Advances in Differential Equations Volume 6, Number 7, July 2001, Pages 873–895

NONLINEAR ELLIPTIC PROBLEMS WITH p-STRUCTURE UNDER MIXED BOUNDARY VALUE CONDITIONS IN POLYHEDRAL DOMAINS

Carsten Ebmeyer
Mathematisches Seminar, Universität Bonn
Nussallee 15, D-53115 Bonn, Germany

(Submitted by: Herbert Amann)

Abstract. Nonlinear elliptic systems with p-structure for $p \geq 2$, such as

$$-\text{div}\left((1+|\nabla u|^{p-2})\nabla u\right) = f(x) + \sum_{i=1}^{n} \partial_i f_i(x),$$

are considered under mixed boundary value conditions on nonsmooth domains. Regularity results in fractional-order Sobolev spaces are proven, e.g., $u \in W^{r,p}(\Omega)$ for all $r < 1 + \frac{1}{p}$ and $|\nabla u|^p \in W^{s,1}(\Omega)$ for some $s > 1$.

0. Introduction

Problems with p-structure arise in many physical situations, e.g., in fluid mechanics or in nonlinear diffusion. We study problems with p-structure in nonsmooth domains under boundary value conditions of mixed type. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded, polyhedral domain and $u : \Omega \to \mathbb{R}^N$ for $N \geq 1$. We treat the elliptic system

$$-\text{div}\left((1+|\nabla u|^{q_1})^{q_2}\, \nabla u\right) = f(x) + \sum_{i=1}^{n} \partial_i f_i(x) \quad \text{in } \Omega, \tag{0.1}$$

where $f, f_i \in \mathbb{R}^N$ and $q_1, q_2 \geq 0$. The system has p-structure for some $p \geq 2$, where $p = q_1 q_2 + 2$. Let ν be the outward normal of $\partial\Omega$ and $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_S$. We consider (0.1) under the mixed boundary value conditions

$$u(x) = a(x) \qquad \text{on } \Gamma_D, \tag{0.2}$$

$$\sum_{i=1}^{n} (1+|\nabla u|^{q_1})^{q_2}\, \partial_i u\, \nu_i = b(x) \qquad \text{on } \Gamma_N, \tag{0.3}$$

Accepted for publication August 2000.
AMS Subject Classifications: 35J55, 35J65; 35J25.

$$\sum_{i=1}^{n}(1+|\nabla u|^{q_1})^{q_2}\,\partial_i u\,\nu_i + c(x)u(x) = d(x) \qquad \text{on } \Gamma_S, \tag{0.4}$$

where $a, b, d \in \mathbb{R}^N$, $c \in \mathbb{R}$, and $c \geq 0$.

The boundary value conditions on Γ_D and Γ_N are nonhomogeneous Dirichlet and Neumann conditions, whereas on Γ_S a boundary value condition of the third kind is given.

The purpose of this paper is to study the global regularity of weak solutions. It is well-known that they may have singularities at points in $\partial\Omega$ where the boundary condition changes or where $\partial\Omega$ is not smooth. We consider polyhedral domains which may be nonconvex. Further, we assume that the inner angle between two adjacent faces is less than π, if one face is a Γ_D– and the other a Γ_N– or a Γ_S–face. For linear problems, i.e., $p = 2$, there are various investigations of such singularities; cf. [4, 13, 14, 17, 20]. It is well-known that weak solutions are $W^{s,2}(\Omega)$–functions for all $s < \frac{3}{2}$. But, in general, this is not satisfied for $s > \frac{3}{2}$.

In this paper we consider the case $p \geq 2$. We prove that a weak solution u is a $W^{s,p}(\Omega)$–function for all $s < 1 + \frac{1}{p}$. Moreover, we investigate the regularity of $|\nabla u|^\alpha$ for $\alpha \geq \frac{p}{2}$ and show $|\nabla u|^\alpha \in W^{\frac{1}{2}-\varepsilon,q}(\Omega)$ for some number q depending on α.

The method of proof is a difference-quotient technique, first applied in [6, 10]. It provides regularity results in Nikolskii spaces. Then the claim follows due to the imbedding theorem of Nikolskii spaces into Sobolev spaces.

The literature concerning nonlinear problems on nonsmooth domains under mixed boundary value conditions is very rare. Quasilinear and nonlinear problems are studied in [1, 5, 8, 11, 19]. Problems with p–structure are treated in [2, 7, 9, 12, 18, 21].

This paper is organized as follows. In Section 1 the assumptions on the data and the main results are stated. Section 2 contains some notation. In Section 3 we prove some weighted estimates for difference quotients of ∇u. The proofs of the main results are given in Section 4.

1. The main results

Let $p = q_1 q_2 + 2$ and $p' = \frac{p}{p-1}$. Due to $q_1 q_2 \geq 0$ it follows that $p \geq 2$ and $p' \leq 2$. We make the following assumptions on the boundary data and the right-hand side of system (0.1).

(A1) $f(x) \in L^{p'}(\Omega; \mathbb{R}^N)$ and $f_i(x) \in W^{1,p'}(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$ for $1 \leq i \leq n$.

(A2) $a(x) \in W^{1,p}(\Gamma_D; \mathbb{R}^N)$.
(A3) $b(x) \in L^{p'}(\Gamma_N; \mathbb{R}^N)$, $c(x) \in L^{p'}(\Gamma_S)$, and $d(x) \in L^{p'}(\Gamma_S; \mathbb{R}^N)$.
(A4) $c(x) \geq 0$ almost everywhere on Γ_S.
(A5) If $p < n$, then $c(x) \in L^{\frac{np}{n(p-2)+p}+\delta}(\Gamma_S)$ for some $\delta > 0$.

We consider classical polyhedrons, but not domains with a slit. Each inner angle between two adjacent faces Γ^i and Γ^j satisfies $0 < \text{angle}(\Gamma^i, \Gamma^j) < 2\pi$. If there is a change of the boundary value condition between a face $\Gamma^i \subset \Gamma_D$ and some adjacent face $\Gamma^j \not\subset \Gamma_D$ we suppose $\text{angle}(\Gamma^i, \Gamma^j) < \pi$. More precisely we assume

i) $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded polyhedral domain.

ii) $\partial\Omega = \bigcup_{1 \leq i \leq M} \overline{\Gamma^i}$, where each Γ^i is an open subset of a hyperplane, and $\partial\Gamma^i$ is polyhedral.

iii) $\Gamma^i \cap \Gamma^j = \emptyset$ for $i \neq j$.

iv) There are disjoint index sets Λ_D, Λ_N, Λ_S such that $\Lambda_D \cup \Lambda_N \cup \Lambda_S = \{1, \dots, M\}$, $\Gamma^i \subset \Gamma_D$ for $i \in \Lambda_D$, $\Gamma^i \subset \Gamma_N$ for $i \in \Lambda_N$, and $\Gamma^i \subset \Gamma_S$ for $i \in \Lambda_S$.

v) $\overline{\Gamma^{i_1}} \cap \cdots \cap \overline{\Gamma^{i_k}} = \emptyset$ if $k > n$ and $i_1 < \cdots < i_k$.

vi) Each inner angle between a pair of boundary manifolds Γ^i, Γ^j satisfies $\text{angle}(\Gamma^i, \Gamma^j) < \pi$ if $i \in \Lambda_D$, $j \in \Lambda_N \cup \Lambda_S$, and $\partial\Gamma^i \cap \partial\Gamma^j \neq \emptyset$.

Remark. Our method is not restricted to polyhedral domains. We are able to treat more general Lipschitz domains; cf. [6, 9, 10].

Let us define the spaces

$$W^{1,p}_\Gamma = \{v \in W^{1,p}(\Omega; \mathbb{R}^N) : v = a \text{ on } \Gamma_D\},$$
$$V = \{v \in W^{1,p}(\Omega; \mathbb{R}^N) : v = 0 \text{ on } \Gamma_D\}.$$

We call $u \in W^{1,p}_\Gamma$ a weak solution of problem (0.1), if

$$\sum_{i=1}^n \int_\Omega (1 + |\nabla u|^{q_1})^{q_2}\, \partial_i u \cdot \partial_i \varphi = \int_\Omega f \cdot \varphi - \sum_{i=1}^n \int_\Omega f_i \cdot \partial_i \varphi + \int_{\Gamma_N} g_0 \cdot \varphi + \int_{\Gamma_S} (g_1 - cu) \cdot \varphi \qquad (1.1)$$

for all $\varphi \in V$, where $g_0 = b + \sum_{i=1}^n f_i \nu_i$ and $g_1 = d + \sum_{i=1}^n f_i \nu_i$.

It is easy to see that there exists a weak solution $u \in W^{1,p}_\Gamma$, if $c(x) \geq 0$. Further, it is well-known that the weak solution is unique.

Now we state the main results.

Theorem 1.1. *There holds*

$$u \in W^{1+\frac{1}{p}-\varepsilon,p}(\Omega;\mathbb{R}^N) \qquad \text{for all } \varepsilon > 0. \tag{1.2}$$

Remark 1.1. i) For $p = 2$ there holds $u \in W^{\frac{3}{2}-\varepsilon,2}(\Omega;\mathbb{R}^N)$ for all $\varepsilon > 0$. This result is well-known for linear problems, and proven in [6, 10] for nonlinear problems. ii) Let $\Omega \subset \mathbb{R}^2$. Then a weak solution u of the Dirichlet problem with homogeneous boundary data fulfils $|u| \le cr^{1-\frac{1}{p}}$, where r is the distance to the corner point of $\partial\Omega$; see [5]. Let us note that (1.2) is sharp for $r^{1-\frac{1}{p}}$; i.e., $r^{1-\frac{1}{p}}$ is a $W^{1+\frac{1}{p}-\varepsilon,p}(\Omega)$–function for $\varepsilon > 0$, but not for $\varepsilon = 0$.

Theorem 1.2. *If* $\frac{p}{2} \le \alpha \le \frac{(2n-1)p}{2n-2}$, *then*

$$|\nabla u|^\alpha \in W^{\frac{1}{2}-\varepsilon,q}(\Omega) \qquad \text{for } q = \frac{np}{(n-1)\alpha + \frac{1}{2}p} \text{ and all } \varepsilon > 0. \tag{1.3}$$

Remark 1.2. i) Theorem 1.2 yields $|\nabla u|^\alpha \in W^{\frac{1}{2}-\varepsilon,\frac{p}{\alpha}}(\Omega)$. In particular, it holds that $|\nabla u|^{p-1} \in W^{\frac{1}{2}-\varepsilon,p'}(\Omega)$ and $|\nabla u|^p \in W^{\frac{1}{2}-\varepsilon,1+\delta}(\Omega)$ for some $\delta > 0$. Further, it follows that $|\nabla u|^{\frac{p}{2}} \in W^{\frac{1}{2}-\varepsilon,2}(\Omega)$. This latter result has been proven in [9] for $p > 1$ in the case of homogeneous boundary data. ii) Our method is not restricted to the case $p \ge 2$. Under suitable conditions on the boundary data similar results for $1 < p < 2$ can be proven. Further, we are able to deal with more general systems; cf. [9, 12], where only some growth conditions on the coefficients of the system are given. Moreover, therein the degenerate case, i.e., the p-Laplacian, is treated as well.

If a weak solution u is Hölder continuous we are able to improve Theorem 1.2. The Hölder continuity for elliptic equations, i.e., $N = 1$, is well-known under suitable assumptions on the boundary data. It may be proven as in [16]. Weak solutions of systems, i.e., $N > 1$, may not be Hölder continuous. In [22], however, Hölder continuity is proven for certain systems with p–structure in the case of homogeneous boundary data.

Theorem 1.3. *Let u be Hölder continuous, $p \le n-1$, $n \ge 3$, and*

$$b, d \in L^{\frac{n-1}{p-1}+\delta}(\partial\Omega,\mathbb{R}^N), \quad c \in L^{\frac{n(n-1)}{n(p-2)+p}+\delta}(\partial\Omega), \quad f,\, \partial_k f_i \in L^{\frac{np'}{p+1}+\delta}(\Omega,\mathbb{R}^N)$$

for $1 \le i,k \le n$ and some small $\delta > 0$. If $\alpha \in [\frac{p}{2}, p+\frac{1}{2}]$ there holds

$$|\nabla u|^\alpha \in W^{\frac{1}{2}-\varepsilon,q}(\Omega) \qquad \text{for } q = \frac{2p+2}{2\alpha+1} \text{ and all } \varepsilon > 0. \tag{1.4}$$

Remark 1.3. In [8] the results (1.3) and (1.4) are proven for $p = 2$, where nonlinear systems with coefficients satisfying a quadratic growth condition are considered.

2. Notation

By $W^{s,p}(\Omega)$ we denote the usual Sobolev spaces. Further, we use the Nikolskii spaces $\mathcal{N}^{s,p}(\Omega)$, defined as follows; cf. [15]. Let $m \geq 0$ be an integer, $0 < \sigma < 1$, $s = m + \sigma$, $z \in \mathbb{R}^n$, $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq \delta\}$, and $1 \leq p < \infty$. The space $\mathcal{N}^{s,p}(\Omega)$ consists of all functions $f : \Omega \to \mathbb{R}$ for which the norm

$$\|f\|_{\mathcal{N}^{s,p}(\Omega)} = \Big(\|f\|^p_{L^p(\Omega)} + \sum_{|\alpha|=m} \sup_{\substack{\delta>0\\ 0<|z|<\delta}} \int_{\Omega_\delta} \frac{|\partial^\alpha f(x+z) - \partial^\alpha f(x)|^p}{|z|^{\sigma p}}\, dx \Big)^{\frac{1}{p}}$$

is finite.

By u^ν $(1 \leq \nu \leq N)$ we denote the ν–th component of u, and $\nabla u \in \mathbb{R}^{nN}$ has the components $\partial_i u^\nu$ $(1 \leq i \leq n,\ 1 \leq \nu \leq N)$. Thus, $|\nabla u|^2 = \sum_{\nu=1}^N \sum_{i=1}^n |\partial_i u^\nu|^2$.

Let us introduce the function $F(r)$. Let $r \in \mathbb{R}^{nN}$ be a vector with components r_i^ν, where $1 \leq i \leq n$ and $1 \leq \nu \leq N$. We set

$$F(r) = \int_0^{|r|^2} \frac{1}{2} \left(1 + s^{\frac{q_1}{2}}\right)^{q_2}\, ds. \tag{2.1}$$

Further, let $F_i^\nu(r) = \frac{\partial}{\partial r_i^\nu} F(r)$ and $F_{i,k}^{\nu\mu}(r) = \frac{\partial}{\partial r_k^\mu} F_i^\nu(r)$ for $1 \leq i,k \leq n$, $1 \leq \nu, \mu \leq N$. Let us note that $F_i^\nu(r) = (1 + |r|^{q_1})^{q_2} r_i^\nu$. Here, $F_i^\nu(r)$ denotes the ν–th component of $F_i(r) = (1 + |r|^{q_1})^{q_2} r_i \in \mathbb{R}^N$. Moreover, we have

$$\frac{\partial}{\partial r_k^\mu} \frac{\partial}{\partial r_i^\nu} F(r) = q_1 q_2 (1 + |r|^{q_1})^{q_2 - 1} |r|^{q_1 - 2} r_k^\mu r_i^\nu + (1 + |r|^{q_1})^{q_2} \delta_{ik} \delta_{\nu\mu}$$

and

$$\sum_{i,j} \sum_{\nu,\mu} F_{i,k}^{\nu\mu}(r) \xi_{kl} \xi_{ij} \geq (1 + |r|^{q_1})^{q_2} |\xi|^2 \geq k_0 \left(1 + |r|^{p-2}\right) |\xi|^2 \tag{2.2}$$

for all $\xi \in \mathbb{R}^{nN}$ and some constant $k_0 > 0$. It is easy to see that

$$\frac{1}{p} |r|^p \leq F(r) \leq C(1 + |r|^p) \tag{2.3}$$

and

$$|F_i(r)| \leq C(1 + |r|^{p-1}). \tag{2.4}$$

Next, there is a $W^{1+\frac{1}{p},p}(\Omega; \mathbb{R}^N)$–function $w(x)$ with the trace $a(x)$ on Γ_D. We decompose u such that $u = v + w$. It follows that $v \in W^{1,p}(\Omega; \mathbb{R}^N)$ and $v = 0$ on Γ_D.

In the following section we investigate the local regularity of ∇u. Therefore, we consider some appropriate ball $B_{5R}(P)$. Let $P \in \partial\Omega$ be fixed, $R > 0$, $B_R(P) = \{x \in \mathbb{R}^n : |P - x| < R\}$, and $B_R = B_R(P)$. We define the sets $B_1, \dots, B_5$ by $B_1 = B_R \cap \Omega$, $B_2 = B_{2R} \cap \Omega$, etc. Further, we shall write Γ_5^i and $\Gamma_{D;4}$ instead of $\Gamma^i \cap B_{5R}(P)$ and $\Gamma_D \cap B_{4R}(P)$.

Let P be the only vertex of $B_{5R}(P) \cap \partial\Omega$ or let there be no vertex of $\partial\Omega$ in $B_{5R}(P)$. Further, let $B_{5R}(P) \cap \partial\Omega$ be simply connected, and $\partial\Gamma^i \cap \partial\Gamma^j \neq \emptyset$ if $\Gamma_5^i \neq \emptyset$ and $\Gamma_5^j \neq \emptyset$.

In what follows, we assume $0 < h < R$. Let $\zeta \in \mathbb{R}^n$ be a vector satisfying $|\zeta| = 1$ and $x + s\zeta \subset \overline{\Omega}$ for $x \in \overline{\Omega} \cap B_{4R}(P),\ s \in (0, R)$. We set $\Delta_\zeta^h f(x) = f(x + h\zeta) - f(x)$, $\Delta_\zeta^{-h} f(x) = f(x - h\zeta) - f(x)$,

$$\Omega_\zeta^h = \{y \in B_4 : y \neq x + h\zeta,\ x \in B_4\}, \tag{2.5}$$

and

$$\Omega_\zeta^{-h} = \{y \in B_{4R}(P) \setminus \Omega : y = x - h\zeta,\ x \in B_4\}. \tag{2.6}$$

To simplify the notation we shall often write Δ^h and Ω^h instead of Δ_ζ^h and Ω_ζ^h, respectively. Let $\partial_i = \frac{\partial}{\partial x_i}$, $\sum_{i,j,k} = \sum_{i,j,k=1}^n$, and $\sum_{\nu,\mu} = \sum_{\nu,\mu=1}^N$. Moreover, let C be a generic constant which may vary from equation to equation.

3. The basic estimates

In this section we give weighted estimates for difference quotients of ∇u; see Proposition 3.1 below. First of all, we choose some appropriate vector ζ and estimate the two difference quotients $\Delta_\zeta^h \nabla u$ and $\Delta_\zeta^{-h} \nabla u$. We obtain two estimates which are needed in order to prove Proposition 3.1. It provides a weighted estimate not depending on the sign of h. To begin with, we consider a direction parallel to Γ_D and across $\Gamma_N \cup \Gamma_S$; see Lemma 3.1 and 3.2. Next, in the Lemmas 3.3–3.4 a direction across Γ_D and parallel to $\Gamma_N \cup \Gamma_S$ is investigated.

Lemma 3.1. *Let ζ be parallel to $\Gamma_{D;4}$, angle$(\zeta, \Gamma_{N;4} \cup \Gamma_{S;4}) \geq \alpha^* > 0$, and $\delta, \varepsilon > 0$ be small. Then there holds*

$$\begin{aligned} &\sup_{0<h<R} \int_{B_2} \omega^h h^{\delta-1} \left| \Delta_\zeta^h \nabla u \right|^2 dx + \sup_{0<h<R} h^{\delta-1} \int_{\Omega^h \cap B_2} |\nabla u|^p dx \\ &\leq C + \varepsilon \, \|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)}, \end{aligned} \tag{3.1}$$

where $\omega^h(x) = \int_0^1 (1-t) \left[1 + |t\nabla u(x + h\zeta) + (1-t)\nabla u(x)|^{p-2}\right] dt$, and the constant C only depends on R, α^*, δ, ε, and the data.

Proof. Let $\tau \in W^{1,\infty}(\mathbb{R}^n)$ be a positive cut-off function with supp $\tau = B_{3R}(P)$ and $\tau = 1$ in $B_{2R}(P)$. Let $0 < h < R$. Notice that $x + h\zeta \in \Gamma_{D;4}$ for all $x \in \Gamma_{D;3}$. Thus, we have

$$\Delta^h v(x) \equiv \Delta^h_\zeta v(x) = v(x+h\zeta) - v(x) = 0 \quad \text{on } \Gamma_{D;3}.$$

Hence, the function $\varphi = \tau h^{\delta-1}\Delta^h v$ is an admissible test function in (1.1). We obtain

$$\begin{aligned} &\sum_i \int_{B_3} (1+|\nabla u|^{q_1})^{q_2}\, \partial_i u \cdot \tau \partial_i h^{\delta-1}\Delta^h v \\ &\quad = \quad -\sum_i \int_{B_3} (1+|\nabla u|^{q_1})^{q_2}\, \partial_i u \cdot \partial_i \tau h^{\delta-1}\Delta^h v \\ &\qquad\quad + \int_{B_3} f \cdot \tau h^{\delta-1}\Delta^h v - \sum_i \int_{B_3} f_i \cdot \partial_i(\tau h^{\delta-1}\Delta^h v) \\ &\qquad\quad + \int_{\Gamma_{N;3}} g_0 \cdot \tau h^{\delta-1}\Delta^h v + \int_{\Gamma_{S;3}} (g_1 - cu)\cdot \tau h^{\delta-1}\Delta^h v. \end{aligned} \tag{3.2}$$

Let us consider the integral on the left-hand side. The Taylor expansion of the function $F(r)$ introduced in (2.1) and the ellipticity condition (2.2) provide

$$\begin{aligned} F(r') - F(r) &= \sum_\nu \sum_i (r'-r)^\nu_i F^\nu_i(r) \\ &+ \sum_{\nu,\mu} \sum_{i,k} (r'-r)^\nu_i (r'-r)^\mu_k \int_0^1 (1-t) F^{\nu\mu}_{i,k}(tr' + (1-t)r)\, dt \\ &\geq \sum_i (r'-r)_i \cdot F_i(r) + k_0 |r'-r|^2 \int_0^1 (1-t)\left[1 + |tr' + (1-t)r|^{p-2}\right] dt. \end{aligned} \tag{3.3}$$

To simplify the notation we shall write $\overline{u}(x)$ instead of $u(x+h\zeta)$. Setting $r = \nabla u$ and $r' = \nabla \overline{u}$ we get

$$\begin{aligned} &\int_{B_3} \tau h^{\delta-1} \left[F(\nabla\overline{u}) - F(\nabla u)\right] \\ &\quad \geq \quad \sum_i \int_{B_3} \tau F_i(\nabla u)\cdot h^{\delta-1}\Delta^h \partial_i u + k_0 \int_{B_3} \tau \omega^h h^{\delta-1} \left|\Delta^h \nabla u\right|^2, \end{aligned} \tag{3.4}$$

where $\omega^h(x) = \int_0^1 (1-t)\left[1 + |t\nabla u(x+h\zeta) + (1-t)\nabla u(x)|^{p-2}\right] dt$. Further, let us note that $F_i(\nabla u) h^{\delta-1}\Delta^h \partial_i u = F_i(\nabla u) h^{\delta-1}\Delta^h \partial_i (v+w)$. Due to (3.2) and (3.4) we find

$$J_0 + J_1 = k_0 \int_{B_3} \tau \omega^h h^{\delta-1} \left| \Delta^h \nabla u \right|^2 - \int_{B_3} \tau h^{\delta-1} \left[F(\nabla \overline{u}) - F(\nabla u) \right] \tag{3.5}$$

$$\begin{aligned} &\leq - \sum_i \int_{B_3} (1 + |\nabla u|^{q_1})^{q_2} \, \partial_i u \cdot \tau h^{\delta-1} \Delta^h \partial_i w \\ &+ \sum_i \int_{B_3} (1 + |\nabla u|^{q_1})^{q_2} \, \partial_i u \cdot \partial_i \tau h^{\delta-1} \Delta^h v \\ &- \int_{B_3} f \cdot \tau h^{\delta-1} \Delta^h v + \sum_i \int_{B_3} f_i \cdot \partial_i \left(\tau h^{\delta-1} \Delta^h v \right) \\ &- \int_{\Gamma_{N;3}} g_0 \cdot \tau h^{\delta-1} \Delta^h v - \int_{\Gamma_{S;3}} (g_1 - cu) \cdot \tau h^{\delta-1} \Delta^h v = J_2 + \cdots + J_7. \end{aligned}$$

Now we estimate J_1 from below and the integrals on the right-hand side from above. The Leibniz rule $-f \Delta^h g = -\Delta^h (fg) + \Delta^h f \overline{g}$ yields

$$\begin{aligned} J_1 &= - \int_{B_3} \tau h^{\delta-1} \Delta^h F(\nabla u) \\ &= -h^{\delta-1} \int_{B_3} \Delta^h \left(\tau F(\nabla u) \right) + h^{\delta-1} \int_{B_3} \Delta^h \tau F(\nabla \overline{u}) = J_{11} + J_{12}. \end{aligned}$$

Let $\widetilde{\Omega}^h = \{ y \in B_4 \setminus B_3 : y = x + h\zeta, \ x \in B_3 \}$. Notice that $\tau = 0$ in $\widetilde{\Omega}^h$. Due to (2.3) we have

$$J_{11} = h^{\delta-1} \int_{\Omega^h} \tau F(\nabla u) - h^{\delta-1} \int_{\widetilde{\Omega}^h} \tau F(\nabla u) \geq \frac{h^{\delta-1}}{p} \int_{\Omega^h} \tau |\nabla u|^p,$$

where Ω^h is given in (2.5). Moreover, using $h^{\delta-1} \Delta^{-h} \tau \in W^{1,\infty}(B_4)$ and (2.3) we get

$$|J_{12}| \leq C \left(1 + \|\nabla \overline{u}\|^p_{L^p(B_3)} \right) \leq C.$$

Further, it follows that

$$\begin{aligned} J_2 &= \sum_i \int_{B_3} \partial_i \left((1 + |\nabla u|^{q_1})^{q_2} \, \partial_i u \right) \cdot \tau h^{\delta-1} \Delta^h w \\ &+ \sum_i \int_{B_3} (1 + |\nabla u|^{q_1})^{q_2} \, \partial_i u \cdot \partial_i \tau h^{\delta-1} \Delta^h w \\ &- \sum_i \int_{\Gamma_3} \nu_i \, (1 + |\nabla u|^{q_1})^{q_2} \, \partial_i u \cdot \tau h^{\delta-1} \Delta^h w = J_{21} + J_{22} + J_{23}, \end{aligned}$$

where ν is the outward normal of ∂B_3 and $\Gamma_3 = \partial\Omega \cap \partial B_3$. The equation (0.1) provides

$$J_{21} = -\int_{B_3} \Big(f + \sum_i \partial_i f_i\Big) \cdot \tau h^{\delta-1} \Delta^h w.$$

Thus, in view of (A1) we find $|J_{21}| + |J_{22}| \leq C$. Let us note that

$$\begin{aligned} J_{23} &= \sum_i \int_{\Gamma_{D;3}} \nu_i \left(1 + |\nabla w|^{q_1}\right)^{q_2} \partial_i w \cdot \tau h^{\delta-1} \Delta^h w \\ &\quad + \int_{\Gamma_{N;3}} b \cdot \tau h^{\delta-1} \Delta^h w + \int_{\Gamma_{S;3}} (d - cu) \cdot \tau h^{\delta-1} \Delta^h w. \end{aligned}$$

It holds that $w \in W^{1,p}(\Gamma_D)$. Utilizing the Hölder inequality and (A3) we get

$$|J_{23}| \leq C\Big(1 + \|cu\|^{p'}_{L^{p'}(\Gamma_{S;3})} + \Big\|h^{\delta-1}\Delta^h w\Big\|^p_{L^p(\Gamma_{N;3} \cup \Gamma_{S;3})}\Big).$$

In the case when $p \geq n$, the function u is bounded; i.e., $u \in L^\infty(\Gamma_{S;3})$. In view of (A3) we obtain

$$\|cu\|^{p'}_{L^{p'}(\Gamma_{S;3})} \leq C.$$

Otherwise, let $p > 2$. We use the Hölder inequality, (A5), and the estimates

$$\|u\|_{L^{\frac{np}{n-p+2\delta}}(\Gamma_{S;3})} \leq C\,\|u\|_{\mathcal{N}^{1-\frac{\delta}{p},p}(\Gamma_{S;3})} \leq C\,\|u\|_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_3)}$$

and get

$$\|cu\|^{p'}_{L^{p'}(\Gamma_{S;3})} \leq C + \varepsilon\,\|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_3)} \tag{3.6}$$

for $\varepsilon > 0$. (In the case $p = 2$ and $n > 2$ we obtain (3.6) using hypothesis (A3) and $cu^{\varepsilon'} \in L^\sigma(\Gamma_{S;3})$ for some $\sigma > \frac{np}{n(p-2)+p}$, where σ is suitable if $\varepsilon' > 0$ is sufficiently small.) Further, let $\gamma = \frac{\delta}{p'}$. Due to imbedding theorems we obtain

$$\begin{aligned} \Big\|h^{\delta-1}\Delta^h w\Big\|_{L^p(\Gamma_{N;3}\cup\Gamma_{S;3})} &\leq C\,\Big\|h^{\delta-1}\Delta^h w\Big\|_{\mathcal{N}^{\gamma+\frac{1}{p},p}(B_3)} \\ \leq C\,\|w\|_{\mathcal{N}^{1-\delta+\gamma+\frac{1}{p},p}(B_4)} &= C\,\|w\|_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)} \leq C. \end{aligned} \tag{3.7}$$

Next, in view of (2.4) and (A1) we have

$$|J_3| \leq C\Big(1 + \|\nabla u\|^p_{L^p(B_3)} + \Big\|h^{\delta-1}\Delta^h v\Big\|^p_{L^p(B_3)}\Big) \leq C$$

and

$$|J_4| \leq C\Big(\,\|f\|^{p'}_{L^{p'}(B_3)} + \Big\|h^{\delta-1}\Delta^h v\Big\|^p_{L^p(B_3)}\Big) \leq C.$$

Utilizing the Leibniz rule $f\Delta^h g = -\Delta^h f\overline{g} + \Delta^h(fg)$ and the fact that $v = u - w$ we get

$$J_5 = \sum_i \int_{B_3} f_i \cdot \partial_i \tau h^{\delta-1}\Delta^h v - \sum_i \int_{B_3} h^{\delta-1}\Delta^h(f_i\tau)\cdot\partial_i\overline{u}$$
$$+\sum_i \int_{B_3} h^{\delta-1}\Delta^h\left(f_i\cdot\tau\partial_i u\right) - \sum_i \int_{B_3} f_i\cdot\tau h^{\delta-1}\Delta^h\partial_i w = J_{51}+\cdots+J_{54}.$$

Due to (A1) it follows that $|J_{51}| + |J_{52}| \le C$. Let $\gamma > 0$. We find

$$\begin{aligned}|J_{53}| &\equiv \Big|\sum_i \int_{B_3} h^{\delta-1}\Delta^h(f_i\cdot\tau\partial_i u)\Big| = \Big|h^{\delta-1}\sum_i \int_{\Omega^h} \tau f_i\cdot\partial_i u\Big| \\ &\le C\gamma^{-\frac{p'}{p}} h^{\delta-1}|\Omega^h|\sum_i \|f_i\|^{p'}_{L^\infty(B_3)} + \gamma h^{\delta-1}\int_{\Omega^h}\tau|\nabla u|^p \\ &\le C + \frac{h^{\delta-1}}{2p}\int_{\Omega^h}\tau|\nabla u|^p\end{aligned}$$

for $\gamma = \frac{1}{2p}$. Moreover, partial integration, (A1), (A2), and (3.7) entail

$$\begin{aligned}|J_{54}| \le& \sum_i \|\partial_i(\tau f_i)\|_{L^{p'}(B_3)}\left\|h^{\delta-1}\Delta^h w\right\|_{L^p(B_3)} \\ &+\sum_i \|f_i\|_{L^{p'}(\partial\Omega\cap B_3)}\left\|h^{\delta-1}\Delta^h w\right\|_{L^p(\partial\Omega\cap B_3)} \le C.\end{aligned}$$

The Hölder and Young inequalities yield for $\varepsilon > 0$

$$\begin{aligned}|J_6| + |J_7| \quad\le\quad & C\varepsilon^{-\frac{p'}{p}}\left(\|g_0\|^{p'}_{L^{p'}(\Gamma_{N;3})} + \|g_1\|^{p'}_{L^{p'}(\Gamma_{S;3})} + \|cu\|^{p'}_{L^{p'}(\Gamma_{S;3})}\right) \\ &+\varepsilon\left\|h^{\delta-1}\Delta^h v\right\|^p_{L^p(\Gamma_{N;3}\cup\Gamma_{S;3})}.\end{aligned}$$

Using (A3), (3.6), and arguing as in (3.7) we may conclude that

$$|J_6| + |J_7| \le C + \varepsilon\,\|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)}.$$

Collecting results we obtain

$$k_0\int_{B_3}\tau\omega^h h^{\delta-1}\left|\Delta^h\nabla u\right|^2 + \frac{h^{\delta-1}}{2p}\int_{\Omega^h}\tau|\nabla u|^p \le C + \varepsilon\,\|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)}.$$

Notice that $\tau \equiv 1$ in B_2. Thus, the assertion follows. □

Lemma 3.2. *Let ζ satisfy the assumptions of Lemma* 3.1 *and let $\delta, \varepsilon > 0$ be small. Then there holds*

$$\sup_{0<h<R} \int_{B_1} \omega^{-h} h^{\delta-1} \left|\Delta_\zeta^{-h} \nabla u\right|^2 dx \le C + \varepsilon \|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)}, \tag{3.8}$$

where $\omega^{-h}(x) = \int_0^1 (1-t)\left[1 + |t\nabla u(x-h\zeta) + (1-t)\nabla u(x)|^{p-2}\right] dt$, and the constant C depends only on R, α^*, δ, ε, and the data.

Proof. Let $\eta \in W^{1,\infty}(\mathbb{R}^n)$ be a positive cut-off function with supp $\eta = B_{2R}(P)$ and $\eta = 1$ in $B_R(P)$. Let $0 < h < R$. We extend the functions v, w, and f_k $(1 \le k \le n)$ into Ω^{-h}, where Ω^{-h} is introduced in (2.6). Let $z \in \partial\Omega \cap \partial\Omega^{-h}$. For $0 \le \lambda \le R$ we set

$$v(z - \lambda\zeta) = v(z + \lambda\zeta), \quad \text{etc.}, \tag{3.9}$$

if $z - \lambda\zeta \in \Omega^{-h}$. Clearly, these extensions are $W^{1,p}$–extensions.

The test function $\varphi = \eta h^{\delta-1}\Delta_\zeta^{-h} v$ is admissible. Thus, we obtain

$$\begin{aligned}
&\sum_i \int_{B_2} (1 + |\nabla u|^{q_1})^{q_2}\, \partial_i u \cdot \eta \partial_i h^{\delta-1} \Delta^{-h} v \\
&\quad = -\sum_i \int_{B_2} (1 + |\nabla u|^{q_1})^{q_2}\, \partial_i u \cdot \partial_i \eta h^{\delta-1} \Delta^{-h} v \\
&\qquad + \int_{B_2} f \cdot \eta h^{\delta-1} \Delta^{-h} v - \sum_i \int_{B_2} f_i \cdot \partial_i (\eta h^{\delta-1} \Delta^{-h} v) \\
&\qquad + \int_{\Gamma_{N;2}} g_0 \cdot \eta h^{\delta-1} \Delta^{-h} v + \int_{\Gamma_{S;2}} (g_1 - cu) \cdot \eta h^{\delta-1} \Delta^{-h} v.
\end{aligned} \tag{3.10}$$

Let $\underline{u}(x) = u(x - h\zeta)$. The Taylor expansion (3.3) implies that

$$\begin{aligned}
\int_{B_2} \eta h^{\delta-1} \left[F(\nabla \underline{u}) - F(\nabla u)\right] \;\ge\; & \sum_i \int_{B_2} \eta F_i(\nabla u) \cdot h^{\delta-1} \Delta^{-h} \partial_i u \\
& + k_0 \int_{B_2} \eta \omega^{-h} h^{\delta-1} \left|\Delta^{-h} \nabla u\right|^2.
\end{aligned} \tag{3.11}$$

From (3.10) and (3.11) it follows that

$$\begin{aligned}
J_0 + J_1 &= k_0 \int_{B_2} \eta \omega^{-h} h^{\delta-1} \left|\Delta^{-h} \nabla u\right|^2 - \int_{B_2} \eta h^{\delta-1} \left[F(\nabla \underline{u}) - F(\nabla u)\right] \\
&\le -\sum_i \int_{B_2} (1 + |\nabla u|^{q_1})^{q_2}\, \partial_i u \cdot \eta h^{\delta-1} \Delta^{-h} \partial_i w
\end{aligned}$$

$$+\sum_i \int_{B_2} (1+|\nabla u|^{q_1})^{q_2}\, \partial_i u \cdot \partial_i \eta h^{\delta-1}\Delta^{-h} v$$
$$-\int_{B_2} f \cdot \eta h^{\delta-1}\Delta^{-h} v + \sum_i \int_{B_2} f_i \cdot \partial_i \left(\eta h^{\delta-1}\Delta^{-h} v\right)$$
$$-\int_{\Gamma_{N;2}} g_0 \cdot \eta h^{\delta-1}\Delta^{-h} v - \int_{\Gamma_{S;2}} (g_1 - cu)\cdot \eta h^{\delta-1}\Delta^{-h} v = J_2 + \cdots + J_7.$$

Now we estimate the integrals $J_1, \ldots, J_7$. We find

$$J_1 = -h^{\delta-1}\int_{B_2} \Delta^{-h}\left(\eta F(\nabla u)\right) + h^{\delta-1}\int_{B_2} \Delta^{-h}\eta F(\nabla \underline{u}) = J_{11} + J_{12}.$$

Due to Lemma 3.1 and the extension of u into Ω^{-h} it holds that

$$h^{\delta-1}\int_{\Omega^{-h}} \eta|\nabla u|^p dx \le C + \varepsilon \|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)}. \tag{3.12}$$

Thus, it follows that

$$J_{11} \ge -h^{\delta-1}\int_{\Omega^{-h}} \eta F(\nabla u) \ge -Ch^{\delta-1}\int_{\Omega^{-h}} \eta|\nabla u|^p \ge -C - \varepsilon\|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)}.$$

Further, using (3.12) and the same notations as above, we get

$$\begin{aligned} |J_{53}| &= \left|\sum_i \int_{B_3} h^{\delta-1}\Delta^{-h}(f_i\cdot \eta\partial_i u)\right| = \left|h^{\delta-1}\sum_i\int_{\Omega^{-h}} \eta f_i \cdot \partial_i u\right| \\ &\le C + \frac{h^{\delta-1}}{2p}\int_{\Omega^{-h}} \eta|\nabla u|^p \le C + \varepsilon\|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)}. \end{aligned}$$

Let us note that the extensions (3.9) of v and w are $\mathcal{N}^{1+\frac{1-\delta}{p},p}$–extensions; thus, e.g.,

$$\|v\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(\Omega^{-h}\cup B_3)} \le C\|v\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_3)}.$$

Hence, estimating all other integrals as above we obtain the assertion. □

Next, we consider a direction ζ which is across the Dirichlet boundary $\Gamma_{D;4}$ and parallel to $\Gamma_{N;4}\cup\Gamma_{S;4}$. In the following lemma we shall estimate the difference quotient $h^{\delta-1}|\nabla u(x-h\zeta)-\nabla u(x)|^2$. Therefore, we need extensions of the functions $u(\cdot)$ and $f_k(\cdot)$ into Ω^{-h}. Let $y\in\partial\Omega\cap\partial\Omega^{-h}$, $0<\lambda\le h$, $y+\lambda\zeta\in\Omega$, and $y-\lambda\zeta\in\Omega^{-h}$. We set

$$f_k(y-\lambda\zeta) = f_k(y+\lambda\zeta) \tag{3.13}$$

for $1\le k\le n$. Further, we extend u. Notice that $u = v + w$. We set

$$v(y-\lambda\zeta) = 0 \quad \text{and} \quad w(y-\lambda\zeta) = w(y). \tag{3.14}$$

Let us remark that $v = 0$ on $\partial\Omega \cap \partial\Omega^{-h}$. Thus, the extensions (3.13) and (3.14) are $W^{1,p}$–extensions.

Lemma 3.3. *Let ζ be parallel to $\Gamma_{N;4} \cup \Gamma_{S;4}$, angle$(\zeta, \Gamma_{D;4}) \geq \alpha^* > 0$, and $\delta, \varepsilon > 0$ be small. Then there holds*

$$\sup_{0<h<R} \int_{B_2} \omega^{-h} h^{\delta-1} \left|\Delta_\zeta^{-h} \nabla u\right|^2 dx \leq C + \varepsilon \|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)}, \tag{3.15}$$

where $\omega^{-h}(x) = \int_0^1 (1-t)\left[1 + |t\nabla u(x-h\zeta) + (1-t)\nabla u(x)|^{p-2}\right] dt$, and the constant C depends only on R, α^*, δ, ε, and the data.

Proof. Let τ be the cut-off function from above and $0 < h < R$. Let us verify that the function $\varphi = \tau h^{\delta-1}\Delta_\zeta^{-h} v$ is an admissible test function. Notice that $\Gamma_{D;4} = \partial\Omega \cap \partial\Omega^{-h}$. From (3.14) it follows that

$$\Delta^{-h} v(y) = v(y - h\zeta) - v(y) = 0 \qquad \text{for } y \in \partial\Omega \cap \partial\Omega^{-h}.$$

Thus, $\varphi = 0$ holds on $\Gamma_{D;4}$. Taking $\varphi = \tau h^{\delta-1}\Delta^{-h} v$ in equation (1.1) yields

$$\begin{aligned} &\sum_i \int_{B_3} (1 + |\nabla u|^{q_1})^{q_2}\, \partial_i u \cdot \tau h^{\delta-1}\Delta^{-h}\partial_i v \\ &= -\sum_i \int_{B_3} (1 + |\nabla u|^{q_1})^{q_2}\, \partial_i u \cdot \partial_i \tau h^{\delta-1}\Delta^{-h} v \\ &\quad + \int_{B_3} f \cdot \tau h^{\delta-1}\Delta^{-h} v - \sum_i \int_{B_3} f_i \cdot \partial_i \left(\tau h^{\delta-1}\Delta^{-h} v\right) \\ &\quad + \int_{\Gamma_{N;3}} g_0 \cdot \tau h^{\delta-1}\Delta^{-h} v + \int_{\Gamma_{S;3}} (g_1 - cu) \cdot \tau h^{\delta-1}\Delta^{-h} v. \end{aligned} \tag{3.16}$$

In view of the fact that $v = u - w$ we have

$$-(1+|\nabla u|^{q_1})^{q_2}\partial_i u \Delta^{-h}\partial_i v = (1+|\nabla u|^{q_1})^{q_2}\partial_i u(-\Delta^{-h}\partial_i u + \Delta^{-h}\partial_i w). \tag{3.17}$$

Further, the Taylor expansion of $F(\cdot)$ and the ellipticity condition (2.2) entail

$$\begin{aligned} &h^{\delta-1}\left[F(\nabla \underline{u}) - F(\nabla u)\right] \\ &\quad \geq \sum_i F_i(\nabla u) \cdot h^{\delta-1}(\Delta^{-h}\partial_i u) + k_0 \omega^{-h} h^{\delta-1} |\Delta^{-h}\nabla u|^2, \end{aligned} \tag{3.18}$$

where $\underline{u}(x) = u(x - h\zeta)$. From (3.16), (3.17), and (3.18) we get

$$k_0 \int_{B_3} \tau\omega^{-h} h^{\delta-1} \left|\Delta^{-h}\nabla u\right|^2 \leq \int_{B_3} \tau h^{\delta-1}\left[F(\nabla \underline{u}) - F(\nabla u)\right]$$

$$
\begin{aligned}
&- \sum_i \int_{B_3} (1+|\nabla u|^{q_1})^{q_2}\, \partial_i u \cdot \tau h^{\delta-1} \Delta^{-h} \partial_i w \\
&+ \sum_i \int_{B_3} (1+|\nabla u|^{q_1})^{q_2}\, \partial_i u \cdot \partial_i \tau h^{\delta-1} \Delta^{-h} v \\
&- \int_{B_3} f \cdot \tau h^{\delta-1} \Delta^{-h} v + \sum_i \int_{B_3} f_i \cdot \partial_i \left(\tau h^{\delta-1} \Delta^{-h} v \right) \\
&- \int_{\Gamma_{N;3}} g_0 \cdot \tau h^{\delta-1} \Delta^{-h} v - \int_{\Gamma_{S;3}} (g_1 - cu) \cdot \tau h^{\delta-1} \Delta^{-h} v = J_1 + \cdots + J_7.
\end{aligned}
$$

Let us start estimating the integrals on the right-hand side. Summation by parts entails

$$
\begin{aligned}
J_1 &= \int_{B_3} \tau h^{\delta-1} \Delta^{-h} F(\nabla u) = h^{\delta-1} \int_{B_3} \Delta^{-h} \left(\tau F(\nabla u)\right) - h^{\delta-1} \int_{B_3} \Delta^{-h} \tau F(\nabla \underline{u}) \\
&= J_{11} + J_{12}.
\end{aligned}
$$

Notice that

$$
J_{11} = h^{\delta-1} \int_{\Omega^{-h}} \tau F(\nabla u) - h^{\delta-1} \int_{\widetilde{\Omega}^{-h}} \tau F(\nabla u),
$$

where $\widetilde{\Omega}^{-h} = \{y \in B_3 : y + h\zeta \notin B_3\}$. The second integral is negative. Due to the fact that $v = 0$ in Ω^{-h} we have $\nabla u = \nabla w$ in Ω^{-h}; thus,

$$
|J_{11}| \le C h^{\delta-1} \int_{\Omega^{-h}} \tau (1 + |\nabla w|^p).
$$

The extension (3.14) of w into Ω^{-h} implies that

$$
h^{\delta-1} \int_{\Omega^{-h}} |\nabla w|^p \le C h^{\delta} |w|^p_{W^{1,p}(\partial\Omega \cap \partial\Omega^{-h})} \le C. \tag{3.19}
$$

Hence, the integral J_{11} is bounded. Moreover, it follows that

$$
|J_{12}| \le C \int_{B_3} \left| h^{\delta-1} \Delta^{-h} \tau \right| (1 + |\nabla \underline{u}|^p) \le C.
$$

The integrals J_2, J_3, J_4, J_6, and J_7 may be estimated as in the proof of Lemma 3.1. Further, using again summation by parts we find

$$
\begin{aligned}
J_5 &= \sum_i \int_{B_3} f_i \cdot \partial_i \tau h^{\delta-1} \Delta^{-h} v - \sum_i \int_{B_3} h^{\delta-1} \Delta^{-h} (f_i \tau) \cdot \partial_i \underline{v} \\
&\quad + \sum_i \int_{B_3} h^{\delta-1} \Delta^{-h} \left(f_i \cdot \tau \partial_i v \right) = J_{51} + J_{52} + J_{53}.
\end{aligned}
$$

In view of (A1) it follows that $|J_{51}| + |J_{52}| \le C$. Due to the facts that $\tau = 0$ in $B_4 \setminus B_3$ and $\nabla v = 0$ in Ω^{-h}, we get

$$|J_{53}| = \Big|\sum_i \int_{B_4} h^{\delta-1}\Delta^{-h}(f_i \cdot \tau\partial_i v)\Big| = \Big|h^{\delta-1}\sum_i \int_{\Omega^{-h}} \tau f_i \cdot \partial_i v\Big| = 0.$$

Collecting results we arrive at

$$k_0 \int_{B_3} \tau\omega^{-h}h^{\delta-1}|\Delta^{-h}\nabla u|^2 \le C + \varepsilon \, \|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)} \tag{3.20}$$

for a sufficiently small $\varepsilon > 0$. Thus, the assertion follows. □

Corollary 3.1. *Let ζ satisfy the assumptions of Lemma* 3.3 *and let $\delta, \varepsilon > 0$ be small. Then there holds*

$$\sup_{0<h<R} h^{\delta-1} \int_{\Omega^h \cap B_2} |\nabla u|^p dx \le C + \varepsilon \, \|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)} \tag{3.21}$$

and the constant C depends only on R, α^*, δ, ε, and the data.

Proof. Let $a, b \in \mathbb{R}^{nN}$ be fixed. There is a number $t_0 \in (0, \frac{1}{2})$ such that

$$\int_0^{t_0} (1-t)|ta + (1-t)b|^{p-2}dt \ge \int_0^{t_0} \frac{(1-t)}{2}|b|^{p-2}dt \ge \frac{1}{4}|b|^{p-2}.$$

Using estimates like $2ab \le \varepsilon^{-1}a^2 + \varepsilon b^2$ we find two constants $C_1, C_2 > 0$ such that

$$|a-b|^2|b|^{p-2} \ge ((1-\varepsilon^{-1})|a|^2 + (1-\varepsilon)|b|^2)|b|^{p-2} \ge -C_1|a|^p + C_2|b|^p.$$

Let the extension of v and w into Ω^{-h} be given by (3.14). It follows that $\nabla v = 0$ in Ω^{-h}. Thus, for $x \in \Omega^h$ it holds that

$$\Delta^{-h}\nabla u(x) = \nabla \underline{w}(x) - \nabla u(x)$$

and

$$\omega^{-h}(x) = \int_0^1 (1-t)[1 + |t\nabla \underline{w}(x) - (1-t)\nabla u(x)|^{p-2}]\, dt.$$

This yields

$$\begin{aligned}\int_{\Omega^h} \tau\omega^{-h}h^{\delta-1}|\Delta^{-h}\nabla u|^2 &\ge \frac{1}{4}\int_{\Omega^h} \tau h^{\delta-1}|\nabla u|^{p-2}|\nabla \underline{w} - \nabla u|^2 \\ &\ge -\frac{C_1 h^{\delta-1}}{4}\int_{\Omega^{-h}} \tau|\nabla w|^p + \frac{C_2 h^{\delta-1}}{4}\int_{\Omega^h} \tau|\nabla u|^p.\end{aligned}$$

Due to (3.20) and (3.19) the assertion follows. □

Lemma 3.4. *Let ζ satisfy the assumptions of Lemma 3.3 and let $\delta, \varepsilon > 0$ be small. Then there holds*

$$\sup_{0<h<R} \int_{B_1} \omega^h h^{\delta-1} \left|\Delta_\zeta^h \nabla u\right|^2 dx \le C + \varepsilon \|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)}, \tag{3.22}$$

where $\omega^h(x) = \int_0^1 (1-t)\left[1 + |t\nabla u(x+h\zeta) + (1-t)\nabla u(x)|^{p-2}\right] dt$, and the constant C depends only on R, α^*, δ, ε, and the data.

Proof. Let η be the cut-off function from above and $0 < h < R$. Let $y \in \partial\Omega \cap \partial\Omega^{-h}$, $0 < \lambda \le h$, $y + \lambda\zeta \in \Omega$, and $y - \lambda\zeta \in \Omega^{-h}$. We extend the functions v, w, and f_k into Ω^{-h} by setting

$$v(y-\lambda\zeta) = -v(y+\lambda\zeta), \quad w(y-\lambda\zeta) = w(y+\lambda\zeta), \tag{3.23}$$

and $f_k(y-\lambda\zeta) = f_k(y+\lambda\zeta)$ for $1 \le k \le n$. The Taylor expansion yields (cf. (3.4) and (3.11))

$$\begin{aligned} & k_0 \int_{B_2} \eta\omega^h h^{\delta-1} \left|\Delta^h \nabla u\right|^2 + k_0 \int_{B_2} \eta\omega^{-h} h^{\delta-1} \left|\Delta^{-h}\nabla u\right|^2 \\ \le\ & \int_{B_2} \eta h^{\delta-1}\left[F(\nabla\overline{u}) - F(\nabla u)\right] + \int_{B_2} \eta h^{\delta-1}\left[F(\nabla\underline{u}) - F(\nabla u)\right] \\ & - \sum_i \int_{B_2} \eta\,(1+|\nabla u|^{q_1})^{q_2}\,\partial_i u \cdot h^{\delta-1}(\Delta^h + \Delta^{-h})\partial_i w \\ & - \sum_i \int_{B_2} \eta\,(1+|\nabla u|^{q_1})^{q_2}\,\partial_i u \cdot h^{\delta-1}(\Delta^h + \Delta^{-h})\partial_i v. \end{aligned} \tag{3.24}$$

Let us note that $\Gamma_{D;4} = \partial\Omega\cap\partial\Omega^{-h}$. Due to (3.23), we get, for $y \in \partial\Omega\cap\partial\Omega^{-h}$,

$$v(y+h\zeta) - 2v(y) + v(y-h\zeta) = 0;$$

hence, the function $\eta h^{\delta-1}(\Delta^h + \Delta^{-h})v$ is an admissible test function in (1.1). It follows that

$$\begin{aligned} & - \sum_i \int_{B_2} (1+|\nabla u|^{q_1})^{q_2}\,\partial_i u \cdot \eta\partial_i h^{\delta-1}(\Delta^h + \Delta^{-h})v \\ & = \sum_i \int_{B_2} (1+|\nabla u|^{q_1})^{q_2}\,\partial_i u \cdot \partial_i\eta h^{\delta-1}(\Delta^h + \Delta^{-h})v \\ & - \int_{B_2} f \cdot \eta h^{\delta-1}(\Delta^h + \Delta^{-h})v + \sum_i \int_{B_2} f_i \cdot \partial_i(\eta h^{\delta-1}(\Delta^h + \Delta^{-h})v) \\ & - \int_{\Gamma_{N;2}} g_0 \cdot \eta h^{\delta-1}(\Delta^h + \Delta^{-h})v - \int_{\Gamma_{S;2}} (g_1 - cu) \cdot \eta h^{\delta-1}(\Delta^h + \Delta^{-h})v. \end{aligned} \tag{3.25}$$

Now we replace the last integral on the right-hand side of (3.24) by the right-hand side of (3.25). Next, we estimate all the integrals as above. In particular, we find

$$\int_{B_2} h^{\delta-1}\left[\overline{\eta}F(\nabla\overline{u})-\eta F(\nabla u)\right]=-h^{\delta-1}\int_{\Omega^h}\eta F(\nabla u)\le 0$$

and, due to (3.23) and (3.21),

$$\int_{B_2} h^{\delta-1}[\underline{\eta}F(\nabla\underline{u})-\eta F(\nabla u)]\le h^{\delta-1}\int_{\Omega^{-h}}\eta F(\nabla u)\le Ch^{\delta-1}\int_{\Omega^{-h}}(1+|\nabla u|^p)$$
$$\le Ch^{\delta-1}\int_{\Omega^h}(1+|\nabla u|^p+|\nabla w|^p)\le C+\varepsilon\,\|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)}.$$

Here we have used the fact that

$$h^{\delta-1}\int_{\Omega^h}|\nabla w|^p\le C.$$

This estimate follows by extending w into Ω^{-h} according to (3.14) and utilizing (3.19) and the fact that $h^{\delta-1}\int_{\Omega^h}|\Delta^{-h}\nabla w|^p\le C$.

Altogether we may conclude that

$$k_0\int_{B_2}\eta\omega^h h^{\delta-1}|\Delta^h\nabla u|^2+k_0\int_{B_2}\eta\omega^{-h}h^{\delta-1}|\Delta^{-h}\nabla u|^2\le C+\varepsilon\,\|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)}.$$

This yields the assertion. □

Proposition 3.1. *Let* $\Lambda=\{\zeta\in\mathbb{R}^n : x+\lambda\zeta\in\overline{\Omega}$ *for all* $x\in\overline{B_4}$ *and* $\lambda\in[0,R]\}$ *and* $\delta,\varepsilon>0$. *Then there are* n *linearly independent unit vectors* $\zeta^1,\dots,\zeta^n\in\Lambda$ *and a constant* C *depending only on* R, ε, *and the data such that*

$$\sup_{1\le i\le n}\sup_{0<h<R}\int_{B_1}\omega\left|\frac{\nabla u(x+h\zeta^i)-\nabla u(x)}{h^{\frac{1-\delta}{2}}}\right|^2\le C+\varepsilon\,\|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)},\quad (3.26)$$

where $\omega(x)=\int_0^1[1+|t\nabla u(x+h\zeta^i)+(1-t)\nabla u(x)|^{p-2}]\,dt$.

Proof. Let h and $\zeta\in\Lambda$ be fixed. We have

$$\omega^{-h}(x+h\zeta)=\int_0^1(1-t)\left[1+|t\nabla u(x)+(1-t)\nabla u(x+h\zeta)|^{p-2}\right]\,dt$$
$$=\int_0^1 t\left[1+|(1-t)\nabla u(x)+t\nabla u(x+h\zeta)|^{p-2}\right]\,dt.$$

Let us note that $\omega^h(x) = \int_0^1 (1-t)[1+|(1-t)\nabla u(x) + t\nabla u(x+h\zeta)|^{p-2}]\,dt$. Thus,

$$\omega^{-h}(x+h\zeta)\big|\Delta^{-h}\nabla u(x+h\zeta)\big|^2 + \omega^h(x)\big|\Delta^h\nabla u(x)\big|^2 = \omega(x)\big|\Delta^h\nabla u(x)\big|^2.$$

Hence, if $\zeta \in \mathbb{R}^n$ satisfies the assumptions of either Lemma 3.1 or Lemma 3.3 it follows that

$$\sup_{0<h<R}\int_{B_1}\omega h^{\delta-1}\left|\Delta_\zeta^h\nabla u\right|^2 dx \le C + \varepsilon\,\|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)}. \tag{3.27}$$

More general, we can find basis vectors $\zeta^1,\ldots,\zeta^n$ of $\mathbb{R}^n$ satisfying
(1) $|\zeta^i| = 1$ for $1 \le i \le n$.
(2) $\mathrm{angle}(\zeta^i,\zeta^j) \ge \alpha$ for $1 \le i < j \le n$.
(3) $x + s\zeta^i \in \overline{\Omega}$ for $x \in \overline{\Omega}\cap B_{4R}(P)$ and $0 < s < R$.
(4) If $\Gamma^k \cap B_{5R}(P) \ne \emptyset$ and ζ^i is not parallel to Γ^k, then $\mathrm{angle}(\zeta^i,\Gamma^k) \ge \alpha$.
(5) If $\mathrm{angle}(\zeta^i,\Gamma^k_5) \ge \alpha$, then $x - s\zeta^i \notin \overline{\Omega}$ for all $x \in \Gamma^k_4$ and $0 < s < R$.
(6) There holds at least one of the following conditions:
i) $\mathrm{angle}(\zeta^i,\Gamma_{D;5}) \ge \alpha$ and $\Gamma_{N;5}\cup\Gamma_{S;5} = \emptyset$,
ii) $\mathrm{angle}(\zeta^i,\Gamma_{D;5}) \ge \alpha$ and ζ^i is parallel to $\Gamma_{N;5}\cup\Gamma_{S;5}$,
iii) $\mathrm{angle}(\zeta^i,\Gamma_{N;5}\cup\Gamma_{S;5}) \ge \alpha$ and $\Gamma_{D;5} = \emptyset$,
iv) $\mathrm{angle}(\zeta^i,\Gamma_{N;5}\cup\Gamma_{S;5}) \ge \alpha$ and ζ^i is parallel to $\Gamma_{D;5}$.
(Parallel to $\Gamma_{D;5}$ means parallel to all $\Gamma^k \subset \Gamma_{D;5}$ with $\Gamma^k_5 \ne \emptyset$.)
(7) If $\Gamma^k_5 \subset \Gamma_{D;5}$ and ζ^i is parallel to Γ^k, then $y + s\zeta^i \in \Gamma^k$ for $y \in \Gamma^k_4$ and $0 < s < R$.
(8) The constant $\alpha > 0$ depends only on the geometry of Ω.

Following the proofs of Lemmas 3.1–3.4 we obtain (3.27) for each ζ^i ($1 \le i \le n$). Thus, the assertion follows. □

4. Proofs of the main results

In this section the proofs of Theorems 1.1–1.3 are given.

Proof of Theorem 1.1. Let $\zeta^i \in \mathbb{R}^n$ satisfy the assumptions of Proposition 3.1. The Taylor expansion yields

$$F_i^\nu(\nabla\overline{u}) - F_i^\nu(\nabla u) = \sum_\mu\sum_k\int_0^1 F_{i,k}^{\nu\mu}(t\nabla\overline{u} - (1-t)\nabla u)\left(\partial_k\overline{u}^\mu - \partial_k u^\mu\right)\,dt.$$

Utilizing $|F_{i,k}^{\nu\mu}(r)| \le C\left(1+|r|^{p-2}\right)$, we get

$$\sum_i\int_{B_1} h^{\delta-1}\left(F_i(\nabla\overline{u}) - F_i(\nabla u)\right)\cdot\left(\partial_i\overline{u} - \partial_i u\right)$$

$$\leq C \int_{B_1} h^{\delta-1} \left|\Delta^h \nabla u\right|^2 \int_0^1 \left(1 + |t\nabla\overline{u} + (1-t)\nabla u|^{p-2}\right) dt. \tag{4.1}$$

On the other hand, there is a constant C such that

$$|a-b|^2 \int_0^1 |ta + (1-t)b|^{p-2} dt \geq C|a-b|^p$$

for all $a, b \in \mathbb{R}^{nN}$; cf. [3]. Hence, the ellipticity condition (2.2) entails

$$\sum_i (F_i(\nabla\overline{u}) - F_i(\nabla u)) \cdot (\partial_i\overline{u} - \partial_i u)$$

$$\geq k_0 \left|\Delta^h \nabla u\right|^2 \int_0^1 (1 + |t\nabla\overline{u} + (1-t)\nabla u|^{q_1})^{q_2} dt \geq C \left|\Delta^h \nabla u\right|^p. \tag{4.2}$$

Altogether (4.1), (4.2) and (3.26) yield

$$\sup_{0<h<R} \int_{B_1} h^{\delta-1} \left|\Delta^h \nabla u\right|^p \leq C + \varepsilon \|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)}.$$

Thus, it follows that

$$\sup_{\substack{\mu>0 \\ 0<|z|<\mu}} \int_{B_\mu} \left|\frac{\nabla u(x+z) - \nabla u(x)}{|z|^{\frac{1-\delta}{p}}}\right|^p dx \leq C + \varepsilon \|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_4)},$$

where $B_\mu = \{x \in B_1 : \mathrm{dist}(x, \partial\Omega) \geq \mu\}$. Let us cover $\overline{\Omega}$ by a finite number of balls $B_{R_i}(P_i)$ such that

1) $\overline{\Omega} \subset \bigcup_{i=1}^{N_0} B_{R_i}(P_i)$.

2) The radii R_i $(1 \leq i \leq N_0)$ depend only on the geometry of $\partial\Omega$.

3) If $B_{R_i}(P_i) \cap \partial\Omega \neq \emptyset$, then $B_{R_i}(P_i) \cap \partial\Omega$ is simply connected. Further, $\partial\Gamma^j \cap \partial\Gamma^k \neq \emptyset$ holds if $\Gamma^j \cap B_{R_i}(P_i) \neq \emptyset$ and $\Gamma^k \cap B_{R_i}(P_i) \neq \emptyset$.

4) Each point P_i $(1 \leq i \leq N_0)$ is the only vertex of $B_{R_i}(P_i) \cap \partial\Omega$, or there is no vertex of $\partial\Omega$ in $B_{R_i}(P_i)$.

We may conclude that

$$\|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(\Omega)} \leq C + \varepsilon \sum_{i=1}^{N_0} \|u\|^p_{\mathcal{N}^{1+\frac{1-\delta}{p},p}(B_{4R_i}(P_i)\cap\Omega)}$$

for a sufficiently small $\varepsilon > 0$. Absorbing the last term into the right-hand side we obtain

$$u \in \mathcal{N}^{1+\frac{1-\delta}{p},p}(\Omega; \mathbb{R}^N) \qquad \text{for all } \delta > 0. \tag{4.3}$$

The imbedding theorem of Nikolskii spaces into Sobolev spaces (cf. [15]) provides

$$u \in W^{1+\frac{1-2\delta}{p},p}(\Omega; \mathbb{R}^N) \qquad \text{for all } \delta > 0.$$

Thus, the assertion is proven. □

Let us remark $u \in \mathcal{N}^{1+\frac{1-\delta}{p},p}(\Omega;\mathbb{R}^N)$ for all $\delta > 0$ implies that

$$\nabla u \in L^{\frac{np}{n-1}-\varepsilon}(\Omega;\mathbb{R}^{nN}) \qquad \text{for all } \varepsilon > 0. \tag{4.4}$$

In the case when the assumptions of Theorem 1.3 are satisfied a better result is known. Then there is some number $\gamma > 0$ such that

$$\nabla u \in L^{p+1+\gamma}(\Omega). \tag{4.5}$$

A proof of (4.5) for homogeneous boundary data is given in [12].

Proof of Theorem 1.2. Let $\frac{p}{2} \le \alpha \le \frac{(2n-1)p}{2n-2}$, $s \in \mathbb{R}^{nN}$, $G(s) = |s|^\alpha$, and $G_i^\nu(s) = \frac{\partial}{\partial s_i^\nu} G(s)$. The Taylor expansion yields

$$G(s') - G(s) = \sum_i (s'-s)_i \cdot \int_0^1 G_i(ts + (1-t)s)\; dt.$$

Let ζ^i satisfy the assumptions of Proposition 3.1. We put $s = \nabla u(x)$ and $s' = \nabla \overline{u}(x) \equiv \nabla u(x + h\zeta^i)$. Notice that $G_i(s) = \alpha |s|^{\alpha-2} s_i$. Thus, it follows that

$$\left| h^{\frac{\delta-1}{2}} \Delta^h |\nabla u|^\alpha \right| \le C \left| h^{\frac{\delta-1}{2}} \Delta^h \nabla u \right| \int_0^1 |t\nabla \overline{u} + (1-t)\nabla u|^{\alpha-1} dt.$$

For $\sigma \ge 1$, we obtain

$$J_0 = \int_{B_1} \left| h^{\frac{\delta-1}{2}} \Delta^h |\nabla u|^\alpha \right|^\sigma dx \le C \int_{B_1} \left(\left| h^{\frac{\delta-1}{2}} \Delta^h \nabla u \right| \int_0^1 \gamma^{\alpha-1} dt \right)^\sigma dx,$$

where $\gamma = |t\nabla\overline{u} + (1-t)\nabla u|$. The Hölder inequality yields

$$\int_0^1 \gamma^{\alpha-1} dt = \int_0^1 \gamma^{\frac{p-2}{2}} \gamma^{\frac{2\alpha-p}{2}} dt \le \left(\int_0^1 \gamma^{p-2} dt \right)^{\frac{1}{2}} \left(\int_0^1 \gamma^{2\alpha-p} dt \right)^{\frac{1}{2}};$$

thus,

$$J_0 \le C \int_{B_1} \left(\left| h^{\frac{\delta-1}{2}} \Delta^h \nabla u \right|^2 \int_0^1 \gamma^{p-2} dt \right)^{\frac{\sigma}{2}} \left(\int_0^1 \gamma^{2\alpha-p} dt \right)^{\frac{\sigma}{2}} dx. \tag{4.6}$$

Let $1 \le \sigma < 2$. Using again the Hölder inequality (with $p_1 = \frac{2}{\sigma}$ and $p_2 = \frac{2}{2-\sigma}$), we get

$$J_0 \le C \Big(\int_{B_1} \left| h^{\frac{\delta-1}{2}} \Delta^h \nabla u \right|^2 \int_0^1 \gamma^{p-2} dt\, dx \Big)^{\frac{\sigma}{2}} \Big(\int_{B_1} \Big(\int_0^1 \gamma^{2\alpha-p} dt \Big)^{\frac{\sigma}{2-\sigma}} dx \Big)^{\frac{2-\sigma}{2}}. \tag{4.7}$$

We fix some $\sigma \in [1, \frac{2np}{2(n-1)\alpha+p})$. Notice that $(2\alpha - p)\frac{\sigma}{2-\sigma} < \frac{np}{n-1}$. Due to (4.4) we find

$$\int_{B_1} \Big(\int_0^1 \gamma^{2\alpha-p} dt \Big)^{\frac{\sigma}{2-\sigma}} dx \le C \int_{B_1} \int_0^1 \gamma^{(2\alpha-p)\frac{\sigma}{2-\sigma}} dt\, dx$$
$$\le C \int_{B_1} (|\nabla u| + |\nabla \overline{u}|)^{(2\alpha-p)\frac{\sigma}{2-\sigma}} \le C. \tag{4.8}$$

Using (4.7), (4.8), (3.26), and (4.3) we obtain

$$\sup_{0<h<R} \int_{B_1} \Big| h^{\frac{\delta-1}{2}} \Delta^h |\nabla u|^\alpha \Big|^\sigma \le C \qquad \text{for } 1 \le \sigma < \frac{2np}{2(n-1)\alpha + p}.$$

This implies that

$$\sup_{\substack{\mu>0 \\ 0<|z|<\mu}} \int_{B_\mu} \Big| \frac{|\nabla u(x+z)|^\alpha - |\nabla u(x)|^\alpha}{|z|^{\frac{1-\delta}{2}}} \Big|^\sigma dx \le C,$$

where $B_\mu = \{x \in B_1 : \operatorname{dist}(x, \partial\Omega) \ge \mu\}$. Employing the same cover argument as in the proof of Theorem 1.1 it follows that $|\nabla u|^\alpha \in \mathcal{N}^{\frac{1-\delta}{2},\sigma}(\Omega)$; thus, $|\nabla u|^\alpha \in W^{\frac{1}{2}-\varepsilon,\sigma}(\Omega)$ for all $\varepsilon > 0$, and $|\nabla u|^\alpha \in W^{\frac{1}{2}-2\varepsilon,q}(\Omega)$ for $q = \frac{2np}{2(n-1)\alpha+p}$ and all $\varepsilon > 0$. This yields the assertion (1.3).

Finally, let us remark that we can find a number $1 \le \sigma < 2$ satisfying

$$\sigma < \frac{2np}{2(n-1)\alpha + p},$$

if $\alpha > \frac{p}{2}$. Moreover, in the case when $\alpha = \frac{p}{2}$ we set $\sigma = 2$. Then (4.6) yields

$$J_0 \le C \int_{B_1} \bigg(\Big| h^{\frac{\delta-1}{2}} \Delta^h \nabla u \Big|^2 \int_0^1 \gamma^{p-2} dt \bigg) dx.$$

Due to (3.26) and (4.3) we obtain

$$\sup_{0<h<R} \int_{B_1} \Big| h^{\frac{\delta-1}{2}} \Delta^h |\nabla u|^{\frac{p}{2}} \Big|^2 \le C.$$

Thus, the assertion follows. □

Proof of Theorem 1.3. Let ζ^i fulfill the assumptions of Proposition 3.1. We distinguish two cases.

Case 1: Let $\alpha > \frac{p}{2}$. It holds that $(2\alpha - p)\frac{\sigma}{2-\sigma} = p + 1$ for $\sigma = \frac{2p+2}{2\alpha+1}$. Due to (4.5) we obtain

$$\int_{B_1} \Big(\int_0^1 \gamma^{2\alpha-p} dt \Big)^{\frac{\sigma}{2-\sigma}} dx \le C \int_{B_1} (|\nabla u| + |\nabla \overline{u}|)^{p+1} \le C. \tag{4.9}$$

From (4.7), (3.26), (4.3), and (4.9) we get

$$\sup_{\substack{\mu>0\\0<|z|<\mu}} \int_{B_\mu} \Big|\frac{|\nabla u(x+z)|^\alpha - |\nabla u(x)|^\alpha}{|z|^{\frac{1-\delta}{2}}}\Big|^\sigma dx \le C \quad \text{for } \sigma = \frac{2p+2}{2\alpha+1}, \tag{4.10}$$

where $B_\mu = \{x \in B_1 : \text{dist}(x, \partial\Omega) \ge \mu\}$.

Case 2: Let $\alpha = \frac{p}{2}$. Then we have $\sigma = \frac{2p+2}{2\alpha+1} = 2$. Estimate (4.6) yields

$$J_0 \le \int_{B_1} \Big|h^{\frac{\delta-1}{2}} \Delta^h \nabla u\Big|^2 \int_0^1 \gamma^{p-2} dt\, dx.$$

Hence, in view of (3.26) and (4.3) we obtain (4.10) for $\alpha = \frac{p}{2}$ as well.

From (4.10) it follows as above that $|\nabla u|^\alpha \in \mathcal{N}^{\frac{1-\delta}{2},\sigma}(\Omega)$; thus, $|\nabla u|^\alpha \in W^{\frac{1}{2}-\varepsilon,\sigma}(\Omega)$ for $\sigma = \frac{2p+2}{2\alpha+1}$ and all $\varepsilon > 0$. This entails the assertion (1.4). $\square$

References

[1] M.V. Borsuk, *Estimates of solutions of the Dirichlet problem for a quasilinear nondivergence elliptic equation of second order near a corner boundary point*, St. Petersburg Math. J., 3 (1992), 1281–1302.

[2] M.V. Borsuk and D. Portnyagin, *Barriers on cones for degenerate quasilinear elliptic operators*, Electr. J. Differ. Equations, 11 (1998), 1–8.

[3] Y.Z. Chen and E. DiBenedetto, *Boundary estimates for solutions of nonlinear degenerate parabolic systems*, J. Reine Angew. Math., 395 (1985), 102–131.

[4] M. Dauge, *Elliptic Boundary Value Problems on Corner Domains*, Lecture Notes in Math., 1341, Springer-Verlag, Berlin, 1988.

[5] M. Dobrowolski, *On quasilinear elliptic equations in domains with conical boundary points*, J. Reine Angew. Math., 394 (1989), 186–195.

[6] C. Ebmeyer, *Mixed boundary value problems for nonlinear elliptic systems in n-dimensional Lipschitzian domains*, Zeit. Anal. Anwend., 18 (1999), 539–555.

[7] C. Ebmeyer, *Steady flow of fluids with shear dependent viscosity under mixed boundary value conditions in polyhedral domains*, Math. Models Methods Appl. Sci., 10 (2000), 629–650.

[8] C. Ebmeyer, *Nonlinear elliptic problems under mixed boundary value conditions in nonsmooth domains*, SIAM J. Math. Anal., 32 (2000), 103–118.

[9] C. Ebmeyer, *Mixed boundary value problems for nonlinear elliptic systems with p–structure in polyhedral domains*, Math. Nachr. (to appear).

[10] C. Ebmeyer and J. Frehse, *Mixed boundary value problems for nonlinear elliptic equations in multidimensional non-smooth domains*, Math. Nachr., 203 (1999), 47–74.

[11] C. Ebmeyer and J. Frehse, *Steady Navier-Stokes equations with mixed boundary value conditions in three-dimensional Lipschitzian domains*, Math. Ann. (to appear).

[12] C. Ebmeyer and J. Frehse, *Mixed boundary value problems for nonlinear elliptic equations with p–structure in nonsmooth domains*, Differ. Integral Equ. (to appear).

[13] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman Advanced Publishing Program, Boston, London, Melbourne, 1985.

[14] V.A. Kondrat'ev, *Boundary value problems for elliptic equations in domains with conical and angular points,* Trans. Moscow Math. Soc., 16 (1967), 227–313.

[15] A. Kufner, O. John, and S. Fučik, *Function Spaces*, Academia, Prague, 1977.

[16] O.A. Ladyzhenskaya and N.N. Uraltseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, London, 1968.

[17] V.G. Maz'ya, S.A. Nazarov, and B.A. Plamenevskii, *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten*, Vol., 1, 2, Akademie-Verlag, Berlin, 1991.

[18] E. Miersemann, *Zur gemischten Randwertaufgabe für die Minimalflächengleichung*, Math. Nachr., 115 (1984), 125–136.

[19] E. Miersemann, *Asymptotic expansion of solutions of the Dirichlet problem for quasilinear elliptic equations of second order near a conical point*, Math. Nachr., 135 (1988), 239–274.

[20] S.A. Nazarov and B.A. Plamenevsky, *Elliptic Problems in Domains with Piecewise Smooth Boundaries*, Walter de Gruyter, Berlin, 1994.

[21] P. Tolksdorf, *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Part. Diff. Equations, 8 (1983), 773–817.

[22] K. Uhlenbeck, *Regularity for a class of nonlinear elliptic systems*, Acta Math., 138 (1977), 219–240.