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ON SOME CLASS OF PROBLEMS WITH NONLOCAL SOURCE AND BOUNDARY FLUX

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Abstract. In this paper we study a nonlocal, semilinear, parabolic problem. The existence and uniqueness of a maximal solution is proved for bounded domains, in arbitrary dimensions, using the Schauder fixed-point theorem. In the one-dimensional case, we give a result of positivity and a comparison principle for the integral of the solution. The proofs are based on the decomposition of the solutions in an appropriate spectral basis.

1. Introduction

For the past few years, nonlinear nonlocal problems have arisen in numerous physical situations (see for instance [2], [5], [6], [7], [9], [10] or [14]). The goal of this paper is to investigate a class of such problems of parabolic type.

1.1. **Description of the physical problem.** Consider the following situation. Let Ω be a thin plate. At each time $t_0, t_0+\Delta t, t_0+2\Delta t, \dots,$ one measures the temperature on some part Ω' of Ω. Between t_0 and $t_0+\Delta t$, one cools Ω and heats a part of its border using processes that involve for instance the mean temperature of Ω' at time t_0. Dependence of quantities of the form

$$\frac{1}{|B|}\int_B u(t,x)\,dx,$$

where $u(t,x)$ denotes the temperature of a point $x \in \Omega$ at time t and B is a ball contained in Ω, is very natural. Indeed, we cannot measure the temperature at a precise point but only the mean temperature in a neighbourhood of it.

We will suppose that the nonheated part of the border is maintained at a fixed temperature.

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1.2. **Modelisation.** Knowing the temperature $u(t_0,x)$ at time t_0 for each point x of Ω, the temperature at some instant $t \in [t_0, t_0+\Delta t)$ is a solution to

$$\begin{cases} u_t - \Delta u &= -a\left(\int_{\Omega'} u(t_0,x)\,dx\right) & \text{in } [t_0,t_0+\Delta t)\times\Omega,\\ u(t,\cdot) &= 0 & \text{on } [t_0,t_0+\Delta t)\times\Gamma_0,\\ \partial_n u(t,\cdot) &= b\left(\int_{\Omega'} u(t_0,x)\,dx\right) & \text{on } [t_0,t_0+\Delta t)\times\Gamma_1,\\ u(t_0,\cdot) &= u_0(\cdot) & \text{in } \Omega. \end{cases}$$

These equations will be used for our numerical experiments. In order to study the evolution of the temperature from a theoretical point of view, we will consider the more general continuous problem

$$(\mathbf{P})\begin{cases} u_t + \mathcal{A}u &= -a\,(q(u)) & \text{in } (0,T)\times\Omega,\\ u(t,\cdot) &= 0 & \text{on } (0,T)\times\Gamma_0,\\ \partial_n u(t,\cdot) &= b\,(q(u)) & \text{on } (0,T)\times\Gamma_1,\\ u(0,\cdot) &= u_0(\cdot) & \text{in } \Omega,\\ q(u) &\in D & \text{on } (0,T). \end{cases}$$

Let us now define the above quantities and make our assumptions precise. T is a positive real number, N an integer greater than or equal to 1 and Ω a connected, bounded, Lipschitz domain of $\mathbb{R}^N$. Γ_0 and Γ_1 are two disjoint subsets of $\partial\Omega$, measurable with respect to the measure area on $\partial\Omega$ and satisfying $\overline{\Gamma}_0 \cup \overline{\Gamma}_1 = \partial\Omega$ and $\text{mes}(\Gamma_0) > 0$. Γ_0 is the part of the boundary where the temperature is fixed–here equal to 0, Γ_1 the part where some heating is performed.

In the problem (P), $\mathcal{A}$ is the linear elliptic operator

$$\mathcal{A}u = -\partial_j(a_{i,j}(x)\partial_i u + a_j(x)u) + a_0(x)u,$$

∂_n denotes the conormal derivative associated to the operator $\mathcal{A}$:

$$\partial_n u = a_{i,j}\frac{\partial u}{\partial x_i}n_j + a_j n_j u,$$

where $n = (n_1, \ldots, n_N)$ is the outward unit normal vector to $\partial\Omega$. a and b are numerical functions defined on some interval D of $\mathbb{R}$. q is a functional from $L^2(\Omega)$ into $\mathbb{R}$.

For applications, we have for instance in mind the Model Problem

$$(\mathbf{MP}p,q)\begin{cases} u_t - \Delta u &= -\left(\int_\Omega u(t,x)\,dx\right)^p & \text{in } (0,T)\times\Omega,\\ u(t,\cdot) &= 0 & \text{on } (0,T)\times\Gamma_0,\\ \partial_n u(t,\cdot) &= \left(\int_\Omega u(t,x)\,dx\right)^q & \text{on } (0,T)\times\Gamma_1,\\ u(0,\cdot) &= u_0(\cdot) & \text{in } \Omega,\\ \int_\Omega u(\cdot,x)\,dx &\geq 0 & \text{on (0,T)}, \end{cases}$$

where p and q are real numbers greater than or equal to 1.

In the sequel, we will deal with variational solutions. More precisely, we would like to consider the Variational Problem

$$(\mathbf{VP},\ T)\begin{cases} u \in H^1(0,T;V,V'), \\ \langle u_t, \varphi\rangle + A(u,\varphi) = -a\left(q(u)\right)\displaystyle\int_\Omega \varphi(x)\,dx \\ \qquad\qquad +b\left(q(u)\right)\displaystyle\int_{\Gamma_1}\varphi(\sigma)\,d\sigma & \text{in } \mathcal{D}'(0,T),\ \forall\varphi\in V, \\ u(0,\cdot) = u_0(\cdot) & \text{in } L^2(\Omega), \\ q(u)\in D & \text{on (0,T).} \end{cases}$$

Here and after, $V := \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}$ is endowed with the norm $|u|_V := (\int_\Omega |\nabla u|^2\,dx)^{1/2}$. Due to the Poincaré inequality, V is then a Hilbert space continuously embedded in $H^1(\Omega)$ (see [11] for details). The space $H^1(0,T;V,V')$ is defined by

$$H^1(0,T;V,V') = \left\{u \in L^2(0,T;V) : \frac{du}{dt} \in L^2(0,T;V')\right\}.$$

With the norm $(|u|^2_{L^2(0,T;V)} + |u_t|^2_{L^2(0,T;V')})^{1/2}$, it is a Hilbert space. (We refer for instance to [8] for all the questions regarding functional analysis and Sobolev spaces.)

The bracket $\langle\cdot,\cdot\rangle$ denotes the duality between V and V' ; q is a functional from $L^2(\Omega)$ into $\mathbb{R}$. A denotes the bilinear form canonically associated to $\mathcal{A}$; i.e.,

$$A(u,v) = \int_\Omega a_{i,j}(x)\partial_i u\partial_j v + a_j u\partial_j v + a_0(x)uv\,dx$$

where $a_{i,j}$ and a_i belong to $L^\infty(\Omega)$. We will assume that

the bilinear form $A(\cdot,\cdot)$ is continuous and coercive on $V\times V$; (1.1)

that is to say, there exist two real numbers M, $m > 0$ such that for all $u, v \in V$,

$$|A(u,v)| \le M|u|_V|v|_V \tag{1.2}$$

and

$$m|u|_V^2 \le A(u,u). \tag{1.3}$$

Finally, we will always choose

$$u_0 \in L^2(\Omega). \tag{1.4}$$

1.3. **Plan.** This paper is divided as follows. In Section 2, we address the issue of well-posedness of problem (VP, T). In Section 3, we investigate the existence and uniqueness of a maximal solution to problem (VP) (see below the definition of (VP)). Next we study the positivity of the integral of solutions (Section 4) and give some comparison principle.

2. Existence and uniqueness

2.1. **A local existence result.** Let a and b be two functions from $\mathbb{R}$ into $\mathbb{R}$ such that

$$a \text{ and } b \text{ are continuous on } \mathbb{R}; \tag{2.1}$$

there exist a real number $p \geq 1$ and a constant $c_0 > 0$ such that

$$|a(s)| \leq c_0(1+|s|^{p/2}), \tag{2.2}$$

$$|b(s)| \leq c_0(1+|s|^{p/2}), \quad \forall s \in \mathbb{R}. \tag{2.3}$$

Let q be a mapping from $L^2(\Omega)$ into $\mathbb{R}$ such that

$$q \text{ is globally Lipschitz continuous on } L^2(\Omega), \tag{2.4}$$

that is to say, such that there exists a constant $q_0 > 0$ such that

$$|q(u)-q(v)| \leq q_0 |u-v|_{L^2(\Omega)} \quad \forall u, v \in L^2(\Omega). \tag{2.5}$$

Note that since $H^1(0,T;V,V') \subset C([0,T];L^2(\Omega))$ (see [8]) and $a \circ q : L^2(\Omega) \to \mathbb{R}$ is continuous on $L^2(\Omega)$ (see (2.1) and (2.4)), the right-hand side of the second equation in the problem (VP, T) is continuous on $[0,T]$ and then defines a distribution.

Moreover, the third equation makes sense thanks to the above inclusion. Then the following holds:

Theorem 2.1. *Let the assumptions* (1.1), (1.4), and (2.1)–(2.4) *be satisfied. Then there exists a time* $T>0$ *such that the problem*

$$(\mathbf{VP},\ T)\begin{cases} u \in H^1(0,T;V,V'), \\ \langle u_t, \varphi\rangle + A(u,\varphi) = -a\,(q(u)) \displaystyle\int_\Omega \varphi(x)\,dx \\ \qquad\qquad + b\,(q(u)) \displaystyle\int_{\Gamma_1} \varphi(\sigma)\,d\sigma \quad \text{in } \mathcal{D}'(0,T),\ \forall \varphi \in V, \\ u(0,\cdot) = u_0(\cdot) \qquad\qquad \text{in } L^2(\Omega), \end{cases}$$

admits at least one solution.

Proof of Theorem 2.1. We rely on the Schauder fixed-point theorem. For this, consider the Banach space (recall that $p \geq 1$) $X := L^p(0,T;L^2(\Omega))$ endowed with the norm

$$|u|_X := \Big(\int_0^T |u(t)|^p_{L^2(\Omega)}\, dt \Big)^{1/p}.$$

Let $\mathcal{B}$ be the closed unit ball of X. The proof is divided into five steps.
Step 1. *Definition of a map from $\mathcal{B}$ into X.* For this, we need the following lemma:

Lemma 2.1. *For all $T > 0$ and v belonging to X, the problem*

$$(\mathbf{VP},\ T, v) \begin{cases} u \in H^1(0,T;V,V'), \\ \langle u_t, \varphi\rangle + A(u,\varphi) = -a(q(v)) \displaystyle\int_\Omega \varphi\, dx \\ \qquad\qquad\qquad\qquad +b(q(v)) \displaystyle\int_{\Gamma_1} \varphi\, d\sigma \quad \text{in } \mathcal{D}'(0,T),\ \forall \varphi \in V, \\ u(0,\cdot) = u_0(\cdot) \qquad\qquad\qquad\qquad\quad \text{in } L^2(\Omega), \end{cases}$$

has a unique solution u. Moreover, $u \in X \cap C([0,T];L^2(\Omega))$.

Lemma 2.1 allows us to define a map

$$\begin{array}{rcl} F\ :\ \mathcal{B} & \rightarrow & X \\ v & \mapsto & u, \textit{ the solution to problem } (VP,\ T, v). \end{array}$$

Proof of Lemma 2.1. Using (2.2) and (2.5), we obtain, for all $v \in L^2(\Omega)$,

$$|a(q(v))| \leq c_0(1 + |q(v)|^{p/2}) \leq c_0 \left(1 + (|q(0)| + q_0|v|_{L^2(\Omega)})^{p/2}\right).$$

Thus, for some constant $c > 0$:

$$|a(q(v))| \leq c(1 + |v|^{p/2}_{L^2(\Omega)}). \tag{2.6}$$

Arguing the same way with $b(q(v))$, it follows that, for each $v \in X$, the modulus of the right-hand side of the equation in problem (VP, T, v) is bounded by

$$c\big(1 + |v(t)|^{p/2}_{L^2(\Omega)}\big)|\varphi|_V, \tag{2.7}$$

for some new constant c. Thus, the map

$$\begin{array}{rcl} f_{v(t)}\ :\ V & \rightarrow & \mathbb{R} \\ \varphi & \mapsto & -(a\circ q)(v(t)) \displaystyle\int_\Omega \varphi\, dx + (b \circ q)(v(t)) \int_{\Gamma_1} \varphi\, d\sigma \end{array}$$

belongs to V' for almost every $t \in [0,T]$, and one has by (2.7)

$$|f_{v(t)}|^2_{V'} \le c(|v(t)|^p_{L^2(\Omega)} + 1).$$

Integrating between 0 and T, one gets

$$|f_v|^2_{L^2(0,T;V')} \le c\int_0^T |v(t)|^p_{L^2(\Omega)}\,dt + cT = c|v|^p_X + cT. \tag{2.8}$$

Hence, $f_v \in L^2(0,T;V')$ (note that by our assumptions the measurability in t of $t \mapsto f_{v(t)}$ is easy to establish). Applying the Lions variational theorem (see [8], Chapter XVIII), we conclude that the problem (VP, T, v) has a unique solution u. Furthermore, $u \in C([0,T];L^2(\Omega))$ because

$$H^1(0,T;V,V') \subset C([0,T];L^2(\Omega)), \tag{2.9}$$

and u belongs also to X since (see the above reference) $u \in L^\infty(0,T;\,L^2(\Omega))$ and

$$L^\infty(0,T;L^2(\Omega)) \subset L^p(0,T;L^2(\Omega)) =: X.$$

Step 2. *Estimates.* Let $T \in (0,+\infty)$, $v \in \mathcal{B}$ and $u := F(v)$. Note first that the second equation of (VP, T, v) reads also

$$u_t + \mathcal{A}u = f_{v(\cdot)} \quad \text{in } L^2(0,T;V').$$

Hence, for all $w \in L^2(0,T;V)$, one has by taking the V', V-duality bracket with w

$$\langle u_t(\cdot), w(\cdot)\rangle + A(u(\cdot), w(\cdot)) = \langle f_{v(\cdot)}, w(\cdot)\rangle \quad \text{in } L^1(0,T). \tag{2.10}$$

Choosing $w := u$ in this equation, it becomes

$$\langle u_t(\cdot), u(\cdot)\rangle + A(u(\cdot), u(\cdot)) = \langle f_{v(\cdot)}, u(\cdot)\rangle \quad \text{in } L^1(0,T).$$

But (see [8] or [3])

$$\langle u_t, u\rangle = \frac{1}{2}\frac{d}{dt}|u(t)|^2_{L^2(\Omega)} \quad \text{in } \mathcal{D}'(0,T).$$

Thus, taking into account these two equations, one gets

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2_{L^2(\Omega)} + A(u(t), u(t)) = \langle f_{v(t)}, u(t)\rangle, \quad \text{in } L^1(0,T). \tag{2.11}$$

This relation will allow us to obtain estimates for $|u|_{L^\infty(0,T;L^2(\Omega))}$ and $|u|_{L^2(0,T;V)}$. Indeed, from (1.3) and the Young inequality, we derive from (2.11) that

$$\frac{d}{dt}|u(t)|^2_{L^2(\Omega)} + m|u|^2_V \le \frac{1}{m}|f_v|^2_{V'}. \tag{2.12}$$

Integration on $[0,t]$, for $t \in [0,T]$ leads to

$$|u(t)|^2_{L^2(\Omega)} - |u(0)|^2_{L^2(\Omega)} + m|u|^2_{L^2(0,t;V)} \leq \frac{1}{m}|f_v|^2_{L^2(0,T;V')}. \tag{2.13}$$

With (2.8), the fact that $v \in \mathcal{B}$ and (2.13), one has, for a new constant $c > 0$,

$$|u(t)|^2_{L^2(\Omega)} + m|u|^2_{L^2(0,t;V)} \leq |u_0|^2_{L^2(\Omega)} + c(1+T), \quad \text{for a.e. } t \in [0,T].$$

Thus, we obtain

$$|u|_{L^\infty(0,T;L^2(\Omega))} \leq \left(|u_0|^2_{L^2(\Omega)} + c(1+T)\right)^{1/2} \tag{2.14}$$

and

$$|u|_{L^2(0,T;V)} \leq \frac{1}{\sqrt{m}}\left(|u_0|^2_{L^2(\Omega)} + c(1+T)\right)^{1/2}, \tag{2.15}$$

where c is independent of T, u_0 and $v \in \mathcal{B}$.

We will need also an estimate of $|u|_X$ and $|u_t|_{L^2(0,T;V')}$. Thanks to (2.14),

$$|u|^p_X \leq |u|^p_{L^\infty(0,T;L^2(\Omega))} T \leq \left(|u_0|^2_{L^2(\Omega)} + c(1+T)\right)^{p/2} T.$$

Thus

$$|u|_X \leq \left(|u_0|^2_{L^2(\Omega)} + c(1+T)\right)^{1/2} T^{1/p}. \tag{2.16}$$

Moreover, for all $\varphi \in V$ (see (2.10)),

$$\langle u_t, \varphi\rangle + A(u(t), \varphi) \leq |f_{v(t)}|_{V'}|\varphi|_V \quad \text{in } L^1(0,T).$$

Using (1.2), we obtain

$$|\langle u_t, \varphi\rangle| \leq M|u(t)|_V|\varphi|_V + |f_{v(t)}|_{V'}|\varphi|_V.$$

Thus

$$|u_t|^2_{V'} \leq 2M^2|u(t)|^2_V + 2|f_{v(t)}|^2_{V'} \quad \text{in } L^1(0,T).$$

Integrating between 0 and T and using (2.15) and (2.8), one gets, for a constant C independent of $v \in \mathcal{B}$,

$$|u_t|_{L^2(0,T;V')} \leq C. \tag{2.17}$$

Step 3. *Let us show that $F(\mathcal{B}) \subset \mathcal{B}$.* Let $v \in \mathcal{B}$ and $u := F(v)$. There exists (see (2.16)) a constant $c > 0$ independent of $|u_0|_{L^2(\Omega)}$, T and $v \in \mathcal{B}$, such that

$$|u|_X \leq c\left(|u_0|^2_{L^2(\Omega)} + T + 1\right)^{1/2} T^{1/p}. \tag{2.18}$$

We would like to prove, as claimed at the beginning of this step, that for a suitable T, one has $|u|_X \leq 1$ but also (this will be used in [13] to obtain a blow-up alternative) that the convergence of T towards 0 implies that of

$|u_0|_{L^2(\Omega)}$ towards $+\infty$. For this, considering the function $T \mapsto c(|u_0|^2_{L^2(\Omega)} + T + 1)^{1/2}T^{1/p}$, we see that there is a unique time $T > 0$ such that

$$c(|u_0|^2_{L^2(\Omega)} + T + 1)^{1/2}T^{1/p} = 1. \tag{2.19}$$

With this choice of T and (2.18), we get $F(\mathcal{B}) \subset \mathcal{B}$.

Step 4. *$F(\mathcal{B})$ is relatively compact in X.* Let $(v_n)_{n\geq 0} \subset \mathcal{B}$. Lemma 2.1 tells us that $u_n := F(v_n)$ belongs to $H^1(0,T;V,V')$. Using (2.15) and (2.17), we see that $(u_n)_{n\geq 0}$ is bounded in $H^1(0,T;V,V')$. The embedding of $H^1(0,T;V,V')$ in $L^2(0,T;L^2(\Omega))$ being compact, we may assume that, up to a subsequence, $(u_n)_{n\geq 0}$ converges in $L^2(0,T;L^2(\Omega))$.

Let us show that the convergence takes place in $X := L^p(0,T;L^2(\Omega))$. If $p \leq 2$ then $L^2(0,T;L^2(\Omega))$ is continuously embedded in X. Thus the above assertion is proved. If $p > 2$ then for all nonnegative integers n and m,

$$\begin{aligned} |u_n - u_m|^p_X &\leq \int_0^T |u_n - u_m(t)|^{p-2}_{L^2(\Omega)} |u_n - u_m(t)|^2_{L^2(\Omega)}\, dt \\ &\leq |u_n - u_m|^{p-2}_{L^\infty(0,T;L^2(\Omega))} |u_n - u_m|^2_{L^2(0,T;L^2(\Omega))} \\ &\leq C|u_n - u_m|^2_{L^2(0,T;L^2(\Omega))}, \end{aligned}$$

according to (2.14). Therefore, $(u_n)_{n\geq 0}$ converges in X.

Step 5. *Continuity of the mapping F.* Let $v \in X$ and $(v_n)_{n\geq 0} \subset \mathcal{B}$ be such that $v_n \to v$ in X. Set $u := F(v)$, $u_n := F(v_n)$. From Step 4, the sequence (u_n) is relatively compact in X; thus it is enough to prove that u is the unique accumulation point of $(u_n)_{n\geq 0}$.

For all $\varphi \in L^2(0,T;V)$, one has from (2.10)

$$\langle u_t, \varphi\rangle + A(u(t), \varphi) = \langle f_{v(t)}, \varphi\rangle, \tag{2.20}$$

$$\langle u_{n_t}, \varphi\rangle + A(u_n(t), \varphi) = \langle f_{v_n(t)}, \varphi\rangle \quad \text{in } L^1(0,T). \tag{2.21}$$

Subtracting (2.21) from (2.20) and choosing $\varphi(\cdot) := (u - u_n)(t, \cdot)$ leads to

$$\langle (u-u_n)_t, u - u_n\rangle + A((u-u_n)(t), (u-u_n)(t)) = \langle f_v - f_{v_n}, (u-u_n)(t)\rangle.$$

Arguing as in Step 2 (see (2.12) and (2.13)) and using the identity $u(0,\cdot) \equiv u_n(0,\cdot)$, one gets for almost every $t \in (0,T)$

$$|(u-u_n)(t)|^2_{L^2(\Omega)} + m|u - u_n|^2_{L^2(0,t;V)} \leq c|f_v - f_{v_n}|^2_{L^2(0,T;V')}. \tag{2.22}$$

Let us show that the right-hand side converges to 0 as $n \to +\infty$. One has (see the proof of Lemma 2.1) for some constant c

$$|f_v - f_{v_n}|^2_{L^2(0,T;V')} \le c \int_0^T [a \circ q(v(t)) - a \circ q(v_n(t))]^2 \, dt + c \int_0^T [b \circ q(v(t)) - b \circ q(v_n(t))]^2 \, dt. \quad (2.23)$$

Moreover, $v_n \to v$ in X; thus

$$|(v - v_n)(t)|_{L^2(\Omega)} \to 0 \quad \text{in } L^p(0,T).$$

We infer with the converse of Lebesgue theorem (see [1] Theorem IV.9) that there exists some function $h \in L^p(0,T)$ such that (up to a subsequence)

$$|v_n(t)|_{L^2(\Omega)} \le h(t) \quad \text{for a.e. } t \in (0,T) \quad (2.24)$$

and

$$v_n(t) \to v(t) \quad \text{in } L^2(\Omega), \text{ for a.e. } t \in (0,T).$$

From the assumptions (2.1) and (2.4), the function $a \circ q$ is continuous on $L^2(\Omega)$. Thus

$$(a \circ q(v(t)) - a \circ q(v_n(t)))^2 \to 0, \quad \text{for a.e. } t \in (0,T). \quad (2.25)$$

Moreover (see (2.6) and (2.24)),

$$\begin{aligned}|a \circ q(v(t)) - a \circ q(v_n(t))| &\le |a(q(v(t)))| + |a(q(v_n(t)))| \\ &\le c(1 + |v(t)|^{p/2}_{L^2(\Omega)} + |v_n(t)|^{p/2}_{L^2(\Omega)}) \le c(1 + |v(t)|^{p/2}_{L^2(\Omega)} + h^{p/2}(t)).\end{aligned}$$

Thus,

$$\begin{aligned}[a \circ q(v(t)) - a \circ q(v_n(t))]^2 &\le c^2(1 + |v(t)|^{p/2}_{L^2(\Omega)} + h^{p/2}(t))^2 \\ &\le 4c^2(1 + |v(t)|^p_{L^2(\Omega)} + h^p(t)).\end{aligned}$$

Using (2.25), the latter estimate and the fact that the function

$$t \mapsto 4c^2(1 + |v(t)|^p_{L^2(\Omega)} + h^p(t))$$

is in $L^1(0,T)$ and independent of n, we obtain thanks to the Lebesgue theorem

$$\int_0^T [a \circ q(v(t)) - a(\circ q(v_n(t))]^2 \, dt \to 0 \quad \text{as } n \to +\infty.$$

Arguing similarly with b and going back to (2.23) and (2.22), one obtains (since the above argument holds for any subsequence)

$$\sup_{[0,T]} |u_n - u(t)|_{L^2(\Omega)} \to 0 \quad \text{as } n \to +\infty. \tag{2.26}$$

As the convergence in $L^\infty(0,T;L^2(\Omega))$ implies the convergence in $L^p(0,T;L^2(\Omega))$, it follows that $u_n \to u$ in X, as $n \to +\infty$. Thus F is continuous.

We conclude the proof using the Schauder fixed-point theorem. □

2.2. **Uniqueness.** In order to get a uniqueness result, we need a stronger hypothesis than continuity for the functions a and b. We will assume that

the functions a and b are locally Lipschitz continuous on $\mathbb{R}$; (2.27)

that is to say, for any bounded interval $[-M, M]$ of $\mathbb{R}$, it holds that

$$|a(s) - a(r)| \leq C(M)|s - r| \tag{2.28}$$

and

$$|b(s) - b(r)| \leq C(M)|s - r| \quad \forall (s,r) \in [-M, M]^2, \tag{2.29}$$

where $C(M)$ is a positive constant. Then we have

Theorem 2.2. *If assumptions* (1.1), (1.4), (2.4) *and* (2.27) *hold, then the* problem (VP, T) *has at most one solution.*

Proof of Theorem 2.2. Let u and v be solutions to problem (VP, T). By difference, for all $\varphi \in L^2(0,T;V)$, it holds that

$$\begin{aligned}&\langle (u-v)_t, \varphi\rangle + A(u-v,\varphi) = \\ &\quad - [a\circ q(u(t)) - a\circ q(v(t))] \int_\Omega \varphi\, dx + [b\circ q(u(t)) - b\circ q(v(t))] \int_{\Gamma_1} \varphi\, d\sigma,\end{aligned}$$

in $L^1(0,T)$. Choosing $\varphi(\cdot) = (u-v)(t,\cdot)$, one gets

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|(u-v)(t)|^2_{L^2(\Omega)} + m|(u-v)(t)|^2_V &\leq c\,|a\circ q(u(t)) - a\circ q(v(t))|\,|u-v|_V \\ &\quad + c\,|b\circ q(u(t)) - b\circ q(v(t))|\,|u-v|_V.\end{aligned}$$

Thanks to the Young inequality, the right-hand side is bounded by

$$\frac{c^2}{2m}(a\circ q(u(t)) - a\circ q(v(t)))^2 + \frac{c^2}{2m}(b\circ q(u(t)) - b\circ q(v(t)))^2 + m|(u-v)(t)|^2_V.$$

Thus, for a new constant c, we obtain

$$\frac{d}{dt}|(u-v)(t)|^2_{L^2(\Omega)} \leq c\,(a \circ q(u(t)) - a \circ q(v(t)))^2 + c\,(b \circ q(u(t)) - b \circ q(v(t)))^2 . \quad (2.30)$$

Set $M := |u|_{L^\infty(0,T;L^2(\Omega))} + |v|_{L^\infty(0,T;L^2(\Omega))}$. Then, thanks to (2.9), M is finite; thus by (2.5)

$$|q(u(t))| \leq |q(0)| + q_0|u(t)|_{L^2(\Omega)} \leq |q(0)| + q_0 M =: R_0$$

and

$$|q(v(t))| \leq R_0.$$

Using now (2.28), one has

$$|a(q(u(t))) - a(q(v(t)))| \leq C(R_0)|q(u(t)) - q(v(t))| \leq C(R_0)q_0|(u-v)(t)|_{L^2(\Omega)}.$$

Having the same estimate for b and going back to (2.30), we obtain

$$\frac{d}{dt}|(u-v)(t)|^2_{L^2(\Omega)} \leq C|(u-v)(t)|^2_{L^2(\Omega)}.$$

Using the Gronwall lemma and the fact that $(u-v)(0,\cdot) = 0$ in $L^2(\Omega)$ leads to the uniqueness for the solution to problem (VP, T). □

3. Maximal solution

First, we would like to make this notion precise.

Theorem 3.1. *Let J be a subinterval of $[0, +\infty)$ containing the origin. A function $u : J \to L^2(\Omega)$ is called a maximal solution to* problem (VP) *if*

i) *u is a solution to* (VP, T) *for all $T > 0$ such that $[0, T] \subset J$,*
ii) *there is no solution v to* problem (VP, T') *with $[0, T'] \supset J$, $[0, T'] \neq J$ and $v \equiv u$ on J.*

The following proposition allows us to extend a local solution.

Proposition 3.1. *Under the assumptions of Theorem* 2.1, *let u be a solution to* problem (VP, T). *Then there exists w a proper extension of u which is a solution to* (VP, $T + T_0$) *where T_0 is the unique positive number satisfying*

$$c\left(|u(T)|^2_{L^2(\Omega)} + T_0 + 1\right)^{1/2} T_0^{1/p} = 1. \quad (3.1)$$

(*Here c is the positive constant in* (2.19).)

Proof of Proposition 3.1. According to Theorem 2.1, there exists a solution v to the problem

$$\begin{cases} v \in H^1(0,T_0;V,V'), \\ \langle v_t,\varphi\rangle + A(v,\varphi) = -a\circ q(v)\int_\Omega \varphi\,dx \\ \qquad +b\circ q(v)\int_{\Gamma_1}\varphi\,d\sigma \quad \text{in } \mathcal{D}'(0,T_0),\ \forall\varphi\in V, \\ v(0,\cdot) = u(T,\cdot) \quad \text{in } L^2(\Omega). \end{cases}$$

Then the function w defined by

$$w(t) = \begin{cases} u(t) & \text{if } 0\le t\le T \\ v(t-T) & \text{if } T<t\le T+T_0 \end{cases}$$

is obviously a proper extension of u and belongs to $L^2(0,T+T_0;V)$. Moreover, thanks to the lemma below, one proves that w belongs to $H^1(0,T+T_0;V,V')$. Then it is clear that w is a solution to (VP, $T+T_0$). $\square$

Lemma 3.1. (see [12] for the proof) *Let $0<T<T'$, $w\in L^2(0,T';V)$ and denote by u and v the restrictions of w to $(0,T)$ and (T,T'), respectively. Moreover assume*

i) $u\in H^1(0,T;V,V')$,
ii) $v\in H^1(T,T';V,V')$,
iii) $w\in C([0,T'];L^2(\Omega))$. *Then* $w\in H^1(0,T';V,V')$.

Now we can prove now the main result of this section

Theorem 3.1. *If assumptions* (1.1), (1.4), (2.2)–(2.4) *and* (2.27) *hold, then the* problem (VP) *has a unique maximal solution. Moreover it is defined on some interval* $[0,T_{max}(u_0))$, *open on the right.*

Proof of Theorem 3.1. We begin with the uniqueness. Let u and v be two maximal solutions defined on J and J' respectively. Without loss of generality we may assume $J\subset J'$.

From condition i) of Definition 3.1, u and v are solutions to (VP, T) for all $[0,T]\subset J$. Thus, according to Theorem 2.2, $u\equiv v$ on J.

Thus, it remains only to prove that $J=J'$. We argue by contradiction assuming $J\neq J'$. If J is closed, v provides an extension of u, which contradicts the maximality of u. If J is open in $[0,+\infty)$ then $J\neq\overline{J}\subset J'$, and from i), v is a solution to (VP, T) with $\overline{J}=[0,T]$ and is an extension of u. (Note that one can have u solution to (VP, T') for any $T'<T$ without u being a solution to (VP, T).) We have again the same contradiction.

Let us now investigate the question of existence of the maximal solution. Fix $u_0\in L^2(\Omega)$. Thanks to Theorem 2.1, the set $\{T>0$: (VP, T) has a solution} is a nonempty interval. Let T_{max} be its upper bound. Let $u : [0, T_{max}) \to L^2(\Omega)$ be defined as follows. For each $t \in [0, T_{max})$, $u(t)$ is the value at the point t of the solution to problem (VP, t). Note that u is well defined thanks to (2.9) and the uniqueness result of Theorem 2.2.

Let us show that u is a maximal solution in the sense of Definition 3.1. Condition i) is obviously satisfied. Suppose that ii) does not hold. Then, there exist $[0, T'] \supset [0, T_{max})$ and v a solution to (VP, T') such that v coincides with u on $[0, T_{max})$. Of course, as above, v can be extended on the right of T', and this contradicts the definition of T_{max}. □

This theorem applies directly to a problem close to problem (PM$p, p$). Indeed, one has

Corollary 3.1. *Let p and q be real numbers greater than or equal to 1 and u_0 a function of $L^2(\Omega)$. Then the problem*

$$\begin{cases} u_t - \Delta u &= -\left(\left(\int_\Omega u(t,x)\,dx\right)^+\right)^p & \textit{in } (0,T)\times\Omega,\\ u(t,\cdot) &= 0 & \textit{on } (0,T)\times\Gamma_0,\\ \partial_n u(t,\cdot) &= \left(\left(\int_\Omega u(t,x)\,dx\right)^+\right)^q & \textit{on } (0,T)\times\Gamma_1,\\ u(0,\cdot) &= u_0(\cdot) & \textit{on } \Omega, \end{cases}$$

admits a unique maximal solution.

$(\cdot)^+ : \mathbb{R} \to [0, +\infty)$ denotes the Lipschitz-continuous function defined by

$$(s)^+ = \begin{cases} s & \text{if } s > 0,\\ 0 & \text{otherwise.} \end{cases}$$

4. Qualitative results for the integral

In this section, we will assume that $N = 1$ and that $\Omega := (0, l)$ where $l \in (0, +\infty)$.

4.1. **A result of positivity of the integral.** We would like to consider the one-dimensional problem

$$(\mathbf{P1}, \mathbf{u_0}) \begin{cases} u_t - u'' &= -a(\int_\Omega u(t,x)\,dx) & \text{in } (0,T)\times\Omega,\\ u(t,0) &= 0 & \text{on } (0,T),\\ u'(t,l) &= b(\int_\Omega u(t,x)dx) & \text{on } (0,T),\\ u(0,\cdot) &= u_0(\cdot) & \text{in } \Omega. \end{cases}$$

First, note that, as we can see by numerical simulations, u is not necessarily positive in $(0,T)\times\Omega$ even if $u_0 > 0$ on Ω.

However, even if u becomes negative, its integral $\int_\Omega u(t,x)\,dx$ can remain positive. This is what is shown in the theorem below:

Theorem 4.1. *Assume that*

i) $\Omega := (0, l)$ *with* $l \in (0, \frac{3\pi}{10}]$,
ii) *the functions* a *and* b *satisfy* (2.2), (2.3) *and* (2.27),
iii) *for all* $s > 0$, $a(s) \leq b(s)$ *and* $b(s) \geq 0$,
iv) *the initial condition* u_0 *belongs to* $L^2(\Omega)$ *and* $u_0 \geq 0, u_0 \not\equiv 0$ *almost everywhere in* Ω.

Then the integral of the maximal variational solution to problem (P1,u_0) *is positive in* $[0, T_{max}(u_0))$.

Proof of Theorem 4.1. Let $(\varphi_k)_{k\geq 1}$ be the Hilbertian basis of $L^2(\Omega)$ defined by

$$\begin{cases} -\varphi_k'' = \lambda_k \varphi_k & \text{in } \Omega, \\ \varphi_k(0) = 0, \quad \varphi_k'(l) = 0, \\ \int_\Omega \varphi_k \, dx > 0, \quad |\varphi_k|_{L^2(\Omega)} = 1. \end{cases}$$

Recall that the embedding of V into $L^2(\Omega)$ is compact. An easy computation shows that

$$\lambda_k = \frac{\pi^2}{4l^2}(2k-1)^2, \quad \varphi_k(x) = \sqrt{\frac{2}{l}} \sin(\sqrt{\lambda_k} x) = \sqrt{\frac{2}{l}} \sin(\frac{\pi}{2l}(2k-1)x). \tag{4.1}$$

Let us denote by u the maximal variational solution to problem (P1,u_0) (see Theorem 3.1) and by T a real number in $(0, T_{max}(u_0))$. Taking $\varphi = \varphi_k$ in the variational form of (P1,u_0), we get

$$\begin{aligned} &\frac{d}{dt}\Big(\int_\Omega u(t)\varphi_k \, dx\Big) + \int_\Omega u'(t)\varphi_k' \, dx \\ &= -a\Big(\int_\Omega u(t)\, dx\Big) \int_\Omega \varphi_k \, dx + b\Big(\int_\Omega u(t)\, dx\Big)\varphi_k(l), \end{aligned}$$

in $L^1(0, T)$. Now, since φ_k is an eigenfunction, this equation reads also

$$\frac{d}{dt}\Big(\int_\Omega u(t)\varphi_k \, dx\Big) + \lambda_k \int_\Omega u(t)\varphi_k \, dx = -a\Big(\int_\Omega u\Big) \int_\Omega \varphi_k \, dx + b\Big(\int_\Omega u\Big)\varphi_k(l), \tag{4.2}$$

in $C([0, T])$. Since $u_0 \geq 0$ and $u_0 \not\equiv 0$, it holds that

$$\int_\Omega u_0(x)\, dx > 0.$$

Using an argument of continuity, the theorem will be proved if we show that for each t_0 belonging to $[0, T_{max}(u_0))$ and satisfying

$$\int_\Omega u(t, x)\, dx > 0, \qquad \forall t \in [0, t_0), \tag{4.3}$$

one has

$$\int_\Omega u(t_0,x)\,dx > 0.$$

For a t_0 satisfying (4.3) one has by iii),

$$a\left(\int_\Omega u(t)\,dx\right) \le b\left(\int_\Omega u(t)\,dx\right) \qquad \forall t\in[0,t_0].$$

Thus setting

$$u_k(t) := \int_\Omega u(t)\varphi_k\,dx$$

and

$$D(\varphi_k) := \varphi_k(l) - \int_\Omega \varphi_k\,dx = \sqrt{\frac{2}{l}}\left((-1)^{k+1} - \frac{1}{\sqrt{\lambda_k}}\right), \tag{4.4}$$

we obtain from (4.2), since $\int_\Omega \varphi_k\,dx > 0$, by iii),

$$u_k'(t) + \lambda_k u_k(t) \ge b\left(\int_\Omega u(t)\,dx\right)D(\varphi_k), \qquad k=1,2,\ldots,\ \forall t\in[0,t_0]. \tag{4.5}$$

Let $v_k : [0, T_{max}(u_0)) \to \mathbb{R}$ be the solution to the problem

$$v_k'(t) + \lambda_k v_k(t) = b\left(\int_\Omega u(t)\,dx\right)D(\varphi_k), \tag{4.6}$$

$$v_k(0) = u_{0_k}, \tag{4.7}$$

where u_{0_k} denotes the k$^{\text{th}}$ coordinate of the initial condition; i.e.,

$$u_{0_k} = \int_\Omega u_0(x)\varphi_k(x)\,dx.$$

It is easy to see that

$$u_k(t) \ge v_k(t), \qquad \forall t\in[0,t_0],\ \forall k=1,2,\ldots \tag{4.8}$$

We want to show now that $\sum_{k=1}^\infty v_k(t_0)\int_\Omega \varphi_k\,dx$ is positive. From (4.6)–(4.7), we deduce the following representation of v_k :

$$v_k(t) = e^{-\lambda_k t}u_{0_k} + \int_0^t e^{-\lambda_k(t-s)}b(\int_\Omega u(s)\,dx)\,ds D(\varphi_k). \tag{4.9}$$

For all integer $n\ge 1$, consider the series of functions

$$S_n(t) := \sum_{k=1}^n v_k(t)\int_\Omega \varphi_k\,dx$$

defined on $[0, T_{max}(u_0))$. With the notation

$$E_k := D(\varphi_k) \int_\Omega \varphi_k \, dx = \frac{2}{l} \left(\frac{(-1)^{k+1}}{\sqrt{\lambda_k}} - \frac{1}{\lambda_k} \right) \tag{4.10}$$

and

$$I_k := \int_0^{t_0} e^{-\lambda_k(t_0 - s)} b(\int_\Omega u(s) \, dx) \, ds,$$

we write $S_n(t_0)$ in the form

$$S_n(t_0) = \sum_{k=1}^n e^{-\lambda_k t_0} u_{0_k} \int_\Omega \varphi_k \, dx + \sum_{k=1}^n I_k E_k =: S_n^1(t_0) + S_n^2(t_0), \tag{4.11}$$

where $S_n^1(t_0)$ and $S_n^2(t_0)$ are defined in an obvious way.

Let us show first that for all even integers $n \geq 2$, $S_n^2(t_0)$ is nonnegative. For each odd integer $k \geq 1$, one has (see (4.10) and (4.1))

$$E_k + E_{k+1} = \frac{\pi^2}{2l^4 \lambda_k \lambda_{k+1}} \left(\pi(4k^2 - 1) - 2(4k^2 + 1)l \right).$$

Thus

$$l \leq \frac{3\pi}{10} \Longrightarrow E_k + E_{k+1} \geq 0 \quad \forall k \geq 1, \text{ odd.} \tag{4.12}$$

Let us write $S_n^2(t_0)$ in the form

$$S_n^2(t_0) = \sum_{k=1,odd}^{n-1} I_k E_k + I_{k+1} E_{k+1}.$$

Using the fact that for all $t \in [0, t_0]$, $\int_\Omega u(t) \, dx \geq 0$ and the hypothesis iii), one deduces

$$b(\int_\Omega u(\cdot) \, dx) \geq 0 \quad \text{ in } [0, t_0].$$

Thus, $k \mapsto I_k$ decreases. It follows that for all odd integers $k \geq 1$,

$$I_{k+1} E_{k+1} \geq I_k E_{k+1}, \tag{4.13}$$

since $E_{k+1} \leq 0$. Then from (4.13), (4.12) and the positivity of I_k, it follows that $I_k E_k + I_{k+1} E_{k+1} \geq I_k (E_k + E_{k+1}) \geq 0$. Thus,

$$S_n^2(t_0) \geq 0, \quad \forall n = 2, 4, \ldots. \tag{4.14}$$

Let us consider now the sum $S_n^1(t_0)$. It is well known that the variational solution to the problem

$$\begin{cases} w_{n_t} - w_n'' &= 0 & \text{in } (0,+\infty)\times\Omega, \\ w_n(t,0) = 0, & w_n'(t,l) = 0, & \text{on } (0,+\infty), \\ w_n(0,x) &= \sum_{k=1}^n u_{0_k}\varphi_k & \text{in } \Omega, \end{cases}$$

converges in $C([0,t_0], L^2(\Omega))$ towards the variational solution w to the problem

$$\begin{cases} w_t - w'' &= 0 & \text{in } (0,+\infty)\times\Omega, \\ w(t,0) = 0, & w'(t,l) = 0, & \text{on } (0,+\infty), \\ w(0,x) &= u_0 & \text{in } \Omega. \end{cases}$$

Now, it is clear that

$$w_n(t) = \sum_{k=1}^n e^{-\lambda_k t} u_{0_k}\varphi_k.$$

Thus, by the continuity of the integral,

$$\sum_{k=1}^n e^{-\lambda_k t_0} u_{0_k}\int_\Omega \varphi_k\,dx \to \int_\Omega w(t_0)\,dx \quad \text{ as } n\to+\infty.$$

Going back to u, one has, from (4.8), (4.11) and (4.14),

$$\sum_{k=1}^n u_k(t_0)\int_\Omega \varphi_k\,dx \geq \sum_{k=1}^n v_k(t_0)\int_\Omega \varphi_k\,dx = S_n(t_0) \geq \sum_{k=1}^n e^{-\lambda_k t_0}u_{0_k}\int_\Omega \varphi_k\,dx.$$

Passing to the limit, one deduces

$$\int_\Omega u(t_0)\,dx \geq \int_\Omega w(t_0)\,dx.$$

Now, since $u_0 \geq 0$ almost everywhere in Ω and $u_0 \not\equiv 0$, we have by the maximum principle (see [C], Theorem 13.6) $w(t_0) > 0$ almost everywhere in Ω. In particular, $\int_\Omega w(t_0)\,dx > 0$; thus

$$\int_\Omega u(t_0)\,dx > 0.$$

This completes the proof of the theorem. □

Let us apply our results to problem (MP$p,p$). Considering the one-dimensional problem

$$(\mathbf{P}+,u_0)\begin{cases} u_t-u'' &= -\Big(\big(\int_\Omega u(t,x)\,dx\big)^+\Big)^p & \text{in } (0,T)\times\Omega,\\ u(t,0) &= 0 & \text{on } (0,T),\\ u'(t,l) &= \Big(\big(\int_\Omega u(t,x)\,dx\big)^+\Big)^p & \text{on } (0,T),\\ u(0,\cdot) &= u_0(\cdot) & \text{in } \Omega,\end{cases}$$

we can show

Corollary 4.1. *Assume that*
i) $\Omega := (0,l)$ *with* $l \in (0, \frac{3\pi}{10}]$,
ii) *the initial condition* u_0 *belongs to* $L^2(\Omega)$ *and* $u_0 \geq 0, u_0 \not\equiv 0$ *almost everywhere in* Ω.

Then the problem (MP$p,p$) admits a unique maximal variational solution. Moreover, it coincides with the maximal variational solution to (P$+,u_0$).

Proof of Corollary 4.1. According to Definition 3.1 and Corollary 3.1, it is enough to prove that these problems have the same local solutions. Indeed, every variational solution to (MP$p,p$) is clearly a solution to (P$+,u_0$). Conversely, by Theorem 4.1, every solution to (P$+,u_0$) is also a solution to (MP$p,p$). □

We would like now, following a question of Prof. F. Conrad, to extend Theorem 4.1 to larger domains. Note that we have been unable to extend it to an arbitrary domain. First, we need

Lemma 4.1. *If the problem (P1,u_0) admits a variational solution, then, for all* $l_0 > 0$, *the problem*

$$(\mathbf{P}_\lambda)\begin{cases} v_t-v'' &= -A(\int_{\Omega_0} v(t,x)\,dx) & \text{in } (0,T_0)\times\Omega_0,\\ v(t,0) &= 0 & \text{on } (0,T_0),\\ v'(t,l_0) &= B(\int_{\Omega_0} v(t,x)\,dx) & \text{on } (0,T_0),\\ v(0,x) &= u_0(\lambda x) & \text{for a.e. } x\in\Omega_0,\end{cases}$$

has also a variational solution for $\Omega_0 := (0,l_0)$, $T_0 := \frac{T}{\lambda^2}$ *where* $\lambda := \frac{l}{l_0}$ *and* A *and* B *defined by*

$$A(s) = \lambda^2 a(\lambda s), \quad B(s) = \lambda b(\lambda s), \quad \forall s\in\mathbb{R}. \tag{4.15}$$

Proof of Lemma 4.1. For all φ_0 belonging to $V_0 := \{v\in H^1(\Omega_0) : v(0) = 0\}$ and all α_0 in $\mathcal{D}(0,T_0)$, the functions φ and α defined by

$$\varphi(x) := \varphi_0(\lambda^{-1}x) \quad \text{for a.e. } x\in\Omega \tag{4.16}$$

and

$$\alpha(t) := \alpha_0(\lambda^{-2}t) \quad \forall t \in [0,T] \tag{4.17}$$

belong respectively to V and $\mathcal{D}(0,T)$.

Choosing φ and α as test functions, a variational solution u to problem (P1,u_0) satisfies

$$\int_0^T \int_0^l u(t,x)\varphi(x)\alpha'(t)\,dx\,dt + \int_0^T \int_0^l u'(t,x)\varphi'(x)\alpha(t)\,dx\,dt = \\ \int_0^T \left(-a\left(\int_0^l u\,dx\right)\int_0^l \varphi\,dx + b\left(\int_0^l u\,dx\right)\varphi(l)\right)\alpha(t)\,dt.$$

Setting

$$v(t,x) := u(\lambda^2 t, \lambda x) \quad \forall t \in [0,T_0],\ \text{for a.e. } x \in (0,l_0) \tag{4.18}$$

and using (4.16) and (4.17), one obtains

$$\int_0^T \int_0^l v(\lambda^{-2}t,\lambda^{-1}x)\varphi_0(\lambda^{-1}x)\lambda^{-2}\alpha_0'(\lambda^{-2}t)\,dx\,dt + \\ \int_0^T \int_0^l \lambda^{-1}v'(\lambda^{-2}t,\lambda^{-1}x)\lambda^{-1}\varphi_0'(\lambda^{-1}x)\alpha_0(\lambda^{-2}t)\,dx\,dt = \\ \int_0^T \left(-a\left(\int_0^l v(\lambda^{-2}t,\lambda^{-1}x)\,dx\right)\int_0^l \varphi_0(\lambda^{-1}x)\,dx \right. \\ \left. +b\left(\int_0^l v(\lambda^{-2}t,\lambda^{-1}x)\,dx\right)\varphi_0(\lambda^{-1}l)\right)\alpha_0(\lambda^{-2}t)\,dt.$$

The change of variables $\tau = \lambda^{-2}t, y = \lambda^{-1}x$ gives

$$\int_0^{T_0} \int_0^{l_0} v(\tau,y)\varphi_0(y)\alpha_0'(\tau)\lambda\,dy\,d\tau + \int_0^{T_0} \int_0^{l_0} v'(\tau,y)\varphi_0'(y)\alpha_0(\tau)\lambda\,dy\,d\tau \\ = \int_0^{T_0} \Big(-\lambda^2 a\Big(\lambda\int_0^{l_0} v\,dy\Big)\int_0^{l_0}\varphi_0(y)\,dy + \lambda b\Big(\lambda\int_0^{l_0} v\,dy\Big)\varphi_0(l_0)\Big)\alpha_0(\tau)\lambda\,d\tau.$$

Next we simplify by λ and introduce the functions A and B given by (4.15). Then the result follows. □

Let us introduce the set

$$I_+ := \{l > 0 : \text{Theorem 4.1 holds under the assumptions ii), iii), iv),} \\ \text{without constraint on } l \text{ for } \Omega = (0,l)\}.$$

Note that we suppose here that Theorem 4.1 holds for *every* a and b satisfying the assumptions imposed. Then, the following holds:

Corollary 4.2. *I_+ is a subinterval of $(0,+\infty)$ which contains $(0,\frac{3\pi}{10}]$.*

Proof of Corollary 4.2. The second part of the assertion follows from Theorem 4.1. In order to establish the first part, it is enough to prove that for all $l_0 \in I_+$, $(0, l_0)$ is contained in I_+.

Let $l_0 \in I_+$, $l \in (0, l_0)$ and a, b, u_0 satisfy the assumptions of Theorem 4.1. Set $\lambda := \frac{l}{l_0}$. Then the functions A, B defined by (4.15) and $v_0 \in L^2(0, l_0)$ defined by

$$v_0(x) := u_0(\lambda x) \quad \text{for a.e. } x \in \Omega_0 := (0, l_0)$$

satisfy also these assumptions. The only point to be checked is perhaps iii). For $s > 0$, $A(s) \leq B(s)$ if $a(s) \leq 0$; otherwise, since $\lambda < 1$ and $a \leq b$ on $(0, +\infty)$, one has

$$A(s) := \lambda^2 a(\lambda s) \leq \lambda a(\lambda s) \leq \lambda b(\lambda s) =: B(s).$$

Denote by u (respectively v) the maximal variational solution to problem (P1, u_0) (respectively (P_λ)) defined on $[0, T_{max}(u_0))$ (respectively $[0, T_{max}(v_0))$). According to Lemma 4.1 and (4.18),

$$\int_\Omega u(\lambda^2 t, x)\, dx = \lambda \int_{\Omega_0} v(t, x)\, dx \quad \forall t \in [0, T_{max}(v_0)).$$

Thus iii) holds. Since l_0 belongs to I_+, the right-hand side is positive. Moreover, Lemma 4.1 and uniqueness for the above problems imply

$$\lambda^2 T_{max}(v_0) = T_{max}(u_0).$$

Then, clearly,

$$\int_\Omega u(t, x)\, dx > 0 \quad \forall t \in [0, T_{max}(u_0)).$$

This is just what we had to prove. □

Set $l_1 := \sup I_+$ (recall that I_+ is not empty) and for a and b defined on $[0, +\infty)$ with values in $\mathbb{R}$,

$$\lambda(a, b) := \inf \left\{ \frac{b(s)}{a(s)} : s > 0, a(s) > 0 \right\}.$$

If the above set is empty, we put $\lambda(a, b) := +\infty$.

Then we can prove

Theorem 4.2. *Let a and b be two functions satisfying the assumptions* (2.2), (2.3) *and* (2.27). *In addition, we assume*

i) $\Omega := (0, l)$ *with* $0 < l < \lambda(a, b) l_1$,

ii) *the initial condition* u_0 *belongs to* $L^2(\Omega)$ *and* $u_0 \geq 0, u_0 \not\equiv 0$ *almost everywhere in* Ω.

Then the integral of the maximal variational solution to problem (P1,u_0) *is positive in* $[0, T_{max}(u_0))$.

Remarks 4.1. (i) If $a \leq 0$ on $[0, +\infty)$ then $\lambda(a, b) = +\infty$ and Theorem 4.2 holds for all bounded domains $\Omega = (0, l)$. Note that this can be proved directly by the maximum principle .

(ii) The assumption "$a \leq b$ on $[0, +\infty)$" of Theorem 4.1 is here no more needed since the condition $\lambda(a, b) < 1$ is allowed. For instance, if

$$a(s) = s^p, \quad b(s) = s^{p+1} + \alpha s^p \quad \forall s > 0,$$

where p and α are two given real numbers such that $p \geq 1$ and $0 < \alpha < 1$ then $\lambda(a, b) = \alpha$ and $b < a$ on $(0, 1-\alpha)$. The assumptions of Theorem 4.2 can also be satisfied if a dominates b for large values of the argument. Indeed, if

$$a(s) = s^p, \quad b(s) = \alpha s^p + s^{p-1} \quad \forall s > 0$$

(where p and α are as above), one has $\lambda(a, b) = \alpha$ and $b < a$ on $(\frac{1}{1-\alpha}, +\infty)$.

Proof of Theorem 4.2. From assumption i), there exists (even if $\lambda(a, b) = +\infty$) some real number $l_0 \in (0, l_1)$ such that $l \leq \lambda(a, b) l_0$. Set $\lambda := l/l_0$. We use the notation and arguments of the previous corollary. For all $s > 0$, $A(s) \leq B(s)$ if $a(s) \leq 0$; otherwise,

$$A(s) := \lambda^2 a(\lambda s) \leq \lambda \cdot \lambda(a, b) a(\lambda s) \leq \lambda b(\lambda s) =: B(s),$$

Therefore, A, B and $v_0 \in L^2(0, l_0)$ satisfy the assumptions of Theorem 4.1. Since l_0 belongs to I_+ (see Corollary 4.2), we conclude as in Corollary 4.2. □

4.2. **A comparison principle for the integral.** In this section, we assume that the functions a and b are equal. More precisely, we deal with the one-dimensional problem

$$(\mathbf{P}1, u_0) \begin{cases} u_t - u'' &= -a(\int_\Omega u(t,x)\,dx) \quad \text{in } (0,T) \times \Omega, \\ u(t,0) &= 0 \quad \text{on } (0,T), \\ u'(t,l) &= a(\int_\Omega u(t,x)\,dx) \quad \text{on } (0,T), \\ u(0,\cdot) &= u_0(\cdot) \quad \text{in } \Omega. \end{cases}$$

Then we have

Theorem 4.3. *Assume*

i) $\Omega = (0, l)$ *where* $l \in (0, \frac{3\pi}{10}]$,

ii) *the function* a *satisfies assumptions* (2.1) *and* (2.2) *and is nondecreasing on* $\mathbb{R}$,

iii) *the initial conditions* u_0 *and* v_0 *belong to* $L^2(\Omega)$ *and satisfy*

$$u_0 \le v_0, \quad \text{a.e. in } \Omega.$$

Then the maximal variational solutions u *and* v *to problem* (P1,u_0) *and* (P1,v_0) *satisfy*

$$\int_\Omega u(t,x)\,dx \le \int_\Omega v(t,x)\,dx \quad \forall t \in [0, T_{max}(u_0) \wedge T_{max}(v_0)).$$

Proof of Theorem 4.3. Let T be any real number in $(0, T_{max}(u_0) \wedge T_{max}(v_0))$. Using the notation of Paragraph 4.1, (4.2) and the fact that $a \equiv b$, one deduces that the coordinates of $u(t)$ and $v(t)$ in the basis (φ_k) fulfill

$$\frac{d}{dt}\left(\int_\Omega u(t)\varphi_k\,dx\right) + \lambda_k \int_\Omega u(t)\varphi_k\,dx = \left(\varphi_k(l) - \int_\Omega \varphi_k\,dx\right) a\left(\int_\Omega u(t)\,dx\right)$$

and

$$\frac{d}{dt}\left(\int_\Omega v(t)\varphi_k\,dx\right) + \lambda_k \int_\Omega v(t)\varphi_k\,dx = \left(\varphi_k(l) - \int_\Omega \varphi_k\,dx\right) a\left(\int_\Omega v(t)\,dx\right),$$

in $C^1(0,T)$. By difference, the function $w_k := v_k - u_k$ is a solution to

$$w_k'(t) + \lambda_k w_k(t) = \left(a(\int_\Omega v) - a(\int_\Omega u)\right) D(\varphi_k), \quad w_k(0) = v_{0_k} - u_{0_k}.$$

Hence, it holds for all $t \in [0,T]$ that

$$w_k(t) = e^{-\lambda_k t} w_k(0) + \int_0^t e^{-\lambda_k(t-s)} \left(a(\int_\Omega v) - a(\int_\Omega u)\right) ds D(\varphi_k).$$

Moreover, for $w := v - u$, one has

$$\int_\Omega w(t)\,dx = \sum_{k=1}^\infty w_k(t) \int_\Omega \varphi_k\,dx = \sum_{k=1}^\infty e^{-\lambda_k t} w_k(0) \int_\Omega \varphi_k\,dx + \sum_{k=1}^\infty I_k(t) E_k, \tag{4.19}$$

where E_k is defined by (4.10) and $I_k(t)$ denotes the quantity

$$I_k(t) := \int_0^t e^{-\lambda_k(t-s)} \left(a(\int_\Omega v) - a(\int_\Omega u)\right) ds.$$

Since $u_0 \le v_0$, we can assume

$$\int_\Omega u_0(x)\,dx < \int_\Omega v_0(x)\,dx. \tag{4.20}$$

The theorem will be proved if we can show that, for all t_0 belonging to $(0, T_{max}(u_0) \wedge T_{max}(v_0))$ and satisfying

$$\int_\Omega u(t,x)\,dx < \int_\Omega v(t,x)\,dx \qquad \forall t \in [0,t_0), \tag{4.21}$$

one has

$$\int_\Omega u(t_0,x)\,dx < \int_\Omega v(t_0,x)\,dx$$

or equivalently

$$\int_\Omega w(t_0,x)\,dx > 0.$$

Consider such a t_0. Since a is nondecreasing on $\mathbb{R}$, one deduces from (4.21) that

$$I_k(t_0) \geq 0, \quad \forall k \geq 1 \tag{4.22}$$

and $k \mapsto I_k(t_0)$ decreases. We can show as in the proof of Theorem 4.1 that the series $\sum_{k=1}^\infty I_k(t_0)E_k$ (which by (4.19) converges) is nonnegative. It remains to show that the first sum in (4.19) is positive. It is clear that the function

$$z(t,x) := \sum_{k=1}^\infty e^{-\lambda_k t} w_k(0)\varphi_k(x)$$

is a solution to the problem

$$\begin{cases} z_t - z'' &= 0 & \text{in } (0,+\infty)\times\Omega, \\ z(t,0)=0, & z'(t,l)=0, & \\ z(0,x) &= v_0 - u_0 & \text{in } \Omega. \end{cases}$$

Since $z(0) = v_0 - u_0 \geq 0$ and $v_0 - u_0 \not\equiv 0$ according to (4.20), one deduces from the maximum principle that $z(t_0) > 0$ almost everywhere in Ω. Hence

$$\int_\Omega z(t_0)\,dx = \sum_{k=1}^\infty e^{-\lambda_k t_0} w_k(0) \int_\Omega \varphi_k\,dx > 0. \tag{4.23}$$

Finally, we wish to apply this result to problem (MP$p,p$).

Corollary 4.3. *Assume*

i) $\Omega := (0,l)$ *with* $l \in (0, \frac{3\pi}{10}]$,

ii) $p \geq 1$,

iii) *the initial conditions* u_0 *and* v_0 *belong to* $L^2(\Omega)$ *and satisfy*

$$0 \leq u_0 \leq v_0, \qquad \text{a.e. in } \Omega.$$

Then the variational solutions u and v of (MP$p, p$) *corresponding respectively to u_0 and v_0 satisfy*

$$0 \le \int_\Omega u(t,x)\,dx \le \int_\Omega v(t,x)\,dx, \qquad \forall t \in [0, T_{max}(u_0) \wedge T_{max}(v_0)).$$

Proof of Corollary 4.3. We apply Theorem 4.3 to the problem

$$\begin{cases} u_t - u'' &= -\left(\left(\int_\Omega u(t,x)\,dx\right)^+\right)^p & \text{in } (0,T)\times\Omega, \\ u(t,0) &= 0 & \text{on } (0,T), \\ u'(t,l) &= \left(\left(\int_\Omega u(t,x)\,dx\right)^+\right)^p & \text{on } (0,T), \\ u(0,\cdot) &= u_0(\cdot) & \text{in } \Omega, \end{cases}$$

and come back to problem (MP$p, p$) by the positivity of the integrals. □

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