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SINGULARITIES AND NONUNIQUENESS IN CYLINDRICAL FLOW OF NEMATIC LIQUID CRYSTALS

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Abstract. The subject of this paper is the behavior of the director field of a nematic liquid crystal in flow through a tube with circular cross-section. Both the flow and the director field are assumed to have cylindrical symmetry. The requirement of finite Frank-Oseen energy forces "admissible" director fields to be axially directed at the location of the symmetry axis. Thus, the angle between the axis and the director field at the location of the axis amounts to $k\pi$, k being an integer. In the steady case, it is shown that k is largely (but not uniquely) determined by the (Dirichlet) boundary conditions. In particular, this may give rise to line singularities with finite energy density. Moreover, the associated Dirichlet problem may have several distinct solutions with identical value of k. Analogously to the heat flow of harmonic mappings, finite-time blow-up phenomena for the associated parabolic problem are established.

1. Introduction

This paper addresses some aspects of the behavior of the director field of a nematic liquid crystal flowing through a tube with circular cross-section. The steady situation with cylindrical symmetry has been studied in [1] and [9], where different ways have been chosen to avoid infinite energies. The first paper considers the flow between two coaxial cylinders and studies in particular what happens when the inner cylinder shrinks. The second paper

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is based on a variational formulation of the problem, which has many analogies with the problem of finding a harmonic mapping from the unit disc D^2 into the unit sphere S^2.

A natural setting for this variational approach is the collection of director fields with finite Frank-Oseen energy. In the one constant approximation (see [5]), it reads $H^1(D^2; S^2) = \{\mathbf{u} \in H^1(D^2; \mathbb{R}^3) : |\mathbf{u}(\mathbf{x})| = 1$ almost everywhere$\}$. In particular, we are interested in the set of *symmetric* director fields with finite energy

$$\{\mathbf{u} \in H^1(D^2; S^2) : \ \mathbf{u} = (u_1, u_2, u_3) = (\frac{x_1}{r}\sin\theta, \frac{x_2}{r}\sin\theta, \cos\theta)\},$$

where $r = |\mathbf{x}|$ and θ is a function of r, which satisfies

$$\mathcal{E}(\theta) = \int_0^1 \Big\{\theta'^2 + \frac{\sin^2\theta}{r^2}\Big\} r\,dr < \infty. \tag{1}$$

This finite-energy requirement implies that $\theta \in C([0,1])$, and $\theta(0) = k\pi$ for some integer k (see [9]). In what follows, we identify a director field and the associated function θ. The fact that $\theta(0)$ assumes discrete values gives rise to an interesting question.

Given $b \in \mathbb{R}$, does there exist a steady director field $\theta \in C^1([0,1])$ with finite energy, such that $\theta(0) = 0$ and $\theta(1) = b$? Such a function would be a (finite energy) solution of the following problem, considered in [9]:

Problem $(\mathcal{P})$.

$$\theta'' + \frac{\theta'}{r} - \frac{\sin 2\theta}{2r^2} + v'(r)g(\theta) = 0 \text{ in } [0,1), \tag{2}$$

$$\theta(1) = b, \tag{3}$$

with the additional condition

$$\theta(0) = 0. \tag{4}$$

Here $v(r)$ represents the flow field, of which we require that

$$v \in C^\infty([0,1]), \ v'(r) \geq 0, \text{ and } v^{(2n+1)}(0) = 0 \text{ for all } n, \tag{5}$$

$$\text{and } \ g(\theta) = \frac{1}{K}(\alpha_2 \sin^2\theta - \alpha_3\cos^2\theta), \tag{6}$$

with Frank-Oseen constant $K > 0$ and Leslie-Ericksen parameters $\alpha_2 < \alpha_3 < 0$.

In [9], it has been shown that a finite energy solution to problem $(\mathcal{P})$ exists if the inner boundary condition is relaxed to $\theta(0) = k\pi$ with (integer) k free. When $k \neq 0$, a line singularity occurs in the sense that k complete rotations of the director field are concentrated on the axis of the tube (or in the origin of D^2), and this gives rise to a finite line energy density. When no flow is present, it has been shown in [9] that k must be $\neq 0$ as soon as $|b| \geq \pi$, but in the presence of nontrivial flow the question whether such a singularity really develops has not been answered.

Another interesting question concerns *uniqueness* of solutions of problem $(\mathcal{P})$ in the case that existence is guaranteed. In the case that no flow is present, uniqueness is easily established, but the case with nontrivial flow has been left unanswered.

In the present paper, we shall provide answers to both questions. For the situation with flow, we shall first show that line singularities do indeed occur, and, second, that there are situations in which problem $(\mathcal{P})$ admits at least three different finite-energy solutions.

Finally, the occurrence of singularities suggests studying the corresponding "heat flow" problem, similar to the case of harmonic mappings (see [11] for an overview). Thus, we shall study the parabolic version of problem $(\mathcal{P})$ and address the question whether finite-time blow-up phenomena are to be expected, similar to those found in the case of the heat flow for harmonic mappings $D^2 \to S^2$ (see [4]). Our results are analogous to these earlier results. We remark that the heat flow, considered here, is a simplification of the "physical" heat flow, in that we assume the flow field to be constant.

This paper is organized as follows. In Section 2, we present some smoothness arguments, from which we deduce that a solution to problem $(\mathcal{P})$ can be obtained as the solution of a shooting problem of the following type:

Problem (P_α).

$$\begin{aligned} \theta_\alpha'' + \frac{\theta_\alpha'}{r} - \frac{\sin 2\theta_\alpha}{2r^2} + v'(r)g(\theta_\alpha) &= 0 \ (0 < r < 1) \\ \theta_\alpha(0) &= 0, \quad \theta_\alpha'(0) = \alpha. \end{aligned} \tag{7}$$

For such problems, we shall prove

Theorem 1.1. (i) *The set* $\{\theta_\alpha\}$ *is bounded in* $C([0,1])$. (ii) *When* α *tends to* ∞ $(-\infty)$, *then* θ_α *tends to* $\bar{\theta}$ $(\underline{\theta})$ *in* $C^2([\delta,1])$ *for each* $\delta \in (0,1)$, *where* $\bar{\theta}$ $(\underline{\theta})$ *is the unique solution of problem* (P):

$$\begin{aligned} \theta'' + \frac{\theta'}{r} - \frac{\sin 2\theta}{2r^2} + v'(r)g(\theta) &= 0 \ (0 < r < 1) \\ \theta(0) &= \pi \ (-\pi) \quad \theta'(0) = 0, \end{aligned} \tag{8}$$

and the set $I = \{\theta_\alpha(1) : \alpha \in \mathbb{R}\} \cup \{\bar{\theta}(1)\} \cup \{\underline{\theta}(1)\}$ *is a closed, bounded interval* $[B_1, B_2]$.

We then deduce that problem (P) cannot have a solution if $|b|$ is too large.

In Section 3, we address the uniqueness problem for problem (P). To this aim we shall use the variational formulation of the problem, as introduced in [9]. In the closed subset $\{\theta(1) = b\}$ of the Hilbert space

$$\mathcal{W} = \{\tilde{\theta} \in L^2(D^2);\ \tilde{\theta}(\mathbf{x}) = \theta(r);\ \tilde{\theta}' \in L^2(D^2);\ \frac{\tilde{\theta}}{r} \in L^2(D^2)\}$$

with norm $||\tilde{\theta}|| = ||\theta|| = \sqrt{\int_0^1 \{\theta'^2 + \frac{\theta^2}{r^2}\} r dr}$, we shall consider the problem of finding (relative) minima of the functional

$$\mathcal{H}(\theta) = \int_0^1 \big\{K\big(\theta'^2 + \frac{\sin^2\theta}{r^2}\big) + v'(r)G(\theta)\big\} r dr,$$

where $G(\theta) = (\alpha_3 - \alpha_2)\theta + \dfrac{\alpha_2 + \alpha_3}{2}\sin 2\theta$.

The function G assumes (relative) minima at $\Theta_k^{\min} = \arctan(\sqrt{\frac{\alpha_3}{\alpha_2}}) + k\pi$, and (relative) maxima at $\Theta_k^{\max} = -\arctan(\sqrt{\frac{\alpha_3}{\alpha_2}}) + k\pi$ (remember that $\alpha_2 < \alpha_3 < 0$). Moreover, $G(\Theta_k^{\min}) < G(\Theta_{k+1}^{\min})$ and $G(\Theta_k^{\max}) < G(\Theta_{k+1}^{\max})$. For suitable values of b, we shall construct multiple solutions for problem (P). To be precise, we shall prove

Theorem 1.2. *Let $\Theta_0^{\max} < b < \Theta_0^{\min}$, and suppose that $v'(r) > 0$ when $r > 0$.*

(i) *There exists a classical solution $\bar{\theta}$ of problem ($\mathcal{P}$), which minimizes $\mathcal{H}(\theta)$ over the set $\{\theta \in \mathcal{W} :\ \theta(1) = b$ and $\Theta_0^{\max} \le \theta \le \Theta_0^{\min}\}$.*

(ii) *If K is small enough, problem (P) admits a second solution $\tilde{\theta} \neq \bar{\theta}$, which minimizes $\mathcal{H}(\theta)$ over the set $\{\theta \in \mathcal{W} :\ \theta(1) = b$ and $-\pi \le \theta \le \Theta_0^{\min}\}$, and a third solution $\hat{\theta}$.*

In Section 4, we study the corresponding parabolic heat flow problem

$$(\mathrm{P_{par}}) \begin{cases} h_t = \Lambda(h, \mathbf{x}, t) & \text{in} \quad D^2 \times (0, T), \\ h(\mathbf{x}, 0) = \tilde{h}_0(\mathbf{x}) = h_0(|\mathbf{x}|) & \text{in} \quad D^2, \\ h(\mathbf{x}, t) = h_0(1) = b & \text{in} \quad \partial D^2 \times (0, T), \end{cases}$$

where $\Lambda(h, \mathbf{x}, t) = \Delta h - \frac{\sin 2h}{2r^2} + g(h)v'(r)$ and where $\tilde{h}_0$ and h_0 satisfy the hypotheses

(**H1**) $\tilde{h}_0 \in C^1(\overline{D^2})$,

(**H2**) $\mathcal{H}(h_0) < \infty$.

This problem is closely related to the study of the heat flow for harmonic mappings; see [11], [10], [4], [6], [7] and [8]. We shall address the question whether blow-up phenomena are to be expected in finite time, following an approach similar to the one in [4]. We shall not consider global existence when the initial/boundary values are appropriate, as has been done in [3] in the case of harmonic mappings.

We start by proving a comparison principle which also applies in the harmonic case. This principle allows us to compare "subsolutions" and "supersolutions," even if jumps of a certain type are allowed. Such jumps will be called "jumps of type I." They are defined as follows:

Definition 1.3. The function h has a jump of type I at time T when there is an integer $k \neq 0$ such that $\lim_{t\uparrow T} \lim_{r\to 0} h(r,t) - \lim_{r\to 0} h(r,T) = k\pi$.

The comparison principle reads

Lemma 1.4 (Comparison Principle). *Let* $\theta \in C([0,1]\times[0,T]\backslash\{(0,T_{11}), \ldots, (0,T_{1n})\})$ *and* $\psi \in C([0,1]\times[0,T]\backslash\{(0,T_{21}),\ldots,(0,T_{2m})\})$ *be classical solutions of*

$$\begin{cases} \theta_t \leq \tilde{\Lambda}(\theta, r, t) & \text{in } (0,1)\times(0,T], \\ \theta(1,t) = \Theta(t) & \text{in } [0,T], \\ \theta(r,0) = \theta_0(r) & \text{in } [0,1], \end{cases}$$

and

$$\begin{cases} \psi_t \geq \tilde{\Lambda}(\psi, r, t) & \text{in } (0,1)\times(0,T], \\ \psi(1,t) = \Psi(t) & \text{in } [0,T], \\ \psi(r,0) = \psi_0(r) & \text{in } [0,1], \end{cases}$$

where $\tilde{\Lambda}(f,r,t) = f''(r) + \dfrac{f'(r)}{r} - \dfrac{\sin 2f(r)}{2r^2} + v'(r)g(f(r))$. *Let* $\mathcal{H}(\theta(\cdot,t)) < \infty$ and $\mathcal{H}(\psi(\cdot,t)) < \infty$ *for all* $0 \leq t \leq T$, *and suppose that* θ *and* ψ *have jumps of type I at the points* $(0,T_{1i})$ *and* $(0,T_{2i})$, *respectively. If* $\theta_0 \leq \psi_0$ *and* $\Theta(t) \leq \Psi(t)$ for $t \leq T$, *then* $\theta(r,t) \leq \psi(r,t)$ *for* $0 \leq r \leq 1,\ 0 \leq t \leq T$.

We stress that singularities different from jumps of type I can be shown to exist. If such singularities are allowed, the comparison principle is not valid, and uniqueness of the heat flow is not guaranteed any more. See [2] for details.

Finally, we shall prove

Theorem 1.5. (i) *There is a maximal time of existence* $T > 0$ (*which may be infinite*), *such that problem* $(\mathrm{P}_{\mathrm{par}})$ *admits a unique solution*

$$h \in C^\infty(\overline{D^2}\times(0,T)) \cap C([0,T), C^1(\overline{D^2})).$$

(ii) *If $b \notin [B_1, B_2]$, then T is finite.*

Remark 1.6. In what follows, we shall freely identify radially symmetric functions, defined on D^2, with their r-dependent counterparts. By Δ we shall always mean either the two-dimensional Laplacian or its radially symmetric one-dimensional expression.

2. The occurrence of singularities

The main result of this section will be the proof of Theorem 1.1.

We first claim that solutions θ of (2), (3) with finite energy satisfy $\theta \in C^\infty([0,1))$. To do that, we first remark that, by (5), v' can be written as

$$v'(r) = r\psi(r^2), \text{ where } \psi \in C^\infty([0,1]). \tag{9}$$

Then, we go back to the formulation of the problem in terms of the vector function $\mathbf{u} \in H^1(D^2; S^2)$:

$$\Delta\mathbf{u} + |\nabla\mathbf{u}|^2\mathbf{u} = f(\mathbf{x}, \mathbf{u}),$$

where $f(\mathbf{x}, \mathbf{u}) = \frac{\{\alpha_2(u_1^2+u_2^2)-\alpha_3 u_3^2\}\psi(|\mathbf{x}|^2)}{K}(-u_3\mathbf{x} + (x_1u_1 + x_2u_2)\mathbf{e}_3)$. Here, both D^2 and S^2 are thought to be imbedded in $\mathbb{R}^3$, and $\mathbf{e}_3$ is a unit vector in an axial direction. From the arguments in [9] we deduce that $\mathbf{u}$ is continuous, and then it follows from Corollary 12B.5 in [12] and a bootstrap argument that $\mathbf{u} \in C^\infty(D^2; S^2)$. It is now easy to prove the claim.

The line of our arguments is as follows. Next to problem (P_α), we consider:

Problem (Q_α):

$$\phi_\alpha'' + \frac{\phi_\alpha'}{r} - \frac{\sin 2\phi_\alpha}{2r^2} = 0 \ (0 < r < 1), \quad \phi_\alpha(0) = 0, \quad \phi_\alpha'(0) = \alpha. \tag{10}$$

Problem (R_α):

$$\zeta_\alpha'' + \frac{\zeta_\alpha'}{r} - \frac{\sin 2\zeta_\alpha}{2r^2}\cos 2\phi_\alpha + \frac{\sin^2\zeta_\alpha}{r^2}\sin 2\phi_\alpha + v'(r)g(\zeta_\alpha + \phi_\alpha) = 0 \tag{11}$$

$$(0 < r < 1) \qquad \zeta_\alpha(0) = 0, \quad \zeta_\alpha'(0) = 0,$$

where $\zeta_\alpha = \theta_\alpha - \phi_\alpha$.

Problem (Q_α) admits a unique solution

$$\phi_\alpha(r) = \arccos\left(\frac{4 - \alpha^2r^2}{4 + \alpha^2r^2}\right) = 2\arctan\left(\frac{\alpha r}{2}\right).$$

Setting

$$\gamma(r, \zeta, \phi) = \frac{\sin 2\zeta}{2r^2}\cos 2\phi - \frac{\zeta}{r^2} - \frac{\sin^2\zeta}{r^2}\sin 2\phi - v'(r)g(\zeta + \phi), \tag{12}$$

we may write (11) as

$$\big(\zeta_\alpha' + \frac{\zeta_\alpha}{r}\big)' = \gamma(r, \zeta_\alpha, \phi_\alpha). \tag{13}$$

Integrating twice we obtain

$$\zeta_\alpha(r) = \frac{1}{2r}\int_0^r (r^2-\rho^2)\gamma(\rho, \zeta_\alpha(\rho), \phi_\alpha(\rho))\ d\rho. \tag{14}$$

Existence and uniqueness for ζ_α will be established by a contraction argument. For $A > 0$ sufficiently large and $R < 1$ sufficiently small, we consider $\mathcal{M} = \big\{z \in C([0,R]) : ||z|| =_{\text{def}} \sup_{(0,R)} |\frac{z}{r^2}| \le A\big\}$, and the operator $\mathcal{F}_\alpha$, defined on $\mathcal{M}$ by

$$\mathcal{F}_\alpha(z) = \frac{1}{2r}\int_0^r (r^2-\rho^2)\gamma(\rho, z(\rho), \phi_\alpha(\rho))\ d\rho.$$

Lemma 2.1. *Let* $B = \dfrac{|\alpha_2| + |\alpha_3|}{K}||v'||_\infty$ *and* $A > B$. *When* R *is so small that* $\frac{2}{9}A^2R^4 + \frac{A}{3}R^2 + \frac{B}{3A} \le \frac{1}{3}$ *and* $\frac{2}{5}(A^2R^4 + AR^2) + BR^2 < 1$, *then* $\mathcal{F}_\alpha$ *is a contraction of* $\mathcal{M}$.

Proof. For $z \in \mathcal{M}$ we have that

$$|\mathcal{F}_\alpha(z)(r)| \le \frac{r^2}{3}\sup_{0<s<r}\Big\{\frac{|\sin 2z - 2z|}{2s^2}|\cos 2\phi_\alpha| + \frac{|z|}{s^2}|\cos 2\phi_\alpha - 1|$$
$$+ \frac{\sin^2 z}{s^2}|\sin 2\phi_\alpha| + |v'(s)g(z+\phi_\alpha)|\Big\}.$$

Estimating the various terms in the right-hand side, we obtain that

$$\frac{|\mathcal{F}_\alpha(z)(r)|}{r^2} \le \frac{1}{3}\big(\frac{2}{3}A^3r^4 + 2A + A^2r^2 + B\big) \le A,$$

so that $\mathcal{F}_\alpha$ indeed maps $\mathcal{M}$ into itself. Moreover, by the definition of $\mathcal{F}_\alpha$ we have that

$$\frac{|\mathcal{F}_\alpha(z_1)(r) - \mathcal{F}_\alpha(z_2)(r)|}{r^2}$$
$$\le \frac{1}{2r^3}\int_0^r (r^2-s^2)\Big\{\frac{|\sin 2z_1 - 2z_1 - \sin 2z_2 + 2z_2|}{2s^2}|\cos 2\phi_\alpha|$$
$$+ \frac{|z_1 - z_2|}{s^2}|\cos 2\phi_\alpha - 1| + \frac{|\sin^2 z_1 - \sin^2 z_2|}{s^2}|\sin 2\phi_\alpha|$$
$$+ |v'(s)g(z_1+\phi_\alpha) - v'(s)g(z_2+\phi_\alpha)|\Big\}\, ds$$
$$\le \Big\{\frac{2}{15}(A^2R^4 + AR^2) + \frac{2}{3} + \frac{BR^2}{3}\Big\}\ ||z_1 - z_2||.$$

Remark 2.2. From now on, we fix A and R *independently of* α such that the conditions of Lemma 2.1 are fulfilled, and choose $\mathcal{M}$ accordingly.

Corollary 2.3. *The problems* (P_α) *and* (R_α) *admit unique solutions.*

Proof. From the above lemma it follows that the shooting problems (R_α) admit unique solutions in the class $\mathcal{M}$. These solutions are restricted to the region $0 \le r \le R$, but by standard ODE theory they can be extended to $[0,1]$. Of course, this provides us with solutions of the problems (P_α) as well. In principle, it still seems possible that solutions to (R_α) exist which do not belong to $\mathcal{M}$. Let ζ be such a solution. Then still there is some $\hat{A} > A$ with the property that $\zeta(r) \le \hat{A}r^2$. Choose $\hat{R} < R$ and $\hat{\mathcal{M}}$ accordingly. It follows that ζ is the unique solution in $\hat{\mathcal{M}}$. On the other hand, $\zeta_\alpha \in \mathcal{M}$ belongs to $\hat{\mathcal{M}}$ as well, and thus $\zeta(r) = \zeta_\alpha(r)$ when $r < \hat{R}$. Thus, by continuation, $\zeta \equiv \zeta_\alpha$, which is a contradiction. □

Lemma 2.4. *There exists a constant* C*, independent of* α*, such that, for all* $r \le R$*,*

(i) $|\zeta'_\alpha(r)| \le Cr$,

(ii) $|\theta'_\alpha(r)| \le Cr + \frac{4|\alpha|}{4+\alpha^2 r^2}$.

Proof. The fact that $\zeta_\alpha \in \mathcal{M}$ implies that $|\zeta_\alpha(r)| \le Ar^2$ when $r \le R$. From (14) we deduce that

$$\zeta'_\alpha(r) = \frac{1}{2}\int_0^r \big(1 + \frac{s^2}{r^2}\big)\gamma(s, \zeta_\alpha(s), \phi_\alpha(s))\, ds. \tag{15}$$

Now, (i) follows from the definition of γ (see (12)), while (ii) follows from (i) and the fact that $\phi'_\alpha(r) = \frac{4\alpha}{4+\alpha^2 r^2}$. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. (i) Let A and R be as in Lemma 2.1. We have that $|\theta_\alpha(r)| \le \pi + Ar^2$ for all $r \le R$. For a fixed $\delta < R$, Lemma 2.4, (ii) gives $|\theta'_\alpha(\delta)| \le C\delta + \frac{1}{\delta}$ for all $\alpha \in \mathbb{R}$. Equation (7) may be written as

$$(r\theta'_\alpha)' = \frac{\sin 2\theta_\alpha}{2r} - rv'(r)g(\theta_\alpha),$$

and integrating yields that

$$\theta'_\alpha(r) = \frac{\delta}{r}\theta'_\alpha(\delta) + \frac{1}{r}\int_\delta^r \big(\frac{\sin\theta_\alpha(s)\cos\theta_\alpha(s)}{s} - sv'(s)g(\theta_\alpha(s))\big)\, ds.$$

So

$$|\theta'_\alpha(r)| \le C\delta + \frac{1}{\delta} + \frac{2}{\delta}|\log(\delta)| + \frac{1-\delta^2}{2\delta}B \quad \text{for all} \quad 1 \ge r \ge \delta, \tag{16}$$

and we obtain

$$||\theta_\alpha||_\infty \le \pi + A\delta^2 + C\delta + \frac{1}{\delta} + \frac{2}{\delta}|\log(\delta)| + \frac{1-\delta^2}{2\delta}B.$$

(ii) First, we study the convergence of $\{\theta_\alpha\}$ in $[\delta, 1]$ for $0 < \delta < R$. By (i), (16) and (7), we deduce that the set $\{\theta''_\alpha\}$ is uniformly bounded on $[\delta, 1]$. So $\{\theta'_\alpha\}$ is equicontinuous, and, by (7), $\{\theta''_\alpha\}$ is equicontinuous as well. We can now construct a subsequence $\{\theta_{\alpha_j}\}$ which converges in $C^2([\delta, 1])$ to, say, $\bar\theta_\delta$, as $\alpha_j \to \infty$, and $\bar\theta_\delta$ satisfies

$$\theta'' + \frac{\theta'}{r} - \frac{\sin 2\theta}{2r^2} + v'(r)g(\theta) \ = 0 \tag{17}$$

in $(\delta, 1)$. Moreover, $\|\bar\theta_\delta\|_{C^2([\delta,1])} \le C(\delta)$.

We now choose a decreasing sequence $\delta_i \to 0$ and consider, for $\delta_j < \delta_i$, the restrictions of $\bar\theta_{\delta_j}$ to $(\delta_i, 1)$. By a diagonal process and the above arguments, we obtain a subsequence which converges to, say, $\bar\theta$ in $C^2([\delta, 1])$ for any $\delta > 0$. The function $\bar\theta$ satisfies (17) in $(0, 1)$.

To see that $\bar\theta$ is the unique solution of problem (P), we first remark that, by Corollary 2.3, this problem, which is equivalent to problem (P_α) with $\alpha = 0$, admits a unique solution.

To show that $\bar\theta$ solves problem (P), we only have to prove that $\bar\theta(0) = \pi$, $\bar\theta'(0) = 0$. Let $\{\theta_{\alpha_n}\}$ be a sequence which converges to $\bar\theta$ in $C^2([\delta, 1])$ for all $\delta > 0$, as $\alpha_n \to \infty$. For any $\varepsilon > 0$ and any fixed $0 < r < R$, Lemma 2.4, (ii) guarantees that $|\theta'_\alpha(r)| < Cr + \varepsilon$ when α is sufficiently large, which implies that $\bar\theta'(0) = 0$. Next, since $|\theta_\alpha(r) - \phi_\alpha(r)| \le Ar^2$, and $\phi_\alpha(r) \to \pi$ as $\alpha \to \infty$, we have $\bar\theta(0) = \pi$.

The uniqueness of the limit implies that no restriction to subsequences is needed in the statement that $\theta_\alpha \to \bar\theta$ in $C^2([\delta, 1])$ for all $\delta \in (0, 1)$.

We can repeat the same argument when $\alpha \to -\infty$. Finally, from (i) we know that I is a bounded set. The proof is completed by the next lemma. □

Lemma 2.5. *The mappings $\alpha \mapsto \theta_\alpha$ and $\alpha \mapsto \theta'_\alpha : \mathbb{R} \to C([0, 1])$ are continuous.*

Proof. Since $\theta_\alpha = 2\arctan\left(\frac{\alpha r}{2}\right) + \zeta_\alpha$, it is sufficient to prove the assertion for the mappings $\alpha \mapsto \zeta_\alpha$ and $\alpha \mapsto \zeta'_\alpha$. In the region $0 \le r \le R$ we have

$$\sup_{0\le r\le R} \frac{|\zeta_\alpha(r) - \zeta_\beta(r)|}{r^2} = \sup_{0\le r\le R} \frac{|\mathcal{F}_\alpha(\zeta_\alpha)(r) - \mathcal{F}_\beta(\zeta_\beta)(r)|}{r^2} \le$$
$$\sup_{0\le r\le R} \frac{|\mathcal{F}_\alpha(\zeta_\alpha)(r) - \mathcal{F}_\alpha(\zeta_\beta)(r)|}{r^2} + \sup_{0\le r\le R} \frac{|\mathcal{F}_\alpha(\zeta_\beta)(r) - \mathcal{F}_\beta(\zeta_\beta)(r)|}{r^2}.$$

By the contraction property of $\mathcal{F}_\alpha$ (Lemma 2.1) we deduce that there exists some $0 < \delta < 1$ such that

$$(1-\delta) \sup_{0\le r\le R} \frac{|\zeta_\alpha(r) - \zeta_\beta(r)|}{r^2} \le \sup_{0\le r\le R} \frac{|\mathcal{F}_\alpha(\zeta_\beta)(r) - \mathcal{F}_\beta(\zeta_\beta)(r)|}{r^2}. \tag{18}$$

On the other hand,

$$\begin{aligned}&|\mathcal{F}_\alpha(\zeta_\beta)(r) - \mathcal{F}_\beta(\zeta_\beta)(r)| \\ &= \frac{1}{2r}\Big| \int_0^r (r^2-\rho^2)\Big\{ \frac{\sin 2\zeta_\beta(\rho)}{2\rho^2}[\cos 2\phi_\alpha(\rho) - \cos 2\phi_\beta(\rho)] \\ &+ v'(\rho)\{g(\zeta_\beta(\rho)+\phi_\alpha(\rho)) - g(\zeta_\beta(\rho)+\phi_\beta(\rho))\} \\ &- \frac{\sin^2\zeta_\beta}{\rho^2}[\sin 2\phi_\alpha(\rho) - \sin 2\phi_\beta(\rho)]\Big\}\ d\rho\Big|.\end{aligned}$$

Because $|\zeta_\beta(r)| \le Ar^2$, there is some $\Gamma > 0$ such that

$$|\mathcal{F}_\alpha(\zeta_\beta)(r) - \mathcal{F}_\beta(\zeta_\beta)(r)| \le r^2\Gamma \sup_{0\le\rho\le r} |\phi_\alpha(\rho) - \phi_\beta(\rho)| \le \Gamma r^3|\alpha-\beta|. \tag{19}$$

Combining (18) and (19), we conclude that

$$\sup_{0\le r\le R} |\zeta_\alpha(r) - \zeta_\beta(r)| \le \frac{\Gamma R^3}{1-\delta}|\alpha-\beta|.$$

Thus, the mapping $\alpha \mapsto \zeta_\alpha :\ \mathbb{R} \to C([0,R])$ is continuous.

Next we note that, when $r \le R$,

$$\begin{aligned}&|\gamma(r,\zeta_\alpha,\phi_\alpha) - \gamma(r,\zeta_\alpha,\phi_\beta) + \gamma(r,\zeta_\alpha,\phi_\beta) - \gamma(r,\zeta_\beta,\phi_\beta)| \\ &\le C_1|\phi_\alpha-\phi_\beta| + \Big|\frac{\sin 2\zeta_\alpha - 2\zeta_\alpha - \sin 2\zeta_\beta + 2\zeta_\beta}{2r^2}\cos 2\phi_\beta\Big| \\ &+ 2|\zeta_\alpha-\zeta_\beta|\Big|\frac{\sin^2\phi_\beta}{r^2}\Big| + \Big|\frac{(\sin\zeta_\alpha - \sin\zeta_\beta)(\sin\zeta_\alpha+\sin\zeta_\beta)}{r^2}\Big||\sin 2\phi_\beta| \\ &+ v'(r)|g(\zeta_\alpha+\phi_\beta) - g(\zeta_\beta-\phi_\beta)| \le C_1|\phi_\alpha-\phi_\beta| + (C_2+2\beta^2)|\zeta_\alpha-\zeta_\beta|.\end{aligned}$$

Combining this with the previous result and (15) we obtain that the mapping $\alpha \mapsto \zeta'_\alpha : \mathbb{R} \to C([0,R])$ is continuous. The proof is completed by arguments from elementary ODE theory. □

3. Nonuniqueness of the steady director field in the presence of flow

The main result of this section will be the proof of Theorem 1.2. Throughout this section we require that $v'(r) > 0$ when $r > 0$.

It has been shown in [9] that $\mathcal{W} \hookrightarrow C([0,1])$, that $\theta(0) = 0$ for all $\theta \in \mathcal{W}$, and that $\mathcal{H}$ is bounded from below in the set $\mathcal{W}_b = \{\theta \in \mathcal{W} : \ \theta(1) = b\}$. Minimizing sequences, however, may be unbounded in $\mathcal{W}$ and, in view of the previous section, it is not evident that a minimizer can be found inside $\mathcal{W}$. To avoid this difficulty, we shall need

Lemma 3.1. *Let $\{\theta_k\} \subset \mathcal{W}$ be a sequence with the properties*
(i) *$\{\mathcal{H}(\theta_k)\}$ is bounded in $\mathcal{W}$ and*
(ii) *$||\theta_k||_\infty \leq C < \pi$ for all k. Then $\{\theta_k\}$ is bounded in $\mathcal{W}$.*

Proof. It has been shown in [9] that $\{\mathcal{H}(\theta_k)\}$ is bounded if and only if $\{\mathcal{E}(\theta_k)\}$ is bounded. Because of (ii), there is some $\delta > 0$ such that $|\sin\theta_k| \geq \delta|\theta_k|$, and so $\{\theta_k\}$ is bounded. □

Proof of Theorem 1.2. (i) We recall (see [9]) that, in the absence of flow, the solution $\mu(r) = 2\arctan(r\tan(\frac{b}{2}))$ of problem (P) minimizes $\mathcal{E}(\theta)$ over $\mathcal{W}_b$. For $0 < \varepsilon < \frac{\pi}{2} - \Theta_0^{\min}$ we consider the problem of finding a minimizer $\tilde{\theta}$ of $\mathcal{H}$ among the set $\{\theta \in \mathcal{W}_b : \ \Theta_0^{\max} \leq \theta \leq \Theta_0^{\min} + \varepsilon\}$. Let $\{\theta_i\} \subset \mathcal{W}_b$ be a minimizing sequence. By the above minimizing property of μ and the behavior of G, it follows that $\mathcal{H}(\max\{\min\{\theta_i, \Theta_0^{\min}\}, \mu\}) \leq \mathcal{H}(\theta_i)$, so that we may suppose that $\mu \leq \theta_i \leq \Theta_0^{\min}$.

By Lemma 3.1, $\{\theta_i\}$ is a bounded set in the Hilbert space $\mathcal{W}$, and therefore also in $H^1([r_0,1])$ for every $r_0 > 0$. So, there is a function $\bar{\theta} \in \mathcal{W}_b$ and a subsequence, called $\{\theta_i\}$ again, such that $\theta_i \rightharpoonup \bar{\theta}$ weakly in $\mathcal{W}$ and uniformly on compact subsets of $(0,1]$. The limit $\bar{\theta}$ is easily seen to be a minimizer. Because $\Theta_0^{\max} < \mu(r) \leq \bar{\theta}(r) \leq \Theta_0^{\min} < \Theta_0^{\min} + \varepsilon$ for $r \in (0,1)$, we conclude that $\bar{\theta}$ is a weak, and therefore also a strong, solution of problem (P).

(ii) For $\varepsilon > 0$ we define

$$\begin{aligned} D(\varepsilon) = &\int_0^1 v'(r)G(\Theta_0^{\min})r dr - \int_0^\varepsilon v'(r)G(\Theta_0^{\max})r\,dr \\ &- \int_\varepsilon^{1-\varepsilon} v'(r)G(\Theta_{-1}^{\min})r dr - \int_{1-\varepsilon}^1 v'(r)G(\Theta_0^{\max})r\,dr. \end{aligned}$$

Note that $D(\varepsilon) > 0$ when ε is sufficiently small. Let $\chi_\varepsilon \in \mathcal{W}_b$ be defined by

$$\chi_\varepsilon(r) = \begin{cases} 2\arctan\left(\frac{r}{\varepsilon}\tan\left(\frac{\Theta_{-1}^{\min}}{2}\right)\right) & (r \leq \varepsilon), \\ \Theta_{-1}^{\min} & (\varepsilon \leq r \leq 1-\varepsilon), \\ \frac{(r-1+\varepsilon)b + (1-r)\Theta_{-1}^{\min}}{\varepsilon} & (r \geq 1-\varepsilon), \end{cases}$$

and suppose that K is so small that $K\mathcal{E}(\chi_{\bar\varepsilon}) < D(\bar\varepsilon)$ for some $\bar\varepsilon > 0$ such that $D(\bar\varepsilon) > 0$. It is clear that

$$\mathcal{H}(\bar\theta) > \int_0^1 v'(r)G(\Theta_0^{\min})r dr > \mathcal{H}(\chi_{\bar\varepsilon}).$$

We now consider an $\mathcal{H}$-minimizing sequence $\{\theta_j\}$ in the set $\{\theta \in \mathcal{W}_b : \ -\pi \leq \theta \leq \Theta_0^{\min}\}$. Because of the minimizing property of $\bar\theta$, it is no restriction to assume that $\theta_j \leq \bar\theta$, which we shall do from now on. The next step is to show that there is a $\nu > 0$ such that we may suppose that $\theta_j \geq -\pi + \nu$. To do this, we first remark that the function

$$\psi(r) = -\pi + 2\arctan\big(r\tan\big(\frac{\Theta_0^{\min}}{2}\big)\big)$$

is the unique minimizer of $\mathcal{E}(\phi)$ in the set $\{\phi : \ \phi + \pi \in \mathcal{W}, \ \ \phi(1) = \Theta_{-1}^{\min}\}$. By the same reasoning as before we may assume that $\theta_j \geq \psi$. Next, for $r > 0$ and $\theta \in \mathcal{W}$ we define

$$\mathcal{H}_r(\theta) = \int_0^r \Big\{K\big((\theta')^2 + \frac{\sin^2\theta}{\rho^2}\big) + v'(\rho)G(\theta)\Big\}\rho\, d\rho.$$

By (9) there is a constant $C > 0$ such that $0 \leq v'(r) \leq Cr$, and upon choosing r_0 so small that

$$\Gamma = 8K - \frac{8K}{1 + r_0^2\tan^2(\frac{\Theta_0^{\min}}{2})} - r_0^4 + \{CG(\Theta_{-1}^{\min}) - CG(\Theta_0^{\max})\}\frac{r_0^3}{3} > 0,$$

we define

$$\tilde\theta_j(r) = \begin{cases}\theta_j(r) & (r \geq r_0)\\ 2\arctan\big(\frac{r}{r_0}\tan\big(\frac{\theta_j(r_0)}{2}\big)\big) & (r \leq r_0).\end{cases}$$

Using the results of [9] we find that

$$\mathcal{H}_{r_0}(\tilde\theta_j) \leq 4K\sin^2\big(\frac{\psi(r_0)}{2}\big) + CG\left(\Theta_0^{\max}\right)\frac{r_0^3}{3}. \tag{20}$$

To construct a contradiction, let us now assume that there is a subsequence, called $\{\theta_j\}$ again, and a sequence $r_j \to 0$, such that $\theta_j(r_j) \to -\pi$. It follows easily that

$$\liminf_{j\to\infty}\int_0^{r_0}\big\{(\theta_j')^2 + \frac{\sin^2\theta_j}{r^2}\big\}r dr \geq 4K + 4K\cos^2\big(\frac{\psi(r_0)}{2}\big),$$

so that

$$\mathcal{H}_{r_0}(\theta_j) \geq 4K + 4K\cos^2\big(\frac{\psi(r_0)}{2}\big) - r_0^4 + CG(\Theta_{-1}^{\min})\frac{r_0^3}{3} \tag{21}$$

when j is sufficiently large. Subtracting (20) from (21) we obtain that

$$\mathcal{H}_{r_0}(\theta_j)-\mathcal{H}_{r_0}(\tilde{\theta}_j)\geq 4K+4K\cos(\psi(r_0))-r_0^4+\{CG(\Theta_{-1}^{\min})-CG(\Theta_0^{\max})\}\frac{r_0^3}{3}.$$

On the other hand, by the definition of ψ, we have

$$\cos(\psi(r_0))=1-\frac{2}{1+r_0^2\tan^2\left(\frac{\Theta_0^{\min}}{2}\right)},$$

so that $\mathcal{H}_{r_0}(\theta_j)-\mathcal{H}_{r_0}(\tilde{\theta}_j)\geq\Gamma$. This contradicts the fact that $\{\theta_j\}$ is a minimizing sequence. Therefore, we may conclude that there is some $\nu>0$ such that $\theta_j(r)+\pi\geq\nu$ for all r, when j is sufficiently large. Lemma 3.1 now implies that $\{\theta_j\}$ is bounded in $\mathcal{W}$. Therefore, there is a function $\tilde{\theta}\in\mathcal{W}$ and a subsequence, called $\{\theta_j\}$ again, such that $\theta_j\rightharpoonup\tilde{\theta}$ weakly in $\mathcal{W}$. Now it is easy to see that $\tilde{\theta}$ is a solution to problem (P), with the property that $\mathcal{H}(\tilde{\theta})<\mathcal{H}(\bar{\theta})$.

To conclude the proof of (ii), we shall now show the existence of $\hat{\theta}$. We shall write $\bar{\theta}_b$ and $\tilde{\theta}_b$ to indicate the dependence on b of the above solutions. Choosing $b^+\in(b,\Theta_0^{\min}]$ and $b^-\in[\Theta_0^{\max},b)$, we remark that the above results remain true when b is replaced by b^+ or by b^-. With reference to the notation of Section 2, we write $\theta_{\alpha_1}=\bar{\theta}_b$, $\theta_{\alpha_2}=\bar{\theta}_{b^-}$, $\theta_{\alpha_3}=\tilde{\theta}_{b^+}$, and $\theta_{\alpha_4}=\tilde{\theta}_b$. We claim that $\alpha_1>\alpha_2>\alpha_3>\alpha_4$. To see this, we first remark that all these solutions are different. Corollary 2.3 therefore guarantees that all α's are pairwise different. Because of the minimizing properties of θ_{α_1} and θ_{α_2}, and the uniqueness of solutions of the corresponding initial value problems, it is obvious that θ_{α_1} and θ_{α_2} cannot intersect, which implies that $\alpha_1>\alpha_2$; a similar reasoning yields that $\alpha_3>\alpha_4$. Finally, suppose for contradiction that $\alpha_2<\alpha_3$. The above results concerning $\bar{\theta}$ and $\tilde{\theta}$ imply that there is some $\bar{r}\in(0,1)$ such that $\theta_{\alpha_3}(\bar{r})<\Theta_0^{\max}<\theta_{\alpha_2}(\bar{r})$. Therefore, there exists some $\tilde{r}\in(0,\bar{r})$ such that $\theta_{\alpha_2}(\rho)<\theta_{\alpha_3}(\rho)$ for all $\rho\in(0,\tilde{r})$, and $\theta_{\alpha_2}(\tilde{r})=\theta_{\alpha_3}(\tilde{r})$. Again, this intersection leads to a contradiction. This proves the claim.

Next, we consider the shooting problem (P_α), where $\alpha_2>\alpha>\alpha_3$. By Lemma 2.5 and the construction of α_2 and α_3, we find that there is some $\alpha^*\in(\alpha_3,\alpha_2)$ such that $\theta_{\alpha^*}(1)=b$. Thus, we have constructed a third solution $\hat{\theta}=\theta_{\alpha^*}$. □

Remark 3.2. The existence of $\hat{\theta}$ is strongly suggested by the fact that $\tilde{\theta}$ and $\bar{\theta}$ are "local" minimizers, so that a mountain pass-like reasoning comes into mind. The reason to look for a shooting argument lies in the fact that direct application of the mountain pass lemma is not so easy, because the Palais-Smale condition seems to fail in this case.

4. Finite time blow-up

We begin this section by proving the comparison principle, announced in Section 1.

Proof of Lemma 1.4. Because $\mathcal{H}(\theta)$ and $\mathcal{H}(\psi)$ are finite, there are integers $k(t)$ and $m(t)$ such that $\theta(0,t) = k(t)\pi$, $\psi(0,t) = m(t)\pi$. To begin with, we assume that $k(0) = m(0)$, which implies that $k(t) = m(t)$ up to the first jumping time t^* (which may be infinite: no jump occurs at all; this case is dealt with in the same way as what follows now). Suppose, without loss of generality, that the supersolution ψ is the first one to jump (simultaneity included). Let $t < t^*$. Define

$$\hat{r}(t) = \max\{r : |\theta(\rho,t) + \psi(\rho,t) - (k(t)+m(t))\pi| \leq \tfrac{\pi}{2} \text{ for all } \rho \leq r\}.$$

Obviously, $\hat{r}(t) > 0$. We claim that $R(t) \equiv \inf_{\tau \leq t} \hat{r}(\tau) > 0$. Suppose, to the contrary, that there is a sequence $\{\tau_i\}$ such that $0 \leq \tau_i \leq t$ and $\hat{r}(\tau_i) \to 0$ as $i \to \infty$. Then, by compactness, there is a subsequence $\{\tau_{ij}\}$ which converges to, say, $\bar{\tau}$ and, by continuity, we have that

$$|\psi(0,\bar{\tau}) + \theta(0,\bar{\tau}) - (k(\bar{\tau}) + m(\bar{\tau}))\pi| = \frac{\pi}{2}.$$

So, $\psi(0,\bar{\tau}) + \theta(0,\bar{\tau})$ is not an integer multiple of π, which is a contradiction. This proves our claim.

Subtracting the differential equations, multiplying the result by $(\theta-\psi)^+$ and integrating over the disc we obtain

$$\frac{d}{2dt}\int_0^1 ((\theta-\psi)^+)^2 \, r\,dr \leq -\int_0^1 \frac{\cos(\theta+\psi)\sin(\theta-\psi)}{r^2}(\theta-\psi)^+ r\,dr \qquad (22)$$
$$+\int_0^1 v'(r)(g(\theta)-g(\psi))(\theta-\psi)^+ r dr.$$

Our next claim says that there is a $0 < \bar{t} < t^*$ such that $(\theta-\psi)^+(r,t) \leq \frac{\pi}{2}$ for all r, when $t \leq \bar{t}$. Suppose, to the contrary, that there are sequences $\{t_i\}$, $\{r_i\}$ such that $t_i \to 0$ and $\theta(r_i,t_i) - \psi(r_i,t_i) > \frac{\pi}{2}$. By compactness there is a subsequence $\{r_{ij}\}$, which converges to, say, $\bar{r}$. By continuity we have that $\theta_0(\bar{r}) - \psi_0(\bar{r}) \geq \frac{\pi}{2}$, which is a contradiction. This proves the claim.

For $t \leq \bar{t}$ we write (22) as follows:

$$\frac{d}{2dt}\int_0^1 ((\theta-\psi)^+)^2 r\,dr \leq -\int_0^{R(\bar{t})} \frac{\cos(\theta+\psi)\sin(\theta-\psi)}{r^2}(\theta-\psi)^+ \, rdr$$

$$- \int_{R(\bar{t})}^{1} \frac{\cos(\theta+\psi)\sin(\theta-\psi)}{r^2}(\theta-\psi)^+ \; rdr + C_1 \int_0^1 ((\theta-\psi)^+)^2 r\, dr.$$

Since $k(t) = m(t)$, we know that $\cos(\theta+\psi)$ in the first integral on the right-hand side is nonnegative. On the other hand, because $\theta - \psi \leq \frac{\pi}{2}$, we have that $(\theta-\psi)^+ \sin(\theta-\psi) \geq 0$. So, the contribution of the first integral on the right-hand side is nonpositive, and we have

$$\frac{d}{2dt}\int_0^1 ((\theta-\psi)^+)^2 \; rdr \leq - \int_{R(\bar{t})}^{1} \frac{\cos(\theta+\psi)\sin(\theta-\psi)(\theta-\psi)^+}{r} \; dr \\ + C_1 \int_0^1 ((\theta-\psi)^+)^2 r\, dr.$$

Thus, we have obtained that

$$\frac{d}{2dt}\int_0^1 ((\theta-\psi)^+)^2 \; rdr \leq C_2 \int_0^1 ((\theta-\psi)^+)^2 \; rdr,$$

from which we derive that $\theta(\cdot,t) \leq \psi(\cdot,t)$ for $t \leq \bar{t}$. It is easy to extend this to $t \leq t^*$. Thus, we have established a comparison principle up to the first jump.

It is easy to see that $k(t^{*+}) \leq m(t^{*+})$. Equality can occur when θ and ψ jump simultaneously with identical jumps; in that case, we can repeat the above argument up to the next jump. So we may assume that, after the jump, we have that $k(t) < m(t)$. Up to the next jump, for each t there is a maximal $\bar{r}(t)$ such that $\theta(r,t) < \psi(r,t)$ for $r < \bar{r}(t)$; the same argument as before shows that $\bar{r}(t)$ is bounded away from 0 as long as we stay away from the next jumping time; this gives us a new $R(t)$. We can repeat the same arguments as before, with the proviso that the first integral on the right-hand side vanishes, because $(\theta-\psi)^+$ vanishes where $r < R(t)$. When $k(0) < m(0)$ we have the same argument. □

The remainder of this section is devoted to the

Proof of Theorem 1.5. (i) As in the steady case, we go back to the formulation of the problem in terms of mappings $D^2 \to S^2$:

$$\mathbf{u}_t = \Delta\mathbf{u} + |\nabla\mathbf{u}|^2\mathbf{u} - f(\mathbf{x},\mathbf{u}),$$

with f as in Section 2 and appropriate boundary and initial conditions. Upon a simple shift, we apply [12], Chapter 15, Propositions 3.1 and 3.2, to deduce that a unique local solution $\mathbf{u}$ with the required smoothness properties exists. Extending this result to h is straightforward. □

Because of the apparent symmetry, we shall write $h(\mathbf{x},t) = \hat{h}(r,t)$ and omit the hat immediately.

Before proving (ii), we first present some auxiliary results, the first one being the construction of a local subsolution which blows up in finite time. We follow the arguments of [4], but we must restrict ourselves to a region $0 \le r \le r_0$, for some suitable $r_0 < R$.

Lemma 4.1. *Let $0 < \epsilon < 1$ and $T_0 > 0$ be given. Then there exist $\mu > 0$, $r_0 \in (0,1)$ and a function $\alpha(t) > 0$ (< 0) such that the function*

$$f(r,t) = \phi_{\alpha(t)}(r) + (-)\arccos\big(\frac{\mu^2 - r^{2(1+\epsilon)}}{\mu^2 + r^{2(1+\epsilon)}}\big)$$

has a jump of type I at $t = T_0$ and satisfies $f_t \le (\ge)\Lambda(f,r,t)$ in $(0,r_0)\times(0,T_0)$. (Here ϕ_α is as in Section 2.)

Proof. We consider only the case that $\alpha(t) > 0$. Writing

$$\Theta_\mu(r) = \arccos\big(\frac{\mu^2 - r^{2(1+\epsilon)}}{\mu^2 + r^{2(1+\epsilon)}}\big),$$

we choose μ so large that $\cos\Theta_\mu(r) \ge \frac{1}{1+\epsilon}$ for all $r \le 1$, and r_0 so small that $\epsilon_1 = \frac{2\mu\epsilon}{\mu^2+1} - r_0^{1-\epsilon}B > 0$, where B is as in Lemma 2.1. Next, when $M(\epsilon) = \max_{s\in(0,\infty)} \frac{s^{2-\epsilon}}{4+s^2}$, let $\delta \le \frac{\epsilon_1}{M(\epsilon)}$ and define $\alpha(t)$ by

$$\alpha'(t) = \delta\alpha^{2-\epsilon}, \quad \alpha(0) = \alpha_0,$$

where $T_0 = \frac{1}{\alpha_0^{1-\epsilon}\delta(1-\epsilon)}$. With these choices, the arguments of [4] are easily reconstructed. □

Remark 4.2. When $\xi \in C^1([0,\hat{r}])$ has the property that $\xi(0) = 0$, $\xi'(0) > 0$ and $\xi(r) > 0$ for all $0 < r \le \hat{r}$, then ϵ and α_0 (or T_0) can be chosen such that $f(r,0) \le \xi(r)$ for all $r \le \hat{r}$.

We shall also consider the steady problem

$$(\mathrm{P_{st}}) \quad \begin{cases} \Delta\theta - \frac{\sin 2\theta}{2r^2} + v'(r)g(\theta) = 0 & \text{in } [0,1), \\ \theta(r=1) \ = b. \end{cases}$$

Lemma 4.3. *Let h be a global solution of problem $(\mathrm{P_{par}})$. Then there exist a solution $\tilde{h} \in C^\infty([0,1)) \cap C([0,1])$ of problem $(\mathrm{P_{st}})$ and a sequence $t_k \to \infty$ such that $h(\cdot,t_k)$ converges to $\tilde{h}$, uniformly on sets $[\hat{r},1]$ for all $0 < \hat{r} < 1$.*

Proof. Multiplying the differential equation in problem (P_{par}) by h_t and integrating partially, we obtain that

$$\frac{1}{2K}\frac{d}{dt}\mathcal{H}(h) = -\int_{D^2} h_t^2. \tag{23}$$

Because $\mathcal{H}(h)$ is bounded from below (see [9]), it follows that

$$h_t \in L^2(D^2 \times (0,\infty)). \tag{24}$$

In particular, we conclude that there exists a sequence $\{t_k \to \infty\}$ with the property that

$$\lim_{k\to\infty}\int_{D^2} h_t^2(t_k) = 0.$$

This statement is equivalent to

$$\Lambda(h(t_k)) = \Delta h(t_k) - \frac{\sin 2h(t_k)}{2r^2} + v'(r)g(h(t_k)) \to 0 \text{ in } L^2(D^2). \tag{25}$$

On the other hand, by (23), $\mathcal{H}(h)$ is not increasing in time. We now follow the arguments in [9], Section 3.2, to deduce that there is a function $\tilde{h}$ with the property that $\tilde{h} - k\pi \in \mathcal{W}$, such that (along a subsequence) $h(t_k) \to \tilde{h}$ uniformly on sets $[\tilde{r}, 1]$ for $\tilde{r} > 0$, and such that $h_r(\cdot, t_k) \rightharpoonup \tilde{h}'$ weakly in $L^2(D^2)$. Moreover, a slight adaptation of the arguments in [9] yields that

$$\frac{\sin 2h(r,t_k)}{r} \rightharpoonup \frac{\sin 2\tilde{h}(r)}{r} \text{ weakly in } L^2(D^2).$$

It is easy to conclude that $\tilde{h}$ is a weak solution of the steady problem. Smoothness follows from the arguments given in the beginning of Section 2.

Lemma 4.4. *If $\tilde{h}$ is a solution of problem* (P_{st}) *with $\mathcal{H}(\tilde{h}) < \infty$ and $b > B_2$ ($b < B_1$), then $\tilde{h}'(0) > 0$ or $\tilde{h}(0) > \pi$ ($\tilde{h}'(0) < 0$ or $\tilde{h}(0) < -\pi$).*

Proof. We consider only the case that $b > B_2$. First, we remark that $\tilde{h}(0) = k\pi > 0$, and we shall assume that $k = 1$. Let us first suppose that $\tilde{h}'(0) = 0$. Then, by Theorem 1.1 (ii), $\tilde{h} = \lim_{\alpha\to\infty}\theta_\alpha$, so $b \le B_2$, which is a contradiction.

Next, suppose that $\tilde{h}'(0) < 0$. From the fact that $\tilde{h}$ is a steady solution, we deduce that there is some $\beta > 0$ such that $\tilde{h} = \pi + \theta_{-\beta}$. Writing

$$\mathcal{A} = \{\alpha \in \mathbb{R} : \exists\, r \in (0,1) : \theta_\alpha(r) > \tilde{h}(r)\},$$

we deduce from Theorem 1.1 (ii) that $\mathcal{A} \ne \emptyset$ and from Lemma 2.5 that $\mathcal{A}$ is open. On the other hand, $-\beta \notin \mathcal{A}$ and we choose some $\bar{\alpha} \in \partial\mathcal{A}$. Note that $\theta_{\bar{\alpha}}(r) \le \tilde{h}(r)$ for all r. Let $\{\alpha_i\} \subset \mathcal{A}$ such that $\alpha_i \to \bar{\alpha}$. Then, by Lemma

2.5, $\theta_{\alpha_i} \to \theta_{\bar{\alpha}}$ uniformly on $[0,1]$. Therefore, there exists a point $\bar{r}$ such that $\theta_{\bar{\alpha}}(\bar{r}) = \tilde{h}(\bar{r})$. Because $\tilde{h}(0) = \pi$ and $\tilde{h}(1) > B_2$, it is clear that $\bar{r} \in (0,1)$, and we find that $\theta'_{\bar{\alpha}}(\bar{r}) = \tilde{h}'(\bar{r})$. By the uniqueness property of the corresponding initial value problem, we deduce that $\tilde{h} = \theta_{\bar{\alpha}}$, which is a contradiction. $\square$

Lemma 4.5. *If h is a global solution of problem* $(\mathrm{P}_{\mathrm{par}})$*, with $b > B_2$ ($b < B_1$), then there exist $\bar{\varepsilon} > 0$ and $\bar{r} > 0$ with the property that, for any $0 < r < \bar{r}$, there is some $\hat{t}(r)$ such that $h(r,t) \geq \pi + \bar{\varepsilon} r$ ($h(r,t) \leq -\pi - \bar{\varepsilon} r$) when $t \geq \hat{t}(r)$.*

Proof. We consider only the case that $b > B_2$. Suppose, to the contrary, that for all $\varepsilon > 0$ there is a sequence $r_i \to 0$ and, for each i, a sequence $t_{ij} \to \infty$ such that $h(r_i, t_{ij}) < \pi + \varepsilon r_i$ for all j. Let ε be given and let $\tilde{r}$ and $\tau_i \to \infty$ be such that $h(\tilde{r}, \tau_i) < \pi + \frac{\varepsilon \tilde{r}}{3}$. In what follows, it is convenient to have that $\tau_{i+1} \geq \tau_i + 1$, and we shall assume this from now on. Applying the arguments in [9] and the fact that $\mathcal{H}(h(\cdot,t))$ is bounded, we find that there is a continuous function h^* such that, along an appropriate subsequence, which we call $\{\tau_i\}$ again, $h(\cdot, \tau_i) \to h^*$, uniformly on sets $[\rho, 1]$ for all $\rho > 0$. In particular, we have that $h^*(\tilde{r}) \leq \pi + \frac{\varepsilon \tilde{r}}{3}$, and we deduce that there are numbers $0 < p < \tilde{r} < q$ such that $h(r, \tau_i) \leq \pi + \frac{\varepsilon r}{2}$ $(p \leq r \leq q)$ when τ_i is sufficiently large. On the other hand, we have, by Schwarz's inequality and (24), that

$$\int_{\tau_i}^{\tau_i+1} \int_p^q |h_t| r dr dt \leq \sqrt{\frac{q^2 - p^2}{2}} \sqrt{\int_{\tau_i}^{\tau_i+1} \int_{D^2} h_t^2} \to 0 \text{ as } \tau_i \to \infty.$$

This implies in particular that, when τ_i is sufficiently large, there exists some $r_i \in [p,q]$ with the property that $h(r_i, \tau_i + \tilde{\tau}) < \pi + \varepsilon r_i$ for all $0 \leq \tilde{\tau} \leq 1$. Furthermore, we know that

$$\sum_i \int_{\tau_i}^{\tau_i+1} \int_{D^2} h_t^2 < \infty,$$

from which we deduce that there exists a sequence $t_i^* \in [\tau_i, \tau_i + 1]$ with the property that

$$\lim_{t_i^* \to \infty} \int_{D^2} h_t^2(r, t_i^*) = 0.$$

Again taking a subsequence, we repeat the arguments in the proof of Lemma 4.3 to show that $h(r, t_i^*) \to \tilde{h}(r)$, uniformly on sets $[\hat{r}, 1]$, where $\tilde{h}$ is a solution of (P_{st}). Moreover, by compactness, there is a subsequence $r_i \to r^* \in [p,q]$, and we conclude that

$$\tilde{h}(r^*) \leq \pi + \varepsilon r^*. \tag{26}$$

By Lemma 4.4 and the results of Section 2, we know that $\tilde{h}(r) = \pi + \theta_\alpha(r)$ for some $\alpha > 0$, and, consequently, that $\tilde{h}(r) \geq \pi + 2\arctan\left(\frac{\alpha r}{2}\right) - Ar^2$ when $r \leq R$. Because ε and r^* in (26) may be constructed as small as we want, this implies that there is a sequence $\{\tilde{h}_j\}$ of steady solutions of problem $(\mathrm{P_{st}})$, such that $\tilde{h}'_j(0) \to 0$ as $j \to \infty$. By Lemma 2.5 and Theorem 1.1 (ii), we find that $b \leq B_2$, which is a contradiction. □

We are now ready to complete the proof of Theorem 1.5.

Proof of Theorem 1.5 (ii). Again, we consider the case that $b > B_2$. Suppose, to the contrary, that h is a global smooth solution. The construction of a contradiction takes place in several steps. We shall often use the arguments and notation of the proof of Lemma 4.1. To begin with, we remark that

$$\Theta_\mu(r) = \arccos\big(\frac{\mu^2 - r^{2(1+\epsilon)}}{\mu^2 + r^{2(1+\epsilon)}}\big) = 2\arctan\big(\frac{r^{1+\epsilon}}{\mu}\big).$$

Step 1. Claim 1: There exist $r^* > 0$ and $t^* \geq 0$ such that $h(r,t) \geq 0$ when $t \geq t^*$ and $r \leq r^*$. To prove this, let $0 < \epsilon < 1$ be given; ϵ will be fixed throughout the proof of Theorem 1.5 (ii). Choose $\mu_1 > 0$ so large that

$$\cos\Theta_{\mu_1}(1) \geq \frac{1}{1+\epsilon},$$

and choose $r_1 < \bar{r}$ so small that

$$\frac{2\mu_1\epsilon}{\mu_1^2 + 1} - r_1^{1-\epsilon}B > 0.$$

Here, $\bar{r}$ is as in Lemma 4.5, and for $t > \hat{t}(r_1)$, we have that $h(r_1,t) \geq \pi + \bar{\varepsilon} r_1$. Let $-n\pi \leq \min_{r \leq r_1} h(r, \hat{t}(r_1)) < -(n-1)\pi$. We now consider

$$(\mathrm{P_{par1}}) \quad \begin{cases} h_{1,t} = \Lambda(h_1, r, t) & \text{in} \quad D^2_{r_1} \times (0,\infty), \\ h_1(r,0) = h(r, \hat{t}(r_1)) & \text{in} \quad D^2_{r_1}, \\ h_1(r_1,t) = h(r_1, \hat{t}(r_1) + t) & \text{in} \quad (0,\infty). \end{cases}$$

Here, $D^2_{r_1}$ is the disc with radius r_1. Following the proof of Lemma 4.1, we construct, on the basis of ϵ and r_1, a function $f_1(r,t)$ with corresponding $\alpha_1(t) > 0$ and with the property that $f_{1,t} \leq \Lambda(f_1, r, t)$ on $(0, r_1) \times [0, T_1)$, with blow-up time T_1, and such that $f_1(r,0) < \pi$ for all $0 \leq r \leq r_1$. Next, we set $\tilde{f}_1 = f_1 - (n+1)\pi$. It follows by comparison that

$$h_1(r, T_1) \geq \Theta_{\mu_1}(r) - n\pi \text{ for all } 0 \leq r \leq r_1.$$

For $0 \le r \le r_1$ and $0 < \tilde{\tau} < T_1$, let $\tilde{f}_{1,\tilde{\tau}}(r,t) = \tilde{f}_1(r, t - \tilde{\tau})$. It is easy to see that $\tilde{f}_{1,\tilde{\tau}}(r,t) < h_1(r,t)$ for all $\tilde{\tau} \le t < T_1 + \tilde{\tau}$. In particular, we have that $h(r, \hat{t}(r_1) + T_1 + \tilde{\tau}) \ge \Theta_{\mu_1}(r) - n\pi$. Extending this argument beyond $\hat{t}(r_1) + 2T_1$, we prove that

$$h(r,t) \ge \Theta_{\mu_1}(r) - n\pi \quad \text{for all } t \ge \hat{t}(r_1) + T_1,\ 0 \le r \le r_1. \tag{27}$$

When $n \le 0$, the claim is true. Otherwise, we choose $\mu_2 > \mu_1$ and $0 < r_2 \le r_1$ such that

$$\frac{2\mu_2\epsilon}{\mu_2^2 + 1} - r_2^{1-\epsilon} B > 0.$$

Let $\bar{t} = \max\{T_1 + \hat{t}(r_1), \hat{t}(r_2)\}$, and consider the problem

$$(\mathrm{P}_{\mathrm{par2}}) \quad \begin{cases} h_{2,t} = \Lambda(h_2, r, t) & \text{in} \quad D^2_{r_2} \times (0, \infty), \\ h_2(r, 0) = h(r, \bar{t}) & \text{in} \quad D^2_{r_2}, \\ h_2(r_2, t) = h(r_2, \bar{t} + t) & \text{in} \quad (0, \infty). \end{cases}$$

In view of (27), the fact that $\Theta_{\mu_2}(r) < \Theta_{\mu_1}(r)$ for $r > 0$, and the fact that $h(0,t) = 0 > -n\pi$, we find that there is a constant $\Gamma_1 > 0$ with the property that

$$h_2(r, 0) \ge \Gamma_1 + \Theta_{\mu_2}(r) - n\pi \quad \text{for all } 0 \le r \le r_2. \tag{28}$$

It is now easy to construct a subsolution $f_2 = \phi_{\alpha(t)} + \Theta_{\mu_2}$ in the sense of Lemma 4.1, defined on $[0, r_2] \times [0, T_2)$, with blow-up time T_2, where α_0 is so small that

$$f_2(r, 0) - n\pi \le h_2(r, 0). \tag{29}$$

Note that, by the definition of $\hat{t}$, we have that $f_2(r_2, t) - n\pi \le \Theta_{\mu_2}(r_2) - (n-1)\pi < h_2(r_2, t)$. We deduce by comparison that, when $t \ge T_2$,

$$\begin{aligned} &h_2(r,t) \ge -(n-1)\pi + \Theta_{\mu_2}(r), \text{ and} \\ &h(r,t) \ge \Theta_{\mu_2}(r) - (n-1)\pi \quad \text{for all } t \ge \bar{t} + T_2,\ 0 \le r \le r_2. \end{aligned} \tag{30}$$

Repeating this procedure a finite number of times, we prove that there exist $r^* > 0$ and $t^* \ge 0$ such that $h(r,t) \ge 0$ when $t \ge t^*$ and $r \le r^*$, which is Claim 1.

Step 2. Claim 2: There is a time $\tilde{t} \ge t^*$ such that $h_r(0,t) > 0$ when $t \ge \tilde{t}$. To see this, we first use Claim 1 and Lemma 4.5 to deduce that it is no restriction to suppose that $h(r,0) \ge 0$ for $0 \le r \le r^*$ and $h(r^*, t) \ge \pi + \bar{\varepsilon} r^*$ for all $t > 0$. Then, in a way similar to the proof of Claim 1, we find that there are a $\bar{\mu} > 0$ and a time $\tilde{t}$ such that

$$h(r,t) \ge \Theta_{\bar{\mu}}(r) = 2\arctan\Big(\frac{r^{1+\epsilon}}{\bar{\mu}}\Big) \quad \text{for all } 0 \le r \le r^*,\ t \ge \tilde{t}. \tag{31}$$

Finally, we suppose, to the contrary, that $h_r(0,\tau) = 0$ for some $\tau \geq \tilde{t}$. By the smoothness of h we then have that

$$h_{rr}(0,\tau) = 2\lim_{r\to 0}\frac{h(r,\tau)}{r^2} < \infty.$$

On the other hand,

$$\lim_{r\to 0}\frac{2}{r^2}\arctan\big(\frac{r^{1+\epsilon}}{\bar{\mu}}\big) = \infty,$$

which gives a contradiction. This proves Claim 2.

Step 3. Let $\bar{\mu}$ and $\tilde{t}$ be as in (31). Choose $\mu > \bar{\mu}$ and choose $r_0 < r^*$ so small that

(i) $\frac{2\mu\epsilon}{\mu^2+1} - r_0^{1-\epsilon}B > 0$ and (ii) $\Theta_\mu(r_0) < \bar{\varepsilon}r_0$.

Let $t^0 = \max\{\tilde{t}, \hat{t}(r_0)\}$, and choose α_0 and δ so small that the corresponding $f(r,t)$ (see the proof of Lemma 4.1) satisfies $f(r,0) < h(r,t^0)$ for all $0 < r < r_0$ and $f(r_0,t) < h(r_0,t^0+t)$ for all $t < T_0$, the time of blow-up of f. (Note that this can be done because $\mu > \bar{\mu}$.) Comparison implies that $h(0,t^0+T_0) \geq \pi$, which is a contradiction. Thus, the proof is finished. □

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