Advances in Differential Equations Volume 6, Number 9, September 2001, Pages 1049–1114

SINGULAR LIMIT OF A FOURTH-ORDER PROBLEM ARISING IN THE MICROPHASE SEPARATION OF DIBLOCK COPOLYMERS

M. Henry
Analyse Numérique et EDP Université de Paris-Sud, 91405 Orsay Cedex, France

(Submitted by: Michel Chipot)

Abstract. We study the limiting behavior as ε tends to zero of the solution of a system arising in the microphase separation of diblock copolymers. This system involves a fourth-order parabolic equation. We consider the case of spherical symmetry, and we show the convergence to a free-boundary Hele–Shaw-type problem.

1. Introduction

In this paper we consider a model for the microphase separation of diblock copolymers which has been proposed by Nishiura and Ohnishi [5]. Their starting point is the energy functional

$$E^\varepsilon(u^\varepsilon) = \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon}F(u^\varepsilon) + \frac{1}{2}|\nabla v^\varepsilon|^2\Big)dx \tag{1.1}$$

where $F(s) = \frac{(1-s^2)^2}{2}$ and v^ε is the unique solution of the elliptic problem

$$(Q) \quad \begin{cases} -\Delta v = u^\varepsilon - \fint_\Omega u^\varepsilon \, dx & \text{in } \Omega \\ \dfrac{\partial v}{\partial n} = 0 & \text{in } \partial\Omega \\ \quad \fint_\Omega v \, dx = 0, & \end{cases}$$

where $\Omega \subset R^N$ $(N \geq 2)$ is a smooth, bounded domain. In what follows we use the notation $v^\varepsilon = Q[u^\varepsilon]$. The functional (1.1) has been introduced by [1] and [6] in order to describe the microphase separation of diblock copolymers where two different homopolymers are connected; this connectivity is responsible for introducing the long-range interactions, which are expressed

Accepted for publication May 1999.
AMS Subject Classifications: 35B25, 35K30, 82C24, 82C26.

in the nonlocal term $\int_\Omega |\nabla v^\varepsilon|^2$, where $v^\varepsilon = Q[u^\varepsilon]$. Whereas the first part of the functional

$$E_1^\varepsilon(u^\varepsilon) = \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon}F(u^\varepsilon)\Big)$$

leads to minimization of the area of the interface between the regions where $u = 1$ and those where $u = -1$, where u denotes the limit of u^ε as ε tends to zero, the nonlocal part $E_2(u^\varepsilon) = \int_\Omega |\nabla v^\varepsilon|^2$ tends to create oscillations in the function u^ε, which in turn increase the area of the interface. We thus have to deal with two opposite tendencies. Nishiura and Ohnishi are interested in solutions of the time evolution equation

$$u_t^\varepsilon + \Delta\partial E^\varepsilon(u^\varepsilon) = 0 \ \text{ in } \Omega\times(0,T), \tag{1.2}$$

which satisfy the Neumann boundary condition

$$\frac{\partial u^\varepsilon}{\partial n} = 0 \ \text{ on } \partial\Omega\times(0,T) \tag{1.3}$$

and the conservation law

$$\int_\Omega u^\varepsilon(x,t)\,dx = \int_\Omega u_0^\varepsilon(x)\,dx \text{ for all t} \in (0,T); \tag{1.4}$$

the condition (1.4) implies in turn that

$$\frac{\partial \Delta u^\varepsilon}{\partial n} = 0 \ \text{ on } \partial\Omega\times(0,T). \tag{1.5}$$

This leads us to consider the fourth-order parabolic evolution problem

$$(P_1^\varepsilon)\begin{cases} u_t^\varepsilon + \Delta(\varepsilon\Delta u^\varepsilon + \frac{1}{\varepsilon}f(u^\varepsilon)) = -u^\varepsilon + \displaystyle{⨍_\Omega} u^\varepsilon dx & \text{in } \Omega\times(0,T) \quad (1.6)\\ \dfrac{\partial u^\varepsilon}{\partial n} = \dfrac{\partial\Delta u^\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega\times(0,T) \quad (1.7)\\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & \text{in } \Omega, \quad (1.8)\end{cases}$$

where $f(s) := -F'(s) = 2s(1-s^2)$. Throughout this paper we suppose that the initial value $u_0^\varepsilon \in H^2(\Omega)$ satisfies the hypothesis H_0^ε,

$$H_0^\varepsilon\begin{cases}\text{There exists a positive constant C such that } E^\varepsilon(u_0^\varepsilon) \le C;\\ ⨍_\Omega u_0^\varepsilon(x)\ dx \in (-1,+1).\end{cases}$$

As is done in the case of Cahn–Hilliard equation, it is useful to introduce the auxiliary function $w^\varepsilon = -(\varepsilon\Delta u^\varepsilon + \frac{1}{\varepsilon}f(u^\varepsilon) - v^\varepsilon)$, where $v^\varepsilon = Q[u^\varepsilon]$. When ε tends to zero, the solution u^ε converges to $u = \pm 1$ almost everywhere in $\Omega\times(0,T)$, so that the limiting problem as ε tends to zero is a free-boundary problem. Nishiura and Ohnishi [5] formally show that this problem has the form of a two-phase Hele–Shaw problem, namely,

$$(P_0)\begin{cases} \Delta w = 0 & \text{in } \Omega_t^{\pm},\ t \in (0,T) \\ V_n = [\dfrac{\partial w}{\partial n}]_{\Gamma_t} & \text{on } \Gamma_t,\ t \in (0,T), \\ K = 3/2(w - v) & \text{on } \Gamma_t,\ t \in (0,T) \\ \Gamma_{t|_{t=0}} = \Gamma_0, & \end{cases}$$

where $v = Q[u]$, $u(\cdot,t) = \pm 1$ on $\Omega_t^{\pm}$, $\Omega = \Omega_t^+ \cup \Omega_t^- \cup \Gamma_t$, K is the mean curvature of Γ_t taking the sign convention that convex hypersurfaces have positive mean curvature, V_n is the normal velocity of the interface taking the sign convention that the normal velocity of expanding hypersurfaces is positive.

The purpose of this paper is to give a rigorous derivation of the limit problem (P_0) in the case of spherical symmetry. This paper is organized as follows. In Section 2 we state some preliminary estimates and a first convergence result. In particular, we prove that u^ε tends to $u = \pm 1$ in $L^1(\Omega \times (0,T))$ and almost everywhere. From Section 3 on we assume that Ω is the unit ball in R^N and rewrite problem (P_1^ε) in the radial variable $r = |x|$. We prove in Section 3 a key estimate, which implies in particular that far away from the origin, the shape of the solution u^ε is close to that of the function $\pm\tanh(\frac{r}{\varepsilon})$. In Section 4 we give a definition of "jumps" of the limit function u and present some its properties. In particular, we prove that a jump is the limit of an odd number of zeros of u^ε. In Section 5 we establish by means of an error estimate that u^ε is close to the function $\pm\tanh(\frac{r}{\varepsilon})$ in the neighborhood of its zeros. In Section 6 we approximate u^ε by a first-order asymptotic expansion with respect to ε. We obtain in Section 7 the equation of the interface; more precisely, we prove the following result.

Theorem 1.1. *Assume that H_0^ε is satisfied.*

(i) *There exist a sequence $\{\varepsilon_n\}$ and functions u, w, v such that $u^{\varepsilon_n} \to u$ in $L^1(\Omega \times (0,T))$ almost everywhere, where $u = \pm 1$ almost everywhere in $\Omega \times (0,T)$, $w^{\varepsilon_n} \rightharpoonup w$ in $L^2(0,T,H^1(\Omega))$, and $v^{\varepsilon_n} \rightharpoonup v$ in $L^2(0,T,H^1(\Omega))$, as ε_n tends to 0. The functions u, w and v are such that*

$$\int_0^T \int_\Omega (u\varphi_t - \nabla w \nabla \varphi)(x,t)\,dx\,dt + \int_\Omega u_0(x)\varphi(x,0)\,dx = 0,$$

for all $\varphi \in C^1(\overline{\Omega} \times [0,T])$ such that $\varphi(T) = 0$ and

$$\begin{cases} -\Delta v = u - \fint_\Omega u & \textit{a.e. in } \Omega \times (0,T) \\ \fint_\Omega v\,dx = 0 & \textit{for a.e. } t \in (0,T) \\ \dfrac{\partial v}{\partial n} = 0 & \textit{a.e. on } \partial\Omega \times (0,T). \end{cases}$$

(ii) *Suppose that* Ω *is the unit ball in* R^N *and that* u_0^ε *is radially symmetric so that* u^ε *is also radially symmetric. Moreover, for almost every* $t \in [0,T]$ *if* $u(\cdot,t)$ *has a jump point between* -1 *and* 1 *at the point* $\overline{r}(t) > 0$*, then* $\overline{r}(t)$ *satisfies*

$$-(N-1)\frac{1}{\overline{r}(t)} = \frac{3}{2}\nu(\overline{r}(t))(w-v)(\overline{r}(t),t),$$

where

$$\nu(\overline{r}(t)) = \begin{cases} 1 & \textit{if } u \textit{ jumps from } -1 \textit{ to } 1 \textit{ across } \overline{r}(t), \\ -1 & \textit{if } u \textit{ jumps from } 1 \textit{ to } -1 \textit{ across } \overline{r}(t). \end{cases}$$

Future work will involve the study of problem (P_1^ε) in the case of arbitrary space dimension without symmetry. This study extends a similar study of Stoth [2] in the case of the three-dimensional Cahn–Hilliard equation with Dirichlet boundary conditions.

2. A priori estimates and first convergence results

We show below some estimates, which imply in particular the compactness of (u^ε) in $L^1(\Omega\times(0,T))$. We use an equivalent formulation of problem (P_1^ε), namely,

$$(P^\varepsilon) \begin{cases} u_t^\varepsilon = \Delta w^\varepsilon & \text{in } \Omega\times(0,T) & (2.1) \\ w^\varepsilon = -(\varepsilon\Delta u^\varepsilon + \dfrac{1}{\varepsilon} f(u^\varepsilon) - v^\varepsilon) & \text{in } \Omega\times(0,T) & (2.2) \\ -\Delta v^\varepsilon = u^\varepsilon - \fint_\Omega u^\varepsilon dx & \text{in } \Omega\times(0,T) & (2.3) \\ \fint_\Omega v^\varepsilon dx = 0 & \text{for } t\in(0,T) & (2.4) \\ \dfrac{\partial u^\varepsilon}{\partial n} = \dfrac{\partial v^\varepsilon}{\partial n} = \dfrac{\partial w^\varepsilon}{\partial n} = 0 & \text{in } \partial\Omega\times(0,T) & (2.5) \\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & \text{for } x\in\Omega. & (2.6) \end{cases}$$

Lemma 2.1. *Under the assumption* H_0^ε, *the solution* $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ *of problem* (P^ε) *satisfies the estimate*

$$E^\varepsilon(u^\varepsilon)(t) + \int_0^T \int_\Omega |\nabla w^\varepsilon|^2 \, dx \, dt = E^\varepsilon(u_0^\varepsilon) \leq C. \tag{2.7}$$

Moreover, there exists $\rho_0 > 0$ *such that for all* $\rho \leq \rho_0$

$$\sup_{t\in(0,T)} \sup_{|z|\leq\rho} \int_{\Omega_\rho} |u^\varepsilon(x+z,t) - u^\varepsilon(x,t)| \, dx \leq C\rho^{1/2}, \tag{2.8}$$

where $\Omega_\rho = \{x \in \Omega : d(x, \partial\Omega) \geq \rho\}$ *and*

$$\sup_{t\in(0,T)} \int_{\Omega\setminus\Omega_\rho} |u^\varepsilon(x,t)| \, dx \leq C\rho^{1/2}. \tag{2.9}$$

Moreover, there exists $h_0 > 0$ *such that*

$$\int_0^{T-h} \int_\Omega |u^\varepsilon(x,t+h) - u^\varepsilon(x,t)| \, dx \, dt \leq C \ h^{1/4}, \tag{2.10}$$

for all $h \in (0, h_0)$.

Corollary 2.2. u^ε *is bounded in* $L^\infty(0,T,L^4(\Omega))$.

Corollary 2.2 is a consequence of (2.7) and of the definition of F.

Proof of Lemma 2.1. We multiply (2.2) by u_t^ε, and (2.1) by w^ε. Subtracting both equations and integrating on Ω, we obtain

$$\int_\Omega \Delta w^\varepsilon w^\varepsilon \ dx = \int_\Omega \Big(- \varepsilon \Delta u^\varepsilon u_t^\varepsilon - \frac{1}{\varepsilon} f(u^\varepsilon) u_t^\varepsilon + v^\varepsilon u_t^\varepsilon \Big) dx.$$

This implies that

$$-\int_\Omega |\nabla w^\varepsilon|^2 \ dx = \varepsilon \int_\Omega \nabla u^\varepsilon \nabla u_t^\varepsilon \ dx + \frac{1}{\varepsilon}\frac{d}{dt}\Big(\int_\Omega F(u^\varepsilon) \ dx \Big) + \int_\Omega v^\varepsilon u_t^\varepsilon \ dx. \tag{2.11}$$

Next we estimate the last term of (2.11). Differentiating (2.3) in time and using (1.4), we obtain that $u_t^\varepsilon = -\Delta v_t^\varepsilon$. Multiplying this by v^ε and integrating the result on Ω, we obtain that

$$\int_\Omega v^\varepsilon u_t^\varepsilon \ dx = \int_\Omega v^\varepsilon(-\Delta v_t^\varepsilon) \ dx = \int_\Omega \nabla v^\varepsilon . \nabla v_t^\varepsilon \ dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v^\varepsilon|^2 dx. \tag{2.12}$$

Substituting (2.12) into (2.11), we deduce that

$$-\int_\Omega |\nabla w^\varepsilon|^2 \ dx = \frac{d}{dt}\Big[\int_\Omega \frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon}F(u^\varepsilon) + \frac{1}{2}|\nabla v^\varepsilon|^2 dx \Big],$$

which we integrate on the time interval $(0,t)$ to conclude in view of the definition (1.1) of $E^\varepsilon(u^\varepsilon)$ that

$$E^\varepsilon(u^\varepsilon)(t) + \int_0^t \int_\Omega |\nabla w^\varepsilon|^2 \; dx\, dt = E^\varepsilon(u^\varepsilon)(0).$$

This completes the proof of (2.7). In order to prove (2.8), we set $g(s) := \int_0^s \sqrt{2F(\tau)}\, d\tau$, for all $s \in R$, and we apply Lemma A.1 in the Appendix with $\mu = u^\varepsilon(x+z,t)$ and $\lambda = u^\varepsilon(x,t)$. It follows that

$$\begin{aligned}&\int_{\Omega_\rho} |u^\varepsilon(x+z,t) - u^\varepsilon(x,t)|\, dx \\ &\leq \delta \; meas(\Omega_\rho) + \frac{1}{K\delta}\int_{\Omega_\rho} |g(u^\varepsilon(x+z,t)) - g(u^\varepsilon(x,t))|\, dx.\end{aligned} \tag{2.13}$$

Furthermore, we have

$$g(u^\varepsilon(x+z,t)) - g(u^\varepsilon(x,t)) = \int_0^1 g'(u^\varepsilon(x+\lambda z,t))\nabla u^\varepsilon(x+\lambda z,t) z \; d\lambda,$$

which we integrate on Ω_ρ, to obtain

$$\begin{aligned}&\int_{\Omega_\rho} |g(u^\varepsilon(x+z,t)) - g(u^\varepsilon(x,t))|\, dx \\ &\leq |z| \int_0^1 \int_{\Omega_\rho} (\sqrt{2F(u^\varepsilon)}|\nabla u^\varepsilon|)(x+\lambda z,t)\, dx\, d\lambda \\ &\leq |z| \int_{\Omega'_\rho} (\sqrt{2F(u^\varepsilon)}|\nabla u^\varepsilon|)(y,t)\, dy,\end{aligned}$$

where $\Omega'_\rho \subset \Omega$. This implies that

$$\begin{aligned}\int_{\Omega_\rho} |g(u^\varepsilon(x+z,t)) - g(u^\varepsilon(x,t))|\, dx &\leq |z| \int_\Omega \sqrt{2F(u^\varepsilon)}|\nabla u^\varepsilon| \; dx \\ &\leq |z| \int_\Omega (\frac{1}{\varepsilon}F(u^\varepsilon) + \frac{\varepsilon}{2}|\nabla u^\varepsilon|^2)\, dx.\end{aligned} \tag{2.14}$$

Substituting (2.14) into (2.13) and also using (2.7) we deduce that for $\delta = \rho^{1/2}$ we have

$$\int_{\Omega_\rho} |u^\varepsilon(x+z,t) - u^\varepsilon(x,t)|\, dx \leq C[\delta + \frac{1}{\delta}\rho] = C\rho^{1/2}, \tag{2.15}$$

for all $z \in \Omega$ such that $|z| \le \rho$, which coincides with (2.8). Furthermore, using (2.2) and the fact that $meas(\Omega \setminus \Omega_\rho) \le C\rho^{1/2}$ we also have that

$$\int_{\Omega\setminus\Omega_\rho} |u^\varepsilon(x,t)| \le (meas(\Omega\setminus\Omega_\rho))^{1/2}\Big(\int_\Omega |u^\varepsilon(x,t)|^2\Big)^{1/2} \le C\rho^{1/2}.$$

This completes the proof of (2.9). Next we prove (2.10). To obtain the bound on the differences of time translates we set $\chi^\epsilon(x,t) := u^\varepsilon(x,t+h) - u^\varepsilon(x,t)$, where h is positive. Let $\psi \in C^\infty(R^N)$ such that $\psi \ge 0$, $supp\ \psi \subset B_1(0)$, where $B_r(0)$ denotes the ball of center 0 and radius r, and $\int_{R^N}\psi = 1$. We set $\psi_\rho(x) = \frac{1}{\rho^N}\psi(\frac{x}{\rho})$, and consequently we have that $\int_{R^N}\psi_\rho = 1$. Setting

$$\chi^\epsilon * \psi_\rho(x,t) := \int_{B_\rho(0)} \chi^\epsilon(x+z,t)\psi_\rho(z)\,dz, \tag{2.16}$$

we first note that

$$\int_0^{T-h}\!\!\int_\Omega|\chi^\epsilon| \le \int_0^{T-h}\!\!\int_{\Omega_\rho}|\chi^\epsilon*\psi_\rho - \chi^\epsilon| + \int_0^{T-h}\!\!\int_{\Omega_\rho}|\chi^\epsilon*\psi_\rho| + \int_0^{T-h}\!\!\int_{\Omega\setminus\Omega_\rho}|\chi^\epsilon|, \tag{2.17}$$

where $\Omega_\rho = \{x \in \Omega : d(x,\partial\Omega) \ge \rho\}$. Next we estimate the terms on the right-hand side of (2.17). We have

$$\begin{aligned}
&\int_0^{T-h}\int_{\Omega_\rho} |\chi^\epsilon*\psi_\rho - \chi^\epsilon|\,dx\,dt\\
&= \int_0^{T-h}\int_{\Omega_\rho}\Big|\int_{B_\rho(0)}\psi_\rho(z)(\chi^\epsilon(x+z,t)-\chi^\epsilon(x,t))\ dz\Big|\,dx\,dt\\
&\le \int_{B_\rho(0)}\psi_\rho(z)\Big\{\int_0^{T-h}\int_{\Omega_\rho}|\chi^\epsilon(x+z,t)-\chi^\epsilon(x,t)|\,dx\,dt\Big\}dz\\
&\le \int_{B_\rho(0)}\psi_\rho(z)\Big\{\sup_{z\in B_\rho(0)}\int_0^{T-h}\int_{\Omega_\rho}|\chi^\epsilon(x+z,t)-\chi^\epsilon(x,t)|\,dx\,dt\Big\}dz\\
&\le \sup_{z\in B_\rho(0)}\int_o^{T-h}\int_{\Omega_\rho}|\chi^\epsilon(x+z,t)-\chi^\epsilon(x,t)|\,dx\,dt.
\end{aligned}$$

Thus, using the definition of χ^ϵ we deduce that

$$\begin{aligned}
&\int_0^{T-h}\int_{\Omega_\rho}|\chi^\epsilon*\psi_\rho - \chi^\epsilon|\,dx\,dt\\
&\le \sup_{z\in B_\rho(0)}\Big[\int_0^{T-h}\int_{\Omega_\rho}|u^\varepsilon(x+z,t+h)-u^\varepsilon(x,t+h)|\,dx\,dt
\end{aligned}$$

$$+ \int_0^{T-h} \int_{\Omega_\rho} |u^\varepsilon(x+z,t) - u^\varepsilon(x,t)|\, dx\, dt\Big]$$
$$\le 2 \sup_{z\in B_\rho(0)} \int_0^T \int_{\Omega_\rho} |u^\varepsilon(x+z,t) - u^\varepsilon(x,t)|\, dx\, dt,$$

which in view of (2.8) implies that

$$\int_0^{T-h} \int_{\Omega_\rho} |\chi^\epsilon * \psi_\rho - \chi^\epsilon|\, dx\, dt \le C\rho^{1/2}. \tag{2.18}$$

Furthermore, noting that

$$\chi^\epsilon(y,t) = h \int_0^1 u_t^\varepsilon(y, t+\lambda h)\ d\lambda$$

and using (2.16) we deduce that

$$\begin{aligned}\int_0^{T-h} |\chi^\epsilon * \psi_\rho|(x,t)\, dt &= h \int_0^{T-h} \Big| \int_0^1 \int_{B_\rho(0)} u_t^\varepsilon(x+z, t+\lambda h)\psi_\rho(z)\, dz\, d\lambda \Big| dt \\ &\le h \int_0^1 \int_0^{T-h} \Big| \int_{B_\rho(0)} u_t^\varepsilon(x+z, t+\lambda h)\psi_\rho(z)\ dz \Big| dt\, d\lambda \\ &\le h \int_0^1 \int_{\lambda h}^{T+(\lambda-1)h} \Big| \int_{B_\rho(0)} u_t^\varepsilon(x+z, t)\psi_\rho(z)\ dz \Big| dt\, d\lambda.\end{aligned}$$

This implies that

$$\int_0^{T-h} |\chi^\epsilon * \psi_\rho|(x,t)\, dt \le h \int_0^T \Big| \int_{B_\rho(0)} u_t^\varepsilon(x+z,t)\psi_\rho(z)\ dz \Big|\, dt. \tag{2.19}$$

Moreover, using (2.1) and the fact that ψ_ρ vanishes on $\partial B_\rho(0)$ we obtain that

$$\int_{B_\rho(0)} u_t^\varepsilon(x+z,t)\psi_\rho(z) = -\int_{B_\rho(0)} \nabla w^\varepsilon(x+z,t)\nabla\psi_\rho(z). \tag{2.20}$$

Substituting (2.20) into (2.19) we deduce that

$$\begin{aligned}&\int_0^{T-h} \int_{\Omega_\rho} |\chi^\epsilon * \psi_\rho(x,t)|\, dx\, dt \\ &\le h \int_0^T \int_{\Omega_\rho} \Big(\int_{B_\rho(0)} |\nabla w^\varepsilon(x+z,t)||\nabla\psi_\rho(z)|\, dz\Big)\, dx\, dt \\ &\le h \int_0^T \int_{B_\rho(0)} |\nabla\psi_\rho(z)| \Big(\int_\Omega |\nabla w^\varepsilon(y,t)|\, dy\Big)\, dz\, dt.\end{aligned}$$

Moreover, using the fact that $\int_{B_\rho(0)} |\nabla\psi_\rho| = \frac{1}{\rho}\int_{B_1(0)} |\nabla\psi| \le \frac{C_1}{\rho}$, we obtain

$$\int_0^{T-h}\int_{\Omega_\rho} |\chi^\epsilon * \psi_\rho(x,t)|\,dx\,dt \le C_1\frac{h}{\rho}(T\ meas(\Omega))^{1/2}(\int_0^T\int_\Omega |\nabla w^\varepsilon|^2\,dy\,dt)^{1/2}.$$

In view of (2.7) this implies that

$$\int_0^{T-h}\int_{\Omega_\rho} |\chi^\epsilon * \psi_\rho|\,dx\,dt \le C_2\frac{h}{\rho}. \tag{2.21}$$

Next we estimate the last term on the right-hand side of (2.17).

$$\int_0^{T-h}\int_{\Omega\backslash\Omega_\rho} |\chi^\epsilon(x,t)|\,dx\,dt \le 2\int_0^{T}\int_{\Omega\backslash\Omega_\rho} |u^\varepsilon(x,t)|\,dx\,dt$$
$$\le 2(T\ meas(\Omega\backslash\Omega_\rho))^{1/2}\left(\int_0^T\int_\Omega |u^\varepsilon(x,t)|^2\,dx\,dt\right)^{1/2}.$$

Using Corollary 2.2 and the fact that $meas(\Omega\backslash\Omega_\rho) \le C(\Omega)\rho$ we deduce that

$$\int_0^{T-h}\int_{\Omega\backslash\Omega_\rho} |\chi^\epsilon(x,t)|\,dx\,dt \le C_3\rho^{1/2}. \tag{2.22}$$

Substituting (2.18), (2.21), and (2.22) into (2.17) we conclude that

$$\int_0^{T-h}\int_\Omega |u^\varepsilon(x,t+h) - u^\varepsilon(x,t)|\,dx\,dt \le C_4[\rho^{1/2} + \frac{h}{\rho}] \le C\ h^{1/4},$$

if ρ is chosen to be equal to $h^{1/2}$. This completes the proof of (2.10). □

Before proving a first convergence result we state some more a priori estimates:

Lemma 2.3. *There exists a positive constant C, depending on Ω, such that*

$$\sup_{t\in(0,T)}\int_\Omega \left(|v^\varepsilon|^2 + |\nabla v^\varepsilon|^2\right)(x,t)\,dx \le C \tag{2.23}$$

$$\int_0^T\int_\Omega (|w^\varepsilon|^2 + |\nabla w^\varepsilon|^2)(x,t)\,dx\,dt \le C. \tag{2.24}$$

Proof. We deduce from (2.7) that

$$\sup_{t\in(0,T)}\int_\Omega |\nabla v^\varepsilon|^2(x,t)\,dx \le C_1, \tag{2.25}$$

which together with the generalized Poincaré inequality, namely

$$\|h\|_{L^2(\Omega)} \le C(\|\nabla h\|_{L^2(\Omega)} + |\int_\Omega h|),\ \text{for all } h\in H^1(\Omega), \tag{2.26}$$

and the fact that $\int_\Omega v^\varepsilon = 0$, implies that

$$\int_\Omega |v^\varepsilon|^2\,dx \le C.$$

This completes the proof of (2.23). In order to prove (2.24) we recall a result due to X. Chen (Lemma 3.4, [4]), namely the following: Let u^ε satisfy

$$\begin{cases} w^\varepsilon = -(\varepsilon\Delta u^\varepsilon + \frac{1}{\varepsilon}f(u^\varepsilon) - v^\varepsilon) & \text{in } \Omega \\ \frac{\partial u^\varepsilon}{\partial n} = 0 & \text{in } \partial\Omega, \end{cases}$$

where $w^\varepsilon - v^\varepsilon \in H^1(\Omega)$. Then there exists a constant $C(\Omega) > 0$ such that for ε small enough we have

$$\|(w^\varepsilon - v^\varepsilon)(\cdot,t)\|_{H^1(\Omega)} \le C(\Omega)(E^\varepsilon(u^\varepsilon)(t) + \|(\nabla w^\varepsilon - \nabla v^\varepsilon)(\cdot,t)\|_{L^2(\Omega)}), \quad (2.27)$$

for all $t \in (0,T)$. We first deduce from (2.7) that

$$\int_0^T \int_\Omega |\nabla w^\varepsilon|^2(x,t)\,dx \le C_2. \quad (2.28)$$

Using (2.27) and the fact that the function $E^\varepsilon(u^\varepsilon)$ is decreasing we deduce that

$$\begin{aligned}\|w^\varepsilon(\cdot,t)\|_{H^1(\Omega)} \le C(\Omega)(E^\varepsilon(u^\varepsilon)(0) + \|\nabla w^\varepsilon(\cdot,t)\|_{L^2(\Omega)} \\ + \|\nabla v^\varepsilon(\cdot,t)\|_{L^2(\Omega)}) + \|v^\varepsilon(\cdot,t)\|_{H^1(\Omega)}.\end{aligned} \quad (2.29)$$

Integrating (2.29) in time and using (2.23), (2.28) and the hypothesis H_0^ε we obtain (2.24). This completes the proof of Lemma 2.3. Next we state the following convergence result:

Theorem 2.4. *There exist functions u, w, and v such that $u^{\varepsilon_n} \to u$ in $L^1(\Omega \times (0,T))$ almost everywhere and moreover, $u = \pm 1$ almost everywhere in $(0,T)$, $w^{\varepsilon_n} \rightharpoonup w$ in $L^2(0,T,H^1(\Omega))$, and $v^{\varepsilon_n} \rightharpoonup v$ in $L^2(0,T,H^1(\Omega))$. As a consequence $u^{\varepsilon_n}(\cdot,t) \to u(\cdot,t)$ in $L^1(\Omega)$ almost everywhere in Ω, for almost every $t \in (0,T)$.*

Proof. We deduce from the Riesz–Fréchet–Kolmogorov lemma (see for instance [2]) that there exist a sequence $\{\varepsilon_n\}$ and a function $u \in L^1(\Omega \times (0,T))$ such that $u^{\varepsilon_n} \to u$ in $L^1(\Omega \times (0,T))$ and almost everywhere. Next we check that $u = \pm 1$. In view of (2.7) and Fatou's lemma we deduce that

$$\int_0^T \int_\Omega \liminf_{\varepsilon_n \to 0} F(u^\varepsilon)\,dx\,dt \le \liminf_{\varepsilon_n \to 0} \int_0^T \int_\Omega F(u^\varepsilon)\,dx\,dt \le 0.$$

This implies that $F(u) = 0$ and consequently that $u = \pm 1$ almost everywhere in $\Omega \times (0,T)$. Next we prove the convergence of the subsequences $\{w^{\varepsilon_n}\}$ and $\{v^{\varepsilon_n}\}$. We deduce from (2.23) and (2.24) that $\{w^{\varepsilon_n}\}$ and $\{v^{\varepsilon_n}\}$

are bounded in $L^2(0,T,H^1(\Omega))$. This in turn implies that there exist two functions w, v and a subsequence $\{\varepsilon_{n_m}\}$ such that $\{w^{\varepsilon_{n_m}}\}$ tends to w weakly in $L^2(0,T,H^1(\Omega))$ and $\{v^{\varepsilon_{n_m}}\}$ tends to v weakly in $L^2(0,T,H^1(\Omega))$ as ε_{n_m} tends to zero. This completes the proof of Theorem 2.4.

Consequently, we deduce that the functions (u, v, w) are such that

$$\int_0^T \int_\Omega (u\varphi_t - \nabla w \nabla \varphi)(x,t)\,dx\,dt + \int_\Omega u_0(x)\varphi(x,0)\,dx = 0$$

for all $\varphi \in C^1(\overline{\Omega} \times [0,T])$ such that $\varphi(T) = 0$ and

$$\begin{cases} -\Delta v = u - \fint_\Omega u & \text{a.e. in } \Omega \times (0,T), \\ \fint_\Omega v\,dx = 0 & \text{for a.e. } t \in (0,T), \\ \dfrac{\partial v}{\partial n} = 0 & \text{a.e. on } \partial\Omega \times (0,T). \end{cases}$$

Thus we have shown the first part of Theorem 1.1. In the following we will denote the sequence $\{\varepsilon_n\}$ by $\{\varepsilon\}$.

3. The approximation in the radial case

From now on we suppose that Ω is the unit ball in R^N and we interpret the solutions $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ in the radial variable $r = |x|$. Moreover, we suppose that u_0^ε is radially symmetric, and thus the solutions $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ satisfy the following property:

$$(Or)\ u_r^\varepsilon(0,t) = v_r^\varepsilon(0,t) = w_r^\varepsilon(0,t) = 0, \quad \text{for all } t \in (0,T).$$

Problem (P^ε) takes the form

$$(P_r^\varepsilon) \begin{cases} u_t^\varepsilon = w_{rr}^\varepsilon + \dfrac{N-1}{r} w_r^\varepsilon & \text{in } (0,1)\times(0,T) & (3.1) \\ w^\varepsilon = -\varepsilon u_{rr}^\varepsilon - \varepsilon \dfrac{N-1}{r} u_r^\varepsilon - \dfrac{1}{\varepsilon} f(u^\varepsilon) + v^\varepsilon & \text{in } (0,1)\times(0,T) & (3.2) \\ -v_{rr}^\varepsilon - \dfrac{N-1}{r} v_r^\varepsilon = u^\varepsilon - \fint_\Omega u^\varepsilon\,dx & \text{in } (0,1)\times(0,T) & (3.3) \\ \fint_0^1 v^\varepsilon(r,t) r^{N-1} dr = 0 & \text{for } t \in (0,T) & (3.4) \\ u_r^\varepsilon(0,t) = w_r^\varepsilon(0,t) = v_r^\varepsilon(0,t) = 0 & \text{for } t \in (0,T) & (3.5) \\ u_r^\varepsilon(1,t) = w_r^\varepsilon(1,t) = v_r^\varepsilon(1,t) = 0 & \text{for } t \in (0,T) & (3.6) \\ u^\varepsilon(r,0) = u_0^\varepsilon(r) & \text{for } r \in (0,1). & (3.7) \end{cases}$$

The energy estimate becomes

$$\sup_{t\in[0,T]}\Big[\int_0^1 (\frac{\varepsilon}{2}|u_r^\varepsilon|^2+\frac{1}{\varepsilon}F(u^\varepsilon)+\frac{1}{2}|v_r^\varepsilon|^2)r^{N-1}\,dr\Big] + \int_0^T\int_0^1 (w_r^\varepsilon)^2 r^{N-1}\,dr\,dt \le K. \tag{3.8}$$

We remark that in spherical symmetry the limit functions w and v are such that $w, v \in L^2(0,T;C([R_0,1]))$ for all $R_0 > 0$. Thus, for almost $t\in(0,T)$ $w(\cdot,t)$ and $v(\cdot,t)$ are continuous on $[R_0,1]$ for all $R_0>0$.

In this section we prove a result which implies that far away from the origin, the solution u^ε is close to the function $\pm q(\frac{\xi}{\varepsilon})$ where q satisfies

$$q_{\xi\xi}+f(q)=0,\quad q(-\infty)=-1,\ q(0)=0,\ q(+\infty)=1;$$

that is, $q(\xi)=\tanh(\xi)$. We first give some preliminary estimates.

Lemma 3.1. *Let $0<R_0<1$; there exists a function $C(R_0)$ independent of ε such that $(u^\varepsilon,v^\varepsilon,w^\varepsilon)$ satisfy the following estimates:*

$$\int_0^T\int_{R_0}^1 |w^\varepsilon(r,t)|^2\,dr\,dt\le C(R_0), \tag{3.9}$$

$$\int_{R_0}^1 |v^\varepsilon(r,t)|^2\,dr\le C(R_0), \tag{3.10}$$

$$\varepsilon\int_{R_0}^1 |u_r^\varepsilon(r,t)|^2\,dr+\frac{1}{\varepsilon}\int_{R_0}^1 F(u^\varepsilon(r,t))\,dr\le C(R_0), \tag{3.11}$$

$$\varepsilon\int_{R_0}^1 \frac{|u_r^\varepsilon(r,t)|^2}{r}\,dr\le C(R_0), \tag{3.12}$$

$$\|u^\varepsilon(\cdot,t)\|_{L^\infty(R_0,1)}\le C(R_0)\varepsilon^{-1/2}, \tag{3.13}$$

for all $t\in[0,T]$.

Proof. In order to prove (3.9), we note that

$$\int_{R_0}^1 |w^\varepsilon(r,t)|^2\,dr\le\frac{1}{R_0^{N-1}}\int_{R_0}^1 |w^\varepsilon(r,t)|^2 r^{N-1}\,dr. \tag{3.14}$$

Integrating (3.14) in $(0,T)$ and using (2.24) we deduce (3.9). In a similar way we obtain (3.10). Next we prove (3.11). The energy estimate (3.8) implies that

$$\varepsilon\int_{R_0}^1 |u_r^\varepsilon|^2\,dr\le\frac{\varepsilon}{R_0^{N-1}}\int_0^1 |u_r^\varepsilon|^2 r^{N-1}\,dr\le\frac{2K}{R_0^{N-1}},$$

and similarly

$$\frac{1}{\varepsilon}\int_{R_0}^{1} F(u^\varepsilon)\,dr \le \frac{1}{\varepsilon R_0^{N-1}}\int_0^1 F(u^\varepsilon) r^{N-1}\,dr \le \frac{K}{R_0^{N-1}},$$

for all $t \in (0,T)$. This in turn implies (3.11). Next we prove (3.12); in view of (3.8) we have that

$$\varepsilon\int_{R_0}^{1}\frac{|u_r^\varepsilon|^2}{r}\,dr \le \frac{\varepsilon}{R_0^N}\int_0^1 |u_r^\varepsilon|^2 r^{N-1}\,dr \le \frac{2K}{R_0^N}.$$

Finally we prove (3.13). We first note that

$$u^\varepsilon(r,t) - u^\varepsilon(s,t) = \int_s^r u_r^\varepsilon(\xi,t)\,d\xi \quad \text{ for all } R_0 \le s \le r \le 1,$$

which we integrate in s on $(R_0, 1)$ to obtain

$$\begin{aligned}
(1-R_0)u^\varepsilon(r,t) &= \int_{R_0}^1 u^\varepsilon(s,t)\,ds + \int_{R_0}^1\int_s^r u_r^\varepsilon(\xi,t)\,d\xi\,ds \\
(1-R_0)|u^\varepsilon(r,t)| &\le (1-R_0)^{1/2}\left(\int_{R_0}^1 |u^\varepsilon|^2\,ds\right)^{1/2} + (1-R_0)\int_{R_0}^1 |u_r^\varepsilon|\,ds \\
&\le \Big[\frac{(1-R_0)}{R_0^{N-1}}\Big]^{1/2}\Big(\int_{R_0}^1 |u^\varepsilon|^2 s^{N-1}\,ds\Big)^{1/2} \\
&\quad + \Big[\frac{(1-R_0)^3}{R_0^{N-1}}\Big]^{1/2}\Big(\int_{R_0}^1 (u_r^\varepsilon)^2 s^{N-1}\,dsv)^{1/2}.
\end{aligned}$$

Consequently, we deduce from Corollary 2.2 and from (3.8) that

$$|u^\varepsilon(r,t)| \le C(R_0)\varepsilon^{-1/2}, \text{ for all } (r,t) \in [R_0,1]\times[0,T].$$

This completes the proof of Lemma 3.1. □

Before proving the key estimate of this section we show some inequalities which hold for almost every $t \in (0,T)$.

Lemma 3.2. *Let γ be a positive constant. There exist a set of measure zero $D(R_0,T) \subset (0,T)$ and a subsequence of $\{\varepsilon_n\}$ which we denote again by $\{\varepsilon\}$ such that for all $t \in (0,T)\setminus D(R_0,T)$ there exists a positive constant $C(R_0,t)$ satisfying*

$$\int_{R_0}^1 |w^\varepsilon(r,t)|^2\,dr \le C(R_0,t)\varepsilon^{-\gamma}, \tag{3.15}$$

$$\int_{R_0}^1 |w_r^\varepsilon(r,t)|^2 r^{N-1}\,dr \le C(R_0,t)\varepsilon^{-\gamma}. \tag{3.16}$$

Proof. By (3.9) we have that

$$\varepsilon^\gamma \int_0^T \int_{R_0}^1 |w^\varepsilon|^2 \, dr \, dt \le C(R_0)\varepsilon^\gamma.$$

This implies that the sequence $\{\varepsilon^\gamma \int_{R_0}^1 |w^\varepsilon|^2 \, dr\}$ converges to zero in $L^1(0,T)$. Thus, we deduce that there exists a subsequence which converges to zero almost everywhere in (0,T), which in turn implies (3.15). We obtain (3.16) in a similar way. □

We are now in a position to prove the fundamental estimate, namely

Theorem 3.3. *For each $t \in (0,T)\setminus D(R_0,T)$, there exists a constant $\tilde{C}(R_0,t)$ such that u^ε satisfies*

$$\|\frac{\varepsilon^2}{2}(u_r^\varepsilon(\cdot,t))^2 - F(u^\varepsilon(\cdot,t))\|_{L^\infty(R_0,1)} \le \tilde{C}(R_0,t)\varepsilon^{1/2-\gamma},$$

for all $\gamma \in (0,1/2)$.

Proof. Fix η and ρ in the interval $[R_0,1]$. We multiply the equation (3.2) by $(-\varepsilon u_r^\varepsilon)$, and we integrate from η to ρ. This gives

$$\int_\eta^\rho \Big(\varepsilon^2 u_r^\varepsilon u_{rr}^\varepsilon + (N-1)\varepsilon^2 \frac{(u_r^\varepsilon)^2}{r} + f(u^\varepsilon)u_r^\varepsilon\Big)\, dr = \int_\eta^\rho \varepsilon(-w^\varepsilon + v^\varepsilon)u_r^\varepsilon \, dr. \quad (3.17)$$

Moreover, we have that

$$\int_\eta^\rho u_r^\varepsilon u_{rr}^\varepsilon \, dr = \frac{1}{2}[(u_r^\varepsilon)^2]_\eta^\rho \quad \text{and} \quad \int_\eta^\rho f(u^\varepsilon)u_r^\varepsilon \, dr = -[F(u^\varepsilon)]_\eta^\rho,$$

which we substitute in (3.17) to obtain

$$\frac{\varepsilon^2}{2}(u_r^\varepsilon)^2(\rho,t) - F(u^\varepsilon(\rho,t)) = \frac{\varepsilon^2}{2}(u_r^\varepsilon)^2(\eta,t) - F(u^\varepsilon(\eta,t))$$
$$-(N-1)\varepsilon^2 \int_\eta^\rho \frac{(u_r^\varepsilon)^2}{r}\, dr + \varepsilon \int_\eta^\rho (-w^\varepsilon + v^\varepsilon)u_r^\varepsilon \, dr. \quad (3.18)$$

We integrate (3.18) in η in $(R_0,1)$; this gives

$$\big(\frac{\varepsilon^2}{2}(u_r^\varepsilon(\rho,t))^2 - F(u^\varepsilon(\rho,t))\big)(1-R_0) = \int_{R_0}^1 \frac{\varepsilon^2}{2}(u_r^\varepsilon(\eta,t))^2 - F(u^\varepsilon(\eta,t))\, d\eta$$
$$-(N-1)\varepsilon^2 \int_{R_0}^1 \int_\eta^\rho \frac{(u_r^\varepsilon)^2}{r}\, dr\, d\eta + \varepsilon \int_{R_0}^1 \int_\eta^\rho (-w^\varepsilon + v^\varepsilon)u_r^\varepsilon \, dr\, d\eta. \quad (3.19)$$

Furthermore, we have

$$\int_{R_0}^1 \int_\eta^\rho \frac{(u_r^\varepsilon)^2}{r}\, dr\, d\eta \le \int_{R_0}^1 \int_{R_0}^1 \frac{(u_r^\varepsilon)^2}{r}\, dr\, d\eta \le (1-R_0)\int_{R_0}^1 \frac{(u_r^\varepsilon)^2}{r}\, dr, \quad (3.20)$$

and also that

$$\int_{R_0}^1 \int_\eta^\rho (-w^\varepsilon + v^\varepsilon) u_r^\varepsilon \, dr d\eta \le \int_{R_0}^1 \int_{R_0}^1 |w^\varepsilon u_r^\varepsilon| + |v^\varepsilon u_r^\varepsilon| \, dr d\eta \tag{3.21}$$

$$\le (1-R_0)\Big(\int_{R_0}^1 |u_r^\varepsilon|^2 \, dr\Big)^{1/2} \Big[\Big(\int_{R_0}^1 |w^\varepsilon|^2 \, dr\Big)^{1/2} + \Big(\int_{R_0}^1 |v^\varepsilon|^2 \, dr\Big)^{1/2}\Big].$$

Substituting (3.20) and (3.21) into (3.19), and also using the estimates (3.10), (3.11), (3.12), and (3.15), we obtain that

$$|\frac{\varepsilon^2}{2}(u_r^\varepsilon(\rho,t))^2 - F(u^\varepsilon(\rho,t))|$$

$$\le C_1(R_0)\varepsilon + [C(R_0,t)^{1/2}\varepsilon^{-\gamma/2} + C(R_0)^{1/2}]C(R_0)^{1/2}\varepsilon^{1/2} \le \tilde{C}(R_0,t)\varepsilon^{1/2-\gamma},$$

for all ρ in $[R_0, 1]$. This completes the proof of Theorem 3.3.

4. A definition of "jumps" of u and properties

In the following we suppose that $\gamma \in (0, 1/4)$. Let $\{R_0^n\}$ be a sequence of $(0,1)$, which converges to zero as n tends to $+\infty$. We set $D(T) := \cup_n D(R_0^n, t)$, where $D(R_0^n,t)$ has been introduced in Lemma 3.2. Thus $D(T)$ is a set of measure zero. In this section we fix $t \in (0,T) \setminus D(T)$ such that $u^\varepsilon(\cdot,t)$ tends to ± 1 in $L^1(\Omega)$ almost everywhere in Ω. We set $A_+(t) := \{r \in (0,1) : u(r,t) = 1\}$, $A_-(t) := \{r \in (0,1) : u(r,t) = -1\}$.

Definition 4.1. We call $\overline{r} = \overline{r}(t)$ a jump point of $u(\cdot,t)$ in $(0,1)$ if

$$\begin{cases} meas([\overline{r}-\rho, \overline{r}+\rho] \cap A_+(t)) > 0 \\ meas([\overline{r}-\rho, \overline{r}+\rho] \cap A_-(t)) > 0 \end{cases}$$

for all $\rho > 0$ small enough.

Next we state some preliminary results.

Lemma 4.2. *Let $\overline{r}(t)$ be a jump of $u(\cdot,t)$. For all $\rho > 0$, there exists $\varepsilon_0 > 0$ such that there exists a zero $r^\varepsilon = r^\varepsilon(t) \in (\overline{r}-\rho, \overline{r}+\rho)$ of $u^\varepsilon(\cdot,t)$, for all $\varepsilon \le \varepsilon_0$.*

Proof. Since $\overline{r}(t) > 0$ there exists $R_0 > 0$ such that $R_0 < \overline{r}(t) < 1$. We first set $A_+(\rho,t) := \{r \in (\overline{r}-\rho, \overline{r}+\rho) \cap (R_0, 1) : u(r,t) = +1\}$, and thus we have $meas(A_+(\rho,t)) > 0$. Since $u^\varepsilon(\cdot,t)$ converges to $u(\cdot,t)$ in $L^1(\Omega)$ we have that

$$\int_{R_0}^1 |(u-u^\varepsilon)(r,t)| r^{N-1} dr < R_0^{N-1} meas(A_+(\rho,t)), \tag{4.1}$$

for ε small enough. Next we prove that there exists $r_1 \in (\overline{r}-\rho, \overline{r}+\rho)$ such that $u^\varepsilon(r_1,t) > 0$. Suppose that $u^\varepsilon(r,t) \leq 0$ for all $r \in (\overline{r}-\rho, \overline{r}+\rho)$; then we have that $(u-u^\varepsilon)(\cdot,t) \geq 1$ in $A_+(\rho,t)$. This in turn implies that

$$\int_{R_0}^1 |(u-u^\varepsilon)(r,t)| r^{N-1} dr \geq \int_{A_+(\rho,t)} r^{N-1} dr \geq R_0^{N-1} meas(A_+(\rho,t)). \quad (4.2)$$

In view of (4.1) we deduce that (4.2) is impossible. Thus we deduce that there exists $r_1 \in (\overline{r}-\rho, \overline{r}+\rho)$ such that $u^\varepsilon(r_1,t) > 0$. Similarly one can prove that there exists $r_2 \in (\overline{r}-\rho, \overline{r}+\rho)$ such that $u^\varepsilon(r_2,t) < 0$. Therefore, we conclude that there exists $r^\varepsilon \in (\overline{r}-\rho, \overline{r}+\rho)$ such that $u^\varepsilon(r^\varepsilon,t) = 0$. This completes the proof of Lemma 4.2.

Lemma 4.3. *There exists $\varepsilon_0(R_0,t)$ such that for all $\varepsilon < \varepsilon_0(R_0,t)$ we have*

$$M^\varepsilon(t) = M^\varepsilon := \sharp\{r \in (R_0,1) : u^\varepsilon(r,t) = 0\} \leq C(R_0).$$

Thus, there exists a subsequence $\{\varepsilon_n\}$, which we denote again by $\{\varepsilon\}$ such that $M^\varepsilon = M$ for ε small enough.

Proof. Let r^ε be a zero of $u^\varepsilon(\cdot,t)$. Thus, we have $F(u^\varepsilon(r^\varepsilon,t)) = 1/2$, which in view of Theorem 3.3 implies that $u_r^\varepsilon(r^\varepsilon,t) \neq 0$. Moreover, using the definition of the derivative we have that $\lim_{h\to 0} \frac{u^\varepsilon(r^\varepsilon+h,t)}{h} = u_r^\varepsilon(r^\varepsilon,t)$. Since $u_r^\varepsilon(r^\varepsilon,t) \neq 0$ this gives that $u^\varepsilon(r^\varepsilon+h,t) \neq 0$ for all $h \neq 0$ small enough. Thus r^ε is an isolated zero. Therefore, we have shown that the zeros of u^ε are isolated in $[R_0,1]$, and thus the number M^ε of zeros of u^ε is finite and consequently we also have that u^ε change of sign at r^ε. Next we prove that M^ε is uniformly bounded. Let r_i^ε and r_{i+1}^ε be two consecutive zeros of u^ε; then there exists $d_i^\varepsilon \in (r_i^\varepsilon, r_{i+1}^\varepsilon)$ such that $u_r^\varepsilon(d_i^\varepsilon,t) = 0$. Using Theorem 3.3 we deduce that $|F(u^\varepsilon(d_i^\varepsilon,t))| \leq 1/8$, which implies that

$$1/2 \leq (u^\varepsilon(d_i^\varepsilon,t))^2 \leq 3/2, \quad (4.3)$$

for ε small enough. Since u^ε changes sign at each zero we have that $u^\varepsilon(d_i^\varepsilon,t)$ and $u^\varepsilon(d_{i+1}^\varepsilon,t)$ are of opposite sign. This with (4.3) and the fact that the function g is odd implies that there exists a positive constant C such that

$$|g(u^\varepsilon(d_i^\varepsilon,t)) - g(u^\varepsilon(d_{i+1}^\varepsilon,t))| \geq C,$$

for all $i \in [1,\ldots,M^\varepsilon-1]$. Therefore, we deduce that

$$(M^\varepsilon-1)C \leq \Sigma_{i=1}^{i=M^\varepsilon-1} \int_{d_i^\varepsilon}^{d_{i+1}^\varepsilon} |(g(u^\varepsilon(r,t))_r|\, dr \leq \int_{R_0}^1 |(g(u^\varepsilon(r,t))_r|\, dr. \quad (4.4)$$

Furthermore, we have that

$$\int_{R_0}^1 |g(u^\varepsilon(r,t))_r|\, dr = \int_{R_0}^1 |u_r^\varepsilon(r,t)\sqrt{2F(u^\varepsilon(r,t))}|\, dr$$

$$\leq \frac{\varepsilon}{4}\int_{R_0}^{1}(u_r^\varepsilon)^2\,dr+\frac{2}{\varepsilon}\int_{R_0}^{1}F(u^\varepsilon)\,dr.$$

Using (3.11) this implies that

$$\int_{R_0}^{1}|(g(u^\varepsilon(r,t))_r|\,dr\leq C(R_0). \tag{4.5}$$

Substituting (4.5) into (4.4) we obtain $(M^\varepsilon-1)C\leq C(R_0)$, which implies that M^ε is uniformly bounded in ε. Thus there exists $M\in N^*$ and a subsequence $\{M^{\varepsilon_n}\}$ such that M^{ε_n} tends to M as $\varepsilon_n\downarrow 0$, and consequently $M^{\varepsilon_n}=M$ for ε_n small enough.

Corollary 4.4. *The number of jumps of $u(\cdot,t)$ in $(0,1)$ is finite.*

Proof. Let $\overline{r}_1>\overline{r}_2>\cdots>\overline{r}_L$ be an arbitrary subsequence of L jumps of $u(\cdot,t)$. Using Lemma 4.2, we deduce that for all $\rho>0$ small enough and for all ε small enough, $u^\varepsilon(\cdot,t)$ has at least one zero in $(\overline{r}_i-\rho,\overline{r}_i+\rho)$, for all $i\in[1,L]$. This implies that $L\leq M$, where M is defined in Lemma 4.3. We deduce that the number of jumps of $u(\cdot,t)$ is finite.

Lemma 4.5. *Let $\overline{r}_1$ and $\overline{r}_2$ be two consecutive jumps of $u(\cdot,t)$; then either $u(\cdot,t)=1$ almost everywhere in $(\overline{r}_1,\overline{r}_2)$ or $u(\cdot,t)=-1$ almost everywhere in $(\overline{r}_1,\overline{r}_2)$.*

Proof. Let I be an interval of $(0,1)$ without jump of $u(\cdot,t)$. We prove that $u(\cdot,t)$ is constant on I. Therefore, Lemma 4.5 will follow. Let $I:=[p,q]$ and $I_\alpha:=[p+\alpha,q-\alpha]$ for $\alpha>0$ small enough. We set

$\phi_\rho^+(r)=meas([r-\rho,r+\rho]\cap A_+(t))$ and $\phi_\rho^-(r)=meas([r-\rho,r+\rho]\cap A_-(t))$,

for all $0<\rho<\alpha$ and

$$W^-=\{r\in I_\alpha,\ \text{there exists}\ \rho>0:\phi_\rho^+(r)=0\},$$
$$W^+=\{r\in I_\alpha,\ \text{there exists}\ \rho>0:\phi_\rho^-(r)=0\}.$$

We first note that since there is no jump of u in I_α we have

$$W^+\cup W^-=I_\alpha. \tag{4.6}$$

Next we prove that W^+ and W^- are two open sets. Let $r\in W^-$; there exists ρ_0 such that $\phi_{\rho_0}^+(r)=0$. Let $y\in(r-\rho_0,r+\rho_0)$; we have

$$[y-\rho,y+\rho]\subset(r-\rho_0,r+\rho_0),\ \text{for}\ \rho\ \text{small enough}.$$

This implies that $\phi_\rho^+(y)=meas([y-\rho,y+\rho]\cap A_+(t))=0$. This in turn gives that $y\in W^-$, and thus W^- is an open set. Similarly one can prove that W^+ is also an open set. We deduce from this, from (4.6) and from the fact that I_α is connected that $W^+=I_\alpha$ or $W^-=I_\alpha$. Suppose that

$W^- = I_\alpha$. This implies that for all $r \in I_\alpha$ there exists $\rho > 0$ such that $meas([r-\rho, r+\rho] \cap A_+(t)) = 0$. Therefore, since $I_\alpha = \cup_{finite}(r-\rho, r+\rho)$ we obtain that $meas(I_\alpha \cap A_+(t)) = 0$, for all α small enough. Thus we deduce that $meas(I \cap A_+(t)) = 0$. Therefore, $u(\cdot,t) = -1$ almost everywhere in I. Similarly we have that if $W^+ = I_\alpha$, then $u(\cdot,t) = 1$ almost everywhere in I. This completes the proof of Lemma 4.5. □

Lemma 4.5 leads us to introduce the following definition.

Definition 4.6. Let $\overline{r}$ be a jump of $u(\cdot,t)$ and η be small enough such that there is no other jump of $u(\cdot,t)$ in $[\overline{r}-\eta, \overline{r}+\eta]$; we define

$$\nu(\overline{r}) = \begin{cases} 1 & \text{if } u(\cdot,t) = -1 \text{ on } [\overline{r}-\eta, \overline{r}) \text{ and } u(\cdot,t) = 1 \text{ on } (\overline{r}, \overline{r}+\eta], \\ -1 & \text{if } u(\cdot,t) = 1 \text{ on } [\overline{r}-\eta, \overline{r}) \text{ and } u(\cdot,t) = -1 \text{ on } (\overline{r}, \overline{r}+\eta]. \end{cases}$$

We are now in a position to relate the convergence of zeros of u^ε to the jump points of u:

Theorem 4.7. *Let $\overline{r}(t)$ be a jump of $u(\cdot,t)$ and $R_0 \in (0,1)$ such that $R_0 < \overline{r}(t)$; then there exist m zeros of $u^\varepsilon(\cdot,t)$, which we denote by $1 > r_1^\varepsilon(t) > r_2^\varepsilon(t) > \cdots > r_m^\varepsilon(t) > R_0$, such that*

(*i*) *m is odd;*

(*ii*) $\lim_{\varepsilon\to 0} r_i^\varepsilon(t) = \overline{r}(t)$ *for all $i \in [1,m]$;*

(*iii*) *either $m = 1$ or $|r_i^\varepsilon(t) - r_{i+1}^\varepsilon(t)| \le \varepsilon^{1/4}$, for all $i \in [1, m-1]$;*

(*iv*) *$\nu(\overline{r})$ and $u_r^\varepsilon(r_m^\varepsilon, t)$ have the same sign*;

(*v*) *if ρ is a zero of u^ε such that $\rho > r_1^\varepsilon$ or $\rho < r_m^\varepsilon$ or if ρ is equal R_0 or 1, then we have $|r_1^\varepsilon - \rho| \ge \varepsilon^{1/4}$ and $|r_m^\varepsilon - \rho| \ge \varepsilon^{1/4}$.*

Proof. We first note that (i) implies (iv). Suppose for instance that $\nu(\overline{r}) = 1$. By symmetry one can obtain the same result in the case that $\nu(\overline{r}) = -1$. Moreover, since the time is fixed we omit the variable t in the proof. Let $R_0 < s_M^\varepsilon < \cdots < s_1^\varepsilon < 1$ be the zeros of $u^\varepsilon(\cdot,t)$. Extracting a subsequence, which we denote again by ε, we may suppose that $\{s_i^\varepsilon\}$ converges to a limit λ_i for all $i \in [1,M]$. We denote by $r_1^\varepsilon > \cdots > r_l^\varepsilon$ the zeros of $u^\varepsilon(\cdot,t)$, which converge to $\overline{r}$. Thus there exists $\eta > 0$ such that $\overline{r}$ is the only limit in $[\overline{r}-\eta, \overline{r}+\eta] \subset [R_0, 1]$ and such that for ε small enough $r_1^\varepsilon, \ldots, r_l^\varepsilon$ are the only zeros of $u^\varepsilon(\cdot,t)$ in $[\overline{r}-\frac{\eta}{2}, \overline{r}+\frac{\eta}{2}]$. Suppose that $\frac{\eta}{2} \ge \varepsilon^{1/4}$; this in particular implies that $r_1^\varepsilon \le 1 - \varepsilon^{1/4}$ and $r_l^\varepsilon \ge R_0 + \varepsilon^{1/4}$.

Using the fact that $u(\cdot,t)$ is constant between two jumps and that $\nu(\overline{r}) = 1$, we deduce that $u^\varepsilon(\cdot,t) \to -1$ almost everywhere on $[\overline{r}-\eta, \overline{r})$ and $u^\varepsilon(\cdot,t) \to 1$ almost everywhere on $(\overline{r}, \overline{r}+\eta]$. Using the fact that u^ε has a constant sign on $(r_1^\varepsilon, \overline{r}+\eta]$ and that u^ε converges to 1 we have that $u^\varepsilon > 0$ on $(r_1^\varepsilon, \overline{r}+\eta]$. Similarly we have that $u^\varepsilon < 0$ on $[\overline{r}-\eta, r_l^\varepsilon)$. This in particular implies that l is odd and $u_r^\varepsilon(r_l^\varepsilon) > 0$. Next we prove that we can find m zeros of u^ε satisfying the properties (i), (ii), (iii), (iv) and (v) of Theorem 4.7.

If $l = 1$, then we set $m := l = 1$, and moreover, r_1^ε satisfies the properties (i)–(v). If $|r_1^\varepsilon - r_2^\varepsilon| > \varepsilon^{1/4}$, we set $m = 1$; thus, properties (i)–(v) are obvious. If we have $|r_i^\varepsilon - r_{i+1}^\varepsilon| \le \varepsilon^{1/4}$ for all $i \in [1, l-1]$, then we set $m := l$ and thus we deduce that the zeros $r_1^\varepsilon > \cdots > r_{l=m}^\varepsilon$ satisfy the properties (i)–(v). Otherwise, let k be such that

$$|r_i^\varepsilon - r_{i+1}^\varepsilon| \le \varepsilon^{1/4} \text{ for all } i \in [1, k-1], \quad \text{and} \quad |r_k^\varepsilon - r_{k+1}^\varepsilon| > \varepsilon^{1/4}.$$

There are two subcases. We first consider the case where k is odd. Setting $m := k$ we have thus found m zeros $r_1^\varepsilon > \cdots > r_{k=m}^\varepsilon$ of u^ε satisfying the properties (i)–(v) of Theorem 4.7. Furthermore, if k is even, then $k < l$, and so $k+1$ is odd and $u_r^\varepsilon(r_{k+1}^\varepsilon) > 0$. Moreover, there are $(l-k)$ zeros left over, and since $(l-k)$ is odd, we may begin again starting from r_{k+1}^ε, and we argue in the same way. Since the zeros of u^ε are finite this process has to stop. Thus there exist m zeros of u^ε, which we denote by $r_1^\varepsilon > \cdots > r_m^\varepsilon$, satisfying the properties (i)–(v). This completes the proof of Theorem 4.7.

5. First approximation

Theorem 5.1. 1. *Suppose that there exist two successive zeros $r_-^\varepsilon(t)$ and $r_+^\varepsilon(t)$ of $u^\varepsilon(\cdot,t)$ such that $[r_-^\varepsilon(t), r_+^\varepsilon(t)] \subset (R_0, 1)$, and the sign of u^ε is constant in $(r_-^\varepsilon(t), r_+^\varepsilon(t))$. Then*

$$(i) \quad \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[r_+^\varepsilon(t) - r_-^\varepsilon(t)] = +\infty; \tag{5.1}$$

$$(ii) \quad \lim_{\varepsilon\to 0} \sup_{r_-^\varepsilon \le r \le r_+^\varepsilon} [sgn(u^\varepsilon(r,t))\,(u^\varepsilon(r,t) - u_0^\varepsilon(r,t))]_- = 0, \tag{5.2}$$

where u_0^ε is defined by $u_0^\varepsilon(r,t) := sgn(u^\varepsilon(r,t))\Big\{[1-\xi^\varepsilon(r,t)]\tanh(\frac{r-r_-^\varepsilon(t)}{\varepsilon}) - [\xi^\varepsilon(r,t)]\tanh(\frac{r-r_+^\varepsilon(t)}{\varepsilon})\Big\}$, where ξ^ε is a smooth function on $(r_-^\varepsilon(t), r_+^\varepsilon(t))$, such that $0 \le \xi^\varepsilon \le 1$ and

$$\xi^\varepsilon(r,t) = \begin{cases} 0 & \text{if } \; r_-^\varepsilon < r < \frac{r_-^\varepsilon + r_+^\varepsilon}{2} - \varepsilon \\ 1 & \text{if } \; \frac{r_-^\varepsilon + r_+^\varepsilon}{2} + \varepsilon < r < r_+^\varepsilon. \end{cases}$$

2. *Suppose that $r^\varepsilon(t)$ is a zero of $u^\varepsilon(\cdot,t)$, that there exists $a^\varepsilon \in (r^\varepsilon(t), 1)$ such that $a^\varepsilon - r^\varepsilon(t) \ge \varepsilon^{1/4}$, and that u^ε does not vanish on the interval $(r^\varepsilon(t), a^\varepsilon)$.*

Then

$$\lim_{\varepsilon\to 0}\sup_{[r^\varepsilon(t),r^\varepsilon(t)+\varepsilon^{2/5}]}\Big[sgn(u^\varepsilon(r,t))\Big(u^\varepsilon(r,t) - sgn(u^\varepsilon(r,t))\tanh(\frac{r-r^\varepsilon(t)}{\varepsilon})\Big)\Big]_- = 0. \tag{5.3}$$

3. *Suppose that* $r^\varepsilon(t)$ *is a zero of* $u^\varepsilon(\cdot,t)$*, that there exists* $a^\varepsilon \in (R_0, r^\varepsilon(t))$ *such that* $r^\varepsilon(t) - a^\varepsilon \geq \varepsilon^{1/4}$*, and that* u^ε *does not vanish on the interval* $(a^\varepsilon, r^\varepsilon(t))$*. Then*

$$\lim_{\varepsilon\to 0}\sup_{[r^\varepsilon(t)-\varepsilon^{2/5},r^\varepsilon(t)]}\Big[sgn(u^\varepsilon(r,t))\Big(u^\varepsilon(r,t) + sgn(u^\varepsilon(r,t))\tanh(\frac{r-r^\varepsilon(t)}{\varepsilon})\Big)\Big]_- = 0. \tag{5.4}$$

Note that ξ^ε is well defined since by (i) we have $r_+^\varepsilon - r_-^\varepsilon > 2\varepsilon$ for ε small enough.

In view of Theorem 3.3 the function $U^\varepsilon(z,t) = u^\varepsilon(r,t)$ where $z = \frac{r}{\varepsilon}$ satisfies

$$\|\frac{1}{2}(U_z^\varepsilon(\cdot,t))^2 - F(U^\varepsilon(\cdot,t))\|_{L^\infty(\frac{R_0}{\varepsilon},\frac{1}{\varepsilon})} \leq \tilde{C}(R_0,t)\varepsilon^{1/2-\gamma}.$$

In order to prove Theorem 5.1 we first state some properties of the solutions of the differential equation

$$(E)\ (\varphi'(z))^2 - 2F(\varphi(z)) = g(z),$$

where g is a smooth function such that $\|g\|_\infty \leq \delta^2$ and $0 < \delta \leq \delta_0 < 1/4$.

Lemma 5.2. *Suppose that* ϕ *is a solution of equation (E) such that* ϕ *has two successive zeros* z_- *and* z_+ *and that* ϕ *is positive on* (z_-, z_+)*. Then there exists* $b_0 = b_0(\delta)$*, a function of* δ *such that* $\lim_{\delta\to 0} b_0(\delta) = +\infty$*, and a positive constant* K *such that*

$$z_+ - z_- > 2(b_0+1) \to +\infty \quad \textit{as } \delta \downarrow 0, \tag{5.5}$$

$$|\phi(z) - \tanh(z-z_-)| \leq K\sqrt{\delta}, \quad \textit{for all } z \in [z_-, z_- + b_0], \tag{5.6}$$

$$|\phi(z) + \tanh(z-z_+)| \leq K\sqrt{\delta}, \quad \textit{for all } z \in [z_+ - b_0, z_+], \tag{5.7}$$

$$\phi(z) \geq 1 - K\sqrt{\delta}, \quad \textit{for all } z \in [z_- + b_0, z_+ - b_0]. \tag{5.8}$$

Moreover, for all points c *such that* $\phi_z(c) = 0$ *we have that*

$$c \in [z_- + b_0, z_+ - b_0]. \tag{5.9}$$

Proof. In order to simplify the proof we first suppose that $\phi(0) = 0$ and will later perform a translation to insure that ϕ satisfies the hypotheses of Lemma 5.2. We denote by (E_+) the equation

$$(E_+)\ \varphi'(z) = +\sqrt{2F(\varphi(z)) + g(z)}.$$

ϕ is the solution of (E_+) such that $\phi(0) = 0$ in a maximal interval (β', β). Thus, $\phi'(0) = \sqrt{1 + g(0)}$ and ϕ is strictly increasing on (β', β). We set $X_\delta := \{z \in [0, \beta) : F(\phi(z)) > \delta\}$ and $b := \sup\{x < \beta : (0, x) \subset X_\delta\}$. Then as z varies from 0 to b

$$\begin{aligned} &\phi \text{ increases from 0 to } \lim_{z\to b} \phi(z) \text{ and} \\ &F(\phi) \text{ decreases from 1 to } \lim_{z\to b} F(\phi(z)) \geq \delta. \end{aligned} \tag{5.10}$$

Thus, we have either $\phi^2(z) \leq 1 - \sqrt{2\delta}$, for all $z \in [0, b)$, or $\phi^2(z) \geq 1 + \sqrt{2\delta}$, for all $z \in [0, b)$. Since $\phi(0) = 0$ and ϕ is increasing on $[0, b)$ we deduce that

$$0 < \phi(z) < \sqrt{1 - \sqrt{2\delta}}, \text{ for all } z \in (0, b). \tag{5.11}$$

Moreover, we deduce from (E_+) that

$$\phi'(z) \geq \sqrt{2\delta - \delta^2}, \text{ for all } z \in (0, b). \tag{5.12}$$

Using the mean value theorem, (5.11) and (5.12) we obtain that $z\sqrt{2\delta - \delta^2} \leq \sqrt{1 - \sqrt{2\delta}}$ for all $z \in (0, b)$. Letting z tend to b we deduce that

$$b \leq \frac{\sqrt{1 - \sqrt{2\delta}}}{\sqrt{2\delta - \delta^2}}, \tag{5.13}$$

which implies that b is finite. Therefore, using the continuity of both the functions ϕ' and $F(\phi)$ at the point b and the results (5.12) and (5.10) we deduce that

$$\phi'(b) \geq \sqrt{2\delta - \delta^2} \quad \text{and} \quad F(\phi(b)) \geq \delta. \tag{5.14}$$

Furthermore, since X_δ is open we deduce that $b \notin X_\delta$, which using (5.14) implies that $F(\phi(b))) = \delta$. Thus, we deduce that

$$\phi(b) = \sqrt{1 - \sqrt{2\delta}}. \tag{5.15}$$

Next we prove the inequality

$$|\phi(z) - \tanh(z)| \leq \frac{1}{2}\delta z, \text{ for all } z \in [0, b]. \tag{5.16}$$

Using equality (E_+) we deduce that

$$\frac{\phi'(z)}{\sqrt{2F(\phi(z))}} = \sqrt{1 + \frac{g(z)}{2F(\phi(z))}}. \tag{5.17}$$

Integrating this on $[0, z]$ we obtain that

$$\phi(z) = \tanh\Big(\int_0^z \sqrt{1 + \frac{g(\xi)}{2F(\phi(\xi))}}\, d\xi\Big).$$

Thus, using the fact that $|\tanh'(s)| < 1$ for all $s \in R$, we have

$$|\phi(z) - \tanh(z)| \le \Big|\int_0^z \Big[\sqrt{1 + \frac{g(\xi)}{2F(\phi(\xi))}} - 1\Big]\, d\xi\Big|, \quad \text{for all } z \in [0, b].$$

This in turn implies

$$\begin{aligned} |\phi(z) - \tanh(z)| &\le z \sup_{[0,b]} \Big|\sqrt{1 + \frac{g(\xi)}{2F(\phi(\xi))}} - 1\Big| \\ &\le z\|g\|_{L^\infty(0,b)} \sup_{[0,b]} \Big|\frac{1}{2F(\phi(\cdot))}\Big|. \end{aligned}$$

Using (5.10) and the fact that $\|g\|_\infty \le \delta^2$, we deduce

$$|\phi(z) - \tanh(z)| \le \frac{z}{2}\delta, \text{ for all } z \in [0, b],$$

which coincides with (5.16). In view of (5.13) this implies that

$$|\phi(z) - \tanh(z)| \le C_1\sqrt{\delta} \text{ for all } z \in [0, b], \tag{5.18}$$

for some constant $C_1 > 0$. Letting z tend to b in (5.18) and using (5.15) we deduce that $|\sqrt{1 - \sqrt{2\delta}} - \tanh(b)| \le C_1\sqrt{\delta}$. This implies that

$$\tanh(b) \ge \sqrt{1 - \sqrt{2\delta}} - C_1\sqrt{\delta} \ge 1 - C_2\sqrt{\delta}. \tag{5.19}$$

We set

$$b_0 := \arg\tanh(1 - C_2\sqrt{\delta}) - 1. \tag{5.20}$$

We first note that this definition implies that

$$b_0 \to +\infty \text{ as } \delta \downarrow 0. \tag{5.21}$$

Using (5.19) and (5.20) we deduce that $b_0 + 1 \le b$. Therefore, using (5.18) we have $|\phi(z) - \tanh(z)| \le C_1\sqrt{\delta}$ for all $z \in [0, b_0]$. Next we apply the results above, supposing that the zeros of the function ϕ are given by z_- and z_+

and that ϕ is strictly positive on $[z_-, z_+]$. This gives that there exist b_- and b_+ and b_0 a function of δ defined by (5.20) such that

$$z_+ - z_- > 2(b_0 + 1), \tag{5.22}$$

$$|\phi(z) - \tanh(z - z_-)| \leq C_1\sqrt{\delta}, \text{ for all } z \in [z_-, z_- + b_0], \tag{5.23}$$

$$|\phi(z) + \tanh(z - z_+)| \leq C_1\sqrt{\delta}, \text{ for all } z \in [z_+ - b_0, z_+], \tag{5.24}$$

$$\phi(z_- + b_-) = \phi(z_+ - b_+) = \sqrt{1 - \sqrt{2\delta}}. \tag{5.25}$$

In view of (5.22) and (5.21) we obtain (5.5). Moreover, (5.23) and (5.24) coincide with (5.6) and (5.7). Next we prove (5.8); more precisely, we first prove that

$$\phi(z) \geq 1 - K\sqrt{\delta}, \text{ for all } z \in [z_- + b_0, z_- + b_-] \cup [z_+ - b_+, z_+ - b_0]. \tag{5.26}$$

Using the fact that ϕ increases on $[z_- + b_0, z_- + b_-]$ and (5.23) we have that

$$\phi(z) \geq \phi(z_- + b_0) \geq \tanh(z_- + b_0 - z_-) - C_1\sqrt{\delta} = \tanh(b_0) - C_1\sqrt{\delta} \tag{5.27}$$

for all $z \in [z_- + b_0, z_- + b_-]$. Moreover, we have by (5.20) that

$$\begin{aligned} 1 - \tanh(b_0) &= 1 - \tanh(b_0 + 1 - 1) \\ &= \frac{1 - \tanh(1)[1 - C_2\sqrt{\delta}] - [1 - C_2\sqrt{\delta}] + \tanh(1)}{1 - \tanh(1)[1 - C_2\sqrt{\delta}]} \\ &\leq \frac{C_2\sqrt{\delta}(\tanh(1) + 1)}{1 - \tanh(1)}. \end{aligned}$$

Thus, we deduce that $\tanh(b_0) \geq 1 - C_3\sqrt{\delta}$. Substituting this into (5.27) we deduce that $\phi(z) \geq 1 - C_4\sqrt{\delta}$ for all $z \in [z_- + b_0, z_- + b_-]$. Similarly, using (5.24) one can show that $\phi(z) \geq 1 - C_4\sqrt{\delta}$ for all $z \in [z_+ - b_+, z_+ - b_0]$. Therefore, we deduce (5.26). Next we prove a result, which will be useful to obtain a lower bound of ϕ on $[z_- + b_-, z_+ - b_+]$; namely, suppose that there exist z_1 and z_2 such that $\phi(z_1) = \phi(z_2) = \sqrt{1 - \sqrt{2\delta}}$ and that ϕ does not vanish on $[z_1, z_2]$; then

$$\phi(z) \geq \sqrt{1 - \sqrt{2\delta}} \quad \text{for all } z \in [z_1, z_2]. \tag{5.28}$$

We prove (5.28) by contradiction. Let $c \in [z_1, z_2]$ satisfy

$$\phi(c) = \inf_{z \in [z_1, z_2]} \phi(t).$$

We suppose that $c \in (z_1, z_2)$; then $\phi'(c) = 0$. In view of equality (E) and the fact that $\|g\|_\infty \leq \delta^2$, we have $|1 - \phi(c)^2|^2 \leq \delta^2$. By the definition of c we have $0 < \phi(c) \leq \phi(z_1) < 1$, and thus we deduce that $\phi(c) \geq \sqrt{1 - \delta}$. This implies a contradiction with the fact that $\phi(c) \leq \phi(z_1) \leq \sqrt{1-\sqrt{2\delta}}$. Therefore, we conclude that $c = z_1$ or $c = z_2$, and thus $\inf_{z\in[z_1,z_2]} \phi(z) = \sqrt{1-\sqrt{2\delta}}$. This completes the proof of (5.28).

Applying (5.28) with the points $z_- + b_-$ and $z_+ - b_+$ and using (5.25) we obtain that $\phi(z) \geq \sqrt{1-\sqrt{2\delta}}$ for all $z \in [z_- + b_-, z_+ - b_+]$. This with (5.26) implies (5.8). Moreover, we have also shown that for all points c such that $\phi_z(c) = 0$, we have $c \in [z_- + b_0, z_+ - b_0]$, which implies (5.9). This completes the proof of Lemma 5.2. □

Next we prove two lemmas, which will be useful in proving results 2 and 3 of Theorem 5.1.

Lemma 5.3. *Let ϕ be a solution of (E) and I an interval such that $0 < \phi(z) \leq \sqrt{1-\sqrt{2\delta}}$ for all $z \in I$. Then we have meas$(I) \leq \frac{2}{\sqrt{\delta}}$ for δ small enough.*

Proof. We prove Lemma 5.3 in the case that $\phi > 0$ on I. We set z_0 the middle of I and $\mu_0 = \tanh^{-1}(\phi(z_0))$. Integrating (5.17) on $[z_0, z_0 + z]$ we obtain an equivalent result of (5.16), namely,

$$|\phi(z_0 + z) - \tanh(\mu_0 + z)| \leq \frac{\delta}{2}|z|, \text{ for all } z \text{ such that } z + z_0 \in I. \quad (5.29)$$

Let z_1 be such that $\tanh(\mu_0+z_1) = -\sqrt{\delta}$. Since $0 \leq \tanh(\mu_0) \leq \sqrt{1-\sqrt{2\delta}} \leq \tanh(\frac{1}{2\sqrt{\delta}})$ we have $\mu_0 \leq \frac{1}{2\sqrt{\delta}}$; moreover, we have $|\mu_0 + z_1| = \tanh^{-1}(\sqrt{\delta}) \leq 2\sqrt{\delta}$. This gives that $|z_1| \leq \frac{1}{\sqrt{\delta}}$ for $\delta \in (0, 1/4)$. Supposing that $z_0 + z_1 \in I$ and applying (5.29) at the point z_1 we deduce that

$$|\phi(z_0 + z_1) + \sqrt{\delta}| \leq \frac{\sqrt{\delta}}{2}.$$

This implies that $\phi(z_0+z_1) < 0$, and thus ϕ vanishes in I, which is impossible. Therefore, we conclude that $z_0 + z_1 \notin I$, and then $meas(I) < 2|z_1| \leq \frac{2}{\sqrt{\delta}}$. This completes the proof of Lemma 5.3. □

Lemma 5.4. *Let ϕ be a solution of (E) and let z_- be a zero of ϕ. Suppose that ϕ is strictly positive on $[z_-, A]$ where $A - z_- > \frac{2}{\sqrt{\delta}}$; then there exists a positive constant K_1 such that*

$$\sup_{[z_-, A-\frac{2}{\sqrt{\delta}}]} [\phi(z) - \tanh(z - z_-)]_- \leq K_1\sqrt{\delta}. \quad (5.30)$$

Proof. In view of the proof of Lemma 5.2 (see (5.15) and (5.18)) we deduce that there exists $b(\delta) \to \infty$ as $\delta \downarrow 0$ such that

$$\begin{aligned}&|\phi(z) - \tanh(z - z_-)| \leq C_1\sqrt{\delta}, \text{ for all } z \in [z_-, z_- + b]\\ &\text{and } \ \phi(b) = \sqrt{1 - \sqrt{2\delta}}.\end{aligned}$$

Therefore, if $A \leq z_- + b$, then Lemma 5.4 follows. Moreover, if $A > z_- + b$ we have only to prove that

$$\sup_{[z_- + b, A - \frac{2}{\sqrt{\delta}}]} [\phi(z) - \tanh(z - z_-)]_- \leq K_1\sqrt{\delta}. \tag{5.31}$$

We first consider the case that $\phi(z) \geq \sqrt{1 - \sqrt{2\delta}}$ for all $z \in [z_- + b, A]$. Since $[\phi(z) - 1]_- = 0$ or $[\phi(z) - 1]_- = -\phi(z) + 1$ we have that $[\phi(z) - 1]_- \leq 1 - \sqrt{1 - \sqrt{2\delta}}$. This gives since $\tanh(z - z_-) \leq \tanh(b)$ that

$$\begin{aligned}&(\phi(z) - \tanh(z - z_-))_- \leq [\phi(z) - 1]_- + |1 - \tanh(z - z_-)|\\ &\leq 1 - \sqrt{1 - \sqrt{2\delta}} + 1 - \tanh(b) \leq 2(1 - \sqrt{1 - \sqrt{2\delta}}) \leq D\sqrt{\delta}\end{aligned}$$

for all $z \in [z_- + b, A]$, which implies (5.31).

We now consider the case that there exists a point $z_1 \in [z_- + b, A]$ such that $\phi(z_1) < \sqrt{1 - \sqrt{2\delta}}$, and we set z_2 the first point of $(z_- + b, A]$ such that $\phi(z_2) = \sqrt{1 - \sqrt{2\delta}}$. Using (5.28) we deduce that $\phi(z) \geq \sqrt{1 - \sqrt{2\delta}}$ for all $z \in [z_- + b, z_2]$, and we conclude as previously that

$$\sup_{[z_- + b, z_2]} [\phi(z) - \tanh(z - z_-)]_- \leq D\sqrt{\delta}. \tag{5.32}$$

Next we check by contradiction that $0 < \phi(z) < \sqrt{1 - \sqrt{2\delta}}$ for all $z \in [z_2, A]$. Suppose that there exists $z_3 \in (z_2, A]$ such that $\phi(z_3) = \sqrt{1 - \sqrt{2\delta}}$. Thus, using the result (5.28) we obtain that $\phi(z) \geq \sqrt{1 - \sqrt{2\delta}}$ for all $z \in [z_2, z_3]$, which implies a contradiction with the fact that $\phi'(z_2) < 0$. Therefore, we deduce that $0 < \phi(z) < \sqrt{1 - \sqrt{2\delta}}$ for all $z \in [z_2, A]$, which in view of Lemma 5.3 gives that $A - z_2 \leq \frac{2}{\sqrt{\delta}}$. Since by assumption we have $A - z_- > \frac{2}{\sqrt{\delta}}$ this implies that $A - \frac{2}{\sqrt{\delta}} \in (z_-, z_2]$, which in view of (5.32) gives (5.31). This completes the proof of Lemma 5.4. □

We are now in position to prove Theorem 5.1.

Proof of Theorem 5.1. We consider only the case where the sign of u^ε is positive on $(r_-^\varepsilon(t), r_+^\varepsilon(t))$. We set $U^\varepsilon(z, t) := u^\varepsilon(r, t)$ where $z = \frac{r}{\varepsilon}$. In view of Theorem 3.3 we deduce that we can apply Lemma 5.2 with the function

U^ε and with $\delta = [\tilde{C}(R_0,t)\varepsilon^{1/2-\gamma}]^{1/2}$. Thus we obtain that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[r_+^\varepsilon(t) - r_-^\varepsilon(t)] = +\infty,$$

which coincides with (5.1). Next we check that

$$\lim_{\varepsilon\to 0} \sup_{r_-^\varepsilon \le r \le r_+^\varepsilon} [u^\varepsilon(r,t) - u_0^\varepsilon(r,t)]_- = 0.$$

We first consider the case where $r \in [r_-^\varepsilon, r_-^\varepsilon + \varepsilon b_0]$. Using (5.5) we deduce that $[r_-^\varepsilon, r_-^\varepsilon + \varepsilon b_0] \subset [r_-^\varepsilon, \frac{r_-^\varepsilon + r_+^\varepsilon}{2} - \varepsilon]$, which implies that $u_0^\varepsilon(r,t) = \tanh(\frac{r - r_-^\varepsilon}{\varepsilon})$ for all $r \in [r_-^\varepsilon, r_-^\varepsilon + \varepsilon b_0]$. So we deduce from (5.6) that

$$\lim_{\varepsilon\to 0} |u^\varepsilon(r,t) - u_0^\varepsilon(r,t)| = 0, \text{ uniformly in } [r_-^\varepsilon, r_-^\varepsilon + \varepsilon b_0]. \tag{5.33}$$

Similarly we deduce from (5.7) that

$$\lim_{\varepsilon\to 0} |u^\varepsilon(r,t) - u_0^\varepsilon(r,t)| = 0, \text{ uniformly in } [r_+^\varepsilon - \varepsilon b_0, r_+^\varepsilon]. \tag{5.34}$$

Next we prove that

$$\lim_{\varepsilon\to 0} \sup_{[r_-^\varepsilon + \varepsilon b_0, r_+^\varepsilon - \varepsilon b_0]} |u_0^\varepsilon - 1| = 0. \tag{5.35}$$

Using the definition of u_0^ε, we deduce that

$$|u_0^\varepsilon(r,t) - 1| \le \Big|\tanh(\frac{r - r_-^\varepsilon(t)}{\varepsilon}) - 1\Big| + \Big|\tanh(\frac{r - r_-^\varepsilon(t)}{\varepsilon}) + \tanh(\frac{r - r_+^\varepsilon(t)}{\varepsilon})\Big|. \tag{5.36}$$

Moreover, we have $\tanh(\frac{r - r_-^\varepsilon(t)}{\varepsilon}) \ge \tanh(b_0)$ and $\tanh(\frac{r - r_+^\varepsilon(t)}{\varepsilon}) \le -\tanh(b_0)$, for all $r \in [r_-^\varepsilon + \varepsilon b_0, r_+^\varepsilon - \varepsilon b_0]$. This together with (5.5) implies that

$$\lim_{\varepsilon\to 0} \sup_{[r_-^\varepsilon + \varepsilon b_0, r_+^\varepsilon - \varepsilon b_0]} |\tanh(\frac{r - r_-^\varepsilon(t)}{\varepsilon}) - 1| = 0, \tag{5.37}$$

$$\lim_{\varepsilon\to 0} \sup_{[r_-^\varepsilon + \varepsilon b_0, r_+^\varepsilon - \varepsilon b_0]|} |\tanh(\frac{r - r_+^\varepsilon(t)}{\varepsilon}) + 1| = 0. \tag{5.38}$$

Letting ε tend to zero in (5.36) and using (5.37) and (5.38) we deduce (5.35). Moreover, we have that

$$[u^\varepsilon(r,t) - u_0^\varepsilon(r,t)]_- \le [u^\varepsilon(r,t) - 1]_- + |u_0^\varepsilon(r,t) - 1|. \tag{5.39}$$

Letting ε tend to zero in (5.39) and using (5.8), (5.35) we obtain that

$$\lim_{\varepsilon\to 0} [u^\varepsilon(r,t) - u_0^\varepsilon(r,t)]_- = 0, \text{ for all } r \in [r_-^\varepsilon + \varepsilon b_0, r_+^\varepsilon - \varepsilon b_0]. \tag{5.40}$$

Finally, using (5.33), (5.34) and (5.40) we deduce (5.2). This completes the proof of the first part of Theorem 5.1. Since the proofs of the results 2 and

3 of Theorem 5.1 are very similar we only prove result 2, and moreover we suppose that u^ε is positive on $[r^\varepsilon(t), a^\varepsilon]$. By assumption we have $\frac{a^\varepsilon - r^\varepsilon(t)}{\varepsilon} \geq \varepsilon^{-3/4}$, and since $U^\varepsilon(z,t) = u^\varepsilon(r,t)$ is strictly positive on $[\frac{r^\varepsilon(t)}{\varepsilon}, \frac{a^\varepsilon}{\varepsilon}]$ we may apply Lemma 5.4 with $A = \frac{a^\varepsilon}{\varepsilon}$ and $\delta = C(R_0,t)\varepsilon^{1/4-\gamma/2}$, and we deduce that

$$\left[U^\varepsilon(z,t) - \tanh(z - \frac{r^\varepsilon(t)}{\varepsilon})\right]_- \leq K_1 \varepsilon^{1/8-\gamma/4}$$

for all $z \in [\frac{r^\varepsilon(t)}{\varepsilon}, \frac{r^\varepsilon(t)}{\varepsilon} + \varepsilon^{-3/5}]$. After rescaling this implies that

$$[u^\varepsilon(r,t) - \tanh(r - r^\varepsilon(t))]_- \leq K_1 \varepsilon^{1/8-\gamma/4}$$

for all $r \in [r^\varepsilon(t), r^\varepsilon(t) + \varepsilon^{2/5}]$, which coincides with (5.3). This completes the proof of Theorem 5.1. □

Next we give two corollaries, which will be useful in the following.

Corollary 5.5. *We set $U^\varepsilon(z,t) = u^\varepsilon(r,t)$ where $z = \frac{r}{\varepsilon}$, and we suppose that U^ε has two successive zeros z_-^ε and z_+^ε; then there exists a point b_0^ε such that $z_+ - z_- > 2(b_0^\varepsilon + 1)$ and $\lim_{\varepsilon\to 0} b_0^\varepsilon = +\infty$. Moreover, for all points c^ε satisfying $u_r^\varepsilon(c^\varepsilon, t) = 0$ we have $c^\varepsilon \in [z_-^\varepsilon + b_0^\varepsilon, z_+^\varepsilon - b_0^\varepsilon]$.*

Proof. Applying (5.9) with the function U^ε and with $\delta = [C(R_0,t)\varepsilon^{1/2-\gamma}]^{1/2}$ we deduce that there exists $b_0(\delta)$, which we denote by b_0^ε, satisfying $z_+ - z_- > 2(b_0^\varepsilon + 1)$ and $\lim_{\varepsilon\to 0} b_0^\varepsilon = +\infty$ and such that for all points c^ε satisfying $u_r^\varepsilon(c^\varepsilon, t) = 0$ we have $c^\varepsilon \in [z_-^\varepsilon + b_0^\varepsilon, z_+^\varepsilon - b_0^\varepsilon]$, which completes the proof of Corollary 5.5.

Corollary 5.6. *u_0^ε satisfies the following estimates:*

$$\lim_{\varepsilon\to 0} \sup_{[\frac{r_-^\varepsilon + r_+^\varepsilon}{2} - \varepsilon, \frac{r_-^\varepsilon + r_+^\varepsilon}{2} + \varepsilon]} |u_0^\varepsilon - 1| = 0, \tag{5.41}$$

$$\lim_{\varepsilon\to 0} \sup_{[r_-^\varepsilon + \varepsilon b_0, r_+^\varepsilon - \varepsilon b_0]} |(u_0^\varepsilon)_r| = 0. \tag{5.42}$$

Consequently, we also have

$$\lim_{\varepsilon\to 0} \sup_{[\frac{r_-^\varepsilon + r_+^\varepsilon}{2} - \varepsilon, \frac{r_-^\varepsilon + r_+^\varepsilon}{2} + \varepsilon]} |(u_0^\varepsilon)_r| = 0. \tag{5.43}$$

Proof. First we note that

$$[\frac{r_-^\varepsilon + r_+^\varepsilon}{2} - \varepsilon, \frac{r_-^\varepsilon + r_+^\varepsilon}{2} + \varepsilon] \subset [r_-^\varepsilon + \varepsilon b_0, r_+^\varepsilon - \varepsilon b_0]. \tag{5.44}$$

This together with (5.35) implies (5.41). Next we prove (5.42). By definition of u_0^ε we have that

$$|(u_0^\varepsilon)_r(r,t)| \leq \|\xi_r^\varepsilon\|_{L^\infty} \Big| \tanh(\frac{r-r_-^\varepsilon(t)}{\varepsilon}) + \tanh(\frac{r-r_+^\varepsilon(t)}{\varepsilon}) \Big| + \Big| 1-\tanh^2(\frac{r-r_-^\varepsilon(t)}{\varepsilon}) \Big| + \Big| 1-\tanh^2(\frac{r-r_+^\varepsilon(t)}{\varepsilon}) \Big|.$$

Using (5.37) and (5.38) we deduce that $\lim_{\varepsilon\to 0}\sup_{[r_-^\varepsilon+\varepsilon b_0, r_+^\varepsilon-\varepsilon b_0]} |(u_0^\varepsilon)_r| = 0$, which coincides with (5.42). Finally, using (5.44) we obtain (5.43). This completes the proof of Corollary 5.6.

6. Construction of the approximation

Let $t \in (0,T)\setminus D(T)$, where $D(T)$ has been defined in Section 4 and such that $u^\varepsilon(\cdot,t)$ tends to ± 1 in $L^1(\Omega)$ and almost everywhere in Ω. Let $\overline{r}(t)$ be a jump of $u(\cdot,t)$; there exists $R_0 > 0$ such that $\overline{r}(t) > R_0$. In view of Section 4 there exist m zeros $r_1^\varepsilon > r_2^\varepsilon > \cdots > r_m^\varepsilon$ of $u^\varepsilon(\cdot,t)$ satisfying the properties (i)–(v) of Theorem 4.7. We introduce a new variable, namely, $z := \frac{r-r_1^\varepsilon}{\varepsilon}$. Moreover, we set $z_-^\varepsilon := \frac{R_0-r_1^\varepsilon}{\varepsilon}$, $z_+^\varepsilon := \frac{1-r_1^\varepsilon}{\varepsilon}$ and $z_i^\varepsilon := \frac{r_i^\varepsilon-r_1^\varepsilon}{\varepsilon}$; thus, we have $z_-^\varepsilon < z_m^\varepsilon < z_{m-1}^\varepsilon < \cdots < z_1^\varepsilon = 0 < z_+^\varepsilon$. Furthermore by the property (iii) of Theorem 4.7 we have that

$$|z_i^\varepsilon - z_{i+1}^\varepsilon| \leq \varepsilon^{-3/4}, \text{ for all } i \in [1, m-1]. \tag{6.1}$$

From now on we use capital letters for functions defined in this variable, so that $U^\varepsilon(z) := U^\varepsilon(z,t) = u^\varepsilon(r,t)$, $W^\varepsilon(z) := W^\varepsilon(z,t) = w^\varepsilon(r,t)$, and $V^\varepsilon(z) := V^\varepsilon(z,t) = v^\varepsilon(r,t)$. In the z variable, equation (3.2) becomes

$$-U_{zz}^\varepsilon - f(U^\varepsilon) - (N-1)\frac{\varepsilon}{\varepsilon z + r_1^\varepsilon} U_z^\varepsilon + \varepsilon V^\varepsilon - \varepsilon W^\varepsilon = 0. \tag{6.2}$$

Next we give some definitions:

1. Let $\{E_i^\varepsilon\}_{1\leq i\leq m}$ be a partition of unity such that the function E_i^ε has support in $(\frac{z_{i+1}^\varepsilon+z_i^\varepsilon}{2}-1, \frac{z_i^\varepsilon+z_{i-1}^\varepsilon}{2}+1)$ and $E_i^\varepsilon = 1$ on $[\frac{z_{i+1}^\varepsilon+z_i^\varepsilon}{2}+1, \frac{z_i^\varepsilon+z_{i-1}^\varepsilon}{2}-1]$ for all $i \in [2, m-1]$, while E_1^ε has support in $(\frac{z_2}{2}-1, \infty)$ and $E_1^\varepsilon = 1$ on $(\frac{z_2}{2}+1, \infty)$; E_m has support in $(-\infty, \frac{z_m^\varepsilon+z_{m-1}^\varepsilon}{2}+1)$ and $E_m = 1$ on $(-\infty, \frac{z_m^\varepsilon+z_{m-1}^\varepsilon}{2}-1)$. Moreover, we suppose that $(E_i^\varepsilon)_z$ and $(E_i^\varepsilon)_{zz}$ are uniformly bounded in ε.

2. We set $U_0^\varepsilon(z) := \sum_{i=1}^{i=m} E_i^\varepsilon(z) U_{0i}^\varepsilon(z)$, where $U_{0i}^\varepsilon(z) = (-1)^{i+1}\tanh(z - z_i^\varepsilon)$ and $U_1^\varepsilon(z) := \sum_{i=1}^{i=m} E_i^\varepsilon(z) U_{1i}^\varepsilon(z)$, where U_{1i}^ε is a solution of the system

$$\begin{cases} -(U_{1i}^\varepsilon)_{zz} - f'(U_{0i}^\varepsilon) U_{1i}^\varepsilon = W^\varepsilon - V^\varepsilon \\ U_{1i}^\varepsilon(z_i^\varepsilon) = 0. \end{cases} \tag{6.3}$$

3. Finally we set for all $z \in [z_-^\varepsilon, z_+^\varepsilon]$, $\Theta^\varepsilon(z) := \nu(\overline{r})(U_0^\varepsilon + \varepsilon U_1^\varepsilon)(z)$ and $\Theta_i^\varepsilon(z) := \nu(\overline{r})(U_{0i}^\varepsilon + \varepsilon U_{1i}^\varepsilon)(z)$, where $\nu(\overline{r})$ has been defined in Definition 4.6 and

$$\Psi^\varepsilon(z) := U^\varepsilon(z) - \Theta^\varepsilon(z). \tag{6.4}$$

Next we prove that $U^\varepsilon(\cdot, t)$ is well approximated by $\Theta^\varepsilon(\cdot)$; more precisely, we prove the following result:

Theorem 6.1. *Let ξ^ε be a smooth function such that*

$$\xi^\varepsilon(z) := \begin{cases} 1 \ in \ (-\varepsilon^{-\frac{1}{2}} + z_m^\varepsilon, \varepsilon^{-\frac{1}{2}}) \\ 0 \ in \ R \setminus (-\varepsilon^{-3/5} + z_m^\varepsilon, \varepsilon^{-3/5}), \end{cases}$$

$0 \le \xi^\varepsilon \le 1$ *and* $\|\xi_z^\varepsilon\|_{L^\infty} \le C\varepsilon^{3/5}$. *Then we have*

$$\int_{(-\infty, z_m^\varepsilon) \cup (0,\infty)} (|\Psi_z^\varepsilon|^2 + |\Psi^\varepsilon|^2)(\xi^\varepsilon)^2 \, dz \le C\varepsilon^{6/5},$$

$$\lim_{\varepsilon \to 0} \int_{(z_m^\varepsilon, 0)} (|\Psi_z^\varepsilon|^2 + |\Psi^\varepsilon|^2) \, dz = 0.$$

We first note that $z_-^\varepsilon < z_m^\varepsilon - \varepsilon^{-3/5} < z_m^\varepsilon - \varepsilon^{-1/2} < z_m^\varepsilon < z_{m-1}^\varepsilon < \cdots < z_1^\varepsilon = 0 < \varepsilon^{-1/2} < \varepsilon^{-3/5} < z_+^\varepsilon$ and that there is no zero of U^ε in $[z_m^\varepsilon - \varepsilon^{-3/5}, z_m^\varepsilon)$ and in $(z_1^\varepsilon = 0, \varepsilon^{-3/5}]$. Next we prove preliminary lemmas, which will be useful to prove Theorem 6.1.

Lemma 6.2. *Setting*

$$H^\varepsilon(z) := -\Theta_{zz}^\varepsilon - f(\Theta^\varepsilon) + \varepsilon V^\varepsilon - \varepsilon W^\varepsilon, \tag{6.5}$$

$$G^\varepsilon(z) := (N-1)\frac{\varepsilon}{\varepsilon z + r_1^\varepsilon} U_z^\varepsilon = -U_{zz}^\varepsilon - f(U^\varepsilon) + \varepsilon V^\varepsilon - \varepsilon W^\varepsilon, \tag{6.6}$$

we have

$$\begin{aligned} H^\varepsilon = {} & \varepsilon^2 \sum_{i=1}^{i=m} E_i^\varepsilon \left(6U_{0i}^\varepsilon (U_{1i}^\varepsilon)^2 + 2\varepsilon (U_{1i}^\varepsilon)^3\right) \\ & + \sum_{i=1}^{i=m-1} (\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon) \left[(\zeta_i^\varepsilon)_{zz} + 2E_i^\varepsilon E_{i+1}^\varepsilon \left((1+E_i^\varepsilon)(\Theta_i^\varepsilon)^2\right.\right. \\ & \left.\left. +(E_i^\varepsilon - 2)(\Theta_{i+1}^\varepsilon)^2 + (1-2E_i^\varepsilon)\Theta_i^\varepsilon \Theta_{i+1}^\varepsilon\right)\right] + \sum_{i=1}^{i=m-1} (\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)_z (\zeta_i^\varepsilon)_z, \end{aligned} \tag{6.7}$$

where $\zeta_i^\varepsilon = \sum_{k=1}^{k=i} E_k^\varepsilon$. *Moreover, the function* Ψ_ε *defined by* (6.4) *satisfies*

$$-\Psi_{zz}^\varepsilon - f'(\Theta^\varepsilon)\Psi^\varepsilon = -2\left(3\Theta^\varepsilon(\Psi^\varepsilon)^2 + (\Psi^\varepsilon)^3\right) + G^\varepsilon(z) - H^\varepsilon(z), \tag{6.8}$$

$$\Psi^\varepsilon(z_i^\varepsilon) = 0.$$

Proof. We refer to the Appendix (see Lemma A.2 in the Appendix) for the proof of (6.7). Next we prove (6.8). Subtracting (6.5) from (6.6) we obtain

$$G^\varepsilon - H^\varepsilon = -U_{zz}^\varepsilon + \Theta_{zz}^\varepsilon - f(U^\varepsilon) + f(\Theta^\varepsilon). \tag{6.9}$$

Moreover, we have

$$f(U^\varepsilon) - f(\Theta^\varepsilon) = \Psi^\varepsilon f'(\Theta^\varepsilon) + 2(\Psi^\varepsilon)^2(-3\Theta^\varepsilon - \Psi^\varepsilon). \tag{6.10}$$

Substituting (6.10) into (6.9) we conclude that

$$-\Psi_{zz}^\varepsilon - f'(\Theta^\varepsilon)\Psi^\varepsilon = -2(3\Theta^\varepsilon(\Psi^\varepsilon)^2 + (\Psi^\varepsilon)^3) + G^\varepsilon - H^\varepsilon,$$

which coincides with (6.8).

Next we give a bound for U_1^ε.

Lemma 6.3. *There exists a function $C_1(R_0, t)$ independent of ε such that U_1^ε satisfies*

$$\|U_1^\varepsilon\|_{H^{1,\infty}(z_-^\varepsilon, z_+^\varepsilon)} \le C_1(R_0, t)\varepsilon^{-\gamma/2}. \tag{6.11}$$

Proof. Applying Lemma A.3 in the Appendix with $k(z) = (V^\varepsilon - W^\varepsilon)(z + z_i^\varepsilon, t)$, we obtain for a suitable choice of $U_{1,i}^\varepsilon$ (see (A.14)) that

$$\|U_{1i}^\varepsilon\|_{H^{1,\infty}(z_-^\varepsilon, z_+^\varepsilon)} \le 12\|V^\varepsilon - W^\varepsilon\|_{L^\infty(z_-^\varepsilon, z_+^\varepsilon)} \text{ for all } i \in [1, m]. \tag{6.12}$$

Next we give a bound of W^ε and V^ε. We have

$$|w^\varepsilon(r,t)| \le |w^\varepsilon(s,t)| + \int_{R_0}^1 |w_r^\varepsilon(\sigma, t)|\, d\sigma, \text{ for all } r, s \in [R_0, 1].$$

Integrating this in s on $(R_0, 1)$, we deduce that

$$|w^\varepsilon(r,t)|(1 - R_0) \le (1 - R_0)^{\frac{1}{2}} \Big(\int_{R_0}^1 (w^\varepsilon)^2 \Big)^{\frac{1}{2}} + \frac{(1 - R_0)^{\frac{3}{2}}}{R_0^{\frac{N-1}{2}}} \Big(\int_{R_0}^1 (w_r^\varepsilon)^2 r^{N-1} \Big)^{\frac{1}{2}}. \tag{6.13}$$

In view of (3.15) and (3.16) this implies that

$$\|W^\varepsilon(\cdot, t)\|_{L^\infty(z_-^\varepsilon, z_+^\varepsilon)} \le C_1(R_0, t)\varepsilon^{-\gamma/2}. \tag{6.14}$$

Similarly we obtain an analogous result for v^ε, namely,

$$|v^\varepsilon(r,t)|(1 - R_0) \le (1 - R_0)^{\frac{1}{2}} \Big(\int_{R_0}^1 (v^\varepsilon)^2 \Big)^{\frac{1}{2}} + \frac{(1 - R_0)^{\frac{3}{2}}}{R_0^{\frac{N-1}{2}}} \Big(\int_{R_0}^1 (v_r^\varepsilon)^2 r^{N-1} \Big)^{\frac{1}{2}}, \tag{6.15}$$

which in view of (3.8) and (3.10) implies that

$$\|V^\varepsilon(\cdot, t)\|_{L^\infty(z_-^\varepsilon, z_+^\varepsilon)} \le C_2(R_0). \tag{6.16}$$

Substituting (6.14) and (6.16) into (6.12) we conclude that

$$\|U_{1i}^\varepsilon\|_{H^{1,\infty}(z_-^\varepsilon,z_+^\varepsilon)} \le C_3(R_0,t)\varepsilon^{-\gamma/2}, \text{ for all } i\in[1,m].$$

Finally, using the definition of U_1^ε and the fact that E_i^ε and $(E_i^\varepsilon)_z$ are uniformly bounded in ε we deduce that $\|U_1^\varepsilon\|_{H^{1,\infty}(z_-^\varepsilon,z_+^\varepsilon)} \le C(R_0,t)\varepsilon^{-\gamma/2}$. This completes the proof of Lemma 6.3.

Next we state a lemma, which will be useful in proving Theorem 6.1.

Lemma 6.4.

$$\lim_{\varepsilon\to 0} z_{i+1}^\varepsilon - z_i^\varepsilon = -\infty \tag{6.17}$$

for all $i\in[1,m-1]$. *Moreover,* $\Psi^\varepsilon\Theta^\varepsilon$ *satisfies*

$$\lim_{\varepsilon\to 0}\sup_{[-\varepsilon^{-3/5}+z_m^\varepsilon,\varepsilon^{-3/5}]}[\Psi^\varepsilon\Theta^\varepsilon]_- = 0. \tag{6.18}$$

Proof. (6.17) is a direct consequence of (5.1) (cf. Theorem 5.1). Next we prove (6.18). By definitions of Ψ^ε and Θ^ε we first note that

$$[\Psi^\varepsilon\Theta^\varepsilon]_- \le [(U^\varepsilon-\nu(\overline{r})U_0^\varepsilon)\nu(\overline{r})U_0^\varepsilon]_- + \varepsilon|U^\varepsilon-\nu(\overline{r})U_0^\varepsilon||U_1^\varepsilon| + \varepsilon|U_1^\varepsilon||U_0^\varepsilon| + \varepsilon^2|U_1^\varepsilon|^2.$$

Using (3.13), Lemma 6.3 and the fact that $|U_0^\varepsilon|\le 1$ this implies that

$$[\Psi^\varepsilon\Theta^\varepsilon]_- \le [(U^\varepsilon-\nu(\overline{r})U_0^\varepsilon)\nu(\overline{r})U_0^\varepsilon]_- + \le C(R_0,t)\varepsilon^{\frac{1-\gamma}{2}}. \tag{6.19}$$

Next we prove in the case that $\nu(\overline{r})=1$ that $[(U^\varepsilon-\nu(\overline{r})U_0^\varepsilon)\nu(\overline{r})U_0^\varepsilon]_-$ tends to zero uniformly on $[-\varepsilon^{-3/5}+z_m^\varepsilon,\varepsilon^{-3/5}]$ as $\varepsilon\downarrow 0$; similarly, one can check the case that $\nu(\overline{r})=-1$. By definition of U_0^ε we have, for all $z\in[z_{i+1}^\varepsilon,z_i^\varepsilon]$, and for all $i\in[1,m-1]$,

$$(-1)^i U_0^\varepsilon(z) = E_{i+1}^\varepsilon\tanh(z-z_{i+1}^\varepsilon) - E_i^\varepsilon\tanh(z-z_i^\varepsilon)\ge 0. \tag{6.20}$$

This with the fact that $[ab]_- = [a]_- b$ for $b>0$ gives

$$[(U^\varepsilon-U_0^\varepsilon)U_0^\varepsilon]_- = [(-1)^i(U^\varepsilon-U_0^\varepsilon)(-1)^iU_0^\varepsilon]_- = [(-1)^i(U^\varepsilon-U_0^\varepsilon)]_-|U_0^\varepsilon|.$$

Noting that $(-1)^i$ is equal to the sign of U^ε on $[z_{i+1}^\varepsilon,z_i^\varepsilon]$, which we denote by τ_i, and using (6.20) and the fact that $E_i^\varepsilon+E_{i+1}^\varepsilon=1$ on $[z_{i+1}^\varepsilon,z_i^\varepsilon]$ we deduce that

$$\begin{aligned}&[((U^\varepsilon-U_0^\varepsilon)U_0^\varepsilon)(z)]_-\\ &\le \left[\tau_i(U^\varepsilon(z)-\tau_i[(1-E_i^\varepsilon)\tanh(z-z_{i+1}^\varepsilon)-E_i^\varepsilon\tanh(z-z_i^\varepsilon)])\right]_-.\end{aligned}$$

Therefore, we deduce from the first part of Theorem 5.1

$$[(U^\varepsilon-\nu(\overline{r})U_0^\varepsilon)\nu(\overline{r})U_0^\varepsilon]_- \to 0 \text{ uniformly on } \cup_{i=1}^{i=m-1}[z_{i+1}^\varepsilon,z_i^\varepsilon] = [z_m^\varepsilon,z_1^\varepsilon], \tag{6.21}$$

as $\varepsilon \downarrow 0$. Moreover, since $U_0^\varepsilon(z) = \tanh(z - z_1^\varepsilon) \geq 0$ and $U^\varepsilon \geq 0$ on $[z_1^\varepsilon, \varepsilon^{-3/5}]$ we have

$$[((U^\varepsilon - U_0^\varepsilon)U_0^\varepsilon)(z)]_- = [sgn(U^\varepsilon)(U^\varepsilon(z) - sgn(U^\varepsilon)\tanh(z - z_1^\varepsilon))]_- |U_0^\varepsilon|,$$

for all $z \in [z_1^\varepsilon, \varepsilon^{-3/5}]$. Similarly, since $U_0^\varepsilon(z) = \tanh(z - z_m^\varepsilon) \leq 0$ and $U^\varepsilon \leq 0$ on $[-\varepsilon^{-3/5} + z_m^\varepsilon, z_m^\varepsilon]$, we have

$$\begin{aligned}[((U^\varepsilon - U_0^\varepsilon)U_0^\varepsilon)(z)]_- &= [-1(U^\varepsilon(z) - \tanh(z - z_m^\varepsilon))]_- | - U_0^\varepsilon| \\ &= [sgn(U^\varepsilon)(U^\varepsilon + sgn(U^\varepsilon)\tanh(z - z_m^\varepsilon))]_- |U_0^\varepsilon|\end{aligned}$$

for all $z \in [-\varepsilon^{-3/5} + z_m^\varepsilon, z_m^\varepsilon]$. Therefore, we deduce from the results 2 and 3 of Theorem 5.1

$$[(U^\varepsilon - \nu(\overline{r})U_0^\varepsilon)\nu(\overline{r})U_0^\varepsilon]_- \to 0 \text{ uniformly on } [z_1^\varepsilon, \varepsilon^{-3/5}] \cup [-\varepsilon^{-3/5} + z_m^\varepsilon, z_m^\varepsilon],$$

as $\varepsilon \downarrow 0$. This with (6.21) implies that

$$\lim_{\varepsilon \to 0} \sup_{[-\varepsilon^{-3/5} + z_m^\varepsilon, \varepsilon^{-3/5}]} [\Psi^\varepsilon \Theta^\varepsilon]_- = 0.$$

This completes the proof of Lemma 6.4.

Next we give a bound for G^ε. More precisely we prove the following result:

Lemma 6.5. *There exists a constant $C_2(R_0)$ such that G^ε satisfies*

$$\int_{z_-^\varepsilon}^{z_+^\varepsilon} |G^\varepsilon(z)|^2 dz \leq C_2(R_0)\varepsilon^2. \tag{6.22}$$

Proof. By definition of G^ε in (6.6), we have

$$\begin{aligned}\int_{z_-^\varepsilon}^{z_+^\varepsilon} |G^\varepsilon(z)|^2 \, dz &\leq (N-1)^2 \varepsilon^2 \int_{z_-^\varepsilon}^{z_+^\varepsilon} \frac{(U_z^\varepsilon)^2}{(\varepsilon z + r_1^\varepsilon)^2} \, dz = (N-1)^2 \varepsilon^3 \int_{R_0}^1 \frac{(u_r^\varepsilon)^2}{r^2} \, dr \\ &\leq (N-1)^2 \frac{\varepsilon^3}{R_0^{N+1}} \int_{R_0}^1 (u_r^\varepsilon)^2 r^{N-1} dr.\end{aligned}$$

Finally using (3.8) we deduce that $\int_{z_-^\varepsilon}^{z_+^\varepsilon} |G^\varepsilon(z)|^2 \, dz \leq 2(N-1)^2 \frac{\varepsilon^2}{R_0^{N+1}} K$. This completes the proof of Lemma 6.5. □

Next we give a bound for H^ε. More precisely, we prove the following result:

Lemma 6.6. *There exists a function $C_3(R_0, t)$ independent of ε such that H^ε satisfies*

$$\int_{(-\varepsilon^{-3/5} + z_m^\varepsilon, z_m^\varepsilon) \cup (z_1^\varepsilon, \varepsilon^{-3/5})} |H^\varepsilon(z)|^2 \, dz \leq C_3(R_0, t)\varepsilon^{6/5}. \tag{6.23}$$

Moreover, we also have that

$$\lim_{\varepsilon\to 0}\int_{z_m^\varepsilon}^{z_1^\varepsilon} |H^\varepsilon(z)|^2\, dz = 0. \tag{6.24}$$

Proof. Using the definition of the partition of unity $\{E_i^\varepsilon\}$ we have that $E_i^\varepsilon = 0$ for all $i \in [1, m-1]$ and $E_m^\varepsilon = 1$, on the interval $[z_-^\varepsilon, z_m^\varepsilon]$. This with (6.7) implies that

$$H^\varepsilon(z) = \varepsilon^2\left(6U_{0m}^\varepsilon(U_{1m}^\varepsilon)^2 + 2\varepsilon(U_{1m}^\varepsilon)^3\right)(z),\ \text{for all } z\in[z_-^\varepsilon, z_m^\varepsilon]. \tag{6.25}$$

Similarly, we have that $E_i^\varepsilon = 0$ for all $i \in [2, m]$ and $E_1^\varepsilon = 1$, on $[z_1^\varepsilon, z_+^\varepsilon]$, which implies that

$$H^\varepsilon(z) = \varepsilon^2\left(6U_{01}^\varepsilon(U_{11}^\varepsilon)^2 + 2\varepsilon(U_{11}^\varepsilon)^3\right)(z),\ \text{for all } z\in[z_1^\varepsilon, z_+^\varepsilon]. \tag{6.26}$$

Moreover, we have

$$|U_{0i}^\varepsilon|_{L^\infty(z_-^\varepsilon, z_+^\varepsilon)} \le 1,\ \text{for all } i\in[1,m]. \tag{6.27}$$

Using (6.25), (6.26) and (6.27), we deduce that

$$\int_{(-\varepsilon^{-3/5}+z_m^\varepsilon, z_m^\varepsilon)\cup(z_1^\varepsilon, \varepsilon^{-3/5})} |H^\varepsilon(z)|^2 dz$$

$$\le C\varepsilon^4\left[\varepsilon^2|U_1^\varepsilon|^6_{L^\infty(z_-^\varepsilon, z_+^\varepsilon)} + \varepsilon|U_1^\varepsilon|^5_{L^\infty(z_-^\varepsilon, z_+^\varepsilon)} + |U_1^\varepsilon|^4_{L^\infty(z_-^\varepsilon, z_+^\varepsilon)}\right](\varepsilon^{-3/5}).$$

Using Lemma 6.3 we deduce that

$$\int_{(-\varepsilon^{-3/5}+z_m^\varepsilon, z_m^\varepsilon)\cup(z_1^\varepsilon, \varepsilon^{-3/5})} |H^\varepsilon(z)|^2 dz \ \le C(R_0)\varepsilon^{4-2\gamma-3/5} \le C(R_0)\varepsilon^{6/5}.$$

This completes the proof of (6.23). Next we prove (6.24). We set

$$\begin{aligned}
H_1^\varepsilon(z) &= \varepsilon^2\sum_{i=1}^{i=m} E_i^\varepsilon(6U_{0i}^\varepsilon(U_{1i}^\varepsilon)^2 + 2\varepsilon(U_{1i}^\varepsilon)^3),\\
H_2^\varepsilon(z) &= \sum_{i=1}^{i=m-1}(\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)\left[(\zeta_i^\varepsilon)_{zz} + 2E_i^\varepsilon E_{i+1}^\varepsilon\left((1+E_i^\varepsilon)(\Theta_i^\varepsilon)^2\right.\right.\\
&\qquad \left.\left.+(E_i^\varepsilon-2)(\Theta_{i+1}^\varepsilon)^2 + (1-2E_i^\varepsilon)\Theta_i^\varepsilon\Theta_{i+1}^\varepsilon\right)\right],\\
H_3^\varepsilon(z) &= \sum_{i=1}^{i=m-1}(\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)_z(\zeta_i^\varepsilon)_z.
\end{aligned}$$

Therefore, we have

$$H^\varepsilon(z) = H_1^\varepsilon(z) + H_2^\varepsilon(z) + H_3^\varepsilon(z), \tag{6.28}$$

and thus

$$\int_{z_m^\varepsilon}^0 |H^\varepsilon(z)|^2 \le C\Big[\int_{z_m^\varepsilon}^0 |H_1^\varepsilon(z)|^2 + \int_{z_m^\varepsilon}^0 |H_2^\varepsilon(z)|^2 + \int_{z_m^\varepsilon}^0 |H_3^\varepsilon(z)|^2\Big]. \qquad (6.29)$$

We first prove that

$$\int_{z_m^\varepsilon}^0 |H_1^\varepsilon(z)|^2 dz \to 0, \quad \text{as } \varepsilon \downarrow 0. \qquad (6.30)$$

Using (6.27), Lemma 6.3 and the fact that $|z_m^\varepsilon| \le \frac{1}{\varepsilon}$, we obtain for all $i \in [1, m]$

$$\varepsilon^4\Big[\int_{z_m^\varepsilon}^0 (E_i^\varepsilon)^2((U_{0i}^\varepsilon)^2(U_{1i}^\varepsilon)^4 + \varepsilon^2(U_{1i}^\varepsilon)^6)\Big] \le \varepsilon^4 C(R_0,t)[\varepsilon^{-2\gamma}]\frac{1}{\varepsilon} \le C(R_0,t)\varepsilon^{3-2\gamma}.$$

This in turn implies (6.30). Next we prove that

$$\int_{z_m^\varepsilon}^0 |H_2^\varepsilon(z)|^2 \, dz \to 0, \quad \text{as } \varepsilon \downarrow 0. \qquad (6.31)$$

We set $A_i(z) = \big((1+E_i^\varepsilon)(\Theta_i^\varepsilon)^2 + (E_i^\varepsilon - 2)(\Theta_{i+1}^\varepsilon)^2 + (1-2E_i^\varepsilon)\Theta_i^\varepsilon\Theta_{i+1}^\varepsilon\big)(z)$. Using (6.27) and Lemma 6.3, we obtain

$$\|\Theta_i^\varepsilon\|_{L^\infty(z_-^\varepsilon, z_+^\varepsilon)} \le K_1(R_0,t), \text{ for all } i \in [1,m], \qquad (6.32)$$

which implies

$$\|A_i\|_{L^\infty(z_-^\varepsilon, z_+^\varepsilon)} \le K_2(R_0,t), \text{ for all } i \in [1,m]. \qquad (6.33)$$

Using (6.33) and the definition of H_2^ε, we deduce

$$\int_{z_m^\varepsilon}^0 |H_2^\varepsilon(z)|^2 \, dz \le \qquad (6.34)$$
$$K_3(R_0,t)\Big[\int_{z_m^\varepsilon}^0 (\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)^2((\zeta_i^\varepsilon)_{zz})^2 + \int_{z_m^\varepsilon}^0 (\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)^2(E_i^\varepsilon E_{i+1}^\varepsilon)^2\Big].$$

Noting $\{(\zeta_i^\varepsilon)_{zz} \neq 0\} \cap [z_{i+1}^\varepsilon, z_i^\varepsilon] \subset [\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1]$, we have

$$\int_{z_m^\varepsilon}^0 (\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)^2((\zeta_i^\varepsilon)_{zz})^2 = \sum_{i=1}^{i=m-1} \int_{z_{i+1}^\varepsilon}^{z_i^\varepsilon} (\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)^2((\zeta_i^\varepsilon)_{zz})^2$$
$$\le K_1 \sum_{i=1}^{i=m-1} \int_{\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1}^{\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1} |\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon|^2. \qquad (6.35)$$

In the same way noting $\{E_i^\varepsilon E_{i+1}^\varepsilon \neq 0\} \cap [z_{i+1}^\varepsilon, z_i^\varepsilon] \subset [\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1]$, we obtain

$$\int_{z_m^\varepsilon}^0 (\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)^2 (E_i^\varepsilon E_{i+1}^\varepsilon)^2 \leq \sum_{i=1}^{i=m-1} \int_{\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1}^{\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1} |\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon|^2. \tag{6.36}$$

Substituting (6.35) and (6.36) into (6.34) and using the fact that the measure of the interval $[\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1]$ is equal to 2, we deduce

$$\int_{z_m^\varepsilon}^0 |H_2^\varepsilon(z)|^2 \, dz \leq K_4(R_0, t) \sum_{i=1}^{i=m-1} \|\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon\|^2_{L^\infty(\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1)}. \tag{6.37}$$

Furthermore, using the definition of Θ_i^ε and Lemma 6.3, we have

$$\begin{aligned} &\|\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon\|_{L^\infty(\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1)} \\ &\quad \leq \|U_{0i+1}^\varepsilon - U_{0i}^\varepsilon\|_{L^\infty(\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1)} + C(R_0, t)\varepsilon^{1-\gamma/2}. \end{aligned} \tag{6.38}$$

Moreover, by the definition of U_{0i}^ε, we obtain

$$(U_{0i+1}^\varepsilon - U_{0i}^\varepsilon)(z) = (-1)^{i+2} \left(\tanh(z - z_{i+1}^\varepsilon) + \tanh(z - z_i^\varepsilon) \right). \tag{6.39}$$

Furthermore, we have

$$\frac{z_{i+1}^\varepsilon - z_i^\varepsilon}{2} - 1 \leq z - z_i^\varepsilon \leq \frac{z_{i+1}^\varepsilon - z_i^\varepsilon}{2} + 1 \quad \text{and} \tag{6.40}$$

$$-(\frac{z_{i+1}^\varepsilon - z_i^\varepsilon}{2}) - 1 \leq z - z_{i+1}^\varepsilon \leq -(\frac{z_{i+1}^\varepsilon - z_i^\varepsilon}{2}) + 1, \tag{6.41}$$

for all $z \in [\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1]$. Using (6.39), (6.40), (6.41) and (6.17) we deduce

$$\lim_{\varepsilon \to 0} \|U_{0i+1}^\varepsilon - U_{0i}^\varepsilon\|_{L^\infty(\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1)} = 0. \tag{6.42}$$

In view of (6.42) and (6.38) we have that the quantity

$$\|\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon\|_{L^\infty(\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1)}$$

tends to 0 as $\varepsilon \downarrow 0$. Thus, letting ε tend to zero in (6.37) we obtain (6.31). Next we prove that the last term of (6.29) tends to zero; namely,

$$\int_{z_m^\varepsilon}^0 |H_3^\varepsilon(z)|^2 dz \to 0, \quad \text{as } \varepsilon \downarrow 0. \tag{6.43}$$

As in the proof of (6.31), noting $\{(\zeta_i^\varepsilon)_z \neq 0\} \cap [z_{i+1}^\varepsilon, z_i^\varepsilon] \subset [\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1]$, we obtain

$$\int_{z_m^\varepsilon}^0 |H_3^\varepsilon(z)|^2 dz \le C \sum_{i=1}^{i=m-1} \int_{\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1}^{\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1} |(\Theta_{i+1}^\varepsilon)_z - (\Theta_i^\varepsilon)_z|^2 dz$$

$$\le 2C \sum_{i=1}^{i=m-1} \|(\Theta_{i+1}^\varepsilon)_z - (\Theta_i^\varepsilon)_z\|^2_{L^\infty(\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1)}. \tag{6.44}$$

Furthermore, using Lemma 6.3, we have

$$\|(\Theta_{i+1}^\varepsilon)_z - (\Theta_i^\varepsilon)_z\|_{L^\infty(\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1)}$$

$$\le \|(U_{0i+1}^\varepsilon)_z - (U_{0i}^\varepsilon)_z\|_{L^\infty(\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1)} + C(R_0, t)\varepsilon^{1-\gamma/2}. \tag{6.45}$$

By (6.39), we have $((U_{0i+1}^\varepsilon)_z - (U_{0i}^\varepsilon)_z)(z) = (-1)^{i+2}[(1 - \tanh^2(z - z_{i+1}^\varepsilon)) + (1 - \tanh^2(z - z_i^\varepsilon))]$. Therefore, using (6.17), (6.40), and (6.41), we deduce

$$\lim_{\varepsilon \to 0} \|(U_{0i+1}^\varepsilon)_z - (U_{0i}^\varepsilon)_z\|_{L^\infty(\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1)} = 0. \tag{6.46}$$

Letting ε tend to zero in (6.44) and also using (6.45) and (6.46), we deduce (6.43). In view of (6.30), (6.31), (6.43), and (6.29), we conclude

$$\lim_{\varepsilon \to 0} \int_{z_m^\varepsilon}^0 |H^\varepsilon(z)|^2 dz = 0.$$

This completes the proof of Lemma 6.6. □

We are now in a position to prove Theorem 6.1.

Proof of Theorem 6.1. Multiplying (6.8) by $|\xi^\varepsilon|^2 \Psi^\varepsilon$ and integrating the result on $I = (-\infty, z_m^\varepsilon) \cup (0, +\infty)$ or $(z_m^\varepsilon, 0)$, we obtain

$$\int_I \left(-\Psi_{zz}^\varepsilon - f'(\Theta^\varepsilon)\Psi^\varepsilon\right) \Psi^\varepsilon |\xi^\varepsilon|^2 = -6 \int_I \Theta^\varepsilon (\Psi^\varepsilon)^3 |\xi^\varepsilon|^2 - 2 \int_I (\Psi^\varepsilon)^4 |\xi^\varepsilon|^2 + \int_I G^\varepsilon \Psi^\varepsilon |\xi^\varepsilon|^2 - \int_I H^\varepsilon \Psi^\varepsilon |\xi^\varepsilon|^2.$$

Using Lemma A.4, we deduce that

$$S_1 \int_I |\Psi_z^\varepsilon|^2 |\xi^\varepsilon|^2 + S_2 \int_I |\Psi^\varepsilon|^2 |\xi^\varepsilon|^2$$

$$\le \frac{1}{S_1} \int_I |\Psi^\varepsilon|^2 |\xi_z^\varepsilon|^2 + \frac{1}{S_2} \Big(\int_I |G^\varepsilon|^2 |\xi^\varepsilon|^2 + \int_I |H^\varepsilon|^2 |\xi^\varepsilon|^2 \Big)$$

$$+ \frac{S_2}{2} \int_I |\Psi^\varepsilon|^2 |\xi^\varepsilon|^2 + 6([\Theta^\varepsilon \Psi^\varepsilon]_-)_{L^\infty([-\varepsilon^{-3/5} + z_m^\varepsilon, \varepsilon^{-3/5}])} \int_I |\Psi^\varepsilon|^2 |\xi^\varepsilon|^2.$$

This implies

$$
\begin{aligned}
&S_1\int_I|\Psi_z^\varepsilon|^2|\xi^\varepsilon|^2+\Big(\frac12 S_2-6([\Theta^\varepsilon\Psi^\varepsilon]_-)_{L^\infty([-\varepsilon^{-3/5}+z_m^\varepsilon,\varepsilon^{-3/5}])}\Big)\int_I|\Psi^\varepsilon|^2|\xi^\varepsilon|^2\\
&\le\frac{1}{S_1}\int_I|\Psi^\varepsilon|^2|\xi_z^\varepsilon|^2+\frac{1}{S_2}\Big(\int_I|G^\varepsilon|^2|\xi^\varepsilon|^2+\int_I|H^\varepsilon|^2|\xi^\varepsilon|^2\Big). \qquad (6.47)
\end{aligned}
$$

Since by (6.18) $\sup_{[-\varepsilon^{-3/5}+z_m^\varepsilon,\varepsilon^{-3/5}]}[\Theta^\varepsilon\Psi^\varepsilon]_-$ tends to zero as ε tends to zero, we choose ε small enough such that

$$
\sup_{[-\varepsilon^{-3/5}+z_m^\varepsilon,\varepsilon^{-3/5}]}[\Theta^\varepsilon\Psi^\varepsilon]_-<\frac{1}{24}S_2. \qquad (6.48)
$$

Substituting (6.48) into (6.47), we obtain

$$
\begin{aligned}
&S_1\int_I|\Psi_z^\varepsilon|^2|\xi^\varepsilon|^2+\frac{S_2}{4}\int_I|\Psi^\varepsilon|^2|\xi^\varepsilon|^2 \qquad (6.49)\\
&\le\frac{1}{S_1}\int_I|\Psi^\varepsilon|^2|\xi_z^\varepsilon|^2+\frac{1}{S_2}\Big(\int_I|G^\varepsilon|^2|\xi^\varepsilon|^2+\int_I|H^\varepsilon|^2|\xi^\varepsilon|^2\Big).
\end{aligned}
$$

If $I=(z_m^\varepsilon,0)$, then using (6.49), Lemmas 6.5 and 6.6, and the fact that $\xi^\varepsilon=1$ on $[z_m^\varepsilon,0]$, we deduce that

$$
\lim_{\varepsilon\to0}\int_{(z_m^\varepsilon,0)}(|\Psi_z^\varepsilon|^2+|\Psi^\varepsilon|^2)\,dz=0.
$$

Next we consider the case that $I=(-\infty,z_m^\varepsilon)$ or $(0,+\infty)$. By (6.49) and Lemmas 6.5 and 6.6, we have

$$
S_1\int_I|\Psi_z^\varepsilon|^2|\xi^\varepsilon|^2+\frac{S_2}{4}\int_I|\Psi^\varepsilon|^2|\xi^\varepsilon|^2\le\frac{1}{S_1}\int_I|\Psi^\varepsilon|^2|\xi_z^\varepsilon|^2+C(R_0,t)(\varepsilon^2+\varepsilon^{6/5}). \qquad (6.50)
$$

Next we estimate $\int_I|\Psi^\varepsilon|^2|\xi_z^\varepsilon|^2$. Since $|\xi_z^\varepsilon|\le C\varepsilon^{3/5}$, we have

$$
\int_I|\Psi^\varepsilon|^2|\xi_z^\varepsilon|^2\le C\varepsilon^{\frac65}\int_{\{\xi_z^\varepsilon\neq0\}}|\Psi^\varepsilon|^2\le C\varepsilon^{\frac65}\Big[\int_{-\varepsilon^{-\frac35}+z_m^\varepsilon}^{-\varepsilon^{-\frac12}+z_m^\varepsilon}|\Psi^\varepsilon|^2+\int_{\varepsilon^{-\frac12}}^{\varepsilon^{-\frac35}}|\Psi^\varepsilon|^2\Big]. \qquad (6.51)
$$

First we remark that

$$
|s-sign(s)1|^2\le|s^2-1|^2 \text{ for all } s\in R, \qquad (6.52)
$$

and we prove that both integrals of the right-hand side of (6.51) are bounded. We consider only the case where $\nu(\overline{r})=1$ since by symmetry one can obtain the same result in the case $\nu(\overline{r})=-1$. Since $\Psi^\varepsilon(z)=U^\varepsilon(z)-U_0^\varepsilon(z)-\varepsilon U_1^\varepsilon(z)$, we have

$$\int_{\varepsilon^{-\frac{1}{2}}}^{\varepsilon^{-\frac{3}{5}}} |\Psi^\varepsilon|^2 \le C\Big(\int_{\varepsilon^{-\frac{1}{2}}}^{\varepsilon^{-\frac{3}{5}}} |U^\varepsilon - 1|^2 + \int_{\varepsilon^{-\frac{1}{2}}}^{\varepsilon^{-\frac{3}{5}}} |1 - U_0^\varepsilon|^2 + \varepsilon^{2-\gamma-\frac{3}{5}}\Big). \tag{6.53}$$

Using (6.52) and the fact that $U^\varepsilon > 0$ on $[\varepsilon^{-1/2}, \varepsilon^{-3/5}]$, we deduce

$$\begin{aligned}\int_{\varepsilon^{-\frac{1}{2}}}^{\varepsilon^{-\frac{3}{5}}} |U^\varepsilon - 1|^2 dz &\le \int_{\varepsilon^{-\frac{1}{2}}}^{\varepsilon^{-\frac{3}{5}}} |(U^\varepsilon)^2 - 1|^2 dz \\ &\le \int_{z_-^\varepsilon}^{z_+^\varepsilon} |(U^\varepsilon)^2 - 1|^2 dz = \frac{1}{\varepsilon}\int_{R_0}^{1} |(u^\varepsilon)^2 - 1|^2 dr.\end{aligned}$$

This with (3.11) implies

$$\int_{\varepsilon^{-1/2}}^{\varepsilon^{-3/5}} |U^\varepsilon - 1|^2 dz \le C_1(R_0). \tag{6.54}$$

Moreover, by the definition of U_0^ε we have $U_0^\varepsilon(z) = \tanh(z)$ on $[\varepsilon^{-1/2}, \varepsilon^{-3/5}]$, and thus

$$\int_{\varepsilon^{-1/2}}^{\varepsilon^{-3/5}} |1 - U_0^\varepsilon|^2 dz \le [1 - \tanh(\varepsilon^{-1/2})]^2 \varepsilon^{-3/5}.$$

Therefore, we deduce that $\int_{\varepsilon^{-1/2}}^{\varepsilon^{-3/5}} |1 - U_0^\varepsilon|^2 dz \to 0$ as $\varepsilon \downarrow 0$. This in view of (6.53) and (6.54) implies that

$$\int_{\varepsilon^{-1/2}}^{\varepsilon^{-3/5}} |\Psi^\varepsilon|^2 \le C_2(R_0), \tag{6.55}$$

for ε small enough. Similarly we have $U^\varepsilon < 0$ on $[-\varepsilon^{-3/5} + z_m^\varepsilon, -\varepsilon^{-1/2} + z_m^\varepsilon]$. Thus, using (6.52) and the fact that $U_0^\varepsilon(z) = \tanh(z - z_m^\varepsilon)$, we deduce

$$\begin{aligned}&\int_{-\varepsilon^{-\frac{3}{5}}+z_m^\varepsilon}^{-\varepsilon^{-\frac{1}{2}}+z_m^\varepsilon} |\Psi^\varepsilon|^2 \\ &\le C\Big(\int_{-\varepsilon^{-\frac{3}{5}}+z_m^\varepsilon}^{-\varepsilon^{-\frac{1}{2}}+z_m^\varepsilon} |U^\varepsilon + 1|^2 + \int_{-\varepsilon^{-\frac{3}{5}}+z_m^\varepsilon}^{-\varepsilon^{-\frac{1}{2}}+z_m^\varepsilon} |-1 - U_0^\varepsilon|^2 + \varepsilon^{2-\gamma-\frac{3}{5}}\Big) \\ &\le C\Big(\int_{-\varepsilon^{-\frac{3}{5}}+z_m^\varepsilon}^{-\varepsilon^{-\frac{1}{2}}+z_m^\varepsilon} |(U^\varepsilon)^2 - 1|^2 dz + \int_{-\varepsilon^{-\frac{3}{5}}+z_m^\varepsilon}^{-\varepsilon^{-\frac{1}{2}}+z_m^\varepsilon} |1 + \tanh(z - z_m^\varepsilon)|^2 + \varepsilon^{2-\gamma-\frac{3}{5}}\Big) \\ &\le C\Big(\frac{1}{\varepsilon}\int_{R_0}^{1} |(u^\varepsilon)^2 - 1|^2 dr + [1 - \tanh(\varepsilon^{-\frac{1}{2}})]^2 \varepsilon^{-\frac{3}{5}} + \varepsilon^{2-\gamma-\frac{3}{5}}\Big).\end{aligned}$$

Using (3.11) we deduce that

$$\int_{-\varepsilon^{-3/5}+z_m^\varepsilon}^{-\varepsilon^{-1/2}+z_m^\varepsilon} |\Psi^\varepsilon|^2 \le C_2(R_0,t). \tag{6.56}$$

Substituting (6.55) and (6.56) into (6.51) we obtain

$$\int_I |\Psi^\varepsilon|^2 |\xi_z^\varepsilon|^2 \le C_3(R_0,t)\varepsilon^{6/5}. \tag{6.57}$$

Finally, using (6.50) and (6.57), we conclude

$$\int_{(-\varepsilon^{-1/2}+z_m^\varepsilon,z_m^\varepsilon)\cup(0,\varepsilon^{-1/2})} |\Psi_z^\varepsilon|^2|\xi^\varepsilon|^2 + \int_{(-\varepsilon^{-1/2}+z_m^\varepsilon,z_m^\varepsilon)\cup(0,\varepsilon^{-1/2})} |\Psi^\varepsilon|^2|\xi^\varepsilon|^2 \le C(R_0,t)\varepsilon^{6/5}. \tag{6.58}$$

This completes the proof of Theorem 6.1.

Corollary 6.7. *Let* $J := [-\varepsilon^{-1/2}+z_m^\varepsilon, z_m^\varepsilon]\cup[0,\varepsilon^{-1/2}]$; *we have*

$$\|\Psi^\varepsilon(\cdot,t)\|_{H^{1,\infty}(J)} \le C(R_0,t)\varepsilon^{3/5}, \tag{6.59}$$

$$\lim_{\varepsilon\to 0} \|\Psi^\varepsilon(\cdot,t)\|_{H^{1,\infty}(0,z_m^\varepsilon)} = 0, \tag{6.60}$$

$$\lim_{\varepsilon\to 0} \|U^\varepsilon(\cdot,t) - \nu(\overline{r})U_0^\varepsilon(\cdot,t)\|_{H^{1,2}(-\varepsilon^{-1/2}+z_m^\varepsilon,\varepsilon^{-1/2})} = 0, \tag{6.61}$$

$$\|U^\varepsilon(\cdot,t) - \nu(\overline{r})U_0^\varepsilon(\cdot,t)\|_{H^{1,\infty}(J)} \le C(R_0,t)\varepsilon^{3/5}, \tag{6.62}$$

$$\lim_{\varepsilon\to 0} \|U^\varepsilon(\cdot,t) - \nu(\overline{r})U_0^\varepsilon(\cdot,t)\|_{H^{1,\infty}(0,z_m^\varepsilon)} = 0. \tag{6.63}$$

Proof. We note that

$$(\Psi^\varepsilon(z))^2 = (\Psi^\varepsilon(y))^2 + 2\int_y^z \Psi_z^\varepsilon \Psi^\varepsilon.$$

This implies that

$$|\Psi^\varepsilon(z)|^2 \le |\Psi^\varepsilon(y)|^2 + \int_S |\Psi_z^\varepsilon|^2 + \int_S |\Psi^\varepsilon|^2, \tag{6.64}$$

where $y = 0$ if $S = [0,\varepsilon^{-1/2}]$ or $S = [z_m^\varepsilon, 0]$ and where $y = z_m^\varepsilon$ if $S = [-\varepsilon^{-1/2}+z_m^\varepsilon, z_m^\varepsilon]$. Let $z \in [z_m^\varepsilon, 0]$, applying (6.64) with $y = 0$, so that $\Psi^\varepsilon(0) = 0$ and the fact that $\xi^\varepsilon(z) = 1$ on $[z_m^\varepsilon, 0]$, and Theorem 6.1, we deduce

$$\lim_{\varepsilon\to 0} \|\Psi^\varepsilon(\cdot,t)\|_{L^\infty(0,z_m^\varepsilon)} = 0. \tag{6.65}$$

Moreover, applying (6.64) with $y = 0$ and using Theorem 6.1 and the fact that $\xi^\varepsilon(z) = 1$ on $[0,\varepsilon^{-1/2}]$, we deduce

$$\|\Psi^\varepsilon(\cdot,t)\|_{L^\infty(0,\varepsilon^{-1/2})}^2 \le C(R_0,t)\varepsilon^{6/5}. \tag{6.66}$$

Similarly, applying (6.64) with $y = z_m^\varepsilon$ and using Theorem 6.1 and the fact that $\xi^\varepsilon(z) = 1$ on $[-\varepsilon^{-1/2} + z_m^\varepsilon, z_m^\varepsilon]$, we deduce

$$\|\Psi^\varepsilon(\cdot,t)\|^2_{L^\infty(-\varepsilon^{-1/2}+z_m^\varepsilon, z_m^\varepsilon)} \le C(R_0,t)\varepsilon^{6/5}. \tag{6.67}$$

Furthermore, we have

$$(\Psi_z^\varepsilon(z))^2 \le (\Psi_z^\varepsilon(y))^2 + \int_S |\Psi_{zz}^\varepsilon|^2 + \int_S |\Psi_z^\varepsilon|^2, \tag{6.68}$$

where $S = [0, \varepsilon^{-1/2}]$ or $S = [-\varepsilon^{-1/2} + z_m^\varepsilon, z_m^\varepsilon]$ or $S = [z_m^\varepsilon, 0]$. Using the differential equation of Ψ^ε (6.8), we have

$$\begin{aligned}\int_S |\Psi_{zz}^\varepsilon|^2 \le C\Big[&\int_S |f'(\Theta^\varepsilon)\Psi^\varepsilon|^2 + \int_S |(\Theta^\varepsilon(\Psi^\varepsilon))\Psi^\varepsilon|^2 + \int_S (\Psi^\varepsilon)^6 \\ &+ \int_S |G^\varepsilon(z)|^2 + \int_S |H^\varepsilon(z)|^2\Big] \\ \le \tilde{C}\Big\{&\Big(\int_S |\Psi^\varepsilon|^2\Big)\Big[\|f'(\Theta^\varepsilon)\|^2_{L^\infty(S)} + \|\Theta^\varepsilon\Psi^\varepsilon\|^2_{L^\infty(S)} + \|\Psi^\varepsilon\|^4_{L^\infty(S)}\Big] \\ &+ \int_S |G^\varepsilon(z)|^2 + \int_S |H^\varepsilon(z)|^2\Big\}. \end{aligned} \tag{6.69}$$

Using (6.65) and (6.66) we deduce that

$$\|\Psi^\varepsilon\|^2_{L^\infty(S)} \le K_1(R_0,t). \tag{6.70}$$

Moreover, since $f'(s) = -6s^2 + 2$, we also have

$$\begin{aligned}&\|\Theta^\varepsilon\|_{L^\infty(S)} \le \|U_0^\varepsilon\|_{L^\infty(S)} + \varepsilon\|U_1^\varepsilon\|_{L^\infty(S)} \le K_2(R_0,t) \quad \text{and} \\ &\|f'(\Theta^\varepsilon)\|^2_{L^\infty(S)} \le K_3(R_0,t).\end{aligned} \tag{6.71}$$

Therefore, substituting (6.70) and (6.71) into (6.69) and also using Lemma 6.5 we deduce that

$$\int_S |\Psi_{zz}^\varepsilon|^2 \le K_4(R_0,t)\Big[\int_S |\Psi^\varepsilon|^2 + \varepsilon^2 + \int_S |H^\varepsilon(z)|^2\Big]. \tag{6.72}$$

In view of (6.68), we deduce that

$$(\Psi_z^\varepsilon(z))^2 \le (\Psi_z^\varepsilon(z))^2 + \int_S |\Psi_z^\varepsilon|^2 + K_4(R_0,t)\Big[\int_S |\Psi^\varepsilon|^2 + \varepsilon^2 + \int_S |H^\varepsilon(z)|^2\Big]. \tag{6.73}$$

Integrating (6.73) in y on S, we obtain

$$(\Psi_z^\varepsilon(z))^2 \le (1 + \frac{1}{meas(S)})\int_S |\Psi_z^\varepsilon|^2 + K_4(R_0,t)\Big[\int_S |\Psi^\varepsilon|^2 + \varepsilon^2 + \int_S |H^\varepsilon(z)|^2\Big]. \tag{6.74}$$

For $S = [0, \varepsilon^{-1/2}]$ or $S = [-\varepsilon^{-1/2} + z^\varepsilon_m, z^\varepsilon_m]$, we have $meas(S) = \varepsilon^{-1/2}$. Moreover, using (6.74), Lemma 6.6 and Theorem 6.1, we have

$$(\Psi^\varepsilon_z(z))^2 \leq (1+\varepsilon^{1/2})\varepsilon^{6/5} + K_5(R_0,t)\big[\varepsilon^{6/5} + \varepsilon^2 + \varepsilon^{6/5}\big],$$

for all $z \in [-\varepsilon^{-1/2} + z^\varepsilon_m, z^\varepsilon_m] \cup [0, \varepsilon^{-1/2}]$. This with (6.66) and (6.67) implies (6.59). Next we prove (6.60). Since $S = [z^\varepsilon_m, 0]$ and

$$|z^\varepsilon_m| \leq \sum_1^{m-1} |z^\varepsilon_{i+1} - z^\varepsilon_i| = \sum_1^{m-1} \frac{|r^\varepsilon_{i+1} - r^\varepsilon_i|}{\varepsilon} \leq (m-1)\varepsilon^{-3/4} \tag{6.75}$$

we have $meas(S) \leq (m-1)\varepsilon^{-3/4}$. Therefore, using (6.74), Lemma 6.6 and Theorem 6.1 we deduce that $\lim_{\varepsilon\to 0} \|\Psi^\varepsilon_z\|_{L^\infty([z^\varepsilon_m,0])} = 0$, which with (6.68) implies (6.60). Next we prove (6.61). We first note that

$$\begin{aligned} &|U^\varepsilon - \nu(\overline{r})U^\varepsilon_0| \leq |\Psi^\varepsilon| + \varepsilon|U^\varepsilon_1| \quad \text{ and} \\ &|(U^\varepsilon - \nu(\overline{r})U^\varepsilon_0)_z| \leq |\Psi^\varepsilon_z| + \varepsilon|(U^\varepsilon_1)_z|. \end{aligned} \tag{6.76}$$

Adding both inequalities of (6.76) and integrating the result on $[-\varepsilon^{-1/2} + z^\varepsilon_m, \varepsilon^{-1/2}]$, we deduce that

$$\begin{aligned} &\int_{\varepsilon^{-\frac{1}{2}}+z^\varepsilon_m}^{\varepsilon^{\frac{1}{2}}} |U^\varepsilon - \nu(\overline{r})U^\varepsilon_0|^2 + |(U^\varepsilon - \nu(\overline{r})U^\varepsilon_0)_z|^2 \\ &\leq 2\Big[\int_{\varepsilon^{-\frac{1}{2}}+z^\varepsilon_m}^{\varepsilon^{-\frac{1}{2}}} |\Psi^\varepsilon|^2 + |\Psi^\varepsilon_z|^2 + \varepsilon^2\Big(\int_{\varepsilon^{-\frac{1}{2}}+z^\varepsilon_m}^{\varepsilon^{-\frac{1}{2}}} |(U^\varepsilon_1)_z|^2 + \int_{\varepsilon^{-\frac{1}{2}}+z^\varepsilon_m}^{\varepsilon^{-\frac{1}{2}}} |(U^\varepsilon_1)_z|^2\Big)\Big] \\ &\leq 2\Big[\|\Psi^\varepsilon\|^2_{H^{1,2}(-\varepsilon^{-\frac{1}{2}}+z^\varepsilon_m,\varepsilon^{-\frac{1}{2}})} + \varepsilon^2\|U^\varepsilon_1\|^2_{H^{1,\infty}(-\varepsilon^{-\frac{1}{2}}+z^\varepsilon_m,\varepsilon^{-\frac{1}{2}})}|z^\varepsilon_m|\Big]. \end{aligned}$$

Using Lemma 6.3 and (6.75), we obtain

$$\begin{aligned} &\|U^\varepsilon(\cdot,t) - \nu(\overline{r})U^\varepsilon_0(\cdot,t)\|^2_{H^{1,2}(-\varepsilon^{-1/2}+z^\varepsilon_m,\varepsilon^{-1/2})} \\ &\leq C(R_0,t)\left[\|\Psi^\varepsilon\|^2_{H^{1,2}(-\varepsilon^{-1/2}+z^\varepsilon_m,\varepsilon^{-1/2})} + \varepsilon^{1-\gamma}\right]. \end{aligned}$$

Moreover, in view of Theorem 6.1, we deduce that

$$\lim_{\varepsilon\to 0} \|U^\varepsilon(\cdot,t) - \nu(\overline{r})U^\varepsilon_0(\cdot,t)\|_{H^{1,2}(-\varepsilon^{-1/2}+z^\varepsilon_m,\varepsilon^{-1/2})} = 0,$$

which coincides with (6.61). Next we prove (6.62). In view of (6.76) we have

$$\|U^\varepsilon - \nu(\overline{r})U^\varepsilon_0\|_{H^{1,\infty}(S)} \leq \|\Psi^\varepsilon\|_{H^{1,\infty}(S)} + \varepsilon\|U^\varepsilon_1\|_{H^{1,\infty}(S)} \tag{6.77}$$

where $S = [-\varepsilon^{-1/2} + z^\varepsilon_m, z^\varepsilon_m]$, $[0, \varepsilon^{-1/2}]$ or $[z^\varepsilon_m, 0]$. Using Lemma 6.3,

$$\|U^\varepsilon - \nu(\overline{r})U^\varepsilon_0\|_{H^{1,\infty}(S)} \leq \|\Psi^\varepsilon\|_{H^{1,\infty}(S)} + \varepsilon^{1-\gamma/2}. \tag{6.78}$$

Finally using (6.78) with $S = [-\varepsilon^{-1/2} + z_m^\varepsilon, z_m^\varepsilon]$, $[0, \varepsilon^{-1/2}]$ and also using (6.59) we deduce (6.62). Similarly, using (6.78) with $S = [z_m^\varepsilon, 0]$ and also using (6.60), we deduce (6.63). This completes the proof of Corollary 6.7.

7. The limit equation

7.1. **Passage to the limit.** In this section we prove the equation for the interface; more precisely, we prove

Theorem 7.1. *Let $t \in (0,T)\backslash D(T)$, where $D(T)$ has been defined in Section 4 and such that $u^\varepsilon(\cdot,t)$ tends to ± 1 in $L^1(\Omega)$ and almost everywhere in Ω and let $\overline{r}(t)$ be a jump of $u(\cdot,t)$. There exists a subsequence ε_n, which we denote again by ε, such that*

$$\lim_{\varepsilon\to 0}(W^\varepsilon - V^\varepsilon)(0) = Y_0.$$

Moreover, we have the following limit equation:

$$-(N-1)\frac{4}{3}\frac{1}{\overline{r}(t)} = 2\nu(\overline{r})Y_0,$$

where $\nu(\overline{r})$ has been defined in Definition 4.6.

Proof. We rewrite the equation (3.2) in the variable $z := \frac{r-r_1^\varepsilon}{\varepsilon}$. This gives

$$W^\varepsilon - V^\varepsilon = -\frac{1}{\varepsilon}U_{zz}^\varepsilon - \frac{N-1}{\varepsilon z + r_1^\varepsilon}U_z^\varepsilon - \frac{1}{\varepsilon}f(U^\varepsilon). \tag{7.1}$$

Multiplying (7.1) by U_z^ε and integrating the result on $[-\varepsilon^{-1/2} + z_m^\varepsilon, \varepsilon^{-1/2}]$ we obtain

$$\begin{aligned}&\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} (W^\varepsilon - V^\varepsilon)U_z^\varepsilon \\ &= -\frac{1}{\varepsilon}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} U_{zz}^\varepsilon U_z^\varepsilon - \int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \frac{N-1}{\varepsilon z + r_1^\varepsilon}(U_z^\varepsilon)^2 - \frac{1}{\varepsilon}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} f(U^\varepsilon)U_z^\varepsilon.\end{aligned}$$

This in turn implies that

$$\begin{aligned}&\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} (W^\varepsilon - V^\varepsilon)U_z^\varepsilon \\ &= -\frac{1}{\varepsilon}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \left[\frac{1}{2}(U_z^\varepsilon)^2 - F(U^\varepsilon)\right]_z - (N-1)\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \frac{1}{\varepsilon z + r_1^\varepsilon}(U_z^\varepsilon)^2.\end{aligned} \tag{7.2}$$

In order to pass to the limit in (7.2), we first prove the following lemma:

Lemma 7.2.

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}}\Big[\frac{1}{2}(U_z^\varepsilon)^2-F(U^\varepsilon)\Big]_z=0.$$

Proof. We prove Lemma 7.2 in the case where $\nu(\overline{r})=1$; similarly, one can check the case where $\nu(\overline{r})=-1$. We first note that

$$\Big|\frac{1}{\varepsilon}\int_{-\varepsilon^{-\frac{1}{2}}+z_m^\varepsilon}^{\varepsilon^{-1/2}}\Big[\frac{1}{2}(U_z^\varepsilon)^2-F(U^\varepsilon)\Big]_z\Big|\le\frac{1}{\varepsilon}\Big(\frac{1}{2}\Big|(U_z^\varepsilon)^2(\varepsilon^{-\frac{1}{2}})-(U_z^\varepsilon)^2(z_m^\varepsilon-\varepsilon^{-\frac{1}{2}})\Big| \\ +\Big|-F(U^\varepsilon(\varepsilon^{-\frac{1}{2}}))+F(U^\varepsilon(z_m^\varepsilon-\varepsilon^{-\frac{1}{2}}))\Big|\Big). \tag{7.3}$$

Moreover, we have

$$|U_z^\varepsilon|^2\le(|U_z^\varepsilon-(U_0^\varepsilon)_z|+|(U_0^\varepsilon)_z|)^2\le 2[|U_z^\varepsilon-(U_0^\varepsilon)_z|^2+|(U_0^\varepsilon)_z|^2]. \tag{7.4}$$

Using the definition of U_0^ε, we obtain

$$|(U_0^\varepsilon)_z(\varepsilon^{-1/2})|=1-\tanh^2(\varepsilon^{-1/2}),\quad\text{and also} \\ |(U_0^\varepsilon)_z(z_m^\varepsilon-\varepsilon^{-1/2})|=1-\tanh^2(\varepsilon^{-1/2}). \tag{7.5}$$

Substituting (7.5) into (7.4) and also using the fact that the points $\varepsilon^{-1/2}$ and $z_m^\varepsilon-\varepsilon^{-1/2}$ are in the interval J defined in Corollary 6.7, we deduce that

$$|U_z^\varepsilon(\varepsilon^{-1/2})|^2+|U_z^\varepsilon(z_m^\varepsilon-\varepsilon^{-1/2})|^2 \\ \le 4[\|U_z^\varepsilon-(U_0^\varepsilon)_z\|^2_{L^\infty(J)}+(1-\tanh^2(\varepsilon^{-1/2}))^2]. \tag{7.6}$$

Substituting (7.6) into (7.3), we conclude that

$$\Big|\frac{1}{\varepsilon}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}}\Big[\frac{1}{2}(U_z^\varepsilon)^2-F(U^\varepsilon)\Big]_z\Big| \\ \le\frac{2}{\varepsilon}\|U_z^\varepsilon-(U_0^\varepsilon)_z\|^2_{L^\infty(J)}+\frac{2}{\varepsilon}(1-\tanh^2(\varepsilon^{-1/2}))^2 \\ +\frac{1}{\varepsilon}\Big|-F(U^\varepsilon(\varepsilon^{-1/2}))+F(U^\varepsilon(z_m^\varepsilon-\varepsilon^{-1/2}))\Big|. \tag{7.7}$$

Furthermore, we also have that $U_0^\varepsilon(\varepsilon^{-1/2})=U_0^\varepsilon(-\varepsilon^{-1/2}+z_m^\varepsilon)=\tanh(\varepsilon^{-1/2})$, and thus $F(U_0^\varepsilon(\varepsilon^{-1/2}))=F(U_0^\varepsilon(z_m^\varepsilon-\varepsilon^{-1/2}))=1/2[1-\tanh^2(\varepsilon^{-1/2})]^2$. This implies that

$$F(U^\varepsilon(\varepsilon^{-1/2}))\le|(F(U^\varepsilon)-F(U_0^\varepsilon))(\varepsilon^{-1/2})|+\tfrac{1}{2}|1-\tanh^2(\varepsilon^{-1/2})|^2, \tag{7.8}$$

$$F(U^\varepsilon(z_m^\varepsilon-\varepsilon^{-\frac{1}{2}}))\le|(F(U^\varepsilon)-F(U_0^\varepsilon))(z_m^\varepsilon-\varepsilon^{-\frac{1}{2}})|+\tfrac{1}{2}|1-\tanh^2(\varepsilon^{-1/2})|^2. \tag{7.9}$$

Substituting (7.8) and (7.9) into (7.7), we obtain

$$\begin{aligned}
&\Big|\frac{1}{\varepsilon}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}}\Big[\frac{1}{2}(U_z^\varepsilon)^2 - F(U^\varepsilon)\Big]_z\Big| \\
&\le \frac{2}{\varepsilon}\|U_z^\varepsilon - (U_0^\varepsilon)_z\|^2_{L^\infty(J)} + \frac{3}{\varepsilon}(1-\tanh^2(\varepsilon^{-1/2}))^2 \\
&+ \frac{1}{\varepsilon}\Big|F(U^\varepsilon(\varepsilon^{-1/2})) - F(U_0^\varepsilon(\varepsilon^{-1/2}))\Big| \\
&+ \frac{1}{\varepsilon}\Big|F(U^\varepsilon(z_m^\varepsilon - \varepsilon^{-1/2})) - F(U_0^\varepsilon(z_m^\varepsilon - \varepsilon^{-1/2}))\Big|. \qquad (7.10)
\end{aligned}$$

Furthermore, using the fact that $U_0^\varepsilon(\varepsilon^{-1/2}) = \tanh(\varepsilon^{-1/2})$ and (6.62) we obtain that both points $U^\varepsilon(\varepsilon^{-1/2})$ and $U_0^\varepsilon(\varepsilon^{-1/2})$ belong to an interval $[1 - L\varepsilon^{3/5}, 1 + L\varepsilon^{3/5}]$ for some constant $L > 0$. This implies that there exists $\xi \in (1 - L\varepsilon^{3/5}, 1 + L\varepsilon^{3/5})$ such that

$$\begin{aligned}
\Big|(F(U^\varepsilon) - F(U_0^\varepsilon))\,(\varepsilon^{-1/2})\Big| &= |2\xi(\xi-1)(\xi+1)|\,\Big|(U^\varepsilon - U_0^\varepsilon)(\varepsilon^{-1/2})\Big| \\
&\le K\varepsilon^{3/5}\|U^\varepsilon - U_0^\varepsilon\|_{L^\infty(J)}. \qquad (7.11)
\end{aligned}$$

Similarly, we have

$$\Big|(F(U^\varepsilon) - F(U_0^\varepsilon))\,(z_m^\varepsilon - \varepsilon^{-1/2})\Big| \le K\varepsilon^{3/5}\|U^\varepsilon - U_0^\varepsilon\|_{L^\infty(J)}. \qquad (7.12)$$

Substituting (7.11) and (7.12) into (7.10), we obtain that

$$\begin{aligned}
\Big|\frac{1}{\varepsilon}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}}\Big[\frac{1}{2}(U_z^\varepsilon)^2 - f(U^\varepsilon)\Big]_z\Big| &\le \frac{2}{\varepsilon}\|U_z^\varepsilon - (U_0^\varepsilon)_z\|^2_{L^\infty(J)} \\
+ \frac{3}{\varepsilon}(1-\tanh^2(\varepsilon^{-1/2}))^2 &+ \tilde{K}\varepsilon^{-2/5}\|U^\varepsilon - U_0^\varepsilon\|^2_{L^\infty(J)}. \qquad (7.13)
\end{aligned}$$

Letting ε tend to zero in (7.13) and using (6.62) of Corollary 6.7, we conclude that $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}}\big[\frac{1}{2}(U_z^\varepsilon)^2 - F(U^\varepsilon)\big]_z = 0$, which coincides with Lemma 7.2.

Next we give the limit as ε tends to zero of the last term of (7.2), namely,

Lemma 7.3.

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}}\frac{1}{\varepsilon z + r_1^\varepsilon}(U_z^\varepsilon)^2 = \frac{4}{3}\frac{m}{\overline{r}}.$$

Proof. First we show a preliminary lemma, which will be useful to prove Lemma 7.3.

Lemma 7.4.

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}}[(U_0^\varepsilon)_z]^2=\frac{4}{3}m.$$

Proof. As is done in [3], we divide the integration interval into subintervals, each of which contains one interface. More precisely, we have that

$$\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}}[(U_0^\varepsilon)_z]^2=\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\frac{z_m^\varepsilon+z_{m-1}^\varepsilon}{2}-1}[(U_0^\varepsilon)_z]^2+\sum_{i=2}^{i=m-1}\int_{\frac{z_{i+1}^\varepsilon+z_i^\varepsilon}{2}-1}^{\frac{z_i^\varepsilon+z_{i-1}^\varepsilon}{2}-1}[(U_0^\varepsilon)_z]^2$$
$$+\int_{\frac{z_2^\varepsilon+z_1^\varepsilon}{2}-1}^{\varepsilon^{-1/2}}[(U_0^\varepsilon)_z]^2. \tag{7.14}$$

This in turn implies that

$$\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}}[(U_0^\varepsilon)_z]^2=\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\frac{z_m^\varepsilon+z_{m-1}^\varepsilon}{2}-1}[(U_0^\varepsilon)_z]^2+\sum_{i=1}^{i=m-1}\int_{\frac{z_{i+1}^\varepsilon+z_i^\varepsilon}{2}-1}^{\frac{z_{i+1}^\varepsilon+z_i^\varepsilon}{2}+1}[(U_0^\varepsilon)_z]^2$$
$$+\sum_{i=2}^{i=m-1}\int_{\frac{z_{i+1}^\varepsilon+z_i^\varepsilon}{2}+1}^{\frac{z_i^\varepsilon+z_{i-1}^\varepsilon}{2}-1}[(U_0^\varepsilon)_z]^2+\int_{\frac{z_2^\varepsilon+z_1^\varepsilon}{2}+1}^{\varepsilon^{-1/2}}[(U_0^\varepsilon)_z]^2. \tag{7.15}$$

Moreover, we set

$$I_i^\varepsilon:=\int_{\frac{z_{i+1}^\varepsilon+z_i^\varepsilon}{2}-1}^{\frac{z_{i+1}^\varepsilon+z_i^\varepsilon}{2}+1}[(U_0^\varepsilon)_z]^2. \tag{7.16}$$

After performing the change of variables $z'=z-z_i^\varepsilon$, we obtain

$$\int_{\frac{z_{i+1}^\varepsilon+z_i^\varepsilon}{2}+1}^{\frac{z_i^\varepsilon+z_{i-1}^\varepsilon}{2}-1}[(U_0^\varepsilon)_z]^2=\int_{\frac{z_{i+1}^\varepsilon-z_i^\varepsilon}{2}+1}^{\frac{z_{i-1}^\varepsilon-z_i^\varepsilon}{2}-1}[(U_0^\varepsilon)_z(z'+z_i^\varepsilon)]^2. \tag{7.17}$$

Similarly, we have

$$\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\frac{z_m^\varepsilon+z_{m-1}^\varepsilon}{2}-1}[(U_0^\varepsilon)_z]^2=\int_{-\varepsilon^{-1/2}}^{\frac{z_{m-1}^\varepsilon-z_m^\varepsilon}{2}-1}[(U_0^\varepsilon)_z(z'+z_m^\varepsilon)]^2. \tag{7.18}$$

Substituting (7.16), (7.17), and (7.18) into (7.15), we deduce

$$\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}}[(U_0^\varepsilon)_z]^2=\int_{-\varepsilon^{-1/2}}^{\frac{z_{m-1}^\varepsilon-z_m^\varepsilon}{2}-1}[(U_0^\varepsilon)_z(z+z_m^\varepsilon)]^2+\sum_{i=1}^{i=m-1}I_i^\varepsilon$$
$$+\sum_{i=2}^{i=m-1}\int_{\frac{z_{i+1}^\varepsilon-z_i^\varepsilon}{2}+1}^{\frac{z_{i-1}^\varepsilon-z_i^\varepsilon}{2}-1}[(U_0^\varepsilon)_z(z+z_i^\varepsilon)]^2+\int_{\frac{z_2^\varepsilon+z_1^\varepsilon}{2}+1}^{\varepsilon^{-1/2}}[(U_0^\varepsilon)_z]^2. \tag{7.19}$$

Moreover, for all $z \in [\frac{z^\varepsilon_{i+1}-z^\varepsilon_i}{2}+1, \frac{z^\varepsilon_{i-1}-z^\varepsilon_i}{2}-1]$, we have $z+z^\varepsilon_i \in [\frac{z^\varepsilon_{i+1}+z^\varepsilon_i}{2}+1, \frac{z^\varepsilon_{i-1}+z^\varepsilon_i}{2}-1]$, and thus

$$U^\varepsilon_0(z+z^\varepsilon_i) = (-1)^{i+1}\tanh(z), \text{ for all } i \in [2,m]. \tag{7.20}$$

Similarly, we have

$$U^\varepsilon_0(z+z^\varepsilon_m) = (-1)^{m+1}\tanh(z), \text{ for all } z \in [-\varepsilon^{-\frac{1}{2}}, \frac{z^\varepsilon_{m-1}-z^\varepsilon_m}{2}-1], \tag{7.21}$$

$$\text{and } U^\varepsilon_0(z) = (-1)^2\tanh(z), \text{ for all } z \in [\frac{z^\varepsilon_2+z^\varepsilon_1}{2}+1, \varepsilon^{-\frac{1}{2}}]. \tag{7.22}$$

Substituting (7.20), (7.21), and (7.22) into (7.19), we obtain

$$\begin{aligned}\int_{-\varepsilon^{-1/2}+z^\varepsilon_m}^{\varepsilon^{-1/2}} [(U^\varepsilon_0)_z]^2 &= \sum_{i=1}^{i=m-1} I^\varepsilon_i + \int_{-\varepsilon^{-1/2}+z^\varepsilon_m}^{\frac{z^\varepsilon_{m-1}-z^\varepsilon_m}{2}-1} [1-\tanh^2(z)]^2 \\ &+ \sum_{i=2}^{i=m-1} \int_{\frac{z^\varepsilon_{i+1}-z^\varepsilon_i}{2}+1}^{\frac{z^\varepsilon_{i-1}-z^\varepsilon_i}{2}-1} [1-\tanh^2(z)]^2 + \int_{\frac{z^\varepsilon_2+z^\varepsilon_1}{2}+1}^{\varepsilon^{-1/2}} [1-\tanh^2(z)]^2. \end{aligned} \tag{7.23}$$

In order to pass to the limit in (7.23) as ε tends to zero we next prove the following result:

$$\lim_{\varepsilon\to 0} \sum_{i=1}^{i=m-1} I^\varepsilon_i = 0. \tag{7.24}$$

First we note that by definition of the partition of unity (E^ε_i) we have

$$[\frac{z^\varepsilon_{i+1}+z^\varepsilon_i}{2}-1, \frac{z^\varepsilon_{i+1}+z^\varepsilon_i}{2}+1] \subset \{E^\varepsilon_i \neq 0,1\}, \text{ for all } i \in [1, m-1]. \tag{7.25}$$

In view of (7.25), we deduce that

$$\sum_{i=1}^{i=m-1} I^\varepsilon_i \leq \sum_{i=1}^{i=m-1} \int_{\{E^\varepsilon_i \neq 0,1\}} [(U^\varepsilon_0)_z]^2. \tag{7.26}$$

Next, we prove that

$$\lim_{\varepsilon\to 0} \int_{\{E^\varepsilon_i\neq 0,1\}} [(U^\varepsilon_0)_z]^2 = 0, \text{ for all } i \in [1, m-1]. \tag{7.27}$$

In order to prove (7.27), we first note that

$$\int_{\{E^\varepsilon_i\neq 0,1\}} [(U^\varepsilon_0)_z]^2 \leq \sup_{\{E^\varepsilon_i\neq 0,1\}} |(U^\varepsilon_0)_z|^2 |\{E^\varepsilon_i \neq 0,1\}|. \tag{7.28}$$

Moreover, we have that

$$\sup_{[\frac{r_i^\varepsilon+r_{i-1}^\varepsilon}{2}-\varepsilon,\frac{r_i^\varepsilon+r_{i-1}^\varepsilon}{2}+\varepsilon]} \varepsilon|(u_0^\varepsilon)_r| = \sup_{[\frac{z_i^\varepsilon+z_{i-1}^\varepsilon}{2}-1,\frac{z_i^\varepsilon+z_{i-1}^\varepsilon}{2}+1]} |(U_0^\varepsilon)_z|. \tag{7.29}$$

Using (5.43) (see Corollary 5.6) and (7.29) we deduce that

$$\lim_{\varepsilon\to 0} \sup_{[\frac{z_i^\varepsilon+z_{i-1}^\varepsilon}{2}-1,\frac{z_i^\varepsilon+z_{i-1}^\varepsilon}{2}+1]} (U_0^\varepsilon)_z = 0,$$

and thus

$$\lim_{\varepsilon\to 0} \sup_{\{E_i^\varepsilon\neq 0,1\}} (U_0^\varepsilon)_z = 0. \tag{7.30}$$

Letting ε tend to zero in (7.28) and using (7.30) and the fact that $|\{E_i^\varepsilon \neq 0,1\}| \leq 4$ we conclude that $\lim_{\varepsilon\to 0}\int_{\{E_i^\varepsilon\neq 0,1\}}[(U_0^\varepsilon)_z]^2 = 0$, which coincides with (7.27). Moreover, (7.27) and (7.26) imply (7.24). We are now in a position to prove Lemma 7.4. By Theorem 5.1 we have that

$$\lim_{\varepsilon\to 0} z_{i+1}^\varepsilon - z_i^\varepsilon = -\infty \ \text{ for all } i \in [1, m-1]. \tag{7.31}$$

Since $z_1^\varepsilon = 0$ we have $z_m^\varepsilon = \sum_{i=1}^{i=m-1} z_{i+1}^\varepsilon - z_i^\varepsilon$. Using (7.31) we obtain that

$$\lim_{\varepsilon\to 0} z_m^\varepsilon = -\infty. \tag{7.32}$$

Letting ε tend to zero in (7.23) and using (7.24), (7.31), and (7.32), we deduce that

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} [(U_0^\varepsilon)_z]^2 = m\int_{-\infty}^{+\infty}(1-\tanh^2(z))^2 = \frac{4}{3}m.$$

This completes the proof of Lemma 7.4. □

We now return to the proof of Lemma 7.3; moreover, we consider only the case where $\nu(\overline{r}) = 1$. One can prove the case $\nu(\overline{r}) = -1$ in a similar way. We have

$$\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \frac{1}{\varepsilon z + r_1^\varepsilon}(U_z^\varepsilon)^2 - \int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \frac{1}{\overline{r}}[(U_0^\varepsilon)_z]^2 = \tag{7.33}$$

$$\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \frac{1}{\varepsilon z + r_1^\varepsilon}\left[[U_z^\varepsilon]^2 - [(U_0^\varepsilon)_z]^2\right] + \int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \left[\frac{1}{\varepsilon z + r_1^\varepsilon} - \frac{1}{\overline{r}}\right][(U_0^\varepsilon)_z]^2.$$

Since $\varepsilon z + r_1^\varepsilon = r \geq R_0$, we have that

$$\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \frac{1}{\varepsilon z + r_1^\varepsilon}\left|(U_z^\varepsilon)^2 - [(U_0^\varepsilon)_z]^2\right| \tag{7.34}$$

$$\le \frac{1}{R_0}\Big[\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \big|U_z^\varepsilon-(U_0^\varepsilon)_z\big|^2\Big]^{1/2}\Big[\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \big|U_z^\varepsilon+(U_0^\varepsilon)_z\big|^2\Big]^{1/2}$$

$$\le \frac{\sqrt{2}}{R_0}\Big[\int_{-\varepsilon^{-\frac12}+z_m^\varepsilon}^{\varepsilon^{-\frac12}} \big|U_z^\varepsilon-(U_0^\varepsilon)_z\big|^2\Big]^{\frac12}\Big[\int_{-\varepsilon^{-\frac12}+z_m^\varepsilon}^{\varepsilon^{-\frac12}} |U_z^\varepsilon|^2+\int_{-\varepsilon^{-\frac12}+z_m^\varepsilon}^{\varepsilon^{-\frac12}} |(U_0^\varepsilon)_z|^2\Big]^{\frac12}.$$

Moreover, we have that

$$\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} |U_z^\varepsilon|^2 dz \le \int_{z_-^\varepsilon}^{z_+^\varepsilon} |U_z^\varepsilon|^2 dz = \varepsilon\int_{R_0}^1 |u_r^\varepsilon|^2\ dr. \tag{7.35}$$

In view of (7.35) and the result (3.11) of Lemma 3.1, we deduce that

$$\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} |U_z^\varepsilon|^2 dz \le C(R_0). \tag{7.36}$$

Using (7.34), (7.36), and Lemma 7.4, we deduce that

$$\int_{-\varepsilon^{-\frac12}+z_m^\varepsilon}^{\varepsilon^{-\frac12}} \frac{1}{\varepsilon z+r_1^\varepsilon}\left[(U_z^\varepsilon)^2-[(U_0^\varepsilon)_z]^2\right] \le \tilde{C}(R_0)\Big[\int_{-\varepsilon^{-\frac12}+z_m^\varepsilon}^{\varepsilon^{-\frac12}} |U_z^\varepsilon-(U_0^\varepsilon)_z|^2\Big]^{\frac12}.$$

By (6.61) of Corollary 6.7, we deduce that

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \frac{1}{\varepsilon z+r_1^\varepsilon}\left|(U_z^\varepsilon)^2-[(U_0^\varepsilon)_z]^2\right| = 0.$$

Furthermore, we have for all $z\in[-\varepsilon^{-1/2}+z_m^\varepsilon,\varepsilon^{-1/2}]$

$$\Big|\frac{1}{\varepsilon z+r_1^\varepsilon}-\frac{1}{\overline{r}}\Big| = \Big|\frac{\varepsilon z+r_1^\varepsilon-\overline{r}}{\overline{r}(\varepsilon z+r_1^\varepsilon)}\Big| \le \frac{1}{R_0^2}[\varepsilon^{1/2}+\varepsilon|z_m^\varepsilon|+|r_1^\varepsilon-\overline{r}|]$$
$$= \frac{1}{R_0^2}[\varepsilon^{1/2}+|r_m^\varepsilon-r_1^\varepsilon|+|r_1^\varepsilon-\overline{r}|].$$

This implies that

$$\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \Big[\frac{1}{\varepsilon z+r_1^\varepsilon}-\frac{1}{\overline{r}}\Big][(U_0^\varepsilon)_z]^2$$
$$\le \frac{1}{R_0^2}[\varepsilon^{1/2}+|r_m^\varepsilon-r_1^\varepsilon|+|r_1^\varepsilon-\overline{r}|]\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} |(U_0^\varepsilon)_z|^2. \tag{7.37}$$

Letting ε tend to zero in (7.37) and using Lemma 7.4 we deduce that

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \frac{1}{\varepsilon z+r_1^\varepsilon}[(U_0^\varepsilon)_z]^2 = \frac{4}{3}m\frac{1}{\overline{r}}. \tag{7.38}$$

From (7.36) and (7.38) we deduce that

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \frac{1}{\varepsilon z + r_1^\varepsilon}(U_z^\varepsilon)^2 = \frac{4}{3}m\frac{1}{\overline{r}}.$$

This completes the proof of Lemma 7.3. □

In order to estimate the limit as $\varepsilon \downarrow 0$ of $\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} (W^\varepsilon - V^\varepsilon)U_z^\varepsilon$ we first prove that

$$\lim_{\varepsilon\to 0}\int_{z_m^\varepsilon}^{0}(W^\varepsilon - V^\varepsilon)U_z^\varepsilon = 0. \tag{7.39}$$

Since $U^\varepsilon(0) = U^\varepsilon(z_m^\varepsilon) = 0$ we have that $\int_{z_m^\varepsilon}^0 (W^\varepsilon - V^\varepsilon)U_z^\varepsilon = -\int_{z_m^\varepsilon}^0 (W_z^\varepsilon - V_z^\varepsilon)U^\varepsilon$, which implies that

$$\begin{aligned}\Big|\int_{z_m^\varepsilon}^0 (W^\varepsilon - V^\varepsilon)U_z^\varepsilon\Big| &\le C\Big[\int_{z_m^\varepsilon}^0 |W_z^\varepsilon|^2 dz + \int_{z_m^\varepsilon}^0 |V_z^\varepsilon|^2 dz\Big]^{1/2}\\ &\quad\times\Big[\Big(\int_{z_m^\varepsilon}^0 |U^\varepsilon - U_0^\varepsilon|^2 dz\Big)^{1/2} + \Big(\int_{z_m^\varepsilon}^0 |U_0^\varepsilon|^2 dz\Big)^{1/2}\Big]. \end{aligned}\tag{7.40}$$

Moreover, we have that

$$\begin{aligned}\int_0^z |(W^\varepsilon - V^\varepsilon)_z|^2 &\le C\Big|\int_0^z |W_z^\varepsilon|^2 + |V_z^\varepsilon|^2\Big|\\ &\le C\Big[\int_{z_-^\varepsilon}^{z_+^\varepsilon} |W_z^\varepsilon|^2 + |V_z^\varepsilon|^2\, dz\Big] \le \frac{C}{R_0^{N-1}}\varepsilon\Big[\int_{R_0}^1 (|w_r^\varepsilon|^2 + |v_r^\varepsilon|^2) r^{N-1} dr\Big],\end{aligned}$$

for all $z \in [z_-^\varepsilon, z_+^\varepsilon]$. This with (3.16) and (3.8) implies

$$\int_0^z |(W^\varepsilon - V^\varepsilon)_z|^2 \le C_1(R_0,t)\varepsilon^{1-\gamma}. \tag{7.41}$$

Using (7.41) and the fact that $|U_0^\varepsilon| \le 1$, we obtain that

$$\Big|\int_{z_m^\varepsilon}^0 (W^\varepsilon - V^\varepsilon)U_z^\varepsilon\Big| \le C(R_0,t)\varepsilon^{\frac{1-\gamma}{2}}\Big[\Big(\int_{z_m^\varepsilon}^0 |U^\varepsilon - U_0^\varepsilon|^2 dz\Big)^{\frac12} + |z_m^\varepsilon|^{\frac12}\Big]. \tag{7.42}$$

Substituting (6.75) into (7.42), we obtain

$$\begin{aligned}&\Big|\int_{z_m^\varepsilon}^0 (W^\varepsilon - V^\varepsilon)U_z^\varepsilon\Big| \\ &\le C(R_0,t)\varepsilon^{\frac{1-\gamma}{2}}\Big[\Big(\int_{z_m^\varepsilon}^0 |U^\varepsilon - U_0^\varepsilon|^2 dz\Big)^{1/2} + (m-1)^{1/2}\varepsilon^{-3/8}\Big].\end{aligned}\tag{7.43}$$

Finally, letting ε tend to zero in (7.42) and using (6.61) of Corollary 6.7 we deduce (7.39). Next we prove the following result:

Lemma 7.5. *There exists a subsequence* $\{\varepsilon_n\}$ *and a real* Y_0 *such that*

$$\lim_{\varepsilon_n\to 0}(W^{\varepsilon_n}-V^{\varepsilon_n})(0)=Y_0. \tag{7.44}$$

Moreover, we have

$$\lim_{\varepsilon_n\to 0}\int_{-\varepsilon_n^{-1/2}+z_m^{\varepsilon_n}}^{\varepsilon_n^{-1/2}}(W^{\varepsilon_n}-V^{\varepsilon_n})U_z^{\varepsilon}=2\nu(\overline{r})Y_0. \tag{7.45}$$

Proof. We first prove that the sequence $\{(W^\varepsilon-V^\varepsilon)(0)\}$ is bounded. Thus (7.44) will follow. Suppose that there exists a subsequence $\{(W^{\varepsilon_n}-V^{\varepsilon_n})(0)\}$, which converges to $\pm\infty$. We consider only the case that the sequence $\{(W^{\varepsilon_n}-V^{\varepsilon_n})(0)\}$ converges to $+\infty$; by symmetry one can check the other case. We note that

$$|(W^\varepsilon-V^\varepsilon)(z)-(W^\varepsilon-V^\varepsilon)(0)|\le|z|^{1/2}\Big(\int_0^z|(W^\varepsilon-V^\varepsilon)_z|^2\Big)^{1/2}$$

for all $z\in[\frac{R_0-r_1^\varepsilon}{\varepsilon},\frac{1-r_1^\varepsilon}{\varepsilon}]$. Using (7.41), we obtain

$$|(W^\varepsilon-V^\varepsilon)(z)-(W^\varepsilon-V^\varepsilon)(0)|\le C(R_0,t)|z|^{1/2}\varepsilon^{\frac{1-\gamma}{2}},$$

for all $z\in[\frac{R_0-r_1^\varepsilon}{\varepsilon},\frac{1-r_1^\varepsilon}{\varepsilon}]$. This gives that

$$|(W^\varepsilon-V^\varepsilon)(z)-(W^\varepsilon-V^\varepsilon)(0)|\le C(R_0,t)\varepsilon^{\frac{1-2\gamma}{4}}\text{ for all }z\in[0,\varepsilon^{-\frac12}], \tag{7.46}$$

and similarly since $|z_m^\varepsilon|\le C\varepsilon^{-3/4}$, then

$$|(W^\varepsilon-V^\varepsilon)(z)-(W^\varepsilon-V^\varepsilon)(0)|\le C(R_0,t)\varepsilon^{\frac{1-4\gamma}{8}} \tag{7.47}$$

for all $z\in[-\varepsilon^{-1/2}+z_m^\varepsilon,z_m^\varepsilon]$. Applying (7.46) and (7.47) to the subsequence $\{W^{\varepsilon_n}-V^{\varepsilon_n}\}$, we obtain that

$$\lim_{\varepsilon_n\to 0}\Big(\inf_{[-\varepsilon_n^{-1/2}+z_m^{\varepsilon_n},z_m^{\varepsilon_n}]\cup[0,\varepsilon_n^{-1/2}]}(W^{\varepsilon_n}-V^{\varepsilon_n})(z)\Big)=+\infty. \tag{7.48}$$

We suppose that $W^{\varepsilon_n}-V^{\varepsilon_n}$ is positive on $[0,\varepsilon_n^{-1/2}]$, and similarly one can check the case that $W^{\varepsilon_n}-V^{\varepsilon_n}$ is negative. Moreover, since $U_z^{\varepsilon_n}$ is also positive on this interval, we have

$$\begin{aligned}&\Big|\int_0^{\varepsilon_n^{-1/2}}(W^{\varepsilon_n}-V^{\varepsilon_n})U_z^{\varepsilon_n}\Big|=\int_0^{\varepsilon_n^{-1/2}}(W^{\varepsilon_n}-V^{\varepsilon_n})U_z^{\varepsilon_n}\\&\ge\inf_{[0,\varepsilon_n^{-1/2}]}(W^{\varepsilon_n}-V^{\varepsilon_n})\int_0^{\varepsilon_n^{-1/2}}U_z^{\varepsilon_n}\\&\ge\inf_{[0,\varepsilon_n^{-1/2}]}(W^{\varepsilon_n}-V^{\varepsilon_n})\{[U^{\varepsilon_n}(\varepsilon_n^{-1/2})-U_0^{\varepsilon_n}(\varepsilon_n^{-1/2})]+U_0^{\varepsilon_n}(\varepsilon_n^{-1/2})\}.\end{aligned}$$

Using the definition of U_0^ε, we deduce

$$\Big| \int_0^{\varepsilon_n^{-1/2}} (W^{\varepsilon_n} - V^{\varepsilon_n}) U_z^{\varepsilon_n} \Big| \geq \inf_{[0, \varepsilon_n^{-1/2}]} (W^{\varepsilon_n} - V^{\varepsilon_n}) \{ U^{\varepsilon_n}(\varepsilon^{-1/2}) - U_0^{\varepsilon_n}(\varepsilon^{-1/2}) + \tanh(\varepsilon_n^{-1/2}) \}.$$

Letting ε tend to zero and using (7.48) we deduce that

$$\lim_{\varepsilon \to 0} \int_0^{\varepsilon_n^{-1/2}} (W^{\varepsilon_n} - V^{\varepsilon_n}) U_z^{\varepsilon_n} = +\infty.$$

Similarly, one can prove that

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon_n^{-1/2} + z_m^{\varepsilon_n}}^{z_m^{\varepsilon_n}} (W^{\varepsilon_n} - V^{\varepsilon_n}) U_z^{\varepsilon_n} = +\infty.$$

Therefore, using (7.39) we obtain

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon_n^{-1/2} + z_m^{\varepsilon_n}}^{\varepsilon_n^{-1/2}} (W^{\varepsilon_n} - V^{\varepsilon_n}) U_z^{\varepsilon_n} = +\infty. \tag{7.49}$$

Letting ε_n tend to zero in (7.2) using Lemmas 7.2, 7.3, and (7.49) we obtain that each integral of (7.2) except one converge, which is impossible. Thus we conclude that the sequence $\{(W^\varepsilon - V^\varepsilon)\}$ is bounded, which implies (7.44).

Next we prove (7.45). In this proof we omit the indices n and replace ε_n by ε; moreover, we prove only the case where $\nu(\overline{r}) = 1$. Similarly, one can check the case where $\nu(\overline{r}) = -1$. We first note that

$$\begin{aligned} \Big| \int_{-\varepsilon^{-1/2} + z_m^\varepsilon}^{\varepsilon^{-1/2}} (W^\varepsilon - V^\varepsilon) U_z^\varepsilon - 2Y_0 \Big| &\leq \Big| \int_{-\varepsilon^{-1/2} + z_m^\varepsilon}^{z_m^\varepsilon} (W^\varepsilon - V^\varepsilon - Y_0) U_z^\varepsilon \Big| \\ &+ \Big| \int_0^{\varepsilon^{-1/2}} (W^\varepsilon - V^\varepsilon - Y_0) U_z^\varepsilon \Big| + \Big| \int_{z_m^\varepsilon}^0 (W^\varepsilon - V^\varepsilon) U_z^\varepsilon \Big| \\ &+ \Big| Y_0 \Big[\int_{-\varepsilon^{-1/2} + z_m^\varepsilon}^{z_m^\varepsilon} U_z^\varepsilon + \int_0^{\varepsilon^{-1/2}} U_z^\varepsilon - 2 \Big] \Big|. \end{aligned} \tag{7.50}$$

Next we estimate the limit as ε tend to zero of all the terms of (7.50). We first prove that

$$\lim_{\varepsilon \to 0} \int_0^{\varepsilon^{-1/2}} (W^\varepsilon - V^\varepsilon - Y_0) U_z^\varepsilon = 0. \tag{7.51}$$

We have

$$[(W^\varepsilon - V^\varepsilon)(z) - Y_0] U_z^\varepsilon = [(W^\varepsilon - V^\varepsilon)(z) - (W^\varepsilon - V^\varepsilon)(0)] U_z^\varepsilon \tag{7.52}$$

$$+ [(W^\varepsilon - V^\varepsilon)(0) - Y_0]U_z^\varepsilon = \Big(\int_0^z (W^\varepsilon - V^\varepsilon)_z\Big) U_z^\varepsilon + [(W^\varepsilon - V^\varepsilon)(0) - Y_0]U_z^\varepsilon.$$

Integrating (7.52) on $[0, \varepsilon^{-1/2}]$, we obtain

$$\Big|\int_0^{\varepsilon^{-\frac{1}{2}}} [(W^\varepsilon - V^\varepsilon)(z) - Y_0]U_z^\varepsilon \, dz\Big| \le \int_0^{\varepsilon^{-\frac{1}{2}}} \Big|\int_0^z |(W^\varepsilon - V^\varepsilon)_z|^2\Big|^{\frac{1}{2}} |z|^{\frac{1}{2}} |U_z^\varepsilon|$$

$$+ \Big|\int_0^{\varepsilon^{-\frac{1}{2}}} [(W^\varepsilon - V^\varepsilon)(0) - Y_0]U_z^\varepsilon \, dz\Big|. \tag{7.53}$$

Substituting (7.41) into (7.53) we obtain

$$\Big|\int_0^{\varepsilon^{-1/2}} [(W^\varepsilon - V^\varepsilon)(z) - Y_0]U_z^\varepsilon \, dz\Big| \tag{7.54}$$

$$\le \Big[C_2(R_0,t)\varepsilon^{\frac{1-2\gamma}{4}} + |(W^\varepsilon - V^\varepsilon)(0) - Y_0|\Big] \int_0^{\varepsilon^{-1/2}} |U_z^\varepsilon| \, dz.$$

Furthermore, we have

$$\int_0^{\varepsilon^{-1/2}} |U_z^\varepsilon| \, dz \le \int_0^{\varepsilon^{-1/2}} |U_z^\varepsilon - (U_0^\varepsilon)_z| \, dz + \int_0^{\varepsilon^{-1/2}} |(U_0^\varepsilon)_z| \, dz$$

$$\le C\|U_z^\varepsilon - (U_0^\varepsilon)_z\|_{L^\infty(J)} \varepsilon^{-1/2} + \int_0^{\varepsilon^{-1/2}} |(U_0^\varepsilon)_z| \, dz.$$

Using (6.62) and the fact that $(U_0^\varepsilon)_z(z) = 1 - \tanh^2(z) \ge 0$ on $[0, \varepsilon^{-1/2}]$, we deduce

$$\int_0^{\varepsilon^{-1/2}} |U_z^\varepsilon| \, dz \le C\varepsilon^{1/10} + \tanh(\varepsilon^{-1/2}). \tag{7.55}$$

Substituting (7.55) into (7.54), we obtain

$$\Big|\int_0^{\varepsilon^{-1/2}} [(W^\varepsilon - V^\varepsilon)(z) - Y_0]U_z^\varepsilon \, dz\Big| \tag{7.56}$$

$$\le \Big[C_2(R_0,t)\varepsilon^{\frac{1-2\gamma}{4}} + [(W^\varepsilon - V^\varepsilon)(0) - Y_0]\Big]\Big[C\varepsilon^{1/10} + \tanh(\varepsilon^{-1/2})\Big],$$

which implies (7.51). Similarly we prove that

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon^{-1/2} + z_m^\varepsilon}^{z_m^\varepsilon} (W^\varepsilon - V^\varepsilon - Y_0)U_z^\varepsilon = 0. \tag{7.57}$$

Integrating (7.52) on $[-\varepsilon^{-1/2} + z_m^\varepsilon, z_m^\varepsilon]$, we have

$$\Big|\int_{-\varepsilon^{-1/2} + z_m^\varepsilon}^{z_m^\varepsilon} [(W^\varepsilon - V^\varepsilon)(z) - Y_0]U_z^\varepsilon \, dz\Big| \tag{7.58}$$

$$\leq \int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{z_m^\varepsilon} \Big| \int_0^z |(W^\varepsilon - V^\varepsilon)_z|^2 \Big|^{1/2} |z|^{1/2} |U_z^\varepsilon|$$
$$+ \Big| \int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{z_m^\varepsilon} [(W^\varepsilon - V^\varepsilon)(0) - Y_0] U_z^\varepsilon \, dz \Big|.$$

This together with (7.41) gives

$$\Big| \int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{z_m^\varepsilon} [(W^\varepsilon - V^\varepsilon)(z) - Y_0] U_z^\varepsilon \, dz \Big| \leq \Big[C_2(R_0,t) \varepsilon^{\frac{1-\gamma}{2}} (\varepsilon^{-1/2} + |z_m^\varepsilon|)^{1/2}$$
$$+ [(W^\varepsilon - V^\varepsilon)(0) - Y_0] \Big] \int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{z_m^\varepsilon} |U_z^\varepsilon| \, dz. \tag{7.59}$$

Moreover, we have

$$\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{z_m^\varepsilon} |U_z^\varepsilon| \, dz \leq \int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{z_m^\varepsilon} |U_z^\varepsilon - (U_0^\varepsilon)_z| \, dz + \int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{z_m^\varepsilon} |(U_0^\varepsilon)_z| \, dz$$
$$\leq C \|U_z^\varepsilon - (U_0^\varepsilon)_z\|_{L^\infty(J)} \varepsilon^{-1/2} + \int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{z_m^\varepsilon} |(U_0^\varepsilon)_z| dz.$$

Using (6.62) and the fact that $(U_0^\varepsilon)_z(z) = 1 - \tanh^2(z - z_m^\varepsilon) \geq 0$ on $[-\varepsilon^{-1/2} + z_m^\varepsilon, z_m^\varepsilon]$, we deduce

$$\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{z_m^\varepsilon} |U_z^\varepsilon| \, dz \leq C\varepsilon^{1/10} + \tanh(\varepsilon^{-1/2}). \tag{7.60}$$

Substituting (7.60) and (6.75) into (7.59), we obtain

$$\Big| \int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{z_m^\varepsilon} [(W^\varepsilon - V^\varepsilon)(z) - Y_0] U_z^\varepsilon \, dz \Big| \tag{7.61}$$
$$\leq \Big[C_2(R_0,t) \varepsilon^{\frac{1-4\gamma}{8}} + [(W^\varepsilon - V^\varepsilon)(0) - Y_0] \Big] \Big[C\varepsilon^{1/10} + \tanh(\varepsilon^{-1/2}) \Big].$$

Letting ε tend to zero in (7.61) we obtain (7.57). Next we estimate the limit of the last term of (7.50), namely,

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{z_m^\varepsilon} U_z^\varepsilon + \int_0^{\varepsilon^{-1/2}} U_z^\varepsilon = 2. \tag{7.62}$$

We have

$$\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{z_m^\varepsilon} U_z^\varepsilon + \int_0^{\varepsilon^{-1/2}} U_z^\varepsilon - 2 = -U^\varepsilon(-\varepsilon^{-1/2} + z_m^\varepsilon) + U_0^\varepsilon(-\varepsilon^{-1/2} + z_m^\varepsilon)$$
$$+ U^\varepsilon(\varepsilon^{-1/2}) - U_0^\varepsilon(\varepsilon^{-1/2}) - U_0^\varepsilon(-\varepsilon^{-1/2} + z_m^\varepsilon) + U_0^\varepsilon(\varepsilon^{-1/2}) - 2. \tag{7.63}$$

Since m is odd, we have

$$U_0^\varepsilon(-\varepsilon^{-1/2}+z_m^\varepsilon)=(-1)^{m+1}\tanh(-\varepsilon^{-1/2})\to_{\varepsilon\to 0}(-1)^m=-1 \tag{7.64}$$

$$\text{and}\quad U_0^\varepsilon(\varepsilon^{-1/2})=\tanh(\varepsilon^{-1/2})\to_{\varepsilon\to 0}1. \tag{7.65}$$

We deduce from (7.63), (7.64), (7.65) and from (6.62) of Corollary 6.7 that $\lim_{\varepsilon\to 0}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{z_m^\varepsilon}U_z^\varepsilon+\int_0^{\varepsilon^{-1/2}}U_z^\varepsilon=2$. This completes the proof of (7.62). □

Finally, letting ε tend to zero in (7.50) and using (7.39), (7.51), (7.57) and (7.62) we deduce that

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}}(W^\varepsilon-V^\varepsilon)U_z^\varepsilon=2Y_0,$$

which coincides with (7.45). This completes the proof of Lemma 7.5. □

We now prove Theorem 7.1. Let $\{\varepsilon_n\}$ be the sequence defined in Lemma 7.5. Letting $\varepsilon_n\to 0$ in (7.2) and using Lemmas 7.2, 7.3 and 7.5 we deduce

$$-(N-1)\frac{4}{3}\frac{m}{\overline{r}}=2\nu Y_0. \tag{7.66}$$

Next we prove by contradiction that $m=1$, and moreover we omit the indices n and replace ε_n by ε in this proof. Suppose that there exist $m>1$ zeros $z_m^\varepsilon<\cdots<z_2^\varepsilon<z_1^\varepsilon=0$ of U^ε; thus, there exists $c^\varepsilon\in[z_2^\varepsilon,z_1^\varepsilon]$ such that $U_z^\varepsilon(c^\varepsilon)=0$ and such that U^ε is of constant sign on $[c^\varepsilon,z_1^\varepsilon]$. Using Corollary 5.5, there exists b_0^ε such that

$$z_2^\varepsilon+b_0^\varepsilon\le c^\varepsilon\le z_1^\varepsilon-b_0^\varepsilon=-b_0^\varepsilon,\ z_2^\varepsilon-z_1^\varepsilon<-2(b_0^\varepsilon+1)\ \text{ and }\ \lim_{\varepsilon\to 0}b_0^\varepsilon=+\infty. \tag{7.67}$$

Multiplying (7.1) by U_z^ε and integrating the result on $[c^\varepsilon,\varepsilon^{-1/2}]$ we obtain an equivalent result of (7.2), namely,

$$\int_{c^\varepsilon}^{\varepsilon^{-\frac12}}(W^\varepsilon-V^\varepsilon)U_z^\varepsilon=-\frac{1}{\varepsilon}\int_{c^\varepsilon}^{\varepsilon^{-\frac12}}\big[\tfrac12(U_z^\varepsilon)^2-F(U^\varepsilon)\big]_z-(N-1)\int_{c^\varepsilon}^{\varepsilon^{-\frac12}}\frac{1}{\varepsilon z+r_1^\varepsilon}(U_z^\varepsilon)^2. \tag{7.68}$$

In order to pass to the limit in (7.68), we first prove that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{c^\varepsilon}^{\varepsilon^{-1/2}}\big[\frac12(U_z^\varepsilon)^2-F(U^\varepsilon)\big]_z=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}F(U^\varepsilon(c^\varepsilon)). \tag{7.69}$$

Using the fact that $U_z^\varepsilon(c^\varepsilon)=0$, we obtain

$$\frac{1}{\varepsilon}\int_{c^\varepsilon}^{\varepsilon^{-\frac12}}\big[\frac12(U_z^\varepsilon)^2-F(U^\varepsilon)\big]_z=\frac{1}{\varepsilon}\big[\frac12U_z^\varepsilon(\varepsilon^{-\frac12})-F(U^\varepsilon(\varepsilon^{-\frac12}))+F(U^\varepsilon(c^\varepsilon))\big]. \tag{7.70}$$

Letting ε tend to zero in (7.70) and using (7.6), (7.8) and (7.11) we deduce (7.69). Next we prove that

$$\lim_{\varepsilon\to 0}\int_{c^\varepsilon}^{\varepsilon^{-1/2}} \frac{1}{\varepsilon z + r_1^\varepsilon}(U_z^\varepsilon)^2 = \frac{4}{3}\frac{1}{\overline{r}}. \tag{7.71}$$

We first show that

$$\lim_{\varepsilon\to 0}\int_{c^\varepsilon}^{\varepsilon^{-1/2}} \frac{1}{\varepsilon z + r_1^\varepsilon}(U_z^\varepsilon)^2 = \lim_{\varepsilon\to 0}\frac{1}{\overline{r}}\int_{c^\varepsilon}^{\varepsilon^{-1/2}} [(U_0^\varepsilon)_z]^2. \tag{7.72}$$

We have

$$\Big|\int_{c^\varepsilon}^{\varepsilon^{-1/2}} \frac{1}{\varepsilon z + r_1^\varepsilon}(U_z^\varepsilon)^2 - \int_{c^\varepsilon}^{\varepsilon^{-1/2}} \frac{1}{\overline{r}}[(U_0^\varepsilon)_z]^2\Big| \leq \tag{7.73}$$
$$\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \frac{1}{\varepsilon z + r_1^\varepsilon}\Big|(U_z^\varepsilon)^2 - [(U_0^\varepsilon)_z]^2\Big| + \int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{\varepsilon^{-1/2}} \Big|\frac{1}{\varepsilon z + r_1^\varepsilon} - \frac{1}{\overline{r}}\Big|[(U_0^\varepsilon)_z]^2.$$

Using (7.36), (7.38) and (7.73) we deduce (7.72). Moreover, we prove that

$$\lim_{\varepsilon\to 0}\int_{c^\varepsilon}^{\varepsilon^{-1/2}} [(U_0^\varepsilon)_z]^2 = \frac{4}{3}. \tag{7.74}$$

We have

$$\int_{c^\varepsilon}^{\varepsilon^{-1/2}} [(U_0^\varepsilon)_z]^2 = \int_{c^\varepsilon}^{-b_0^\varepsilon}[(U_0^\varepsilon)_z]^2 + \int_{-b_0^\varepsilon}^{\varepsilon^{-1/4}} [(U_0^\varepsilon)_z]^2. \tag{7.75}$$

In view of (7.67), we have that $[-b_0^\varepsilon, \varepsilon^{-1/4}] \subset [\frac{z_2^\varepsilon - z_1^\varepsilon}{2}+1, \varepsilon^{-1/4}]$; thus, $U_0^\varepsilon(z) = \tanh(z)$ on $[-b_0^\varepsilon, \varepsilon^{-1/4}]$. This gives

$$\int_{-b_0^\varepsilon}^{\varepsilon^{-1/4}} [(U_0^\varepsilon)_z]^2 = \int_{-b_0^\varepsilon}^{\varepsilon^{-1/4}} [1-\tanh^2(z)]^2 \to \frac{4}{3} \tag{7.76}$$

as $\varepsilon \downarrow 0$. Moreover, using (7.67) we have that

$$\int_{c^\varepsilon}^{-b_0^\varepsilon}[(U_0^\varepsilon)_z]^2 \leq \int_{z_2^\varepsilon+b_0^\varepsilon}^{-b_0^\varepsilon}[(U_0^\varepsilon)_z]^2 \leq |z_2^\varepsilon| \sup_{[z_2^\varepsilon+b_0^\varepsilon, -b_0^\varepsilon]}[(U_0^\varepsilon)_z]^2 \leq \varepsilon \sup_{[r_2^\varepsilon+\varepsilon b_0^\varepsilon, r_1^\varepsilon-\varepsilon b_0^\varepsilon]}[(u_0^\varepsilon)_r]^2.$$

Using (5.42), we have

$$\lim_{\varepsilon\to 0}\int_{z_2^\varepsilon+b_0^\varepsilon}^{-b_0^\varepsilon}[(U_0^\varepsilon)_z]^2 = 0. \tag{7.77}$$

Finally using (7.75), (7.76), and (7.77) we obtain (7.74). Next we prove

$$\lim_{\varepsilon\to 0}\int_{c^\varepsilon}^{\varepsilon^{-1/2}} (W^\varepsilon - V^\varepsilon)U_z^\varepsilon = 2\nu Y_0. \tag{7.78}$$

As in the proof of Lemma 7.5, we note that

$$\Big|\int_{c^\varepsilon}^{\varepsilon^{-1/2}}(W^\varepsilon - V^\varepsilon)U_z^\varepsilon - 2\nu Y_0\Big| \le \Big|\int_{c^\varepsilon}^{z_1^\varepsilon=0}(W^\varepsilon - V^\varepsilon - Y_0)U_z^\varepsilon\Big|$$
$$+\Big|\int_0^{\varepsilon^{-1/2}}|(W^\varepsilon - V^\varepsilon - Y_0)U_z^\varepsilon\Big| + \Big|Y_0\Big[\int_{c^\varepsilon}^{\varepsilon^{-1/2}}U_z^\varepsilon - 2\nu(\overline{r})\Big]\Big|, \tag{7.79}$$

and we prove in the same way that all the terms of (7.79) tend to zero. We first prove that

$$\lim_{\varepsilon\to 0}\int_{c^\varepsilon}^{z_1^\varepsilon=0}(W^\varepsilon - V^\varepsilon - Y_0)U_z^\varepsilon = 0. \tag{7.80}$$

Integrating (7.52) on $[c^\varepsilon, z_1^\varepsilon = 0]$, we obtain an equivalent result of (7.53), namely,

$$\Big|\int_{c^\varepsilon}^0[(W^\varepsilon - V^\varepsilon)(z) - Y_0]U_z^\varepsilon\,dz\Big| \le \int_{c^\varepsilon}^0\Big|\int_0^z|(W^\varepsilon - V^\varepsilon)_z|^2\Big|^{1/2}|z|^{1/2}|U_z^\varepsilon|$$
$$+\Big|\int_0^{\varepsilon^{-1/2}}[(W^\varepsilon - V^\varepsilon)(0) - Y_0]U_z^\varepsilon\,dz\Big|. \tag{7.81}$$

Substituting (7.41) into (7.81), we obtain

$$\Big|\int_{c^\varepsilon}^0[(W^\varepsilon - V^\varepsilon)(z) - Y_0]U_z^\varepsilon\,dz\Big| \tag{7.82}$$
$$\le \Big[C_1(R_0,t)\varepsilon^{\frac{1-\gamma}{2}}|c^\varepsilon|^{1/2} + [(W^\varepsilon - V^\varepsilon)(0) - Y_0]\Big]\int_{c^\varepsilon}^0|U_z^\varepsilon|\,dz.$$

Since U_z^ε is of constant sign on $[c^\varepsilon, 0]$, we have $\int_{c^\varepsilon}^0|U_z^\varepsilon|\,dz = |U^\varepsilon(c^\varepsilon)|$. Moreover,

$$\begin{aligned}U_0^\varepsilon(c^\varepsilon) &= E_1^\varepsilon(c^\varepsilon)\tanh(c^\varepsilon) + E_2^\varepsilon(c^\varepsilon)\tanh(z_2^\varepsilon - c^\varepsilon)\\ &= \tanh(c^\varepsilon) + E_2^\varepsilon(c^\varepsilon)[\tanh(z_2^\varepsilon - c^\varepsilon) - \tanh(c^\varepsilon)].\end{aligned}$$

Using (7.67) we have that $z_2^\varepsilon - c^\varepsilon$ and c^ε tend to $-\infty$ as $\varepsilon\downarrow 0$; this gives that

$$\lim_{\varepsilon\to 0}U_0^\varepsilon(c^\varepsilon) = -1. \tag{7.83}$$

Using (7.82), (7.83) and the fact that (6.75) implies $|c^\varepsilon| \le |z_2^\varepsilon| \le \varepsilon^{-3/4}$, we obtain

$$\Big|\int_{c^\varepsilon}^0[(W^\varepsilon - V^\varepsilon)(z) - Y_0]U_z^\varepsilon\,dz\Big|$$
$$\le \Big[C_1(R_0,t)\varepsilon^{\frac{1-\gamma}{8}} + [(W^\varepsilon - V^\varepsilon)(0) - Y_0]\Big]\Big[|U^\varepsilon(c^\varepsilon) - U_0^\varepsilon(c^\varepsilon)| + |U_0^\varepsilon(c^\varepsilon)|\Big].$$

This with (7.44) and (6.63) implies (7.80). Next we show that in the case $\nu(\overline{r}) = 1$,

$$\lim_{\varepsilon\to 0}\left|Y_0\Big[\int_{c^\varepsilon}^{\varepsilon^{-1/2}} U_z^\varepsilon - 2\nu(\overline{r})\Big]\right| = 0, \tag{7.84}$$

$$\Big|\int_{c^\varepsilon}^{\varepsilon^{-\frac{1}{2}}} U_z^\varepsilon - 2\Big| \le |U^\varepsilon(\varepsilon^{-\frac{1}{2}}) - U_0^\varepsilon(\varepsilon^{-\frac{1}{2}})| + |U^\varepsilon(c^\varepsilon) - U_0^\varepsilon(c^\varepsilon)| + |U_0^\varepsilon(\varepsilon^{-\frac{1}{2}}) - U_0^\varepsilon(c^\varepsilon) - 2|.$$

Using (7.83), the fact that $U_0^\varepsilon(\varepsilon^{-1/2}) = \tanh(\varepsilon^{-1/2})$, and (6.62) we deduce (7.84). In view of (7.79), (7.80), (7.84), and (7.51) we deduce (7.78).

Letting ε tend to zero in (7.68) and using (7.69), (7.71), and (7.78) we obtain

$$-\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}F(U^\varepsilon(c^\varepsilon)) - (N-1)\frac{4}{3}\frac{1}{\overline{r}} = 2\nu Y_0. \tag{7.85}$$

Similarly, let $d^\varepsilon \in [z_m^\varepsilon, z_{m-1}^\varepsilon]$ be such that $U_z^\varepsilon(d^\varepsilon) = 0$ and U_z^ε does not vanish on $[z_m^\varepsilon, d^\varepsilon]$. Multiplying (7.1) by U_z^ε and integrating on $[-\varepsilon^{-1/2} + z_m^\varepsilon, d^\varepsilon]$, we obtain an equivalent result of (7.68), namely,

$$\begin{aligned} &\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{d^\varepsilon} (W^\varepsilon - V^\varepsilon)U_z^\varepsilon \\ &= -\frac{1}{\varepsilon}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{d^\varepsilon} [\tfrac{1}{2}(U_z^\varepsilon)^2 - F(U^\varepsilon)]_z - (N-1)\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{d^\varepsilon} \frac{1}{\varepsilon z + r_1^\varepsilon}(U_z^\varepsilon)^2. \end{aligned} \tag{7.86}$$

Following the proofs of (7.69), (7.71), and (7.76) one can show that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{d^\varepsilon} \big[\frac{1}{2}(U_z^\varepsilon)^2 - F(U^\varepsilon)\big]_z = -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}F(U^\varepsilon(d^\varepsilon)), \\ &\lim_{\varepsilon\to 0}\int_{-\varepsilon^{-1/2}+z_m^\varepsilon}^{d^\varepsilon} \frac{1}{\varepsilon z + r_1^\varepsilon}(U_z^\varepsilon)^2 = \frac{4}{3}\frac{1}{\overline{r}}, \\ &\lim_{\varepsilon\to 0}\int_{-\varepsilon^{-1/2}+z_2^\varepsilon}^{d^\varepsilon} (W^\varepsilon - V^\varepsilon)U_z^\varepsilon = 2\nu Y_0. \end{aligned}$$

This with (7.86) implies that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}F(U^\varepsilon(d^\varepsilon)) - (N-1)\frac{4}{3}\frac{1}{\overline{r}} = 2\nu Y_0. \tag{7.87}$$

In view of (7.85) and (7.87) and the fact that $F \ge 0$, we obtain

$$-(N-1)\frac{4}{3}\frac{1}{\overline{r}} = 2\nu Y_0. \tag{7.88}$$

Using (7.88) and (7.66) we deduce that $m = 1$. Moreover, we have also shown that Y_0 defined by the equation (7.88) is the only cluster point of $\{W^\varepsilon(0) - V^\varepsilon(0)\}$. This completes the proof of Theorem 7.1.

We summarize the results of this section as follows:

Corollary 7.6. *Let $t \in [0,T] \setminus D(T)$; then for all $\overline{r}(t)$ a jump of $u(\cdot,t)$ there exists $r^\varepsilon(t)$ a zero of $u^\varepsilon(\cdot,t)$ such that*

$$\begin{cases} \lim_{\varepsilon\to 0} r^\varepsilon(t) = \overline{r}(t) & (7.89) \\ \lim_{\varepsilon\to 0} (w^\varepsilon - v^\varepsilon)(r^\varepsilon(t),t) = Y_0(\overline{r}(t)) & (7.90) \end{cases}$$

where $Y_0(\overline{r}(t))$ satisfies

$$-(N-1)\frac{2}{3}\frac{1}{\overline{r}(t)} = \nu(\overline{r}(t))Y_0(\overline{r}(t)). \tag{7.91}$$

7.2. **Identification of Y_0.** We prove in this section the following result:

Lemma 7.7. *Let $I \subset [0,T] \setminus D(T)$ be the domain of definition of a jump $\overline{r}(t)$; then $Y_0(\overline{r}(t)) = (w-v)(\overline{r}(t),t)$ almost everywhere in I.*

Proof. In order to prove Lemma 7.7, we first note that (7.89), (7.90) and Egoroff's lemma imply that for all $\delta > 0$ there exists $I_\delta \subset I$ such that $meas(I \setminus I_\delta) \le \delta$ and such that

$$\lim_{\varepsilon\to 0} r^\varepsilon(t) = \overline{r}(t) \text{ and } \lim_{\varepsilon\to 0}(w^\varepsilon - v^\varepsilon)(r^\varepsilon(t),t) = Y_0(\overline{r}(t)) \tag{7.92}$$

uniformly on I_δ. Next we show a preliminary result, namely,

$$\lim_{\varepsilon\to 0} \int_{I_\delta} \rho\,[(w^\varepsilon - v^\varepsilon)(r^\varepsilon(t),t) - (w-v)(\overline{r}(t),t)]\,dt = 0, \tag{7.93}$$

for all $\delta > 0$ and $\rho \in L^\infty(0,T)$. We have

$$\begin{aligned} &(w^\varepsilon - v^\varepsilon)(r^\varepsilon(t),t) - (w-v)(\overline{r}(t),t) && (7.94) \\ &= (w^\varepsilon - v^\varepsilon)(r^\varepsilon(t),t) - (w^\varepsilon - v^\varepsilon)(\overline{r}(t),t) + [(w^\varepsilon - v^\varepsilon) - (w-v)](\overline{r}(t),t). \end{aligned}$$

Moreover, we note that

$$\begin{aligned} &\int_{\overline{r}(t)}^{\overline{r}(t)+\mu} \Big[\int_r^{\overline{r}(t)} [(w^\varepsilon - v^\varepsilon) - (w-v)]_r(s,t)\,ds\Big]\,dr && (7.95) \\ &= \int_{\overline{r}(t)}^{\overline{r}(t)+\mu} ([(w^\varepsilon - v^\varepsilon) - (w-v)](\overline{r}(t),t) - [(w^\varepsilon - v^\varepsilon) - (w-v)](r,t))\,dr \\ &= \mu[(w^\varepsilon - v^\varepsilon) - (w-v)](\overline{r}(t),t) - \int_{\overline{r}(t)}^{\overline{r}(t)+\mu} [(w^\varepsilon - v^\varepsilon) - (w-v)](r,t)\,dr, \end{aligned}$$

for all μ small enough. Substituting (7.94) into (7.2), we obtain

$$(w^\varepsilon - v^\varepsilon)(r^\varepsilon(t),t) - (w-v)(\overline{r}(t),t) = \int_{\overline{r}(t)}^{r^\varepsilon(t)} (w^\varepsilon - v^\varepsilon)_r(s,t)\,ds$$
$$+ \frac{1}{\mu}\int_{\overline{r}(t)}^{\overline{r}(t)+\mu}\int_r^{\overline{r}(t)} [(w^\varepsilon - v^\varepsilon) - (w-v)]_r\,ds\,dr$$
$$+ \frac{1}{\mu}\int_{\overline{r}(t)}^{\overline{r}(t)+\mu} [(w^\varepsilon - v^\varepsilon) - (w-v)](r,t)\,dr. \tag{7.96}$$

Multiplying (7.96) by ρ and integrating the result on I_δ, we obtain

$$\Big|\int_{I_\delta} \rho(t)\,[(w^\varepsilon - v^\varepsilon)(r^\varepsilon(t),t) - (w-v)(\overline{r}(t),t)]\ dt\Big| \tag{7.97}$$
$$\le H_{I_\delta} + J_{I_\delta} + \frac{1}{\mu}\Big|\int_{I_\delta}\int_{\overline{r}(t)}^{\overline{r}(t)+\mu} \rho(t)[(w^\varepsilon - v^\varepsilon) - (w-v)](r,t)\,dr\Big|,$$

where we have used

$$H_{I_\delta} := 2\int_{I_\delta} |\rho(t)||r^\varepsilon(t) - \overline{r}(t)|^{1/2}\Big(\int_{\overline{r}(t)}^{r^\varepsilon(t)} (|w_r^\varepsilon|^2 + |v_r^\varepsilon|^2)\,dr\Big)^{1/2} dt \tag{7.98}$$

and

$$J_{I_\delta} := \frac{2}{\mu}\int_{I_\delta}|\rho(t)|\int_{\overline{r}(t)}^{\overline{r}(t)+\mu}|r-\overline{r}(t)|^{\frac{1}{2}}\Big(\int_{\overline{r}(t)}^{r}(|w_r^\varepsilon|^2 + |v_r^\varepsilon|^2 + |w_r|^2 + |v_r|^2)\,dr\Big)^{\frac{1}{2}}dr\,dt. \tag{7.99}$$

Next we estimate H_{I_δ}.

$$H_{I_\delta} \le \Big(\int_{I_\delta} |\rho(t)|^2|r^\varepsilon(t) - \overline{r}(t)|\,dt\Big)^{1/2}\Big(\int_{I_\delta}\int_{\overline{r}(t)}^{r^\varepsilon(t)} (|w_r^\varepsilon|^2 + |v_r^\varepsilon|^2)\,dr\,dt\Big)^{1/2}.$$

Moreover, since $\overline{r}(t)$ and $r^\varepsilon(t)$ are in $(R_0,1)$, we have

$$\int_{\overline{r}(t)}^{r^\varepsilon(t)} (|w_r^\varepsilon|^2 + |v_r^\varepsilon|^2)\,dr \le \frac{1}{R_0^{N-1}}\int_0^1 (|w_r^\varepsilon|^2 + |v_r^\varepsilon|^2)r^{N-1}\,dr.$$

Integrating on $I_\delta \subset (0,T)$ and using (3.8), we obtain

$$\int_{I_\delta}\int_{\overline{r}(t)}^{r^\varepsilon(t)} (|w_r^\varepsilon|^2 + |v_r^\varepsilon|^2)\,dr \le C_1(R_0), \tag{7.100}$$

and thus

$$H_{I_\delta} \le C_1(R_0)^{1/2}\sup_{I_\delta}|\rho|\sup_{I_\delta}|r^\varepsilon(t) - \overline{r}(t)|^{1/2} meas(I_\delta)^{1/2}. \tag{7.101}$$

Next we estimate J_{I_δ}. For all $1 > \mu > 0$ and $r \in [\overline{r}, \overline{r} + \mu]$, we have

$$\int_{\overline{r}(t)}^{r} (|w_r^\varepsilon|^2 + |v_r^\varepsilon|^2 + |w_r|^2 + |v_r|^2)\, dr \tag{7.102}$$
$$\leq \frac{1}{R_0^{N-1}} \int_0^1 (|w_r^\varepsilon|^2 + |v_r^\varepsilon|^2 + |w_r|^2 + |v_r|^2) r^{N-1}\, dr.$$

Using (7.102) and (7.99), we obtain

$$J_{I_\delta} \leq \frac{1}{\mu}\Big[\int_{I_\delta} |\rho(t)|^2 \Big(\int_{\overline{r}(t)}^{\overline{r}(t)+\mu} |r - \overline{r}(t)|^{1/2} dr \Big)^2 \Big]^{1/2}$$
$$\times \Big[\frac{1}{R_0^{N-1}} \int_0^T \int_0^1 (|w_r^\varepsilon|^2 + |v_r^\varepsilon|^2 + |w_r|^2 + |v_r|^2) r^{N-1} dr\, dt \Big]^{1/2}. \tag{7.103}$$

Moreover, we have

$$\int_{\overline{r}(t)}^{\overline{r}(t)+\mu} |r - \overline{r}(t)|^{1/2} dr \leq \mu^{3/2}. \tag{7.104}$$

Substituting (7.104) into (7.103) and using (3.8) and the fact that $w, v \in L^2(0, T, H^1(\Omega))$, we obtain

$$J_{I_\delta} \leq C_2(R_0) \mu^{1/2} \sup_{I_\delta} |\rho| meas(I_\delta)^{1/2}. \tag{7.105}$$

Substituting (7.101) and (7.105) into (7.97), we deduce

$$\Big| \int_{I_\delta} \rho(t) \Big[(w^\varepsilon - v^\varepsilon)(r^\varepsilon(t), t) - (w - v)(\overline{r}(t), t) \Big]\, dt \Big|$$
$$\leq C_3(R_0, T) |I_\delta|^{1/2} \sup_{I_\delta} |\rho| \Big[\sup_{I_\delta} |r^\varepsilon(t) - \overline{r}(t)|^{1/2} + \mu^{1/2} \Big]$$
$$+ \frac{1}{\mu} \Big| \int_{I_\delta} \int_{\overline{r}(t)}^{\overline{r}(t)+\mu} \rho[(w^\varepsilon - v^\varepsilon) - (w - v)](r, t)\, dr\, dt \Big|. \tag{7.106}$$

Let μ be fixed. By (7.92), we have $\lim_{\varepsilon \to 0} \sup_{I_\delta} |r^\varepsilon(t) - \overline{r}(t)|^{1/2} = 0$. Moreover, since $w^\varepsilon - v^\varepsilon$ tends to $w - v$ weakly in $L^2(0, T, L^2(0, 1))$, we have

$$\lim_{\varepsilon \to 0} \int_{I_\delta} \int_{\overline{r}(t)}^{\overline{r}(t)+\mu} \rho[(w^\varepsilon - v^\varepsilon) - (w - v)](r, t)\, dr\, dt = 0.$$

Letting ε tend to zero in (7.106), we conclude that

$$\limsup_{\varepsilon \to 0} \Big| \int_{I_\delta} \rho(t)\, [(w^\varepsilon - v^\varepsilon)(r^\varepsilon(t), t) - (w - v)(\overline{r}(t), t)]\, dt \Big|$$

$$\leq C_3(R_0) meas(I_\delta)^{1/2} \sup_{I_\delta} |\rho| \mu^{1/2}, \quad \text{for all } \mu > 0,$$

which implies (7.93). We are now in a position to prove Lemma 7.7:

$$\begin{aligned} &\int_{I_\delta} \rho(t)\left[(w-v)(\overline{r}(t),t) - Y_0(\overline{r}(t))\right] dt \\ &= \int_{I_\delta} \rho(t)\left[(w-v)(\overline{r}(t),t) - (w^\varepsilon - v^\varepsilon)(r^\varepsilon(t),t)\right] dt \\ &+ \int_{I_\delta} \rho(t)\left[(w^\varepsilon - v^\varepsilon)(r^\varepsilon(t),t) - Y_0(\overline{r}(t))\right] dt. \end{aligned} \tag{7.107}$$

Furthermore, we have

$$\begin{aligned} &\Big| \int_{I_\delta} \rho(t)\left[(w^\varepsilon - v^\varepsilon)(r^\varepsilon(t),t) - Y_0(\overline{r}(t))\right] \, dt \Big| \\ &\leq \sup_{I_\delta} |(w^\varepsilon - v^\varepsilon)(r^\varepsilon(\cdot),\cdot) - Y_0(\overline{r}(\cdot))| meas(I_\delta) \sup_{I_\delta} |\rho|. \end{aligned} \tag{7.108}$$

This together with (7.92) gives that

$$\lim_{\varepsilon\to 0} \int_{I_\delta} \rho(t)\left[(w^\varepsilon - v^\varepsilon)(r^\varepsilon(t),t) - Y_0(\overline{r}(t))\right] dt = 0.$$

Thus letting ε tend to zero in (7.107) and (7.93) we conclude that

$$\int_{I_\delta} \rho(t)\left[(w-v)(\overline{r}(t),t) - Y_0(\overline{r}(t))\right] dt = 0$$

for all $\delta > 0$ and $\rho \in L^\infty((0,T))$. This implies that

$$(w-v)(\overline{r}(t),t) = Y_0(\overline{r}(t)), \text{ a.e. in } I.$$

This completes the proof of Lemma 7.7. □

Moreover, Corollary 7.6 and Lemma 7.7 imply the following limit result

Corollary 7.8. *Let $\overline{r}$ be a jump of u defined on I; then we have*

$$-(N-1)\frac{2}{3}\frac{1}{\overline{r}(t)} = \nu(\overline{r}(t))(w-v)(\overline{r}(t),t) \text{ a.e. in } I.$$

Finally, from the results (2.4) and (7.8), we deduce Theorem 1.1.

Appendix A. Appendix

Lemma A.1. *There exists a constant $K > 0$ such that for all $\delta > 0$*

$$|\mu - \lambda| \leq \delta + \frac{1}{K\delta}|g(\mu) - g(\lambda)| \tag{A.1}$$

for all μ, $\lambda \in R$.

Proof. If $\mu = \lambda$ the result is obvious; then we suppose that $\mu > \lambda$, and we first prove that there exists a constant $K > 0$ such that

$$\frac{|g(\mu) - g(\lambda)|}{|\mu - \lambda|^2} \geq K \tag{A.2}$$

for all $\mu,\ \lambda \in R$. We estimate $\psi(u) = \inf_{x\in R} H_u(x)$, where $H_u(x) := \int_{x-u}^{x+u} |1-s^2|\, ds$ for $u > 0$. Since H_u is even we consider only $x \in [0, +\infty)$. Moreover, we have that $H'_u(x) = |1 - (x+u)^2| - |1 - (x-u)^2|$ is of the sign of

$$A_u(x) = [1 - (x+u)^2]^2 - [1 - (x-u)^2]^2 = 4xu[x^2 - (1 - u^2)].$$

If $u > 1$, we have $A_u(x) > 0$, which implies that $x \to H_u(x)$ is increasing on R. Thus, we have

$$\psi(u) = \inf_{x\geq 0} H_u(x) = H_u(0) = \frac{2}{3}u^3 - 2u + \frac{8}{3} \geq \frac{1}{4}u^2 + (\frac{5}{12}u^2 - 2u + \frac{8}{3}) \geq \frac{1}{4}u^2.$$

If $0 < u \leq 1$, then $H - u$ has a minimum at a point $x_0 = \sqrt{1-u^2}$. Setting $u = \sin\alpha$, where $\alpha \in (0, \frac{\pi}{2}]$, we obtain that $x_0 = \cos\alpha$, and thus

$$\psi(u) = H_u(x_0) = \int_{\cos\alpha - \sin\alpha}^{\cos\alpha+\sin\alpha} |1 - s^2|\, ds = \frac{2}{3}[2 - 2\cos^3\alpha]$$
$$= \frac{4}{3}(1 - \cos^2\alpha) + \frac{4}{3}\cos^2\alpha(1 - \cos\alpha) \geq \frac{4}{3}u^2.$$

Therefore, we have that $\psi(u) \geq Cu^2$, for all $u \in (0, \infty)$. Applying this result with $x = \frac{\mu+\lambda}{2}$ and $u = \frac{\mu-\lambda}{2}$ we deduce that

$$\int_\lambda^\mu |1 - s^2|\, ds \geq K|\mu - \lambda|^2.$$

This implies $g(\mu) - g(\lambda) \geq K|\mu - \lambda|^2$ for all $\mu,\ \lambda \in R$, which coincides with (A.2). Let $\delta > 0$ if $|\mu - \lambda| \leq \delta$; then the result is obvious. If $|\mu - \lambda| \geq \delta$, then we have

$$\frac{|g(\mu) - g(\lambda)|}{|\mu - \lambda|} \geq K|\mu - \lambda| \geq K\delta,$$

and so $|\mu - \lambda| \leq \delta + \frac{1}{K\delta}|g(\mu) - g(\lambda)|$. This completes the proof of Lemma A.1. □

Lemma A.2. *H^ε defined in Lemma 6.2 by*

$$H^\varepsilon(z) := -\Theta^\varepsilon_{zz} - f(\Theta^\varepsilon) + \varepsilon V^\varepsilon - \varepsilon W^\varepsilon \tag{A.3}$$

satisfies

$$H^\varepsilon(z) = \varepsilon^2 \sum_{i=1}^{i=m} E_i^\varepsilon \left(6U^\varepsilon_{0i}(U^\varepsilon_{1i})^2 + 2\varepsilon(U^\varepsilon_{1i})^3\right) \tag{A.4}$$
$$+ \sum_{i=1}^{i=m-1} (\Theta^\varepsilon_{i+1} - \Theta^\varepsilon_i)\Big[(\zeta^\varepsilon_i)_{zz} + 2E^\varepsilon_i E^\varepsilon_{i+1}((1 + E^\varepsilon_i)(\Theta^\varepsilon_i)^2 + (E^\varepsilon_i - 2)(\Theta^\varepsilon_{i+1})^2$$

$$+ (1-2E_i^\varepsilon)\Theta_i^\varepsilon\Theta_{i+1}^\varepsilon)\Big] + \sum_{i=1}^{i=m-1}(\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)_z(\zeta_i^\varepsilon)_z.$$

Proof. One can show that

$$\begin{aligned} f(\Theta_i^\varepsilon) &= 2(U_{0i}^\varepsilon + \varepsilon U_{1i}^\varepsilon)[1-(U_{0i}^\varepsilon+\varepsilon U_{1i}^\varepsilon)^2] \\ &= f(U_{0i}^\varepsilon) + 2\varepsilon U_{1i}^\varepsilon(-3(U_{0i}^\varepsilon)^2+1) + 2\varepsilon^2(U_{1i}^\varepsilon)^2(-3U_{0i}^\varepsilon - \varepsilon u_{1i}^\varepsilon). \end{aligned} \tag{A.5}$$

Using (6.3) and the fact that $(U_0^\varepsilon)_{zz} + f(U_0^\varepsilon) = 0$ we deduce that

$$-(\Theta_i^\varepsilon)'' - f(\Theta_i^\varepsilon) = W^\varepsilon - V^\varepsilon + 2\varepsilon^2(U_{1i}^\varepsilon)^2(-3U_{0i}^\varepsilon - \varepsilon u_{1i}^\varepsilon). \tag{A.6}$$

Moreover, we have

$$\begin{aligned} -(\Theta^\varepsilon)_{zz} - f(\Theta^\varepsilon) &= \sum[(\Theta_i^\varepsilon)_{zz} + f(\Theta_i^\varepsilon)]E_i^\varepsilon - 2\sum(\Theta_i^\varepsilon)_z(E_i^\varepsilon)' - \sum(\Theta_i^\varepsilon)(E_i^\varepsilon)_z \\ &\quad + 2\Big[\sum(\Theta_i^\varepsilon)^3E_i^\varepsilon - \sum(\Theta_i^\varepsilon)^3(E_i^\varepsilon)^3 \\ &\quad - 3\sum(\Theta_i^\varepsilon)^2\Theta_{i+1}^\varepsilon(E_i^\varepsilon)^2E_{i+1}^\varepsilon - 3\sum\Theta_i^\varepsilon(\Theta_{i+1}^\varepsilon)^2Ee_i(E_{i+1}^\varepsilon)^2\Big]. \end{aligned} \tag{A.7}$$

Using (A.3), (A.6), and (A.7), we deduce

$$\begin{aligned} H^\varepsilon(z) &= 2\varepsilon^2\sum(U_{1i}^\varepsilon)^2(3U_{0i}^\varepsilon + \varepsilon u_{1i}^\varepsilon)E_i^\varepsilon + 2\sum(\Theta_i^\varepsilon)^3E_i^\varepsilon(1-(E_i^\varepsilon)^2) \\ &\quad + 6\sum E_i^\varepsilon E_{i+1}^\varepsilon\left((\Theta_i^\varepsilon)^2\Theta_{i+1}^\varepsilon E_i^\varepsilon + \Theta_i^\varepsilon(\Theta_{i+1}^\varepsilon)^2E_{i+1}^\varepsilon\right) \\ &\quad - 2\sum(\Theta_i^\varepsilon)_z(E_i^\varepsilon)_z - \sum(\Theta_i^\varepsilon)(E_i^\varepsilon)_{zz}. \end{aligned} \tag{A.8}$$

Using the fact that $\sum E_i^\varepsilon = 1$, one can prove that

$$\sum(\Theta_i^\varepsilon)^3E_i^\varepsilon(1-(E_i^\varepsilon)^2) = -\sum E_{i+1}^\varepsilon E_i^\varepsilon[(\Theta_{i+1}^\varepsilon)^3(2-E_i^\varepsilon) + (\Theta_i^\varepsilon)^3(1+E_i^\varepsilon)] \tag{A.9}$$

and that

$$\sum E_i^\varepsilon E_{i+1}^\varepsilon\Theta_i^\varepsilon(\Theta_{i+1}^\varepsilon)^2E_{i+1}^\varepsilon = \sum E_i^\varepsilon E_{i+1}^\varepsilon\Theta_i^\varepsilon(\Theta_{i+1}^\varepsilon)^2(1-E_i^\varepsilon). \tag{A.10}$$

Furthermore, we set $\zeta_i^\varepsilon = \sum_{k=1}^{k=i}E_k^\varepsilon$, and we obtain

$$\begin{aligned} \sum(\Theta_i^\varepsilon)(E_i^\varepsilon)_{zz} &= -\sum(\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)(\zeta_i^\varepsilon)_{zz}, \\ \sum(\Theta_i^\varepsilon)_z(E_i^\varepsilon)_z &= -\sum(\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)_z(\zeta_i^\varepsilon)_z. \end{aligned} \tag{A.11}$$

Substituting (A.9), (A.10), and (A.11) into (A.8), we obtain

$$\begin{aligned} &H^\varepsilon(z) = 2\varepsilon^2\sum(U_{1i}^\varepsilon)^2(3U_{0i}^\varepsilon + \varepsilon u_{1i}^\varepsilon)E_i^\varepsilon \\ &+ 2\sum E_{i+1}^\varepsilon E_i^\varepsilon\left[3(\Theta_i^\varepsilon)^2\Theta_{i+1}^\varepsilon E_i^\varepsilon + 3\Theta_i^\varepsilon(\Theta_{i+1}^\varepsilon)^2(1-E_i^\varepsilon) - (\Theta_{i+1}^\varepsilon)^3(2-E_i^\varepsilon)\right. \\ &\left.-(\Theta_i^\varepsilon)^3(1+E_i^\varepsilon)\right] + \sum(\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)(\zeta_i^\varepsilon)_{zz} + \sum(\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)_z(\zeta_i^\varepsilon)_z. \end{aligned} \tag{A.12}$$

We note that

$$3(\Theta_i^\varepsilon)^2\Theta_{i+1}^\varepsilon E_i^\varepsilon + 3\Theta_i^\varepsilon(\Theta_{i+1}^\varepsilon)^2(1-E_i^\varepsilon) - (\Theta_{i+1}^\varepsilon)^3(2-E_i^\varepsilon) - (\Theta_i^\varepsilon)^3(1+E_i^\varepsilon)$$

$$= (1 + E_i^\varepsilon)(\Theta_i^\varepsilon)^2 + (E_i^\varepsilon - 2)(\Theta_{i+1}^\varepsilon)^2 + (1 - 2E_i^\varepsilon)\Theta_i^\varepsilon\Theta_{i+1}^\varepsilon. \tag{A.13}$$

Finally, substituting (A.13) into (A.12), we obtain (A.4). This completes the proof of Lemma A.2.

Lemma A.3. *There exists a solution y of the system*

$$(S)\begin{cases} -Y'' + 2(3(U_{0i}^\varepsilon)^2 - 1)Y = -k(z) \\ Y(0) = 0, \end{cases}$$

where k is continuous on an interval I, of the form

$$y(z) = A(z)\int_0^z B(x)\tilde{k}(x)\,dx + B(z)\int_z^{+\infty} A(x)\tilde{k}(x), \tag{A.14}$$

where $\tilde{k}$ is continuous and bounded on R and such that $\|\tilde{k}\|_{L^\infty(R)} = \|k\|_{L^\infty(I)}$. Moreover, y satisfies the following estimate:

$$|y(z)| + |y'(z)| \le 12\|k\|_{L^\infty(I)} \quad \textit{for all } z \in I.$$

Proof. We first prove that a solution of (S) is given by

$$y_\beta(z) = A(z)\int_0^z \tilde{k}(x)B(x)\,dx - B(z)\int_0^z \tilde{k}(x)A(x)\,dx + \beta B(z) \tag{A.15}$$

where $A(z) = 1 - \tanh^2(z)$ and $B(z) = -A(z)\int_0^z \frac{1}{A^2(x)}\,dx$. First we consider the following equation associated to (S):

$$(E) \qquad -Y'' + 2(3(U_{0i}^\varepsilon)^2 - 1)Y = 0.$$

Since $A'(z) = -2\tanh(z)A(z)$ and $A''(z) = -2A^2(z) + 4\tanh^2(z)A(z)$ we note that A is a solution of (E). Furthermore, we look for a solution y of (E) such that $y = \phi A$. This gives $\phi'' + 2\phi'\frac{A'}{A} = 0$. Integrating this, we obtain $\phi = C\int_0^z \frac{1}{A^2(x)}\,dx + K$, and setting $B(z) := -A(z)\int_0^z \frac{1}{A^2(x)}\,dx$ we obtain that (A, B) is a base of solutions of (E). Thus a solution of (S) is of the form

$$y(z) = \lambda(z)A(z) + \mu(z)B(z), \tag{A.16}$$

for particular $\lambda(z)$ and $\mu(z)$ satisfying

$$\lambda' A + \mu' B = 0, \quad \lambda' A' + \mu' B' = \tilde{k}.$$

This gives $\lambda' = \tilde{k}B$ and $\mu' = -A\tilde{K}$; thus, we have

$$\lambda(z) = \int_0^z \tilde{k}(x)B(x)\,dx + \alpha \text{ and } \mu(z) = -\int_0^z \tilde{k}(x)A(x)\,dx + \beta.$$

Therefore, using (A.16) and the fact that $y(0) = 0$ we deduce that a solution of (S) is of the form

$$y_\beta(z) = A(z)\int_0^z \tilde{k}(x)B(x)\,dx - B(z)\int_0^z \tilde{k}(x)A(x)\,dx + \beta B(z).$$

Finally, choosing $\beta = \int_0^{+\infty} A(x)\tilde{k}(x)\,dx$, we obtain a solution of the form

$$Y(z) = A(z)\int_0^z B(x)\tilde{k}(x)\,dx + B(z)\int_z^{+\infty} A(x)\tilde{k}(x),$$

which coincides with (A.14). Furthermore, we have

$$|A(z)|\int_0^z |B(\xi)|\,d\xi \le 2 \quad \text{and} \quad |B(z)|\int_z^{+\infty} |A(\xi)|\,d\xi \le 1 \tag{A.17}$$

and

$$|A'(z)|\int_0^z |B(\xi)|\,d\xi \le 4 \quad \text{and} \quad |B'(z)|\int_z^{+\infty} |A(\xi)|\,d\xi \le 3. \tag{A.18}$$

Using (A.14), (A.17), and (A.18), we deduce

$$|Y(z)| + |Y'(z)| \le 12\|\tilde{k}\|_{L^\infty(R)} \ \text{ for all } z \in R.$$

Finally, since $\|\tilde{k}\|_{L^\infty(R)} = \|k\|_{L^\infty(I)}$, we deduce

$$|Y(z)| + |Y'(z)| \le 12\|k\|_{L^\infty(I)} \quad \text{for all } z \in I.$$

This completes the proof of Lemma A.3.

Lemma A.4. *There exist two positive constants S_1 and S_2 such that*

$$\int_I (-\Psi^\varepsilon_{zz} - f'(\Theta^\varepsilon)\Psi^\varepsilon)\,\Psi^\varepsilon|\xi^\varepsilon|^2 \ge S_1\int_I |\Psi^\varepsilon_z|^2|\xi^\varepsilon|^2 + S_2\int_I |\Psi^\varepsilon|^2|\xi^\varepsilon|^2 - \frac{2}{S_1}\int_I |\Psi^\varepsilon|^2|\xi^\varepsilon_z|^2,$$

where $I = (-\infty, z^\varepsilon_m]$, $I = [0, +\infty)$, or $I = [z^\varepsilon_m, 0]$.

Proof. Let I be one of the sets of integration. We have that

$$\int_I (-\Psi^\varepsilon_{zz} - f'(\Theta^\varepsilon)\Psi^\varepsilon)\,\Psi^\varepsilon|\xi^\varepsilon|^2 = \int_I |\Psi^\varepsilon_z|^2|\xi^\varepsilon|^2 - \int_I f'(\Theta^\varepsilon)|\Psi^\varepsilon|^2|\xi^\varepsilon|^2 + 2\int_I \Psi^\varepsilon_z\Psi^\varepsilon\xi^\varepsilon_z\xi^\varepsilon.$$

We set for all $a > 0$, $Q^\varepsilon := \{f'(\Theta^\varepsilon) < 2a\} \cap (-\varepsilon^{-3/5} + z^\varepsilon_m, \varepsilon^{-3/5})$, and we estimate the measure of any connected component of I^ε. We have since $-f'(s) \ge -2$ for all $s \in R$ that

$$\begin{aligned}\int_I (-\Psi^\varepsilon_{zz} - f'(\Theta^\varepsilon)\Psi^\varepsilon)\,\Psi^\varepsilon|\xi^\varepsilon|^2 &\ge \int_I |\Psi^\varepsilon_z|^2|\xi^\varepsilon|^2 + 2a\int_I |\Psi^\varepsilon|^2|\xi^\varepsilon|^2 \\ &- (2+2a)\int_{Q^\varepsilon} |\Psi^\varepsilon|^2|\xi^\varepsilon|^2 + 2\int_I \Psi^\varepsilon_z\Psi^\varepsilon\xi^\varepsilon_z\xi^\varepsilon.\end{aligned} \tag{A.19}$$

Using Lemma 6.3 and the fact that $-f'(\Theta^\varepsilon) < 2a$ implies $|\Theta^\varepsilon| < \sqrt{\frac{a+1}{3}}$, we obtain

$$\begin{aligned}Q^\varepsilon &\subset \Big\{\sum_i E^\varepsilon_i|\tanh(z - z^\varepsilon_i)| \le \sqrt{\frac{a+1}{3}} + C(R_0)\varepsilon^{1/4}\Big\} \\ &\subset \cup_i\Big\{\sum_i E^\varepsilon_i|z - z^\varepsilon_i| \le \tanh^{-1}\Big(\sqrt{\frac{a+1}{3}} + C(R_0)\varepsilon^{1/4}\Big)\Big\}.\end{aligned}$$

Thus we have shown that $Q^\varepsilon := \cup_{i=1,\dots,m} Q_i^\varepsilon$ such that the sets Q_i^ε are disjoint and

$$meas(Q_i^\varepsilon) \le \tanh^{-1}\Big(\sqrt{\frac{a+1}{3}} + C(R_0)\varepsilon^{1/4}\Big).$$

Therefore, we have

$$\begin{aligned}\int_{Q^\varepsilon} |\Psi^\varepsilon|^2|\xi^\varepsilon|^2 &= \sum_{i=1}^{i=m}\int_{Q_i^\varepsilon}|\xi^\varepsilon|^2\Big(\int_{z_i^\varepsilon}^z \Psi_z^\varepsilon\Big)^2 \le \sum_{i=1}^{i=m}\int_{Q_i^\varepsilon}|\xi^\varepsilon|^2\Big(\int_{z_i^\varepsilon}^z(\Psi_z^\varepsilon)^2\Big)|z-z_i^\varepsilon| \\ &\le \sum_{i=1}^{i=m}\int_{Q_i^\varepsilon}|\xi^\varepsilon|^2\int_{Q_i^\varepsilon}(\Psi_z^\varepsilon)^2 meas(Q_i^\varepsilon) \\ &\le \Big(\tanh^{-1}(\sqrt{\frac{a+1}{3}} + C(R_0)\varepsilon^{1/4})\Big)^2\int_I|\xi^\varepsilon|^2(\Psi_z^\varepsilon)^2.\end{aligned}$$

Substituting this into (A.19) and setting $C(a,\varepsilon) := 1-(\tanh^{-1}(\sqrt{\frac{a+1}{3}}+C(R_0)\varepsilon^{\frac14}))^2$, which is positive for $a = 0$ and $\varepsilon = 0$, there exist ε_0 and a_0 such that $0 < C(a_0,\varepsilon)$, for all $\varepsilon < \varepsilon_0$. This gives since $0 < C(a_0,\varepsilon) < C(a_0,\varepsilon_0)$ that

$$\begin{aligned}&\int_I(-\Psi_{zz}^\varepsilon - f'(\Theta^\varepsilon)\Psi^\varepsilon)\,\Psi^\varepsilon|\xi^\varepsilon|^2 \\ &\ge C(a_0,\varepsilon_0)\int_I|\Psi_z^\varepsilon|^2|\xi^\varepsilon|^2 + 2a_0\int_I|\Psi^\varepsilon|^2|\xi^\varepsilon|^2 + 2\int_I\Psi_z^\varepsilon\Psi^\varepsilon\xi_z^\varepsilon\xi^\varepsilon, \qquad \text{(A.20)}\end{aligned}$$

for all $\varepsilon < \varepsilon_0$. Furthermore we have that

$$\int_I\Psi_z^\varepsilon\Psi^\varepsilon\xi_z^\varepsilon\xi^\varepsilon \ge -\frac{C(a_0,\varepsilon_0)}{4}\int_I(\Psi_z^\varepsilon)^2|\xi^\varepsilon|^2 - \frac{1}{C(a_0,\varepsilon_0)}\int_I(\Psi_z^\varepsilon)^2|\xi_z^\varepsilon|^2.$$

Substituting this into (A.20) we obtain that there exist two positive constants $S_1 = \frac{C(a_0,\varepsilon_0)}{2}$ and $S_2 = 2a_0$ such that

$$\int_I(-\Psi_{zz}^\varepsilon - f'(\Theta^\varepsilon)\Psi^\varepsilon)\,\Psi^\varepsilon|\xi^\varepsilon|^2 \ge S_1\int_I|\Psi_z^\varepsilon|^2|\xi^\varepsilon|^2 + S_2\int_I|\Psi^\varepsilon|^2|\xi^\varepsilon|^2 - \frac{1}{S_1}\int_I|\Psi^\varepsilon|^2|\xi_z^\varepsilon|^2,$$

for all ε small enough. This completes the proof of Lemma A.4.

References

[1] M. Bahiana and Y. Oono, Phys. Rev., 41 (1990), 6763.
[2] H. Brezis, "Analyse Fonctionnelle Théorie et Applications," Masson, 1983.
[3] L. Bronsard and B. Stoth, *Volume-preserving mean curvature flow as a limit of a nonlocal Ginzburg-Landau equation,* SIAM J. Math. Anal., 28 (1997), 769–807.
[4] X. Chen, *Global asymptotic limit of solutions of the Cahn-Hilliard equation,* J. Diff. Geom., 44 (1996), 262–311.
[5] Y. Nishiura and I. Ohnishi, *Some mathematical aspects of the micro-phase separation in diblock copolymers,* Physica D, 84 (1995), 31–39.
[6] T. Ohta and K. Kawasaki, Macromolecules, 1986, 261.
[7] B. Stoth, *Convergence of the Cahn-Hilliard equation to the Mullins-Sekerka problem in spherical symmetry,* Journal of Differential Equations, 125 (1996), 154-183.