\Big)\lambda + O(\lambda^2 |\log\lambda|) + O(\lambda H). \qquad (4.86)
$$

By (4.75), (4.76), (4.77), (4.80), (4.82) and (4.86), we have ($1 < \forall p < 2$)

$$
\frac{\partial}{\partial a_i} I_H(RPU_{\lambda,a} + w) = -8\pi \Big(\frac{\partial^2 h^1}{\partial x_1 \partial a_i}(a,a) + \frac{\partial^2 h^2}{\partial x_2 \partial a_i}(a,a)\Big)\lambda^2
$$

$$- 8\pi\Big(\frac{\partial^2 h_\gamma}{\partial x_1 \partial x_i}(a)\cdot Re_1 + \frac{\partial^2 h_\gamma}{\partial x_2 \partial x_i}(a)\cdot Re_2\Big) H\lambda \qquad (4.87)$$
$$+ O(H\lambda^{2/p}|\log\lambda|^{1/2}) + O(H^2\lambda^{2/p-1}|\log\lambda|^{1/2}) + O(\lambda^3).$$

Next we estimate $\frac{\partial}{\partial a_i} Q(RPU_{\lambda,a}+w)$.

$$\frac{\partial}{\partial a_i} Q(RPU_{\lambda,a}+w) = \frac{\partial}{\partial a_i}\Big\{\int_\Omega RPU_{\lambda,a}\cdot(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2}\,dx$$
$$+ 3\int_\Omega w\cdot(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2}\,dx$$
$$+ 3\int_\Omega RPU_{\lambda,a}\cdot w_{x_1}\wedge w_{x_2}\,dx + \int_\Omega w\cdot w_{x_1}\wedge w_{x_2}\,dx\Big\}. \qquad (4.88)$$

Here the first term in (4.88) is

$$\frac{\partial}{\partial a_i}\int_\Omega RPU_{\lambda,a}\cdot(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2}\,dx$$
$$= \int_\Omega \frac{\partial}{\partial a_i}PU_{\lambda,a}\cdot(PU_{\lambda,a})_{x_1}\wedge(PU_{\lambda,a})_{x_2}\,dx$$
$$+ \int_\Omega PU_{\lambda,a}\cdot\Big(\Big(\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big)_{x_1}\wedge(PU_{\lambda,a})_{x_2} + (PU_{\lambda,a})_{x_1}\wedge\Big(\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big)_{x_2}\Big)\,dx$$
$$= 3\int_\Omega \frac{\partial}{\partial a_i}PU_{\lambda,a}\cdot(PU_{\lambda,a})_{x_1}\wedge(PU_{\lambda,a})_{x_2}\,dx$$
$$= 3\int_\Omega \frac{\partial}{\partial a_i}PU_{\lambda,a}\cdot(\widehat{U}_{\lambda,a})_{x_1}\wedge(\widehat{U}_{\lambda,a})_{x_2}\,dx$$
$$- 3\int_\Omega \frac{\partial}{\partial a_i}PU_{\lambda,a}\cdot(\widehat{U}_{\lambda,a})_{x_1}\wedge(h_{\lambda,a})_{x_2}\,dx$$
$$- 3\int_\Omega \frac{\partial}{\partial a_i}PU_{\lambda,a}\cdot(h_{\lambda,a})_{x_1}\wedge(\widehat{U}_{\lambda,a})_{x_2}\,dx$$
$$+ 3\int_\Omega \frac{\partial}{\partial a_i}PU_{\lambda,a}\cdot(h_{\lambda,a})_{x_1}\wedge(h_{\lambda,a})_{x_2}\,dx. \qquad (4.89)$$

Here, by (4.8), we have

$$3\int_\Omega \frac{\partial}{\partial a_i}PU_{\lambda,a}\cdot(\widehat{U}_{\lambda,a})_{x_1}\wedge(\widehat{U}_{\lambda,a})_{x_2}\,dx = \frac{3}{2}\int_\Omega \frac{\partial}{\partial a_i}PU_{\lambda,a}\cdot\Delta U_{\lambda,a}\,dx$$
$$= -\frac{3}{2}\int_\Omega \nabla(PU_{\lambda,a})\cdot\nabla\Big(\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big)\,dx$$
$$= 6\pi\Big(\frac{\partial^2 h^1}{\partial x_1\partial a_i}(a,a) + \frac{\partial^2 h^2}{\partial x_2\partial a_i}(a,a)\Big)\lambda^2 + O(\lambda^3). \qquad (4.90)$$

$-\frac{1}{3}\times$(the second term + the third term) is (here $h_{\lambda,a}$ is extended to $\mathbb{R}^2\setminus\Omega$ by $\widehat{U}_{\lambda,a}$)

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial a_i}\widehat{U}_{\lambda,a}\cdot((U_{\lambda,a})_{x_1}\wedge(h_{\lambda,a})_{x_2}+(h_{\lambda,a})_{x_1}\wedge(U_{\lambda,a})_{x_2})\,dx\\
&-\int_\Omega \frac{\partial}{\partial a_i}h_{\lambda,a}\cdot((U_{\lambda,a})_{x_1}\wedge(h_{\lambda,a})_{x_2}+(h_{\lambda,a})_{x_1}\wedge(U_{\lambda,a})_{x_2})\,dx\\
&=\int_\Omega \frac{\partial}{\partial a_i}\widehat{U}_{\lambda,a}\cdot((U_{\lambda,a})_{x_1}\wedge(h_{\lambda,a})_{x_2}+(h_{\lambda,a})_{x_1}\wedge(U_{\lambda,a})_{x_2})\,dx\\
&+O\Big(\lambda^2\int_\Omega|\nabla U_{\lambda,a}|\,dx\Big)\\
&=\int_\Omega \frac{\partial}{\partial a_i}\widehat{U}_{\lambda,a}\cdot((U_{\lambda,a})_{x_1}\wedge(h_{\lambda,a})_{x_2}+(h_{\lambda,a})_{x_1}\wedge(U_{\lambda,a})_{x_2})dx+O(\lambda^3|\log\lambda|)\\
&=\int_{\mathbb{R}^2} \frac{\partial}{\partial a_i}\widehat{U}_{\lambda,a}\cdot((U_{\lambda,a})_{x_1}\wedge(h_{\lambda,a})_{x_2}+(h_{\lambda,a})_{x_1}\wedge(U_{\lambda,a})_{x_2})dx+O(\lambda^3|\log\lambda|)
\end{aligned}$$

(since $\frac{\partial}{\partial a_i}\widehat{U}_{\lambda,a}=-\frac{\partial}{\partial x_i}U_{\lambda,a}$ in $\mathbb{R}^2\setminus\Omega$ and the integrand is 0 in $\mathbb{R}^2\setminus\Omega$)

$$\begin{aligned}
&=\int_{\mathbb{R}^2} h_{\lambda,a}\cdot\Big((U_{\lambda,a})_{x_1}\wedge\Big(\frac{\partial}{\partial a_i}U_{\lambda,a}\Big)_{x_2}+\Big(\frac{\partial}{\partial a_i}U_{\lambda,a}\Big)_{x_1}\wedge(U_{\lambda,a})_{x_2}\Big)\,dx\\
&+O(\lambda^3|\log\lambda|)=\int_{\mathbb{R}^2} h_{\lambda,a}\cdot\frac{\partial}{\partial a_i}\{(U_{\lambda,a})_{x_1}\wedge(U_{\lambda,a})_{x_2}\}\,dx+O(\lambda^3|\log\lambda|)\\
&=-\int_{\mathbb{R}^2} h_{\lambda,a}\cdot\frac{\partial}{\partial x_i}\{(U_{\lambda,a})_{x_1}\wedge(U_{\lambda,a})_{x_2}\}\,dx+O(\lambda^3|\log\lambda|)\\
&=\int_{\mathbb{R}^2}\frac{\partial h_{\lambda,a}}{\partial x_i}\cdot(U_{\lambda,a})_{x_1}\wedge(U_{\lambda,a})_{x_2}\,dx+O(\lambda^3|\log\lambda|)\\
&=-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla U_{\lambda,a}|^2U_{\lambda,a}\cdot\frac{\partial}{\partial x_i}h_{\lambda,a}\,dx+O(\lambda^3|\log\lambda|)\\
&=-2\pi\Big(\frac{\partial^2h^1}{\partial x_1\partial x_i}(a,a)+\frac{\partial^2h^2}{\partial x_2\partial x_i}(a,a)\Big)\lambda^2+O(\lambda^3|\log\lambda|) \qquad (4.91)
\end{aligned}$$

(see the proof of (4.8)). The fourth term in (4.89) is estimated as

$$\begin{aligned}
&3\int_\Omega\frac{\partial}{\partial a_i}PU_{\lambda,a}\cdot(h_{\lambda,a})_{x_1}\wedge(h_{\lambda,a})_{x_2}\,dx\\
&=O\Big(\lambda^2\int_\Omega|\nabla U_{\lambda,a}|\,dx\Big)+O(\lambda^3)=O(\lambda^3|\log\lambda|). \qquad (4.92)
\end{aligned}$$

From (4.90)–(4.92), we have

$$(4.89) = 6\pi\Big(\frac{\partial^2 h^1}{\partial x_1 \partial a_i}(a,a) + \frac{\partial^2 h^2}{\partial x_2 \partial a_i}(a,a) + \frac{\partial^2 h^1}{\partial x_1 \partial x_i}(a,a) + \frac{\partial^2 h^2}{\partial x_2 \partial x_i}(a,a)\Big)\lambda^2 + O(\lambda^3 |\log\lambda|). \tag{4.93}$$

Next we estimate the second term in (4.88). By (4.3), we have

$$\begin{aligned}
&\frac{\partial}{\partial a_i}\int_\Omega w\cdot (RPU_{\lambda,a})_{x_1}\wedge (RPU_{\lambda,a})_{x_2}\,dx \\
&= -\frac{\partial}{\partial a_i}\int_\Omega w\cdot ((R\widehat{U}_{\lambda,a})_{x_1}\wedge (Rh_{\lambda,a})_{x_2} + (Rh_{\lambda,a})_{x_1}\wedge (R\widehat{U}_{\lambda,a})_{x_2})\,dx \\
&\quad + \frac{\partial}{\partial a_i}\int_\Omega w\cdot (Rh_{\lambda,a})_{x_1}\wedge (Rh_{\lambda,a})_{x_2}\,dx.
\end{aligned} \tag{4.94}$$

Here (using Lemma 2.3, (4.25)–(4.27) and (4.78))

$$\begin{aligned}
&\frac{\partial}{\partial a_i}\int_\Omega w\cdot (R\widehat{U}_{\lambda,a})_{x_1}\wedge (Rh_{\lambda,a})_{x_2}\,dx \\
&= \int_\Omega w\cdot \Big(\Big(R\frac{\partial}{\partial a_i}\widehat{U}_{\lambda,a}\Big)_{x_1}\wedge (Rh_{\lambda,a})_{x_2} + (R\widehat{U}_{\lambda,a})_{x_1}\wedge \Big(R\frac{\partial}{\partial a_i}h_{\lambda,a}\Big)_{x_2}\Big)\,dx \\
&= O\Big(\lambda\int_\Omega |w|\Big|\nabla\frac{\partial}{\partial a_i}U_{\lambda,a}\Big|\,dx\Big) + O\Big(\lambda\int_\Omega |w||\nabla U_{\lambda,a}|\,dx\Big) \\
&= O\Big(\lambda\|w\|_{L^{p/(p-1)}(\Omega)}\Big\|\nabla\frac{\partial}{\partial a_i}U_{\lambda,a}\Big\|_{L^p(\Omega)}\Big) + O(\lambda\|\nabla w\|_{L^2(\Omega)})\ (1<\forall p<2) \\
&= O((H+\lambda)\lambda^{2/p}|\log\lambda|^{1/2})\ (1<\forall p<2).
\end{aligned} \tag{4.95}$$

Similarly, we have

$$\frac{\partial}{\partial a_i}\int_\Omega w\cdot (Rh_{\lambda,a})_{x_1}\wedge (R\widehat{U}_{\lambda,a})_{x_2}\,dx = O((H+\lambda)\lambda^{2/p}|\log\lambda|^{1/2})\ (1<\forall p<2) \tag{4.96}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial a_i}\int_\Omega w\cdot (Rh_{\lambda,a})_{x_1}\wedge (Rh_{\lambda,a})_{x_2}\,dx \\
&= \int_\Omega w\cdot \Big(\Big(R\frac{\partial}{\partial a_i}h_{\lambda,a}\Big)_{x_1}\wedge (Rh_{\lambda,a})_{x_2} + (Rh_{\lambda,a})_{x_1}\wedge \Big(R\frac{\partial}{\partial a_i}h_{\lambda,a}\Big)_{x_2}\Big)\,dx \\
&= O(\lambda^2\|\nabla w\|_{L^2(\Omega)}) = O((H+\lambda)\lambda^3|\log\lambda|^{1/2}).
\end{aligned} \tag{4.97}$$

(4.95)–(4.97) imply

$$(4.94) = O((H+\lambda)\lambda^{2/p}|\log\lambda|^{1/2})\ (1<\forall p<2). \tag{4.98}$$

Next we estimate the third term in (4.88).

$$\begin{aligned}&\frac{\partial}{\partial a_i}\int_\Omega RPU_{\lambda,a}\cdot w_{x_1}\wedge w_{x_2}\,dx = \int_\Omega R\frac{\partial}{\partial a_i}PU_{\lambda,a}\cdot w_{x_1}\wedge w_{x_2}\,dx\\ &= O\Big(\Big\|\nabla\frac{\partial}{\partial a_i}PU_{\lambda,a}\Big\|_{L^2(\Omega)}\|\nabla w\|^2_{L^2(\Omega)}\Big) = O((H^2\lambda+\lambda^3)|\log\lambda|).\end{aligned} \tag{4.99}$$

From (4.88)–(4.99), we have $(1<\forall p<2)$

$$\begin{aligned}&\frac{\partial}{\partial a_i}Q(RPU_{\lambda,a}+w) \qquad (4.100)\\ &= 6\pi\Big(\frac{\partial^2 h^1}{\partial x_1\partial x_i}(a,a)+\frac{\partial^2 h^2}{\partial x_2\partial x_i}(a,a)+\frac{\partial^2 h^1}{\partial x_1\partial a_i}(a,a)+\frac{\partial^2 h^2}{\partial x_2\partial a_i}(a,a)\Big)\lambda^2\\ &\quad + O(H\lambda^{2/p}|\log\lambda|^{1/2}) + O(\lambda^{2/p+1}|\log\lambda|^{1/2}) + O(H^2\lambda|\log\lambda|).\end{aligned}$$

From (4.68), (4.72), (4.87) and (4.100), we obtain (4.61).

Proof of (4.62). Note that

$$\Big\langle dJ_H(RPU_{\lambda,a}+w), R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big\rangle = \frac{\partial}{\partial\lambda}J_H(RPU_{\lambda,a}+w).$$

As in the proof of (4.61), we first estimate

$$\frac{\partial}{\partial\lambda}I_H(RPU_{\lambda,a}+w) \quad\text{and}\quad \frac{\partial}{\partial\lambda}Q(RPU_{\lambda,a}+w).$$

By (4.9), we have

$$\begin{aligned}&\frac{\partial}{\partial\lambda}\int_\Omega|\nabla PU_{\lambda,a}|^2\,dx = 2\int_\Omega\nabla PU_{\lambda,a}\cdot\nabla\Big(\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big)\,dx\\ &= -8\pi\Big(\frac{\partial h_a^1}{\partial x_1}(a)+\frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda + O(\lambda^2). \qquad (4.101)\\ &\frac{\partial}{\partial\lambda}\int_\Omega \underline{u}_H\cdot(RPU_{\lambda,a}+w)_{x_1}\wedge(RPU_{\lambda,a}+w)_{x_2}\,dx\\ &= \int_\Omega\underline{u}_H\cdot\frac{\partial}{\partial\lambda}\{(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2}\}\,dx \qquad (4.102)\\ &\quad+\int_\Omega\underline{u}_H\cdot\Big(\Big(R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big)_{x_1}\wedge w_{x_2}+w_{x_1}\wedge\Big(R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big)_{x_2}\Big)\,dx.\end{aligned}$$

Here the second term in (4.102) is estimated as (using Lemma 2.3, (4.28) and (4.29))

$$\begin{aligned}
&\int_\Omega \underline{u}_H \cdot \Big(\Big(R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big)_{x_1} \wedge w_{x_2} + w_{x_1} \wedge \Big(R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big)_{x_2}\Big)\,dx \\
&= \int_\Omega \Big(R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big)\cdot((\underline{u}_H)_{x_1} \wedge w_{x_2} + w_{x_1} \wedge (\underline{u}_H)_{x_2})\,dx \\
&= O\Big(\int_\Omega \Big|\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big||\nabla w|\,dx\Big) = O\Big(\Big\|\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}\Big) \\
&= O(\|\nabla w\|_{L^2(\Omega)}) + O\Big(\Big\|\frac{\partial}{\partial\lambda}U_{\lambda,a}\Big\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}\Big) \\
&= O(\|\nabla w\|_{L^2(\Omega)}) + O\Big(\Big(\int_\Omega \frac{r^2}{(\lambda^2+r^2)^2}\,dx\Big)^{1/2}\|\nabla w\|_{L^2(\Omega)}\Big) \\
&\qquad \Big(\text{since } \Big|\frac{\partial}{\partial\lambda}U_{\lambda,a}\Big| = \frac{2r}{\lambda^2+r^2} \text{ by (4.42)}\Big) \\
&= O((H\lambda+\lambda^2)|\log\lambda|). \qquad (4.103)
\end{aligned}$$

To estimate the first term in (4.102), we write it as

$$\begin{aligned}
&\int_\Omega \underline{u}_H \cdot \frac{\partial}{\partial\lambda}\{(RPU_{\lambda,a})_{x_1} \wedge (RPU_{\lambda,a})_{x_2}\}\,dx \\
&= \int_{\tilde\Omega} \underline{u}_H \cdot \frac{\partial}{\partial\lambda}\{(RU_{\lambda,a})_{x_1} \wedge (RU_{\lambda,a})_{x_2}\}\,dx \\
&\quad - \int_{\tilde\Omega} \underline{u}_H \cdot \frac{\partial}{\partial\lambda}((RU_{\lambda,a})_{x_1} \wedge (Rh_{\lambda,a})_{x_2} + (Rh_{\lambda,a})_{x_1} \wedge (RU_{\lambda,a})_{x_2})\,dx \\
&\quad + \int_{\tilde\Omega} \underline{u}_H \cdot \frac{\partial}{\partial\lambda}\{(Rh_{\lambda,a})_{x_1} \wedge (Rh_{\lambda,a})_{x_2}\}\,dx, \qquad (4.104)
\end{aligned}$$

where $\tilde\Omega$ and $\underline{u}_H$ are as in (4.81).

By (4.28), (4.29) and Lemma A.4 in [1], the second term in (4.104) is estimated as

$$\begin{aligned}
&\int_{\tilde\Omega} \underline{u}_H \cdot \frac{\partial}{\partial\lambda}((RU_{\lambda,a})_{x_1} \wedge (Rh_{\lambda,a})_{x_2} + (Rh_{\lambda,a})_{x_1} \wedge (RU_{\lambda,a})_{x_2})\,dx \\
&= \int_{\tilde\Omega} \underline{u}_H \cdot \Big(\Big(R\frac{\partial}{\partial\lambda}U_{\lambda,a}\Big)_{x_1} \wedge (Rh_{\lambda,a})_{x_2} + (Rh_{\lambda,a})_{x_1} \wedge \Big(R\frac{\partial}{\partial\lambda}U_{\lambda,a}\Big)_{x_2}\Big)\,dx \\
&\quad + \int_{\tilde\Omega} \underline{u}_H \cdot \Big((RU_{\lambda,a})_{x_1} \wedge \Big(R\frac{\partial}{\partial\lambda}h_{\lambda,a}\Big)_{x_2} + \Big(R\frac{\partial}{\partial\lambda}h_{\lambda,a}\Big)_{x_1} \wedge (RU_{\lambda,a})_{x_2}\Big)\,dx
\end{aligned}$$

$$= \int_{\tilde{\Omega}} R\frac{\partial}{\partial\lambda}U_{\lambda,a}\cdot((\underline{u}_H)_{x_1}\wedge(Rh_{\lambda,a})_{x_2}+(Rh_{\lambda,a})_{x_1}\wedge(\underline{u}_H)_{x_2})\,dx$$
$$+\int_{\tilde{\Omega}}\underline{u}_H\cdot\Big((RU_{\lambda,a})_{x_1}\wedge\Big(R\frac{\partial}{\partial\lambda}h_{\lambda,a}\Big)_{x_2}+\Big(R\frac{\partial}{\partial\lambda}h_{\lambda,a}\Big)_{x_1}\wedge(RU_{\lambda,a})_{x_2}\Big)\,dx$$
$$= O\Big(\lambda\int_{\tilde{\Omega}}\Big|\frac{\partial}{\partial\lambda}U_{\lambda,a}\Big|\,dx\Big)+O\Big(\int_{\tilde{\Omega}}|\nabla U_{\lambda,a}|\,dx\Big)$$
$$= O\Big(\lambda\int_{\tilde{\Omega}}\frac{r}{\lambda^2+r^2}\,dx\Big)+O\Big(\int_{\tilde{\Omega}}\frac{\lambda}{\lambda^2+r^2}\,dx\Big)=O(\lambda|\log\lambda|). \tag{4.105}$$

The third term in (4.104) is easily estimated by using (4.28) and (4.29), and we have

$$\int_{\tilde{\Omega}}\underline{u}_H\cdot\frac{\partial}{\partial\lambda}\{(Rh_{\lambda,a})_{x_1}\wedge(Rh_{\lambda,a})_{x_2}\}\,dx=O(\lambda). \tag{4.106}$$

The first term in (4.104) is estimated as

$$\int_{\tilde{\Omega}}\underline{u}_H\cdot\frac{\partial}{\partial\lambda}\{(RU_{\lambda,a})_{x_1}\wedge(RU_{\lambda,a})_{x_2})\,dx$$
$$= -\frac{1}{2}\int_{\tilde{\Omega}}\underline{u}_H\cdot\frac{\partial}{\partial\lambda}(|\nabla U_{\lambda,a}|^2RU_{\lambda,a})\,dx$$
$$= -8\int_{\tilde{\Omega}}\underline{u}_H\cdot\frac{\lambda^2}{(\lambda^2+r^2)^3}R\begin{pmatrix}x_1-a_1\\x_2-a_2\\-\lambda\end{pmatrix}\,dx$$
$$-8\int_{\tilde{\Omega}}\underline{u}_H\cdot\frac{\lambda(r^2-2\lambda^2)}{(\lambda^2+r^2)^4}R\begin{pmatrix}2\lambda(x_1-a_1)\\2\lambda(x_2-a_2)\\r^2-\lambda^2\end{pmatrix}\,dx$$
$$= -8\int_{\tilde{\Omega}}\underline{u}_H\cdot\frac{\lambda^2}{(\lambda^2+r^2)^3}R\begin{pmatrix}x_1-a_1\\x_2-a_2\\0\end{pmatrix}\,dx$$
$$-8\int_{\tilde{\Omega}}\underline{u}_H\cdot\frac{\lambda(r^2-2\lambda^2)}{(\lambda^2+r^2)^4}R\begin{pmatrix}2\lambda(x_1-a_1)\\2\lambda(x_2-a_2)\\0\end{pmatrix}\,dx+8\int_{\tilde{\Omega}}\underline{u}_H\cdot\frac{\lambda^3}{(\lambda^2+r^2)^3}Re_3\,dx$$
$$-8\int_{\tilde{\Omega}}\underline{u}_H\cdot\frac{\lambda(r^2-2\lambda^2)(r^2-\lambda^2)}{(\lambda^2+r^2)^4}Re_3\,dx. \tag{4.107}$$

Here, arguing as the proof of (3.18)–(3.20) and (5.17) in part I [3], we have

$$\int_{\tilde{\Omega}}\underline{u}_H\cdot\frac{\lambda^2}{(\lambda^2+r^2)^3}R\begin{pmatrix}x_1-a_1\\x_2-a_2\\0\end{pmatrix}\,dx \tag{4.108}$$

$$= \frac{\pi}{4}((\underline{u}_H)_{x_1}(a) \cdot Re_1 + (\underline{u}_H)_{x_2}(a) \cdot Re_2) + O(\lambda).$$

$$\begin{aligned} &\int_{\tilde{\Omega}} \underline{u}_H \cdot \frac{\lambda(r^2 - 2\lambda^2)}{(\lambda^2 + r^2)^4} R \begin{pmatrix} 2\lambda(x_1 - a_1) \\ 2\lambda(x_2 - a_2) \\ 0 \end{pmatrix} dx \\ &= \int_{\mathbb{B}_d(a)} \frac{2\lambda^2(r^2 - 2\lambda^2)}{(\lambda^2 + r^2)^4}(x_1 - a_1)\underline{u}_H \cdot Re_1 \, dx \\ &+ \int_{\mathbb{B}_d(a)} \frac{2\lambda^2(r^2 - 2\lambda^2)}{(\lambda^2 + r^2)^4}(x_2 - a_2)\underline{u}_H \cdot Re_2 \, dx + O\Big(\int_{\tilde{\Omega} \setminus \mathbb{B}_d(a)} \frac{\lambda^2 |r^2 - 2\lambda^2| r}{(\lambda^2 + r^2)^4} \, dx \Big). \end{aligned} \tag{4.109}$$

Here, by the oddness of the integral,

$$\begin{aligned} &\int_{\mathbb{B}_d(a)} \frac{2\lambda^2(r^2 - 2\lambda^2)}{(\lambda^2 + r^2)^4}(x_1 - a_1)\underline{u}_H \cdot Re_1 \, dx \\ &= \int_{\mathbb{B}_d(a)} \frac{2\lambda^2(r^2 - 2\lambda^2)}{(\lambda^2 + r^2)^4}(x_1 - a_1)^2(\underline{u}_H)_{x_1}(a) \cdot Re_1 \, dx \\ &+ O\Big(\int_{\mathbb{B}_d(a)} \frac{\lambda^2 |r^2 - 2\lambda^2| r^4}{(\lambda^2 + r^2)^4} \, dx\Big) \\ &= 2(\underline{u}_H)_{x_1}(a) \cdot Re_1 \int_0^{2\pi} \int_o^d \frac{\lambda^2(r^2 - 2\lambda^2)}{(\lambda^2 + r^2)^4} r^3 \cos^2\theta \, dr d\theta \\ &+ O\Big(\lambda^2 \int_0^{d/\lambda} \frac{|s^2 - 2| s^5}{(1 + s^2)^4} \, ds\Big) \\ &= 2\pi(\underline{u}_H)_{x_1}(a) \cdot Re_1 \int_0^{d/\lambda} \frac{(s^2 - 2)s^3}{(1 + s^2)^4} \, ds + O(\lambda^2 |\log \lambda|) \qquad (4.110) \\ &= -2\pi(\underline{u}_H)_{x_1}(a) \cdot Re_1 \int_{d/\lambda}^{\infty} \frac{(s^2 - 2)s^3}{(1 + s^2)^4} \, ds + O(\lambda^2 |\log \lambda|) = O(\lambda^2 |\log \lambda|) \end{aligned}$$

(since $\int_0^\infty \frac{(s^2-2)s^3}{(1+s^2)^4} \, ds = 0$). Similarly, we have

$$\int_{\mathbb{B}_d(a)} \frac{2\lambda^2(r^2 - 2\lambda^2)}{(\lambda^2 + r^2)^4}(x_2 - a_2)\underline{u}_H \cdot Re_2 \, dx = O(\lambda^2 |\log \lambda|). \tag{4.111}$$

The error term in (4.109) is $O(\lambda^2)$ and we have

$$(4.109) = O(\lambda^2 |\log \lambda|). \tag{4.112}$$

The last two terms in (4.107) are estimated as

$$\begin{aligned} &8\int_{\tilde{\Omega}} \frac{\lambda^3}{(\lambda^2+r^2)^3}\underline{u}_H\cdot Re_3\,dx - 8\int_{\tilde{\Omega}} \frac{\lambda(r^2-2\lambda^2)(r^2-\lambda^2)}{(\lambda^2+r^2)^4}\underline{u}_H\cdot Re_3\,dx \\ &= 8\int_{\mathbb{B}_d(a)} \Big\{\frac{\lambda^3}{(\lambda^2+r^2)^3} - \frac{\lambda(r^2-2\lambda^2)(r^2-\lambda^2)}{(\lambda^2+r^2)^4}\Big\}\underline{u}_H\cdot Re_3\,dx \qquad (4.113) \\ &\quad + O\Big(\int_{\tilde{\Omega}\setminus\mathbb{B}_d(a)} \frac{\lambda^3}{(\lambda^2+r^2)^3}\,dx\Big) + O\Big(\int_{\tilde{\Omega}\setminus\mathbb{B}_d(a)} \frac{\lambda|r^2-2\lambda^2||r^2-\lambda^2|}{(\lambda^2+r^2)^4}\,dx\Big). \end{aligned}$$

Here, by the oddness of the integral, we have

$$\begin{aligned} &8\int_{\mathbb{B}_d(a)} \Big\{\frac{\lambda^3}{(\lambda^2+r^2)^3} - \frac{\lambda(r^2-2\lambda^2)(r^2-\lambda^2)}{(\lambda^2+r^2)^4}\Big\}\underline{u}_H\cdot Re_3\,dx \\ &= 8\int_{\mathbb{B}_d(a)} \Big\{\frac{\lambda^3}{(\lambda^2+r^2)^3} - \frac{\lambda(r^2-2\lambda^2)(r^2-\lambda^2)}{(\lambda^2+r^2)^4}\Big\}\underline{u}_H(a)\cdot Re_3\,dx \\ &+ O\Big(\int_{\mathbb{B}_d(a)} \Big|\frac{\lambda^3}{(\lambda^2+r^2)^3} - \frac{\lambda(r^2-2\lambda^2)(r^2-\lambda^2)}{(\lambda^2+r^2)^4}\Big| r^2\,dx\Big) = O(\lambda|\log\lambda|). \end{aligned}$$

The second term in (4.113) is $O(\lambda^3)$, and the third term in (4.113) is $O(\lambda)$. Thus we have

$$(4.113) = O(\lambda|\log\lambda|). \qquad (4.114)$$

From (4.104)–(4.114), we obtain

$$\begin{aligned} (4.104) &= -2\pi((\underline{u}_H)_{x_1}(a)\cdot Re_1 + (\underline{u}_H)_{x_2}(a)\cdot Re_2) + O(\lambda|\log\lambda|) \qquad (4.115) \\ &= -2\pi((h_\gamma)_{x_1}(a)\cdot Re_1 + (h_\gamma)_{x_2}(a)\cdot Re_2) + O(H) + O(\lambda|\log\lambda|). \end{aligned}$$

From (4.101) and (4.115), we have

$$\begin{aligned} &\frac{\partial}{\partial\lambda} I_H(RPU_{\lambda,a}+w) \\ &= -8\pi\Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda - 8\pi((h_\gamma)_{x_1}(a)\cdot Re_1 + (h_\gamma)_{x_2}(a)\cdot Re_2)H \\ &\quad + O(H^2) + O(H\lambda|\log\lambda|) + O(\lambda^2). \qquad (4.116) \end{aligned}$$

Next we calculate $\frac{\partial}{\partial\lambda}Q(RPU_{\lambda,a}+w)$. We have

$$
\begin{aligned}
&\frac{\partial}{\partial\lambda}Q(RPU_{\lambda,a}+w)\\
&=3\int_\Omega R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\cdot(RPU_{\lambda,a}+w)_{x_1}\wedge(RPU_{\lambda,a}+w)_{x_2}\,dx\\
&=3\int_\Omega R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\cdot(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2}\,dx\\
&+3\int_\Omega R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\cdot((RPU_{\lambda,a})_{x_1}\wedge w_{x_2}+w_{x_1}\wedge(RPU_{\lambda,a})_{x_2})\,dx\\
&+3\int_\Omega R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\cdot w_{x_1}\wedge w_{x_2}\,dx.
\end{aligned}\tag{4.117}
$$

Here by (4.20) and Lemma 2.3, the third term in (4.117) is estimated as

$$
\begin{aligned}
\int_\Omega R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\cdot w_{x_1}\wedge w_{x_2}\,dx&=O\Big(\Big\|\nabla\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big\|_{L^2(\Omega)}\|\nabla w\|^2_{L^2(\Omega)}\Big)\\
&=O((H^2+\lambda^2)\lambda|\log\lambda|).
\end{aligned}\tag{4.118}
$$

The second term in (4.117) is estimated as

$$
\begin{aligned}
&\int_\Omega R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\cdot((RPU_{\lambda,a})_{x_1}\wedge w_{x_2}+w_{x_1}\wedge(RPU_{\lambda,a})_{x_2})\,dx\\
&=\int_\Omega w\Big((RPU_{\lambda,a})_{x_1}\wedge\big(R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\big)_{x_2}+\big(R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\big)_{x_1}\wedge(RPU_{\lambda,a})_{x_2}\Big)\,dx\\
&=\frac{\partial}{\partial\lambda}\int_\Omega w\cdot(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2}\,dx\\
&=\frac{\partial}{\partial\lambda}\Big\{-\int_\Omega w\cdot((RU_{\lambda,a})_{x_1}\wedge(Rh_{\lambda,a})_{x_2}+(Rh_{\lambda,a})_{x_1}\wedge(RU_{\lambda,a})_{x_2})\,dx\\
&\qquad+\int_\Omega w\cdot(Rh_{\lambda,a})_{x_1}\wedge(Rh_{\lambda,a})_{x_2}\,dx\Big\}\ \text{(see (4.2) and (4.3))}\\
&=O\Big(\|\nabla w\|_{L^2(\Omega)}\Big\|\nabla\frac{\partial}{\partial\lambda}U_{\lambda,a}\Big\|_{L^2(\Omega)}\|\nabla h_{\lambda,a}\|_{L^2(\Omega)}\Big)\\
&+O\Big(\|\nabla w\|_{L^2(\Omega)}\|\nabla U_{\lambda,a}\|_{L^2(\Omega)}\Big\|\nabla\frac{\partial}{\partial\lambda}h_{\lambda,a}\Big\|_{L^2(\Omega)}\Big)\\
&+O\Big(\|\nabla w\|_{L^2(\Omega)}\Big\|\nabla\frac{\partial}{\partial\lambda}h_{\lambda,a}\Big\|_{L^2(\Omega)}\|\nabla h_{\lambda,a}\|_{L^2(\Omega)}\Big)=O((H\lambda+\lambda^2)|\log\lambda|^{1/2}).
\end{aligned}\tag{4.119}
$$

The first term in (4.117) is estimated as

$$
\begin{aligned}
&3\int_\Omega R\frac{\partial}{\partial\lambda}PU_{\lambda,a}\cdot(RPU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2}\,dx \\
&= 3\int_\Omega \frac{\partial}{\partial\lambda}PU_{\lambda,a}\cdot(\widehat{U}_{\lambda,a})_{x_1}\wedge(\widehat{U}_{\lambda,a})_{x_2}\,dx \\
&\quad - 3\int_\Omega \frac{\partial}{\partial\lambda}PU_{\lambda,a}\cdot((\widehat{U}_{\lambda,a})_{x_1}\wedge(h_{\lambda,a})_{x_2}+(h_{\lambda,a})_{x_1}\wedge(\widehat{U}_{\lambda,a})_{x_2})\,dx \\
&\quad + 3\int_\Omega \frac{\partial}{\partial\lambda}PU_{\lambda,a}\cdot(h_{\lambda,a})_{x_1}\wedge(h_{\lambda,a})_{x_2}\,dx. \qquad (4.120)
\end{aligned}
$$

Here, by the proof of (4.9),

$$
\begin{aligned}
&\int_\Omega \frac{\partial}{\partial\lambda}PU_{\lambda,a}\cdot(\widehat{U}_{\lambda,a})_{x_1}\wedge(\widehat{U}_{\lambda,a})_{x_2}\,dx = \frac{1}{2}\int_\Omega \Delta\widehat{U}_{\lambda,a}\cdot\frac{\partial}{\partial\lambda}PU_{\lambda,a}\,dx \\
&= 2\pi\Big(\frac{\partial h_a^1}{\partial x_1}(a)+\frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda + O(\lambda^2), \qquad (4.121)
\end{aligned}
$$

$$
\begin{aligned}
&\int_\Omega \frac{\partial}{\partial\lambda}PU_{\lambda,a}\cdot((\widehat{U}_{\lambda,a})_{x_1}\wedge(h_{\lambda,a})_{x_2}+(h_{\lambda,a})_{x_1}\wedge(\widehat{U}_{\lambda,a})_{x_2})\,dx \\
&= \int_\Omega h_{\lambda,a}\cdot\Big((\widehat{U}_{\lambda,a})_{x_1}\wedge\Big(\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big)_{x_2}+\Big(\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big)_{x_1}\wedge(\widehat{U}_{\lambda,a})_{x_2}\Big)\,dx \\
&= \int_\Omega h_{\lambda,a}\cdot\frac{\partial}{\partial\lambda}\{(\widehat{U}_{\lambda,a})_{x_1}\wedge(\widehat{U}_{\lambda,a})_{x_2}\}\,dx + O\Big(\lambda\int_\Omega|\nabla U_{\lambda,a}|\,dx\Big) \\
&= -8\int_\Omega h_{\lambda,a}\Big\{\frac{\lambda^2}{(\lambda^2+r^2)^3}\begin{pmatrix} x_1-a_1\\ x_2-a_2\\ -\lambda\end{pmatrix}+\frac{\lambda(r^2-2\lambda^2)}{(\lambda^2+r^2)^4}\begin{pmatrix}2\lambda(x_1-a_1)\\ 2\lambda(x_2-a_2)\\ r^2-\lambda^2\end{pmatrix}\Big\}\,dx \\
&\quad + O(\lambda^2|\log\lambda|) \qquad (4.122) \\
&= -2\pi\Big(\frac{\partial h_a^1}{\partial x_1}(a)+\frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda + O(\lambda^2|\log\lambda|)\ \text{(as in the proof (4.115))},
\end{aligned}
$$

$$
\int_\Omega \frac{\partial}{\partial\lambda}PU_{\lambda,a}\cdot(h_{\lambda,a})_{x_1}\wedge(h_{\lambda,a})_{x_2}\,dx = O\Big(\lambda^2\int_\Omega\Big|\frac{\partial}{\partial\lambda}PU_{\lambda,a}\Big|\,dx\Big) = O(\lambda^2). \qquad (4.123)
$$

Thus we have

$$
(4.120) = 12\pi\Big(\frac{\partial h_a^1}{\partial x_1}(a)+\frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda + O(\lambda^2|\log\lambda|). \qquad (4.124)
$$

From (4.117)–(4.119) and (4.124), we have

$$\frac{\partial}{\partial\lambda}Q(RPU_{\lambda,a}+w) = 12\pi\Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda \\ + O(\lambda^2|\log\lambda|) + O(H\lambda|\log\lambda|^{1/2}) + O(H^2\lambda|\log\lambda|). \tag{4.125}$$

By (4.68), (4.72), (4.116) and (4.125) we have (4.62).

Proof of (4.63). As before we have

$$\langle dJ_H(RPU_{\lambda,a}+w), R\xi_i PU_{\lambda,a}\rangle = |Q(RPU_{\lambda,a}+w)|^{-2/3} \\ \times \Big\{4H\int_\Omega \underline{u}_H\cdot((RPU_{\lambda,a}+w)_{x_1}\wedge(R\xi_i PU_{\lambda,a})_{x_2} \\ + (R\xi_i PU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a}+w)_{x_2})\,dx \\ - \frac{2I_H(RPU_{\lambda,a}+w)}{Q(RPU_{\lambda,a}+w)}\int_\Omega R\xi_i PU_{\lambda,a}\cdot(RPU_{\lambda,a}+w)_{x_1}\wedge(RPU_{\lambda,a}+w)_{x_2}\,dx\Big\}. \tag{4.126}$$

Here we have

$$\int_\Omega \underline{u}_H\cdot((RPU_{\lambda,a}+w)_{x_1}\wedge(R\xi_i PU_{\lambda,a})_{x_2} \\ + (R\xi_i PU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a}+w)_{x_2})\,dx \\ = \int_\Omega \underline{u}_H\cdot((RPU_{\lambda,a})_{x_1}\wedge(R\xi_i PU_{\lambda,a})_{x_2} + (R\xi_i PU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2})\,dx \\ + \int_\Omega \underline{u}_H\cdot(w_{x_1}\wedge(R\xi_i PU_{\lambda,a})_{x_2} + (R\xi_i PU_{\lambda,a})_{x_1}\wedge w_{x_2})\,dx. \tag{4.127}$$

The second term in (4.127) is estimated as

$$\int_\Omega \underline{u}_H\cdot(w_{x_1}\wedge(R\xi_i PU_{\lambda,a})_{x_2} + (R\xi_i PU_{\lambda,a})_{x_1}\wedge w_{x_2})\,dx \\ = \int_\Omega (R\xi_i PU_{\lambda,a})\cdot(w_{x_1}\wedge(\underline{u}_H)_{x_2} + (\underline{u}_H)_{x_1}\wedge w_{x_2})\,dx \\ = O\Big(\int_\Omega |\widehat{U}_{\lambda,a}||\nabla w|\,dx\Big) = O(\|\widehat{U}_{\lambda,a}\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}) \\ = O(H\lambda^2|\log\lambda|) + O(\lambda^3|\log\lambda|). \tag{4.128}$$

The first term in (4.127) is ($\tilde\Omega$ and $\underline{u}_H$ are as in (4.81))

$$\int_\Omega \underline{u}_H\cdot((RPU_{\lambda,a})_{x_1}\wedge(R\xi_i PU_{\lambda,a})_{x_2} + (R\xi_i PU_{\lambda,a})_{x_1}\wedge(RPU_{\lambda,a})_{x_2})\,dx \\ = \int_{\tilde\Omega} \underline{u}_H\cdot((R\widehat{U}_{\lambda,a})_{x_1}\wedge(R\xi_i\widehat{U}_{\lambda,a})_{x_2} + (R\xi_i\widehat{U}_{\lambda,a})_{x_1}\wedge(R\widehat{U}_{\lambda,a})_{x_2})\,dx$$

$$
\begin{aligned}
&- \int_{\tilde\Omega} \underline{u}_H \cdot ((R\widehat{U}_{\lambda,a})_{x_1} \wedge (R\xi_i h_{\lambda,a})_{x_2} + (Rh_{\lambda,a})_{x_1} \wedge (R\xi_i \widehat{U}_{\lambda,a})_{x_2} \\
&\qquad + (R\xi_i \widehat{U}_{\lambda,a})_{x_1} \wedge (Rh_{\lambda,a})_{x_2} + (R\xi_i h_{\lambda,a})_{x_1} \wedge (R\widehat{U}_{\lambda,a})_{x_2})\, dx \\
&+ \int_{\tilde\Omega} \underline{u}_H \cdot ((Rh_{\lambda,a})_{x_1} \wedge (R\xi_i h_{\lambda,a})_{x_2} + (R\xi_i h_{\lambda,a})_{x_1} \wedge (Rh_{\lambda,a})_{x_2})\, dx \\
&= \int_{\tilde\Omega} \underline{u}_H \cdot ((R\widehat{U}_{\lambda,a})_{x_1} \wedge (R\xi_i \widehat{U}_{\lambda,a})_{x_2} + (R\xi_i \widehat{U}_{\lambda,a})_{x_1} \wedge (R\widehat{U}_{\lambda,a})_{x_2})\, dx \\
&+ O\Big(\lambda \int_{\tilde\Omega} |\nabla U_{\lambda,a}|\, dx\Big) + O(\lambda^2). \qquad (4.129)
\end{aligned}
$$

Here, by an argument similar to the proof of (3.21) in part I [3],

$$
\begin{aligned}
&\int_{\tilde\Omega} \underline{u}_H \cdot ((R\widehat{U}_{\lambda,a})_{x_1} \wedge (R\xi_i \widehat{U}_{\lambda,a})_{x_2} + (R\xi_i \widehat{U}_{\lambda,a})_{x_1} \wedge (R\widehat{U}_{\lambda,a})_{x_2})\, dx \\
&= \frac{d}{dt}\Big|_{t=0} \int_{\tilde\Omega} \underline{u}_H \cdot (R\exp(t\xi_i)\widehat{U}_{\lambda,a})_{x_1} \wedge (R\exp(t\xi_i)\widehat{U}_{\lambda,a})_{x_2}\, dx \\
&= \frac{d}{dt}\Big|_{t=0} \int_{\tilde\Omega} ({}^t(R\exp(t\xi_i))\underline{u}_H) \cdot (\widehat{U}_{\lambda,a})_{x_1} \wedge (\widehat{U}_{\lambda,a})_{x_2}\, dx \\
&= \int_{\tilde\Omega} ({}^t(R\xi_i)\underline{u}_H) \cdot (\widehat{U}_{\lambda,a})_{x_1} \wedge (\widehat{U}_{\lambda,a})_{x_2}\, dx \qquad (4.130) \\
&= -2\pi({}^t(R\xi_i)(\underline{u}_H)_{x_1}(a) \cdot e_1 + {}^t(R\xi_i)(\underline{u}_H)_{x_2}(a) \cdot e_2)\lambda + O(\lambda^2) \\
&= -2\pi((h_\gamma)_{x_1}(a) \cdot (R\xi_i e_1) + (h_\gamma)_{x_2}(a) \cdot (R\xi_i e_2))\lambda + O(\lambda^2) + O(\lambda H).
\end{aligned}
$$

From (4.128)–(4.130), we have

$$
\begin{aligned}
&4H \times (4.127) \qquad (4.131) \\
&= -8\pi((h_\gamma)_{x_1}(a) \cdot (R\xi_i e_1) + (h_\gamma)_{x_2}(a) \cdot (R\xi_i e_2))\lambda H + O(\lambda^2 H) + O(\lambda H^2).
\end{aligned}
$$

To estimate the second term in (4.126), we estimate the integral

$$
\begin{aligned}
&\int_\Omega R\xi_i PU_{\lambda,a} \cdot (RPU_{\lambda,a} + w)_{x_1} \wedge (RPU_{\lambda,a} + w)_{x_2}\, dx \\
&= \int_\Omega R\xi_i PU_{\lambda,a} \cdot (RPU_{\lambda,a})_{x_1} \wedge (RPU_{\lambda,a})_{x_2}\, dx \\
&\quad + \int_\Omega R\xi_i PU_{\lambda,a} \cdot ((RPU_{\lambda,a})_{x_1} \wedge w_{x_2} + w_{x_1} \wedge (RPU_{\lambda,a})_{x_2})\, dx \\
&\quad + \int_\Omega R\xi_i PU_{\lambda,a} \cdot w_{x_1} \wedge w_{x_2}\, dx. \qquad (4.132)
\end{aligned}
$$

Here

$$\int_\Omega R\xi_i PU_{\lambda,a} \cdot w_{x_1} \wedge w_{x_2}\, dx = O(\|\nabla w\|^2_{L^2(\Omega)}) = O((H^2\lambda^2 + \lambda^4)|\log\lambda|), \tag{4.133}$$

and by an argument similar argument to the proof of (4.130) and (3.10) in [3],

$$\begin{aligned}
&\int_\Omega R\xi_i PU_{\lambda,a} \cdot ((RPU_{\lambda,a})_{x_1} \wedge w_{x_2} + w_{x_1} \wedge (RPU_{\lambda,a})_{x_2})\, dx \\
&= \int_\Omega {}^t(R\xi_i) w \cdot (PU_{\lambda,a})_{x_1} \wedge (PU_{\lambda,a})_{x_2}\, dx \\
&= -\int_\Omega {}^t(R\xi_i) w \cdot ((\widehat{U}_{\lambda,a})_{x_1} \wedge (h_{\lambda,a})_{x_2} + (h_{\lambda,a})_{x_1} \wedge (\widehat{U}_{\lambda,a})_{x_2})\, dx \\
&\quad + \int_\Omega {}^t(R\xi_i) w \cdot (h_{\lambda,a})_{x_1} \wedge (h_{\lambda,a})_{x_2}\, dx \\
&= O(\lambda\|\nabla w\|_{L^2(\Omega)}) = O((H\lambda^2 + \lambda^3)|\log\lambda|^{1/2}).
\end{aligned} \tag{4.134}$$

The first term in (4.132) is zero. In fact, when $\xi_i = \xi_1$ (write $PU_{\lambda,a} = {}^t(PU^1, PU^2, PU^3)$ for simplicity), we have

$$\begin{aligned}
&\int_\Omega R\xi_i PU_{\lambda,a} \cdot (RPU_{\lambda,a})_{x_1} \wedge (RPU_{\lambda,a})_{x_2}\, dx \qquad (4.135)\\
&= \int_\Omega PU^2((PU^2)_{x_1}(PU^3)_{x_2} - (PU^3)_{x_1}(PU^2)_{x_2})\, dx \\
&\quad - \int_\Omega PU^1((PU^3)_{x_1}(PU^1)_{x_2} - (PU^1)_{x_1}(PU^3)_{x_2})\, dx.
\end{aligned}$$

Here, by integration by parts,

$$\begin{aligned}
&\int_\Omega PU^2((PU^2)_{x_1}(PU^3)_{x_2} - (PU^3)_{x_1}(PU^2)_{x_2})\, dx \qquad (4.136)\\
&= \int_\Omega PU^2((PU^2(PU^3)_{x_2})_{x_1} - ((PU^3)_{x_1} PU^2)_{x_2})\, dx \\
&= -\int_\Omega PU^2((PU^2)_{x_1}(PU^3)_{x_2} - (PU^3)_{x_1}(PU^2)_{x_2})\, dx = 0.
\end{aligned}$$

Similarly, the second term in (4.135) is zero. The cases $\xi_i = \xi_2$ and $\xi_i = \xi_3$ are treated by a similar argument. From (4.133)–(4.136), we have

$$(4.132) = O(H\lambda^2|\log\lambda|^{1/2}) + O(H^2\lambda^2|\log\lambda|) + O(\lambda^3|\log\lambda|^{1/2}). \tag{4.137}$$

(4.68), (4.72), (4.131) and (4.137) imply (4.63). □

Lemma 4.4. *We have the following estimates:*

$$\Big\|\nabla\Big(\frac{\partial^2}{\partial a_i\partial a_j}PU_{\lambda,a}\Big)\Big\|_{L^2(\Omega)} = O(\lambda^{-2}) \ (1\le i,j\le 2), \tag{4.138}$$

$$\Big\|\nabla\Big(\frac{\partial^2}{\partial \lambda\partial a_i}PU_{\lambda,a}\Big)\Big\|_{L^2(\Omega)} = O(\lambda^{-2}) \ (1\le i\le 2), \tag{4.139}$$

$$\Big\|\nabla\Big(\frac{\partial^2}{\partial \lambda^2}PU_{\lambda,a}\Big)\Big\|_{L^2(\Omega)} = O(\lambda^{-2}). \tag{4.140}$$

Proof. By direct computation, we obtain

$$\Big|\nabla\Big(\frac{\partial^2}{\partial a_i\partial a_j}U_{\lambda,a}\Big)\Big| \le C\frac{\lambda}{(\lambda^2+r^2)^2}, \tag{4.141}$$

$$\Big|\nabla\Big(\frac{\partial^2}{\partial \lambda\partial a_i}U_{\lambda,a}\Big)\Big| \le \frac{C}{(\lambda^2+r^2)^{3/2}}, \tag{4.142}$$

$$\Big|\nabla\Big(\frac{\partial^2}{\partial \lambda^2}U_{\lambda,a}\Big)\Big| \le \frac{C}{(\lambda^2+r^2)^{3/2}}. \tag{4.143}$$

By (4.141), (4.142), (4.143) and the fact that

$$\Big\|\nabla\Big(\frac{\partial^2}{\partial a_i\partial a_j}h_{\lambda,a}\Big)\Big\|_{L^2(\Omega)} = O(\lambda),$$

$$\Big\|\nabla\Big(\frac{\partial^2}{\partial \lambda\partial a_i}h_{\lambda,a}\Big)\Big\|_{L^2(\Omega)} = O(1), \quad \Big\|\nabla\Big(\frac{\partial^2}{\partial \lambda^2}h_{\lambda,a}\Big)\Big\|_{L^2(\Omega)} = O(1),$$

we obtain (4.138), (4.139) and (4.140). □

Lemma 4.5. *We have*

$$\frac{\partial^2 h^1}{\partial x_1\partial x_i}(a,a) = \frac{\partial^2 h^1}{\partial x_1\partial a_i}(a,a), \quad \frac{\partial^2 h^2}{\partial x_2\partial x_i}(a,a) = \frac{\partial^2 h^2}{\partial x_2\partial a_i}(a,a) \ (i=1,2).$$

Proof. Let $H(a,x)$ be 4π times the regular part of the Green's function of $-\Delta$ under the Dirichlet boundary condition

$$\begin{cases} \Delta_x H(a,x) = 0 & \text{in } \Omega \\ H(a,x) = \log|x-a|^2 & \text{on } \partial\Omega. \end{cases}$$

Then

$$h^1(a,x) = -\frac{\partial}{\partial a_1} H(a,x), \quad h^2(a,x) = -\frac{\partial}{\partial a_2} H(a,x).$$

Since $H(a,x) = H(x,a)$, we have

$$\frac{\partial H}{\partial x_i}(a,x) = \frac{\partial H}{\partial a_i}(x,a), \tag{4.144}$$

and we have (using (4.144))

$$\begin{aligned} \frac{\partial^2 h^1}{\partial x_1 \partial x_i}(a,x) &= -\frac{\partial^3 H}{\partial x_1 \partial x_i \partial a_1}(a,x) = -\frac{\partial^2}{\partial x_1 \partial a_1}\Big(\frac{\partial H}{\partial x_i}(a,x)\Big) \\ &= -\frac{\partial^2}{\partial x_1 \partial a_1}\Big(\frac{\partial H}{\partial a_i}(x,a)\Big) = -\frac{\partial^3 H}{\partial a_1 \partial x_1 \partial a_i}(x,a) = \frac{\partial^2 h^1}{\partial x_1 \partial a_i}(x,a). \end{aligned} \tag{4.145}$$

Similarly, we have

$$\frac{\partial^2 h^2}{\partial x_2 \partial x_i}(a,x) = \frac{\partial^2 h^2}{\partial x_2 \partial a_i}(x,a). \tag{4.146}$$

(4.145) and (4.146) give the result. □

In the rest of this paper, we prove the uniqueness assertion of Lemma 5.1 in part I [3].

Proof of the uniqueness assertion of Lemma 5.1 in part I. We prove the uniqueness by contradiction. So assume that there exist a sequence $\{\epsilon_k\}$, $\epsilon_k \downarrow 0$, $v_k \in M(\epsilon_k)$, $(\alpha_k, R_k, \lambda_k, a_k) \neq (\tilde{\alpha}_k, \tilde{R}_k, \tilde{\lambda}_k, \tilde{a}_k)$ with $\frac{1}{2} < \alpha_k, \tilde{\alpha}_k < 2$, $R_k, \tilde{R}_k \in SO(3)$, $\lambda_k, \tilde{\lambda}_k > 0$, $a_k, \tilde{a}_k \in \Omega$, $\lambda_k/d_k < 2\epsilon_k$, $\tilde{\lambda}_k/\tilde{d}_k < 2\epsilon_k$ ($d_k := d(a_k, \partial\Omega)$, $\tilde{d}_k = d(\tilde{a}_k, \partial\Omega)$) such that $(\alpha_k, R_k, \lambda_k, a_k)$ and $(\tilde{\alpha}_k, \tilde{R}_k, \tilde{\lambda}_k, \tilde{a}_k)$ are solutions to the problem (3.8) in part I [3].

Set $w_k = v_k - \alpha_k R_k P U_{\lambda_k, a_k}$ and $\tilde{w}_k = v_k - \tilde{\alpha}_k \tilde{R}_k P U_{\tilde{\lambda}_k, \tilde{a}_k}$. Then (3.10) in part I [3] holds for w_k and $\tilde{w}_k$. As in the proof of the existence part of Lemma 5.1 in part I [3], we have

$$R_k^{-1}\tilde{R}_k = I + o(1), \quad \frac{\lambda_k}{\tilde{\lambda}_k} = 1 + o(1), \quad \alpha_k - \tilde{\alpha}_k = o(1), \quad \frac{a_k - \tilde{a}_k}{\lambda_k} = o(1). \tag{4.147}$$

By (3.10) in part I, we have

$$\begin{aligned} &\int_\Omega \{\alpha_k \nabla(R_k P U_{\lambda_k, a_k}) - \tilde{\alpha}_k \nabla(\tilde{R}_k P U_{\tilde{\lambda}_k, \tilde{a}_k})\} \cdot \nabla(R_k U_{\lambda_k, a_k})\, dx \\ &= \int_\Omega \nabla \tilde{w}_k \cdot \nabla(R_k U_{\lambda_k, a_k} - \tilde{R}_k U_{\tilde{\lambda}_k, \tilde{a}_k})\, dx. \end{aligned} \tag{4.148}$$

Here the left-hand side of (4.148) is written as

$$\begin{aligned}
&\int_\Omega \{\alpha_k \nabla(R_k PU_{\lambda_k,a_k}) - \tilde{\alpha}_k \nabla(\tilde{R}_k PU_{\tilde{\lambda}_k,\tilde{a}_k})\} \cdot \nabla(R_k U_{\lambda_k,a_k})\, dx \\
&= (\alpha_k - \tilde{\alpha}_k) \int_\Omega \nabla(R_k PU_{\lambda_k,a_k}) \cdot \nabla(R_k U_{\lambda_k,a_k})\, dx \\
&\quad + \tilde{\alpha}_k \int_\Omega \nabla(R_k PU_{\lambda_k,a_k} - \tilde{R}_k PU_{\tilde{\lambda}_k,\tilde{a}_k}) \cdot \nabla(R_k U_{\lambda_k,a_k})\, dx \\
&= 8\pi(\alpha_k - \tilde{\alpha}_k) + o(|\alpha_k - \tilde{\alpha}_k|) \\
&\quad + \tilde{\alpha}_k \int_\Omega |\nabla U_{\lambda_k,a_k}|^2 (R_k U_{\lambda_k,a_k}) \cdot (R_k PU_{\lambda_k,a_k} - \tilde{R}_k PU_{\tilde{\lambda}_k,\tilde{a}_k})\, dx.
\end{aligned} \tag{4.149}$$

Since $\tilde{w}_k \to 0$ in $H^1(\Omega)$, (4.148) and (4.149) imply

$$\begin{aligned}
&8\pi(\alpha_k - \tilde{\alpha}_k) + \tilde{\alpha}_k \int_\Omega |\nabla U_{\lambda_k,a_k}|^2 (R_k U_{\lambda_k,a_k}) \cdot (R_k PU_{\lambda_k,a_k} - \tilde{R}_k PU_{\tilde{\lambda}_k,\tilde{a}_k})\, dx \\
&= o(|\alpha_k - \tilde{\alpha}_k|) + o(\|\nabla(R_k U_{\lambda_k,a_k} - \tilde{R}_k U_{\tilde{\lambda}_k,\tilde{a}_k})\|_{L^2(\Omega)}).
\end{aligned} \tag{4.150}$$

Here

$$\begin{aligned}
&\|\nabla(R_k U_{\lambda_k,a_k} - \tilde{R}_k U_{\tilde{\lambda}_k,\tilde{a}_k})\|^2_{L^2(\Omega)} \\
&\le 2\int_\Omega |\nabla U_{\lambda_k,a_k} - \nabla U_{\tilde{\lambda}_k,\tilde{a}_k}|^2\, dx + 2|R_k^{-1}\tilde{R}_k - I|^2 \int_\Omega |\nabla U_{\tilde{\lambda}_k,\tilde{a}_k}|^2\, dx.
\end{aligned} \tag{4.151}$$

Set $T_k := R_k^{-1}\tilde{R}_k - I$, $A_k = \frac{a_k - \tilde{a}_k}{\lambda_k}$, $\eta_k = \frac{\tilde{\lambda}_k}{\lambda_k} - 1$ and $B_k = \alpha_k - \tilde{\alpha}_k$. Then the first term of (4.151) is

$$\begin{aligned}
&\int_\Omega |\nabla U_{\lambda_k,a_k} - \nabla U_{\tilde{\lambda}_k,\tilde{a}_k}|^2\, dx \le \int_{\mathbb{R}^2} |\nabla U_{1+\eta_k,-A_k} - \nabla U_{1,0}|^2\, dx \\
&= 16\pi - 2\int_{\mathbb{R}^2} \nabla U_{1+\eta_k,-A_k} \cdot \nabla U_{1,0}\, dx \\
&= 16\pi - 2\int_{\mathbb{R}^2} |\nabla U_{1,0}|^2 U_{1,0} \cdot U_{1+\eta_k,-A_k}\, dx.
\end{aligned} \tag{4.152}$$

The integral in (4.152) is

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla U_{1,0}|^2 U_{1,0} \cdot U_{1+\eta_k,-A_k}\, dx \\
&= 8\int_{\mathbb{R}^2} \frac{4(1+\eta_k)x \cdot (x+A_k) + (|x|^2-1)(|x+A_k|^2 - (1+\eta_k)^2)}{(1+|x|^2)^3((1+\eta_k)^2 + |x+A_k|^2)}\, dx.
\end{aligned} \tag{4.153}$$

Since

$$(1+\eta_k)^2+|x+A_k|^2=(1+|x|^2)\Big\{1+\frac{2\eta_k+2x\cdot A_k+\eta_k^2+|A_k|^2}{1+|x|^2}\Big\}, \quad (4.154)$$

$$((1+\eta_k)^2+|x+A_k|^2)^{-1}=(1+|x|^2)^{-1}\Big\{1-\frac{2\eta_k+2x\cdot A_k}{1+|x|^2}+O\Big(\frac{|\eta_k|^2+|A_k|^2}{1+|x|^2}\Big)\Big\}, \quad (4.155)$$

we have

$$(4.153)=8\pi+O(|A_k|^2+|\eta_k|^2). \quad (4.156)$$

(4.151)–(4.156) imply

$$\|\nabla(R_kU_{\lambda_k,a_k}-\tilde R_kU_{\tilde\lambda_k,\tilde a_k})\|_{L^2(\Omega)}=O(|T_k|+|A_k|+|\eta_k|). \quad (4.157)$$

On the other hand, we have

$$\begin{aligned}&\int_\Omega|\nabla U_{\lambda_k,a_k}|^2(R_kU_{\lambda_k,a_k})\cdot(R_kPU_{\lambda_k,a_k}-\tilde R_kPU_{\tilde\lambda_k,\tilde a_k})\,dx\\&=\int_{\mathbb R^2}|\nabla U_{\lambda_k,a_k}|^2(R_kU_{\lambda_k,a_k})\cdot(R_k\widehat U_{\lambda_k,a_k}-\tilde R_k\widehat U_{\tilde\lambda_k,\tilde a_k})\,dx\\&-\int_{\mathbb R^2}|\nabla U_{\lambda_k,a_k}|^2(R_kU_{\lambda_k,a_k})\cdot(R_kh_{\lambda_k,a_k}-\tilde R_kh_{\tilde\lambda_k,\tilde a_k})\,dx.\end{aligned} \quad (4.158)$$

Here $h_{\lambda,a}$ is extended to $\mathbb R^2$ by $\widehat U_{\lambda,a}$. Then

$$\|R_kh_{\lambda_k,a_k}-\tilde R_kh_{\tilde\lambda_k,\tilde a_k}\|_{L^\infty(\mathbb R^2)}=O(\|h_{\lambda_k,a_k}-h_{\tilde\lambda_k,\tilde a_k}\|_{L^\infty(\mathbb R^2)})+o(|T_k|). \quad (4.159)$$

One can verify that $\|\widehat U_{\lambda_k,a_k}-\widehat U_{\tilde\lambda_k,\tilde a_k}\|_{L^\infty(\mathbb R^2\setminus\Omega)}=o(|\eta_k|+|A_k|)$, and by the maximum principle we have $\|h_{\lambda_k,a_k}-h_{\tilde\lambda_k,\tilde a_k}\|_{L^\infty(\mathbb R^2)}=o(|\eta_k|+|A_k|)$. Thus from (4.159),

$$\|R_kh_{\lambda_k,a_k}-\tilde R_kh_{\tilde\lambda_k,\tilde a_k}\|_{L^\infty(\mathbb R^2)}=o(|\eta_k|+|A_k|+|T_k|). \quad (4.160)$$

The first integral in (4.158) is

$$\begin{aligned}&\int_{\mathbb R^2}|\nabla U_{\lambda_k,a_k}|^2(R_kU_{\lambda_k,a_k})\cdot(R_k\widehat U_{\lambda_k,a_k}-\tilde R_k\widehat U_{\tilde\lambda_k,\tilde a_k})\,dx \quad (4.161)\\&=\int_{\mathbb R^2}|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k}\cdot(\widehat U_{\lambda_k,a_k}-\widehat U_{\tilde\lambda_k,\tilde a_k})\,dx\\&+\int_{\mathbb R^2}|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k}\cdot((I-R_k^{-1}\tilde R_k)\widehat U_{\tilde\lambda_k,\tilde a_k})\,dx\\&=\int_{\mathbb R^2}|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k}\cdot((I-R_k^{-1}\tilde R_k)\widehat U_{\tilde\lambda_k,\tilde a_k})\,dx+O(|\eta_k|^2+|A_k|^2)\end{aligned}$$

(by (4.156)). Since $T_k \to 0$, there exists $\zeta_k \in \mathfrak{so}(3)$ such that $R_k^{-1}\tilde{R}_k = \exp\zeta_k$. Then

$$
\begin{aligned}
(4.161) &= \int_{\mathbb{R}^2} |\nabla U_{1,0}|^2 U_{1,0}\cdot((I-\exp\zeta_k)\widehat{U}_{1+\eta_k,-A_k})\,dx + O(|\eta_k|^2+|A_k|^2)\\
&= -\int_{\mathbb{R}^2} |\nabla U_{1,0}|^2 U_{1,0}\cdot(\zeta_k\widehat{U}_{1+\eta_k,-A_k})\,dx + O(|\eta_k|^2+|A_k|^2+|\zeta_k|^2)\\
&= -\int_{\mathbb{R}^2} |\nabla U_{1,0}|^2 U_{1,0}\cdot(\zeta_k\widehat{U}_{1,0})\,dx \qquad (4.162)\\
&+ \int_{\mathbb{R}^2} |\nabla U_{1,0}|^2 U_{1,0}\cdot(\zeta_k(\widehat{U}_{1,0}-\widehat{U}_{1+\eta_k,-A_k}))\,dx + O(|\eta_k|^2+|A_k|^2+|\zeta_k|^2)\\
&= \int_{\mathbb{R}^2} |\nabla U_{1,0}|^2 U_{1,0}\cdot(\zeta_k(\widehat{U}_{1,0}-\widehat{U}_{1+\eta_k,-A_k}))\,dx + O(|\eta_k|^2+|A_k|^2+|\zeta_k|^2)\\
&\quad (\text{since } \zeta_k \text{ is antisymmetric}) = o(|\eta_k|+|A_k|+|\zeta_k|).
\end{aligned}
$$

(4.150), (4.157), (4.158), (4.160) and (4.162) imply

$$B_k = o(|A_k|+|B_k|+|\eta_k|+|T_k|). \tag{4.163}$$

Similarly, by (3.10) in part I, we have

$$
\begin{aligned}
&\int_\Omega \{\alpha_k\nabla(R_kPU_{\lambda_k,a_k}) - \tilde{\alpha}_k\nabla(\tilde{R}_kPU_{\tilde{\lambda}_k,\tilde{a}_k})\}\cdot\nabla\Big(R_k\frac{\partial}{\partial\lambda}U_{\lambda_k,a_k}\Big)\,dx\\
&= \int_\Omega \nabla\tilde{w}_k\cdot\nabla\Big(R_k\frac{\partial}{\partial\lambda}U_{\lambda_k,a_k} - \tilde{R}_k\frac{\partial}{\partial\lambda}U_{\tilde{\lambda}_k,\tilde{a}_k}\Big)\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
&(\alpha_k-\tilde{\alpha}_k)\int_\Omega \nabla(R_kPU_{\lambda_k,a_k})\cdot\nabla\Big(R_k\frac{\partial}{\partial\lambda}U_{\lambda_k,a_k}\Big)\,dx\\
&+\tilde{\alpha}_k\int_\Omega \nabla(R_kPU_{\lambda_k,a_k} - \tilde{R}_kPU_{\tilde{\lambda}_k,\tilde{a}_k})\cdot\nabla\Big(R_k\frac{\partial}{\partial\lambda}U_{\lambda_k,a_k}\Big)\,dx\\
&= o\Big(\Big\|\nabla\Big(R_k\frac{\partial}{\partial\lambda}U_{\lambda_k,a_k} - \tilde{R}_k\frac{\partial}{\partial\lambda}U_{\tilde{\lambda}_k,\tilde{a}_k}\Big)\Big\|_{L^2(\Omega)}\Big). \qquad (4.164)
\end{aligned}
$$

Here, as in (4.9),

$$
\begin{aligned}
&\int_\Omega \nabla(R_kPU_{\lambda_k,a_k})\cdot\nabla\Big(R_k\frac{\partial}{\partial\lambda}U_{\lambda_k,a_k}\Big)\,dx\\
&= \int_\Omega \nabla(PU_{\lambda_k,a_k})\cdot\nabla\Big(\frac{\partial}{\partial\lambda}PU_{\lambda_k,a_k}\Big)\,dx = O\Big(\frac{\lambda_k}{d_k^2}\Big).
\end{aligned}
\tag{4.165}
$$

By an argument similar to the proof of (4.157), we have

$$\Big\|\nabla\Big(R_k\frac{\partial}{\partial\lambda}U_{\lambda_k,a_k}-\tilde{R}_k\frac{\partial}{\partial\lambda}U_{\tilde{\lambda}_k,\tilde{a}_k}\Big)\Big\|_{L^2(\Omega)}=O\Big(\frac{|\eta_k|+|A_k|+|T_k|}{\lambda_k}\Big). \quad (4.166)$$

Next we estimate the integral

$$\begin{aligned}&\int_\Omega \nabla(R_kPU_{\lambda_k,a_k}-\tilde{R}_kPU_{\tilde{\lambda}_k,\tilde{a}_k})\cdot\nabla\Big(R_k\frac{\partial}{\partial\lambda}U_{\lambda_k,a_k}\Big)\,dx \quad (4.167)\\ &=\int_{\mathbb{R}^2}(R_k\widehat{U}_{\lambda_k,a_k}-\tilde{R}_k\widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k})\cdot\frac{\partial}{\partial\lambda}(|\nabla U_{\lambda_k,a_k}|^2R_kU_{\lambda_k,a_k})\,dx\\ &\quad+O\Big(\|R_kh_{\lambda_k,a_k}-\tilde{R}_kh_{\tilde{\lambda}_k,\tilde{a}_k}\|_{L^\infty(\mathbb{R}^2)}\Big\|\frac{\partial}{\partial\lambda}(|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k})\Big\|_{L^1(\mathbb{R}^2)}\Big).\end{aligned}$$

Direct computation shows that $\left|\frac{\partial}{\partial\lambda}(|\nabla U_{\lambda,a}|^2U_{\lambda,a})\right|\le C\frac{\lambda}{(\lambda^2+r^2)^2}$ and

$$\Big\|\frac{\partial}{\partial\lambda}(|\nabla U_{\lambda,a}|^2U_{\lambda,a})\Big\|_{L^1(\mathbb{R}^2)}=O\Big(\int_{\mathbb{R}^2}\frac{\lambda}{(\lambda^2+r^2)^2}\,dx\Big)=O(\lambda_k^{-1}). \quad (4.168)$$

On the other hand,

$$\begin{aligned}&\int_{\mathbb{R}^2}(R_k\widehat{U}_{\lambda_k,a_k}-\tilde{R}_k\widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k})\cdot\frac{\partial}{\partial\lambda}(|\nabla U_{\lambda_k,a_k}|^2R_kU_{\lambda_k,a_k})\,dx\\ &=\int_{\mathbb{R}^2}(\widehat{U}_{\lambda_k,a_k}-\widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k})\cdot\frac{\partial}{\partial\lambda}(|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k})\,dx \quad (4.169)\\ &\quad+\int_{\mathbb{R}^2}((I-R_k^{-1}\tilde{R}_k)\widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k})\cdot\frac{\partial}{\partial\lambda}(|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k})\,dx.\end{aligned}$$

The first integral in (4.169) is estimated as (by using an argument similar to the proof of (4.156))

$$\begin{aligned}&\int_{\mathbb{R}^2}(\widehat{U}_{\lambda_k,a_k}-\widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k})\cdot\frac{\partial}{\partial\lambda}(|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k})\,dx\\ &=-\int_{\mathbb{R}^2}\widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k}\cdot\frac{\partial}{\partial\lambda}(|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k})\,dx\\ &=\int_{\mathbb{R}^2}\frac{1}{\tilde{\lambda}_k^2+|x-\tilde{a}_k|^2}\begin{pmatrix}2\tilde{\lambda}_k(x_1-\tilde{a}_k^1)\\ 2\tilde{\lambda}_k(x_2-\tilde{a}_k^2)\\ |x-\tilde{a}_k|^2-\tilde{\lambda}_k^2\end{pmatrix}\cdot\Big(\frac{16\lambda_k^2}{(\lambda_k^2+|x-a_k|^2)^3}\begin{pmatrix}x_1-a_k^1\\ x_2-a_k^2\\ -\lambda_k\end{pmatrix}\\ &\qquad+\frac{16\lambda_k(|x-a_k|^2-2\lambda_k^2)}{(\lambda_k^2+|x-a_k|^2)^4}\begin{pmatrix}2\lambda_k(x_1-a_k^1)\\ 2\lambda_k(x_2-a_k^2)\\ |x-a_k|^2-\lambda_k^2\end{pmatrix}\Big)\,dx\end{aligned}$$

$$= \int_{\mathbb{R}^2} \frac{16\lambda_k^2(2\tilde{\lambda}_k(x-\tilde{a}_k)\cdot(x-a_k) - \lambda_k(|x-\tilde{a}_k|^2 - \tilde{\lambda}_k^2))}{(\tilde{\lambda}_k^2 + |x-\tilde{a}_k|^2)(\lambda_k^2 + |x-a_k|^2)^3}\, dx$$
$$+ \int_{\mathbb{R}^2} \Big(16\lambda_k(|x-a_k|^2 - 2\lambda_k^2)(4\lambda_k\tilde{\lambda}_k(x-\tilde{a}_k)\cdot(x-a_k)$$
$$+ (|x-\tilde{a}_k|^2 - \tilde{\lambda}_k^2)(|x-a_k|^2 - \lambda_k^2))\Big) \Big/ \Big((\tilde{\lambda}_k^2 + |x-\tilde{a}_k|^2)(\lambda_k^2 + |x-a_k|^2)^4\Big) dx$$
$$= \frac{16}{\lambda_k}\int_{\mathbb{R}^2} \frac{1}{(1+|x|^2)^3}\, dx + \frac{32\eta_k}{\lambda_k}\int_{\mathbb{R}^2} \frac{|x|^2}{(1+|x|^2)^4}\, dx \tag{4.170}$$
$$+ \frac{16}{\lambda_k}\int_{\mathbb{R}^2} \frac{|x|^2-2}{(1+|x|^2)^3}\, dx + O\Big(\frac{|\eta_k|^2 + |A_k|^2}{\lambda_k}\Big) = \frac{16\pi}{3}\frac{\eta_k}{\lambda_k} + O\Big(\frac{|\eta_k|^2 + |A_k|^2}{\lambda_k}\Big).$$

The second integral in (4.169) is estimated as

$$\int_{\mathbb{R}^2} ((I - R_k^{-1}\tilde{R}_k)\widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k}) \cdot \frac{\partial}{\partial\lambda}(|\nabla U_{\lambda_k,a_k}|^2 U_{\lambda_k,a_k})\, dx$$
$$= -\int_{\mathbb{R}^2} (\zeta_k \widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k}) \cdot \frac{\partial}{\partial\lambda}(|\nabla U_{\lambda_k,a_k}|^2 U_{\lambda_k,a_k})\, dx + o\Big(\frac{|\zeta_k|}{\lambda_k}\Big)$$
$$= -\int_{\mathbb{R}^2} (\zeta_k(\widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k} - \widehat{U}_{\lambda_k,a_k})) \cdot \frac{\partial}{\partial\lambda}(|\nabla U_{\lambda_k,a_k}|^2 U_{\lambda_k,a_k})\, dx \tag{4.171}$$
$$- \int_{\mathbb{R}^2} (\zeta_k U_{\lambda_k,a_k}) \cdot \frac{\partial}{\partial\lambda}(|\nabla U_{\lambda_k,a_k}|^2 U_{\lambda_k,a_k})\, dx + o\Big(\frac{|T_k|}{\lambda_k}\Big)$$
$$= -\int_{\mathbb{R}^2} (\zeta_k U_{\lambda_k,a_k}) \cdot \frac{\partial}{\partial\lambda}(|\nabla U_{\lambda_k,a_k}|^2 U_{\lambda_k,a_k})\, dx + o\Big(\frac{|T_k|}{\lambda_k}\Big).$$

By the calculation of the proof of (4.21) and (4.22), we have

$$\int_{\mathbb{R}^2} (\xi_i U_{\lambda_k,a_k}) \cdot \frac{\partial}{\partial\lambda}(|\nabla U_{\lambda_k,a_k}|^2 U_{\lambda_k,a_k})\, dx = 0 \ (i = 1,2,3)$$

and

$$(4.171) = o\Big(\frac{|T_k|}{\lambda_k}\Big). \tag{4.172}$$

From (4.164)–(4.172), we have

$$\eta_k = o(|A_k| + |B_k| + |\eta_k| + |T_k|). \tag{4.173}$$

Next by (3.10) in part I, we have

$$\int_{\Omega} \{\alpha_k \nabla(R_k P U_{\lambda_k,a_k}) - \tilde{\alpha}_k \nabla(\tilde{R}_k P U_{\tilde{\lambda}_k,\tilde{a}_k})\} \cdot \nabla\Big(R_k \frac{\partial}{\partial a_1} U_{\lambda_k,a_k}\Big)\, dx$$
$$= \int_{\Omega} \nabla\tilde{w}_k \cdot \nabla\Big(R_k \frac{\partial}{\partial a_1} U_{\lambda_k,a_k} - \tilde{R}_k \frac{\partial}{\partial a_1} U_{\tilde{\lambda}_k,\tilde{a}_k}\Big)\, dx$$

and

$$\begin{aligned}(\alpha_k - \tilde{\alpha}_k) &\int_\Omega \nabla(R_k P U_{\lambda_k,a_k}) \cdot \nabla\Big(R_k \frac{\partial}{\partial a_1} U_{\lambda_k,a_k}\Big)\,dx \\ &+ \tilde{\alpha}_k \int_\Omega \nabla(R_k P U_{\lambda_k,a_k} - \tilde{R}_k P U_{\tilde{\lambda}_k,\tilde{a}_k}) \cdot \nabla\Big(R_k \frac{\partial}{\partial a_1} U_{\lambda_k,a_k}\Big)\,dx \\ = o\Big(&\Big\|\nabla\Big(R_k \frac{\partial}{\partial a_1} U_{\lambda_k,a_k} - \tilde{R}_k \frac{\partial}{\partial a_1} U_{\tilde{\lambda}_k,\tilde{a}_k}\Big)\Big\|_{L^2(\Omega)}\Big). \end{aligned} \tag{4.174}$$

As the proof of (4.157) and (4.166),

$$\Big\|\nabla\Big(R_k \frac{\partial}{\partial a_1} U_{\lambda_k,a_k} - \tilde{R}_k \frac{\partial}{\partial a_1} U_{\tilde{\lambda}_k,\tilde{a}_k}\Big)\Big\|_{L^2(\Omega)} = O\Big(\frac{|\eta_k| + |A_k| + |T_k|}{\lambda_k}\Big). \tag{4.175}$$

As in (4.8)

$$\int_\Omega \nabla(R_k P U_{\lambda_k,a_k}) \cdot \nabla\Big(R_k \frac{\partial}{\partial a_1} U_{\lambda_k,a_k}\Big)\,dx = O\Big(\frac{\lambda_k^2}{d_k^3}\Big). \tag{4.176}$$

The second integral in (4.174) is written as

$$\begin{aligned}&\int_\Omega \nabla(R_k P U_{\lambda_k,a_k} - \tilde{R}_k P U_{\tilde{\lambda}_k,\tilde{a}_k}) \cdot \nabla\Big(R_k \frac{\partial}{\partial a_1} U_{\lambda_k,a_k}\Big)\,dx \\ &= \int_\Omega \nabla(P U_{\lambda_k,a_k} - P U_{\tilde{\lambda}_k,\tilde{a}_k}) \cdot \nabla\Big(\frac{\partial}{\partial a_1} U_{\lambda_k,a_k}\Big)\,dx \\ &\quad + \int_\Omega (I - R_k^{-1}\tilde{R}_k) \nabla P U_{\tilde{\lambda}_k,\tilde{a}_k} \cdot \nabla\Big(\frac{\partial}{\partial a_1} U_{\lambda_k,a_k}\Big)\,dx. \end{aligned} \tag{4.177}$$

Here, by (4.160),

$$\begin{aligned}&\int_\Omega \nabla(P U_{\lambda_k,a_k} - P U_{\tilde{\lambda}_k,\tilde{a}_k}) \cdot \nabla\Big(\frac{\partial}{\partial a_1} U_{\lambda_k,a_k}\Big)\,dx \\ &= \int_\Omega (\widehat{U}_{\lambda_k,a_k} - \widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k}) \cdot \frac{\partial}{\partial a_1}(|\nabla U_{\lambda_k,a_k}|^2 U_{\lambda_k,a_k})\,dx \\ &\quad + o\Big((|\eta_k| + |A_k|) \int_{\mathbb{R}^2} \Big|\frac{\partial}{\partial a_1}(|\nabla U_{\lambda_k,a_k}|^2 U_{\lambda_k,a_k})\Big|\,dx\Big). \end{aligned} \tag{4.178}$$

Since $\big|\frac{\partial}{\partial a_1}(|\nabla U_{\lambda_k,a_k}|^2 U_{\lambda_k,a_k})\big| \le C\frac{\lambda_k^2}{(\lambda_k^2+r^2)^{5/2}}$,

$$\int_{\mathbb{R}^2} \Big|\frac{\partial}{\partial a_1}(|\nabla U_{\lambda_k,a_k}|^2 U_{\lambda_k,a_k})\Big|\,dx = O\Big(\int_{\mathbb{R}^2} \frac{\lambda_k^2}{(\lambda_k^2 + r^2)^{5/2}}\,dx\Big) = O(\lambda_k^{-1}). \tag{4.179}$$

The first term in (4.178) is

$$\begin{aligned}
&\int_{\mathbb{R}^2}(\widehat{U}_{\lambda_k,a_k}-\widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k})\cdot\frac{\partial}{\partial a_1}(|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k})\,dx\\
&=-\int_{\mathbb{R}^2}\widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k}\cdot\frac{\partial}{\partial a_1}(|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k})\,dx\\
&=-\int_{\mathbb{R}^2}\frac{2\tilde{\lambda}_k}{\tilde{\lambda}_k^2+|x-\tilde{a}_k|^2}\begin{pmatrix}x_1-\tilde{a}_k^1\\ x_2-\tilde{a}_k^2\\ -\tilde{\lambda}_k\end{pmatrix}\cdot\Big(\frac{-16\lambda_k^2}{(\lambda_k^2+|x-a_k|^2)^3}\begin{pmatrix}\lambda_k\\ 0\\ x_1-a_k^1\end{pmatrix}\\
&+\frac{48\lambda_k^2(x_1-a_k^1)}{(\lambda_k^2+|x-a_k|^2)^4}\begin{pmatrix}2\lambda_k(x_1-a_k^1)\\ 2\lambda_k(x_2-a_k^2)\\ |x-a_k|^2-\lambda_k^2\end{pmatrix}\Big)\,dx\\
&=32\int_{\mathbb{R}^2}\frac{\lambda_k^2\tilde{\lambda}_k(\lambda_k(x_1-\tilde{a}_k^1)-\tilde{\lambda}_k(x_1-a_k^1))}{(\tilde{\lambda}_k^2+|x-\tilde{a}_k|^2)(\lambda_k^2+|x-a_k|^2)^3}\,dx \qquad (4.180)\\
&-96\int_{\mathbb{R}^2}\frac{\lambda_k^2\tilde{\lambda}_k(x_1-a_k^1)(2\lambda_k(x-a_k)\cdot(x-\tilde{a}_k)-\tilde{\lambda}_k(|x-a_k|^2-\lambda_k^2))}{(\tilde{\lambda}_k^2+|x-\tilde{a}_k|^2)(\lambda_k^2+|x-a_k|^2)^4}\,dx\\
&=\frac{32}{\lambda_k}A_k^1\int_{\mathbb{R}^2}\frac{dx}{(1+|x|^2)^4}+O\Big(\frac{|\eta_k|^2+|A_k|^2}{\lambda_k}\Big)=\frac{32\pi}{3}\frac{A_k^1}{\lambda_k}+O\Big(\frac{|\eta_k|^2+|A_k|^2}{\lambda_k}\Big),
\end{aligned}$$

where $A_k=(A_k^1,A_k^2)$.

Writing $\zeta_k=\zeta_k^1\xi_1+\zeta_k^2\xi_2+\zeta_k^3\xi_3$, the second integral in (4.177) is estimated as

$$\begin{aligned}
&\int_\Omega(I-R_k^{-1}\tilde{R}_k)\nabla PU_{\tilde{\lambda}_k,\tilde{a}_k}\cdot\nabla\Big(\frac{\partial}{\partial a_1}U_{\lambda_k,a_k}\Big)\,dx\\
&=\int_\Omega(I-R_k^{-1}\tilde{R}_k)\widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k}\cdot\frac{\partial}{\partial a_1}(|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k})\,dx\\
&\quad+o\Big(|T_k|\Big\|\frac{\partial}{\partial a_1}(|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k})\Big\|_{L^1(\Omega)}\Big)\\
&=-\int_{\mathbb{R}^2}(\zeta_k\widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k})\cdot\frac{\partial}{\partial a_1}(|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k})\,dx+o\Big(\frac{|T_k|}{\lambda_k}\Big)\\
&=-\int_{\mathbb{R}^2}\zeta_k(\widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k}-\widehat{U}_{\lambda_k,a_k})\cdot\frac{\partial}{\partial a_1}(|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k})\,dx\\
&\quad-\int_{\mathbb{R}^2}\zeta_k\widehat{U}_{\lambda_k,a_k}\cdot\frac{\partial}{\partial a_1}(|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k})\,dx+o\Big(\frac{|T_k|}{\lambda_k}\Big)\\
&=-\int_{\mathbb{R}^2}\zeta_k\widehat{U}_{\lambda_k,a_k}\cdot\frac{\partial}{\partial a_1}(|\nabla U_{\lambda_k,a_k}|^2U_{\lambda_k,a_k})\,dx+o\Big(\frac{|T_k|}{\lambda_k}\Big)
\end{aligned}$$

$$= \zeta_k^1 \int_{\mathbb{R}^2} \nabla(\xi_1 U_{\lambda_k,a_k}) \cdot \nabla\Big(\frac{\partial}{\partial a_1} U_{\lambda_k,a_k}\Big)\,dx \tag{4.181}$$
$$+ \zeta_k^2 \int_{\mathbb{R}^2} \nabla(\xi_2 U_{\lambda_k,a_k}) \cdot \nabla\Big(\frac{\partial}{\partial a_1} U_{\lambda_k,a_k}\Big)\,dx$$
$$+ \zeta_k^3 \int_{\mathbb{R}^2} \nabla(\xi_3 U_{\lambda_k,a_k}) \cdot \nabla\Big(\frac{\partial}{\partial a_1} U_{\lambda_k,a_k}\Big)\,dx + o\Big(\frac{|T_k|}{\lambda_k}\Big)$$
$$= \frac{16\pi}{3}\frac{1}{\lambda_k}\zeta_k^2 + o\Big(\frac{|T_k|}{\lambda_k}\Big) \text{ (by (4.14), (4.15) and (4.16)).}$$

From (4.174)–(4.181), we obtain

$$2A_k^1 + \zeta_k^2 = o(|\eta_k| + |A_k| + |B_k| + |T_k|). \tag{4.182}$$

Similarly, by (4.17), (4.18) and (4.19), we obtain

$$2A_k^2 + \zeta_k^3 = o(|\eta_k| + |A_k| + |B_k| + |T_k|). \tag{4.183}$$

As before, by (3.10) in part I, we have

$$\int_\Omega \nabla(\alpha_k R_k P U_{\lambda_k,a_k} - \tilde\alpha_k \tilde R_k P U_{\tilde\lambda_k,\tilde a_k}) \cdot \nabla(R_k \xi_i U_{\lambda_k,a_k})\,dx$$
$$= \int_\Omega \nabla \tilde w_k \cdot \nabla(R_k \xi_i U_{\lambda_k,a_k} - \tilde R_k \xi_i U_{\tilde\lambda_k,\tilde a_k})\,dx$$

and

$$(\alpha_k - \tilde\alpha_k)\int_\Omega \nabla(R_k P U_{\lambda_k,a_k}) \cdot \nabla(R_k \xi_i U_{\lambda_k,a_k})\,dx$$
$$+ \tilde\alpha_k \int_\Omega \nabla(R_k P U_{\lambda_k,a_k} - \tilde R_k P U_{\tilde\lambda_k,\tilde a_k}) \cdot \nabla(R_k \xi_i U_{\lambda_k,a_k})\,dx$$
$$= o(\|\nabla(R_k \xi_i U_{\lambda_k,a_k} - \tilde R_k \xi_i U_{\tilde\lambda_k,\tilde a_k})\|_{L^2(\Omega)}). \tag{4.184}$$

Here

$$\int_\Omega \nabla(R_k P U_{\lambda_k,a_k}) \cdot \nabla(R_k \xi_i U_{\lambda_k,a_k})\,dx = O\Big(\frac{\lambda_k}{d_k}\Big), \tag{4.185}$$

and

$$\|\nabla(R_k \xi_i U_{\lambda_k,a_k} - \tilde R_k \xi_i U_{\tilde\lambda_k,\tilde a_k})\|_{L^2(\Omega)} = O(|\eta_k| + |A_k| + |T_k|). \tag{4.186}$$

The second integral in (4.184) is written as

$$\begin{aligned}
&\int_\Omega \nabla(R_k PU_{\lambda_k,a_k} - \tilde{R}_k PU_{\tilde{\lambda}_k,\tilde{a}_k}) \cdot \nabla(R_k \xi_i U_{\lambda_k,a_k})\, dx \\
&= \int_\Omega \nabla(PU_{\lambda_k,a_k} - PU_{\tilde{\lambda}_k,\tilde{a}_k}) \cdot \nabla(\xi_i U_{\lambda_k,a_k})\, dx \\
&\quad + \int_\Omega (I - R_k^{-1}\tilde{R}_k)\nabla(PU_{\tilde{\lambda}_k,\tilde{a}_k}) \cdot \nabla(\xi_i U_{\lambda_k,a_k})\, dx \\
&= \int_{\mathbb{R}^2} \nabla(\widehat{U}_{\lambda_k,a_k} - \widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k}) \cdot \nabla(\xi_i U_{\lambda_k,a_k})\, dx - \int_{\mathbb{R}^2} \zeta_k \nabla U_{\tilde{\lambda}_k,\tilde{a}_k} \cdot \nabla(\xi_i U_{\lambda_k,a_k})\, dx \\
&\quad + o(|\eta_k| + |A_k| + |T_k|). \qquad (4.187)
\end{aligned}$$

Here

$$\begin{aligned}
&\int_{\mathbb{R}^2} \nabla(\widehat{U}_{\lambda_k,a_k} - \widehat{U}_{\tilde{\lambda}_k,\tilde{a}_k}) \cdot \nabla(\xi_i U_{\lambda_k,a_k})\, dx = -\int_{\mathbb{R}^2} \nabla U_{\tilde{\lambda}_k,\tilde{a}_k} \cdot \nabla(\xi_i U_{\lambda_k,a_k})\, dx \\
&= -\int_{\mathbb{R}^2} |\nabla U_{\lambda_k,a_k}|^2 U_{\tilde{\lambda}_k,\tilde{a}_k} \cdot (\xi_i U_{\lambda_k,a_k})\, dx \qquad (4.188) \\
&= -\int_{\mathbb{R}^2} |\nabla U_{1,0}|^2 U_{1+\eta_k,-A_k} \cdot (\xi_i U_{1,0}) dx.
\end{aligned}$$

When $\xi_i = \xi_1$,

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla U_{1,0}|^2 U_{1+\eta_k,-A_k} \cdot (\xi_1 U_{1,0})\, dx \\
&= \int_{\mathbb{R}^2} \frac{8}{(1+|x|^2)^2} \Big(\frac{1}{(1+\eta_k)^2 + |x+A_k|^2} \begin{pmatrix} 2(1+\eta_k)(x_1 + A_k^1) \\ 2(1+\eta_k)(x_2 + A_k^2) \\ |x+A_k|^2 - (1+\eta_k)^2 \end{pmatrix} \\
&\times \frac{1}{1+|x|^2} \begin{pmatrix} 2x_2 \\ -2x_1 \\ 0 \end{pmatrix} \Big)\, dx \\
&= 32(1+\eta_k) \int_{\mathbb{R}^2} \frac{x_2(x_1 + A_k^1) - x_1(x_2 + A_k^2)}{(1+|x|^2)^3} \Big(1 - \frac{2\eta_k + 2x \cdot A_k}{1+|x|^2} \Big)\, dx \\
&+ O(|\eta_k|^2 + |A_k|^2) = O(|\eta_k|^2 + |A_k|^2). \qquad (4.189)
\end{aligned}$$

When $\xi_i = \xi_2$,

$$\int_{\mathbb{R}^2} |\nabla U_{1,0}|^2 U_{1+\eta_k,-A_k} \cdot (\xi_2 U_{1,0})\, dx$$

$$= \int_{\mathbb{R}^2} \frac{8}{(1+|x|^2)^2}\Big(\frac{1}{(1+\eta_k)^2+|x+A_k|^2}\begin{pmatrix} 2(1+\eta_k)(x_1+A_k^1) \\ 2(1+\eta_k)(x_2+A_k^2) \\ |x+A_k|^2-(1+\eta_k)^2 \end{pmatrix}$$

$$\times \frac{1}{1+|x|^2}\begin{pmatrix} |x|^2-1 \\ 0 \\ -2x_1 \end{pmatrix}\Big)\,dx$$

$$= 16\int_{\mathbb{R}^2} \frac{(1+\eta_k)(x_1+A_k^1)(|x|^2-1)-x_1(|x+A_k|^2-(1+\eta_k)^2)}{(1+|x|^2)^4}$$

$$\times \Big(1-\frac{2\eta_k+2x\cdot A_k}{1+|x|^2}\Big)\,dx + O(|\eta_k|^2+|A_k|^2)$$

$$= 16A_k^1\int_{\mathbb{R}^2}\frac{|x|^2-1}{(1+|x|^2)^4}\,dx - 32A_k^1\int_{\mathbb{R}^2}\frac{x_1^2}{(1+|x|^2)^4}\,dx + O(|\eta_k|^2+|A_k|^2)$$

$$= -\frac{16\pi}{3}A_k^1 + O(|\eta_k|^2+|A_k|^2). \tag{4.190}$$

When $\xi_i=\xi_3$ (as in the case $\xi_i=\xi_2$),

$$\int_{\mathbb{R}^2}|\nabla U_{1,0}|^2 U_{1+\eta_k,-A_k}\cdot(\xi_3 U_{1,0})\,dx$$

$$= \int_{\mathbb{R}^2} \frac{8}{(1+|x|^2)^2}\Big(\frac{1}{(1+\eta_k)^2+|x+A_k|^2}\begin{pmatrix} 2(1+\eta_k)(x_1+A_k^1) \\ 2(1+\eta_k)(x_2+A_k^2) \\ |x+A_k|^2-(1+\eta_k)^2 \end{pmatrix}$$

$$\times \frac{1}{1+|x|^2}\begin{pmatrix} 0 \\ |x|^2-1 \\ -2x_2 \end{pmatrix}\Big)\,dx = -\frac{16\pi}{3}A_k^2 + O(|\eta_k|^2+|A_k|^2). \tag{4.191}$$

On the other hand,

$$-\int_{\mathbb{R}^2}\zeta_k\nabla U_{\tilde\lambda_k,\tilde a_k}\cdot\nabla(\xi_i U_{\lambda_k,a_k})\,dx$$

$$= -\int_{\mathbb{R}^2}|\nabla U_{\lambda_k,a_k}|^2(\zeta_k U_{\tilde\lambda_k,\tilde a_k})\cdot(\xi_i U_{\lambda_k,a_k})\,dx$$

$$= \int_{\mathbb{R}^2}|\nabla U_{1,0}|^2 U_{1,0}\cdot(\xi_i\zeta_k U_{1+\eta_k,-A_k})\,dx = \int_{\mathbb{R}^2}|\nabla U_{1,0}|^2 U_{1,0}\cdot(\xi_i\zeta_k U_{1,0})\,dx$$

$$+ \int_{\mathbb{R}^2}|\nabla U_{1,0}|^2 U_{1,0}\cdot(\xi_i\zeta_k(U_{1+\eta_k,-A_k}-U_{1,0}))\,dx$$

$$= \int_{\mathbb{R}^2}|\nabla U_{1,0}|^2 U_{1,0}\cdot(\xi_i\zeta_k U_{1,0})\,dx + o(|T_k|)$$

$$= -\frac{16\pi}{3}\zeta_k^i + o(|T_k|) \text{ (by (4.23) and (4.24))}. \tag{4.192}$$

From (4.184)–(4.192), we have

$$\zeta_k^1 = o(|\eta_k| + |A_k| + |B_k| + |T_k|) \tag{4.193}$$

$$A_k^1 - \zeta_k^2 = o(|\eta_k| + |A_k| + |B_k| + |T_k|) \tag{4.194}$$

$$A_k^2 - \zeta_k^3 = o(|\eta_k| + |A_k| + |B_k| + |T_k|). \tag{4.195}$$

By (4.163), (4.173), (4.182), (4.183), (4.193), (4.194) and (4.195), we have

$$\eta_k = 0, \quad A_k = 0, \quad B_k = 0, \quad T_k = 0$$

for large k. But this is a contradiction. Thus the proof is complete. □

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