Advances in Differential Equations Volume 6, Number 7, July 2001, Pages 847–872

SLOW DYNAMICS OF INTERIOR SPIKES IN THE SHADOW GIERER-MEINHARDT SYSTEM

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Abstract. This paper is devoted to the rigorous analysis of the dynamics of single-spike solutions in the shadow Gierer-Meinhardt system. We derive a system of ODEs which governs the motion of the spike. Careful analysis of this system shows that the spike moves exponentially slowly towards the point on the boundary that is the closest to the spike.

1. Introduction

1.1. **The shadow Gierer-Meinhardt system.** In this paper we consider the shadow Gierer-Meinhardt system, for $A = A(y,t)$ and $H = H(t)$,

$$\begin{cases} A_t = \varepsilon^2 \Delta_y A - A + A^2/H, & y \in \Omega,\ t > 0, \\ \tau H_t = \varepsilon^{-N} \langle A^2 \rangle_1 - H, & t > 0, \\ \partial_n A = 0, & y \in \partial\Omega,\ t > 0 \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \le 3$, is a bounded domain with C^3 boundary, $\langle f \rangle_1 = \int_\Omega f(x)\,dx$, Δ_y is the Laplace operator with respect to y, ∂_n is the exterior normal derivative, and ε and τ are small positive parameters.

The shadow Gierer-Meinhardt system (1.1) is obtained from the following system of reaction-diffusion equations (Gierer-Meinhardt system),

$$\begin{cases} A^D_t = \varepsilon^2 \Delta_y A^D - A^D + (A^D)^2/H^D, & y \in \Omega,\ t > 0, \\ \tau H^D_t = D\Delta H^D + (A^D)^2 - H^D, & y \in \Omega,\ t > 0, \\ \partial_n A^D = \partial_n H^D = 0, & y \in \partial\Omega,\ t > 0, \end{cases} \tag{1.2}$$

The first author is thankful for the financial support of the National Science Foundation grant DMS-9622872. The second author's research was partially supported by the NSF Mathematical Sciences Postdoctoral Fellowship DMS-9705972.

Accepted for publication July 2000.

AMS Subject Classifications: 35B25, 35C20, 35J60, 35K99, 92C15, 92C40.

after taking the limit $(A^D, H^D) \to (A^\infty, H^\infty)$, as $D \to \infty$, and rescaling $(A^\infty, H^\infty) = |\Omega|\varepsilon^{-N}(A, H)$ (see [25, 16] and the references therein).

The system (1.2) was proposed in [13] (see also [21]) as a model for biochemical reaction of activator-inhibitor type in which a short-range substance, the activator, promotes its own production as well as that of a rapidly diffusing antagonist, the inhibitor. If the diffusion coefficient of the activator, ε^2, is much smaller than that of the inhibitor, D, then local increase in the concentration of the activator will be further amplified, forming regions with a high concentration of the activator surrounded by the "sea" of, essentially uniformly distributed, inhibitor. We speak of *spikes* if the activator concentrates near a single point or a set of isolated points.

In recent years there has been much interest in studying (1.1), (1.2) and especially the associated steady state problems. Most of the results available so far deal with the shadow system (1.1), which represents qualitative behavior similar to (1.2), at least for large D, but it is somewhat easier to treat. In a series of papers [22, 23, 24], Ni and Takagi (also Lin [20]) established the existence of stationary spikes for (1.1) concentrating at points of maximal mean curvature of $\partial\Omega$. Their work is based on the observation that the steady state problem for the shadow system can be reduced to the following semilinear elliptic problem:

$$\varepsilon^2 \Delta w - w + w^2 = 0 \quad \text{in } \Omega, \qquad \partial_n w = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

Subsequently many mathematicians have established the existence of similar types of stationary solutions with one or multiple peaks located at the critical points of the mean curvature H of $\partial\Omega$ ([10, 14, 28, 19] for (1.1) and [9] for (1.2)) and with spikes located in the the interior of Ω ([15, 18, 26], for (1.1)). Other steady state problems with a similar type of localized solutions were also considered, most notably the stationary Cahn-Hilliard equation (see [5, 4, 29] for spike-like solutions, and [1, 8] for transition layers enclosing a small area) and the Gray-Scott system [27].

In this paper we consider the problem of evolution of single-spike solutions to (1.1). We should mention that formal analysis of the dynamics of the spike in (1.1) was carried out by Iron and Ward in [17], and our work here provides rigorous proof of some of their results. Also, we remark that we study only the dynamics of single-spike solutions since initial data with a multiple-spike profile do not seem to persist for a long time.

One obvious difference between the stationary problem associated with (1.1) and its evolution counterpart is that there is no a simple way of reducing the evolutionary problem for the shadow system to a single parabolic equation. Nevertheless, the idea that (1.1) is, in some sense, a perturbation of the parabolic version of (1.3) is the guiding principle in the present work. To make it work, we need to take τ small enough.

The main idea of our approach goes back to works of Carr and Pego [6, 7], and Fusco and Hale [11], who studied exponentially slow dynamics of the transition layers for the Allen-Cahn equation in one space dimension. Their method was further developed (the so-called *invariant manifold approach*) by Alikakos and Fusco [2, 3] for studying the circular fronts (bubbles) in solutions to the Cahn-Hilliard equation. In their work they carried out detailed analysis of the dynamics of such fronts by deriving a system of nonlinear ordinary differential equations governing the motion of the centers of the bubbles. In particular they showed that a bubble moves at an exponentially slow speed (in ε) toward the closest point on the boundary of Ω. In the present paper we establish an analogous result, though our system is completely different from the one analyzed by Alikakos and Fusco. The similarity of the qualitative behavior between the two systems can be explained by the presence of exponentially small eigenvalues both in the spectrum of the Cahn-Hilliard operator "linearized" around bubbles and the spectrum of a "linearized" (around single-spike solutions) operator associated with (1.1).

1.2. **Statement of the main result.** It is convenient to introduce a stretched variable $x = \varepsilon^{-1}y \in \Omega_\varepsilon \stackrel{\text{def}}{=} \{\varepsilon^{-1}y : y \in \Omega\}$. If we set $a(x,t) = A(\varepsilon x, t)$, $h(t) = H(t) - \varepsilon^{-N}\langle A^2\rangle_1$, then system (1.1) can be written as

$$\begin{cases} a_t = \Delta a - a + a^2/[\langle a^2\rangle_\varepsilon + h], \qquad x \in \Omega_\varepsilon,\ t > 0, \\ h_t = -h/\tau + 2\langle |\nabla a|^2 + a^2\rangle_\varepsilon - 2\langle a^3\rangle_\varepsilon/[\langle a^2\rangle_\varepsilon + h], \qquad t > 0, \\ \partial_n a = 0, \qquad x \in \partial\Omega_\varepsilon,\ t > 0, \end{cases} \tag{1.4}$$

where $\langle f\rangle_\varepsilon = \int_{\Omega_\varepsilon} f(x)\,dx$.

Let W be a positive, radially symmetric solution (ground state) of the equation

$$\Delta W - W + W^2 = 0 \quad \text{in } \mathbb{R}^N, \qquad \lim_{|x|\to\infty} W = 0.$$

It is known that W is unique. For each $\xi \in \mathbb{R}^N$, we define $V(\cdot,\xi)$ to be the solution to

$$-\Delta V + V = 0 \quad \text{in } \Omega_\varepsilon, \qquad \partial_n V = -\partial_n W(\cdot - \xi) \quad \text{on } \partial\Omega_\varepsilon.$$

We set

$$\begin{aligned} u(x,\xi) &\stackrel{\text{def}}{=} (W(x-\xi) + V(x,\xi))/\sigma(\xi), \\ \sigma(\xi) &\stackrel{\text{def}}{=} \langle\, (W(\cdot - \xi) + V(\cdot,\xi))^2\,\rangle_\varepsilon = 1/\langle u^2\rangle_\varepsilon, \end{aligned} \tag{1.5}$$

$$f_1(x,\xi) \stackrel{\text{def}}{=} \Delta u - u + u^2/\langle u^2\rangle_\varepsilon = \Delta u - u + \sigma u^2 = V[2W+V]/\sigma \ . \quad (1.6)$$

We let

$$\mathrm{d}_\varepsilon(\xi) \stackrel{\text{def}}{=} \mathrm{dist}(\xi, \partial\Omega_\varepsilon) \qquad \forall \xi \in \bar{\Omega}_\varepsilon,$$

and define the approximate manifold $\mathcal{M}$ by

$$\mathcal{M} \stackrel{\text{def}}{=} \{u(\cdot,\xi) : \xi \in \Omega_\varepsilon, \mathrm{d}_\varepsilon(\xi) > 2|\ln\varepsilon|\} \ . \quad (1.7)$$

Observe that $\mathcal{M}$ is a smooth, N-dimensional manifold in $L^2(\Omega_\varepsilon)$ consisting of spike-like functions with spikes located at interior points of Ω_ε. The function V added to the definition of u is extremely small and is used to make u satisfy the homogeneous Neumann boundary condition.

Our goal is to show that $\mathcal{M}$ is almost invariant; that is, if initially $\mathrm{dist}(a(\cdot,0),\mathcal{M}) + |h(0)|$ is small then $\mathrm{dist}(a(\cdot,t),\mathcal{M}) + |h(t)|$ remains small until the projection $u(\cdot,\xi(t))$ of $a(\cdot,t)$ on $\mathcal{M}$ is on $\partial\mathcal{M} := \{u(\cdot,\xi) : \xi \in \Omega_\varepsilon,\ \mathrm{dist}(\xi,\partial\Omega) = 2|\ln\varepsilon|\}$. Also, we would like to study the dynamics of the projection of (1.4) on $\mathcal{M}$, i.e., the dynamics of $\xi(t)$.

In the sequel we will work with the local coordinates near $\mathcal{M}$ and thus it is convenient to introduce $N_\xi\mathcal{M}$, the normal space of $\mathcal{M}$ at ξ; namely,

$$N_\xi\mathcal{M} = \{\phi \in H^1(\Omega) : \langle \phi u_{\xi_i}\rangle_\varepsilon = 0 \ \text{ for all } \ i\}. \quad (1.8)$$

We decompose the component a of the solution of (1.4) in the form

$$a(x,t) = u(x,\xi(t)) + \phi(x,t), \qquad \phi \in N_\xi\mathcal{M}. \quad (1.9)$$

It is easy to show that there is a $\delta > 0$ such that when $\mathrm{dist}(a,\mathcal{M}) \le \delta$, such decomposition exists and is unique. We can now state our first theorem.

Theorem 1.1. *There exist positive constants $\delta_0, \tau_0, \varepsilon_0$, and K_0 such that for every $\varepsilon \in (0,\varepsilon_0], \tau \in (0,\tau_0]$, and $\delta \in (\varepsilon^7, \delta_0]$, if the initial values $a(x,0) = u(x,\xi(0)) + \phi(x,0)$ and $h(0)$ satisfy*

$$\mathrm{d}_\varepsilon(\xi(0)) > 2|\ln\varepsilon|, \qquad \langle \phi^2(\cdot,0)\rangle_\varepsilon + h^2(0) < \delta, \quad (1.10)$$

then there is a $T^* \in (0,\infty) \cup \{\infty\}$ such that $(\mathrm{dist}(a(\cdot,t),\mathcal{M}))^2 + h^2(t) < \delta$ for all $t \in (0,T^*)$ and

$$\text{either (i) } T^* = \infty \text{ or (ii) } T^* < \infty \text{ and } \mathrm{d}_\varepsilon(\xi(T^*)) = 2|\ln\varepsilon|. \quad (1.11)$$

Furthermore, if in addition we assume that

$$\langle \phi^2(\cdot,0)\rangle_\varepsilon + h^2(0) \le K_0\, \mathrm{d}_\varepsilon(\xi(0))\, e^{-4\mathrm{d}_\varepsilon(\xi(0))}, \quad (1.12)$$

then for all $t \in (0,T^*)$,

$$\langle \phi^2(t)\rangle_\varepsilon + h^2(t) \ < \ K_0\, \mathrm{d}_\varepsilon(\xi(t))\, e^{-4\mathrm{d}_\varepsilon(\xi(t))}, \quad (1.13)$$

$$\begin{aligned}\dot{\xi}(t) &= \sigma^2\beta_0\langle f_1 u_\xi\rangle_\varepsilon + O(1)\ \mathrm{d}_\varepsilon(\xi)^{N+1}e^{-4\mathrm{d}_\varepsilon(\xi)} \qquad (1.14)\\ &= -\beta_0\nabla_\xi J(W(\cdot-\xi)) + O(\varepsilon)\ \|W(\cdot-\xi)\|^2_{L^2(\partial\Omega_\varepsilon)}\\ &= \beta_0\int_{\partial\Omega_\varepsilon}\Big\{n_{\partial\Omega_\varepsilon}(x)+O(\varepsilon)\Big\}\Big(|\nabla W(x-\xi)|^2+W^2(x-\xi)\Big)dS_x \qquad (1.15)\\ &= O(1)e^{-2\mathrm{d}_\varepsilon(\xi)}\end{aligned}$$

where f_1 is as in (1.6), $\beta_0 = N/\langle|\nabla W|^2\rangle$, $n_{\partial\Omega_\varepsilon}$ is the exterior normal to $\partial\Omega_\varepsilon$, and

$$J(w) \overset{\text{def}}{=} \int_{\Omega_\varepsilon}\Big(|\nabla w|^2 + w^2 - \tfrac{2}{3}w^3\Big)dx \qquad \forall\, w\in H^1(\Omega_\varepsilon). \qquad (1.16)$$

Throughout this paper, $O(1)$ represents a quantity that is bounded uniformly in ε, x and ξ; also, for all positive f, $O(f) = O(1)f$.

Remark 1.1. 1. The first half of the theorem states that a trajectory (solution of (1.4)) which initially is close to the manifold $\mathcal{M}\times\{0\}$ will stay close to this manifold, till the center of the spike, characterized by $\xi(t)$, is close to the boundary (cf. (1.11)).

2. In the original variable $y=\varepsilon x\in\Omega$, the center of the spike, $\eta(t)=\varepsilon\xi(t)$, moves with velocity $\dot\eta = \varepsilon\dot\xi(t) = O(1)\varepsilon e^{-2d(\eta)/\varepsilon}$ since $d(\eta) = \mathrm{d}_\varepsilon(\xi)/\varepsilon$; that is, the spike moves exponentially slowly, when it is away from the boundary $\partial\Omega$. Our analysis is valid until the spike is a distance of $2\varepsilon|\ln\varepsilon|$ away from $\partial\Omega$. Integrating the equation $\frac{d}{dt}e^{2d(\eta(t))/\varepsilon} = O(1)$, we see that $T^* > \frac{1}{O(1)}[e^{2d(\eta(0))/\varepsilon} - \varepsilon^{-4}]$.

3. A critical point of the "energy" $J(w)$ defined in (1.16) is a solution to $\Delta w - w + w^2 = 0$ in Ω_ε with boundary condition $\partial_n w = 0$ on $\partial\Omega_\varepsilon$, from which one can obtain an equilibrium of (1.4) or (1.1).

Since $W(x)$ decays exponentially fast as $|x|\to\infty$, the main contribution of the integrand to the integral in (1.15) comes from those regions on $\partial\Omega_\varepsilon$ which are the closest to ξ. When such regions are a singleton, it is possible to calculate the integral more precisely. For this purpose, we denote by $d(\cdot)$ the signed distance to $\partial\Omega$; i.e., $d(y) = \mathrm{dist}(y,\partial\Omega)$ if $y\in\bar\Omega$ and $d(y) = -\mathrm{dist}(y,\partial\Omega)$ if $y\notin\Omega$. Also, if d is differentiable at y,

$$Q(y) \overset{\text{def}}{=} \det\ \Big(\delta_{ij} - dd_{y_iy_j}\Big)_{N\times N} = \Pi_{i=1}^{N-1}[\,1-\ d(y)\ H_i(q(y))\,] \qquad (1.17)$$

where det represents the determinant of a matrix, $H_i(q), i=1,\dots,N-1$, are the principal curvatures of $\partial\Omega$ at q, and $q(y)$ is the projection of y to $\partial\Omega$ along the normal of $\partial\Omega$.

Theorem 1.2. *Assume that $\partial\Omega \in C^3$ and let $s > 0$ be any fixed constant. Then for every $\tau \in (0, \tau_0]$ and $\varepsilon \in (0, \varepsilon_0]$, if $(\phi(\cdot, 0), h(0))$ satisfies* (1.12) *and $d(\cdot)$ is differentiable in an s-neighborhood of $\eta = \varepsilon\xi(0)$, then the function $\xi(t)$ in Theorem* 1.1 *satisfies*

$$\begin{aligned}\dot{\xi}(t) &= \Big\{ - \beta_0 \nabla_\xi \mathrm{d}_\varepsilon(\xi) + O(1)\varepsilon \Big\} \int_{\partial\Omega_\varepsilon} \big(|\nabla W(\cdot - \xi)|^2 + W^2(\cdot - \xi)\big) dS \qquad (1.18)\\ &= \Big\{ - \nabla_\xi \mathrm{d}_\varepsilon(\xi) + O(1)\varepsilon \Big\} \Big\{ c_0\beta_0 + O(1)/\mathrm{d}_\varepsilon(\xi) \Big\}\, Q(\varepsilon\xi)^{-\frac{1}{2}} \mathrm{d}_\varepsilon(\xi)^{\frac{1-N}{2}} e^{-2\mathrm{d}_\varepsilon(\xi)}\end{aligned}$$

where $O(1)$ depends on s and c_0 is a positive constant depending only on the space dimension N.

Corollary 1.3. *There exists a positive constant M such that for every sufficiently small positive ε, the function $\xi(t)$, $t \in (0, T^*)$, in Theorem* 1.2 *lies in the cone $\mathcal{K}$ with vertex $\xi_0 := \xi(0)$, open angle $M\varepsilon$ and in the direction $-\nabla_\xi \mathrm{d}_\varepsilon(\xi_0)$; i.e.,*

$$\xi(t) \in \mathcal{K} \stackrel{\text{def}}{=} \{x \in \Omega_\varepsilon : (\xi_0 - \xi)\cdot\nabla_\xi \mathrm{d}_\varepsilon(\xi_0) > \cos(M\varepsilon)\, |\xi - \xi_0|\,\} \qquad \forall\, t \in (0, T^*).$$

Remark 1.2. The formula (1.18) states that a spike moves towards the boundary in a way which is the most efficient; namely, it moves towards the point on the boundary $\partial\Omega_\varepsilon$ which is the closest to ξ.

Our condition that $d(\cdot)$ is differentiable in an s-neighborhood of η implies that Ω contains a ball of radius $d(\eta) + s$ whose closure intersects $\partial\Omega$ at $q(\eta)$, the projection of η on $\partial\Omega$.

If there is more than one point on the boundary $\partial\Omega_\varepsilon$ that is the closest to ξ, then it is still possible to derive formulae analogous to (1.18) from (1.15). In such a case, one replaces $-\nabla_\xi \mathrm{d}_\varepsilon(\xi)$ by $\int_{\partial\Omega} \frac{y-\eta}{|y-\eta|}\, d\nu_\varepsilon^\eta$ where $\eta = \varepsilon\xi$ and ν_ε^η is a probability measure supported on $\{y \in \partial\Omega : |y - \eta| < d(\eta) + N\varepsilon|\ln\varepsilon|\}$. In particular, if the set of the closest points, $\{y \in \partial\Omega : |y - \eta| = d(\eta)\}$, are finite, then for ε sufficiently small, ν_ε^η can be replaced by a summation of Dirac measures supported on these points, i.e., by the measure

$$\sum_{y\in\partial\Omega, |y-\eta|=d(\eta)} \frac{\Pi_{i=1}^{N-1}[1 - d(\eta)H_i(y)]^{-1/2}}{S}\delta(\cdot - y),$$

$$S = \sum_{y\in\partial\Omega, |y-\eta|=d(\eta)} \Pi_{i=1}^{N-1}\Big[1 - d(\eta)H_i(y)\Big]^{-1/2}.$$

We should add here that since there are interior stationary spikes (see [15, 18, 26]), not all of the interior spikes will be removed from Ω by the dynamics of (1.4). In fact, if D is a smooth subdomain of Ω satisfying $\nabla d(y) \cdot n_D(y) < 0$

on ∂D (n_D the exterior normal to ∂D), then by a geometrical argument, for any sufficiently small positive ε, there is an $\eta \in D$, such that if one starts with $a(\cdot, 0) = u(\cdot, \varepsilon\eta)$ and $h(0) = 0$, then $T^* = \infty$ and $\varepsilon\xi(t) \in D$ for all $t > 0$. In principle the information provided in the above theorems could be utilized to show the existence of a true invariant manifold for (1.1) near $\mathcal{M}$; however, we shall not pursue this here.

There are two ingredients in the proof of Theorem 1.1. One is a quite standard procedure, i.e., the geometric invariant manifold approach [2, 3], and the other involves detailed analysis about the manifold $\mathcal{M}$ explicitly constructed in (1.7). In the next section we shall present the manifold approach to prove our main results, whereas properties of $\mathcal{M}$ that are needed for the manifold approach will only be stated there. Proofs of these properties are given in the subsequent sections.

2. The invariant manifold approach

2.1. **Projection of the trajectories on $\mathcal{M}$.** Decomposing a as in (1.9), we can write (1.4) as follows:

$$\begin{cases} \dot{\xi} \cdot \nabla_\xi u + \phi_t = f_1(x, \xi(t)) + L(\phi) - \sigma^2 u^2 h + \mathcal{N}_1(\phi, h), \\ h_t = f_2(\xi) + [2\sigma^2 \langle u^3 \rangle_\varepsilon - 1/\tau] h + \langle g(u)\phi \rangle_\varepsilon + \mathcal{N}_2(\phi, h) \end{cases} \tag{2.1}$$

where f_1 is as in (1.6),

$$\begin{aligned} L(\phi) &\stackrel{\text{def}}{=} \Delta\phi - \phi + 2\sigma u\phi - 2\sigma^2 u^2 \langle u\phi \rangle_\varepsilon, \\ \mathcal{N}_1(\phi, h) &\stackrel{\text{def}}{=} \frac{(u+\phi)^2}{\langle (u+\phi)^2 \rangle_\varepsilon + h} - \sigma u^2 - 2\sigma u\phi + \sigma^2 u^2 (2\langle u\phi \rangle_\varepsilon + h) \ , \\ f_2(\xi) &\stackrel{\text{def}}{=} 2\langle |\nabla u|^2 + u^2 \rangle_\varepsilon - 2\sigma \langle u^3 \rangle_\varepsilon = -2\langle u f_1 \rangle_\varepsilon \text{ (by (1.6))} \ , \\ g(u) &\stackrel{\text{def}}{=} -4f_1 - 2\sigma u^2 + 4\sigma^2 \langle u^3 \rangle_\varepsilon u \ , \\ \mathcal{N}_2(\phi, h) &\stackrel{\text{def}}{=} 2\langle |\nabla\phi|^2 + \phi^2 \rangle_\varepsilon - \frac{2\langle (u+\phi)^3 \rangle_\varepsilon}{\langle (u+\phi)^2 \rangle_\varepsilon + h} + 2\sigma \langle u^3 \rangle_\varepsilon + 6\sigma \langle u^2\phi \rangle_\varepsilon \\ &\quad - 2\sigma^2 \langle u^3 \rangle_\varepsilon (2\langle u\phi \rangle_\varepsilon + h) \ . \end{aligned}$$

To proceed with the analysis of the dynamics of (2.1), we first list a few properties of u, f_1 and L.

2.2. **Certain properties of u, f_1 and L.**

Lemma 2.1. *There exists a positive constant C depending only on Ω such that for every $\varepsilon \in (0, 1]$ and every $\xi \in \Omega_\varepsilon$, the following estimates hold:*

(1) $0 \le \sigma, 1/\sigma, u, \langle u\rangle_\varepsilon, \langle\, |\nabla_\xi u|^2\,\rangle_\varepsilon, \langle\, |D_{\xi\xi} u|^2\,\rangle_\varepsilon \le C$;
(2) $\langle f_1^2\rangle_\varepsilon, \langle|\nabla_\xi f_1|^2\rangle_\varepsilon \le\, C\,[1+\mathrm{d}_\varepsilon(\xi)]\, e^{-4\mathrm{d}_\varepsilon(\xi)}$;
(3) $|\langle u\nabla_\xi u\rangle_\varepsilon|, |\langle u^2\nabla_\xi u\rangle_\varepsilon| \le\, C\,[1+\mathrm{d}_\varepsilon(\xi)]^{(N+1)/2}\, e^{-2\mathrm{d}_\varepsilon(\xi)}$;
(4) *For every* $i, j = 1, \dots, N$,

$$|\langle u_{\xi_i} u_{\xi_j}\rangle_\varepsilon - \beta_0^{-1}\sigma^{-2}\delta_{ij}| \le\, C\,[1+\mathrm{d}_\varepsilon(\xi)]^{(N+1)/2} e^{-2\mathrm{d}_\varepsilon(\xi)}.$$

Lemma 2.2. *Let* $J(w)$ *be defined as in* (1.16). *Then for every* $\varepsilon \in (0,1]$ *and* $\xi \in \partial\Omega_\varepsilon$,

$$\begin{aligned}
&\sigma^2\langle f_1 \nabla_\xi u\rangle_\varepsilon \\
&= -\nabla_\xi J(W(\cdot-\xi)) + O(1)\Big\{\varepsilon + [1+\mathrm{d}_\varepsilon(\xi)]^{N/2} e^{-\mathrm{d}_\varepsilon(\xi)}\Big\}\, \|W\|^2_{L^2(\partial\Omega_\varepsilon)} \quad (2.2)\\
&= \int_{\partial\Omega_\varepsilon} \Big\{ n_{\partial\Omega_\varepsilon} + O(\varepsilon) + O(1)[1+\mathrm{d}_\varepsilon(\xi)]^{N/2} e^{-\mathrm{d}_\varepsilon(\xi)}\Big\}\big(|\nabla W|^2 + W^2\big) \quad (2.3)\\
&= O(e^{-2\mathrm{d}_\varepsilon(\xi)}). \quad (2.4)
\end{aligned}$$

Lemma 2.3. *Let* $s > 0$ *be any fixed constant. Then for every sufficiently small positive* ε *(depending on* s*), if* $\xi \in \Omega_\varepsilon$ *satisfies* $\mathrm{d}_\varepsilon(\xi) > 2|\ln\varepsilon|$ *and the distance function* $d(y) = \mathrm{dist}(y, \partial\Omega)$ *is differentiable in an* s*-neighborhood of* $\eta = \varepsilon\xi$, *then*

$$\begin{aligned}
&\int_{\partial\Omega_\varepsilon} n_{\partial\Omega_\varepsilon}\big(|\nabla W|^2 + W^2\big) = [-\nabla_\xi \mathrm{d}_\varepsilon(\xi) + O(\varepsilon)] \int_{\partial\Omega_\varepsilon} \big(|\nabla W|^2 + W^2\big) \quad (2.5)\\
&= [-\nabla_\xi \mathrm{d}_\varepsilon(\xi) + O(\varepsilon)][c_0 + O(1/\mathrm{d}_\varepsilon(\xi))] Q(\varepsilon\xi)^{-\frac{1}{2}} \mathrm{d}_\varepsilon(\xi)^{(1-N)/2} e^{-2\mathrm{d}_\varepsilon(\xi)}, \quad (2.6)
\end{aligned}$$

where $O(1)$ *depends on* s *but not on* ε *and* ξ, $Q(y)$ *is as in* (1.17), *and* c_0 *is a positive constant depending only on the space dimension* N.

Lemma 2.4. *Let* $\xi \in \Omega_\varepsilon$ *be such that* $\mathrm{d}_\varepsilon(\xi) \ge 2|\ln\varepsilon|$. *Let* $u = u(\cdot, \xi)$ *be defined as in* (1.5) *and* $\mathcal{L}(\phi, \psi)$ *be a bilinear form defined as*

$$\mathcal{L}(\phi,\psi) \stackrel{\text{def}}{=} \langle -\nabla\phi\cdot\nabla\psi - \phi\psi + 2\sigma u\phi\psi\rangle_\varepsilon - 2\sigma^2\langle u\phi\rangle_\varepsilon\langle u^2\psi\rangle_\varepsilon. \quad (2.7)$$

Then there exists a positive constant ν *which is independent of* ε *such that for every sufficiently small positive* ε,

$$\mathcal{L}(\phi,\phi) \le -\nu\langle|\nabla\phi|^2 + \phi^2\rangle_\varepsilon \quad \forall\, \phi \in N_\xi\mathcal{M}. \quad (2.8)$$

We shall prove Lemmas 2.1–2.3 in the next section, and Lemma 2.4 in Section 4.

2.3. **Attraction property of $\mathcal{M}$.** Now we estimate the distance

$$\mathrm{dist}(a(\cdot,t),\mathcal{M}) = \sqrt{\langle \phi^2(\cdot,t)\rangle_\varepsilon}.$$

For this purpose, we define an "excess energy" $E(t)$ by

$$E(t) = \langle \phi^2(\cdot,t)\rangle_\varepsilon + h^2(t). \tag{2.9}$$

Lemma 2.5. *There exist positive constants ε_0, δ_0, τ_0, and K such that for every $\tau \in (0,\tau_0]$ and $\varepsilon \in (0,\varepsilon_0]$, if for some $t \geq 0$,*

$$E(t) \leq \delta_0, \quad \mathrm{d}_\varepsilon(\xi(t)) \geq 2|\ln\varepsilon|, \tag{2.10}$$

then

$$\frac{d}{dt}E(t) \leq -\frac{\nu}{4}E(t) + K[1+\mathrm{d}_\varepsilon(\xi)]\, e^{-4\mathrm{d}_\varepsilon(\xi)} \tag{2.11}$$

where ν is as in Lemma 2.4.

Remark 2.1. 1. The nonhomogeneous term $K[1+\mathrm{d}_\varepsilon(\xi)]e^{-4\mathrm{d}_\varepsilon(\xi)}$ in (2.11) originates from $f_1 = \Delta u - u - u^2/\langle u^2\rangle_\varepsilon$, an error term which measures how far away the function $u(\cdot,\xi)$ is from an equilibrium.

2. The two key arguments in the proof of Lemma 2.5 are the following: (i) the eigenvalue estimate (2.8) for the operator L (which is not self-adjoint), and (ii) the smallness of τ.

3. Note that (2.11) implies that if

$$\delta \in \Big(\frac{4K}{\nu}\max_{\mathrm{d}_\varepsilon(\xi)>2|\ln\varepsilon|} K[1+\mathrm{d}_\varepsilon(\xi)]e^{-4\mathrm{d}_\varepsilon(\xi)}\ ,\ \delta_0\Big],$$

then any trajectory $(a(\cdot,t),h(t))$ of (1.4) cannot exit from the lateral boundary $\{\mathrm{dist}(a,\mathcal{M})^2 + h^2 = \delta\}$ of the tube $\{(u+\phi,h) : u \in \mathcal{M}, \langle\phi^2\rangle_\varepsilon + h^2 < \delta\}$ since the condition $E(t) \leq \delta_0$ will be then guaranteed by (2.11).

Proof. Integrating over Ω_ε the first equation in (2.1) multiplied by ϕ, and adding the second equation multiplied by h, we obtain, since $\phi \perp u_{\xi_i}$ for each i and $\partial_n\phi = 0$ on $\partial\Omega_\varepsilon$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E(t) &= [\mathcal{L}(\phi,\phi) + (2\sigma^2\langle u^3\rangle_\varepsilon - 1/\tau)h^2] \\ &\quad + [\langle f_1\phi\rangle_\varepsilon - 2\langle f_1 u\rangle_\varepsilon h - \sigma^2 h\langle u^2\phi\rangle_\varepsilon + \langle g(u)\phi\rangle_\varepsilon h] + (\langle\mathcal{N}_1\phi\rangle_\varepsilon + \mathcal{N}_2 h).\end{aligned} \tag{2.12}$$

To estimate the right-hand side we notice that the first term in the brackets can be estimated from above by $-\mu E(t)$ (see Remark 2.1). Our goal is to estimate the remaining terms in (2.12) from above by expressions involving $E(t)$ and known functions which are exponentially small in ε.

By Cauchy's inequality and the estimates for $\langle u^2\rangle_\varepsilon$ in Lemma 2.1,

$$\langle f_1\phi\rangle_\varepsilon - 2\langle f_1 u\rangle_\varepsilon h \le \Big(\frac{1}{\nu}+\frac{4\langle u^2\rangle_\varepsilon}{\nu}\Big)\langle f_1^2\rangle_\varepsilon + \frac{\nu}{4}(\langle\phi^2\rangle + h^2) \le K\langle f_1^2\rangle_\varepsilon + \frac{\nu}{4}(\langle\phi^2\rangle_\varepsilon + h^2).$$

Similarly,

$$-\sigma^2\langle u^2\phi\rangle_\varepsilon h + \langle g(u)\phi\rangle_\varepsilon h \le \frac{\nu}{4}\langle\phi^2\rangle_\varepsilon + K_2 h^2.$$

It remains to estimate $\langle\mathcal{N}_1\phi\rangle_\varepsilon + \mathcal{N}_2 h$. Although these terms grow more than quadratically we shall show that they are bounded by

$$K_4\sqrt{E(t)}(\langle|\nabla\phi|^2+\phi^2\rangle_\varepsilon + h^2),$$

where K_4 is a positive constant. Since $1/C \le \langle u^2\rangle_\varepsilon \le C$, we can take δ_0 small so that $1/(2C) < \langle(u+\phi)^2\rangle_\varepsilon + h^2 \le C+1$. As $\sigma = 1/\langle u^2\rangle_\varepsilon$, it follows that $\mathcal{N}_1(\phi,h)$ is bounded from above by a quadratic function in ϕ and h with bounded coefficients. Similarly $\mathcal{N}_2(\phi,h)$ is bounded from above by a linear function in $\langle\phi^3\rangle_\varepsilon, \langle|\nabla\phi|^2\rangle_\varepsilon$ and h^2 with bounded coefficients. Also, since $N \le 3$, we have the Sobolev embedding

$$\langle|\psi|^3\rangle_\varepsilon \le K_3\sqrt{\langle\psi^2\rangle_\varepsilon}\langle|\nabla\psi|^2+\psi^2\rangle_\varepsilon, \qquad \forall\,\psi\in H^1(\Omega_\varepsilon)$$

where K_3 is independent of ε. Hence

$$|\langle\mathcal{N}_1(\phi,h)\phi\rangle_\varepsilon + \mathcal{N}_2(\phi,h)h| \le K_4\sqrt{\langle\phi^2\rangle_\varepsilon + h^2}\left(\langle|\nabla\phi|^2+\phi^2\rangle_\varepsilon + h^2\right).$$

Finally, using Lemma 2.4 to estimate $\mathcal{L}(\phi,\phi)$ we obtain from (2.12) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}E(t)\\ &\le -\Big[\frac{\nu}{2} - K_4\sqrt{E(t)}\Big]\langle|\nabla\phi|^2+\phi^2\rangle_\varepsilon - \Big[\frac{1}{\tau} - K_4\sqrt{E(t)} - K_5\Big]h^2 + K\langle f_1^2\rangle_\varepsilon\ .\end{aligned}$$

Taking τ_0 and δ_0 small enough (but independent of ε) we then obtain the assertion of the lemma. □

2.4. **The flow projected on $\mathcal{M}$.** Now we will study the dynamics of $\xi(t)$. Integrating the first equation in (2.1) multiplied by u_{ξ_j} and using the relation

$$\langle\phi_t u_{\xi_j}\rangle_\varepsilon = \frac{d}{dt}\langle\phi u_{\xi_j}\rangle_\varepsilon - \sum_{i=1}^N\langle\phi u_{\xi_j\xi_i}\rangle_\varepsilon\dot\xi_i = -\sum_{i=1}^N\langle\phi u_{\xi_i\xi_j}\rangle_\varepsilon\dot\xi_i,$$

we obtain, for every $j=1,\dots,N$,

$$\sum_{i=1}^N\langle u_{\xi_i}u_{\xi_j} - \phi u_{\xi_i\xi_j}\rangle_\varepsilon\ \dot\xi_i\ =\ \langle f_1 u_{\xi_j}\rangle_\varepsilon + \mathcal{L}(\phi,u_{\xi_j}) - \sigma^2 h\langle u^2 u_{\xi_j}\rangle_\varepsilon + \langle u_{\xi_j}\mathcal{N}_1\rangle_\varepsilon. \tag{2.13}$$

We now simplify this equation. First we estimate the right-hand side. As $\mathcal{N}_1$ is quadratic in ϕ and h,

$$|\langle u_{\xi_j}\mathcal{N}_1\rangle_\varepsilon| \le C\langle|\mathcal{N}_1|\rangle_\varepsilon \le CE(t). \tag{2.14}$$

To estimate $\mathcal{L}(\phi, u_{\xi_j})$ we note that, as $\partial_n u_{\xi_j} = 0$ on $\partial\Omega_\varepsilon$,

$$\mathcal{L}(\phi, u_{\xi_j}) = \langle\ \phi L(u_{\xi_j})\ \rangle_\varepsilon - 2\sigma^2\langle u\phi\rangle_\varepsilon\ \langle u^2 u_{\xi_j}\rangle_\varepsilon + 2\sigma^2\langle uu_{\xi_j}\rangle_\varepsilon\ \langle u^2\phi\rangle_\varepsilon.$$

Differentiating (1.6) with respect to ξ_j we obtain $L(u_{\xi_j}) = f_{1\xi_j}$. Using Lemma 2.1 to estimate $\langle|\nabla_\xi f_1|^2\rangle_\varepsilon$, $\langle uu_{\xi_i}\rangle_\varepsilon$, and $\langle u^2 u_{\xi_j}\rangle_\varepsilon$ we conclude that the last three terms on the right-hand side of (2.13) satisfy

$$\mathcal{L}(\phi, u_{\xi_j}) - \sigma^2 h\langle u^2 u_{\xi_j}\rangle_\varepsilon + \langle u_{\xi_j}\mathcal{N}_1\rangle_\varepsilon = O(1)\big\{[1+\mathrm{d}_\varepsilon(\xi)]^{N+1}e^{-4\mathrm{d}_\varepsilon(\xi)} + E(t)\big\}. \tag{2.15}$$

From the fact that $|\langle\phi u_{\xi_i\xi_j}\rangle_\varepsilon| \le \big(\langle u^2_{\xi_i\xi_j}\rangle_\varepsilon\langle\phi^2\rangle_\varepsilon\big)^{1/2}$ and the estimates for $\langle u_{\xi_i}u_{\xi_j}\rangle_\varepsilon$ and $\langle -u^2_{\xi_i\xi_j}\rangle_\varepsilon$ in Lemma 2.1 we get

$$\langle u_{\xi_i}u_{\xi_j} - \phi u_{\xi_i\xi_j}\rangle_\varepsilon = \sigma^{-2}\beta_0^{-1}\delta_{ij} + O(1)\big\{[1+\mathrm{d}_\varepsilon(\xi)]^{(N+1)/2}e^{-2\mathrm{d}_\varepsilon(\xi)} + \sqrt{E(t)}\big\}. \tag{2.16}$$

Hence, multiplying both sides of (2.13) by the inverse of the matrix $\big(\langle u_{\xi_i\xi_j} + \phi u_{\xi_i\xi_j}\rangle_\varepsilon\big)_{N\times N}$, using estimates (2.15), (2.16), and

$$|\langle\ f_1\nabla_\xi u\ \rangle_\varepsilon| \le C\langle f_1^2\rangle_\varepsilon^{1/2} \le C[1+\mathrm{d}_\varepsilon(\xi)]^{1/2}\ e^{-2\mathrm{d}_\varepsilon(\xi)},$$

we then obtain the following:

Lemma 2.6. *There exists a positive constant δ_0 such that if $E(t) < \delta_0$, then*

$$\dot\xi(t) = \sigma^2\beta_0\langle f_1\nabla_\xi u\rangle_\varepsilon + O(1)\Big\{[1+\mathrm{d}_\varepsilon(\xi)]^{N+1}\ e^{-4\mathrm{d}_\varepsilon(\xi)} + E(t)\Big\}\ . \tag{2.17}$$

2.5. **Proofs of the main results. Proof of Theorem 1.1.** Let δ_0 be chosen so that Lemma 2.5 and Lemma 2.6 hold true. We fix $\delta \in (\varepsilon^7, \delta_0]$ and assume that (1.10) is satisfied. Define

$$T^* = \sup\{T > 0 : E(t) < \delta, \mathrm{d}_\varepsilon(\xi(t)) > 2|\ln\varepsilon| \ \text{ for all }\ t \in [0,T]\}.$$

If $T^* < \infty$, then by continuity, either $\mathrm{d}_\varepsilon(\xi(T^*)) = 2|\ln\varepsilon|$ or $E(T^*) = \delta$. We now rule out the possibility $E(T^*) = \delta$. In fact, if $E(T^*) = \delta$, then $\dot E(T^*) \ge 0$, but this is impossible since from (2.11) we have

$$\frac{d}{dt}E(t) \le -\nu\delta/4 + K[1+\mathrm{d}_\varepsilon(\xi)]e^{-4\mathrm{d}_\varepsilon(\xi)} \le -\nu\varepsilon^7/4 + K(1+2|\ln\varepsilon|)\varepsilon^8 < 0$$

if we take ε_0 sufficiently small. Hence, (1.11) holds.

Now assume that (1.12) is satisfied. Then (1.10) holds with $\delta = \varepsilon^3$, so that $E(t) < \varepsilon^3$ in $[0, T^*)$. Consequently, by (2.17), $|\dot\xi| \le \varepsilon^2$.

Consider the function $\tilde{E}(t) = E(t) - K_0 \mathrm{d}_\varepsilon(\xi(t)) e^{-4\mathrm{d}_\varepsilon(\xi(t))}$. We can calculate

$$\begin{aligned} \frac{d}{dt}\tilde{E}(t) &= \dot{E}(t) - K_0[1 - 4\mathrm{d}_\varepsilon(\xi)]e^{-4\mathrm{d}_\varepsilon(\xi)}\nabla_\xi d(\xi)\cdot\dot{\xi} \\ &\le -\frac{\nu}{4}E(t) + [2K + 4\varepsilon^2 K_0]\,\mathrm{d}_\varepsilon(\xi)\,e^{-4\mathrm{d}_\varepsilon(\xi)} < -\frac{\nu}{4}\tilde{E}(t), \quad \forall t\in[0,T^*) \end{aligned}$$

if we take K_0 large and ε_0 small such that $\frac{\nu}{4}K_0 > (2K + 4K_0\varepsilon_0^2)$. Since initially $\tilde{E} = 0$, we must have $\tilde{E} < 0$ in $(0, T^*)$. That proves (1.13). Consequently, by Lemma 2.6, (1.14) holds. The remaining assertion of Theorem 1.1 then follows from Lemma 2.2. □

Proof of Theorem 1.2. The proof of Theorem 1.2 follows from Theorem 1.1 and Lemma 2.3.

Proof of Corollary 1.3. In order to compare small quantities, it is convenient to work in the original domain Ω via the transformation $y = \varepsilon x$ and $\eta(t) = \varepsilon\xi(t)$. Then $\dot{\eta} = |\dot{\eta}|\,[-d_y(\eta) + O(\varepsilon)\vec{\tau}]/|-d_y(\eta) + O(\varepsilon)\vec{\tau}|$ where $|\dot{\eta}| > 0$ and $\vec{\tau}\perp d_y(\eta)$. Consequently, $\dot{d}(\eta(t)) = d_y(\eta)\cdot\dot{\eta} = [-1 + O(\varepsilon)]|\dot{\eta}| < 0$. Thus, $d(\eta(t))$ is strictly decreasing. Without loss of generality, we can assume that $\eta(0) = 0$ and $d_y(0) = -\vec{e}_N$.

Let $\theta = M\varepsilon$ where M is an ε-independent, large constant to be determined. Consider the function $w(t) := \eta(t)\cdot\vec{e}_N - |\eta(t)|\cos\theta$. For sufficiently small positive t, $\eta(t) = [|\dot{\eta}(0)|\;t + o(t^2)][\vec{e}_N + O(\varepsilon)\vec{\tau}]/|\vec{e}_N + O(\varepsilon)\vec{\tau}|$ so that $w(t) = [|\dot{\eta}(0)|\;t + o(t^2)]\{[1 + O(\varepsilon^2)]^{-1/2} - \cos\theta\} > 0$ if $\theta = M\varepsilon$ with M large enough.

We now show that $w > 0$ in $(0, T^*)$. Suppose this is not true. Then there is a first $t^*\in(0,T^*)$ such that $w(t^*) = 0$. Then at $t = t^*$, $\eta = |\eta|(\cos\theta\,\vec{e}_N + \sin\theta\,\vec{\tau})$ for some unit vector $\vec{\tau}\perp\vec{e}_N$. Hence

$$\begin{aligned} \dot{w}(t^*) &= \dot{\eta}\cdot[\vec{e}_N - \eta|\eta|^{-1}\cos\theta\,\vec{\tau}] = \dot{\eta}\cdot[(1-\cos^2\theta)\,\vec{e}_N - \cos\theta\sin\theta\,\vec{\tau}\,] \\ &= \sin\theta\,|\dot{\eta}|\,[-d_y(\eta) + O(\varepsilon)]\cdot[\sin\theta\,\vec{e}_N - \cos\theta\,\vec{\tau}] \\ &= \sin\theta\,|\dot{\eta}|\,\Big\{O(\varepsilon) + \sin\theta - [d_y(\eta) + \vec{e}_N]\cdot[\sin\theta\,\vec{e}_N - \cos\theta\vec{\tau}]\Big\}. \end{aligned}$$

Since $d_y(0) = -\vec{e}_N$, we have $d(a\,\vec{e}_N) = d(0) - a$ and $d_y(a\,\vec{e}_N) = -\vec{e}_N$ for all $a\in(-s, d(0)]$. Hence, setting $\hat{\eta} = |\eta|\cos\theta\,\vec{e}_N$ we have

$$\begin{aligned} d_y(\eta) + \vec{e}_N &= d_y(\eta) - d_y(\hat{\eta}) = d_{yy}(\hat{\eta})(\eta - \hat{\eta}) + O(1)\;|\eta - \hat{\eta}|^2 \\ &= |\eta|\sin\theta\; d_{yy}(\hat{\eta})\cdot\vec{\tau} + O(1)|\eta|^2\sin^2\theta. \end{aligned} \tag{2.18}$$

Consequently, since $\vec{e}_N \cdot d_{yy}(\hat{\eta}) = -d_y \cdot d_{yy}|_{y=\hat{\eta}} = \frac{1}{2}(|d_y|^2)_y|_{y=\hat{\eta}} = 0$, we have

$$\begin{aligned}\dot{w}(t^*) &= \sin\theta\,|\dot{\eta}|\big\{O(\varepsilon) + \sin\theta + |\eta|\sin\theta\cos\theta\vec{\tau}\cdot d_{yy}(\hat{\eta})\cdot\vec{\tau} + O(1)|\eta|^2\sin^2\theta\big\}\\ &= \sin^2\theta|\dot{\eta}|\big\{O(\varepsilon)/\sin\theta + \vec{\tau}\cdot(\mathrm{I} + |\hat{\eta}|\ d_{yy}(\hat{\eta}))\cdot\vec{\tau} + O(1)|\eta|^2\sin\theta\big\}\end{aligned}$$

where I is the identity matrix. Now let $q = d(0)\vec{e}_N \in \partial\Omega$ be the point on $\partial\Omega$ that is closest to 0. Rotating Ω if necessary we can always achieve that $y_1, \dots, y_{N-1}$ are the principal directions of $\partial\Omega$ with principal curvatures $H_1(q), \dots, H_{N-1}(q)$ at q. Then $d_{y_iy_j}(\hat{\eta}) = -\delta_{ij}H_i/[1 - d(\hat{\eta})H_i]$, so that

$$\vec{\tau}\cdot\Big(\mathrm{I} + |\hat{\eta}|\ d_{yy}(\hat{\eta})\Big)\cdot\vec{\tau} = \sum_{i=1}^{N-1}\frac{1-(d(\hat{\eta})+|\hat{\eta}|)H_i(q)}{1-d(\hat{\eta})H_i(q)}\tau_i^2 = \sum_{i=1}^{N-1}\frac{1-d(0)H_i(q)}{1-d(\hat{\eta})H_i(q)}\tau_i^2.$$

Recall that $d(\cdot)$ is differentiable in an s-neighborhood of the origin. It implies that Ω contains the ball of radius $d(0)+s$ with center $-s\vec{e}_N$. Since near q, $\partial\Omega$ has a local representation $y_N = d(0) - \Sigma_{i=1}^{N-1}H_i(q)y_i^2 + O(|y'|^3)$, it then follows that $1-(d(0)+s)H_i(q) \geq 0$ for all $i = 1, \dots, N_1$. Consequently,

$$1 - \ell H_i(q) \geq \frac{s}{d(0)+s} \geq \frac{2s}{\text{diameter }\Omega} \qquad \forall\ell\in[0, d(0)]. \tag{2.19}$$

In particular, the quantity $O(1)$ in (2.18), which involves the third-order derivatives of $d(\cdot)$, is bounded by a quantity depending only on s and the C^3 norm of $\partial\Omega$. Therefore for some positive constant $c(s)$, $\vec{\tau}\cdot(\mathrm{I}-|\hat{\eta}|\ d_{yy}(\hat{\eta}))\cdot\vec{\tau} \geq c(s)$, and $\dot{w}(t^*) \geq \sin^2\theta\ |\dot{\eta}|\ [c(s) - O(\varepsilon)/\sin\theta - O(1)|\eta|^2\sin\theta] > 0$ for every sufficiently small positive ε, provided that we take $\theta = M\varepsilon$ and $M = M(s)$ large enough. However, t^* is the first time that w vanishes, so we should have $\dot{w}(t^*) \leq 0$. Hence, we get a contradiction. Consequently $w > 0$ in $(0, T^*)$; i.e., $\eta(t)\cdot\vec{e}_N > \cos(M\varepsilon)\ |\eta(t)|$ for all $t\in(0, T^*]$. Thus, for all $t\in(0,T^*)$, $\xi(t) = \eta(t)/\varepsilon$ is in the cone $\mathcal{K}$ with vertex ξ_0, opening angle $\theta = M\varepsilon$, and direction $-\nabla_\xi \mathrm{d}_\varepsilon(\xi_0)$. This completes the proof. □

Remark 2.2. *Observe that using $\dot{\eta} = |\dot{\eta}|\ [-d_y(\eta)+O(\varepsilon)\vec{\tau}]/|-d_y(\eta)+O(\varepsilon)\vec{\tau}|$ and rescaling time we obtain $\eta' = -d_y(\eta) + O(\varepsilon)\vec{\tau}$. Since η, as a solution to an ODE, depends continously on $O(\varepsilon)\vec{\tau}$ we infer that η is $o(1)$-close to the solution of $\eta' = -d_y(\eta)$. Clearly the statement of Corollary 1.3 gives a more precise description of the trajectories $\xi(t) = \varepsilon^{-1}\eta(t)$.*

3. Properties of $\mathcal{M}$

This section is devoted to the proofs of Lemmas 2.1–2.3.

3.1. **Basic facts.** Let $J(r)$ and $K(r)$ be the modified Bessel functions defined by

$$J'' + \tfrac{N-1}{r}J' - J = 0 \text{ in } (0,\infty), \quad J(0)=1,\ J'(0)=0,$$

$$K'' + \tfrac{N-1}{r}K' - K = 0 \text{ in } (0,\infty), \quad K(\infty)=0,\ K(r) = \frac{r^{2-N}+O(r^{3-N})}{(N-2)\omega_{N-1}}$$

as $r \searrow 0$, where ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$ and when $N=2$, $\frac{1}{N-2}r^{2-N}$ should be replaced by $-\ln r$. One can show that there exist positive constants c_1 and c_2 such that

$$K(r) = c_1 r^{-(N-1)/2}[1+O(1/r)]e^{-r} \quad \text{and} \quad J(r) = c_2 r^{-(N-1)/2}[1+O(1/r)]e^{r} \tag{3.1}$$

as $r \to \infty$. These expansions are also valid for any derivatives of J and K.

Note that W^2 in the equation $-\Delta W + W = W^2$ can be neglected when $|x|$ becomes very large, so that for some constant w_0

$$W(x) = w_0\, K(|x|)[1+O(e^{-|x|})] \qquad \text{as } |x| \to \infty, \tag{3.2}$$

$$\nabla W(x) = w_0\, x\, |x|^{-1}\, K(|x|)\,[1+O(|x|^{-1})] \qquad \text{as } x \to \infty, \tag{3.3}$$

$$CW(x) \geq |D_x^3 W(x)|, |D_x^2 W(x)|, |\nabla W(x)| \qquad \forall x \in \mathbb{R}^N. \tag{3.4}$$

3.2. **Estimates for V.**

Lemma 3.1. *There exists a positive constant C such that for every $\varepsilon \in (0,1]$ and every $\xi \in \Omega_\varepsilon$,*

$$|V(x,\xi)|, |\nabla_\xi V(x,\xi)|, |D_\xi^2 V| \leq C\Big(\frac{1+\mathrm{d}_\varepsilon(x)}{1+\mathrm{d}_\varepsilon(\xi)}\Big)^{(N-1)/2} e^{-\mathrm{d}_\varepsilon(\xi)-\mathrm{d}_\varepsilon(x)}, \tag{3.5}$$

$$\langle V^{2m}(\cdot,\xi)\rangle_\varepsilon, \langle |\nabla_\xi V(\cdot,\xi)|^{2m}\rangle_\varepsilon \leq C[1+\mathrm{d}_\varepsilon(\xi)]^{-(m-1)(N-1)} e^{-2m\mathrm{d}_\varepsilon(\xi)} \tag{3.6}$$

for $m = 1, 2$.

Proof. We first estimate the L^∞ norm of V. Let M be the maximum of $|V|$ on $\bar{\Omega}_\varepsilon$ and $x_1 \in \partial\Omega_\varepsilon$ be a point such that $|V(x_1)| = M$. Without loss of generality, we assume that $V(x_1) = M$. By $n_{\partial\Omega_\varepsilon}(x_1)$ we denote the exterior normal to $\partial\Omega_\varepsilon$ at x_1. Take $x_0 = x_1 - R\, n_{\partial\Omega_\varepsilon}(x_1)$ where $R > 0$ is the largest number such that $B(x_0,R)$, the ball centered at x_0 with radius R, is contained in Ω_ε. Note that $R = \mathrm{d}_\varepsilon(x_0) = |x_0 - x_1|$. Consider the function $Z(x) = MJ(|x-x_0|)/J(R)$. We have $\Delta Z - Z = 0$ in Ω_ε and $Z \geq M$ on $\partial\Omega_\varepsilon$, so by comparison, $V \leq Z$ on $\bar{\Omega}_\varepsilon$. As $V = Z = M$ at x_1, there holds $|\partial_n V(x_1,\xi)| \geq |\partial_n Z(x_1)|$, which implies, since $\partial_n V(\cdot,\xi) = -\partial_n W(\cdot - \xi)$ on $\partial\Omega_\varepsilon$, that

$$M \leq \max_{\partial\Omega_\varepsilon} |\partial_n W(\cdot - \xi)|\ J(R)/J'(R).$$

Since $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$, we have $R \geq R_0/\varepsilon$ for some R_0 independent of ε; hence $J(R)/J'(R) = 1 + O(\varepsilon)$. In view of (3.1)–(3.3), we then obtain that

$$\max_{\bar{\Omega}_\varepsilon} |V(\cdot, \xi)| \leq C[1 + \mathrm{d}_\varepsilon(\xi)]^{-(N-1)/2} e^{-\mathrm{d}_\varepsilon(\xi)}.$$

Next, we estimate $V(x_0, \xi)$ for every $x_0 \in \bar{\Omega}_\varepsilon$. Consider the function $Z(x) = MJ(|x - x_0|)/J(\mathrm{d}_\varepsilon(x_0))$. We have $\Delta Z - Z = 0$ in Ω_ε and $Z \geq M \geq V$ on $\partial\Omega_\varepsilon$, so by the comparison principle, $V \leq Z$ in Ω_ε; a similar argument with $-Z$ in place of Z leads to $|V(x_0, \xi)| \leq M/J(\mathrm{d}_\varepsilon(x_0))$, $\forall x_0 \in \Omega_\varepsilon$. The estimate for $V(x, \xi)$ in (3.5) then follows by the asymptotic expansion of J in (3.1).

We proceed to prove (3.6) for V. Note that the volume of Ω_ε is $\varepsilon^{-N}|\Omega|$, so (3.6) is not a direct consequence of (3.5).

Integrating $V^{2m-1}(V - \Delta V) = 0$ over Ω_ε yields

$$\int_{\Omega_\varepsilon} \Big(\frac{2m-1}{m^2} |\nabla V^m|^2 + (V^m)^2 \Big) = -\int_{\partial\Omega_\varepsilon} V^{2m-1} \partial_n W$$
$$\leq \|V^m\|^{2-1/m}_{L^2(\partial\Omega_\varepsilon)} \|\partial_n W\|_{L^{2m}(\partial\Omega_\varepsilon)}.$$

Using the Sobolev embedding $\int_{\partial\Omega_\varepsilon} f^2 \leq C \int_{0<\mathrm{d}_\varepsilon(x)<1} (|\nabla f|^2 + f^2)$ where C is independent of ε, we then have

$$\int_{\Omega_\varepsilon} (|\nabla V^m|^2 + V^{2m}) \leq C \int_{\partial\Omega_\varepsilon} |\partial_n W(x - \xi)|^{2m} dS_x$$
$$\leq C[1 + \mathrm{d}_\varepsilon(\xi)]^{-(m-1)(N-1)}\ e^{-2m\mathrm{d}_\varepsilon(\xi)},$$

where the second inequality is obtained by dividing the domain of integration $\partial\Omega_\varepsilon$ into two regions, one of which satisfies $|x - \xi| \leq 2\mathrm{d}_\varepsilon(\xi)$ and has area $O(1)\mathrm{d}_\varepsilon(\xi)^{N-1}$, and the other $|x - \xi| \geq 2\mathrm{d}_\varepsilon(\xi)$. This proves (3.6) for V.

The estimate for $\nabla_\xi V(x, \xi)$ and $D^2_\xi V$ can be established in a similar manner since for example, $V_i := V_{\xi_i}$ satisfies $\Delta V_i = V_i$ and $\partial_n V_i = -\partial_n W_{\xi_i}(x - \xi)$ on $\partial\Omega_\varepsilon$, and $\sup_{\mathbb{R}^N}(|D^2 W| + |\nabla W|)/|W| \leq C$. □

Next we study the behavior of V on $\partial\Omega_\varepsilon$.

Lemma 3.2. *There exists a positive constant C such that for every $\varepsilon \in (0, 1]$ and every $\xi \in \Omega_\varepsilon$,*

$$\|V - W\|^2_{L^2(\partial\Omega_\varepsilon)} \leq C\varepsilon^2 \|W\|^2_{L^2(\partial\Omega_\varepsilon)} + C[1 + \mathrm{d}_\varepsilon(\xi)]^{1-N}\ e^{-4\mathrm{d}_\varepsilon(\xi)}.$$

Proof. Without loss of generality we assume $\xi = 0$. For any $z \in \mathbb{R}^N$, we denote by $\Gamma^z(x) = K(|x - z|)$ the fundamental solution to the operator

$-\Delta + I$. Then, when $z \in \bar{\Omega}_\varepsilon^c$,

$$W(z) = \int_{\Omega_\varepsilon^c} W(-\Delta\Gamma^z + \Gamma^z) = \int_{\Omega_\varepsilon^c} (-\Delta W + W)\Gamma^z + \int_{\partial\Omega_\varepsilon} (W\partial_n\Gamma^z - \Gamma^z\partial_n W),$$

$$0 = \int_{\Omega_\varepsilon} V(-\Delta\Gamma^z + \Gamma^z) = \int_{\partial\Omega_\varepsilon} (-V\partial_n\Gamma^z + \Gamma^z\partial_n V).$$

Taking the difference of the two equations and using the facts that $\partial_n V = -\partial_n W$ on $\partial\Omega_\varepsilon$ and $-\Delta W + W = W^2$ in $\mathbb{R}^N$, we obtain

$$W(z) = \int_{\partial\Omega_\varepsilon} (W+V)\partial_n\Gamma^z + \int_{\Omega_\varepsilon^c} W^2\Gamma^z \ .$$

For any fixed $x_0 \in \partial\Omega_\varepsilon$, letting $z \to x_0$ along the exterior normal of $\partial\Omega_\varepsilon$ at x_0, we get

$$W(x_0) = \tfrac{1}{2}(W(x_0) + V(x_0)) + \text{p.v.}\int_{\partial\Omega_\varepsilon} (W(x) + V(x))\partial_n\Gamma^{x_0}\, dS_x + \int_{\Omega_\varepsilon^c} W^2\Gamma^{x_0} \tag{3.7}$$

where p.v. represents the principal value. We now need the following well-known result:

Let D, $\tilde{D}$ be two regions and $Tf(y) := \int_D f(x)L(x,y)\,dx$, $y \in \tilde{D}$. Then

$$\|Tf\|^2_{L^2(D)} \le \Big(\sup_{x\in D} \|L(x,\cdot)\|_{L^1(\tilde{D})}\Big)\Big(\sup_{y\in\tilde{D}} \|L(\cdot,y)\|_{L^1(D)}\Big)\ \|f\|^2_{L^2(D)}.$$

Applying the above estimate to the right-hand side of (3.7) we then obtain

$$\|W - V\|^2_{L^2(\partial\Omega_\varepsilon)} \le 4c_\varepsilon\|W + V\|^2_{L^2(\partial\Omega_\varepsilon)} + C\|W\|^4_{L^4(\Omega_\varepsilon^c)}$$

where

$$c_\varepsilon = \sup_{x\in\partial\Omega_\varepsilon} \int_{\partial\Omega_\varepsilon} |\partial_{n_x} K(|x-z|)|\, dS_z \times \sup_{z\in\partial\Omega_\varepsilon} \int_{\partial\Omega_\varepsilon} |\partial_{n_x} K(|x-z|)|\, dS_x \ .$$

To estimate c_ε, we use the fact that $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$. Since $\partial\Omega$ is C^3, there exists a constant M such that $|(y_1 - y_2)\cdot n(y_1)| \le M|y_1 - y_2|^2$ for every $y_1, y_2 \in \partial\Omega$. Here $n(y_1)$ is the exterior normal to $\partial\Omega$ at y_1. In terms of $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$, we then have

$$|(x - z)\cdot n_{\partial\Omega_\varepsilon}(x)| \le \varepsilon M|x - z|^2, \qquad \forall\, x, z \in \partial\Omega_\varepsilon.$$

It then follows that

$$|\partial_{n_x}\Gamma^z(x)| = |K'(|x-z|)|\ |(x-z)\cdot n(x)|/|x-z| \le \varepsilon M r|K'(r)|_{|r=|x-z|}\,.$$

With this estimate it is easy to show that $c_\varepsilon = O(\varepsilon^2)$. Note that

$$\int_{\Omega_\varepsilon^c} W^4 \le C\int_{|x|>\mathrm{d}_\varepsilon(\xi)} [1+|x|]^{2(1-N)}e^{-4|x|} \le C[1 + \mathrm{d}_\varepsilon(\xi)]^{1-N}e^{-4\mathrm{d}_\varepsilon(\xi)}.$$

The assertion of the lemma thus follows. □

3.3. **Proof of Lemma 2.1.** The proof of Lemma 2.1 is done in several steps.

1. Set $\bar W = W + V$. Then $-\Delta \bar W + \bar W = W^2$ in Ω_ε and $\partial_n \bar W = 0$ on $\partial\Omega_\varepsilon$, so that $W^2(0) \geq \bar W > 0$ in $\bar\Omega$, $\langle \bar W\rangle_\varepsilon = \langle W^2\rangle_\varepsilon \leq \langle W^2\rangle$, where $\langle f\rangle = \int_{\mathbb{R}^N} f(x)dx$. It then follows that $\sigma = \langle \bar W^2\rangle_\varepsilon$ is bounded independently of ξ and ε.

To find a positive lower bound for σ, we note that $\bar W > W_1$ in $\Omega_\varepsilon \cap \{|x-\xi| < 1\}$ where W_1 is the solution to $-\Delta W_1 + W_1 = W^2$ in $\Omega_\varepsilon \cap \{|x-\xi| < 1\}$ and $W_1 = 0$ on $\partial(\ \Omega_\varepsilon \cap \{|x-\xi| < 1\}\ )$. It then follows that $\sigma = \langle \bar W^2\rangle_\varepsilon$ is positive, uniformly in $\xi \in \bar\Omega_\varepsilon$ and $\varepsilon \in (0,1]$. As $u = \bar W/\sigma$, the first assertion of Lemma 2.1 thus follows.

2. First we estimate $\langle f_1^2\rangle_\varepsilon$. By definition,

$$f_1 = \Delta u - u + \sigma u^2 = \sigma^{-1} V(2W + V).$$

From the L^∞ estimate for V,

$$|VW| \leq C\Big(\frac{1 + \mathrm{d}_\varepsilon(x)}{(1+\mathrm{d}_\varepsilon(\xi))(1 + |x-\xi|)}\Big)^{(N-1)/2} e^{-\mathrm{d}_\varepsilon(\xi) - \mathrm{d}_\varepsilon(x) - |x-\xi|}. \tag{3.8}$$

Dividing the domain Ω_ε into two regions according to the signs of $|x-\xi| - 2\mathrm{d}_\varepsilon(\xi)$ and using the facts that $\mathrm{d}_\varepsilon(x) + |x-\xi| \geq \mathrm{d}_\varepsilon(\xi)$ in the region where $|x-\xi| - 2\mathrm{d}_\varepsilon(\xi) < 0$, we then obtain

$$\int_{\Omega_\varepsilon} V^2 W^2 \leq C[1 + \mathrm{d}_\varepsilon(\xi)] e^{-4\mathrm{d}_\varepsilon(\xi)}.$$

The estimate for $\langle V^4\rangle_\varepsilon$ in (3.6) then yields the desired estimate for $\langle f_1^2\rangle_\varepsilon$. In a similar manner, we can estimate $\langle |\nabla_\xi f_1|^2\rangle_\varepsilon$. This proves the second assertion of Lemma 2.1.

3. Note that (3.8), together with the division of Ω_ε according to the signs of $|x-\xi| - 2\mathrm{d}_\varepsilon(\xi)$, yields

$$\langle |VW|\rangle_\varepsilon \leq C[1 + \mathrm{d}_\varepsilon(\xi)]^{(N+1)/2}\ e^{-2\mathrm{d}_\varepsilon(\xi)}.$$

Similar estimates also hold for $\langle |VW_\xi|\rangle_\varepsilon$ and $\langle |V_\xi W|\rangle_\varepsilon$. Observe that

$$|\nabla_\xi \langle W^2\rangle_\varepsilon| = |\nabla_\xi \int_{\Omega_\varepsilon^c} W^2| \leq 2\int_{|x-\xi| > \mathrm{d}_\varepsilon(\xi)} W|\nabla W| \leq C\ e^{-2\mathrm{d}_\varepsilon(\xi)}.$$

It then follows that

$$|\nabla_\xi \sigma| = |\nabla_\xi \langle W^2\rangle_\varepsilon + \langle\ 2V_\xi(W+V) + VW_\xi\ \rangle_\varepsilon| \leq C[1 + \mathrm{d}_\varepsilon(\xi)]^{(N+1)/2} e^{-2\mathrm{d}_\varepsilon(\xi)}.$$

With this estimate and a similar argument, one can establish the third assertion of Lemma 2.1.

4. The last assertion of Lemma 2.1 follows from an argument similar to that used before and, since W is radially symmetric, the identities

$$\int_{\mathbb{R}^N} W_{\xi_i} W_{\xi_j} = \delta_{ij} \frac{1}{N} \int_{\mathbb{R}^N} |\nabla W(x)|^2 \ = \ \delta_{ij}/\beta_0.$$

This completes the proof of Lemma 2.1. □

3.4. **Proof of Lemma 2.2.** The definition $u = (W+V)/\sigma$ and the fact that $f_1 = V(2W+V)/\sigma$ give the expression, for every $j = 1, \dots, N$,

$$\begin{aligned} \langle u_{\xi_j} f_1 \rangle_\varepsilon &= \sigma^{-2} \langle \ (W_{\xi_j} + V_{\xi_j}) V(2W+V) \ \rangle_\varepsilon - \sigma^{-1} \sigma_{\xi_j} \langle u f_1 \rangle_\varepsilon \\ &= 2\sigma^{-2} \langle \ V W W_{\xi_j} \ \rangle_\varepsilon + \sigma^{-2} \langle W_{\xi_j} V V \rangle_\varepsilon + \sigma^{-1} \langle V_{\xi_j} f_1 \rangle_\varepsilon - \sigma^{-1} \sigma_{\xi_j} \langle u f_1 \rangle_\varepsilon \\ &= \sigma^{-2} \langle \ V(W^2)_{\xi_j} \ \rangle_\varepsilon + O\Big([1 + \mathrm{d}_\varepsilon(\xi)]^{1/2} e^{-3\mathrm{d}_\varepsilon(\xi)} \Big) \end{aligned} \tag{3.9}$$

by the inequalities

$$|\langle \ V_\xi f_1 \ \rangle_\varepsilon| \le \sqrt{\langle V_\xi{}^2 \rangle_\varepsilon \langle f_1^2 \rangle_\varepsilon} \quad \text{and} \quad |\langle V^2 W_\xi \rangle_\varepsilon| \le \sqrt{\langle V^2 \rangle_\varepsilon \langle (V W_\xi)^2 \rangle_\varepsilon},$$

and the previous estimates.

Using the equations $W^2 = W - \Delta W$ and $\Delta V - V = 0$ we can calculate

$$\begin{aligned} \langle \ V(W^2)_{\xi_j} \ \rangle_\varepsilon \ &= \ \langle \ V(W_{\xi_j} - \Delta W_{\xi_j}) \ \rangle_\varepsilon = \int_{\partial\Omega_\varepsilon} \big(W_{\xi_j} \partial_n V - V \partial_n W_{\xi_j} \big) \\ &= \ - \int_{\partial\Omega_\varepsilon} \Big(W_{\xi_j} \partial_n W + W \partial_n W_{\xi_j} \Big) + \int_{\partial\Omega_\varepsilon} (V - W) \ \partial_n W_{\xi_j} \\ &= \ - \frac{\partial}{\partial \xi_j} \int_{\partial\Omega_\varepsilon} W \partial_n W + \int_{\partial\Omega_\varepsilon} (V - W) \partial_n W_{\xi_j} \ . \end{aligned} \tag{3.10}$$

By Lemma 3.2, the second term is bounded by

$$\begin{aligned} &\|V - W\|_{L^2(\partial\Omega_\varepsilon)} \|D^2 W\|_{L^2(\partial\Omega_\varepsilon)} \\ &\le C\varepsilon \|W\|^2_{L^2(\partial\Omega_\varepsilon)} + C[1 + \mathrm{d}_\varepsilon(\xi)]^{(1-N)/2} e^{-2\mathrm{d}_\varepsilon(\xi)} \|W\|_{L^2(\partial\Omega_\varepsilon)}. \end{aligned}$$

To evaluate the first term in (3.10), we note that

$$\begin{aligned} \int_{\partial\Omega_\varepsilon} W \partial_n W \ &= \ \int_{\Omega_\varepsilon} \mathrm{div}(W \nabla W) = \langle |\nabla W|^2 + W \Delta W \rangle_\varepsilon \\ &= \ \langle |\nabla W|^2 + W^2 - W^3 \rangle_\varepsilon = J(W(\cdot - \xi)) - \tfrac{1}{3} \langle W^3 \rangle_\varepsilon. \end{aligned}$$

As $\nabla_\xi \langle W^3 \rangle_\varepsilon = -\nabla_\xi \int_{\Omega_\varepsilon^c} W^3 = O(1)[1 + \mathrm{d}_\varepsilon(\xi)]^{(1-N)/2} e^{-3\mathrm{d}_\varepsilon(\xi)}$, we then conclude that

$$\begin{aligned} \sigma^2 \ \langle f_1 \nabla_\xi u \rangle_\varepsilon = -\nabla_\xi J(W(\cdot - \xi)) + O(1)[1 + \mathrm{d}_\varepsilon(\xi)]^{1/2} e^{-3\mathrm{d}_\varepsilon(\xi)} \\ + O(\varepsilon) \|W\|^2_{L^2(\partial\Omega_\varepsilon)} + O(1)[1 + \mathrm{d}_\varepsilon(\xi)]^{(1-N)/2} e^{-2\mathrm{d}_\varepsilon(\xi)} \|W\|_{L^2(\partial\Omega_\varepsilon)}. \end{aligned} \tag{3.11}$$

Notice that if $p \in \partial\Omega_\varepsilon$ is the point such that $|p-\xi| = \mathrm{d}_\varepsilon(\xi)$, then $|x-\xi| < \mathrm{d}_\varepsilon(\xi)+C$ for all $x \in \partial\Omega_\varepsilon$ satisfying $|x-p| \le \sqrt{\mathrm{d}_\varepsilon(\xi)+1}$. Also note that the region $\{x \in \partial\Omega : |x-q| \le \sqrt{1+\mathrm{d}_\varepsilon(\xi)}\}$ has area at least $c[1+\mathrm{d}_\varepsilon(\xi)]^{(N-1)/2}$ and inside this region, $W^2 > c[1+\mathrm{d}_\varepsilon(\xi)]^{(1-N)}e^{-2\mathrm{d}_\varepsilon(\xi)}$. Hence there exists a positive constant c independent of $\varepsilon \in (0,1]$ and $\xi \in \Omega_\varepsilon$ such that

$$\int_{\partial\Omega_\varepsilon} W^2\, dS \ge \; c\, \mathrm{d}_\varepsilon(\xi)^{(1-N)/2} e^{-2\mathrm{d}_\varepsilon(\xi)}.$$

It then follows from (3.11) that

$$\sigma^2\langle f_1 \nabla_\xi u\rangle_\varepsilon = -\nabla_\xi J + O(1)\left\{ \varepsilon \; + [1+\mathrm{d}_\varepsilon(\xi)]^{N/2} e^{-\mathrm{d}_\varepsilon(\xi)} \right\} \|W\|^2_{L^2(\partial\Omega_\varepsilon)}.$$

This proves (2.2).

Equation (2.3) can be obtained by changing variables $z = x-\xi$:

$$\begin{aligned}
&\nabla_\xi J(W(\cdot-\xi)) = -\nabla_\xi \int_{\Omega_\varepsilon-\xi} \Big(|\nabla_z W(z)|^2 + W^2(z) - \tfrac{2}{3}W^3(z)\Big) dz \\
&= \int_{\partial\Omega_\varepsilon} n_{\partial\Omega_\varepsilon} \Big(\; |\nabla W(x-\xi)|^2 + W^2(x-\xi) - \tfrac{2}{3}W^3(x-\xi) \; \Big) \\
&= \int_{\partial\Omega_\varepsilon} n_{\partial\Omega_\varepsilon} (|\nabla W^2| + W^2)\, dS + O(1)[1+\mathrm{d}_\varepsilon(\xi)]^{(1-N)/2} e^{-\mathrm{d}_\varepsilon(\xi)} \int_{\partial\Omega_\varepsilon} W^2.
\end{aligned}$$

Dividing the domain $\partial\Omega_\varepsilon$ into parts where $|x-\xi| > 2\mathrm{d}_\varepsilon(\xi)$ and $|x-\xi| < 2\mathrm{d}_\varepsilon(\xi)$, we can prove (2.4). This completes the proof of Lemma 2.2. □

3.5. **Proof of Lemma 2.3.** Another version slightly weaker than Lemma 2.3 has already been established in [18, Lemma 4.5]. The proof given below is adapted from there.

Proof. Set $r = |x-\xi|$ and $F(r) = |\nabla W(x-\xi)|^2 + W^2(x-\xi)$. Then by the expansion of W and $|\nabla W|$,

$$F(r) = 2w_0^2 r^{1-N} e^{-2r}[1+O(1/r)] \qquad \text{as} \;\; r \to \infty. \tag{3.12}$$

Write $\mathrm{d}_\varepsilon(\xi)$ as d_ε and let $p = \xi - \mathrm{d}_\varepsilon \nabla \mathrm{d}_\varepsilon(\xi) \in \partial\Omega_\varepsilon$ be the projection of ξ on $\partial\Omega_\varepsilon$. Then $\nabla_\xi \mathrm{d}_\varepsilon(\xi) = -n_{\partial\Omega_\varepsilon}(p)$. Without loss of generality we assume that $\xi = 0$ and $p = \mathrm{d}_\varepsilon \vec{e}_N$. We transfer Ω_ε back to the original domain Ω via $y = \varepsilon x$, $\eta = \varepsilon\xi$ and $q = \varepsilon p = d(0)\vec{e}_N$.

By $n(y)$ we denote the exterior normal of $\partial\Omega$ at $y \in \partial\Omega$. Then $n_{\partial\Omega_\varepsilon}(x) = n(\varepsilon x)$. Let $y_N = d(0) + \varphi(y')$, and $y' \in \mathbb{R}^{N-1}$ be a local representation of $\partial\Omega$ near q. Rotating the coordinates if necessary, we can assume that near the origin, φ has the expansion

$$\varphi(y') = -\Sigma_{i=1}^{N-1} \tfrac{1}{2} H_i y_i^2 + O(|y'|^3),$$

where $H_i = H_i(q)$, $i = 1, \dots, N-1$ are the principal curvatures of $\partial\Omega$ at q. Set $\varphi^\varepsilon(z) = \varepsilon^{-1}\varphi(\varepsilon z)$. Then $x_N = \mathrm{d}_\varepsilon + \varphi^\varepsilon(x') = \mathrm{d}_\varepsilon + \varepsilon^{-1}\varphi(\varepsilon x')$ is a local representation of $\partial\Omega_\varepsilon$ at p. Let $r(z) = |x|_{x=(z,\mathrm{d}_\varepsilon+\varphi^\varepsilon(z))} = \sqrt{|z|^2 + (\mathrm{d}_\varepsilon + \varphi^\varepsilon(z))^2}$. Using the expression for φ, we have

$$r^2(z) = {\mathrm{d}_\varepsilon}^2 + \Sigma_{i=1}^{N-1}(1 - \varepsilon \mathrm{d}_\varepsilon H_i) z_j^2 + O(\varepsilon^2 \mathrm{d}_\varepsilon |z|^3) + O(\varepsilon^2 |z|^4).$$

From (2.19), $1 - \varepsilon \mathrm{d}_\varepsilon H_i \geq \frac{2s}{\text{diameter } \ \Omega}$. Hence,

$$r(z) = \mathrm{d}_\varepsilon \Big\{ 1 + \Sigma_{i=1}^{N-1} (1 - \varepsilon \mathrm{d}_\varepsilon H_i) {\mathrm{d}_\varepsilon}^{-2} z_i^2 [1 + O(\varepsilon^2 \mathrm{d}_\varepsilon |z| + \varepsilon^2 |z|^2) \,] \Big\}^{1/2}.$$

Thus, there exists $k = k(\eta)$ such that if we define $\Gamma^1 = \{(z, \mathrm{d}_\varepsilon + \varphi^\varepsilon(z) : |z| \leq k\sqrt{\mathrm{d}_\varepsilon |\ln \varepsilon|}\}$ and $\Gamma^2 = \partial\Omega_\varepsilon \setminus \Gamma^1$, then $|x| \geq \mathrm{d}_\varepsilon + N|\ln\varepsilon|$ for all $x \in \Gamma^2$. Consequently, $F(|x|) \leq CF(\mathrm{d}_\varepsilon) e^{-2|x-\xi|+2\mathrm{d}_\varepsilon} = O(\varepsilon^{2N}) F(\mathrm{d}_\varepsilon)$ for all $x \in \Gamma^2$. It then follows that

$$\int_{\Gamma^2} F(|x|)\, dS_x = O(\varepsilon^{N+1}) F(\mathrm{d}_\varepsilon).$$

In the set Γ^1, we have the expansion

$$\begin{aligned} r(z) &= \mathrm{d}_\varepsilon + \Sigma_{i=1}^{N-1}(1 - \varepsilon \mathrm{d}_\varepsilon H_i) {\mathrm{d}_\varepsilon}^{-1} z_i^2 \Big\{ 1/2 + O(\varepsilon^2 \mathrm{d}_\varepsilon |z| + \varepsilon^2 |z|^2 + |z|^2/{\mathrm{d}_\varepsilon}^2) \Big\} \\ &= \mathrm{d}_\varepsilon + \Sigma_{i=1}^{N-1}(1 - \varepsilon \mathrm{d}_\varepsilon H_i) Z_i^2 \Big\{ 1/2 + O(\varepsilon^2 d^{3/2} |Z| + \varepsilon^2 \mathrm{d}_\varepsilon |Z|^2 + |Z|^2/\mathrm{d}_\varepsilon) \Big\} \end{aligned}$$

where $z = \sqrt{\mathrm{d}_\varepsilon} Z$, $|Z| \leq k|\sqrt{|\ln\varepsilon|}$. (We remark that above we implicitly assume that $1/2 + O(|Z|^2/\mathrm{d}_\varepsilon) > 1/8$ for all Z satisfying $|Z| \leq k\sqrt{|\ln\varepsilon|}$, which can be proven directly when $2|\ln\varepsilon| \leq \mathrm{d}_\varepsilon = O(|\ln\varepsilon|)$.) Hence

$$\begin{aligned} &F(|x|)\, dS_x = \sqrt{1 + |{\varphi^\varepsilon}_z|^2} F(r(z))\, dz \\ &= [1 + O(\varepsilon^2 |z|^2)] F(\mathrm{d}_\varepsilon) e^{\mathrm{d}_\varepsilon - r(z)} [r(z)/\mathrm{d}_\varepsilon]^{1-N} [1 + O(1/\mathrm{d}_\varepsilon)]\, dz \\ &= \big(1 + O(\tfrac{|Z|^2+1}{\mathrm{d}_\varepsilon})\big) \exp\Big(- \sum_{i=1}^{N-1} (1 - \varepsilon \mathrm{d}_\varepsilon H_i) Z_i^2 [\tfrac{1}{2} + O(\tfrac{|Z|^2}{\mathrm{d}_\varepsilon})] \Big) F(\mathrm{d}_\varepsilon) {\mathrm{d}_\varepsilon}^{\frac{N-1}{2}}\, dZ, \end{aligned}$$

and

$$\begin{aligned} &\int_{\Gamma^1} F(x)\, dS_x \\ &= F(\mathrm{d}_\varepsilon) {\mathrm{d}_\varepsilon}^{(N-1)/2} \Big[1 + O(1/\mathrm{d}_\varepsilon) \Big] \int_{\mathbb{R}^N} \exp\Big\{ - \sum_{i=1}^{N-1} (1 - \varepsilon \mathrm{d}_\varepsilon H_i) Z_i^2/2 \Big\}\, dZ \\ &= \Big\{ 2 w_0^2 (2\pi)^{(N-1)/2} + O(1/\mathrm{d}_\varepsilon) \Big\} {\mathrm{d}_\varepsilon}^{(1-N)/2} e^{-2\mathrm{d}_\varepsilon(\xi)} \prod_{i=1}^{N-1} (1 - \varepsilon \mathrm{d}_\varepsilon H_i)^{-1/2} \end{aligned}$$

by using the expression (3.12) for F. Equation (2.6) then follows from the fact that $\Pi_{i=1}^{N-1}(1-\varepsilon \mathrm{d}_\varepsilon H_i) = \det(\delta_{ij} - dd_{y_i y_j})_{N\times N}|_{y=\varepsilon\xi} = Q(\varepsilon\xi)$.

Next, we prove (2.5). When $x = (z, \mathrm{d}_\varepsilon + \varphi^\varepsilon(z)) \in \Gamma^1$,

$$\Big(n_{\partial\Omega_\varepsilon} + \nabla_\xi \mathrm{d}_\varepsilon(\xi)\Big) F(|x|)\, dS_x = \Big({\varphi^\varepsilon}_z, \sqrt{|{\varphi^\varepsilon}_z|^2+1} - 1\Big) F(r(z))\, dz.$$

Note that

$$\begin{aligned} {\varphi^\varepsilon}_z(z) &= \varepsilon\varphi_{y'y'}(0)z + O(\varepsilon^2|z|^2), \\ rr_z &= z + (\varphi^\varepsilon + \mathrm{d}_\varepsilon){\varphi^\varepsilon}_z = (\mathrm{I} + \varepsilon \mathrm{d}_\varepsilon \varphi_{y'y'}(0))z + O(\varepsilon^2|z|^3 + \varepsilon^2 \mathrm{d}_\varepsilon |z|^2). \end{aligned}$$

It then follows that

$${\varphi^\varepsilon}_z(z) = \varepsilon\varphi_{y'y'}(0)(\mathrm{I} + \varepsilon \mathrm{d}_\varepsilon\varphi_{y'y'}(0))^{-1} rr_z + O(\varepsilon^2|z|^2).$$

Therefore, on Γ^1,

$$\begin{aligned} &\Big(n_{\partial\Omega_\varepsilon} + \nabla \mathrm{d}_\varepsilon(\xi)\Big) F(|x-\xi|)\, dS_x \\ &= (\varphi_{y'y'}(0)(\mathrm{I} + \varepsilon \mathrm{d}_\varepsilon \varphi_{yy}(0)) f(r(z))_z, 0)\, dz + O(\varepsilon^2)|z|^2 F(r(z))\, dz, \end{aligned}$$

where $f(r) = \int_\infty^r sF(s)\, ds = O(rF(r))$. By the fundamental theorem of calculus, the integral on Γ^1 of the first integrand on the right-hand side can be neglected. Since for some positive $l(s)$, $r(z) \ge \mathrm{d}_\varepsilon + l|z|^2/\mathrm{d}_\varepsilon$, we have $F(r) \le F(\mathrm{d}_\varepsilon) C e^{-l|z|^2/\mathrm{d}_\varepsilon}$ so that

$$\begin{aligned} \int_{|z|\le k\sqrt{d_\varepsilon |\ln\varepsilon|}} z^2 F(r(z))\, dz &\le F(\mathrm{d}_\varepsilon){\mathrm{d}_\varepsilon}^{(N-1)/2} \int_{\mathbb{R}^{N-1}} \mathrm{d}_\varepsilon Z^2 e^{-l|Z|^2} dZ \\ &\le C \mathrm{d}_\varepsilon \int_{\Gamma^1} F\, dS_x. \end{aligned}$$

As $\varepsilon^2 \mathrm{d}_\varepsilon = O(\varepsilon)$, we then obtain

$$\int_{\partial\Omega_\varepsilon} \Big\{ n_{\partial\Omega_\varepsilon}(x) + \nabla_\xi \mathrm{d}_\varepsilon(\xi) \Big\}\, F(|x-\xi|)\, dS_x = O(\varepsilon) \int_{\partial\Omega_\varepsilon} F dS_x.$$

This proves (2.5) and completes the proof of Lemma 2.3. □

4. Eigenvalue analysis

In this section, we prove Lemma 2.4. We first consider the problem on $\mathbb{R}^N$. In the sequel, $\langle f\rangle = \int_{\mathbb{R}^N} f(x)\, dx$.

Set $\sigma_0 = \langle W^2\rangle$ and $U(\cdot) = W(\cdot)/\sigma_0$. Then $\sigma_0 = 1/\langle U^2\rangle$ and

$$\Delta U - U + \sigma_0 U^2 = 0 \qquad \text{in } \mathbb{R}^N\ .$$

Lemma 4.1. *Let* L_0 *be an operator defined as*

$$\mathrm{L}_0\phi \stackrel{\text{def}}{=} \Delta\phi - \phi + 2\sigma_0 U\phi.$$

Then the following hold:

(1) *The principal eigenvalue* λ_0 *of* L_0 *is positive, and its associated eigenfunction* ϕ_0 *is positive.*

(2) *Zero is an eigenvalue of* L_0 *with multiplicity* N*; its associated eigenspace is spanned by* $U_{x_1},\dots,U_{x_N}$.

(3) *There exists* $\nu_0 > 0$ *such that*

$$\mathrm{L}_0(\phi,\phi) \stackrel{\text{def}}{=} \langle -|\nabla\phi|^2 - \phi^2 + 2\sigma_0 U\phi^2\rangle \ \le\ \nu_0\langle\phi^2\rangle$$

for all $\phi \in H^1(\mathbb{R}^N)$ *satisfying* $\phi \perp \phi_0, U_{x_1},\dots,U_{x_N}$ *(in the* $L^2(\mathbb{R}^N)$ *sense).*

This lemma follows directly from more general results of [24].

Lemma 4.2. *Let* L_1 *be an operator defined by*

$$\mathrm{L}_1\phi \stackrel{\text{def}}{=} \mathrm{L}_0\phi - \sigma_0^2\langle U\phi\rangle U^2 - \sigma_0^2\langle U^2\phi\rangle U. \tag{4.1}$$

Then L_1 *has the following properties:*

(1) *The operator* L_1 *is self-adjoint;*

(2) *The function* U *is an eigenfunction of* L_1 *with eigenvalue* $-\sigma_0^2\langle U^3\rangle$*;*

(3) *For each* $i = 1,\dots,N$, U_{x_i} *is an eigenfunction of* L_1 *with eigenvalue zero.*

(4) *Assume that* $N \le 3$*. Then there exists a positive constant* $\nu_1 \in (0,1]$ *such that*

$$\mathrm{L}_1(\phi,\phi) \stackrel{\text{def}}{=} \langle -|\nabla\phi|^2 - \phi^2\rangle + 2\sigma_0\langle U\phi^2\rangle - 2\sigma_0^2\langle U\phi\rangle\langle U^2\phi\rangle \ \le\ -\nu_1\langle\phi^2\rangle$$

for all $\phi \in H^1(\mathbb{R}^N)$ *satisfying* $\phi \perp U_{x_1},\dots,U_{x_N}$.

Proof. The first three assertions follow by direct verification. To prove (4), we need only to consider those ϕ which are orthogonal to $U, U_{x_1},\dots,U_{x_N}$. We will argue by contradiction. Since the essential spectrum of L_1 lies in $(-\infty,-1]$, if (4) is not true, then there exists (λ,ϕ) such that

(i) λ is real and nonnegative,

(ii) $\phi \perp U, U_{x_1},\dots,U_{x_N}$, and

(iii) $\mathrm{L}_1\phi = \lambda\phi$.

We will show that (i)–(iii) can not hold simultaneously.

From the definition of L_1, (ii), and (iii), we have

$$(\mathrm{L}_0 - \lambda)\phi = \sigma_0^2\langle U^2\phi\rangle U. \tag{4.2}$$

We first claim that $\lambda \ne \lambda_0$. In fact, if $\lambda = \lambda_0$, then $(\mathrm{L}_0 - \lambda_0)\phi \perp \phi_0$ so that $\langle U^2\phi\rangle\langle U\phi_0\rangle = 0$. Consequently, as $\phi_0 > 0$ and $U > 0$, $\langle U^2\phi\rangle = 0$, so that

$(\mathrm{L}_0 - \lambda_0)\phi = 0$. Thus ϕ is a multiple of ϕ_0. But this contradicts the fact that $\langle U^2\phi\rangle = 0$. Hence, $\lambda \neq \lambda_0$.

Restricted to the space orthogonal to $U_{x_1}, \ldots, U_{x_N}$, $\mathrm{L}_0 - \lambda$ is invertible, so that (4.2) implies that $\phi = \alpha(\mathrm{L}_0 - \lambda)^{-1}U$, $\alpha = \sigma_0^2\langle U^2\phi\rangle$. Hence, $\alpha \neq 0$. Taking the inner product with $\sigma_0^2 U^2/\alpha$, we obtain

$$\begin{aligned} 1 &= \sigma_0^2(U^2, (\mathrm{L}_0 - \lambda)^{-1}U) = \sigma_0(\mathrm{L}_0 U, (\mathrm{L}_0 - \lambda)^{-1}U) \text{ (as } \mathrm{L}_0 U = \sigma_0 U^2) \\ &= \sigma_0((\mathrm{L}_0 - \lambda)U, (\mathrm{L}_0 - \lambda)^{-1}U) + \sigma_0\lambda(U, (\mathrm{L}_0 - \lambda)^{-1}U) \\ &= \sigma_0\langle U^2\rangle + \sigma_0\lambda(U, (\mathrm{L}_0 - \lambda)^{-1}U) = 1 + \sigma_0\lambda(U, (\mathrm{L}_0 - \lambda)^{-1}U). \end{aligned}$$

Consider the function $F(z) = (U, (\mathrm{L}_0 - z)^{-1}U)$ for $z \in (0, \lambda_0) \cup (\lambda_0, \infty)$. We have

$$F'(z) = (U, (\mathrm{L}_0 - z)^{-2}U) = ((\mathrm{L}_0 - z)^{-1}U, (\mathrm{L}_0 - z)^{-1}U) > 0.$$

Since $\mathrm{L}_0(U + \frac{1}{2}x \cdot \nabla U) = U$, ${\mathrm{L}_0}^{-1}U = U + \frac{1}{2}x \cdot \nabla U + \Sigma_{i=1}^N c_i U_{x_i}$. It then follows that $F(0) = (U, U + \frac{1}{2}x \cdot \nabla U) = (1 - \frac{N}{4})\langle U^2\rangle > 0$ as $N \leq 3$. Thus, $F(z) > 0$ for all $z \in (0, \lambda_0)$. As $F(\infty) = 0$, we also have $F(z) < 0$ for all $z \in (\lambda_0, \infty)$. Hence, we have $\lambda \notin (0, \infty)$.

Finally, we show that $\lambda \neq 0$. In fact, if $\lambda = 0$, then as $\phi \perp U_{x_i}$ for all i, $\phi = \alpha {\mathrm{L}_0}^{-1}U = \alpha(U + \frac{1}{2}x \cdot \nabla U)$. But this implies that $(\phi, U) = \alpha(1 - \frac{N}{4})\langle U^2\rangle > 0$, contradicting the assumption $\phi \perp U$. This completes the proof the the lemma. □

Remark 4.1. If $N = 4$, one sees that $\phi = U + \frac{1}{2}x \cdot \nabla U$ is an eigenfunction of L_1 with eigenvalue zero.

Next, we extend Lemma 4.2 to large balls $B_R = \{x : |x| < R\}$.

Lemma 4.3. *Assume $N \leq 3$. There exist positive constants R_0 and ν_2 such that for each $R > R_0$ and each $\phi \in H^1(B_R)$ satisfying $\phi \perp U_{x_i}$, $i = 1, \ldots, N$ in $L^2(B_R)$ there holds*

$$\begin{aligned} {\mathrm{L}_1}^R(\phi, \phi) &\stackrel{\text{def}}{=} \int_{B_R} (-|\nabla\phi|^2 - \phi^2 + 2\sigma_0 U\phi^2) - 2\sigma_0^2 \int_{B_R} U\phi \int_{B_R} U^2\phi \\ &\leq -\nu_2 \int_{B_R} (|\nabla\phi|^2 + \phi^2). \end{aligned} \tag{4.3}$$

Proof. We will argue by contradiction. Suppose the assertion is not true. Then there exists a sequence $\{R_k, \phi_k\}_{k=1}^\infty$ such that $R_k > k, \phi_k \in H^1(B_{R_k})$, $\phi_k \perp U_{x_i}$ in $L^2(B_{R_k})$ for all i, $\int_{B_{R_k}} (|\nabla\phi_k|^2 + \phi_k^2) = 1$, and

$$\limsup_{k\to\infty} {\mathrm{L}_1}^{R_k}(\phi_k, \phi_k) \geq 0. \tag{4.4}$$

Since H^1 is weakly compact and the embedding $H^1 \to L^2$ is compact, we can assume, by taking a subsequence if necessary, that there exists $\phi \in H^1(\mathbb{R}^N)$ such that $\lim_{k\to\infty}\phi_k = \phi$, weakly in $H^1(B_R)$ and strongly in $L^2(B_R)$ for every $R > 0$. In addition, $\|\phi\|_{H^1(\mathbb{R}^N)} \le 1$.

Since U decays exponentially fast, we have

$$\langle \phi U_{x_i}\rangle = \lim_{k\to\infty}\int_{B_{R_k}} \phi_k U_{x_i} = 0 \quad \text{for all } i = 1, \dots, N.$$

In addition, as $k \to \infty$,

$$\begin{aligned}\gamma_k &\stackrel{\text{def}}{=} \int_{B_k} 2\sigma_0 U\phi_k - 2\sigma_0^2\int_{B_{R_k}} U\phi_k \int_{B_{R_k}} U^2\phi_k \to \gamma \\ &\stackrel{\text{def}}{=} 2\sigma_0\langle U\phi^2\rangle - 2\sigma_0^2\langle U\phi\rangle\langle U^2\phi\rangle.\end{aligned}$$

If $\phi \equiv 0$, then $\gamma = 0$ so that

$$\mathrm{L_1}^{R_k}(\phi_k, \phi_k) = \gamma_k - 1 < -1/2$$

for all large k. But this contradicts (4.4).

If $\phi \not\equiv 0$, then by Lemma 4.2, as $\phi \perp U_{x_j}$ for all j, $\mathrm{L}_1(\phi, \phi) \le -\nu_1\langle\phi^2\rangle$. As $\phi \in H^1(\mathbb{R}^N)$, there exists a large M such that $\|\phi\|^2_{H^1(\mathbb{R}^N\setminus B_M)} \le \frac{1}{2}\nu_1\langle\phi^2\rangle$. Consequently,

$$\gamma - \int_{B_M} (|\nabla\phi|^2 + \phi^2)\, dx \le -\tfrac{1}{2}\nu_1\langle\phi^2\rangle.$$

It then follows that

$$\begin{aligned}\limsup_{k\to\infty} \mathrm{L_1}^{R_k}(\phi_k, \phi_k) &\le \limsup_{k\to\infty}\Big\{\gamma_k - \int_{B_M}(|\nabla\phi_k|^2 + \phi_k^2)\Big\} \\ &\le \gamma - \int_{B_M}(|\nabla\phi|^2 + \phi^2) < 0\ .\end{aligned}$$

Again, we obtain a contradiction. The proof is now complete. □

Proof of Lemma 2.4. Let ξ and ϕ be as in the statement of the lemma. By scaling, we can assume that $\langle\phi^2 + |\nabla\phi|^2\rangle_\varepsilon = 1$. Set $R = \mathrm{d}_\varepsilon(\xi)$ (note that $R \ge 2|\ln\varepsilon| \ge R_0$ if ε is sufficiently small). Translating Ω_ε if necessary, we can always achieve $\xi = 0$.

Let $\mathrm{L_1}^R(\phi, \phi)$ be defined as in (4.3). As $|u - U|_{L^\infty} + |\sigma - \sigma_0| = O(\varepsilon)$ and $|U| = O(\varepsilon)$ outside B_R, we have

$$\mathcal{L}(\phi, \phi) = L_1^R(\phi, \phi) - \int_{\Omega_\varepsilon\setminus B_R}(|\nabla\phi|^2 + \phi^2) + O(\varepsilon). \tag{4.5}$$

Now let

$$\phi^R = \phi - \sum_{i=1}^{N} c_i u_{x_i}$$

be the $L^2(B_R)$ orthogonal projection of ϕ on $\{U_{x_1}, \ldots, U_{x_N}\}^\perp$. Then

$$0 = \langle \phi U_{x_i} \rangle_\varepsilon = \int_{B_R} \phi U_{x_i} + O(\varepsilon),$$

so that $c_i = O(\varepsilon)$ for all i. Hence, $\|\phi - \phi^R\|_{H^1(B_R)} = O(\varepsilon)$ and

$$\mathrm{L_1}^R(\phi, \phi) = \mathrm{L_1}^R(\phi^R, \phi^R) + O(\varepsilon). \tag{4.6}$$

From Lemma 4.2 we have

$$\mathrm{L_1}^R(\phi^R, \phi^R) \leq -\nu_1 \int_{B_R} \left(|\nabla \phi^R|^2 + (\phi^R)^2 \right) = -\nu_1 \int_{B_R} (|\nabla \phi|^2 + \phi^2) + O(\varepsilon).$$

Combining this with (4.5) and (4.6) then gives

$$\begin{aligned} \mathcal{L}(\phi, \phi) &\leq -\int_{\Omega_\varepsilon \setminus B_R} (|\nabla \phi|^2 + \phi^2) - \nu_1 \int_{B_R} (|\nabla \phi|^2 + \phi^2) + O(\varepsilon) \\ &\leq -\nu_1 \langle \phi^2 + |\nabla \phi|^2 \rangle_\varepsilon + O(\varepsilon). \end{aligned}$$

Taking ε sufficiently small, we then obtain the assertion of the lemma.

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