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INTERMEDIATE ASYMPTOTICS OF THE POROUS MEDIUM EQUATION WITH SIGN CHANGES*

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Abstract. We study the porous medium equation with sign changes and examine the way sign changes disappear. We give a formal classification of self-similar and non-self-similar scenarios for their disappearance, for $N > 1$, restricting attention to the radial case. The results we present on the classification of similarity solutions are rigorous except where indicated otherwise.

1. Introduction

In this paper we consider the nonlinear diffusion equation

$$u_t = \frac{1}{m}\Delta|u|^{m-1}u = \operatorname{div}(|u|^{m-1}\operatorname{grad} u), \tag{1.1}$$

for $m > 0$ in dimension $N \geq 1$. This equation is commonly called the porous medium equation with sign changes and was studied in, for instance, [8], [12],

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[25], and [17]. We refer to these papers and also to [3], [26] and [22] for a general background to (1.1). We also mention that equation (1.1) arises in the theory of superconducting materials (see, for example, [9] and [10]), the description of the mixing of fresh and salt groundwater due to mechanical dispersion (see [15]) and that the one-dimensional flow of an incompressible power-law fluid can be written in the form (1.1) with $N = 1$; if the velocity components in (x, y, z) are $(0, 0, w(x, t))$, then $u \equiv \partial w/\partial x$.

We are interested here in the local behaviour of solutions as the number of sign changes decreases. Zero-set analysis, which goes back to Sturm, plays a fundamental role in the theory of linear and nonlinear parabolic equations; see e.g. [24]. Before proceeding, it is instructive to note the behaviour for the linear heat equation, $m = 1$. In this case only, (1.1) is invariant under translations of u, so the value $u = 0$ has no special status. In consequence, when a minimum or maximum crosses $u = 0$ it does so with no change of regularity. Taking this to occur at the origin and at $t = 0$, in the radially symmetric case the behaviour is generically of the form

$$u \sim A_1(r^2 + 2Nt) \text{ as } r,\, t \to 0, \tag{1.2}$$

for some constant A_1. For what follows subsequently, the main point to note about (1.2) is that it can be written in the "backward" and "forward" self-similar forms

$$u \sim (\pm t)U_{1,\pm}\Big(\frac{r}{(\pm t)^{\frac{1}{2}}}\Big), \quad U_{1,\pm}(\eta) = A_1(\eta^2 \pm 2N) \text{ as } t \to 0^{\pm}. \tag{1.3}$$

For nongeneric cases in which $A_1 = 0$ there is an infinite sequence of similarity solutions parametrized by n available to describe the local behaviour, namely

$$u \sim (\pm t)^n U_{n,\pm}\Big(\frac{r}{(\pm t)^{\frac{1}{2}}}\Big) \text{ as } t \to 0^{\pm}, \tag{1.4}$$

where n is a positive integer with, for example, $U_{2,\pm}(\eta) = A_2(\eta^4 \pm 4(N + 2)\eta^2 + 4N(N + 2))$; for $N = 1$ the solutions can be written as Hermite polynomials (see the fundamental results in [11]). We shall see that for $m \neq 1$ the similarity exponents in the analogue of (1.4) must be determined via a nonlinear eigenvalue problem, being similarity solutions of the second kind. It is noteworthy that for $m = 1$ they can be obtained beforehand, not in the usual way for solutions of the first kind (i.e., from a conservation law), but rather by appealing to the analyticity of the solution.

We have just described how sign changes can be lost internally when $m = 1$; they can also persist for all time by propagating to infinity, a WKBJ far-field method analysis implying that

$$u \sim t^{-\frac{N}{2}} g(\frac{r}{t}) e^{-\frac{r^2}{4t}} \text{ as } t \to \infty, \ r = O(t), \tag{1.5}$$

where g depends on the initial data and may contain any number of zeros.

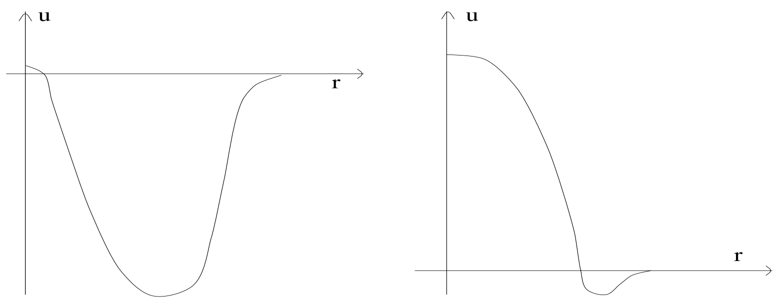

Figure 1. Slow diffusion ($m > 1$): interior extinction at the origin (left) and extinction at the front (right).

For $N = 1$ and $m > 1$, the "slow" diffusion case, there are two distinct ways in which the number of zeros of a compactly supported solution can decrease, and these are illustrated schematically in Figure 1. We show in Section 2 the existence of an infinite sequence of solutions which can describe interior extinction. We restrict attention for $N > 1$ to the radially symmetric case, omitting consideration of additional nonradial possibilities. Theorem 1 in Section 2 gives for each $j = 1, 2, 3, \dots$ a radial similarity solution of (1.1) having j sign changes for $t < 0$, all of which disappear at $t = 0$, where we have $u = Cr^{k_j}$. For $t > 0$ this solution may be continued as a self-similar solution which becomes strictly positive; see also [22]. We note that this is because the exponents satisfy $k_j < \frac{2}{m-1}$. Exponents $k \geq \frac{2}{m-1}$ are discussed in [12] and [25] and do not allow positive solutions. For $k = \frac{2}{m-1}$ (the waiting-time exponent) the solution is separable, and for $k > \frac{2}{m-1}$ the solution has infinitely many sign changes away from $r = 0$ simultaneously coming in from $r = \infty$ as t becomes positive. They stand apart from the other self-similar solutions discussed in this paper.

We believe the solution in Theorem 1 with $j = 1$ to be generic in describing the intermediate asymptotic behaviour of radially symmetric solutions in the case when extinction occurs at the origin; the configuration of Figure 1 (left) corresponds locally to this solution. The higher modes in the sequence contain more zeros and correspond to nongeneric cases in which more sign changes simultaneously disappear at the same location. The corresponding solutions for $N = 1$ are expected to play a similar local role for radially symmetric solutions with $N \geq 1$ when sign changes disappear at $r = r_0$ with $0 < r_0 < s$, $r = s(t)$ being the edge of the support of u; the leading order behaviour is then one-dimensional in the variable $r - r_0$. Odd solutions are also possible, and these are described in Theorem 2.

It follows from the classification in Section 2 that no similarity solutions exist which are appropriate to the configuration of Figure 1 (right), though it is shown in [17] that any solution of nonzero mass becomes one-signed in finite time. In Section 6 we derive by formal asymptotic methods the generic extinction behaviour, showing that the behaviour is not exactly self-similar but may be described by singular perturbation methods.

In Section 4 we consider radially symmetric solutions for the "fast" diffusion case $\max(0, \frac{N-2}{N}) < m < 1$. Sign changes may suffer finite-time interior extinction in a similar manner to those for slow diffusion, see Theorem 3, and the full range $m > 0$ is in fact analyzed in Section 2 (the constraint $m > 0$ is needed for solutions with sign changes to exist). However, in sharp contrast to the slow-diffusion case, a sign change may also persist for all time, in which case it will move out to infinity as $t \to +\infty$. Indeed, if, say, the solution has positive mass but the initial conditions are negative (and decay to zero sufficiently rapidly for large r), then we have

$$u \sim -\Big(\frac{(1-m)}{2\,(2+(m-1)N)}\ \frac{r^2}{t}\Big)^{-\frac{1}{1-m}} \qquad \text{as} \ \ r \to +\infty \tag{1.6}$$

for all t, but as $t \to +\infty$ with $r = O\left(t^{1/(2+(m-1)N)}\right)$ the behaviour is described by the Zel'dovich–Kompaneetz–Barenblatt (ZKB) solution [5]

$$u \sim t^{-\frac{N}{2+(m-1)N}} \Big(\frac{(1-m)}{2\,(2+(m-1)N)}\ \Big(a^2 + \frac{r^2}{t^{\frac{2}{2+(m-1)N}}}\Big)\Big)^{-\frac{1}{1-m}} \tag{1.7}$$

for some positive constant a which depends on the total mass. We anticipate that all sign changes but one disappear in finite time by interior extinction, but a (singular) solution with a single sign change is required to describe the transition between (1.7) and (1.6) for large r and t. This is in some respects the analogue of (1.5) with one sign change of g. In the notation of Section 4 this similarity solution must thus satisfy

$$\begin{aligned} U(\eta) &\sim \Big(\frac{(1-m)}{2(2+(m-1)N)}\ \eta^2\Big)^{-\frac{1}{1-m}} \qquad && \text{as } \eta \to 0^+, \\ U(\eta) &\sim -\Big(\frac{(1-m)}{2(2+(m-1)N)}\ \eta^2\Big)^{-\frac{1}{1-m}} \qquad && \text{as } \eta \to \infty, \end{aligned}$$

where $\eta = r/t^\beta$ for some specific $\beta > 1/(2+(m-1)N)$; this outer solution thus holds as $t \to +\infty$ for r much larger than in (1.7) (which forms the inner solution), namely for $r = O\left(t^\beta\right)$. In Section 4 we prove the existence of such a solution, unique up to rescalings (Theorem 6), and show that no comparable solutions exist with more sign changes. We will refer to this solution as the *tail-switch* solution. If we take the pointwise minimum of a source-type solution and a tail-switch solution we obtain a supersolution which can be used as a barrier, keeping solutions converging to the source-type solution from losing their last sign change.

For the range $0 < m < \frac{N-2}{N}$, the equation (1.1) loses mass at infinity and finite-mass solutions extinguish in finite time. There is a self-similar finite-mass solution without sign changes which describes this behaviour [21]. We shall call this solution the *ground-state* solution. If an arbitrary finite-mass solution has sign changes these may or may not all disappear before the extinction time (we believe that the generic behaviour is that they do all disappear prior to extinction). In addition to possible interior extinction of sign changes, again given in Theorem 3, Theorem 4 in Section 2 gives a family of (singular) self-similar solutions which describe how any finite number of sign changes may disappear at infinity. There is, however, also the possibility (albeit nongeneric) that solutions are, say, negative near $r = 0$ and positive near $r = \infty$ right up to the time of complete extinction. The asymptotic behaviour cannot then be described solely by a self-similar solution, but it may again be adequately understood in terms of matched asymptotic expansions. This case is treated in Section 7.

Finally, in Section 8 we discuss the limit behaviour of the sign-changing similarity solutions and of the exponents appearing in Theorems 1–5 as the number of sign changes j tends to infinity. This part of the analysis relies on numerical computations.

The intermediate asymptotics in this paper fall into two categories, one where the behaviour is self-similar (Sections 2–5) and one where it is not uniformly so and matched asymptotic expansions are needed (Sections 6–7). Examples of the former class have been studied in other contexts using renormalisation group theory; see [5] and references therein. The latter class poses a particular challenge to this approach.

2. The backward equation

In this section we consider radial self-similar solutions of the form

$$u(r,t) = t^\alpha U(rt^{-\beta}), \quad r = |x|, \quad 2\beta = (m-1)\alpha + 1, \tag{2.1}$$

of the backward porous medium equation with sign changes,

$$u_t = -\frac{1}{m}\Delta|u|^{m-1}u. \tag{2.2}$$

We shall write $k = \frac{\alpha}{\beta}$. In the theorems which follow, the similarity exponents are determined via nonlinear eigenvalue problems (i.e., as similarity solutions of the second kind; see [5]).

2.1. **Regular initial data.** We are interested in solutions which have a well-defined nonzero limit as $t \downarrow 0$. As indicated in Section 1, these solutions play a role in the local behaviour as the number of sign changes (of general forward solutions) drops. We shall prove the following two theorems for the slow diffusion case.

Theorem 1. *Let $m > 1$ and $N \geq 1$. There exists a sequence*

$$0 = \alpha_0(m,N) < \alpha_1(m,N) < \alpha_2(m,N) < \alpha_3(m,N) < \cdots < \alpha_{AG}(m,N),$$

such that for every $\alpha = \alpha_j(m,N)$ $(j = 0,1,2,\dots)$ and every constant $C \neq 0$, equation (2.2) *has a radial self-similar solution of the form* (2.1) *with*

$$u(r,0) = \lim_{t\downarrow 0} t^{\alpha} U(rt^{-\beta}) = Cr^{\frac{\alpha}{\beta}} = Cr^k. \tag{2.3}$$

The corresponding profile U has exactly j sign changes in $[0,\infty)$, $U(\eta) \sim C\eta^k$ as $\eta \to \infty$, and $U(0)$ and C are related by a bijection.

This theorem gives us an increasing sequence of exponents $k_j = \alpha_j/\beta_j$ for which (2.2) may be solved with initial data

$$u(x,0) = C|x|^{k_j}. \tag{2.4}$$

Note that $k_0 = 0$ gives the constant solution $u = C$.

The behaviour of a solution to an initial value problem for (1.1) in which sign changes disappear at $r = 0$, $t = 0$ is expected to be given, as $t \uparrow 0$ with $r = O((-t)^{\beta})$, by a similarity solution of this class with $j \geq 1$ (the case $j = 1$ being generic). As $t \uparrow 0$ with $r = O(1)$ we have $u \sim u(r,0)$, which depends on the initial conditions but which satisfies $u(r,0) \sim Cr^k$ as $r \to 0$.

The value $\alpha = \alpha_{AG}(m,N)$ is the Aronson–Graveleau exponent for which there exists a solution with initial data

$$u(x,0) = C|x|^{k_{AG}(m,N)}, \quad k_{AG}(m,N) = \frac{\alpha_{AG}(m,N)}{\beta_{AG}(m,N)}, \tag{2.5}$$

which is supported on the complement of an expanding ball as t increases from zero. In [4] it is shown that there exists a unique value $\alpha = \alpha_{AG}(m,N)$ for which this is the case. In dimension $N = 1$ this is simply a travelling wave solution (together with its mirror image) with $\alpha_{AG}(m,1) = k_{AG}(m,1) = \frac{1}{m-1}$, while $\alpha_{AG}(m,N) < \frac{1}{m-1}$ and $k_{AG}(m,N) < \frac{1}{m-1}$ for $N > 1$. These solutions have been shown to describe what happens locally as the support of a radial nonnegative solution (of the forward equation) fills up a hole; see [2], where it is also shown that for $N > 1$ the "pressure" u^{m-1} in such cases does not have to be Lipschitz in x. An interesting question is whether or not the sequence α_j converges to α_{AG}. A negative answer implies the existence of a "continuous spectrum" of exponents k for which a solution of (2.2) exists with initial data $u(x,0) = C|x|^k$ having infinitely many sign changes near $x = 0$ for $t > 0$. This is briefly discussed in Section 8 where it is found numerically that there is indeed a gap between α_∞ and α_{AG}.

We also note that beyond the Aronson–Graveleau exponent, in fact for every $\alpha > \alpha_{AG}(m,N)$, there is for every C a unique solution U with

$$U(\eta) = O(\eta^{\frac{2}{m-1}}) \text{ as } \eta \to 0 \text{ and } U(\eta) \sim C\eta^k \text{ as } \eta \to \infty. \tag{2.6}$$

This gives backward solutions for initial data

$$u(x,0) = C|x|^k, \quad k_{AG}(m,N) < k < \frac{2}{m-1}, \tag{2.7}$$

with an isolated zero at the origin but no sign changes. These have been noted in [22] in relation to the waiting-time behaviour discovered in [16]. In higher dimensions they provide configurations of the forward equation in which a zero in u does not rise up until after some finite time.

In one space dimension the sequence in Theorem 1 has to be supplemented with a sequence of exponents k_j^* and corresponding initial data

$$u(x,0) = C|x|^{k_j^*-1}x, \tag{2.8}$$

which come from the following theorem.

Theorem 2. *Let $m > 1$ and $N = 1$. There exists a sequence*

$$\frac{1}{m+1} = \alpha_0^*(m) < \alpha_1^*(m) < \alpha_2^*(m) < \alpha_3^*(m) < \cdots < \alpha_{AG}(m,1) = \frac{1}{m-1},$$

such that for every $\alpha = \alpha_j^*(m,N)$ $(j = 1,2,\dots)$ and every constant $C \neq 0$, equation (2.2) has an odd self-similar solution of the form (2.1) with

$$u(x,0) = \lim_{t\downarrow 0} t^\alpha U(xt^{-\beta}) = C|x|^{\frac{\alpha}{\beta}-1}x = C|x|^{k-1}x. \tag{2.9}$$

The corresponding profile U has exactly j sign changes in $(0,\infty)$, $U(\eta) \sim C\eta^k$ as $\eta \to \infty$, and $(U^m)'(0)$ and C are related by a bijection. Finally, the *exponents* α_j^* *are between the exponents* α_j *in Theorem* 1; *i.e.,*

$$\alpha_j(m,1) < \alpha_j^*(m) < \alpha_{j+1}(m,1), \quad j = 1,2,\dots. \tag{2.10}$$

The first of these is the steady-state solution $u(x,t) = C|x|^{\frac{1}{m}-1}x$. These solutions can be used to describe the transition from $2j+1$ sign changes to one sign change.

The powers k in the above two theorems reduce to integers when $m = 1$ (see (1.4) for the even case). This can be deduced directly by solving the similarity ordinary differential equations and relates to the fact that the heat equation can be solved backwards with arbitrary polynomials as initial data by formally letting $\exp(t\Delta)$ act on them.

For fast diffusion, i.e., $0 < m < 1$, the corresponding results are the following:

Theorem 3. *Let* $0 < m < 1$ *and* $N \geq 1$. *There exists a sequence*

$$0 = \alpha_0(m,N) < \alpha_1(m,N) < \alpha_2(m,N) < \alpha_3(m,N) < \cdots < \frac{1}{1-m}$$

such that for every $\alpha = \alpha_j(m,N)$ $(j = 1,2,\dots)$ *and every constant* $C \neq 0$, *equation* (2.2) *has a radial self-similar solution of the form* (2.1) *with*

$$u(r,0) = \lim_{t\downarrow 0} t^{\alpha} U(rt^{-\beta}) = Cr^{\frac{\alpha}{\beta}} = Cr^k. \tag{2.11}$$

The corresponding profile U *has exactly* j *sign changes in* $[0,\infty)$, $U(\eta) \sim C\eta^k$ *as* $\eta \to \infty$, *and* $U(0)$ *and* C *are related by a bijection. For* $0 < m < 1$ *and* $N = 1$ *there exists also a sequence*

$$\frac{1}{m+1} = \alpha_0^*(m) < \alpha_1^*(m) < \alpha_2^*(m) < \alpha_3^*(m) < \cdots \uparrow \frac{1}{1-m},$$

such that for every $\alpha = \alpha_j^*(m,N)$ $(j = 1,2,\dots)$ *and every constant* $C \neq 0$, *equation* (2.2) *has an odd self-similar solution of the form* (2.1) *with*

$$u(r,0) = \lim_{t\downarrow 0} t^{\alpha} U(xt^{-\beta}) = C|x|^{\frac{\alpha}{\beta}-1}x = C|x|^{k-1}x. \tag{2.12}$$

The corresponding profile U *has exactly* j *sign changes in* $(0,\infty)$, $U(\eta) \sim C\eta^{\frac{\alpha}{\beta}}$ *as* $\eta \to \infty$, *and* $(U^m)'(0)$ *and* C *are related by a bijection. Again, the exponents* α_j^* *are between the exponents* α_j; *i.e.,* $\alpha_j(m,1) < \alpha_j^*(m) < \alpha_{j+1}(m,1)$.

The limit behaviour of the exponents α_j depends on m and N. We shall show in Section 3 that

$$0 < \frac{N-2}{N+2} \leq m < 1 \quad \Rightarrow \quad \alpha_j \uparrow \frac{1}{1-m},\ \beta_j \downarrow 0, \tag{2.13}$$

while

$$0 < m < \frac{N-2}{N+2} \quad \Rightarrow \quad \alpha_j < \alpha_{GS} < \frac{1}{1-m},\ \beta_j > \beta_{GS} > 0, \tag{2.14}$$

where $\alpha_{GS} = \alpha_{GS}(m, N)$ is the unique exponent for which there exists a positive ground state solution of the form (2.1) with finite, but nonconstant, mass; see [21] and also [23]. This finite-mass solution has algebraic decay with exponent $\frac{2-N}{m}$ and exists for all $0 < m < \frac{N-2}{N}$, the upper limit of which is the exponent at which the ZKB solution (1.7) loses its meaning. We note that the corresponding exponent β_{GS} has the same sign as $\frac{N-2}{N+2} - m$. There are no solutions of this type with sign changes, a point to which we return in Section 7. The β_j corresponding to α_j in Theorem 3 are all positive; in the case of (2.13) they tend to zero as $j \to \infty$.

We note that the allowable power function initial data for (2.2) in this section may of course also be used as initial data for (1.1). The corresponding (self-similar) forward radial solutions become immediately positive, whereas the odd ones in dimension $N = 1$ have only one zero in $x = 0$ for $t > 0$; such solutions describe the local behaviour immediately after the number of sign changes has decreased. These powers thus give global solutions in back- and forward time; cf. [22].

2.2. **Singular initial data.** In the range $0 < m < \frac{N-2}{N}$ we have also the possibility that sign changes disappear at infinity. We note that in this case there are no finite-mass solutions with sign changes, i.e., there is no sequence analogous to that of Theorem 5 in Section 4. This is partly to be expected since mass is not preserved for this range of m, so even for initial conditions with zero mass u will in general evolve to the similarity solution of [21] (see, however, Section 7 below). In some sense the sequence in the next theorem replaces the finite-mass sequence, providing a second means (in addition to interior extinction) for sign changes to disappear prior to complete extinction of the solution, namely by their moving out to infinity. The solutions in this sequence all have $\beta < 0$ and singular initial data with exponent $k = \frac{\alpha}{\beta} < -\frac{2}{1-m}$.

Theorem 4. *Let $N > 2$ and $0 < m < \frac{N-2}{N}$. There exists a sequence*

$$\frac{N-2}{N(1-m)-2} = \tilde{\alpha}_0(m, N) > \tilde{\alpha}_1(m, N) > \tilde{\alpha}_2(m, N) > \cdots > \frac{1}{1-m},$$

such that for every $\alpha = \tilde{\alpha}_j(m, N)$ $(j = 1, 2, \dots)$ and every constant $C \neq 0$, equation (2.2) *has a radial singular self-similar solution of the form* (2.1) *with*

$$u(r,0) = \lim_{t\downarrow 0} t^{\alpha} U(rt^{-\beta}) = Cr^{\frac{\alpha}{\beta}} = Cr^k,$$

and

$$\lim_{\eta\to 0} \eta^{-k} U(\eta) = C \text{ and } \lim_{\eta\to\infty} \eta^{\frac{N-2}{m}} U(\eta) = D \neq 0, \tag{2.15}$$

with $D = D(C)$ *depending on* C *by scaling. The corresponding profile* U *has exactly* j *sign changes in* $[0,\infty)$.

The first one ($j = 0$) is just the stationary solution $u = Cr^{\frac{2-N}{m}}$. Applying these results to the behaviour of solutions to (1.1) in which sign changes disappear at $r = \infty$, $t = 0$, the extinction time of the solution being greater than 0, we conjecture the following. As $t \uparrow 0$ with $r = O((-t)^{\beta})$, u tends to a similarity solution of the above type with $j \geq 1$, $j = 1$ being generic. As $t \uparrow 0$ with $r = O(1)$, we have $u \sim u(r,0)$, which depends on the initial data but which satisfies $u(r,0) \sim Cr^k$ as $r \to \infty$.

We have

$$\frac{N-2}{N+2} < m < \frac{N-2}{N} \quad \Rightarrow \quad \tilde{\alpha}_j > \alpha_{GS} > \frac{1}{1-m}, \tag{2.16}$$

where $\alpha_{GS} = \alpha_{GS}(m,N)$ is again the exponent of the finite-mass ground-state solution, while, as will be shown in Section 3,

$$0 < m \leq \frac{N-2}{N+2} \quad \Rightarrow \quad \tilde{\alpha}_j \downarrow \frac{1}{1-m}. \tag{2.17}$$

Theorem 4 is in fact equivalent to Theorem 3, as can be seen from a transformation in [20], which maps the ranges $0 < m < \frac{N-2}{N+2}$ and $\frac{N-2}{N+2} < m < 1$ into one another while interchanging the origin and infinity. The bounds in (2.14) and (2.16) again suggest the question as to whether the exponents converge to α_{GS}; see Section 8.

3. PROOFS

The proofs of the theorems in Section 2 are based on a scaling invariance which allows the equation for U to be reduced to a two-dimensional autonomous system. This invariance also gives rise to the free constant C in all the solutions we construct. Writing

$$\eta = rt^{-\beta}, \tag{3.1}$$

substitution of (2.1) in (2.2) gives

$$\eta^{1-N}(\eta^{N-1}|U(\eta)|^{m-1}U'(\eta))' + m\alpha U(\eta) = m\beta\eta U'(\eta), \tag{3.2}$$

which, restricting attention to $\eta > 0$ and $U > 0$ and setting

$$x = \frac{\eta U'}{U}, \quad y = \eta^2 U^{1-m}, \quad s = \log \eta, \tag{3.3}$$

is reduced to the quadratic system to

$$\dot{x} = x(2 - N - mx) - y(\alpha - \beta x), \quad \dot{y} = y(2 - (m-1)x). \tag{3.4}$$

Here dots denote differentiation with respect to s. We restrict our attention to the upper half plane $H = \{y \geq 0\}$. A typical phase plane is shown in Figure 2.

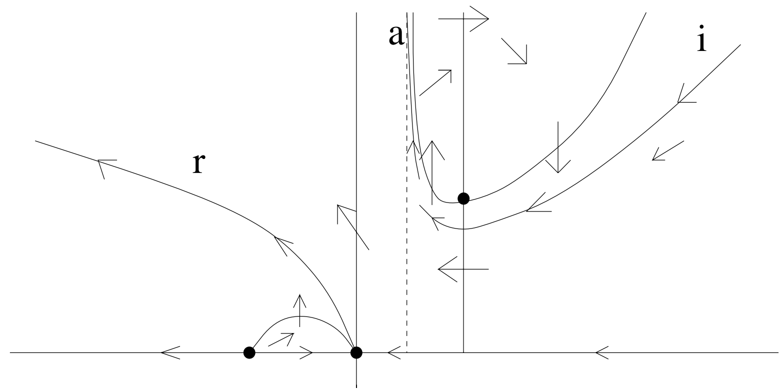

Figure 2. The phase plane for $\alpha, \beta > 0$, $m > 1$, $N > 2$, with orbits labelled by index (see below), together with nullclines. $\alpha, \beta > 0$, $m > 1$, $N > 2$.

Solutions with η and/or U negative first have to be reflected in $\eta = 0$ and/or $U = 0$. For the case of the forward porous-medium equation, this system has been studied in detail in [12], the only difference from (3.2) being that the α and β terms have minus signs. We shall restrict the analysis to the case $\alpha > 0$. The remaining cases are not relevant here and are left to the interested reader.

3.1. **Local analysis for the slow-diffusion case $m > 1$.** The local analysis is much the same as in [12], so we shall be rather sketchy here. The critical points are

$$P_0 = (0,0), \quad P_1 = (\frac{2-N}{m}, 0), \quad P_2 = (\frac{2}{m-1}, 2(\frac{2m}{m-1} + N - 2)),$$

with linearizations

$$A_1 = \begin{bmatrix} 2-N & -\alpha \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} N-2 & -\frac{(Nm-N+2)\alpha+N-2}{2m} \\ 0 & \frac{Nm-N+2}{m} \end{bmatrix},$$

and

$$A_3 = \begin{bmatrix} (Nm - N + 2 - \frac{2m}{m-1})\alpha & \frac{1}{m-1} \\ -2(Nm - N + 2) & 0 \end{bmatrix}.$$

Radial solutions of (3.2) with $U(0) > 0$ correspond to the single orbit γ_r (subscript r for radial) coming out of the critical point P_0 into the upper half plane H along the eigenvector with eigenvalue $\lambda = 2$.

For $N = 1$ (or, more generally, $0 < N < 2$; the restriction to integers is not essential in this paper), the point P_1 is a saddle, and the orbit γ_o (o for odd) coming out into H contains odd solutions U (i.e., those with $U(0) = 0$ and $(U^m)'(0) > 0$). In this case the point P_0 is an unstable node, the other orbits coming out being the solutions with $U(0) > 0$ and $U'(0) \neq 0$. These nonsymmetric solutions do not play a role in the analysis.

For $N > 2$, P_0 is a saddle and P_1 an unstable node, where one finds singular (at $\eta = 0$, with exponent $\frac{2-N}{m}$) solutions of (3.2).

For $N = 2$ the (logarithmic) singular solutions are contained in the centre unstable manifold coming out of $P_0 = P_1$ into H along the negative x-axis.

We pause to remark that for $N > 2$ the orbits coming out of P_1 have a local expansion of the form $U(\eta)^m \sim a + b\eta^{2-N}$. For $m > \frac{N-2}{2}$ there is a fast orbit γ^* coming out, corresponding to solutions with $a = 0$. Every argument in the global analysis of the sections below concerning values of α for which γ_r has some exceptional behaviour can also be applied to γ^*. Thus, we also get, as in Theorem 1, a sequence of exponents with sign-changing solutions having the expansion above with $a = 0$ near $\eta = 0$ and with algebraic growth (exponent k) for $\eta \to \infty$; see also Section 5.

The point P_2 was not discussed in [12]. Its linear structure depends on the parameters. The determinant of its linearization equals $\frac{2}{m-1}(mN - N + 2) > 0$ so it is either a node or a spiral point. It is stable for $\alpha = 0$ and as α increases it undergoes a Hopf bifurcation when $\alpha = \alpha_H = \frac{2m}{(mN-N+2)(m-1)}$, with an unstable periodic orbit appearing for α decreasing from this value, while for α larger than this value P_2 is unstable. These facts follow directly from results in [4]. Orbits coming out of this point give rise to solutions U with algebraic behaviour near $\eta = 0$, the exponent being $\frac{2}{m-1}$. Orbits going in represent solutions with algebraic growth with the same exponent. This does not lead to a proper limit of (2.1) as $t \downarrow 0$.

In addition to the three finite critical points, there are four critical points at infinity in H, as the introduction of polar coordinates,

$$x = \frac{r}{1-r}\cos\phi, \quad y = \frac{r}{1-r}\sin\phi, \tag{3.5}$$

or a Poincaré transformation reveals; see Figure 3. These are given, in terms of (3.5), by

$$r = 1 \text{ and } \phi = 0, \operatorname{arccotan} \beta, \frac{\pi}{2}, \pi. \tag{3.6}$$

The points with $\phi = 0$ and $\phi = \pi$ are, respectively, an unstable node and a stable node, corresponding to solutions $U(\eta)$ changing sign for some nonzero η-value, with $(U^m)' \neq 0$. More precisely, before the sign change the solution corresponds to an orbit escaping to infinity with $\phi \to \pi$, whereas after the sign change it corresponds to a solution coming out of infinity with $\phi \to 0$. Thus, tracing solutions with sign changes we look for “extended” orbits going to infinity along $\phi = \pi$ and coming back out of infinity along $\phi = 0$.

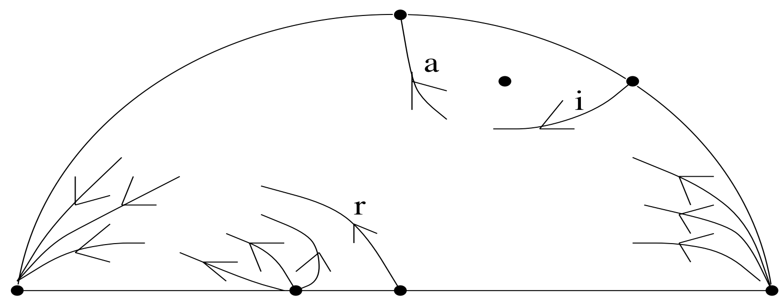

Figure 3. Compactified phase plane for $0 < \alpha < \frac{1}{m-1}$.

The point with $\phi = \operatorname{arccotan} \beta$ is a saddle with one orbit γ_i (i for interface) coming out into H, representing solutions whose support is bounded from below. At the free boundary (interface) which, again using the scaling, we may locate at any $\eta = \eta_0 > 0$, they satisfy the “interface condition”

$$(U^{m-1})'(\eta_0^+) = (m-1)\beta\eta_0. \tag{3.7}$$

Finally the point with $\phi = \frac{\pi}{2}$ is a saddle node, with the nodal part on the “other side of infinity” ($r > 1$). One orbit γ_a (subscript a for algebraic) of (3.4) escapes to infinity with $\phi \to \frac{\pi}{2}$, asymptotic to $x = \frac{\alpha}{\beta} = k$, and it contains solutions with (slow) algebraic growth

$$U(\eta) \sim C\eta^k \text{ as } \eta \to \infty. \tag{3.8}$$

This is the growth needed for the existence of a limit as $t \downarrow 0$ of (2.1).

3.2. **Global analysis for the slow-diffusion case $m > 1$.** In this subsection we prove Theorems 1 and 2. The solutions we seek correspond to values of the parameters for which the orbit γ_r (Theorem 1) and γ_o (Theorem 2) coincide with γ_a. Here we mean orbits in an extended sense: as in [12], we shall see the orbit representing a U-solution before a sign change and the orbit representing it after the sign change as part of one and the same extended orbit.

From [4] we already know that for $\alpha \geq \alpha_H = \frac{2m}{(mN-N+2)(m-1)}$ the orbit γ_a actually comes out of the source P_2. This gives the solutions satisfying (2.6) with an explicit constant, namely $((m-1)/(2(2+(m-1)N)))^{\frac{1}{m-1}}$, near $\eta = 0$. There are no periodic orbits in this range.

As α drops below the Hopf bifurcation value α_H, P_2 becomes stable, and γ_a must have a periodic orbit as its α-limit set as long as $\alpha > \alpha_{AG}(m, N)$ (the Aronson–Graveleau exponent). Thus we obtain again solutions $U(\eta)$ with algebraic growth (exponent $k = \frac{\alpha}{\beta}$) near infinity and a zero at $\eta = 0$, where the solution oscillates between two power functions with exponent $\frac{2}{m-1}$. Numerics in [4] suggest that this periodic orbit is unique and coincides with the periodic orbit born in the Hopf bifurcation, though this is not as obvious as might at first be expected.

For $\alpha = \alpha_{AG}(m, N)$, the orbit γ_a coincides with γ_i, giving the focussing solution. We note we cannot have such an intersection in the extended sense (i.e., with sign changes). Indeed, if γ_i escapes to infinity with $\phi \to \pi$, it blocks all orbits coming out of infinity with $\phi \to 0$ from escaping to infinity with $\phi \to \frac{\pi}{2}$.

Our analysis concerns the range between $\alpha = 0$ and $\alpha = \alpha_{AG}$. To show that the connections we need for the proof of Theorem 1 exist, we observe for $\alpha = 0$ that $\gamma_r = \gamma_a$ is the positive y-axis (the corresponding solution is simply $U(\eta) \equiv C$). On the other hand, we have for $\alpha = \alpha_{AG}(m, N)$ that $\gamma_i = \gamma_a$, which forces (by inspection of the phase plane) the orbit γ_r to escape to infinity with $\phi \to \pi$ (sign change) and, after emerging again from infinity on the other side ($\phi \to 0$), the argument may be repeated. Thus the extended orbit γ_r has infinitely many passages through infinity, corresponding to solutions with infinitely many sign changes, with $f(s)$ (defined by (3.10) below) tending to a periodic solution as $s \to \infty$.

Using continuity arguments, as in [12], the information listed above is in itself sufficient to establish, varying α between 0 and $\alpha_{AG}(m, N)$, the existence of α-values for which, after a prescribed number j of passages through infinity, γ_r coincides with γ_a. The uniqueness of these (bifurcation) values follows from a monotonicity property (see also [4]). As we vary the parameters the arrows in the phase diagram all turn in the same direction, since

$$\frac{dx}{ds}\frac{d}{d\alpha}\frac{dy}{ds} - \frac{dy}{ds}\frac{d}{d\alpha}\frac{dx}{ds} = \frac{y^2}{2}(2 + (1-m)x)^2 \geq 0. \tag{3.9}$$

Note that β depends on α in this calculation; see (2.1).

However, as we consider extended orbits going through infinity, the meaning of (3.9) is not immediately clear. Therefore we use another reduction of (3.2). If we set

$$U(\eta) = \eta^{\frac{2}{m-1}} f(\log \eta)^{\frac{1}{m}} \text{ , i.e., } u(r,t) = (\frac{r^2}{t})^{\frac{1}{m-1}} f(\log r - \beta \log t)^{\frac{1}{m}}, \quad (3.10)$$

we obtain for f the equation

$$(\frac{2m}{m-1} + \frac{d}{ds})(\frac{2m}{m-1} + N - 2 + \frac{d}{ds})f(s) = (\frac{m}{m-1} + \beta m \frac{d}{ds})f(s)^{\frac{1}{m}}, \quad (3.11)$$

which may be written as the system

$$\frac{dg}{ds} = -\frac{2m}{m-1} g - (m\alpha + \beta(N-2) f^{\frac{1}{m}} + \beta |f|^{\frac{1-m}{m}} g, \quad (3.12)$$

$$\frac{df}{ds} = (2 - N - \frac{2m}{m-1}) f + g. \quad (3.13)$$

Here we have used the notation that $f^{\frac{1}{m}} = |f|^{\frac{1}{m}-1} f$. In polar coordinates

$$g = (\frac{\rho}{1-\rho})^{\frac{m}{m-1}} \cos\theta, \quad f = (\frac{\rho}{1-\rho})^{\frac{m}{m-1}} \sin\theta; \quad (3.14)$$

this gives

$$\frac{m}{m-1} \rho \frac{d\rho}{ds} = \rho^2 (1-\rho)(\sin\theta(\cos\theta - (N-2)\sin\theta) - \frac{2m}{m-1})$$
$$+\rho(1-\rho)^2 \cos\theta |\sin\theta|^{\frac{1-m}{m}} (\beta\cos\theta - (m\alpha + \beta(N-2))\sin\theta), \quad (3.15)$$
$$\rho \frac{d\theta}{ds} = \rho\cos\theta(\cos\theta - (N-2)\sin\theta)$$
$$-(1-\rho)\sin^{\frac{1}{m}}\theta(\beta\cos\theta - (m\alpha + \beta(N-2))\sin\theta). \quad (3.16)$$

In terms of (3.14) the orbit γ_r now comes out of $(\rho,\theta) = (1, \text{cotan}(N-2))$, γ_a goes into the saddle $(\rho,\theta) = (0, \text{cotan}(N-2+mk))$ and γ_i comes out of $(\rho,\theta) = (0,0)$. Sign changes correspond to intersections with the negative or positive g-axis, which we identify if we consider θ modulo π.

The system (3.12)–(3.13) again has the monotonicity property that the arrows in the phase plane all turn in the same direction as the parameter α is varied. This follows from the fact that

$$\frac{df}{ds}\frac{d}{d\alpha}\frac{dg}{ds} - \frac{dg}{ds}\frac{d}{d\alpha}\frac{df}{ds} = \frac{m-1}{2}|f|^{\frac{1-m}{m}}(\frac{df}{ds})^2. \quad (3.17)$$

Combining (3.17) with the analysis above, the proof of Theorem 1 is now standard and therefore further details are omitted.

The proof of Theorem 2 is basically the same. Instead of the orbit γ_r one has to consider (with $N = 1$) the orbit γ_o, which for $\alpha = \frac{1}{m+1}$ is just the line $x = \frac{1}{m}$. Varying α between $\frac{1}{m+1}$ and $\frac{1}{m-1}$, the analysis is similar to that above.

3.3. **Fast diffusion** $0 < m < 1$**.** In this section we prove Theorems 3 and 4. The critical point P_2 disappears (into the lower half plane) as m drops below 1, but it reappears again as m drops below $\frac{N-2}{N}$ (we assume $N > 2$ in this latter case). This happens with an exchange of stability between P_1 and P_2.

For $0 < m < \frac{N-2}{N}$ the point P_1 becomes a saddle, with one orbit γ_f (f for fast decay with finite mass) going in from H, representing solutions with fast decay, i.e., $U(\eta) \sim C\eta^{\frac{2-N}{m}}$ as $\eta \to \infty$, which implies finite mass near infinity.

The point P_2 is stable for all $0 < m < \frac{N-2}{N}$ if $\alpha \geq 0$, but there are cases (explained below) for which it is necessarily surrounded by at least one unstable periodic orbit. Orbits going into P_2 give solutions with slow decay, i.e., $U(\eta) \sim C\eta^{-\frac{2}{1-m}}$ with $C^{1-m} = 2(\frac{-2m}{1-m} + N - 2)$ as $\eta \to \infty$ (infinite mass near infinity).

We note that in terms of (3.11) the picture is rather different now. The radial orbit γ_r for instance now comes out of the origin, as can be seen directly from (3.10), and $U(\eta) \sim C \neq 0$ as $\eta \to 0$. Likewise, if α and β are both positive, we have that the orbit γ_a goes to infinity, since $U(\eta) \sim C\eta^k$ as $\eta \to \infty$.

The ratio $k = \frac{\alpha}{\beta}$ varies from 0 to $+\infty$, with β decreasing to 0, as α ranges from 0 to $\frac{1}{1-m}$. As α goes through $\frac{1}{1-m}$ and β drops below 0, there is an exchange of stability between, in terms of (3.5), the critical points $(r, \phi) = (1, \frac{\pi}{2})$ and $(r, \phi) = (1, \text{arccotan}\beta)$. The second becomes an unstable node, while the first becomes a saddle node with a unique orbit coming out of it. We again call this orbit γ_a, but now it contains solutions with $U(\eta) \sim C\eta^k$ as $\eta \to 0$, with $k = \frac{\alpha}{\beta} < -\frac{2}{1-m}$. Thus in terms of (3.12)–(3.13) the orbit γ_a comes out of infinity. The corresponding similarity solutions have $u(r, 0) = Cr^k$. The large η-behaviour of these solutions will now be discussed, it being necessary to divide the analysis into a number of subcases.

3.3.1. $\max(0, \frac{N-2}{N}) \leq m < 1$. Let $\alpha \in (0, \frac{1}{1-m})$ so $\beta > 0$. In terms of (3.5) the critical point $(r, \phi) = (1, \arccos\beta)$ is now a stable node and orbits going into it contain solutions U with a vertical asymptote.

In terms of (3.11) we have a nonlinear oscillator with a negative friction term ($\beta > 0$). This implies that, in terms of (3.11), there are no periodic solutions (in terms of (3.4) there are also no periodic solutions because there are no critical points in the interior of H).

For $\beta = 0$ the negative friction term is linear, while the forcing term is superlinear. Consequently, it is easy to see that all solutions remain oscillatory as $s \to \infty$, or, in terms of (3.4), all orbits make infinitely many passages through infinity and, in terms of f, they oscillate and become arbitrarily large. Instead of using the Aronson–Graveleau focussing solution to force any number of sign changes to occur for γ_r as α is being increased from 0 to $\frac{1}{1-m}$, one thus now has to use this $\beta = 0$ limit case.

Since the monotonicity property (cf. (3.17)) is valid, the existence of the solutions in Theorem 3 follows along the same lines as before, the solutions corresponding to α-values for which γ_r has j passages ($j = 0, 1, 2 \dots$) through infinity before coinciding with γ_a.

In addition, we also conclude that $\alpha_j \to \frac{1}{1-m}$ and $\beta_j \to 0$ as $j \to \infty$. To see this suppose that $\alpha_j \to \alpha_\infty < \frac{1}{1-m}$. Then, in view of monotonicity, for every $\alpha \in (\alpha_\infty, \frac{1}{1-m})$ the extended orbit γ_r has infinitely many passages through infinity. In terms of (3.12)–(3.13) we then have that γ_r, which now comes out of the origin $(g, f) = (0, 0)$ along the positive g-axis, is going around the origin spiralling outwards (because of the negative friction). On the other hand, γ_a goes to infinity in forward s-time but goes spiralling inward as s goes to $-\infty$, while being bounded away from the origin by the outward-spiralling γ_r. Using the Poincaré–Bendixson argument, it follows that the ω-limit set of γ_r and the α-limit set of γ_a are both periodic orbits. But such orbits cannot exist in view of the observation above that the friction is negative. This completes the proof of Theorem 3 and of (2.13) in this range.

Finally, we observe that for every $\alpha > \frac{1}{1-m}$ the orbit γ_a has infinitely many passages through infinity. Thus the corresponding self-similar profiles have infinitely many sign changes for large η and in terms of f they approach a periodic profile. This is because (3.11) is now a Liénard equation [14] with one stable globally attracting periodic orbit. This gives backward self-similar solutions with singular initial data $u(r, 0) = Cr^k$ and sign changes coming in from $r = \infty$.

3.3.2. $\frac{N-2}{N+2} < m < \frac{N-2}{N}$. For $\alpha \in (0, \frac{1}{1-m})$ the friction term in (3.11) is still negative. The reappearance of P_2 and the exchange of stability between P_1 and P_2 have no effect on the proof of Theorem 3. Observe that if we ignore the first-order friction terms in (3.11), we are left with a second-order equation with two stable rest points corresponding to P_2, with one unstable rest point with higher energy, namely the origin, in between. Since γ_r comes out of the origin, it will, because of the negative friction, stay away from the other two rest points. Tracing the orbit γ_r as α runs from 0 to $\frac{1}{1-m}$, the proof of Theorem 3 follows as before. The argument which gives $\alpha_j \to \frac{1}{1-m}$ is also not affected because, as γ_r is going around the origin spiralling outwards in the (g, f)-plane, the critical points corresponding to P_2 are on the inside of γ_r for the reason given above.

To prove Theorem 4 let $\alpha > \frac{1}{1-m}$ so $\beta < 0$. Thus the friction term in (3.11) becomes positive for large f. For $k = \frac{\alpha}{\beta} = \frac{2-N}{m}$, i.e., $\alpha = \tilde{\alpha}_0 = \frac{N-2}{N(1-m)-2}$, we have that $\gamma_a = \gamma_f$ is the line $x = k = \frac{\alpha}{\beta}$. The finite-mass ground-state solution corresponds to $\gamma_r = \gamma_f$, which happens for a unique exponent $\alpha_{GS} \in (\frac{1}{1-m}, \alpha_0)$; see [21] and [23]. For $\frac{1}{1-m} \leq \alpha < \alpha_{GS}$, the orbits γ_r and γ_a escape to infinity with $\phi \to \pi$ and, after coming back out of infinity, this behaviour repeats itself. The corresponding backward similarity solutions have singular initial data $u(r, 0) = Cr^k$ with $k = \frac{\alpha}{\beta} < \frac{\alpha_{GS}}{\beta_{GS}} < 0$ (γ_a) and $u(r, 0) = 0$ (γ_r).

Now decreasing α from $\tilde{\alpha}_0$ to α_{GS} we pick up a decreasing sequence of exponents $\tilde{\alpha}_j$ for which γ_a and γ_f coincide after j sign changes of the corresponding profile U. Thus Theorem 4 follows in this range but there may be a gap between the limit of $\tilde{\alpha}_j$ and α_{GS}.

Finally, for $\alpha > \tilde{\alpha}_0$ we have that γ_a converges to P_2 or possibly a periodic orbit around P_2, without passages through infinity. This gives positive singular solutions with initial data $u(r, 0) = Cr^k$ with $k = \frac{\alpha}{\beta} < -\frac{N-2}{m}$. A similar statement holds for γ_r if $\alpha \geq \tilde{\alpha}_0$, again giving solutions with zero initial trace. All these solutions have $U(\eta) \sim C\eta^{-\frac{N-2}{m}}$ as $\eta \to \infty$.

3.3.3. $m = \frac{N-2}{N+2}$. The difference from the previous case is that $\alpha_{GS} = \frac{1}{1-m}$, $\beta_{GS} = 0$ and the corresponding ground state solution is separable. The proofs of both Theorem 3 and Theorem 4 remain the same. However, the range of α between $\frac{1}{1-m}$ and α_{GS} disappears. In (3.11) there is only one friction term, its sign being the same as that of β. This also allows one to conclude that the $\tilde{\alpha}_j$ converge to $\frac{1}{1-m}$ because periodic f-solutions are impossible.

3.3.4. $0 < m < \frac{N-2}{N+2}$. Let $\alpha < \frac{1}{1-m}$ so $\beta > 0$. Here α_{GS} has dropped below $\frac{1}{1-m}$ (so $\beta_{GS} < 0$). We now pick up the exponents in Theorem 3 by varying α between 0 and α_{GS}. For $\alpha = \alpha_{GS}$ we have $\gamma_r = \gamma_f$, forcing γ_a to have an infinite number of passages through infinity as we trace it backwards. For $\alpha = 0$, on the other hand, we have $\gamma_a = \gamma_r$, and γ_f comes out of a periodic orbit around P_2 without passages through infinity. Varying α between 0 and $\alpha = \alpha_{GS}$ then completes the proof of Theorem 3 in this case.

For $\alpha \in (\alpha_{GS}, \frac{1}{1-m})$ the orbit γ_a has infinitely many passages through infinity as $s \to -\infty$, giving a singular profile with infinitely many sign changes near $\eta = 0$ and a similarity solution with singular initial data Cr^k.

Finally, for Theorem 4 we decrease α from $\tilde{\alpha}_0$ down to $\alpha = \frac{1}{1-m}$ whence $\beta < 0$ and both friction terms in (3.11) are negative. For $\alpha = \frac{1}{1-m}$, we have $\beta = 0$ so the extended orbit γ_f has infinitely many passages through infinity as $s \to -\infty$, while $\gamma_f = \gamma_a$ is the line $x = k = \frac{\alpha}{\beta}$ for $\alpha = \tilde{\alpha}_0$ in Theorem 4. Decreasing α down to $\frac{1}{1-m}$ we pick up the exponents $\tilde{\alpha}_j$ in Theorem 4. Again the fact that there are no periodic solutions of (3.11), together with a monotonicity property (cf. (3.17)), ensures that the exponents converge to $\frac{1}{1-m}$.

Finally for $\alpha > \tilde{\alpha}_0$ the orbit γ_a goes, without a passage through infinity, to the point P_2 or to a periodic orbit around it. The corresponding singular similarity solutions have no sign changes and singular initial data Cr^k.

4. The forward equation

It was shown in [12] that in the case of slow diffusion ($m > 1$), equation (1.1) has a sequence of compactly supported solutions of the form

$$u(r,t) = t^{-\alpha} U(rt^{-\beta}), \quad r = |x|, \quad 2\beta + (m-1)\alpha = 1, \tag{4.1}$$

characterised by their number of sign changes. In fact these are the only solutions of this form which are integrable over the whole space. The first (radial) one is the ZKB instantaneous point-source solution [5], whereas the first odd one ($N = 1$) is the dipole solution [5].

For the fast diffusion case the existence of such a sequence of "eigenfunctions" was established in [6] in dimension $N = 1$, integrability now corresponding to fast algebraic decay (exponent $\frac{2}{1-m}$). The proof, given in terms of the appropriate version of (3.11), is easily extended to higher dimensions, under the natural restriction

$$1 - m < \frac{2}{N}. \tag{4.2}$$

Let us state both cases via one single theorem.

Theorem 5. *Let $N \geq 1$ and $m > \max(0, \frac{N-2}{N})$. There exists a sequence*

$$\frac{N}{2 + N(m-1)} = \alpha_0(m,N) < \alpha_1(m,N) < \alpha_2(m,N) < \alpha_3(m,N) < \cdots$$

(*not to be confused with the sequences in Theorems* 1–3)*, such that for every* $\alpha = \alpha_j(m, N)$ $(j = 1, 2, \dots)$ *and every* $C \neq 0$*, equation* (1.1) *has a one-parameter family of radial self-similar integrable solutions of the form* (4.1)*. These solutions have compact support if* $m > 1$*, exponential decay for* $m = 1$ *and*

$$\lim_{\eta\to\infty} \eta^{\frac{2}{1-m}} U(\eta) = \left(\frac{1-m}{2(2+(m-1)N)}\right)^{\frac{1}{m-1}} \quad \textit{for } m < 1. \qquad (4.3)$$

The corresponding profiles U *have exactly* j *sign changes in* $[0, \infty)$*, and each family is parametrised by* $U(0)$*.*

The corresponding exponents $k_j = \frac{\alpha_j}{\beta_j}$ again increase with j from $k_0 = N$ to ∞ if $m \geq 1$, and it was conjectured in [6] that $k_j \to \frac{2}{1-m}$ for $m < 1$. Our next theorem implies that this is not in fact the case.

Theorem 6. *Let* $N \geq 1$ *and* $\max(0, \frac{N-2}{N}) < m < 1$*. There exists a unique exponent* $\alpha = \alpha_{TS}(m, N)$*, such that equation* (1.1) *has a one-parameter family of radial self-similar tail switch solutions of the form* (4.1) *with*

$$-\lim_{\eta\to\infty} \eta^{\frac{2}{1-m}} U(\eta) = \lim_{\eta\to 0} \eta^{\frac{2}{1-m}} U(\eta) = \left(\frac{1-m}{2(2+(m-1)N)}\right)^{\frac{1}{m-1}}. \qquad (4.4)$$

The corresponding profile U *has exactly one sign change in* $[0, \infty)$*.*

The corresponding $k_{TS} = \frac{\alpha_{TS}}{\beta_{TS}}$ is an upper bound for the exponents k_j from Theorem 5, and again there is the question whether this bound is really a supremum (see Section 8).

The proof of Theorem 5 is omitted here; see [12] and [6]. We do observe however that, just as in the backward case, we have for $m > \frac{N-2}{2}$ and $N > 2$ that singular solutions with $U(\eta)^m = a + b\eta^{2-N}$ are represented by orbits of (3.4) (with the signs of α and β changed) coming out of the critical point $P_1 = (\frac{2-N}{m}, 0)$ and that $a = 0$ corresponds to a fast orbit γ^* coming out of P_1. In this range one also has a family of exponents α_j^* for which this solution $U(\eta)$ with $a = 0$ has j sign changes and finite mass (compactly supported for $m > 1$, decay with exponent $\frac{2}{1-m}$ for $m < 1$).

To prove Theorem 6 we use the appropriate version of (3.11),

$$\left(\frac{2m}{m-1} + \frac{d}{ds}\right)\left(\frac{2m}{m-1} + N - 2 + \frac{d}{ds}\right) f(s) = -\left(\frac{m}{m-1} + \beta m \frac{d}{ds}\right) f(s)^{\frac{1}{m}}. \quad (4.5)$$

We again use the notation $f^{\frac{1}{m}} = |f|^{\frac{1-m}{m}} f$. After a simple rescaling of f to put β in the denominator, (4.5) may be written as

$$f'' + (N - 2\frac{m+1}{1-m} + |f|^{\frac{1-m}{m}})f' + \frac{2m}{1-m}(\frac{2}{1-m} - N)f = \frac{m}{\beta(1-m)} f^{\frac{1}{m}}. \quad (4.6)$$

The solution we are looking for corresponds (in terms of the corresponding phase plane) to a saddle-point connection between the two nonzero equilibria $P_\pm$ of (4.6), which for $\beta = \infty$ is a Liénard equation with a unique (stable globally attracting) periodic orbit; see e.g. [14]. For β sufficiently large, this stable orbit persists, but is no longer globally attracting. The two equilibria above originate from infinity as saddles in the Liénard phase plane. A straightforward perturbation argument then yields that the orbit γ_s (s for singular) coming out of this saddle going towards the origin converges to this stable periodic orbit. On the other hand we know that for $\beta = \beta_0$ from Theorem 5 the orbit γ_r connects the origin to this saddle point, forcing the orbit γ_s to have only one sign change and to escape to infinity. Combining this with the usual monoticity property (cf. (3.17)), Theorem 6 then follows. A transformation from [20] maps between pairs of solutions from Theorem 6 with different values of N.

We note that $\beta_{TS} \to 1$ (and hence $\alpha_{TS} \to \infty$) as $m \uparrow 1$; cf. $g(r/t)$ in (1.5). This can be seen directly from a limiting argument applied to another scaling of (4.5), which reads

$$(1 - \frac{d}{ds})(1 + \frac{(1-m)(2-N)}{2m} - \frac{d}{ds})f(s) = (1 - 2\beta m \frac{d}{ds}) f(s)^{\frac{1}{m}}. \quad (4.7)$$

The tail-switch solutions correspond to a connection between the critical points $\pm(1 + \frac{1-m}{m}\frac{2-N}{2})^{\frac{m}{1-m}}$. In the limit $m \uparrow 1$ we find a connection between the critical points $\pm e^{\frac{2-N}{2}}$. This implies that in the limit $\beta = 1$. The details of this argument are left to the reader.

5. Summary

Let us summarize our knowledge of similarity solutions relevant for the local and global theory of (1.1).

We have three families of solutions which are characterised by their number of sign changes. First of all there is a sequence of forward radial solutions with finite mass ([12] and Theorem 5). These exist in the so-called ZKB range; i.e., $m > \max(0, \frac{N-2}{N})$. They have compact support for $m > 1$, exponential decay for $m = 1$ and algebraic decay (exponent $\frac{2}{1-m}$) for $m < 1$. These solutions are relevant for the behaviour of general solutions as $t \to \infty$.

Then there is, for every $m > 0$, a sequence of backward radial solutions with powers as initial data (Theorems 1 and 3), related to local behaviour as zeros disappear, without the large r-behaviour being felt.

Both sequences are joined in dimension $N = 1$ by a sequence of odd solutions of the same type and for $m > \frac{N-2}{2} > 0$ by a sequence of solutions with $U(\eta)^m \sim a + b\eta^{2-N}$ near $\eta = 0$ where $a = 0$. These are both related to a conservation law for what may be called the first radial moment; see [19]. Since

$$\frac{\partial}{\partial t}(ru) = \frac{\partial}{\partial r}\left(r|u|^{m-1}\frac{\partial u}{\partial r} + \frac{N-2}{m}|u|^{m-1}u \right),$$

we have that the contribution near the origin in

$$\frac{d}{dt}\int_0^\infty ru\,dr = [r|u|^{m-1}u_r + \frac{N-2}{m}|u|^{m-1}u]_0^\infty$$

is zero because $a = 0$, which can be regarded as imposing zero flux of first moment at $r = 0$. The first one of these singular solutions is given by (see [18] and [13])

$$\alpha = \frac{1}{m}, \quad \beta = \frac{1}{2m}, \quad U(\eta) = \eta^{\frac{2-N}{m}}(C - \frac{m-1}{2+(m-1)N}\eta^{N+\frac{2-N}{m}})^{\frac{1}{m-1}}.$$

Finally, outside the ZKB range, i.e., for $0 < m < \frac{N-2}{N}$, we have a sequence of backward radial solutions singular near zero (exponent $\frac{\alpha}{\beta}$) and finite mass near infinity (algebraic decay with exponent $\frac{2-N}{m}$). These describe how sign changes disappear at infinity before the solution extinguishes completely.

These families are supplemented by singleton solutions which have no sign changes, namely the backward finite-mass ground-state solution in the range $0 < m < \frac{N-2}{N}$, relevant again for the description of solutions which extinguish in finite time, and the focussing (Aronson–Graveleau) solutions. Finally there is in the range $\frac{N-2}{N} < m < 1$ a tail-switch solution which changes sign once, is singular near the origin and algebraic near infinity (in both cases with exponent $\frac{2}{1-m}$), and is relevant for the far-field behaviour as $t \to \infty$.

6. Non-self-similar asymptotics: slow diffusion

This section will give formal asymptotic results for the case in which the reduction in the number of sign changes takes place at a moving front, with $m > 1$. Since this concerns the behaviour close to such a front, the required analysis is locally one-dimensional even for $N > 1$. In the absence of an appropriate similarity solution, we base our approach on the conjecture that the generic intermediate asymptotic behaviour is in some sense close to the travelling wave solution

$$u = \pm \left((m-1)q\left(-(x-qt)\right)_+\right)^{\frac{1}{m-1}}, \tag{6.1}$$

where q is a constant. The self-consistency of the assumptions implicit in what follows is readily checked a posteriori. We shall treat the case in which the leading front starts to move before $t = 0$. Cases in which it exhibits waiting-time behaviour right up to $t = 0$ (when it is overrun, with a consequent decrease in the number of sign changes) have to be treated separately. The details are less interesting (the behaviour being similar to that of non-negative solutions at the end of a waiting-time) and are omitted.

We again take the time and location at which the number of sign changes decreases to be $t = 0,\ x = 0$. In view of (6.1), we start our asymptotic analysis by writing u in the form

$$u = (-t)^{\frac{1}{m-1}} f(\eta,\tau), \quad \eta = x/(-t), \quad \tau = -\log(-t); \tag{6.2}$$

the solution will be shown to vary slowly with τ in the limit $\tau \to +\infty$. We thus have

$$\frac{\partial f}{\partial \tau} - \frac{1}{m-1} f + \eta \frac{\partial f}{\partial \eta} = \frac{\partial}{\partial \eta}\left(|f|^{m-1}\frac{\partial f}{\partial \eta}\right). \tag{6.3}$$

We take the leading front to be located at $\eta = -a(\tau)$ and the sign change to be at $\eta = -b(\tau)$ (see Figure 4).

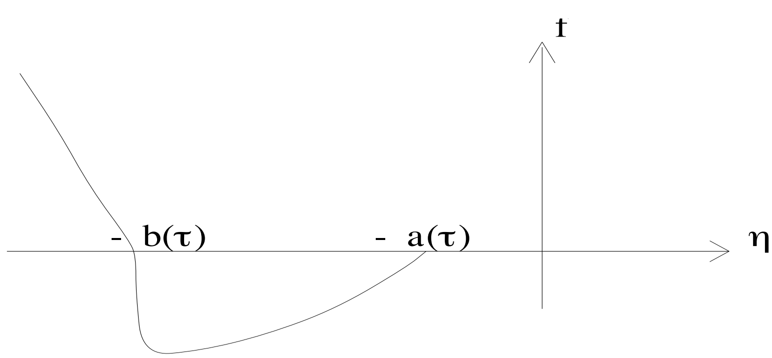

Figure 4. Sign change near the front.

We shall construct expansions valid in $-b(\tau) < \eta < -a(\tau)$ and in $\eta < -b(\tau)$ and then match these into an interior layer located about $\eta = -b(\tau)$, thereby obtaining expressions for $a(\tau)$ and $b(\tau)$.

6.1. $-b(\tau) < \eta < -a(\tau)$**.** Here we have $f < 0$ and we write $f = -v^{1/(m-1)}$ to give

$$\frac{\partial v}{\partial \tau} - v + \eta \frac{\partial v}{\partial \eta} = v \frac{\partial^2 v}{\partial \eta^2} + \frac{1}{m-1} \left(\frac{\partial v}{\partial \eta} \right)^2. \tag{6.4}$$

We now introduce the expansion (cf. (6.1))

$$v \sim -(m-1)\; a(\tau)\,(\eta + a(\tau)) + V_1, \tag{6.5}$$

to give

$$-(m-1)\; \dot{a}(\eta + 2a) - V_1 + \eta \frac{\partial V_1}{\partial \eta} = -(m-1)\; a(\eta + a) \frac{\partial^2 V_1}{\partial \eta^2} - 2a \frac{\partial V_1}{\partial \eta}, \tag{6.6}$$

where $\dot{a}$ denotes $da/d\tau$. Applying the method of reduction of order by writing

$$V_1 = (\eta + 2a) W_1 \tag{6.7}$$

yields

$$\frac{\partial W_1}{\partial \eta} = a^{\frac{m}{m-1}}\; \dot{a} \frac{e^{-\frac{\eta}{(m-1)a}} e^{-\frac{1}{m-1}}}{(\eta + 2a)^2\, (|\eta + a|)^{\frac{1}{m-1}}} \int_0^{-1-\eta/a} (\sigma - 1)^2\; \sigma^{\frac{1}{m-1} - 1}\; e^{-\frac{\sigma}{m-1}}\; d\sigma, \tag{6.8}$$

the constant of integration being fixed by imposing the required behaviour at the front $\eta = -a$, which is that $V_1 = \frac{\partial V_1}{\partial \eta} = 0$.

Since it turns out that $a << b$, we shall require the behaviour of V_1 as $\eta \to -\infty$ in order to perform the matching. We find from (6.8) that

$$V_1 \sim (m-1)\; a^{\frac{2m-1}{m-1}}\; \dot{a} \frac{e^{-\frac{\eta}{(m-1)a}}}{(-\eta)^{\frac{m}{m-1}}}\; e^{-\frac{1}{m}}\; I_m \text{ as } \eta \to -\infty, \tag{6.9}$$

where the constant I_m is defined by

$$I_m = \int_0^\infty (\sigma - 1)^2\; \sigma^{\frac{1}{m-1} - 1}\; e^{-\frac{\sigma}{m-1}}\; d\sigma. \tag{6.10}$$

We can now be more precise about the ranges of validity of the expansion (6.5). Two scalings must be considered. For $\eta = O(a)$ we have from (6.5) that the leading term is $O(a^2)$, whereas the correction term V_1 is $O(a\dot{a})$; consistency thus requires $\dot{a}/a \to 0$ as $\tau \to +\infty$, as is to be expected. For $\eta = O(b)$ with $-1 < \eta/b < 0$, the leading-order term is $O(ab)$ while (6.9) is exponentially growing in $-\eta$; the expansion is nevertheless valid for $\eta/b > -1$ because $\dot{a}$ will turn out to be exponentially small. The expansion will break down as $\eta/b \to (-1)^+$, in which limit it should match into the interior layer around $\eta = -b$. This will occur when V_1 becomes of $O(ab)$, and it thus follows from (6.9) that to leading order we require

$$b^{\frac{2m-1}{m-1}} = -a^{\frac{m}{m-1}}\ \dot{a}\ e^{\frac{b}{(m-1)a}}, \tag{6.11}$$

which provides the first of our equations relating a and b; we note that $\dot{a} < 0$. Note also that we have without loss of generality chosen the constant in (6.11) equal to -1; any other choice of a negative constant will give essentially the same results.

6.2. $\eta < -b(\tau)$**.** We write $f = v^{1/(m-1)}$ to recover (6.4) and expand in the form

$$v \sim -(m-1)\ b(\tau)(\eta + b(\tau)) + V_2. \tag{6.12}$$

Taking $V_2 = (\eta + 2b)W_2$ gives

$$\frac{\partial W_2}{\partial \eta} = -b^{\frac{m}{m-1}}\ \dot{b}\frac{e^{-\frac{\eta}{(m-1)b}}\ e^{-\frac{1}{m-1}}}{(\eta+2b)^2\,(|\eta+b|)^{\frac{1}{m-1}}}\ \int_{-1-\eta/b}^{\infty} (\sigma-1)^2\ \sigma^{\frac{1}{m-1}-1}\ e^{-\frac{\sigma}{m-1}}\ d\sigma, \tag{6.13}$$

the constant of integration being fixed by requiring that V_2 does not grow exponentially as $\eta \to -\infty$. From (6.13) it follows that

$$\frac{\partial V_2}{\partial \eta} \sim -b^{\frac{1}{m-1}}\ \dot{b}\ I_m\,(|\eta+b|)^{-\frac{1}{m-1}} \qquad \text{as } \eta \to (-b)^-. \tag{6.14}$$

The expansion (6.12) applies for $\eta = O(b)$ with $\eta/b < -1$, the leading term being $O(b^2)$ and the correction term V_1 being $O(b\dot{b})$; consistency thus requires that $\dot{b}/b \to 0$ as $\tau \to +\infty$.

6.3. **Interior layer.** The appropriate scalings here are

$$\eta = -b + az, \qquad f = (ab)^{\frac{1}{m-1}}\ F, \tag{6.15}$$

leading to the dominant balance

$$\frac{\partial}{\partial z}(F^{m-1}\frac{\partial F}{\partial z}) \sim -\frac{\partial F}{\partial z}. \tag{6.16}$$

Writing $F \sim F_0$ as $\tau \to +\infty$, matching with (6.5) requires (comparing the leading-order terms)

$$F_0 \sim -(m-1)^{\frac{1}{m-1}} \qquad \text{as } z \to +\infty, \tag{6.17}$$

while matching with (6.12) requires, again comparing the leading-order terms,

$$F_0 \sim \Big((m-1)(-z)\Big)^{\frac{1}{m-1}} \qquad \text{as } z \to -\infty. \tag{6.18}$$

Using (6.16) and (6.17) yields

$$-\left(F_0 + (m-1)^{\frac{1}{m-1}}\right) = |F_0|^{m-1}\,\frac{\partial F_0}{\partial z}, \tag{6.19}$$

and (6.18) is automatically satisfied. From (6.15) and (6.19) we obtain, comparing the derivatives of the inner and the outer expansion up to the correction term, the matching condition

$$\frac{\partial V_2}{\partial \eta} \sim -(m-1)ba^{\frac{1}{m-1}}\left(|\eta + b|\right)^{-\frac{1}{m-1}} \quad \text{as } \eta \to -b,$$

so comparison with (6.14) gives

$$b^{\frac{1}{m-1}} - 1\;\dot{b}\;I_m = (m-1)a^{\frac{1}{m-1}}, \tag{6.20}$$

which is our second equation relating a and b.

The above expansions are each derived in the limit $\tau \to +\infty$, and to complete the analysis we must solve (6.11) and (6.20) in this limit. It is easily seen that as $\tau \to +\infty$ we have

$$\begin{aligned} a &\sim a_0, \quad \dot{a} \sim -a_0\,(\tau/I_m)^{2m-1}\exp\left(-(\tau/I_m)^{m-1}\Big/(m-1)\right) \\ b &\sim a_0\,(\tau/I_m)^{m-1} \end{aligned} \tag{6.21}$$

for some constant a_0 which depends on the initial data and cannot be determined from the asymptotics[1].

6.4. **The limit profile.** The final region to be discussed in the limit $\tau \to +\infty$ is $\zeta = O(\tau)$ where $\zeta = \log(-\eta)$ and the dominant balance in (6.3) is

$$\frac{\partial f}{\partial \tau} \sim \frac{1}{m-1}f - \frac{\partial f}{\partial \zeta}$$

so that

$$f \sim e^{\frac{1}{m-1}\zeta}\;\Phi\;(\tau - \zeta). \tag{6.22}$$

[1]It can be checked by back-substitution that the above formal asymptotic analysis is self-consistent for $1 < m < 2$. However, for larger m the $V_{1\tau}$ term neglected in (6.6) is not in fact negligible (we are grateful to Don Aronson for pointing this out) and the analysis becomes significantly more complicated, in the sense that two additional scalings for η would need to be considered in Section 6.1 with many of the details of the analysis changing (the other sections are unchanged). We omit these details, however, because the modifications they induce in the final result are insignificant - b remains as in (6.21) and, while the expression for $\dot{a}$ is modified somewhat, it is still exponentially small for $m \geq 2$ with the same dominant exponential.

Matching with (6.12) as $-\eta \to \infty$ and using (6.21) gives

$$\Phi(\xi) = \Big((m-1)a_0\Big)^{\frac{1}{m-1}} \; \xi \Big/ I_m.$$

As is to be expected, in the original variables (6.22) corresponds to a solution which depends only on x, and we thus find that

$$u \sim \frac{1}{I_m}\Big((m-1)a_0\Big)^{\frac{1}{m-1}} (-x)^{\frac{1}{m-1}} |\log(-x)|. \tag{6.23}$$

This gives the behaviour of $u(x,0)$ in the limit $x \to 0^-$.

6.5. **Behaviour as $t \to 0^+$.** We conclude this section by noting the behaviour near the front immediately after the decrease in number of sign changes occurs. Writing

$$u = t^{\frac{1}{m-1}} \; f(\eta, \tau), \quad \eta = x/t, \quad \tau = \log t \tag{6.24}$$

in place of (6.2), and using (6.23) as the initial condition, it is not hard to show that

$$f \sim \Big((m-1)\; a_0(-\tau/I_m)^{m-1} \; \big(a_0(-\tau/I_m)^{m-1} - \eta\big)_+\Big)^{\frac{1}{m-1}}$$

as $\tau \to -\infty$ with $\eta = O\left((-\tau)^{m-1}\right)$.

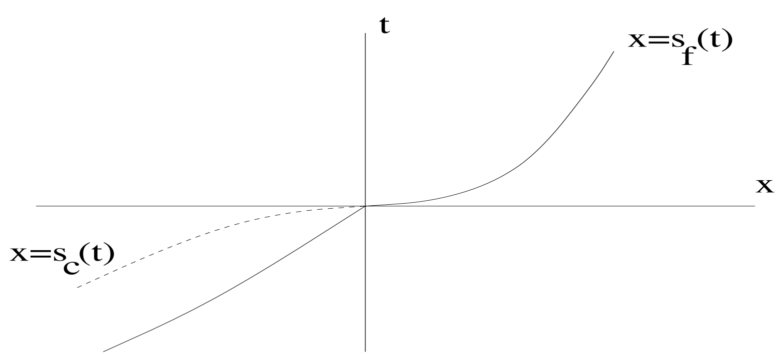

Figure 5. Schematic behaviour of leading front $s_f(t)$ and sign change $s_c(t)$ as the latter disappears.

We can now summarise the above by describing the behaviour of the leading front $x = s_f(t)$ as

$$\begin{aligned} s_f(t) &\sim a_0 t && \text{as } t \to 0^-, \\ s_f(t) &\sim a_0 \Big(|\log t| \Big/ I_m\Big)^{m-1} t && \text{as } t \to 0^+, \end{aligned} \tag{6.25}$$

so that $s_f(t)$ approaches $x = 0$ from below at finite speed before moving off at an unbounded rate. The point of sign change $x = s_c(t)$ satisfies

$$s_c(t) \sim -a_0 \left(|\log(-t)| \Big/ I_m \right)^{m-1} (-t) \quad \text{as } t \to 0^-,$$

and the situation is shown schematically in Figure 5.

The analysis of this section has concerned the (generic) case in which a single sign change disappears at a front. Nongeneric cases in which several sign changes disappear at a given time can similarly be constructed using three or more copies of a "slowly-varying" version of (6.1). Moreover, using multiple copies of the Aronson–Graveleau solution instead, one can generalise the analysis to $N > 1$ to describe the (nongeneric) phenomenon of the simultaneous disappearance at $r = 0$ of the front and of one or more sign changes.

7. Non-self-similar asymptotics: fast diffusion

We are concerned in this section with the extinction behaviour for $0 < m < \frac{N-2}{N}$ in the nongeneric scenario in which the solution contains one or more sign changes at the instant of extinction. The solutions in Theorem 4 can be used as the outer expansion for $t \to 0^-$ of solutions with j sign changes disappearing at infinity at time $t = 0$, assuming that the solution does not extinguish completely at $t = 0$. In this way, or via interior extinction, solutions may lose all their sign changes before they extinguish completely (which they do as the self-similar ground-state solution). There is an interesting borderline case consisting of solutions which have, say, a negative region near $r = 0$ and a positive region near $r = \infty$ for all times smaller than the extinction time. We next give a description of the non-self-similar behaviour of such solutions near their extinction time, which we again assume to be $t = 0$. In this analysis the self-similar ground-state solution still plays a crucial role.

We write

$$u = (-t)^\alpha f(\frac{r}{(-t)^\beta}, \tau), \quad \tau = -\log(-t),$$

where $\alpha = \alpha_{GS}$ and $\beta = \beta_{GS}$ are the exponents of the ground-state solution, see Section 2. For f we thus have

$$\frac{\partial f}{\partial \tau} - \alpha f + \beta \eta \frac{\partial f}{\partial \eta} = \eta^{1-N} \frac{\partial}{\partial \eta}(\eta^{N-1}|f|^{m-1}\frac{\partial f}{\partial \eta}). \tag{7.1}$$

The ground state is a steady-state solution of (7.1) which we denote by $F(\eta)$. It is determined completely by

$$F(0) = 1, \quad F'(0) = 0, \quad F(\eta) \sim A\eta^{-\frac{N-2}{m}} \text{ as } \eta \to \infty,$$

where A is a uniquely determined constant. We first scale this solution and use it as the lowest-order term in an inner and an outer expansion.

7.1. **Inner region.** The inner region of the expansion, in which $f < 0$ will be taken to hold, will be given by $\eta = O(\phi(\tau))$ with f expanded as

$$f \sim -\phi^{-\frac{2}{1-m}}(F(\frac{\eta}{\phi}) + \frac{\dot{\phi}}{\phi}G(\frac{\eta}{\phi})),$$

where $\phi(\tau)$ remains to be determined subject to the condition that $\frac{\dot{\phi}}{\phi} \to 0$ as $\tau \to \infty$.

For $G(\xi)$, where $\xi = \frac{\eta}{\phi}$, we find the inhomogeneous linear equation

$$\xi^{1-N}\frac{d}{d\xi}(\xi^{N-1}\frac{d}{d\xi}(F^{m-1}G)) - \beta\xi\frac{dG}{d\xi} + \alpha G + \xi\frac{dF}{d\xi} + \frac{2}{1-m}F = 0, \quad (7.2)$$

with $\frac{dG}{d\xi} = 0$ at $\xi = 0$. The general solution to this is given by the sum of a particular solution $G(\xi)$ and an arbitrary multiple of $\frac{2}{1-m}F + \xi\frac{dF}{d\xi}$, which has decay rate $\xi^{-\frac{N-2}{m}}$. The (slower) decay rate of the particular solution as $\xi \to \infty$ is given by $B\xi^{-\frac{(N-2)(1-m)}{m}}$, where B is again a uniquely determined constant; all solutions thus have this decay rate.

7.2. **Outer region.** The outer region, with $f > 0$, is taken to be $\eta = O(\psi(\tau))$ with f expanded as

$$f \sim \psi^{-\frac{2}{1-m}}(F(\frac{\eta}{\psi}) + \frac{\dot{\psi}}{\psi}H(\frac{\eta}{\psi})),$$

and $\psi(\tau)$ remains to be determined subject to the conditions that $\frac{\phi}{\psi} \to 0$ and $\frac{\dot{\psi}}{\psi} \to 0$ as $\tau \to \infty$.

For $H(\zeta)$, where $\zeta = \frac{\eta}{\psi}$, we find the same inhomogeneous linear equation

$$\zeta^{1-N}\frac{d}{d\zeta}(\zeta^{N-1}\frac{d}{d\zeta}(F^{m-1}H)) - \beta\zeta\frac{dH}{d\zeta} + \alpha H + \zeta\frac{dF}{d\zeta} + \frac{2}{1-m}F = 0, \quad (7.3)$$

where $H(\zeta)$ is required to have decay rate $\zeta^{-\frac{N-2}{m}}$ as $\zeta \to \infty$. The general solution is then given by the sum of a particular solution, which has $H(\zeta) \sim -C\zeta^{-(N-2)}$ as $\zeta \to 0$, and an arbitrary multiple of $\frac{2}{1-m}F + \zeta\frac{dF}{d\zeta}$, which is bounded as $\zeta \to 0$. Thus all solutions with decay rate $\zeta^{-\frac{N-2}{m}}$ behave the same as $\zeta \to 0$, the constant C being uniquely determined.

7.3. **Transition layer.** In the transition layer, whose scalings we determine below, the leading-order balance is

$$\frac{\partial}{\partial\eta}(\eta^{N-1}|f|^{m-1}\frac{\partial f}{\partial\eta}) = 0. \tag{7.4}$$

The behaviour of the inner expansion for $\frac{\eta}{\phi} \to \infty$ is

$$(-f)^m \sim \phi^{-\frac{2m}{1-m}} A^m((\frac{\eta}{\phi})^{-(N-2)} + \frac{mB}{A}\frac{\dot{\phi}}{\phi}), \tag{7.5}$$

where we must have $\dot{\phi} < 0$, whereas for $\frac{\eta}{\psi} \to 0$ the outer expansion has

$$f^m \sim \psi^{-\frac{2m}{1-m}}(1 - mC\frac{\dot{\psi}}{\psi}(\frac{\eta}{\psi})^{-(N-2)}), \tag{7.6}$$

with $\dot{\psi} > 0$. In view of (7.4), expressions (7.5) and (7.6) each give precisely the leading-order solution in the transition layer on either side of the zero of f, and hence we require that for $\tau \to \infty$ the behaviour of ϕ and ψ be described by

$$mBA^{m-1}\phi^{-\frac{2m}{1-m}-1}\dot{\phi} = -\psi^{-\frac{2m}{1-m}}, \quad mC\psi^{N-3-\frac{2m}{1-m}}\dot{\psi} = A^m\phi^{N-2-\frac{2m}{1-m}}, \tag{7.7}$$

the transition-layer scaling being $\eta^{N-2} = O(\phi^{N-1}/\dot{\phi}) = O(\psi^{N-3}/\dot{\psi})$. In the analysis of (7.7) we have to distinguish between three cases.

7.3.1. $m > \frac{N-2}{N+2}$. Here as $\tau \to \infty$ we have

$$\phi \sim K, \quad \psi^{N-\frac{2}{1-m}} \sim \frac{(N(1-m)-2)A^m K^{N-\frac{2}{1-m}}\tau}{(1-m)mC},$$

where K is a constant which cannot be determined from the asymptotics. The inner solution is thus precisely the ground-state solution, while the outer solution expands logarithmically faster in $-t$.

7.3.2. $m < \frac{N-2}{N+2}$. Now as $\tau \to \infty$,

$$\psi \sim K, \quad \phi^{-\frac{2m}{1-m}} \sim -\frac{2K^{-\frac{2m}{1-m}}\tau}{(1-m)BA^{m-1}},$$

so in this case the outer solution is the ground-state solution, while the inner one contracts logarithmically faster in $-t$.

7.3.3. $m = \frac{N-2}{N+2}$. Here the ground-state solution is separable; the analysis is more delicate and requires the result that for $m = \frac{N-2}{N+2}$

$$BA^{\frac{N-6}{N+2}} = C, \tag{7.8}$$

which follows from applying a transformation (see [20]) which maps the inner region into the outer and vice versa. This transformation also maps the inner and outer regions in Section 7.3.2 to, respectively, the outer and inner regions in Section 7.3.1. In the special case when $m = \frac{N-2}{N+2}$, it exchanges the inner and the outer region and, moreover, $F(\eta)$ is explicitly available, with $A = (4N)^{\frac{N+2}{2}}$. Combining (7.7) and (7.8) we find that as $\tau \to \infty$

$$\psi \sim \frac{K}{\phi}, \quad \phi^{-(N-2)} \sim \frac{(N+2)A^{\frac{4}{N+2}}\tau}{BK^{\frac{N-2}{2}}}.$$

In this case the inner region contracts and the outer expands logarithmically in $-t$.

We note that in each of the three cases the asymptotic behaviour contains a single degree of freedom, K, as is to be expected. Such arbitrary constants arise in all the cases discussed here; their values depend on the initial conditions.

The analysis of Sections 6–7 indicates general approaches which are applicable in the many contexts in which the asymptotic behaviour is non-self-similar but is based on multiple copies of a self-similar solution. The key step in such analyses is the matching together of two copies via a transition layer (see Sections 6.3 and 7.3), thereby determining how such copies interact (as described by (7.7), for example). That non-self-similar families can be constructed in this way relates to the fact that for α close to the exceptional value (just less than α_{AG} or α_{GS}; see Section 8) the trajectory with the required behaviour at infinity spirals into a limit cycle, crossing zero an infinite number of times; our non-self-similar approach enables asymptotic solutions to be constructed which initially follow similar trajectories but contain only a finite number of zeros. The significance of this is that any attempt to pursue a similar analysis using multiple copies of a member of a self-similar family (e.g. from Theorem 5), rather than of a singleton, is doomed to failure; for such families the number of zeros increases by one each time an eigenvalue is crossed rather than abruptly increasing to infinity on crossing the exceptional exponent. If such an attempt is made, it is found that no suitable transition-layer balance is available.

8. Singleton solutions and gaps

In our analysis we have found three singleton similarity solutions of the second kind, each of which has an exponent bounding a sequence of exponents of self-similar solutions with increasing numbers of sign changes. For each of these bounds it is natural to ask whether it is optimal or whether there is a gap between the limit of the sign change exponents and the singleton exponent. In this section we briefly discuss the results of a numerical study of this behaviour. A more extensive report of this study may be found in [7]; we should add, however, that a detailed survey varying m and N has not been undertaken so the conclusions regarding the existence or otherwise of gaps are somewhat tentative.

On the basis of what follows we conjecture that gaps between the sign-change exponents and the singleton exponent occur in the backward cases only. It is noteworthy that the case with no gap corresponds to a forward similarity solution (Theorem 6) which does not form the basis of a non-self-similar family (see Section 9), whereas all those with gaps are backward and do form the building blocks for such families (see Sections 6–7). It is interesting to speculate whether these facts are related; the phase plane analysis seems to provide no insight into such matters.

8.1. **The tail-switch solution.** This solution, see Theorem 6, appears in the range $\frac{N-2}{N} < m < 1$. Its exponents $\alpha_{TS} > 0$ and $\beta_{TS} > \frac{1}{2}$ ensure that there is a solution of (4.6) connecting the two nonzero equilibria P_+ and P_-. In fact, because of symmetry, there are two connections, one connecting P_- to P_+, the other P_+ to P_-. In the phase plane for (4.6), these two connections between the saddle points P_+ and P_- bound a region containing the zero equilibrium, out of which the orbit γ_r representing the profiles with $U(0) \neq 0$ and $U'(0) = 0$ emerges. In the range $0 < \alpha < \alpha_{TS}$ lie α_j for which γ_r connects to P_+ or P_-, yielding the solutions of Theorem 5. The numerical results suggest that in this case $\alpha_j \uparrow \alpha_{TS}$ as $j \to \infty$. In Figure 6 we plot solutions of (4.5), tracing backward the solution which goes into P_-. The connection between P_+ and P_- must occur for α between the two values in Figure 6. The fact that for α just below α_{TS} the solution comes out of the origin and not out of a periodic orbit makes it unlikely that there is a gap, and we conjecture that α_{TS} provides a sharp upper bound for the exponents in Theorem 5 for $m < 1$.

We note that one of the two orbits coming out the saddle P_+ becomes unbounded without having any sign changes; the same holds for P_-. A similar statement holds for one of the two orbits going into P_+ (and P_-). This excludes oscillatory behaviour of "large" solutions. For $0 < \alpha < \alpha_{TS}$ the numerically observed convergence of α_j to α_{TS} excludes oscillatory behaviour of "small" solutions. On the other hand, if $\alpha > \alpha_{TS}$, the connections have broken open giving two orbits emerging from P_+ and P_- spiralling inward to a stable (from the outside) limit cycle. Likewise orbits coming out of the origin spiral outward to a stable (from the inside) limit cycle (in particular, it is noteworthy that for every $\alpha > \alpha_{TS}$ there is a radial solution of zero mass which is oscillatory near infinity, containing an infinite number of zeros). Again assuming that $\alpha_j \uparrow \alpha_{TS}$, these two limit cycles (which we conjecture in fact coincide) must emerge out of the two connections as we increase α from α_{TS}.

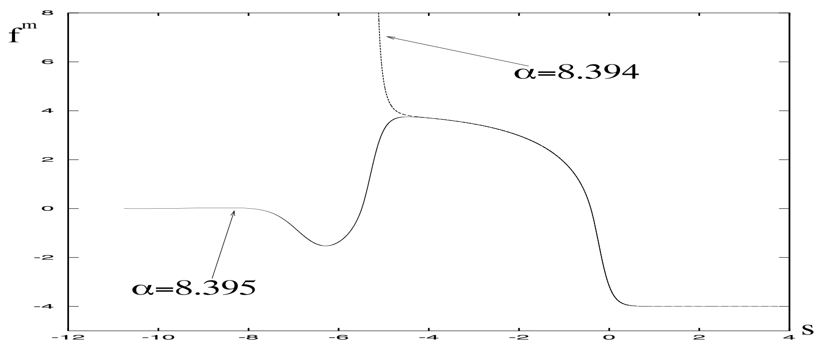

Figure 6. Shooting backwards from the negative equilibrium, $N = 1$, $m = \frac{1}{2}$.

8.2. **The Aronson–Graveleau solution.** This solution appears in the context of Theorem 1 when $m > 1$ (slow diffusion); see Sections 3.1 and 3.2. Here we find numerically that the exponents α_j of Theorem 1 increase to a limiting exponent α_∞ strictly smaller than the Aronson–Graveleau exponent α_{AG}.

In Figure 7 we have plotted the number of sign changes of radial solutions represented by γ_r as we increase α. In all three cases we clearly see a gap between α_∞ and α_{AG}.

It follows that for each $\alpha_\infty < \alpha < \alpha_{AG}$ there must be, in terms of (3.11), at least two periodic orbits with sign changes, one being the ω-limit set of γ_r, the other the α-limit set of γ_a. The latter represents similarity solutions satisfying (2.3) with infinitely many sign changes disappearing at $r = 0$ as $t \to 0$. We conjecture that these two orbits are unique, that the unstable one emerges from the Aronson–Graveleau connection as α drops below α_{AG} and that they merge into a single semistable periodic orbit for $\alpha = \alpha_\infty$.

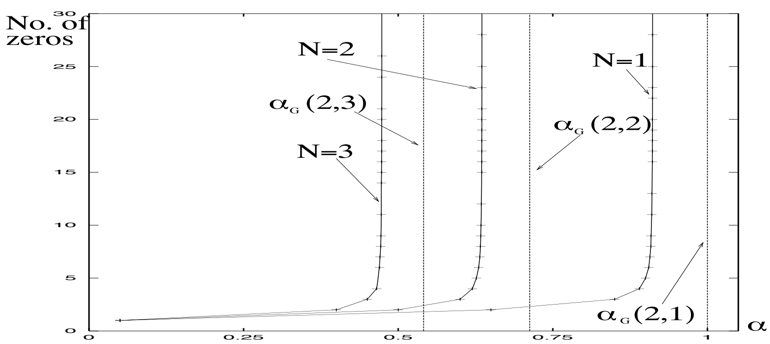

Figure 7. Number of sign changes of the radial solution represented by γ_r for $m = 2$. The values of α_{AG} are taken from [4].

In Figure 8 we plot the solutions of (3.11) corresponding to odd (Theorem 2) similarity profiles when $N = 1$ and $m = 2$. In this case $\alpha_{AG} = 1$, and we see that a stable periodic orbit already exists for $\alpha = 0.95$ but not yet for $\alpha = 0.9$. Thus $0.9 < \alpha_\infty < 0.95$.

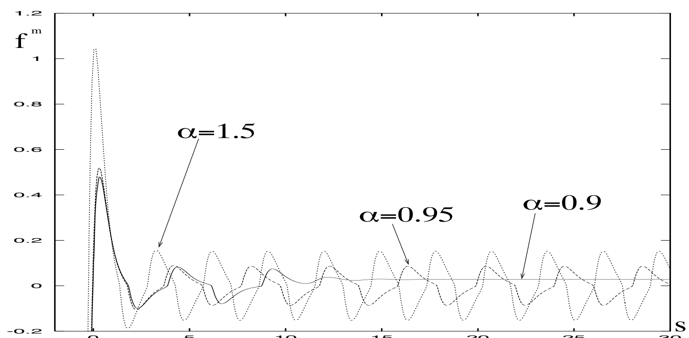

Figure 8. $N = 1$, $m = 2$; graphs of f when U is odd.

8.3. **The ground-state solution.** The exponent of this solution is related to the exponents in Theorem 3 if $0 < m < \frac{N-2}{N+2}$ and in Theorem 4 if $\frac{N-2}{N+2} < m < \frac{N-2}{N}$. The phase planes in these two cases are equivalent because of the transformation in [20].

8.3.1. $0 < m < \frac{N-2}{N+2}$. Here the ground state has $\alpha_{GS} < \frac{1}{1-m}$ and $\beta_{GS} > 0$. As in the Aronson–Graveleau case above, we find numerically that the α_j of Theorem 3 increase to a limit α_∞ strictly smaller than α_{GS}. The discussion is therefore similar to that in Section 8.2. In particular, we have for each $\alpha_\infty < \alpha < \alpha_{GS}$ that, in terms of (3.11), the orbit γ_r converges to a periodic orbit with sign changes, the difference being that $U(\eta)$ now oscillates near infinity and is (in view of (3.10)) of order $O(\eta^{-\frac{2}{1-m}})$ and thus not integrable. The corresponding similarity solution has $u(r,0) = 0$ for all $r \geq 0$.

For $\alpha_{GS} < \alpha < \frac{1}{1-m}$ we have that the orbit γ_f has no sign changes and comes out of a periodic orbit (in terms of (3.4)). It represents finite-mass solutions with a singularity at $\eta = 0$ ($u(r,0) = 0$ for $r > 0$).

In Figure 9 we plot different f-profiles corresponding to different γ_r. We see a periodic orbit with sign changes for $\alpha = 0.466 < \alpha_{GS}$ and a periodic orbit without sign changes for $\alpha = 0.468 > \alpha_{GS}$.

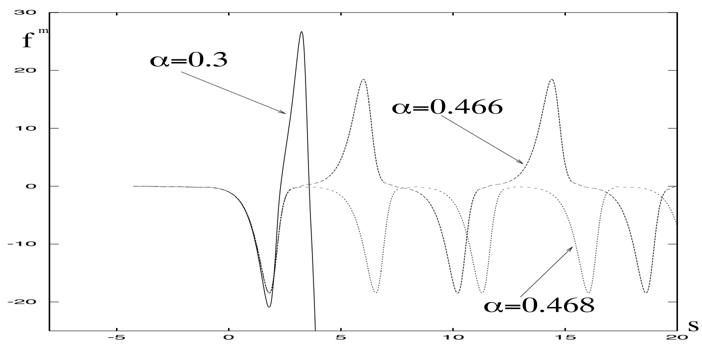

Figure 9. $N = 10$, $m = \frac{1}{2}$; graphs of f for orbit γ_r.

8.3.2. $\frac{N-2}{N+2} < m < \frac{N-2}{N}$. Here the ground state has $\alpha_{GS} > \frac{1}{1-m}$ and $\beta_{GS} < 0$. We have that the $\tilde{\alpha}_j$ of Theorem 4 decrease to a limit $\tilde{\alpha}_\infty$ strictly larger than α_{GS}. For each $\alpha_{GS} < \alpha < \tilde{\alpha}_\infty$, the orbit γ_a representing the singular solution ($U(\eta) \sim C\eta^k$ as $\eta \to 0$) converges to a periodic orbit of (3.11) with sign changes. The corresponding similarity solutions are singular at the origin, oscillating and of order $O(\eta^{-\frac{2}{1-m}})$ (not integrable) near infinity and have $u(r,0) = Cr^k$ for all $r \neq 0$. In this range the orbit γ_f converges backwards to a periodic orbit of (3.11) with sign changes. The corresponding solutions have fast decay $O(\eta^{\frac{N-2}{m}})$ near infinity, oscillate near $\eta = 0$ (of order $O(\eta^{-\frac{2}{1-m}})$, integrable) and have $u(r,0) = 0$ for all $r > 0$.

For $\frac{1}{1-m} < \alpha < \alpha_{GS}$ we have that γ_r and γ_a converge to a periodic orbit with sign changes (with U not integrable near infinity). The orbit γ_f has no sign changes and comes out of a periodic orbit (finite-mass solutions with a singularity, $u(r,0) = 0$ for $r > 0$).

9. Discussion

We conclude by noting some implications of, and conjectures arising from, the preceding analysis. The relevant self-similar solutions are given (for

$N > 2$) in the table below; all play a clear role in the intermediate asymptotic behaviour of broad classes of solutions.

	Family 1	Family 2	Singleton
$m > 1$	ZKB solutions [12], Thm 5	sign change loss at 0, Thm 1,2	AG [4]
$\frac{N-2}{N} < m < 1$	ZKB solutions [6], Thm 5	sign change loss at 0, Thm 3	TS, Thm 6
$0 < m < \frac{N-2}{N}$	sign change loss at ∞, Thm 4	sign change loss at 0, Thm 3	GS [21]

The solutions in Theorems 1–4 and the limit profile (6.23) characterize the optimal regularity the solutions of (1.1) may enjoy before their sign changes have disappeared.

For $m > 1$ it is known, see [17], that compactly supported solutions with nonzero integral lose all their sign changes in finite time; the way they do this is described by Theorems 1–2 and Section 6. For radial solutions containing a single sign change, for example, which scenario actually occurs can immediately be deduced from the initial conditions: in Figure 1 the left-hand side occurs if the mass is negative and the right-hand side if it is positive. For $\max(0, \frac{N-2}{N}) < m < 1$ solutions can maintain a sign change, as the tail-switch solution shows; indeed, the tail-switch solution can be used to provide barriers which show that radial solutions with a finite number of sign changes can sustain at most one as $t \to \infty$. Correspondingly, in an analysis similar to Section 7 using multiple copies of the tail-switch solution, the equations analogous to (7.7) imply that the two copies move together (leading to the disappearance of a zero) rather than apart; hence, unlike the Aronson–Graveleau solutions, the tail-switch solution does not form the basis for an asymptotic non-self-similar family. Solutions for $\alpha > \alpha_{TS}$ which spiral into the limit cycle (these are noted near the end of Section 8.1) can also be used as tail (outer) solutions, matching into a member of the family of Theorem 5 and containing an infinite number of zeros; such solutions separate those with negative tails from those with positive ones. It thus appears for this range that solutions of nonzero mass can contain as $t \to \infty$ no sign changes (mass and tail of same sign), one sign change (mass and tail of opposite sign) or an infinite number of zeros (oscillating tail); which scenario will in fact occur can be determined from the initial conditions. For $m = 1$, however, the expression (1.5) can contain an arbitrary number of zeros; the limit $m \uparrow 1$ is of some interest from the asymptotic point of view, the nonpersistence of zeros being governed by metastable dynamics.

When $0 < m < \frac{N-2}{N}$, finite-mass solutions generically become one-signed before extinction, but Section 7 describes a borderline scenario for solutions "between" those solutions that eventually become positive and those that eventually become negative. Further, even less generic, scenarios with any number of sign changes can be constructed in a similar way using multiple copies of the ground state solution. In the range $\max(0, \frac{N-2}{N}) < m < 1$ sign changes have an infinite amount of time in which to disappear internally; in the current range they do not, a fact to which the distinctions just alluded can be attributed.

Our analysis typifies the behaviour which can occur at values of u at which $D(u)$ in $u_t = \nabla \cdot (D(u)\nabla u)$ is nonanalytic, touching zero or infinity. It applies directly close to $u = 0$ when $D(u) \sim |u|^{m-1}$ but also with minor modifications when $D(u) \sim K_\pm |u|^{m-1}$ as $u \to 0^\pm$.

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