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ON CRITICAL SEMILINEAR ELLIPTIC SYSTEMS

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Abstract. We establish in this paper existence results for critical strongly indefinite semilinear elliptic systems defined on both bounded domains and $\mathbb{R}^N$.

1. Introduction. Our primary objective is to investigate the existence of solutions of the semilinear elliptic system

$$-\Delta u+u = |v|^{q-1}v+g(x,v), \quad -\Delta v+v = |u|^{p-1}u+f(x,u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

$$u(x)\to 0 \quad \text{and} \quad v(x)\to 0 \quad \text{as} \quad |x|\to\infty, \tag{1.2}$$

where $\frac{1}{p+1}+\frac{1}{q+1}=\frac{N-2}{N}$, $p,q>1$, which is known as the critical hyperbola. The system is variational. Critical points of the associated functional

$$I(z) = \int_{\mathbb{R}^N} (\nabla u\cdot\nabla v+uv)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx - \frac{1}{q+1}\int_{\mathbb{R}^N} |v|^{q+1}\,dx - \int_{\mathbb{R}^N} (F(x,u)+G(x,v))\,dx,$$

defined on a suitable function space, are weak solutions of (1.1)–(1.2), where $z=(u,v)$, $F(x,u)=\int_0^u f(x,t)\,dt$, $G(x,v)=\int_0^v g(x,t)\,dt$. Special features of the functional I are that it has a strongly indefinite quadratic part and the growths of u and v in nonlinear terms are mutually complementary. The problem can be studied by linking-type theorems based on a choice of fractional Sobolev spaces. In bounded domains, the compactness will remain in the subcritical case; i.e., p and q satisfy $\frac{1}{p+1}+\frac{1}{q+1}>\frac{N-2}{N}$, $p,q>1$, and f and g contain lower growth terms. The problem in bounded domains has been studied by many authors; particularly, we refer to [16] and [20]. On the other hand, nonexistence results can be found via a Pohozaev's type-identity

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for Hamiltonian systems in [25], [29] for the critical case: $\frac{1}{p+1}+\frac{1}{q+1}=\frac{N-2}{N}$, $p,q>1$. Actually, in this case a lack of compact Sobolev inclusions leads to a failure of the (PS) condition in general. The existence problem becomes delicate. In [10], Brézis and Nirenberg have shown that a positive solution exists for critical scalar semilinear elliptic equations. The crucial point in their arguments is that the $(PS)_c$ condition is valid for c in an interval related to the best Sobolev constant; then solutions can be found by critical-point theory in the interval. Inspired by the work of [10], Hulshof et al. in [19] proved the existence of solutions for the system

$$\begin{cases} -\Delta v = \lambda u + |u|^{p-1}u, & \text{in } \Omega, \\ -\Delta u = \mu v + |v|^{q-1}v, & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

with proper λ and μ. They used a dual variational method originally due to [12]. This approach was also used in [4] as an alternative for the methods in [10]. The main advantage of the argument is that the associated dual functional possesses a geometry of the mountain pass. It is easier to get control of critical values described by the mountain pass theorem than by the linking theorem. An existence result then can be obtained by combining local compactness and the mountain pass theorem.

Our problem is the setting in $\mathbb{R}^N$. There is a lack of compactness due to the fact that $\mathbb{R}^N$ is unbounded, which is other than the critical case. For subcritical autonomous systems, Figueiredo and the author [17] proved the existence of positive radial solutions. We decompose spaces with a spectral family of operators and apply the linking theorem. In the general case, one can only expect local compactness because as we show in Section 4, there are energy levels of associated functionals which are obstacle points of the compactness. So in our case, we encounter two types of the loss of compactness caused by both critical exponents and unbounded domains. To study the existence, we begin with a problem in a bounded domain Ω :

$$-\Delta u + u = |v|^{q-1}v + g(x,v), \quad -\Delta v + v = |u|^{p-1}u + f(x,u) \text{ in } \Omega, \tag{1.3}$$

$$u(x) = 0, \quad v(x) = 0 \quad \text{on} \quad \partial\Omega. \tag{1.4}$$

Let

$$f_1(x,t) := |t|^{p-1}t + f(x,t), \quad g_1(x,t) := |t|^{q-1}t + g(x,t),$$

$$\mathcal{F}(x,t) = \int_0^t f_1(x,s)\,ds, \quad \mathcal{G}(x,t) = \int_0^t g_1(x,s)\,ds.$$

We assume that

(H1) $f, g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are measurable in the first variable, continuous in the second variable, and $f(x,0) = g(x,0) = 0$. Both $\mathcal{F}(x,t)$ and $\mathcal{G}(x,t)$ are increasing and strictly convex in t.

(H2) $\lim_{t\to 0} f(x,t)/t = 0, \quad \lim_{t\to 0} g(x,t)/t = 0, \quad \forall x \in \mathbb{R}^N.$

(H2) $\lim_{t\to\infty} \frac{f(x,t)}{|t|^{p-1}t} = 0, \quad \lim_{t\to\infty} \frac{g(x,t)}{|t|^{q-1}t} = 0, \quad \forall x \in \mathbb{R}^N.$

(H4) There are constants $2 < \alpha \le p+1$, $2 < \beta \le q+1$ such that

$$0 < \alpha F(x,t) \le t f(x,t), \quad 0 < \beta G(x,t) \le t g(x,t), \quad \text{if} \quad |t| > 0.$$

We shall use the ground state (u,v) of

$$-\Delta u = v^q, \quad -\Delta v = u^p, \quad \text{in} \quad \mathbb{R}^N$$

to push the critical value described by the mountain pass below $\frac{1}{N} S_{p,q}^{\frac{N}{2}}$, where $S_{p,q}$ is defined in Section 3. u and v are radial functions. Let $u_\epsilon(x) = \epsilon^{-\frac{N}{p+1}} u(\frac{x}{\epsilon})$, $v_\epsilon(x) = \epsilon^{-\frac{N}{q+1}} v(\frac{x}{\epsilon})$. Denote $\theta(\epsilon) = \|u_\epsilon\|_2^2 + \|v_\epsilon\|_2^2 := \theta_1(\epsilon) + \theta_2(\epsilon)$. The asymptotic behaviours of $\|u_\epsilon\|_2^2$ and $\|v_\epsilon\|_2^2$ as $\epsilon \to 0$ are given in [19].

(H5) There exist functions $\bar{f}(t)$ and $\bar{g}(t)$ such that $f(x,t) \ge \bar{f}(t)$, $g(x,t) \ge \bar{g}(t)$ and both

$$\lim_{\epsilon\to 0} \frac{\epsilon^N}{\theta_1(\epsilon)} \int_0^{1/\epsilon} \bar{F}(\epsilon^{-\frac{N}{p+1}} u(r)) r^{N-1}\, dr = \infty, \quad \text{and}$$
$$\lim_{\epsilon\to 0} \frac{\epsilon^N}{\theta_2(\epsilon)} \int_0^{1/\epsilon} \bar{G}(\epsilon^{-\frac{N}{q+1}} v(r)) r^{N-1}\, dr = \infty$$

if both $\bar{f} \not\equiv 0$ and $\bar{g} \not\equiv 0$. Otherwise, we assume one of the limits holds with θ replacing θ_i.

Assumption (H5) is a Brézis and Nirenberg-type condition; it can be verified in details as in [10].

Theorem A. *Assume* (H1)–(H5); *then problem* (1.3)–(1.4) *possesses at least a nontrivial solution. Furthermore, if* Ω *is a ball and* $f = f(|x|,t)$, $g = g(|x|,t)$*, then problem* (1.3)–(1.4) *has a nontrivial radial solution.*

Using Theorem A we prove the existence result for problem (1.1)–(1.2) by approximation arguments. We construct a Palais-Smale sequence of the functional related to problem (1.1)–(1.2) by Theorem A. In Section 4, we prove a global compact result for Palais-Smale sequences. The result allows us to show Palais-Smale sequences are relatively compact for the values in certain intervals. In Section 6, we verify a condition forcing critical values described by the mountain pass theorem into a given interval. Therefore, the Palais-Smale sequence has a strongly converging subsequence. The limit function will be a solution of (1.1)–(1.2). Before stating the result, we assume further that

(H6) $f(x,t) \to \bar{f}(t)$, $g(x,t) \to \bar{g}(t)$ uniformly for t bounded as $|x| \to \infty$, $|f(x,t) - \bar{f}(t)| \le \epsilon(R)|t|$, $|g(x,t) - \bar{g}(t)| \le \epsilon(R)|t|$, whenever $|x| \ge R$, $|t| \le \delta$ for some constants $R > 0$ and $\delta > 0$, where $\epsilon(R) \to 0$ as $R \to \infty$.

(H7) $\text{meas}\{x \in R^N : f(x,t) \not\equiv \bar{f}(t)\} > 0$, or $\text{meas}\{x \in R^N : g(x,t) \not\equiv \bar{g}(t)\} > 0$.

(H8) $\bar{f}_1(t)/t$ and $\bar{g}_1(t)/t$ are increasing in t.

We put the same $\bar{f}$ and $\bar{g}$ in (H5) and (H6) for simplicity although they may be chosen in a different way.

Theorem B. *Assume* (H1)–(H8); *then problem* (1.1)–(1.2) *possesses at least a nontrivial solution. Furthermore, if* $f = f(|x|,t)$, $g = g(|x|,t)$, *then problem* (1.1)–(1.2) *has a nontrivial radial solution.*

We may see in particular that functions $f(u) = |u|^{\gamma-1}u$ and $g(v) = |v|^{\nu-1}v$, where $1 < \gamma < p$, $1 < \nu < q$, fulfill all assumptions (H1)–(H8). Other examples can be constructed as in [32].

In Section 2, we prove decaying laws for solutions of (1.1)–(1.2) in a special case. Existence results are given in Section 3 for bounded domains and in Section 5 for $\mathbb{R}^N$. We also show in Section 5 that there exists a ground state for the problem

$$-\Delta u + u = |v|^{q-1}v + \bar{g}(v), \quad -\Delta v + v = |u|^{p-1}u + \bar{f}(u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.5}$$

$$u(x) \to 0, \quad v(x) \to 0 \quad \text{as} \quad |x| \to 0. \tag{1.6}$$

The proofs of Theorems A and B are completed in Section 6.

2. Decay of solutions at infinity. In this section we prove a decaying law for strong solutions of problem (1.1)–(1.2) in the case $p = q = 2^* - 1$, where $2^* = \frac{2N}{N-2}, N \ge 3$. By a strong solution of (1.1)–(1.2) we mean a solution (u,v) of (1.1)–(1.2) satisfying u, $v \in W^{2,2^{*'}}$. Moreover, if f and g are independent of x, positive solutions of problem (1.1)–(1.2) are radial and exponentially decaying.

Lemma 2.1. *Assume* (H1)–(H3). *Let* (u,v) *be a strong solution of* (1.1)–(1.2). *Then, it belongs to* L^γ *for* $\gamma \in [2,\infty)$.

Proof. The arguments are similar to that of [17]; we outline the proof.

A bootstrap argument [13] shows that u and v are continuous functions. For each $k > 0$, we define the open set $\Omega_k = \{x \in \mathbb{R}^N : |u(x)| + |v(x)| < k\}$. Now given $x_o \in \mathbb{R}^N$, there exist $k_o > 0$ and $r > 0$ such that the open ball $B_r(x_o) \subset \Omega_k$, for all $k \geq k_o$. Let $R(k) = \sup\{r > 0 : B_r(x_o) \subset \Omega_k\}$. Clearly $R(k) \to +\infty$ as $k \to +\infty$. Let $\phi \in C_o^\infty(\mathbb{R}^N)$ be a function such that

$$\begin{aligned}&\phi(x) = 1, \text{ for } x \in B_{1/2}(0);\ \phi(x) = 0, \text{ for } x \in \mathbb{R}^N \backslash B_1(0);\\ &0 \leq \phi(x) \leq 1 \quad \text{and} \quad |\nabla \phi(x)| \leq \text{ const, for all } x \in \mathbb{R}^N.\end{aligned}$$

Define $\phi_R(x) = \phi(\frac{x-x_o}{R})$ for $R := R(k)$. Multiplying the first equation in (1.1) by $\phi_R^2 |u|^{s-1}u$, with $s > 1$, and integrating by parts, we obtain as in [17] that

$$\begin{aligned}&\int_{R^N} |\nabla(\phi_R u|u|^{\frac{s-2}{2}})|^2\, dx\\ &\leq \frac{3(s+1)^2}{8(s-\epsilon)} \int_{R^N} \phi_R^2 |u|^{s-1} u(|v|^{2^*-2}v + g(x,v))\, dx + C(\epsilon) \int_{R^N} |u|^{s+1} |\nabla \phi_R|^2 dx\\ &=: \frac{3(s+1)^2}{8(s-\epsilon)} I_1 + C(\epsilon) I_2, \qquad (2.1)\end{aligned}$$

where $C(\epsilon)$ is a constant depending on ϵ. We next use Sobolev embedding to estimate the left side of (2.1) from below:

$$(\int_{R^N} (\phi_R^2 |u|^{s+1})^{\frac{N}{N-2}}\, dx)^{\frac{N-2}{N}} \leq \frac{3(s+1)^2}{8(s-\epsilon)} I_1 + C(\epsilon) I_2. \qquad (2.2)$$

To estimate I_1, we denote $\Omega(m) := \{x \in \mathbb{R}^N : |v(x)| \geq m\}$ for some $m > 0$. By Hölder's inequality we obtain

$$\begin{aligned}&|\int_{R^N} \phi_R^2 |u|^s |v|^{2^*-2} v\, dx|\\ &\leq |\int_{\Omega(m)} \phi_R^2 |u|^s |v|^{2^*-2} v\, dx| + |\int_{R^N \backslash \Omega(m)} \phi_R^2 |u|^s |v|^{2^*-2} v\, dx|\\ &\leq \int_{\Omega(m)} \phi_R^2 |u|^s |v|^{2^*-1}\, dx + m^{\frac{4}{N-2}} \int_{R^N \backslash \Omega(m)} \phi_R^2 |u|^s |v|\, dx\\ &\leq \int_{\Omega(m)} \phi_R^2 |u|^s |v|^{2^*-1}\, dx + m^{\frac{4}{N-2}} \int_{\mathbb{R}^N} \phi_R^2 |u|^s |v|\, dx. \qquad (2.3)\end{aligned}$$

Using Hölder's inequality again, we have

$$\int_{R^N} \phi_R^2 |u|^s |v| \, dx \le C \int_{R^N} \phi_R^2 (|u|^{s+1} + |v|^{s+1}) \, dx \tag{2.4}$$

and

$$\int_{\Omega(m)} \phi_R^2 |u|^s |v|^{2^*-1} \, dx \le \Big[\int_{R^N} (\phi_R^2 |u|^{s+1})^{\frac{N}{N-2}} \, dx \Big]^{\frac{s}{s+1} \frac{N-2}{N}} \times \Big[\int_{R^N} (\phi_R^2 |v|^{s+1})^{\frac{N}{N-2}} \, dx \Big]^{\frac{1}{s+1} \frac{N-2}{N}} \Big[\int_{\Omega(m)} |v|^{2^*} \, dx \Big]^{\frac{2}{N}}. \tag{2.5}$$

Let $A = [\int_{R^N} (\phi_R^2 |u|^{s+1})^{\frac{N}{N-2}} \, dx]^{\frac{N-2}{N}}$ and $B = [\int_{R^N} (\phi_R^2 |v|^{s+1})^{\frac{N}{N-2}} \, dx]^{\frac{N-2}{N}}$. It follows from (2.3)–(2.5) that

$$\int_{R^N} \phi_R^2 |u|^s |v|^{2^*-1} \, dx \le A^{s/(s+1)} B^{1/(s+1)} \Big[\int_{\Omega(m)} |v|^{2^*} \, dx \Big]^{\frac{2}{N}} + C(m) \int_{R^N} \phi_R^2 (|u|^{s+1} + |v|^{s+1}) \, dx. \tag{2.6}$$

By (H1)–(H3) we have $|g(x, v)| \le C(|v|^{2^*-1} + |v|)$, which together with (2.4) and (2.6) yield that

$$I_1 \le C A^{\frac{s}{s+1}} B^{\frac{1}{s+1}} \Big[\int_{\Omega(m)} |v|^{2^*} \, dx \Big]^{\frac{2}{N}} + C(m) \int_{R^N} \phi_R^2 (|u|^{s+1} + |v|^{s+1}) \, dx. \tag{2.7}$$

We conclude from (2.1) and (2.7) that

$$A \le C(\epsilon) \Big\{ A^{\frac{s}{s+1}} B^{\frac{1}{s+1}} \Big[\int_{\Omega(m)} |v|^{2^*} \, dx \Big]^{\frac{2}{N}} + \int_{R^N} \phi_R^2 (|u|^{s+1} + |v|^{s+1}) \, dx + I_2 \Big\}. \tag{2.8}$$

A similar expression can be obtained with the roles of A and B exchanged:

$$B \le C(\epsilon) \Big\{ A^{\frac{1}{s+1}} B^{\frac{s}{s+1}} \Big[\int_{\Omega(m)} |u|^{2^*} \, dx \Big]^{\frac{2}{N}} + \int_{R^N} \phi_R^2 (|u|^{s+1} + |v|^{s+1}) \, dx + I_2 \Big\}. \tag{2.9}$$

Assuming that $\int_{R^N} |u|^{s+1} \, dx < \infty$ and $\int_{R^N} |v|^{s+1} \, dx < \infty$ we obtain from (2.8) and (2.9) that

$$A \le C(\epsilon) A^{s/(s+1)} B^{1/(s+1)} \Big[\int_{\Omega(m)} |v|^{2^*} \, dx \Big]^{\frac{2}{N}} + C(\epsilon), \tag{2.10}$$

$$B \leq C(\epsilon)A^{1/(s+1)}B^{s/(s+1)}\Big[\int_{\Omega(m)}|u|^{2^*}\,dx\Big]^{\frac{2}{N}} + C(\epsilon). \tag{2.11}$$

Multiplying (2.10) by (2.11) we obtain

$$\begin{aligned} AB \leq C(\epsilon)\Big\{AB[\int_{\Omega(m)}|u|^{2^*}\,dx]^{\frac{2}{N}}\Big[\int_{\Omega(m)}|v|^{2^*}\,dx\Big]^{\frac{2}{N}} \\ +A^{\frac{s}{s+1}}B^{\frac{1}{s+1}}\Big[\int_{\Omega(m)}|v|^{2^*}\,dx\Big]^{\frac{2}{N}} + A^{\frac{1}{s+1}}B^{\frac{s}{s+1}}\Big[\int_{\Omega(m)}|u|^{2^*}\,dx\Big]^{\frac{2}{N}} + 1\Big\}. \end{aligned} \tag{2.12}$$

Since

$$\int_{R^N}|u|^{2^*}\,dx < \infty \quad \text{and} \quad \int_{R^N}|v|^{2^*}\,dx < \infty,$$

we may choose $m > 0$ large enough such that

$$\int_{\Omega(m)}|u|^{2^*}\,dx \quad \text{and} \quad \int_{\Omega(m)}|v|^{2^*}\,dx$$

are small, and we get

$$AB \leq C(\epsilon)[A^{s/(s+1)}B^{1/(s+1)} + A^{1/(s+1)}B^{s/(s+1)} + 1]. \tag{2.13}$$

Letting $k \to \infty$, we have $R \to \infty$, and it follows from (2.13) that

$$\int_{R^N}|u|^{(s+1)N/(N-2)}\,dx < \infty \quad \text{and} \quad \int_{R^N}|v|^{(s+1)N/(N-2)}\,dx < \infty.$$

Repeating this procedure we see that $u, v \in L^\gamma$ for $\gamma = (s+1)(\frac{N}{N-2})^2$. So we may start with $s = 2^* - 1$ and obtain $u, v \in L^\gamma$ for all $\gamma = 2^*(\frac{2^*}{2})^n$, $n = 1, 2, \ldots$. Using the Riesz-Thorin interpolation theorem [8], we conclude that $u, v \in L^\gamma$ for all $\gamma \geq 2^*$. The assertion follows. □

Using results in Lemma 2.1 we may prove the following decaying laws for strong solutions of (1.1)–(1.2) as in [17].

Proposition 2.2. *Assume* (H1)–(H3) *and* $p = q = 2^* - 1$. *The strong solutions* (u, v) *of* (1.1)–(1.2) *satisfy*

$$\lim_{|x|\to+\infty}|\nabla u(x)| = 0, \qquad \lim_{|x|\to+\infty}|\nabla v(x)| = 0. \tag{2.14}$$

Furthermore, if f and g are independent of x, (u,v) are radially symmetric and satisfy

$$u(r)=o(e^{-\theta r}),\quad v(r)=o(e^{-\theta r}),\quad u_r(r)=o(e^{-\theta_1 r}),$$

$$v_r(r)=o(e^{-\theta_1 r}),\quad u_{rr}(r)=o(e^{-\theta_2 r}),\quad u_{rr}(r)=o(e^{-\theta_2 r}),$$

where $0<\theta,\theta_1,\theta_2<1$.

3. Existence results in bounded domains. Let $T=-\Delta+id$. For $0\le s\le 2$, we define the space E^s as the domain $D(T^{s/2})$ of $\mathcal{A}^s:=T^{s/2}$. It is well known that the inclusions $E^s\to L^\gamma(\Omega)$ are continuous if $2\le\gamma\le\frac{2N}{N-2s}$ and are compact if $2<\gamma<\frac{2N}{N-2s}$ provided that Ω is bounded.

We write $p+1=\frac{2N}{N-2s}$ and $q+1=\frac{2N}{N-2t}$ with $s+t=2$. Denote $E=E^s\times E^t$, $X=L^{p+1}(\Omega)\times L^{q+1}(\Omega)$ and $X^*=L^{\frac{p+1}{p}}(\Omega)\times L^{\frac{q+1}{q}}(\Omega)$. Critical points of the strongly indefinite functional

$$\begin{aligned}I(z)=&\int_\Omega(\nabla u\nabla v+uv)\,dx-\int_\Omega[\frac{1}{p+1}|u|^{p+1}+F(x,u)]\,dx\\&-\int_\Omega[\frac{1}{q+1}|v|^{q+1}+G(x,v)]\,dx\end{aligned}$$

defined on E with $z=(u,v)$ are solutions of (1.3)–(1.4). However, to get control of energy levels of associated functionals, we consider the dual functional J of I. We recall the following facts. For each x, the Legendre-Fenchel transformations $\mathcal{F}^*(x,s)$ of $\mathcal{F}(x,t)$ and $\mathcal{G}^*(x,s)$ of $\mathcal{G}(x,t)$ are defined by

$$\mathcal{F}^*(x,s)=\sup_{t\in R}\{st-\mathcal{F}(x,t)\},\quad \mathcal{G}^*(x,s)=\sup_{t\in R}\{st-\mathcal{G}(x,t)\}\tag{3.1}$$

respectively. Equivalently, we have

$$\mathcal{F}^*(x,s)=st-\mathcal{F}(x,t)\quad\text{with}\quad s=f_1(x,t),\quad t=\mathcal{F}_s^{*'}(x,s),\tag{3.2}$$

$$\mathcal{G}^*(x,s)=st-\mathcal{G}(x,t)\quad\text{with}\quad s=g_1(x,t),\quad t=\mathcal{G}_s^{*'}(x,s).\tag{3.3}$$

In the same way, we define $\bar{\mathcal{F}}^*$ and $\bar{\mathcal{G}}^*$ for $\bar{\mathcal{F}}(t):=\frac{1}{p+1}|t|^{p+1}+\bar{F}(t)$ and $\bar{\mathcal{G}}(t):=\frac{1}{q+1}|t|^{q+1}+\bar{G}(t)$ respectively. By (H6) and properties of Legendre-Fenchel transformation, we have

$$\mathcal{F}^*(x,s)\le\bar{\mathcal{F}}^*(s),\quad \mathcal{G}^*(x,s)\le\bar{\mathcal{G}}^*(s).\tag{3.4}$$

Assume (H1)–(H4). The following properties of $\mathcal{F}^*$ and $\mathcal{G}^*$ can be verified as in [3], [14] and [26].

Lemma 3.1. *$\mathcal{F}^*, \mathcal{G}^* \in C^1$ and*

$$\mathcal{F}^*(x,s) \geq (1-\frac{1}{\alpha})s\mathcal{F}_s^{*'}(x,s), \quad \mathcal{G}^*(x,s) \geq (1-\frac{1}{\beta})s\mathcal{G}_s^{*'}(x,s), \tag{3.5}$$

$$\mathcal{F}^*(x,s) \geq C|s|^{\frac{p+1}{p}} - C, \quad \mathcal{G}^*(x,s) \geq C|s|^{\frac{q+1}{q}} - C. \tag{3.6}$$

Lemma 3.2. *There exist $\delta > 0$, C_δ and $C'_\delta > 0$ such that*

$$\mathcal{F}^*(x,s) \geq \begin{cases} C_\delta |s|^2, & \text{if } |s| \leq \delta \\ C'_\delta |s|^{\frac{p+1}{p}}, & \text{if } |s| \geq \delta \end{cases}, \quad \mathcal{G}^*(x,s) \geq \begin{cases} C_\delta |s|^2, & \text{if } |s| \leq \delta \\ C'_\delta |s|^{\frac{q+1}{q}}, & \text{if } |s| \geq \delta \end{cases},$$

where $C_\delta, C'_\delta \to \infty$ as $\delta \to 0$.

Let

$$A = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}, \quad K = A^{-1} = \begin{pmatrix} 0 & T^{-1} \\ T^{-1} & 0 \end{pmatrix}.$$

The dual functional

$$J(w) = \int_\Omega (\mathcal{F}^*(x,w_1) + \mathcal{G}^*(x,w_2))\, dx - \frac{1}{2}\int_\Omega \langle w, Kw\rangle \, dx,$$

of I is well defined and C^1 on X^*. A critical point w of J satisfies

$$(-\Delta + id)^{-1} w_2 = \mathcal{F}_s^{*'}(x,w_1), \quad (-\Delta + id)^{-1} w_1 = \mathcal{G}_s^{*'}(x,w_2).$$

Let

$$u = (-\Delta + id)^{-1} w_2, \quad v = (-\Delta + id)^{-1} w_1.$$

Then (u,v) satisfies (1.3)–(1.4). We deduce by (3.2) and (3.3) that $I(z) = J(w)$. Such a result is also valid for solutions of (1.1)–(1.2). Now we use the mountain pass theorem to find critical points of J. Following arguments of [6], we know that assumption (H2) implies $\mathcal{F}^*(x,t)/t^2 \to \infty$ and $\mathcal{G}^*(x,t)/t^2 \to \infty$. Thus, 0 is a local minimum of J. Precisely,

Lemma 3.3. *Suppose* (H2). *There exist constants $\alpha, \rho > 0$, independent of Ω, such that $J(w) \geq \alpha > 0$ if $\|w\|_{X^*} = \rho$.*

By (H1), (H2) and (H4), we have

$$\mathcal{F}(x,t) \geq C|t|^\alpha, \quad \mathcal{G}(x,t) \geq C|t|^\beta, \tag{3.7}$$

which yields

$$\mathcal{F}^*(x,s) \leq C|s|^{\frac{\alpha}{\alpha-1}}, \quad \mathcal{G}^*(x,s) \leq C|s|^{\frac{\beta}{\beta-1}}. \tag{3.8}$$

Lemma 3.4. *There exist $T > 0$ and $w \in X^*$ such that $J(tw) \le 0$ whenever $t \ge T$.*

Proof. We take $w \in X^*, w \not\equiv 0$ such that $\int_\Omega \langle w, Kw\rangle\, dx > 0$, whence by (3.8)

$$J(tw) \le t^{\frac{\alpha}{\alpha-1}} \int_\Omega |w_1|^{\frac{\alpha}{\alpha-1}}\, dx + t^{\frac{\beta}{\beta-1}} \int_\Omega |w_2|^{\frac{\beta}{\beta-1}}\, dx - \frac{1}{2} t^2 \int_\Omega \langle w, Kw\rangle\, dx$$

for $t > 0$. Since $\frac{\alpha}{\alpha-1}, \frac{\beta}{\beta-1} < 2$, the assertion follows. $\square$

In order to find critical points of J, the Palais-Smale condition has to be considered. We say that J satisfies the $(PS)_c$ condition if any sequence $\{w_n\} \subset X^*$ such that $J(w_n) \to c, J'(w_n) \to 0$ as $n \to \infty$ has a subsequence converging strongly in X^*. Define

$$S_{p,q} = \inf\{\|\Delta u\|_{L^{\frac{q+1}{q}}} : u \in W^{2,\frac{q+1}{q}}(\Omega) \cap W_o^{1,\frac{q+1}{q}}(\Omega), \|u\|_{L^{p+1}} = 1\}.$$

$S_{p,q}$ is independent of Ω and depends only on p and q.

Lemma 3.5. *Under hypotheses* (H1)–(H4), *the functional J satisfies the $(PS)_c$ condition for*

$$0 < c < \frac{2}{N} S_{p,q}^{\frac{N}{2}}. \tag{3.9}$$

Proof. Let $\{w_n\}$ be a sequence satisfying $J(w_n) \to c < \frac{2}{N} S_{p,q}^{\frac{N}{2}}$, $J'(w_n) \to 0$ as $n \to \infty$, which along with Lemma 3.1 yield

$$\begin{aligned}
&\int_\Omega (\mathcal{F}^*(x, w_n^1) + \mathcal{G}^*(x, w_n^2))\, dx \le \frac{1}{2} \int_\Omega \langle w_n, Kw_n\rangle\, dx + C \\
&\le \frac{1}{2} \int_\Omega (\mathcal{F}_s^{*'}(x, w_n^1) w_n^1 + \mathcal{G}_s^{*'}(x, w_n^2) w_n^2)\, dx + o(1)\|w_n\|_{X^*} + C \\
&\le \frac{1}{2} \frac{\alpha}{\alpha-1} \int_\Omega \mathcal{F}^*(x, w_n^1)\, dx + \frac{1}{2} \frac{\beta}{\beta-1} \int_\Omega \mathcal{G}^*(x, w_n^2)\, dx + C + o(1)\|w_n\|_{X^*}.
\end{aligned}$$

Therefore,

$$\int_\Omega (\mathcal{F}^*(x, w_n^1) + \mathcal{G}^*(x, w_n^2))\, dx \le C + o(1)\|w_n\|_{X^*}.$$

By Lemma 3.2, we obtain

$$\|w_n^1\|_{L^{\frac{p+1}{p}}}^{\frac{p+1}{p}} + \|w_n^2\|_{L^{\frac{q+1}{q}}}^{\frac{q+1}{q}} \leq C + o(1)\|w_n\|_{X^*}.$$

So $\|w_n\|_{X^*}$ is bounded.

Let $z_n = Kw_n$. Since $K : X^* \to X$ is bounded, it follows that $\|z_n\|_X \leq C$; similarly $\|z_n\|_E \leq C\|w_n\|_{X^*} \leq C$.

Solving the equation $Az_n = w_n$ and using elliptic regularity theory, we obtain

$$z_n \in [W^{2,\frac{q+1}{q}}(\Omega) \cap W_o^{1,\frac{q+1}{q}}(\Omega)] \times [W^{2,\frac{p+1}{p}}(\Omega) \cap W_o^{1,\frac{p+1}{p}}(\Omega)]$$

and

$$\|u_n\|_{W^{2,\frac{q+1}{q}}(\Omega)\cap W_o^{1,\frac{q+1}{q}}(\Omega)} \leq C, \quad \|v_n\|_{W^{2,\frac{p+1}{p}}(\Omega)\cap W_o^{1,\frac{p+1}{p}}(\Omega)} \leq C.$$

Hence, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$z_{n_k} \to z \quad \text{weakly in} \quad E \quad \text{and} \quad X, \quad \text{and} \quad z_{n_k} \to z \quad \text{in} \quad L^\tau(\Omega) \times L^\gamma(\Omega)$$

as $n_k \to \infty$, for $2 \leq \tau < \frac{2N}{N-2s}$, $2 \leq \gamma < \frac{2N}{N-2t}$.

Since $\{w_n\}$ is bounded in X^*, it is straightforward that

$$-\Delta u_n + u_n - |v_n|^{q-1}v_n - g(x, v_n) = \epsilon_{1,n} \quad \text{in} \quad L^{\frac{q+1}{q}}, \tag{3.10}$$

$$-\Delta v_n + v_n - |u_n|^{p-1}u_n - f(x, u_n) = \epsilon_{2,n} \quad \text{in} \quad L^{\frac{p+1}{p}} \tag{3.11}$$

with $\|\epsilon_n\|_{X^*} \to 0$, where $\epsilon_n = (\epsilon_{1,n}, \epsilon_{2,n})$. We claim that $z \not\equiv 0$. In fact, if $z \equiv 0$, we would have

$$z_{n_k} \to 0 \quad \text{strongly in} \quad L^\tau(\Omega) \times L^\gamma(\Omega);$$

as $n_k \to \infty$, (3.10) and (3.11) become

$$-\Delta u_n = |v_n|^{q-1}v_n + o(1), \quad -\Delta v_n = |u_n|^{p-1}u_n + o(1). \tag{3.12}$$

So one has

$$\int_\Omega \nabla u_n \cdot \nabla v_n \, dx = \int_\Omega |u_n|^{p+1} \, dx + o(1) = \int_\Omega |v_n|^{q+1} \, dx + o(1).$$

Therefore,

$$\int_\Omega |\Delta u_n|^{\frac{q+1}{q}}\,dx = \int_\Omega |v_n|^q \mathrm{sign}(v_n)(-|\Delta u_n|^{\frac{1}{q}}\mathrm{sign}(-\Delta u_n))\,dx + o(1)$$
$$\leq (\int_\Omega |v_n|^{q+1}\,dx)^{\frac{q+1}{q}}(\int_\Omega |\Delta u_n|^{\frac{q+1}{q}}\,dx)^{\frac{1}{q+1}} + o(1),$$

which gives

$$\int_\Omega |\Delta u_n|^{\frac{q+1}{q}}\,dx \leq \int_\Omega |v_n|^{q+1}\,dx + o(1) = \int_\Omega |u_n|^{p+1}\,dx + o(1).$$

Assuming that

$$\int_\Omega |u_n|^{p+1}\,dx \to k, \quad \int_\Omega |v_n|^{q+1}\,dx \to k,$$

we obtain $k \geq S_{p,q}^{\frac{N}{2}}$. On the other hand, the convergence of $\{z_{n_k}\}$ in $L^\tau(\Omega)\times L^\gamma(\Omega)$ implies that

$$c + o(1) = I(z_n) = \int_\Omega [(\frac{1}{2} - \frac{1}{p+1})|u_n|^{p+1} + (\frac{1}{2} - \frac{1}{q+1})|v_n|^{q+1}]\,dx + o(1)$$
$$= \frac{2}{N}k + o(1).$$

As a result, $c \geq \frac{2}{N}S_{p,q}^{\frac{N}{2}}$, contradicting (3.9) and therefore $z \not\equiv 0$.

Let $\alpha_n = u_n - u$, $\beta_n = v_n - v$. Then $(\alpha_n, \beta_n) \to (0,0)$ weakly in $(W^{2,\frac{q+1}{q}} \cap W_o^{1,\frac{q+1}{q}}) \times (W^{2,\frac{p+1}{p}} \cap W_o^{1,\frac{p+1}{p}})$ and $L^{p+1}\times L^{q+1}$, and strongly in $L^\tau \times L^\gamma$ for $2 \leq \tau < \frac{2N}{N-2s}, 2 \leq \gamma < \frac{2N}{N-2t}$. Using the Brézis–Lieb lemma [9], one has

$$I(z) + \int_\Omega (-\beta_n \Delta\alpha_n - \frac{1}{p+1}|\alpha_n|^{p+1} - \frac{1}{q+1}|\beta_n|^{q+1})\,dx = c + o(1),$$

$$\langle I'(z), z\rangle + \int_\Omega (-2\beta_n\Delta\alpha_n - |\alpha_n|^{p+1} - |\beta_n|^{q+1})\,dx = o(1).$$

Again by (3.12), we may assume that

$$\int_\Omega |\alpha_n|^{p+1}\,dx \to k, \int_\Omega |\beta_n|^{q+1}\,dx \to k, -\int_\Omega \beta_n\Delta\alpha_n\,dx \to k.$$

Thus

$$I(z) - \frac{2}{N}\int_\Omega \beta_n \Delta\alpha_n\,dx = c + o(1).$$

We have either $k = 0$ or $k \geq S_{p,q}^{\frac{N}{2}}$. In the latter case

$$c = I(z) + \frac{2}{N}k \geq I(z) + \frac{2}{N}S_{p,q}^{\frac{N}{2}} > \frac{2}{N}S_{p,q}^{\frac{N}{2}}$$

since $I(z) > 0$. This contradicts (3.9). So $k = 0$.

Finally, we show that $w_n \to w = Az$ in X^*. We know from (3.10) and (3.11) that

$$\begin{aligned}\|w_n - w\|_{X^*} \leq C\{&\||u_n|^p u_n - |u|^p u\|_{L^{\frac{p+1}{p}}} + \||v_n|^q v_n - |v|^q v\|_{L^{\frac{q+1}{q}}}\\ &+ \|f(x,v_n) - f(x,v)\|_{L^{\frac{p+1}{p}}} + \|g(x,u_n) - g(x,u)\|_{L^{\frac{q+1}{q}}} + \|\epsilon_n\|_{X^*}\}.\end{aligned}$$

The right side tends to zero as $n \to \infty$ because $(u_n, v_n) \to (u, v)$ strongly in $(W^{2,\frac{p+1}{p}} \cap W_o^{1,\frac{p+1}{p}}) \times (W^{2,\frac{q+1}{q}} \cap W_o^{1,\frac{q+1}{q}})$ and $L^{p+1} \times L^{q+1}$. The proof is completed. □

Let $\Gamma = \{g \in C([0,1], X^*) : g(0) = 0,\ g(1) = e\}$, where $e = Tw$ is selected in Lemma 3.4. We define

$$c = c_\Omega = \inf_{g\in\Gamma} \sup_{t\in[0,1]} J(g(t)). \tag{3.13}$$

Proposition 3.6. *Suppose* (H1)–(H4). *If there exists a path $e(t)$ in X^* such that $e(0) = 0$ and $J(e(t)) \leq 0$ for $t > 0$ large satisfying*

$$\sup_{t\geq 0} J(e(t)) < \frac{2}{N}S_{p,q}^{\frac{N}{2}}, \tag{3.14}$$

the problem (1.3)–(1.4) *possesses a nontrivial solution.*

Proof. By (3.14), we may verify that the value c defined by (3.13) satisfies $c < \frac{2}{N}S_{p,q}^{\frac{N}{2}}$. The assertion follows by Lemmas 3.3–3.5 and the mountain pass theorem. □

4. Global compactness results. The functionals

$$\mathcal{I}(z) = \int_{R^N} \mathcal{A}^s u \cdot \mathcal{A}^t v\,dx - \int_{R^N} [\mathcal{F}(x,u) + \mathcal{G}(x,v)]\,dx$$

and

$$\mathcal{I}^{\infty}(z) = \int_{R^N} \mathcal{A}^s u \cdot \mathcal{A}^t v \, dx - \int_{R^N} [\bar{\mathcal{F}}(u) + \bar{\mathcal{G}}(v)] \, dx$$

are well defined on $E = E^s \times E^t$. We show in this section that the obstacle energy levels for the compactness of $\mathcal{I}$ are the energy levels of $\mathcal{I}^{\infty}$ corresponding to the solutions of (1.5)–(1.6). Regularity theory shows that critical points of $\mathcal{I}^{\infty}$ are actually strong solutions of (1.5)–(1.6). Furthermore, we have

Lemma 4.1. *Suppose* (H2), (H3) *and* (H6). *There exists a positive constant* $C > 0$ *such that*

$$\|z\|_E \geq C$$

for all nontrivial solutions $z \in E$ *of* (1.5)–(1.6).

Proof. Suppose $z = (u, v)$ is a solution of (1.5)–(1.6). By assumptions (H2), (H3) and (H6), we obtain

$$\bar{f}(u) \leq C_\epsilon |u|^p + \epsilon u, \quad \bar{g}(v) \leq C_\epsilon |v|^q + \epsilon v. \tag{4.1}$$

Using Hölder's inequality, (4.1) and equations, one has

$$|\int_{R^N} \mathcal{A}^s \phi \mathcal{A}^t v \, dx| \leq (C_\epsilon \|u\|^p_{L^{p+1}} + \epsilon \|u\|_{L^2}) \|\phi\|_{E^s}, \forall \phi \in E^s,$$

which implies $\|v\|_{E^t} \leq C_\epsilon \|u\|^p_{E^s} + \epsilon \|u\|_{E^s}$. Similarly, $\|u\|_{E^s} \leq C_\epsilon \|v\|^q_{E^t} + \epsilon \|v\|_{E^t}$. So for ϵ small, it follows that

$$\|u\|_{E^s} + \|v\|_{E^t} \leq C(\|u\|^p_{E^s} + \|v\|^q_{E^t}).$$

Consequently, either $\|u\|_{E^s} \geq C$ or $\|v\|_{E^t} \geq C > 0$, where $C > 0$ is independent of $z = (u, v)$. □

Proposition 4.2. *Assume* (H1)–(H4) *and* (H6). *Let* $\{z_n\} \subset E$ *be a sequence such that*

$$\mathcal{I}(z_n) \to c < \frac{2}{N} S_{p,q}^{\frac{N}{2}} \quad \textit{and} \quad \mathcal{I}'(z_n) \to 0 \quad \textit{in} \quad E^* \quad \textit{as} \quad n \to 0. \tag{4.2}$$

Then there exists a subsequence (*still denoted by* $\{z_n\}$) *for which the following holds: there exist an integer* $k \geq 0$, *sequences* $\{x_n^i\} \subset \mathbb{R}^N, |x_n^i| \to \infty$ *as* $n \to$

∞ for $1 \le i \le k$, a solution z of (1.1)–(1.2) and solutions z^i $(1 \le i \le k)$ of (1.5)–(1.6) such that

$$z_n \to z \quad \text{weakly} \quad \text{in} \quad E, \tag{4.3}$$

$$\mathcal{I}(z_n) \to \mathcal{I}(z) + \sum_{i=1}^{k} \mathcal{I}^\infty(z^i), \tag{4.4}$$

$$z_n - (z + \sum_{i=1}^{k} z^i(x - x_n^i)) \to 0 \quad \text{in} \quad E \tag{4.5}$$

as $n \to \infty$, where we agree that in the case $k = 0$ the above holds without z^i, x_n^i.

Proof. The result will be derived from the arguments of [5] for one equation. First we show the boundedness of $\{z_n\}$ in E. By (4.2), (H2) and (H4) we have

$$\begin{aligned}
c + \epsilon_n\|z_n\|_E &= \int_{R^N} [(\frac{1}{2} - \frac{1}{p+1})|u_n|^{p+1} + (\frac{1}{2} - \frac{1}{q+1})|v_n|^{q+1}]\, dx \\
&+ \frac{1}{2}\int_{R^N}[u_n f(x,u_n) + v_n g(x,v_n)]\, dx - \int_{R^N}[F(x,u_n) + G(x,v_n)]\, dx \\
&\ge \int_{R^N} [(\frac{1}{2} - \frac{1}{p+1})|u_n|^{p+1} + (\frac{1}{2} - \frac{1}{q+1})|v_n|^{q+1}]\, dx \\
&+ (\frac{\alpha}{2} - 1)\int_{R^N} F(x,u_n)\, dx + (\frac{\beta}{2} - 1)\int_{R^N} G(x,v_n)\, dx \\
&\ge \int_{R^N} [(\frac{1}{2} - \frac{1}{p+1})|u_n|^{p+1} + (\frac{1}{2} - \frac{1}{q+1})|v_n|^{q+1}]\, dx + C\int_{R^N}(|u_n|^\alpha + |v_n|^\beta) dx.
\end{aligned} \tag{4.6}$$

On the other hand, we may deduce as in Lemma 4.1 that

$$\|v_n\|_{E^t} \le \epsilon\|u_n\|_{E^s} + C_\epsilon\|u_n\|_{L^\alpha}^p + \|u_n\|_{L^{p+1}}^p + \epsilon_n\|z_n\|_E \tag{4.7}$$

and

$$\|u_n\|_{E^s} \le \epsilon\|v_n\|_{E^t} + C_\epsilon\|v_n\|_{L^\beta}^q + \|v_n\|_{L^{q+1}}^q + \epsilon_n\|z_n\|_E. \tag{4.8}$$

Adding the two inequalities we obtain by (4.6) that

$$\begin{aligned}
\|z_n\|_E &= \|u_n\|_{E^s} + \|v_n\|_{E^t} \\
&\le C[\|u_n\|_{L^\alpha}^p + \|v_n\|_{L^\beta}^q + \|u_n\|_{L^{p+1}}^p + \|v_n\|_{L^{q+1}}^q + (\epsilon + \epsilon_n)\|z_n\|_E] \\
&\le C[(\epsilon + \epsilon_n)\|z_n\|_E + 1].
\end{aligned} \tag{4.9}$$

Selecting $\epsilon > 0$ small and for n large, it follows that $\{z_n\}$ is uniformly bounded in E. So we may assume

$$\begin{aligned} &z_n \to z \quad \text{weakly} \quad \text{in} \quad E, \\ &z_n \to z \quad \text{strongly} \quad \text{in} \quad L^\tau_{loc}(\mathbb{R}^N) \times L^\gamma_{loc}(\mathbb{R}^N), \\ &z_n \to z \quad \text{almost everywhere} \quad \text{in} \quad \mathbb{R}^N \end{aligned}$$

as $n \to \infty$, where $2 \le \tau < \frac{2N}{N-2s}, 2 \le \gamma < \frac{2N}{N-2t}$. Denoting $Q(z) = \int_{\mathbb{R}^N} \mathcal{A}^s u \mathcal{A}^t v \, dx$, we have

$$Q(z_n) = Q(z_n - z) + Q(z) + o(1). \tag{4.10}$$

It follows from Brézis and Lieb's lemma [9] that

$$\int_{\mathbb{R}^N} F(x, u_n) \, dx = \int_{\mathbb{R}^N} F(x, u_n - u) \, dx + \int_{\mathbb{R}^N} F(x, u) \, dx + o(1) \tag{4.11}$$

and

$$\int_{\mathbb{R}^N} G(x, v_n) \, dx = \int_{\mathbb{R}^N} G(x, v_n - v) \, dx + \int_{\mathbb{R}^N} G(x, v) \, dx + o(1). \tag{4.12}$$

Hence we obtain

$$\mathcal{I}(z_n) = \mathcal{I}(z_n - z) + \mathcal{I}(z) + o(1), \tag{4.13}$$

$$\mathcal{I}'(z_n) = \mathcal{I}'(z_n - z) + \mathcal{I}'(z) + o(1) \tag{4.14}$$

as $n \to \infty$. Let $z_n^1 = z_n - z$. We may derive from (H6) as [22] and [32] that

$$\int_{\mathbb{R}^N} u_n^1 [f(x, u_n^1) - \bar{f}(u_n^1)] \, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} v_n^1 [g(x, v_n^1) - \bar{g}(v_n^1)] \, dx \to 0$$

as well as

$$\int_{\mathbb{R}^N} [F(x, u_n^1) - \bar{F}(u_n^1)] \, dx \to 0, \quad \int_{\mathbb{R}^N} [G(x, v_n^1) - \bar{G}(v_n^1)] \, dx \to 0$$

as $n \to \infty$. Whence by (4.13) and (4.14) it follows that

$$\mathcal{I}^\infty(z_n^1) = \mathcal{I}(z_n^1) + o(1) = \mathcal{I}(z_n) - \mathcal{I}(z) + o(1) \tag{4.15}$$

$$\mathcal{I}^{\infty'}(z_n^1) = \mathcal{I}'(z_n^1) + o(1) = \mathcal{I}'(z_n) - \mathcal{I}'(z) + o(1). \tag{4.16}$$

Suppose $z_n^1 = z_n - z \not\to 0$ strongly in E (otherwise we shall have finished). We want to show that there exists $x_n^1 \subset \mathbb{R}^N$ such that $|x_n^1| \to +\infty$ and $z_n^1(x + x_n^1) \to z^1 \not\equiv 0$ weakly in E. We claim that

$$\mathcal{I}^\infty(z_n^1) \geq \alpha > 0. \tag{4.17}$$

Indeed, were it not true, we would have

$$\mathcal{I}^\infty(z_n^1) \to 0 \tag{4.18}$$

and

$$\left\langle \mathcal{I}^{\infty'}(z_n^1), \eta \right\rangle = o(1)\|\eta\|_E \quad \text{as} \quad n \to \infty. \tag{4.19}$$

Taking $\eta = (\frac{\beta}{\alpha+\beta} v_n^1, \frac{\alpha}{\alpha+\beta} u_n^1) =: \eta_n$ in (4.19), it follows from (4.18) and (4.19) that

$$\begin{aligned}
&o(1)\|\eta_n\|_E = \mathcal{I}^\infty(z_n^1) - \left\langle \mathcal{I}^{\infty'}(z_n^1), \eta_n \right\rangle \\
&= (\frac{\beta}{\alpha+\beta} - \frac{1}{p+1}) \int_{R^N} |u_n^1|^{p+1}\, dx + (\frac{\alpha}{\alpha+\beta} - \frac{1}{q+1}) \int_{R^N} |v_n^1|^{q+1}\, dx \\
&+ \frac{\beta}{\alpha+\beta} \int_{\mathbb{R}^N} u_n^1 \bar{f}(u_n^1)\, dx + \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^N} v_n^1 \bar{g}(v_n^1)\, dx - \int_{\mathbb{R}^N} [\bar{F}(u_n^1) + \bar{G}(v_n^1)]\, dx \\
&\geq (\frac{\beta}{\alpha+\beta} - \frac{1}{p+1}) \int_{R^N} |u_n^1|^{p+1}\, dx + (\frac{\alpha}{\alpha+\beta} - \frac{1}{q+1}) \int_{R^N} |v_n^1|^{q+1}\, dx \\
&+ (\frac{\alpha\beta}{\alpha+\beta} - 1) \int_{\mathbb{R}^N} [\bar{F}(u_n^1) + \bar{G}(v_n^1)]\, dx.
\end{aligned} \tag{4.20}$$

As $2 < \alpha \leq p+1,\ 2 < \beta \leq q+1$, we conclude that

$$\int_{R^N} (|u_n^1|^{p+1} + |v_n^1|^{q+1})\, dx = o(1), \quad \int_{\mathbb{R}^N} (\bar{F}(u_n^1) + \bar{G}(v_n^1))\, dx = o(1).$$

Again we may deduce as in (4.9) that

$$\|z_n^1\|_E \leq C(\|u_n^1\|_{L^{p+1}}^p + \|v_n^1\|_{L^{q+1}}^q + o(1))$$

implying $\|z_n^1\|_E \to 0$ as $n \to \infty$, which contradicts the fact $\|z_n^1\|_E \not\to 0$.

We decompose $\mathbb{R}^N$ into N-dimensional unit hypercubes Q_j with vertices having integer coordinates and put

$$d_n = \max_j(\|u_n^1\|_{L^{p+1}(Q_j)} + \|v_n^1\|_{L^{q+1}(Q_j)}).$$

We claim that there is a $\beta > 0$ such that

$$d_n \geq \beta > 0 \quad \forall n \in \mathbb{N}. \tag{4.21}$$

Suppose, for the sake of contradiction, that $d_n \to 0$ as $n \to \infty$. Since

$$\mathcal{I}^{\infty\prime}(z_n^1) \to 0 \quad \text{as} \quad n \to \infty, \tag{4.22}$$

noting that $\|z_n^1\|_E^1$ is bounded and denoting $\nu = \min\{p-1, q-1\}$, we have by (H2) and (H3) that

$$\begin{aligned}
0 < \alpha &\leq \mathcal{I}^\infty(z_n^1) \leq C \int_{\mathbb{R}^N} [|u_n^1|^{p+1} + |v_n^1|^{q+1} + u_n^1 \bar{f}(u_n^1) + v_n^1 \bar{g}(v_n^1)]\, dx + o(1) \\
&\leq C_\epsilon(\|u_n^1\|^{p+1}_{L^{p+1}(R^N)} + \|v_n^1\|^{q+1}_{L^{q+1}(R^N)}) + \epsilon(\|u_n^1\|^2_{L^2(R^N)} + \|v_n^1\|^2_{L^2(R^N)}) \\
&\leq C_\epsilon \sum_j (\|u_n^1\|^{p+1}_{L^{p+1}(Q_j)} + \|v_n^1\|^{q+1}_{L^{q+1}(Q_j)}) + \epsilon(\|u_n^1\|^2_{L^2(R^N)} + \|v_n^1\|^2_{L^2(R^N)}) \\
&\leq C_\epsilon d_n^\nu \sum_j (\|u_n^1\|^2_{L^{p+1}(Q_j)} + \|v_n^1\|^2_{L^{q+1}(Q_j)}) + \epsilon C \\
&\leq C d_n^\nu \sum_j (\|u_n^1\|^2_{E^s(Q_j)} + \|v_n^1\|^2_{E^t(Q_j)}) + \epsilon C \\
&\leq C d_n^\nu (\|u_n^1\|^2_{E^s(R^N)} + \|v_n^1\|^2_{E^t(R^N)}) + \epsilon C.
\end{aligned}$$

Let $n \to \infty$ and then $\epsilon \to 0$, we obtain $\mathcal{I}^\infty(z_n^1) \to 0$ as $n \to \infty$, a contradiction. Hence (4.21) holds true.

Let $\{x_n^1\}$ be the center of a hypercube Q_j in which

$$d_n = \|u_n^1\|_{L^{p+1}(Q_j)} + \|v_n^1\|_{L^{q+1}(Q_j)}.$$

Now we show that

$$|x_n^1| \to \infty \quad \text{as} \quad n \to \infty. \tag{4.23}$$

If $\{x_n^1\}$ were bounded, by passing to a subsequence if necessary we should find that x_n^1 would be in the same Q_j and so they should coincide. Defining

$$\bar{z}_n^1(x) = \begin{cases} z_n^1(x) & x \in Q_j \\ 0 & x \in \mathbb{R}^N \backslash Q_j, \end{cases}$$

we should have

$$\begin{aligned}
&\mathcal{I}^\infty|_{E(Q_j)}(\bar{z}_n^1) = \int_{Q_j} \mathcal{A}^s \bar{u}_n^1 \mathcal{A}^t \bar{v}_n^1 \, dx - \int_{Q_j} (\bar{\mathcal{F}}(\bar{u}_n^1) + \bar{\mathcal{G}}(\bar{v}_n^1)) \, dx + o(1) \\
&\geq \int_{Q_j} [(\frac{1}{2} - \frac{1}{p+1})|\bar{u}_n^1|^{p+1} + (\frac{1}{2} - \frac{1}{q+1})|\bar{v}_n^1|^{q+1}] \, dx \\
&+ (\frac{\alpha}{2} - 1) \int_{R^N} \bar{F}(\bar{u}_n^1) \, dx + (\frac{\beta}{2} - 1) \int_{R^N} \bar{G}(\bar{v}_n^1) \, dx + o(1) \\
&\geq \int_{Q_j} [(\frac{1}{2} - \frac{1}{p+1})|\bar{u}_n^1|^{p+1} + (\frac{1}{2} - \frac{1}{q+1})|\bar{v}_n^1|^{q+1}] \, dx \\
&+ C \int_{Q_j} (|\bar{u}_n^1|^\alpha + |\bar{v}_n^1|^\beta) \, dx + o(1) \\
&\geq C(\|\bar{u}_n^1\|^{p+1}_{L^{p+1}(Q_j)} + \|\bar{v}_n^1\|^{q+1}_{L^{q+1}(Q_j)}) + \|\bar{u}_n^1\|^{\alpha}_{L^{p+1}(Q_j)} + \|\bar{v}_n^1\|^{\beta}_{L^{q+1}(Q_j)}) + o(1) \\
&\geq \delta > 0
\end{aligned}$$

for n large, and $\mathcal{I}^{\infty'}(\bar{z}_n^1) \to 0$ as $n \to 0$. Because $\mathcal{I}(z) > 0$ and

$$0 < \delta \leq \mathcal{I}^\infty|_{E(Q_j)}(\bar{z}_n^1) \leq \mathcal{I}^\infty(z_n) = \mathcal{I}(z_n) - \mathcal{I}(z) + o(1) < \frac{2}{N} S_{p,q}^{\frac{N}{2}},$$

Lemma 3.5 implies that $\bar{z}_n^1$ should converge strongly in $E(Q_j)$ to a nonzero function, contradicting $z_n^1 \to 0$ weakly in E, so we have (4.23). Let $z_n^1(\cdot + x_n^1) \to z^1$ weakly in E. Denoting by $\bar{Q}$ the unit hypercube centered at the origin, we have $\|z_n^1\|_{E(\bar{Q})} \geq \beta > 0$; thus $z^1 \not\equiv 0$ and

$$\left\langle \mathcal{I}^{\infty'}(z^1), \eta \right\rangle = 0, \quad \forall \eta \in E. \tag{4.24}$$

Iterating the procedure, we obtain sequences $x_n^l, |x_n^l| \to \infty$ and

$$z_n^l(x) = z_n^{l-1}(x + x_m) - z^{l-1}(x), \quad j \geq 2$$

$$z_n^l(x + x_n^l) \to z^l(x) \quad \text{weakly} \quad \text{in} \quad E$$

as $n \to 0$, where each z^l satisfies (4.24), and by induction

$$\|z_n^l\|_E^2 = \|z_n^{l-1}\|_E^2 - \|z^{l-1}\|_E^2 = \|z_n\|_E^2 - \|z\|_E^2 - \sum_{i=1}^{l-1} \|z^i\|_E^2 + o(1).$$

$$\mathcal{I}^\infty(z_n^l) = \mathcal{I}^\infty(z_n^{l-1}) - \mathcal{I}^\infty(z^{l-1}) + o(1) = \mathcal{I}(z_n) - \mathcal{I}(z) - \sum_{i=1}^{l-1} \mathcal{I}^\infty(z^i) + o(1).$$

Since z^l is a solution of (1.5)–(1.6) and $z^l \not\equiv 0$, by Lemma 4.1 $\|z^l\|_E \geq C > 0$. Thus the iteration will terminate at some index $k \geq 0$. The assertion follows. $\square$

5. Existence results in $\mathbb{R}^N$. Let $R_n \to \infty$, $B_n = B_{R_n}(0)$. Taking $\Omega = B_n$ in problem (1.3)–(1.4), we infer from Proposition 3.6 that there exists a solution z_n of problem (1.3)–(1.4) defined on B_n for each n if (3.14) holds. Moreover,

$$I(z_n) = J(w_n) = c_n \geq \alpha > 0 \tag{5.1}$$

and

$$I'(z_n) = 0, \quad J'(w_n) = 0, \tag{5.2}$$

where $w_n = Az_n$. In fact, z_n is a strong solution of (1.3)–(1.4). Denote by $\mathcal{J}$ the dual functional of $\mathcal{I}$. Extending z_n to $\mathbb{R}^N$ by setting $z_n = 0$ outside B_n, we have

$$\mathcal{I}(z_n) = \mathcal{J}(w_n) = c_n. \tag{5.3}$$

If f and g are independent of x, the solutions z_n are radial.

Lemma 5.1. *z_n is a* (PS) *sequence of $\mathcal{I}$ in E and*

$$\mathcal{I}(z_n) < \frac{2}{N} S_{p,q}^{\frac{N}{2}}. \tag{5.4}$$

Proof. We may readily verify that $c_n = \mathcal{I}(z_n) \leq c_{n-1} = \mathcal{I}(z_{n-1})$; thus

$$\alpha \leq c_n \leq c_1 < \frac{2}{N} S_{p,q}^{\frac{N}{2}}, \tag{5.5}$$

so we obtain

$$c_n = \mathcal{I}(z_n) \to c, \quad \alpha \leq c < \frac{2}{N} S_{p,q}^{\frac{N}{2}}. \tag{5.6}$$

Now we show that

$$\mathcal{I}'(z_n) \to 0, \quad \text{as} \quad n \to \infty. \tag{5.7}$$

Indeed, $\forall (\phi, \psi) \in C_o^\infty(\mathbb{R}^N) \times C_o^\infty(\mathbb{R}^N)$, there is $n_o > 0$ such that $\operatorname{supp}\phi$, $\operatorname{supp}\psi \subset B_n$, whenever $n \geq n_o$ and $\mathcal{I}'(z_n)(\phi, \psi) = 0$, if $n \geq n_o$. This implies that $\mathcal{I}'(z_n)z \to 0$ as $n \to \infty$ $\forall z \in C_o^\infty(\mathbb{R}^N) \times C_o^\infty(\mathbb{R}^N)$. Hence, (5.7) follows because $C_o^\infty(\mathbb{R}^N) \times C_o^\infty(\mathbb{R}^N)$ is dense in E. $\square$

We begin with problem (1.5)–(1.6). We remark that previous results for $\mathcal{I}$ and $\mathcal{J}$ also hold for $\mathcal{I}^\infty$ and $\mathcal{J}^\infty$, where $\mathcal{J}^\infty$ is the dual functional of $\mathcal{I}^\infty$.

Proposition 5.2. *Suppose* (H1)–(H4) *and* (3.14). *Then* (1.5)–(1.6) *has a nontrivial radial solution.*

Proof. We construct a sequence of radial solutions z_n of

$$\begin{cases} -\Delta u + u = |v|^{q-1}v + \bar{g}(v), & \text{in } B_n, \\ -\Delta v + v = |u|^{p-1}u + \bar{f}(u), & \text{in } B_n, \\ u = v = 0 & \text{on } \partial B_n \end{cases}$$

in balls B_n by Proposition 3.6. Lemma 5.1 implies that z_n is a $(PS)_c$ sequence of $\mathcal{I}^\infty$ with $c < \frac{2}{N} S_{p,q}^{\frac{N}{2}}$ and $z_n \in E_r = E_r^s \times E_r^t$, where E_r is the radial Sobolev space. It is known from [7] that the inclusion $E_r^s(\mathbb{R}^N) \hookrightarrow L^\tau(\mathbb{R}^N)$, $2 < p < \frac{2N}{N-2s}$, is compact. We may deduce as in Lemma 3.5 that there exists a subsequence of z_n converging strongly; the limit function is a nontrivial radial solution of (1.5)–(1.6). □

Next, we consider the variational problem

$$\mathcal{I}^\infty = \inf\{\mathcal{I}^\infty(u,v) : (u,v) \text{ is a solution of (1.5)–(1.6)}, (u,v) \not\equiv (0,0)\}. \tag{5.8}$$

Minimizers of (5.8) are called ground states of (1.5)–(1.6). By Proposition 5.2, the variational problem (5.8) is well defined if (3.14) holds. In this case

$$\mathcal{I}^\infty < \frac{2}{N} S_{p,q}^{\frac{N}{2}}. \tag{5.9}$$

Lemma 5.3. *The variational problem* (5.8) *is assumed by a nontrivial solution of* (1.5)–(1.6).

Proof. Let $z_n = (u_n, v_n)$ be a minimizing sequence of $\mathcal{I}^\infty$. By Proposition 4.2 we have

$$\mathcal{I}^\infty = \mathcal{I}^\infty(z_n) + o(1) = \sum_j \mathcal{I}^\infty(z^j) + o(1),$$

where z_j is a nontrivial solution of (1.5)–(1.6). Therefore, $j = 1$ and the proof is completed. □

Proposition 5.4. *Suppose* (H1)–(H4), (H6) *and* (3.14). *If there exists* $w \in X^*$ *such that*

$$\sup_{t \geq 0} J(tw) < \mathcal{I}^\infty, \tag{5.10}$$

then (1.1)–(1.2) *possesses a nontrivial radial solution.*

Proof. By assumptions (3.14) and (5.10), we always may construct a $(PS)_c$ sequence $\{z_n\}$ of $\mathcal{I}$ by Proposition 3.6 and Lemma 5.1 such that

$$0 < \alpha \le c < \mathcal{I}^\infty. \tag{5.11}$$

By Proposition 4.2 we obtain

$$\mathcal{I}(z_n) = \mathcal{I}(z_o) + \sum_j \mathcal{I}^\infty(z^j) + o(1), \tag{5.12}$$

where z_o is a solution of (1.1)–(1.2) and z^j is a solution of (1.5)–(1.6). We deduce from (5.11) and (5.12) that z_o is a nontrivial solution of (1.1)–(1.2). □

6. Verification of conditions (3.14) and (5.10). We verify condition (5.10) first. Let $B_n = B_{R_n}$, $R_n \to \infty$ as $n \to \infty$. For each element w in $X_n^* := L^{\frac{p+1}{p}}(B_n) \times L^{\frac{q+1}{q}}(B_n)$, where $B_n = B_{R_n}$, $R_n \to \infty$ as $n \to \infty$, we may extend it to $\mathbb{R}^N$ by setting $w = 0$ outside B_n, and we have $J_n(w) = \mathcal{J}(w)$.

Proposition 6.1. *Assume* (H1)–(H4), (H6)–(H8) *and* (3.14). *There exist elements $w_n \in X_n^*$ such that*

$$\sup_{t\ge 0} \mathcal{J}(tw_n) < \mathcal{I}^\infty \quad \textit{for } n \textit{ large.} \tag{6.1}$$

Proof. By Proposition 5.2, $\mathcal{I}^\infty$ is assumed. Let $z_o = (u_o, v_o)$ be a minimizer of problem $\mathcal{I}^\infty$. Choosing $w_1^o = \bar{f}_1(u_o) = |u_o|^{p-1}u_o + \bar{f}(u_o)$, $w_2^o = \bar{g}_1(u_o) = |v_o|^{q+1}v_o + \bar{g}(v_o)$, and using (H4), (H6) and equations (1.5)–(1.6), one has $\int_{R^N} \langle w_o, Kw_o\rangle \, dx > 0$, where $w_o = (w_1^o, w_2^o)$. Moreover, there exist $t_2 > t_1 \ge 0$ such that

$$\max_{t\ge 0} \mathcal{J}(tw_o) = \max_{t_1\le t\le t_2} \mathcal{J}(tw_o).$$

Suppose $t_o \in [t_1, t_2]$ and $\mathcal{J}(t_o w_o) = \max_{t_1\le t\le t_2} \mathcal{J}(tw_o)$. Because $\mathcal{F}(x,t) \ge \bar{\mathcal{F}}(t)$ and $\mathcal{G}(x,t) \ge \bar{\mathcal{G}}(t)$, one has $\mathcal{F}^*(x,s) \le \bar{\mathcal{F}}^*(s)$ and $\mathcal{G}^*(x,s) \le \bar{\mathcal{G}}^*(s)$. By assumption (H7), $\mathcal{J}(t_o w_o) < \mathcal{J}^\infty(t_o w_o)$, it follows that

$$\sup_{t\ge 0} \mathcal{J}(tw_o) < \sup_{t\ge 0} \mathcal{J}^\infty(tw_o). \tag{6.2}$$

The density of the real number field implies that there exists $\epsilon > 0$ such that

$$\sup_{t\geq 0} \mathcal{J}(tw_o) + 2\epsilon < \sup_{t\geq 0} \mathcal{J}^{\infty}(tw_o). \tag{6.3}$$

Let $\phi \in C_o^{\infty}(\mathbb{R}^N)$, $0 \leq \phi \leq 1$ and $\phi \equiv 1$ if $|x| \leq \frac{1}{2}$; $\phi \equiv 0$ if $|x| \geq 1$; $\phi_n(x) = \phi(\frac{x}{R_n})$. Then $z_n := (\phi_n u_o, \phi_n v_o)$ converges to (u_o, v_o) in E. Let $w_1^n = \bar{f}_1(\phi_n u_o)$, $w_2^n = \bar{g}_1(\phi_n v_o)$. We also have $w_n \to w_o$ in X^*. Suppose $\mathcal{J}(t_n w_n) = \sup_{t\geq 0} \mathcal{J}(tw_n)$; then $\{t_n\}$ is bounded. Indeed, if $t_n \to \infty$, arguments in Lemma 3.4 would yield $\sup_{t\geq 0} \mathcal{J}(tw_n) \to -\infty$. This is impossible because the value is not negative. Suppose $t_n \to \bar{t}_o$; the continuity of the functional $\mathcal{J}$ gives

$$\mathcal{J}(t_n w_n) \to \mathcal{J}(\bar{t}_o w_o).$$

We claim that $\mathcal{J}(\bar{t}_o w_o) = \sup_{t\geq 0} \mathcal{J}(tw_o)$. In fact, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mathcal{J}(t_o w_o) - \epsilon \leq \mathcal{J}(tw_o)$$

whenever $|t - t_o| < \delta$. By the continuity of $\mathcal{J}$, we may find $n_o > 0$ such that if $n \geq n_o$

$$\mathcal{J}(tw_o) \leq \mathcal{J}(tw_n) + \epsilon, \quad \mathcal{J}(t_n w_n) \leq \mathcal{J}(\bar{t}_o w_o) + \epsilon.$$

Therefore if $n \geq n_o$ we have

$$\mathcal{J}(t_o w_o) - \epsilon \leq \mathcal{J}(t_n w_n) + \epsilon \leq \mathcal{J}(\bar{t}_o w_o) + 2\epsilon \leq \mathcal{J}(t_o w_o) + 2\epsilon.$$

Because ϵ is arbitrary, the conclusion holds. By the same arguments, we find that there exist s_n such that $s_n \to \bar{s}_o$ and

$$\mathcal{J}^{\infty}(s_n w_n) = \sup_{t\geq 0} \mathcal{J}^{\infty}(tw_n) \to \mathcal{J}^{\infty}(\bar{s}_o w_o) = \sup_{t\geq 0} \mathcal{J}^{\infty}(tw_o) \tag{6.4}$$

as $n \to \infty$. By (6.3), we obtain

$$\mathcal{J}(t_n w_n) + \epsilon < \mathcal{J}^{\infty}(s_n w_n) \tag{6.5}$$

for n large enough. We may assume $s_n > 0$, and then

$$\frac{d\mathcal{J}^{\infty}(tw_n)}{dt} \mid_{t=s_n} = 0; \tag{6.6}$$

that is,

$$\int_{R^N} (\bar{\mathcal{F}}_s^{*'}(s_n w_1^n) w_1^n + \bar{\mathcal{G}}_s^{*'}(s_n w_2^n) w_2^n)\, dx - s_n \int_{R^N} \langle w_n, K w_n \rangle \; dx = 0. \tag{6.7}$$

By the definition of the Legendre-Fenchel transformation, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} (\bar{\mathcal{F}}^*(s_n w_1^n) + \bar{\mathcal{G}}^*(s_n w_2^n))\, dx = \int_{\mathbb{R}^N} (\bar{\mathcal{F}}_s^{*'}(s_n w_1^n) s_n w_1^n \\ &+ \bar{\mathcal{G}}_s^{*'}(s_n w_2^n) s_n w_2^n)\, dx - \int_{\mathbb{R}^N} [\bar{\mathcal{F}}(\bar{f}_1^{-1}(s_n w_1^n)) + \bar{\mathcal{G}}(\bar{g}_1^{-1}(s_n w_2^n))]\, dx \\ &= s_n^2 \int_{R^N} \langle w_n, K w_n \rangle \; dx - \int_{\mathbb{R}^N} [\bar{\mathcal{F}}(\bar{f}_1^{-1}(s_n w_1^n)) + \bar{\mathcal{G}}(\bar{g}_1^{-1}(s_n w_2^n))]\, dx. \end{aligned} \tag{6.8}$$

Considering

$$(-\Delta + id)^{-1} w_2^n = u_o + \sigma_n, \quad (-\Delta + id)^{-1} w_1^n = v_o + \mu_n \quad \text{in} \quad \mathbb{R}^N,$$

we obtain

$$(-\Delta + id)^{-1} \sigma_n = \bar{g}_1(\phi_n v_o) - \bar{g}_1(v_o), \quad (-\Delta + id)^{-1} \mu_n = \bar{f}_1(\phi_n u_o) - \bar{f}_1(u_o)$$

in $\mathbb{R}^N$. By L^p estimates we have $\sigma_n \to 0$ and $\mu_n \to 0$ in $H^{2,2}$ as $n \to \infty$ because the right-hand sides of the above equations go to 0 in L^2. Therefore we infer from this and (6.7) that

$$\begin{aligned} &\int_{\mathbb{R}^N} s_n (w_1^n)^2 \Big[\frac{\bar{f}_1^{-1}(s_n w_1^n)}{s_n w_1^n} - \frac{\bar{f}_1^{-1}(w_1^n)}{w_1^n}\Big]\, dx \\ &+ \int_{\mathbb{R}^N} s_n (w_2^n)^2 \Big[\frac{\bar{g}_1^{-1}(s_n w_2^n)}{s_n w_2^n} - \frac{\bar{g}_1^{-1}(w_2^n)}{w_2^n}\Big]\, dx \\ &= \int_{\mathbb{R}^N} [w_1^n \sigma_n + w_2^n \mu_n + (1 - \phi_n)(w_1^n + w_2^n)]\, dx = o(1) \end{aligned}$$

as $n \to \infty$. The equality and assumption (H8) imply $s_n \to 1$ as $n \to \infty$. Hence we deduce by (6.7) and (6.8) that

$$\begin{aligned} \sup_{t \geq 0} \mathcal{J}^\infty(t w_n) &\leq \frac{1}{2} \int_{\mathbb{R}^N} (u_o \bar{f}_1(u_o) + v_o \bar{g}_1(v_o))\, dx \\ &- \int_{R^N} (\bar{\mathcal{F}}(u_o) + \bar{\mathcal{G}}(v_o))\, dx + \epsilon_n = \mathcal{I}^\infty + \epsilon_n, \end{aligned} \tag{6.9}$$

where

$$\begin{aligned}\epsilon_n &= \frac{1}{2}(s_n^2 - 1)\int_{\mathbb{R}^N}(u_o\bar{f}_1(u_o) + v_o\bar{g}(v_o))\,dx\\&- \int_{\mathbb{R}^N}[(\bar{\mathcal{F}}(\phi_n u_o) - \bar{\mathcal{F}}(u_o)) + (\bar{\mathcal{G}}(\phi_n v_o) - \bar{\mathcal{G}}(v_o))]\,dx\\&+ \int_{\mathbb{R}^N}[(\bar{\mathcal{F}}(\phi_n u_o) - \bar{\mathcal{F}}(\bar{f}_1^{-1}(s_n w_1^n))) + (\bar{\mathcal{G}}(\phi_n v_o) - \bar{\mathcal{G}}(\bar{g}_1^{-1}(s_n w_2^n)))]\,dx.\end{aligned}$$

The above estimates imply $\epsilon_n = o(1)$ as $n \to \infty$. From (6.5)–(6.9) we obtain

$$\sup_{t\geq 0}\mathcal{J}(tw_n) < \sup_{t\geq 0}\mathcal{J}^\infty(tw_n) - \epsilon \leq \mathcal{I}^\infty - \epsilon + o(1);$$

the assertion follows for n large. $\square$

Next, we verify (3.14). It is known from [23] that the system

$$-\Delta u = |v|^{q-1}v \quad \text{in} \quad \mathbb{R}^N, \quad -\Delta v = |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N, \tag{6.10}$$

$$u(x) \to 0 \quad \text{and} \quad v(x) \to 0 \quad \text{as} \quad |x| \to \infty \tag{6.11}$$

has a ground state. The ground state is unique up to scalings and translations and is positive, radially symmetric and decreasing in r. Let (u, v) be the ground state of (6.10)–(6.11). Then all the ground states of (6.10)–(6.11) are given by $u_\epsilon(x) = \epsilon^{-\frac{n}{p+1}}u(\frac{x}{\epsilon})$, $v_\epsilon(x) = \epsilon^{-\frac{n}{q+1}}v(\frac{x}{\epsilon})$. Moreover,

$$\int_{R^N}|u_\epsilon|^{p+1}\,dx = \int_{R^N}|v_\epsilon|^{q+1}\,dx = S_{p,q}^{\frac{N}{2}}.$$

The asymptotic behavior of the ground state of (6.10)–(6.11) was found in [21]. It may be stated as follows.

Lemma 6.2. *Let $p \geq \frac{N+2}{N-2}$. Then there exist constants $a > 0$ and $b > 0$ depending on p and n, such that*

$$\begin{aligned}&\lim_{r\to\infty} r^{N-2}v_1(r) = b;\\&\lim_{r\to\infty} r^{N-2}u_1(r) = a \quad \textit{if} \quad q > \tfrac{N}{N-2};\\&\lim_{r\to\infty} \frac{r^{N-2}}{logr}u_1(r) = a \quad \textit{if} \quad q = \tfrac{N}{N-2};\\&\lim_{r\to\infty} r^{q(N-2)-2}u_1(r) = a \quad \textit{if} \quad q < \tfrac{N}{N-2}.\end{aligned}$$

Suppose $1 < \gamma < p$, $1 < \nu < q$ and $1 < q \leq \frac{N+2}{N-2} \leq q$. Parametrizing the critical hyperbola by $p = \frac{N+2+2\alpha}{N-2-2\alpha}$, $q = \frac{N+2-2\alpha}{N-2+2\alpha}$ and using Lemma 6.2, we obtain that if $q > \frac{N}{N-2}$

$$\|u_\epsilon\|^{\gamma}_{\frac{\gamma(p+1)}{p}} = \begin{cases} O(\epsilon^{\frac{N(p-\gamma)}{p+1}}), & \text{if} \quad \gamma > \frac{N+2+2\alpha}{2(N-2)}, \\ O(\epsilon^{\frac{N(p-\gamma)}{p+1}}|\text{log}\epsilon|^{\frac{p}{p+1}}), & \text{if} \quad \gamma = \frac{N+2+2\alpha}{2(N-2)}, \\ O(\epsilon^{\frac{\gamma}{p+1}[(N-2)p-2]}), & \text{if} \quad \gamma < \frac{N+2+2\alpha}{2(N-2)}; \end{cases}$$

if $1 < q < \frac{N}{N-2}$

$$\|u_\epsilon\|^{\gamma}_{\frac{\gamma(p+1)}{p}} = \begin{cases} O(\epsilon^{\frac{N(p-\gamma)}{p+1}}), & \text{if} \quad 2(\gamma+1) + \frac{N}{q+1} < \gamma q(N-2), \\ O(\epsilon^{\frac{N(p-\gamma)}{p+1}}|\text{log}\epsilon|^{\frac{p}{p+1}}), & \text{if} \quad 2(\gamma+1) + \frac{N}{q+1} = \gamma q(N-2), \\ O(\epsilon^{\frac{\gamma}{p+1}[(N-2)p-2]}), & \text{if} \quad 2(\gamma+1) + \frac{N}{q+1} > \gamma q(N-2); \end{cases}$$

if $q = \frac{N}{N-2}$

$$\|u_\epsilon\|^{\gamma}_{\frac{\gamma(p+1)}{p}} = \begin{cases} O(\epsilon^{\frac{N\gamma}{q+1}}|\text{log}\epsilon|^{\gamma}), & \text{if} \quad \gamma < \frac{N}{N-2}\frac{p}{p+1}, \\ O(\epsilon^{\frac{N(p-\gamma)}{p+1}}|\text{log}\epsilon|^{\gamma+\frac{p}{p+1}}), & \text{if} \quad \gamma = \frac{N}{N-2}\frac{p}{p+1}, \\ O(\epsilon^{\frac{N(p-\gamma)}{p+1}}), & \text{if} \quad \gamma > \frac{N}{N-2}\frac{p}{p+1}, \end{cases}$$

and

$$\|v_\epsilon\|^{\nu}_{\frac{\nu(q+1)}{q}} = \begin{cases} O(\epsilon^{\frac{N(q-\nu)}{q+1}}), & \text{if} \quad \nu > \frac{N+2-2\alpha}{2(N-2)}, \\ O(\epsilon^{\frac{N(q-\nu)}{q+1}}|\text{log}\epsilon|^{\frac{q}{q+1}}), & \text{if} \quad \nu = \frac{N+2-2\alpha}{2(N-2)}, \\ O(\epsilon^{\frac{\nu}{q+1}[(N-2)q-2]}), & \text{if} \quad \nu < \frac{N+2-2\alpha}{2(N-2)}. \end{cases}$$

Proposition 6.3. *Assume* (H1), (H3) *and* (H7). *There exists a path* $w(t) \in X^*$ *such that* $w(0) = 0, J(w(t)) \leq 0$ *for* $t > 0$ *large and*

$$\sup_{t\geq 0} J(w(t)) < \frac{2}{N} S_{p,q}^{\frac{N}{2}}. \tag{6.12}$$

Proof. By the definition of duality

$$J(w) = \int_\Omega \Big[f_1^{-1}(x,w_1)w_1 + g_1^{-1}(x,w_2)w_2 - \mathcal{F}(x, f_1^{-1}(x,w_1)) \\ - \mathcal{G}(x, g_1^{-1}(x,w_1))\Big]dx - \frac{1}{2}\int_\Omega \langle Kw, w\rangle \, dx.$$

Choosing $w_1(s) = w_1(s,\epsilon,x) = f_1(x, su_\epsilon), w_2(t) = w_2(t,\epsilon,x) = g_1(x, tv_\epsilon)$, where (u_ϵ, v_ϵ) is a ground state of (6.10)–(6.11), we remark that $w_1(0) = w_2(0) = 0$, $w_1(s), w_2(t) \to \infty$ as $s,t \to +\infty$. Then

$$\begin{aligned} J(w(s,t)) = &\int_\Omega su_\epsilon[s^p u_\epsilon^p + f(x,su_\epsilon)] + tv_\epsilon[t^q v_\epsilon^q + g(x,tv_\epsilon)] \\ &- \int_\Omega [\mathcal{F}(x,su_\epsilon) + \mathcal{G}(x,tv_\epsilon)]\,dx \\ &- \frac{1}{2}\{\int_\Omega [s^p u_\epsilon^p + f(x,su_\epsilon)](-\Delta + id)^{-1}[t^q v_\epsilon^q + g(x,tv_\epsilon)]\,dx \\ &+ \int_\Omega [t^q v_\epsilon^q + g(x,tv_\epsilon)](-\Delta + id)^{-1}[s^p u_\epsilon^p + f(x,su_\epsilon)]\,dx\}. \end{aligned}$$

Let $(-\Delta)^{-1}u_\epsilon^p = v_\epsilon + \xi_\epsilon \in H_o^1(\Omega)$, $(-\Delta)^{-1}v_\epsilon^q = u_\epsilon + \eta_\epsilon \in H_o^1(\Omega)$, $(-\Delta + id)^{-1}u_\epsilon^p = v_\epsilon + \xi_\epsilon + r_\epsilon^1 := v_\epsilon + \bar{\xi}_\epsilon \in H_o^1(\Omega)$, $(-\Delta + id)^{-1}v_\epsilon^q = u_\epsilon + \eta_\epsilon + r_\epsilon^2 := u_\epsilon + \bar{\eta}_\epsilon \in H_o^1(\Omega)$. Then

$$\Delta\xi_\epsilon = 0 \ \text{ in } \ \Omega,\ \xi_\epsilon = -v_\epsilon \ \text{ on } \ \partial\Omega; \quad \Delta\eta_\epsilon = 0 \ \text{ in } \ \Omega,\ \eta_\epsilon = -u_\epsilon \ \text{ on } \ \partial\Omega.$$

$$(-\Delta + id)r_\epsilon^1 = -v_\epsilon - \xi_\epsilon, \quad (-\Delta + id)r_\epsilon^2 = -u_\epsilon - \eta_\epsilon.$$

By the maximum principle $\|\xi_\epsilon\|_{L^\infty(\Omega)} \le \|v_\epsilon\|_{L^\infty(\partial\Omega)}$, $\|\eta_\epsilon\|_{L^\infty(\Omega)} \le \|u_\epsilon\|_{L^\infty(\partial\Omega)}$; $u_\epsilon + \xi_\epsilon \ge 0$, $v_\epsilon + \eta_\epsilon \ge 0$, $r_\epsilon^1 \le 0$, $r_\epsilon^1 \le 0$. We rewrite

$$\begin{aligned} \frac{1}{2}\int_\Omega \langle Kw, w\rangle\ dx = &\frac{1}{2}s^p t^q \int_\Omega (u_\epsilon^{p+1} + v_\epsilon^{q+1})\,dx \\ &+ s^p \int_\Omega v_\epsilon g(x,tv_\epsilon)\,dx + t^q \int_\Omega u_\epsilon f(x,su_\epsilon)\,dx + \phi_\epsilon(s,t), \end{aligned}$$

where

$$\begin{aligned} \phi_\epsilon(s,t) = &\int_\Omega [f(x,su_\epsilon)(-\Delta + id)^{-1}g(x,tv_\epsilon) + s^p\bar{\xi}_\epsilon g(x,tv_\epsilon) + t^q\bar{\eta}_\epsilon f(x,su_\epsilon)]dx \\ &- \frac{1}{2}s^p t^q \int_\Omega (u_\epsilon^p \bar{\eta}_\epsilon + v_\epsilon^q \bar{\xi}_\epsilon)\,dx. \end{aligned}$$

Then

$$\begin{aligned} J(w(s,t)) = &\frac{p}{p+1}s^{p+1}\int_\Omega u_\epsilon^{p+1}\,dx + \frac{q}{q+1}t^{q+1}\int_\Omega v_\epsilon^{q+1}\,dx \\ &- \frac{1}{2}s^p t^q \int_\Omega (u_\epsilon^{p+1} + v_\epsilon^{q+1})\,dx + (t - s^p)\int_\Omega v_\epsilon g(x,tv_\epsilon)\,dx \\ &+ (s - t^q)\int_\Omega u_\epsilon f(x,su_\epsilon)\,dx - \int_\Omega (F(x,su_\epsilon) + G(x,tv_\epsilon))\,dx - \phi_\epsilon(s,t). \end{aligned}$$

Let $s^{p+1}=t^{q+1}$. The highest order of t in

$$J(w(t))=\frac{p}{p+1}t^{q+1}\int_\Omega u_\epsilon^{p+1}\,dx+\frac{q}{q+1}t^{q+1}\int_\Omega v_\epsilon^{q+1}\,dx$$
$$-\frac{1}{2}t^{\frac{p(q+1)}{p+1}+q}\int_\Omega(u_\epsilon^{p+1}+v_\epsilon^{q+1})\,dx+(t-t^{\frac{p(q+1)}{p+1}})\int_\Omega v_\epsilon g(x,tv_\epsilon)\,dx$$
$$+(t^{\frac{q+1}{p+1}}-t^q)\int_\Omega u_\epsilon f(x,t^{\frac{q+1}{p+1}}u_\epsilon)\,dx-\int_\Omega(F(x,su_\epsilon)+G(x,tv_\epsilon))\,dx-\phi_\epsilon(s,t)$$

is $t^{\frac{p(q+1)}{p+1}+q}$. So $J(w(t))\le 0$ for $t>0$ large. There exists $t_o\ge 0$ such that

$$J(w(t_o))=\max_{0\le t\le t_o}J(w(t)).$$

Since $t^{\frac{q+1}{p+1}}-t^q\le 0,\quad t-t^{\frac{p(q+1)}{p+1}}\le 0$ for $t\ge 1$, and by assumptions (H1)–(H3) there exist $1\le\tau\le p_1<p, 1\le\nu\le q_1<q$ such that

$$|f(x,t)|\le C(|t|^\tau+|t|^{p_1}),\quad |g(x,t)|\le C(|t|^\nu+|t|^{q_1}),$$

we obtain for $t\le 1$

$$\int_\Omega u_\epsilon f(x,t^{\frac{q+1}{p+1}}u_\epsilon)\,dx=O(\|u_\epsilon\|^\tau_{\frac{\tau(p+1)}{p}}+\|u_\epsilon\|^{p_1}_{\frac{p_1(p+1)}{p}}):=k_1(\epsilon)$$

and

$$\int_\Omega v_\epsilon g(x,tv_\epsilon)\,dx=O(\|v_\epsilon\|^\nu_{\frac{\nu(q+1)}{q}}+\|v_\epsilon\|^{q_1}_{\frac{q_1(q+1)}{q}}):=k_2(\epsilon).$$

Noting that $\int_\Omega u_\epsilon^{p+1}\,dx$ and $\int_\Omega v_\epsilon^{q+1}\,dx$ tend to $S_{p,q}^{\frac{N}{2}}$ from below as $\epsilon\to 0$, we define

$$h(\epsilon)=(\frac{N+2}{N}t^{q+1}-t^{\frac{(N+2)(q+1)}{N}})S_{p,q}^{\frac{N}{2}}+(t^{\frac{q+1}{p+1}}-t^q)k_1(\epsilon)+(t-t^{\frac{p(q+1)}{p+1}})k_2(\epsilon).$$

The maximum point $t_\epsilon>0$ of $h(\epsilon)$ satisfies $h'(\epsilon)=0$. Let $t_\epsilon=1+\delta_\epsilon$. We obtain from $h'(\epsilon)=0$ that $\delta_\epsilon=O(k_1(\epsilon)+k_2(\epsilon))$. Because the operator $K^{-1}:X^*\to X$ is bounded, we infer that

$$\int_\Omega f(x,su_\epsilon)(-\Delta+id)^{-1}g(x,tv_\epsilon)\,dx$$
$$\le C(\|u_\epsilon\|^{2\tau}_{\frac{\tau(p+1)}{p}}+\|u_\epsilon\|^{2p_1}_{\frac{p_1(p+1)}{p}}+\|v_\epsilon\|^{2\nu}_{\frac{\nu(q+1)}{q}}+\|v_\epsilon\|^{2q_1}_{\frac{q_1(q+1)}{p}}),$$
$$\int_\Omega u_\epsilon^p(\eta_\epsilon+r_\epsilon^2)\,dx=\int_\Omega[u_\epsilon^p\eta_\epsilon-(u_\epsilon+\eta_\epsilon)(v_\epsilon+\xi_\epsilon)-(u_\epsilon+\eta_\epsilon)r_\epsilon^1]\,dx$$
$$=O(\|u_\epsilon\|_2^2+\|v_\epsilon\|_2^2).$$

By estimates for $\|u_\epsilon\|^{\gamma}_{\frac{\gamma(p+1)}{p}}$ and $\|v_\epsilon\|^{\nu}_{\frac{\nu(q+1)}{q}}$, we find the dominating term in $\phi(s,t)$ is $O(\|u_\epsilon\|_2^2 + \|v_\epsilon\|_2^2)$. Therefore,

$$\begin{aligned} J(w(t)) &\le \frac{2}{N} S_{p,q}^{\frac{N}{2}} - \int_\Omega (F(x, su_\epsilon) + G(x, tv_\epsilon))\, dx + \delta_\epsilon^2 + O(\|u_\epsilon\|_2^2 + \|v_\epsilon\|_2^2) \\ &\le \frac{2}{N} S_{p,q}^{\frac{N}{2}} - \epsilon^N \int_0^{R\epsilon^{-1}} \bar{F}(\epsilon^{-\frac{N}{p+1}} u(r)) r^{N-1}\, dr \\ &\quad - \epsilon^N \int_0^{R\epsilon^{-1}} \bar{G}(\epsilon^{-\frac{N}{q+1}} v(r)) r^{N-1}\, dr + O(\|u_\epsilon\|_2^2 + \|v_\epsilon\|_2^2). \end{aligned}$$

We conclude by assumption (H5). □

The proof of Theorems A and B completed. The proof of Theorem A follows by Propositions 3.6 and 6.3.

The existence results of Theorem B follow by Propositions of 5.4, 6.1 and 6.3. Weak solutions of (1.1)–(1.2) obtained by variational method actually are strong solutions [16], therefore the decaying law is obtained by Proposition 2.2 for the case $p = q = \frac{N+2}{N-2}$. □

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