Advances in Differential Equations Volume 7, Number 2, February 2002, Pages 237–256

DIFFUSIVE LOGISTIC EQUATIONS IN POPULATION DYNAMICS

Kazuaki Taira
Institute of Mathematics, University of Tsukuba, Tsukuba 305–8571, Japan

(Submitted by: Herbert Amann)

Dedicated to the memory of Professor Yoshikazu Hirasawa

Abstract. The purpose of this paper is to prove an existence and uniqueness theorem of positive solutions of diffusive logistic equations with indefinite weights, which model population dynamics in environments with strong spatial heterogeneity. We prove that the most favorable situations will occur if there is a relatively large favorable region (with good resources and without crowding effects) located some distance away from the boundary of the environment. Moreover we discuss the stability properties for positive steady states.

1. Introduction and main results

Let Ω be a bounded domain of Euclidean space $\mathbf{R}^n$, $n \geq 3$, with boundary $\partial\Omega$ of class $C^{2+\theta}$ with exponent $0 < \theta < 1$; its closure $\overline{\Omega} = \Omega \cup \partial\Omega$ is an n-dimensional, compact manifold with boundary. The dynamics of a population inhabiting a strongly heterogeneous environment are modeled by diffusive logistic equations of the form

$$\begin{cases} \frac{\partial w}{\partial t} = d\,\Delta w + (m(x) - h(x)w)w & \text{in } \Omega \times (0,\infty), \\ w = 0 & \text{on } \partial\Omega \times (0,\infty), \\ w|_{t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here

(1) $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_n^2$ is the usual Laplacian.
(2) d is a positive parameter.
(3) $m(x)$ is a real-valued function on $\overline{\Omega}$.
(4) $h(x)$ is a nonnegative function on $\overline{\Omega}$.

We discuss our motivation and some of the modeling process leading to problem (1.1). The basic interpretation of the various terms in problem (1.1)

Accepted for publication: April 2001.
AMS Subject Classifications: 35J65, 35P30, 92D25.

is that the solution $w(x,t;u_0)$ represents the population density of a species inhabiting a region Ω. The members of the population are assumed to move about Ω via the type of random walks occurring in Brownian motion that is modeled by the diffusive term $d\Delta$; hence d represents the rate of diffusive dispersal, so large values of d the population spreads more rapidly than for small values of d. The local rate of change in the population density is described by the density dependent term $m(x) - h(x)u$. In this term, $m(x)$ describes the rate at which the population would grow or decline at the location x in the absence of crowding or limitations on the availability of resources. The sign of $m(x)$ will be positive on favorable habitats for population growth and negative on unfavorable ones. Specifically $m(x)$ may be considered as a food source or any resource that will be good in some areas and bad in others. The term $-h(x)u$ describes the effects of crowding on the growth rate of the population at the location x; these effects are assumed to be independent of those determining the growth rate. The size of $h(x)$ describes the strength of the effects of crowding within the population.

On the other hand, in terms of biology, the homogeneous Dirichlet condition represents that Ω is surrounded by a completely hostile exterior such that any member of the population which reaches the boundary dies immediately; in other words, the exterior of the domain is deadly to the population. If the exterior is hostile but not completely deadly, a mixed or Robin boundary condition results, and the analysis is similar.

To study problem (1.1), we may view it as generating a *dynamical system*. The semilinear parabolic initial boundary value problem (1.1) admits a unique classical solution for sufficiently small times. However, comparison theorems based on the maximum principle guarantee the existence of global solutions in time, since the nonlinearity we are dealing with is sublinear. We show that problem (1.1) admits a unique positive steady state which is a global attractor for nonnegative solutions provided d is sufficiently small, so that the population persists, and further we show that the zero solution is a global attractor for nonnegative solutions if d is sufficiently large, so that the population tends to extinction.

Our models are shown to possess a unique positive steady state, that is, a unique positive solution of the problem

$$\begin{cases} d\,\Delta u + (m(x) - h(x)u)u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

A solution $u \in C^2(\overline{\Omega})$ of problem (1.2) is said to be *nontrivial* if it does not identically equal zero on $\overline{\Omega}$. A nontrivial solution u is called a *positive* *solution* if it is strictly positive everywhere in Ω. The object of the analysis is to determine how the spatial arrangement of favorable and unfavorable habitats affects the population being modeled. As is frequently the case, we find that many of the qualitative aspects of the analysis depend crucially on the size of the first positive eigenvalue $\lambda_1(m)$ for the linearized Dirichlet problem with indefinite weight function $m(x)$ and positive parameter $\lambda := 1/d$

$$\begin{cases} -\Delta\phi = \lambda\, m(x)\, \phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The next theorem asserts the existence of the first positive eigenvalue $\lambda_1(m)$ of problem (1.3), implying persistence for the population (see Manes and Micheletti [14], de Figueiredo [5]):

Theorem 1.1. *If $m(x)$ is a function in $L^\infty(\Omega)$ such that the set $\{x \in \Omega : m(x) > 0\}$ has positive measure, then the first positive eigenvalue $\lambda_1(m)$ of problem* (1.3) *is simple and its corresponding eigenfunction $\phi_1(x)$ may be chosen to be strictly positive everywhere in Ω. Moreover, no other eigenvalues have positive eigenfunctions.*

By the Rayleigh principle (see Manes and Micheletti [14], de Figueiredo [5]), we know that the first positive eigenvalue $\lambda_1(m)$ is given by the variational formula

$$\lambda_1(m) = \inf\left\{ \frac{\int_\Omega |\nabla\phi|^2\, dx}{\int_\Omega m(x)\phi^2\, dx} : \phi \in W_0^{1,2}(\Omega),\ \int_\Omega m(x)\phi^2\, dx > 0 \right\}. \tag{1.4}$$

Here $W_0^{1,2}(\Omega)$ is the closure of smooth functions with compact support in Ω in the Sobolev space $W^{1,2}(\Omega)$. By formula (1.4), we find that $\lambda_1(m)$ is strictly decreasing with respect to $m(x)$ in the sense that if $m_1(x) \le m_2(x)$ almost everywhere in Ω, then the corresponding first positive eigenvalues $\lambda_1(m_1)$ and $\lambda_1(m_2)$ satisfy the relation

$$\lambda_1(m_1) \ge \lambda_1(m_2). \tag{1.5}$$

If the inequality is strict on a set of positive measure, then it follows that $\lambda_1(m_1) > \lambda_1(m_2)$.

A biological interpretation of Theorem 1.1 is that if there is a favorable region, then the models we consider predict persistence for a population, since the existence of the first positive eigenvalue is equivalent to the existence of a positive density function describing the distribution of the population of Ω. The size of $\lambda_1(m)$ is of crucial importance; increasing $\lambda_1(m)$ imposes a more stringent condition on the diffusion rate d if the population is to persist, since $0 < d < 1/\lambda_1(m)$ (see Theorem 1.2 or Figure 1.2 below). It is worthwhile to point out here that the first positive eigenvalue $\lambda_1(m)$ will tend to be smaller in situations where favorable and unfavorable habitats are closely intermingled (producing cancellation effects), and larger when the favorable region consists of a relatively small number of relatively large isolated components.

First we study problem (1.2) with $d := 1/\lambda$

$$\begin{cases} -\Delta u = \lambda(m(x) - h(x)u)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

We assume that $h(x)$ is a nonnegative function in $C^1(\overline{\Omega})$, and let

$$\Omega^+(h) = \{x \in \Omega : h(x) > 0\},$$

and $\Omega_0(h) = \Omega \setminus \overline{\Omega^+(h)}$. Our fundamental hypothesis on the function $h(x)$ is the following (see Figure 1.1 below):

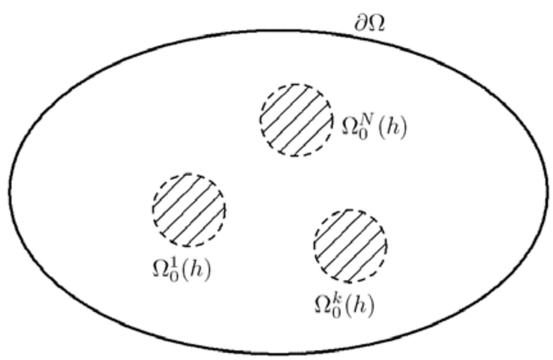

Figure 1.1

(Z) The open set $\Omega_0(h)$ consists of a *finite* number of connected components with boundary of class $C^{2+\theta}$, say $\Omega_0^k(h)$, $1 \leq k \leq N$, that are bounded away from $\partial\Omega$.

We consider the Dirichlet eigenvalue problem with indefinite weight function $m(x)$ in each connected component $\Omega_0^k(h)$

$$\begin{cases} -\Delta\psi = \mu\, m(x)\, \psi & \text{in } \Omega_0^k(h), \\ \psi = 0 & \text{on } \partial\Omega_0^k(h), \end{cases} \tag{1.7}$$

and let

$$\mu_1(\Omega_0^k(h)) = \text{ the } \textit{first positive eigenvalue} \text{ of problem (1.7).}$$

It should be noticed that the first positive eigenvalue $\mu_1(\Omega_0^k(h))$ is given by the variational formula

$$\mu_1(\Omega_0^k(h)) = \inf\left\{ \frac{\int_{\Omega_0^k(h)} |\nabla\psi|^2 dx}{\int_{\Omega_0^k(h)} m(x)\psi^2 dx} : \psi \in W_0^{1,2}(\Omega_0^k(h)), \int_{\Omega_0^k(h)} m(x)\psi^2 dx > 0 \right\}.$$

If we let

$$\mu_1(\Omega_0(h)) = \min\left\{\mu_1(\Omega_0^1(h)), \mu_1(\Omega_0^2(h)), \ldots, \mu_1(\Omega_0^N(h))\right\},$$

then we can state our main result that is a generalization of Cantrell and Cosner [4, Theorems 2.1 and 2.3], Hess and Kato [11, Theorem 2] and Hess [10, Theorem 27.1] to the case where $h(x)$ may vanish in Ω:

Theorem 1.2. *Assume that $h(x) \in C^1(\overline{\Omega})$ satisfies condition (Z). If $m(x)$ is a function in $C^\theta(\overline{\Omega})$, $0 < \theta < 1$, such that each set $\{x \in \Omega_0^k(h) : m(x) > 0\}$, $1 \leq k \leq N$, has positive measure, then problem* (1.6) *has a unique positive solution $u(\lambda) \in C^{2+\theta}(\overline{\Omega})$ for every $\lambda \in (\lambda_1(m), \mu_1(\Omega_0(h)))$. For any $\lambda \geq \mu_1(\Omega_0(h))$, there exists no positive solution of problem* (1.6). *Moreover, we have*

$$\lim_{\lambda \to \mu_1(\Omega_0(h))} \|u(\lambda)\|_{L^2(\Omega)} = +\infty, \tag{1.8}$$

and also

$$\lim_{\lambda \to \lambda_1(m)} \|u(\lambda)\|_{C^{2+\theta}(\overline{\Omega})} = 0. \tag{1.9}$$

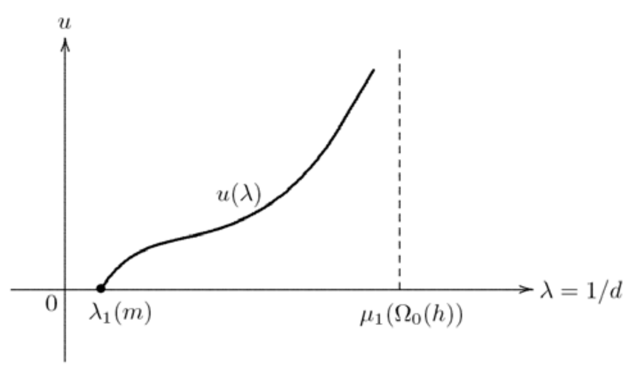

Figure 1.2

A biological interpretation of Theorem 1.2 is that if the environment has a completely hostile boundary, then the models we consider predict persistence for a population if its diffusion rate d is below the critical value $1/\lambda_1(m)$

depending on the coefficient $m(x)$ describing the growth rate and if it is above the critical value $1/\mu_1(\Omega_0(h))$ depending on the coefficient $h(x)$ describing the strength of the crowding effects. Theorem 1.2 also asserts that in a certain sense, the most favorable situations will occur if there is a relatively large favorable region (with good resources and without crowding effects) located some distance away from the boundary of Ω. The situation may be represented schematically by the bifurcation diagram 1.2.

Some important remarks are in order.

Remark 1.1. Theorem 1.2 may be proved by using the super-sub-solution method just as in the proof of Fraile et al. [7, Theorems 3.5 and 4.6], if assertion (1.8) is replaced by a weaker one

$$\lim_{\lambda \to \mu_1(\Omega_0(h))} \|u(\lambda)\|_{C(\overline{\Omega})} = +\infty.$$

Theorem 1.2 asserts that assertion (1.8) holds true if the dimension n is greater than 2 ($n \geq 3$). It should be emphasized that an estimate of the growth rate of the total size $\|u(\lambda)\|_{L^1(\Omega)} = \int_\Omega u(\lambda)\, dx$ of the positive steady states $u(\lambda)$ as $\lambda \uparrow \mu_1(\Omega_0(h))$ is of crucial importance from the viewpoint of population dynamics.

Remark 1.2. López-Gómez and Sabina de Lis [13] analyze the pointwise growth to infinity of positive solutions of the logistic Dirichlet problem in the case where $m(x) \equiv 1$ in Ω (see [13, Theorems 4.2 and 4.3]). Furthermore García-Melián et al. [8] study the pointwise behavior and the uniqueness of positive solutions of nonlinear elliptic boundary value problems of general sublinear type, and give the exact limiting profile of the positive solutions (see [8, Theorem 3.1, Corollary 3.3 and Theorem 6.4]). Their numerical computations confirm and illuminate the above bifurcation diagram 1.2.

Remark 1.3. Assume that $h(x) > 0$ on $\overline{\Omega}$, and that $m(x)$ attains positive values in Ω. In Appendix we replace assertion (1.9) by an estimate of the decay rate of the total size $\|u(\lambda)\|_{L^1(\Omega)} = \int_\Omega u(\lambda)\, dx$ as $\lambda \downarrow \lambda_1(m)$:

$$\int_\Omega u(\lambda)\, dx \leq \left(1 - \frac{\lambda_1(m)}{\lambda}\right) |\Omega|^{2/3} \frac{\left(\int_\Omega (m^+)^3\, dx\right)^{1/3}}{\min_{x \in \overline{\Omega}} h(x)}, \quad \lambda > \lambda_1(m), \quad (1.10)$$

where $|\Omega|$ is the volume of Ω and $m^+(x) = \max\{m(x), 0\}$, $x \in \Omega$.

Secondly we study the asymptotic stability properties for positive solutions of problem (1.6). To do this, we consider the semilinear initial boundary value problem (1.1) with $d := 1/\lambda$

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{1}{\lambda}\,\Delta w + (m(x) - h(x)w)w & \text{in } \Omega \times (0, \infty), \\ w = 0 & \text{on } \partial\Omega \times (0, \infty), \\ w|_{t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{1.11}$$

It is known (see Amann [2, Theorem 4.5]) that problem (1.11) admits a unique classical global solution $w(x, t; u_0)$ for each initial value $u_0 \in C^{2+\theta}(\overline{\Omega})$ which satisfies the conditions

$$\begin{cases} u_0 \geq 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.12}$$

A nonnegative solution $u(x)$ of problem (1.6) is said to be *globally asymptotically stable* if we have

$$\max_{x \in \overline{\Omega}} |w(x, t; u_0) - u(x)| \longrightarrow 0 \quad \text{as } t \to \infty$$

for each nontrivial $u_0 \in C^{2+\theta}(\overline{\Omega})$ which satisfies conditions (1.12).

The next theorem describes the asymptotic stability properties for positive solutions of problem (1.6) (cf. Cantrell and Cosner [4, Theorems 2.1 and 4.9], Fraile et al. [7, Theorem 3.7]):

Theorem 1.3. (*i*) *The zero solution of problem* (1.6) *is* globally asymptotically stable *if λ is so small that $0 < \lambda < \lambda_1(m)$. In this case we can estimate the decay rate of the total size $\|w(\cdot, t; u_0)\|_{L^1(\Omega)} = \int_\Omega w(x, t; u_0)\, dx$ of the population as $t \downarrow 0$:*

$$\begin{aligned} &\int_\Omega w(x, t; u_0)\, dx \\ &\leq \exp\left[-\left(\frac{1}{\lambda} - \frac{1}{\lambda_1(m)}\right)\lambda_1(1)\, t\right] |\Omega|^{1/2} \left(\int_\Omega u_0(x)^2\, dx\right)^{1/2}, \quad t > 0. \end{aligned} \tag{1.13}$$

(*ii*) *The positive solution $u(\lambda)$ of problem* (1.6) *is globally asymptotically stable for each λ satisfying the condition $\lambda_1(m) < \lambda < \mu_1(\Omega_0(h))$.*

(iii) If λ is so large that $\lambda > \mu_1(\Omega_0(h))$, then we have

$$\max_{x \in \overline{\Omega}} |w(x, t; u_0)| \longrightarrow \infty \quad \textit{as } t \to \infty$$

for each nontrivial $u_0 \in C^{2+\theta}(\overline{\Omega})$ which satisfies conditions (1.12).

A biological interpretation of Theorem 1.3 is that a population will grow exponentially until limited by lack of available resources if the diffusion rate $d = 1/\lambda$ is below the critical value $1/\mu_1(\Omega_0(h))$; this idea is generally credited to Thomas Malthus. On the other hand, if the diffusion rate $d = 1/\lambda$ is above the critical value $1/\mu_1(\Omega_0(h))$, then the model obeys the logistic equation introduced by P.F. Verhulst around 1840. The situation may be represented schematically by the following bifurcation diagram 1.3.

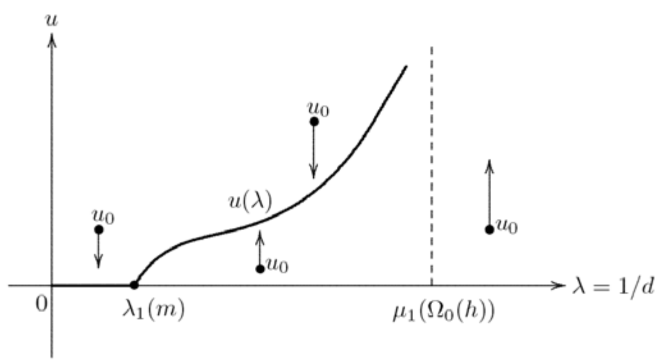

Figure 1.3

The rest of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.2. To prove assertion (1.8), we make use of the implicit function theorem and Sobolev's imbedding theorem just as in Taira [19]. In Section 3 we only prove the decay estimate (1.13), since the other assertions of Theorem 1.3 may be proved by using comparison theorems based on the maximum principle, just as in Fraile et al. [7, Theorem 3.7], Pao [15, Chapter 5, Theorem 4.4] and also Sattinger [16, Theorem 2.6.2]. In the final Section 4 we study the problem with homogeneous Neumann condition

$$\begin{cases} \dfrac{\partial w}{\partial t} = d\,\Delta w + (m(x) - h(x)w)w & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ w = u_0 & \text{in } \Omega, \end{cases} \tag{1.14}$$

where $\mathbf{n}$ is the unit exterior normal to $\partial\Omega$. If the boundary acts as a barrier, so that individuals reaching the boundary simply return to the interior, a Neumann boundary condition results. The analysis may be somewhat different, since the operator $-\Delta$ with homogeneous Neumann condition will have zero as an eigenvalue. However the same general approach can still be used.

I would like to thank Masayasu Mimura for his helpful suggestions on the formulation of Theorem 1.3 from the point of view of mathematical ecology.

I also thank the referee who made many comments that were crucial in improving the exposition, both mathematically and stylistically.

This research is partially supported by Grant-in-Aid for General Scientific Research (No. 10304006), Ministry of Education, Science and Culture of Japan.

2. Proof of Theorem 1.2

Theorem 1.2 except for assertion (1.8) can be proved by using the super-sub-solution method just as in the proof of Fraile et al. [7, Theorems 3.5 and 4.6]. To prove assertion (1.8), we make use of the implicit function theorem as in Taira [19]. To do this, we let

$$C_0^{2+\theta}(\overline{\Omega}) = \{u \in C^{2+\theta}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\},$$

and associate with problem (1.6) a nonlinear mapping $F(\lambda, u)$ of $\mathbf{R} \times C_0^{2+\theta}(\overline{\Omega})$ into $C^\theta(\overline{\Omega})$ as follows:

$$F : \mathbf{R} \times C_0^{2+\theta}(\overline{\Omega}) \longrightarrow C^\theta(\overline{\Omega}), \quad (\lambda, u) \longmapsto -\Delta u - \lambda m(x)\, u + \lambda h(x)\, u^2.$$

It is clear that a function $u \in C^{2+\theta}(\overline{\Omega})$ is a solution of problem (1.6) if and only if $F(\lambda, u) = 0$.

Then we can prove the following existence theorem for the equation $F(\lambda, u) = 0$, just as in the proof of Hess [10, Theorem 27.1] and Hess and Kato [11, Theorem 2]):

Theorem 2.1 *There exists a critical value $\lambda^* \in (\lambda_1(m), +\infty]$ such that one can parametrize the solution curve $(\lambda, u(\lambda))$ of $F(\lambda, u) = 0$ by λ, $\lambda_1(m) < \lambda < \lambda^*$, as a C^1 curve.*

Furthermore, applying Fraile et al. [7, Theorems 3.5 and 4.6] to our situation we can obtain the following result:

Theorem 2.2 *The critical value λ^* is characterized by the formula*

$$\lambda^* = \mu_1(\Omega_0(h)) = \min\left\{\mu_1(\Omega_0^1(h)), \mu_1(\Omega_0^2(h)), \ldots, \mu_1(\Omega_0^N(h))\right\}.$$

Moreover, we have

$$\lim_{\lambda \to \lambda^*} \|u(\lambda)\|_{C(\overline{\Omega})} = +\infty, \tag{2.1}$$

and also

$$\lim_{\lambda \to \lambda_1(m)} \|u(\lambda)\|_{C^{2+\theta}(\overline{\Omega})} = 0.$$

End of Proof of Theorem 1.2. Hence it remains to prove the following stronger assertion (2.2) than assertion (2.1):

Lemma 2.3. *If* $u(\lambda) \in C^2(\overline{\Omega})$*,* $\lambda_1(m) < \lambda < \lambda^*$*, is a solution of problem* (1.6)*, then we have*

$$\lim_{\lambda \to \lambda^*} \|u(\lambda)\|_{L^2(\Omega)} = +\infty. \tag{2.2}$$

Proof. Assume to the contrary that there exists a constant $C > 0$ such that

$$\int_\Omega u(\lambda)^2 \, dx \leq C \quad \text{for all } \lambda \in (\lambda_1(m), \lambda^*). \tag{2.3}$$

Then, by using Green's formula we obtain that

$$\begin{aligned} 0 &= \int_\Omega \left(-\Delta u(\lambda) - \lambda m(x) \, u(\lambda) + \lambda h(x) \, u(\lambda)^2\right) u(\lambda) \, dx \\ &= \int_\Omega |\nabla u(\lambda)|^2 \, dx - \lambda \int_\Omega m(x) \, u(\lambda)^2 \, dx + \lambda \int_\Omega h(x) \, u(\lambda)^3 \, dx. \end{aligned}$$

Thus it follows that

$$\int_\Omega |\nabla u(\lambda)|^2 \, dx + \lambda \int_\Omega h(x) u(\lambda)^3 \, dx = \lambda \int_\Omega m(x) \, u(\lambda)^2 \, dx. \tag{2.4}$$

In particular, we have

$$\int_\Omega |\nabla u(\lambda)|^2 \, dx \leq \lambda \|m^+\|_{L^\infty(\Omega)} \int_\Omega u(\lambda)^2 \, dx,$$

and hence, by condition (2.3),

$$\int_\Omega |\nabla u(\lambda)|^2 \, dx \leq \lambda^* C \|m^+\|_{L^\infty(\Omega)} \quad \text{for all } \lambda \in (\lambda_1(m), \lambda^*). \tag{2.5}$$

Here

$$m^+(x) = \max\{m(x), 0\}, \quad x \in \Omega.$$

On the other hand it follows from an application of Sobolev's inequality (see Adams [1, Theorem 5.4]) that

$$\left(\int_\Omega u(\lambda)^{\frac{2n}{n-2}} \, dx\right)^{\frac{n-2}{n}} \leq C(n)^2 \int_\Omega |\nabla u(\lambda)|^2 \, dx, \quad u(\lambda) \in W^{1,2}(\Omega). \tag{2.6}$$

Here $C(n) > 0$ is a constant depending on the dimension $n \geq 3$. Thus, combining inequalities (2.6) and (2.5) we obtain that

$$\left(\int_\Omega u(\lambda)^{\frac{2n}{n-2}} \, dx\right)^{\frac{n-2}{n}} \leq \lambda \, C(n)^2 \int_\Omega u(\lambda)^2 \, dx,$$

or equivalently

$$\|u(\lambda)\|_{L^{2n/(n-2)}(\Omega)} \leq C(\lambda)^{\frac{1}{2}} \|u(\lambda)\|_{L^2(\Omega)}, \tag{2.7}$$

where

$$C(\lambda) = \lambda\, C(n)^2.$$

Furthermore, if we let

$$\chi = \frac{n}{n-2} > 1 \quad (n \geq 3),$$

then we can write inequality (2.7) in the following form:

$$\|u(\lambda)\|_{L^{2\chi}(\Omega)} \leq C(\lambda)^{\frac{1}{2}} \|u(\lambda)\|_{L^2(\Omega)} . \tag{2.8$_1$}$$

Continuing this procedure as in the proof of [19, Lemma 3.2], we have, after N steps,

$$\begin{aligned} \|u(\lambda)\|_{L^{2\chi^{N+1}}(\Omega)} &\leq C(\lambda)^{\frac{1}{2}(\sum_{i=0}^{N} \chi^{-i})} \|u(\lambda)\|_{L^2(\Omega)} \\ &\leq C(\lambda)^{\frac{n}{4}} \|u(\lambda)\|_{L^2(\Omega)} , \quad \lambda_1(m) < \lambda < \lambda^*. \end{aligned} \tag{2.8$_{N+1}$}$$

Therefore, letting $N \to \infty$ in inequality $(2.8)_{N+1}$ we obtain that

$$\|u(\lambda)\|_{L^\infty(\Omega)} \leq (\lambda C(n)^2)^{\frac{n}{4}} \chi^{\frac{n(n-2)}{4}} \|u(\lambda)\|_{L^2(\Omega)} , \quad \lambda_1(m) < \lambda < \lambda^*. \tag{2.9}$$

By inequalities (2.3) and (2.5), it follows that, for all $\lambda_1(m) < \lambda < \lambda^*$

$$\int_\Omega u(\lambda)^2 \, dx \leq C, \tag{2.10a}$$

$$\int_\Omega |\nabla u(\lambda)|^2 \, dx \leq \lambda^* C \|m^+\|_{L^\infty(\Omega)}. \tag{2.10b}$$

This proves that $u(\lambda)$ are bounded in the Sobolev space $W^{1,2}(\Omega)$, for all $\lambda \in (\lambda_1(m), \lambda^*)$.

However we remark the following two assertions:

(a) Rellich's theorem tells us that the injection of $W^{1,2}(\Omega)$ into $L^2(\Omega)$ is compact (or completely continuous) if the dimension n is greater than 2 $(n \geq 3)$.

(b) It is well known (see Yosida [20, Chapter V, Section 2, Theorem 1]) that the unit ball in the Hilbert space is *sequentially weakly compact*. Therefore, by inequalities (2.10a) and (2.10b) we can find a sequence $\{\lambda_j\}$ and a function $u(\lambda^*) \in W^{1,2}(\Omega)$ such that

$$\lambda_j \longrightarrow \lambda^*, \tag{2.11a}$$

and that

$$u(\lambda_j) \longrightarrow u(\lambda^*) \quad \text{strongly in } L^2(\Omega), \tag{2.11b}$$

$$\nabla u(\lambda_j) \longrightarrow \nabla u(\lambda^*) \quad \text{weakly in } L^2(\Omega). \tag{2.11c}$$

On the other hand, by combining inequalities (2.3) and (2.9) we obtain that

$$\sup_{\Omega} |u(\lambda)| \leq C^{\frac{1}{2}} \left(\lambda^* \, C(n)^2\right)^{\frac{n}{4}} \chi^{\frac{n(n-2)}{4}} \quad \text{for all } \lambda \in (\lambda_1(m), \lambda^*). \tag{2.12}$$

Thus we may assume that the finite limit

$$u(\lambda^*)(x) = \lim_{\lambda_j \to \lambda^*} u(\lambda_j)(x) \tag{2.13}$$

exists for almost all x of Ω.

Now, since $u(\lambda_j)$ is a solution of problem (1.6), we have, for all $\psi \in W_0^{1,2}(\Omega)$,

$$\int_\Omega \nabla u(\lambda_j) \cdot \nabla \psi \, dx + \lambda_j \int_\Omega h(x) \, u(\lambda_j)^2 \, \psi \, dx - \lambda_j \int_\Omega m(x) \, u(\lambda_j) \, \psi \, dx = 0. \tag{2.14}$$

However, we notice that the following assertions hold true for all $\psi \in W_0^{1,2}(\Omega)$:

(i) By assertions (2.11a), (2.11b) and (2.11c), it follows that

$$\int_\Omega u(\lambda_j) \, \psi \, dx \longrightarrow \int_\Omega u(\lambda^*) \, \psi \, dx,$$

and that

$$\int_\Omega \nabla u(\lambda_j) \cdot \nabla \psi \, dx \longrightarrow \int_\Omega \nabla u(\lambda^*) \cdot \nabla \psi \, dx.$$

(ii) By assertions (2.12) and (2.13), it follows from an application of the Lebesgue bounded convergence theorem that

$$\int_\Omega h(x) u(\lambda_j)^2 \, \psi \, dx \longrightarrow \int_\Omega h(x) u(\lambda^*)^2 \, \psi \, dx.$$

Hence, by passing to the limit in formula (2.14) we obtain that the limit function $u(\lambda^*)$ satisfies, for all $\psi \in W_0^{1,2}(\Omega)$, the equation

$$\int_\Omega \nabla u(\lambda^*) \cdot \nabla \psi \, dx + \lambda^* \int_\Omega h(x) \, u(\lambda^*)^2 \, \psi \, dx - \lambda^* \int_\Omega m(x) \, u(\lambda^*) \, \psi \, dx = 0.$$

This proves that the function $u(\lambda^*) \in W^{1,2}(\Omega)$ is a *weak solution* of problem (1.6). Therefore, it follows from an application of the *regularity theorem* in quasilinear elliptic theory (see Ladyzhenskaya and Ural'tseva [12, Chapter 4, Theorem 6.5]) that $u(\lambda^*) \in C^{2+\theta}(\overline{\Omega})$. Furthermore, it should be noticed that the solutions $u(\lambda)$ are strictly positive in Ω and that the continuum of positive solutions of problem (1.6) can not contain a point $(\lambda, 0)$ with $\lambda \neq$

$\lambda_1(m)$ (see Deimling [6, Theorem 29.2]). Thus, by the maximum principle it follows that

$$u(\lambda^*) > 0 \quad \text{in } \Omega.$$

Finally it is easy to see that the Fréchet derivative $F_u(\lambda^*, u(\lambda^*))$ is an algebraic and topological isomorphism. Indeed, it is known (see Gilbarg and Trudinger [9, Theorem 6.15]) that the Fréchet derivative $F_u(\lambda^*, u(\lambda^*))$ is a Fredholm operator with index *zero*. However, arguing as in the proof of Fraile et al. [7, Lemma 3.1] we find that the first eigenvalue $\mu_1(\lambda^*)$ of $F_u(\lambda^*, u(\lambda^*))$ is positive. This proves the injectivity and hence surjectivity of $F_u(\lambda^*, u(\lambda^*))$.

Therefore, by virtue of the implicit function theorem one can extend the bifurcation curve $(\lambda, u(\lambda))$ beyond the point $(\lambda^*, u(\lambda^*))$. This contradicts the definition of the critical value λ^*.

The proof of Lemma 2.3 is complete. □

Remark 2.1. If $u(\lambda) \in C^2(\overline{\Omega})$ is a positive solution of problem (1.6) for $\lambda_1(m) < \lambda < \lambda^*$, we define a function

$$\omega(\lambda) = \frac{u(\lambda)}{\|u(\lambda)\|_{L^2(\Omega)}}, \quad \lambda_1(m) < \lambda < \lambda^*.$$

Then, just as in the proof of [19, Proposition 5.1] we can find a sequence $\{\lambda_j\}$, $\lambda_1(m) < \lambda_j < \lambda^*$, and a function

$$\omega(\lambda^*) \in C^{2+\theta}(\Omega_0(h)) \cap W^{1,2}(\Omega) \cap L^\infty(\Omega)$$

such that $\lambda_j \to \lambda^*$, $\omega(\lambda_j) \to \omega(\lambda^*)$ strongly in $L^2(\Omega)$ and that $\nabla\omega(\lambda_j) \to \nabla\omega(\lambda^*)$ weakly in $L^2(\Omega)$. Furthermore, by using assertion (1.8) we can find a connected component $\Omega_0^k(h)$ of $\Omega_0(h)$ such that the limit function $\omega(\lambda^*)$ satisfies the conditions

$$\begin{cases} -\Delta\omega(\lambda^*) = \lambda^* m(x)\omega(\lambda^*) & \text{in } \Omega_0^k(h), \\ \omega(\lambda^*) = 0 & \text{on } \partial\Omega_0^k(h), \\ \omega(\lambda^*) > 0 & \text{in } \Omega_0^k(h). \end{cases}$$

This implies that the function $\omega(\lambda^*)$ is a positive eigenfunction of problem (1.7) in $\Omega_0^k(h)$.

3. Proof of Theorem 1.3

Theorem 1.3 except for estimate (1.13) can be proved by using comparison theorems based on the maximum principle, just as in Fraile et al. [7, Theorem 3.7], Pao [15, Chapter 5, Theorem 4.4] and also Sattinger [16, Theorem 2.6.2].

To prove the decay estimate (1.13), we introduce the energy function $E(t)$ by the formula

$$E(t) = \frac{1}{2} \int_\Omega w(x,t;u_0)^2\, dx.$$

Then, by Green's formula it follows that

$$\begin{aligned} E'(t) &= \int_\Omega w\, w_t\, dx = \frac{1}{\lambda} \int_\Omega \Delta w \cdot w\, dx + \int_\Omega m(x) w^2\, dx - \int_\Omega h(x) w^3\, dx \\ &= -\frac{1}{\lambda} \int_\Omega |\nabla w|^2\, dx + \int_\Omega m(x) w^2\, dx - \int_\Omega h(x) w^3\, dx \\ &\leq -\frac{1}{\lambda} \int_\Omega |\nabla w|^2\, dx + \frac{1}{\lambda_1(m)} \int_\Omega |\nabla w|^2\, dx. \end{aligned} \tag{3.1}$$

However, we have, by the variational formula (1.4) with $m(x) := 1$,

$$\int_\Omega |\nabla w|^2\, dx \geq \lambda_1(1) \int_\Omega w^2\, dx,$$

and so

$$-\Big(\frac{1}{\lambda} - \frac{1}{\lambda_1(m)}\Big) \int_\Omega |\nabla u|^2\, dx \leq -\Big(\frac{1}{\lambda} - \frac{1}{\lambda_1(m)}\Big) \lambda_1(1) \int_\Omega w^2\, dx. \tag{3.2}$$

Therefore, carrying inequality (3.2) into inequality (3.1) we obtain that

$$E'(t) \leq -\Big(\frac{1}{\lambda} - \frac{1}{\lambda_1(m)}\Big) \lambda_1(1) \int_\Omega w^2\, dx = -2\Big(\frac{1}{\lambda} - \frac{1}{\lambda_1(m)}\Big) \lambda_1(1)\, E(t).$$

By Gronwall's inequality, this implies that

$$\begin{aligned} E(t) &\leq \exp\Big[-2\Big(\frac{1}{\lambda} - \frac{1}{\lambda_1(m)}\Big) \lambda_1(1)\, t\Big] E(0) \\ &= \frac{1}{2} \exp\Big[-2\Big(\frac{1}{\lambda} - \frac{1}{\lambda_1(m)}\Big) \lambda_1(1)\, t\Big] \int_\Omega u_0(x)^2\, dx. \end{aligned} \tag{3.3}$$

On the other hand we have, by Schwarz's inequality,

$$\int_\Omega w\, dx \leq \Big(\int_\Omega w^2\, dx\Big)^{1/2} \Big(\int_\Omega 1^2\, dx\Big)^{1/2} = \sqrt{2}\, |\Omega|^{1/2}\, E(t)^{1/2}. \tag{3.4}$$

The desired decay estimate (1.13) follows by combining inequalities (3.4) and (3.3).

The proof of Theorem 1.3 is complete. □

4. The Neumann case

In this final section we study problem (1.14) with homogeneous Neumann condition. To do this, we consider the linearized Neumann problem with indefinite weight function $m(x)$ and positive parameter $\lambda := 1/d$

$$\begin{cases} -\Delta\phi = \lambda\, m(x)\, \phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial\mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The next theorem (see Brown and Lin [3, Theorem 3.13], Senn and Hess [18, Theorems 2 and 3]) asserts the existence of the first nonnegative eigenvalue of problem (4.1):

Theorem 4.1. *Assume that the function $m(x) \in C^\theta(\overline{\Omega})$, $0 < \theta < 1$, attains both positive and negative values in Ω. Then problem (4.1) admits a unique nonnegative eigenvalue $\mu_1(m)$ having a positive eigenfunction. Moreover, we have*

$$\mu_1(m) > 0 \quad \text{if} \int_\Omega m(x)\, dx < 0,$$

and

$$\mu_1(m) = 0 \quad \text{if} \int_\Omega m(x)\, dx \geq 0.$$

Next we consider the steady state problem (1.14) with $d := 1/\lambda$

$$\begin{cases} -\Delta u = \lambda(m(x) - h(x)u)u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial\mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Then we have the following generalization of Hess [10, Theorem 27.1] to the case where $h(x)$ may vanish in Ω (cf. Fraile et al. [7, Theorem 4.6], Senn [17, Theorem 3.2]):

Theorem 4.2. *Assume that $h(x) \in C^1(\overline{\Omega})$ satisfies condition (Z) and that each set $\{x \in \Omega_0^k(h) : m(x) > 0\}$, $1 \leq k \leq N$, has positive measure. Then problem (4.2) has a unique positive solution $u(\lambda) \in C^{2+\theta}(\overline{\Omega})$ for every $\lambda \in (\mu_1(m), \mu_1(\Omega_0(h)))$. For any $\lambda \geq \mu_1(\Omega_0(h))$, there exists no positive solution of problem (4.2). Moreover we have*

$$\lim_{\lambda \to \mu_1(\Omega_0(h))} \|u(\lambda)\|_{L^2(\Omega)} = +\infty,$$

and also

$$\lim_{\lambda \to \mu_1(m)} \|u(\lambda) - c\|_{C^{2+\theta}(\overline{\Omega})} = 0, \tag{4.3}$$

where

$$c = \max\left\{\frac{\int_\Omega m(x)\,dx}{\int_\Omega h(x)\,dx}, 0\right\}.$$

Remark 4.1. Assume that $h(x) > 0$ on $\overline{\Omega}$, and that $m(x)$ attains both positive and negative values in Ω and $\int_\Omega m(x)\,dx < 0$. Then we can replace assertion (4.3) (with $c = 0$) by an estimate of the decay rate of the total size $\|u(\lambda)\|_{L^1(\Omega)} = \int_\Omega u(\lambda)\,dx$ as $\lambda \downarrow \mu_1(m)$:

$$\int_\Omega u(\lambda)\,dx \leq \Big(1 - \frac{\mu_1(m)}{\lambda}\Big)|\Omega|^{2/3}\,\frac{\big(\int_\Omega (m^+)^3\,dx\big)^{1/3}}{\min_{x\in\overline{\Omega}} h(x)}, \quad \lambda > \mu_1(m). \tag{4.4}$$

Moreover, we have the following generalization of Hess [10, Example 28.6] to the case where $h(x)$ may vanish in Ω (cf. Fraile et al. [7, Theorem 3.7], Senn [17, Proposition 3.7]):

Theorem 4.3. (*i*) *Assume that*

$$\int_\Omega m(x)\,dx < 0.$$

Then the zero solution of problem (4.2) *is globally asymptotically stable if* λ *is so small that*

$$0 < \lambda < \mu_1(m).$$

In this case we can estimate the decay rate of the total size $\|w(\cdot, t, u_0)\|_{L^1(\Omega)} = \int_\Omega w(x, t; u_0)\,dx$ *of the population as* $t \downarrow 0$*:*

$$\begin{aligned}&\int_\Omega w(x, t; u_0)\,dx \\ &\leq \exp\Big[-\Big(\frac{1}{\lambda} - \frac{1}{\mu_1(m)}\Big)\mu_1(1)\,t\Big]|\Omega|^{1/2}\Big(\int_\Omega u_0(x)^2\,dx\Big)^{1/2}, \quad t > 0.\end{aligned}$$

(*ii*) *The positive solution* $u(\lambda)$ *of problem* (4.2) *is globally asymptotically stable for each* λ *satisfying the conditions*

$$\mu_1(m) < \lambda < \mu_1(\Omega_0(h)) \quad \textit{if} \int_\Omega m(x)\,dx < 0,$$

and

$$0 < \lambda < \mu_1(\Omega_0(h)) \quad \textit{if} \int_\Omega m(x)\,dx \geq 0.$$

(*iii*) *If* λ *is so large that* $\lambda > \mu_1(\Omega_0(h))$, *then we have*

$$\max_{x\in\overline{\Omega}} |w(x, t; u_0)| \longrightarrow \infty \quad \textit{as } t \to \infty$$

for each nontrivial $u_0 \in C^{2+\theta}(\overline{\Omega})$ *which satisfies conditions* (1.12).

Theorems 4.2 and 4.3 can be proved essentially in the same way as in the proof of Theorems 1.2 and 1.3, respectively. In fact, the same general approach to problem (1.1) with homogeneous Dirichlet condition can still be used, although the analysis is somewhat different, since the Neumann problem (4.1) may have zero as an eigenvalue (cf. Hess [10, Example 28.6]).

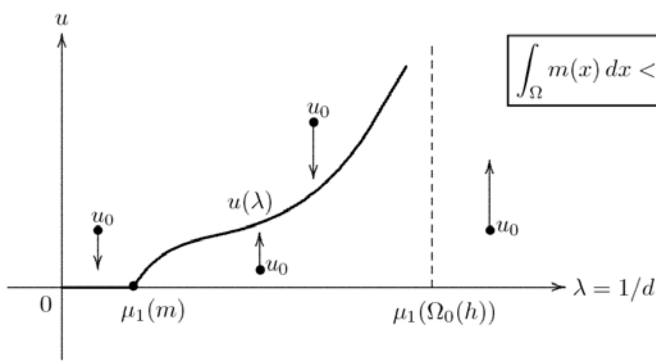

Figure 4.1

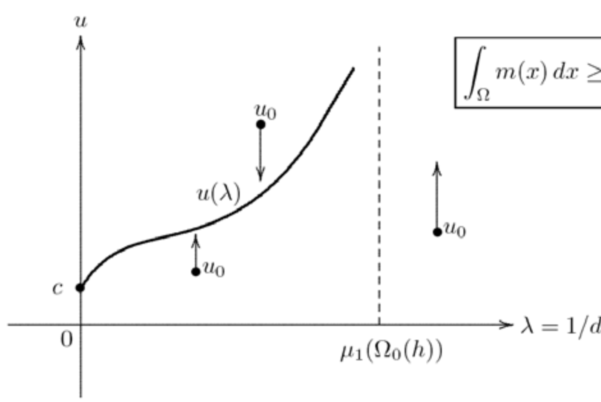

Figure 4.2

A biological interpretation of Theorem 4.3 is that when the environment has an impassable boundary and is on the average unfavorable ($\int_\Omega m(x)\,dx < 0$), then high diffusion rates have the same effect (that is, the ultimate extinction of the population) as they always have the boundary is deadly; but if the boundary is impassable and the environment is on the average neutral or favorable ($\int_\Omega m(x)\,dx \geq 0$), then the population can persist, no matter what its rate of diffusion. The situation may be represented schematically by the above bifurcation diagrams 4.1 and 4.2.

5. Appendix: Proof of Remark 1.3

This appendix is devoted to the proof of Remark 1.3. Namely, we prove that if $h(x) > 0$ on $\overline{\Omega}$ and if $m(x)$ attains positive values in Ω, then the decay estimate (1.10) holds true as $\lambda \downarrow \lambda_1(m)$. Our proof here is inspired by the proof of Cantrell and Cosner [4, Theorem 4.1].

Let $u(\lambda) \in C^2(\overline{\Omega})$ be a positive solution of problem (1.6) for $\lambda_1(m) < \lambda < \mu_1(\Omega_0(h))$. Then it follows from formula (2.4) that

$$\frac{1}{\lambda}\int_\Omega |\nabla u(\lambda)|^2\,dx + \int_\Omega h(x)\,u(\lambda)^3\,dx = \int_\Omega m(x)\,u(\lambda)^2\,dx, \tag{A.1}$$

so that

$$\int_\Omega m(x)\,u(\lambda)^2\,dx > 0.$$

Hence we have, by the variational formula (1.4),

$$\lambda_1(m)\int_\Omega m(x)\,u(\lambda)^2\,dx \le \int_\Omega |\nabla u(\lambda)|^2\,dx. \tag{A.2}$$

By formula (A.1) and inequality (A.2), it follows that

$$\begin{aligned}\int_\Omega h(x)\,u(\lambda)^3\,dx &= \int_\Omega m(x)\,u(\lambda)^2\,dx - \frac{1}{\lambda}\int_\Omega |\nabla u(\lambda)|^2\,dx \\ &\le \int_\Omega m(x)\,u(\lambda)^2\,dx - \frac{\lambda_1(m)}{\lambda}\int_\Omega m(x)\,u(\lambda)^2\,dx \\ &= \Big(1-\frac{\lambda_1(m)}{\lambda}\Big)\int_\Omega m(x)\,u(\lambda)^2\,dx. \qquad \text{(A.3)}\end{aligned}$$

Furthermore, we have, by Hölder's inequality,

$$\begin{aligned}&\int_\Omega m(x)\,u(\lambda)^2\,dx \le \int_\Omega m^+(x)\,u(\lambda)^2\,dx \\ &\le \Big(\int_\Omega m^+(x)^3\,dx\Big)^{1/3}\Big(\int_\Omega u(\lambda)^3\,dx\Big)^{2/3} = \|m^+\|_{L^3(\Omega)}\,\|u(\lambda)\|^2_{L^3(\Omega)}. \qquad \text{(A.4)}\end{aligned}$$

Therefore, by using inequalities (A.3) and (A.4) we obtain that

$$\begin{aligned}\min_{\overline{\Omega}} h\cdot\|u(\lambda)\|^3_{L^3(\Omega)} &\le \int_\Omega h(x)\,u(\lambda)^3\,dx \le \Big(1-\frac{\lambda_1(m)}{\lambda}\Big)\int_\Omega m(x)\,u(\lambda)^2\,dx \\ &\le \Big(1-\frac{\lambda_1(m)}{\lambda}\Big)\|m^+\|_{L^3(\Omega)}\,\|u(\lambda)\|^2_{L^3(\Omega)}.\end{aligned}$$

This proves that

$$\|u(\lambda)\|_{L^3(\Omega)} \leq \Big(1 - \frac{\lambda_1(m)}{\lambda}\Big)\|m^+\|_{L^3(\Omega)}\Big(\frac{1}{\min_{\overline{\Omega}} h}\Big), \qquad (A.5)$$

since $h(x) > 0$ on $\overline{\Omega}$.

On the other hand we have, by Hölder's inequality,

$$\int_\Omega u(\lambda)\,dx \leq \Big(\int_\Omega u(\lambda)^3\,dx\Big)^{1/3}\Big(\int_\Omega dx\Big)^{2/3} = |\Omega|^{2/3}\cdot\|u(\lambda)\|_{L^3(\Omega)}. \qquad (A.6)$$

The desired decay estimate (1.10) follows by combining inequalities (A.6) and (A.5). □

Similarly, in the Neumann case we can prove the decay estimate (4.4) in Remark 4.1, by making use of the variational formula due to Brown and Lin [3, Theorem 3.13].

References

[1] R.A. Adams, "Sobolev Spaces," Academic Press, New York, 1975.

[2] H. Amann, *Periodic solutions of semilinear parabolic equations,* "Nonlinear Analysis," L. Cesari, R. Kannan and H.F. Weinberger (editors) Academic Press, New York San Francisco London, (1978), 1–29.

[3] K.J. Brown and S.S. Lin, *On the existence of positive eigenfunctions for an eigenvalue problem with indefinite weight function,* J. Math. Anal. Appl., 75 (1980), 112–120.

[4] R.S. Cantrell and C. Cosner, *Diffusive logistic equations with indefinite weights: population models in disrupted environments,* Proc. Roy. Soc. Edinburgh, 112A (1989), 293–318.

[5] D.G. de Figueiredo, *Positive solutions of semilinear elliptic problems,* Lecture Notes in Mathematics, No. 957, Springer-Verlag, Berlin Heidelberg, New York, (1982), 34–87.

[6] K. Deimling "Nonlinear Functional Analysis," Springer-Verlag, Berlin Heidelberg, New York, Tokyo, 1985.

[7] J.M. Fraile, P. Koch Medina, J. López-Gómez, and S. Merino, *Elliptic eigenvalue problems and unbounded continua of positive solutions of a semilinear elliptic equation,* J. Differential Equations, 127 (1996), 295–319.

[8] J. García-Melián, R. Gómez-Reñasco, J. López-Gómez, and J.C. Sabina de Lis, *Pointwise growth and uniqueness of positive solutions for a class of sublinear elliptic problems where bifurcation from infinity occurs,* Arch. Rational Mech. Anal., 145 (1998), 261–289.

[9] D. Gilbarg and N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order," second edition, Springer-Verlag, New York Berlin Heidelberg Tokyo (1983).

[10] P. Hess, "Periodic-Parabolic Boundary Value Problems and Positivity," Pitman Research Notes in Mathematical Series, 247, Longman Scientific & Technical, Harlow, Essex (1991).

[11] P. Hess and T. Kato, *On some linear and nonlinear eigenvalue problems with an indefinite weight function,* Comm. Partial Differential Equations, 5 (1980), 999–1030.

[12] O.A. Ladyzhenskaya and N.N. Ural'tseva, "Linear and Quasilinear Elliptic Equations," Academic Press, New York, London, Toronto, Sydney, San Francisco, (1968).

[13] J. López-Gómez and J. C. Sabina de Lis, *First variations of principal eigenvalues with respect to the domain and point-wise growth of positive solutions for problems where bifurcation from infinity occurs,* J. Differential Equations, 148 (1998), 47–64.

[14] A. Manes and A. M. Micheletti, *Un'estensione della teoria variazionale classica degli autovalori per operatori ellitici del secondo ordine,* Boll. Un. Mat. Ital., 7 (1973), 285–301.

[15] C.V. Pao "Nonlinear Parabolic and Elliptic Equations," Plenum Press, New York, London (1992).

[16] D.H. Sattinger, "Topics in Stability and Bifurcation Theory," Lecture Notes in Mathematics, No. 309,Springer-Verlag, New York, Heidelberg, Berlin (1973).

[17] S. Senn, *On a nonlinear elliptic eigenvalue problem with Neumann boundary conditions, with an application to population genetics,* Comm. Partial Differential Equations, 8 (1983), 1199–1228.

[18] S. Senn and P. Hess, *On positive solutions of a linear elliptic eigenvalue problem with Neumann boundary conditions,* Math. Ann., 258 (1982), 459–470.

[19] K. Taira, *The Yamabe problem and nonlinear boundary value problems,* J. Differential Equations, 122 (1995), 316–372.

[20] K. Yosida, "Functional Analysis," sixth edition, Springer-Verlag, Berlin, Heidelberg, New York (1980).