Advances in Differential Equations Volume 7, Number 2, February 2002, Pages 217–236

SPACE–TIME ESTIMATES OF LINEAR FLOW AND APPLICATION TO SOME NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS CORRESPONDING TO FRACTIONAL-ORDER TIME DERIVATIVE

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(Submitted by: H. Ishii)

Abstract. In this paper we study a class of nonlinear integro-differential equations which correspond to a fractional-order time derivative and interpolate nonlinear heat and wave equations. For this purpose we first establish some space–time estimates of the linear flow which is produced by Mittag–Leffler's functions based on Mihlin–Hörmander's multiplier estimates and other harmonic analysis tools. Using these space–time estimates we prove the well-posedness of a local mild solution of the Cauchy problem for the nonlinear integro-differential equation in $C([0,T);L^p(\mathbf{R}^n))$ or $L^q(0,T;L^p(\mathbf{R}^n))$.

1. Introduction

This paper is devoted to the study of the Cauchy problem for nonlinear integro-differential equations

$$\begin{cases} \dfrac{\partial u(t)}{\partial t} = \displaystyle\int_0^t R_\alpha(t-s)\Delta u(s)ds + \int_0^t R_\alpha(t-s)f(u(s))ds, \ 0<\alpha<1, \\ u(0)=\varphi, \end{cases} \tag{1.1}$$

Accepted for publication: April 2001.
AMS Subject Classifications: 35K05, 35L05, 45K05.

where $u(t) = u(x,t)$, $\varphi = \varphi(x)$, $(x,t) \in \mathbf{R}^n \times I$, I is an interval in $(0,\infty)$, $R_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $f(u) \in C^1(\mathbf{R},\mathbf{R})$ is a nonlinear function such that

$$f(0) = 0 \quad \text{and} \quad |f'(u)| \le C(1+|u|^\sigma), \quad \sigma > 0. \tag{1.2}$$

Formally, (1.1) is written as a fractional-order evolution equation

$$\begin{cases} \dfrac{\partial^{1+\alpha} u}{\partial t^{1+\alpha}} = \Delta u + f(u), \quad u = u(x,t), \\ u(0) = \varphi, \quad \varphi = \varphi(x), \end{cases}$$

and this equation interpolates nonlinear heat and wave equations.

In the linear case ($f(u) = 0$) for $n = 1$, (1.1) describes the heat conduction with memory [6, 13], and many authors studied the problem ([3, 4, 6, 13, 15, 18, 19]). Schneider and Wyss [15] and Y. Fujita [3] obtained the representation formula of the fundamental solution for this linear equation for $n = 1$. Furthermore, Y. Fujita [3] proved that the fundamental solution of this linear equation has some properties which are similar to that of heat equations and other properties which are similar to that of wave equations. One example is the points where the fundamental solution takes its maximum propagate with finite speed. Y. Fujita [4] also obtained the representation of the solution of a linear integro-differential equation including the wave equation by probability methods. On the other hand, there is no study for the nonlinear equations of this type. In the heat equation case $\alpha = 0$, the nonlinear term $f(u)$ represents the heat source, and the wave equation case $\alpha = 1$, the interaction term. In our case $0 < \alpha < 1$, we can consider $f(u)$ as the source with the memory. The typical form of nonlinear term is the power type: $f(u) = au^\sigma$. The purpose of this paper is the study of the Cauchy problem for nonlinear integro-differential equation (1.1).

Firstly, we state the representation formula for a linear integro-differential equation with inhomogeneous term $g(x,t)$,

$$\begin{cases} \dfrac{\partial u(t)}{\partial t} = \displaystyle\int_0^t R_\alpha(t-s)\Delta u(s)ds + \int_0^t R_\alpha(t-s)g(\cdot,s)ds, \quad 0 < \alpha < 1, \\ u(0) = \varphi, \end{cases}$$

by using Laplace and Fourier transforms. Secondly, we introduce the concept of an admissible triplet and establish a series of space–time estimates for the solutions of linear integro-differential equations based on Mittag–Leffler's functions, Mihlin–Hörmander's multiplier estimates and other harmonic analysis tools [7, 8, 10, 14, 16, 20]. One can find these space–time estimates are different from those of wave or dispersive wave equations [1, 17], and similar to the case of parabolic equation [5, 11, 12]. Thirdly, using the space–time estimates, we prove the well-posedness of a local mild solution of the Cauchy problem for nonlinear integro-differential equations in $C([0,T);L^r(\mathbf{R}^n))$ or $L^q(0,T;L^p(\mathbf{R}^n))$. Moreover, we also obtain the existence of global small solutions for nonlinear integro-differential equations.

Now, we introduce the concept of admissible triplet.

Definition 1.1. We call (p,q,r) as admissible triplet with respect to α if

$$\frac{1}{q}=\frac{n(1+\alpha)}{2}\left(\frac{1}{r}-\frac{1}{p}\right),\quad 1<r\le p<\begin{cases}\frac{n(1+\alpha)r}{n(1+\alpha)-2}, & \text{if } n\ge 2,\\ \infty, & \text{if } n=1.\end{cases} \tag{1.3}$$

Remark 1.2. It is easy to see that $q=q(p,r)$ is completely determined by p and r, and $q>r$ for any admissible triplet (p,q,r).

Our main results can be stated as follows.

Theorem 1.3. *Assume that $f(u)$ satisfies condition* (1.2). *Let $r\ge r_0=\frac{n(1+\alpha)(1+\sigma)}{2}$. Then, for any $\varphi\in L^r(\mathbf{R}^n)$, where $r\ge r_0$ in the case $r_0>1$ and $r>1$ in the case $r_0\le 1$, there is a maximal existence time $T^*>0$ and a function u satisfying* (1.1), *satisfying following properties.*

(a) $u\in C((0,T^*);L^r(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n))$. *In particular, for any $p\ge r$ and $t\in(0,T^*)$, $u(t)\in L^p(\mathbf{R}^n)$.*

(b) *This solution u is unique in the class*

$$\left\{v\in C((0,T^*);L^p(\mathbf{R}^n)):\sup_{t\in(0,T^*)}t^{1/q}\|v(t)\|_p<\infty\right\}$$

for any $p\ge r$. Here, q is determined as (p,q,r) form an admissible triplet.

(c) *For any $p>r$, u satisfies*

$$t^{\frac{1}{q}}\|u(t)\|_p\to 0,\quad \text{as}\quad t\to 0. \tag{1.4}$$

Again, q is determined as (p,q,r) form an admissible triplet here.

(d) $u\in L^q((0,T^*-\varepsilon);L^p(\mathbf{R}^n))$ *for any $\varepsilon>0$.*

(e) *If $T^*<\infty$, for any $p\ge r$,*

$$\lim_{t\to T^*}\|u(t)\|_p=\infty,\quad r\le p\le\infty.$$

In addition, we have

$$\|u(t)\|_p\ge C(T^*-t)^{(1+\alpha)\left(\frac{n}{2p}-\frac{1}{\sigma}\right)}. \tag{1.5}$$

Remark 1.4. Theorem 1.3 is also valid if the conditions on the nonlinear function $f(u) \in C(\mathbf{R}, \mathbf{R})$ are replaced by the following:

$$|f(u) - f(v)| \leq C(1 + |u|^\sigma + |v|^\sigma)|u - v|, \quad f(0) = 0, \quad \sigma > 0. \tag{1.2$'$}$$

Before concluding this section, we would like to give some notation which we shall use in this paper. We use the notation $\mathcal{F}v$ and $\mathcal{F}^{-1}v$ as the Fourier and Fourier inverse transforms of v in $\mathbf{R}^n$ respectively; that is,

$$\mathcal{F}v(\xi) = \big(\frac{1}{2\pi}\big)^{n/2} \int_{\mathbf{R}^n} e^{-ix\xi} v(x)dx \quad \text{and} \quad \mathcal{F}^{-1}v(x) = \big(\frac{1}{2\pi}\big)^{n/2} \int_{\mathbf{R}^n} e^{ix\xi} v(\xi)d\xi.$$

We also use the more abbreviated notation $\hat{v}$ and $\check{v}$; that is, $\mathcal{F}v = \hat{v}$ and $\mathcal{F}^{-1}v = \check{v}$. The notation $\mathcal{L}$ and $\mathcal{L}^{-1}$ denotes the Laplace and Laplace inverse transforms in $\mathbf{R}^+$. $\Gamma(\cdot)$ and $B(\cdot, \cdot)$ denote usual gamma and beta functions respectively. The notation E_α and $E_{\alpha,\beta}$ denotes Mittag–Leffler's function and generalized Mittag–Leffler's function [11, 12]; that is,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \tag{1.6}$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0. \tag{1.7}$$

For detailed properties of Mittag–Leffler's function, one can see [2, 12]. The notation $*$ denotes usual convolution with respect to x; that is,

$$(\varphi * \psi)(x) = \int_{\mathbf{R}^n} \varphi(x - y)\psi(y)dy.$$

The notation $\mathcal{S}$ denotes the usual Schwartz space, and $\mathcal{S}'$ denotes the tempered distributions space. For $1 \leq p \leq \infty$, L^p denotes the standard Lebesgue space equipped with norm $\|\cdot\|_p$. $C_b(I; X)$ denotes the space of bounded continuous functions which are defined on an interval I and take values in a Banach space X, equipped with norm $\sup_{t \in I} \|\cdot\|_X$. And for $\alpha \notin \mathbf{N}$, we define function spaces $C^\alpha(I; X)$, which are the set of "α-times continuously differentiable functions" on I as follows.

Let $\alpha \notin \mathbf{N}$. We say that a continuous X-valued function u on an interval I is α-times continuously differentiable at the point $t_0 \in I$ if there exists a continuous X-valued function ω on I such that ω is $[\alpha]$-times continuously differentiable at the point $t \in I$ and

$$u(t) = u(0) + \int_0^t R_{\alpha - [\alpha]}(t - s)\omega(s)ds,$$

where $[\alpha]$ is the integer part of $[\alpha]$. Then, the α-th derivative of u at the point t is defined by

$$\frac{d^\alpha}{dt^\alpha}u(t) = \omega^{([\alpha])}(t).$$

If u is α-times continuously differentiable for all $t \in I$, u is called α-times continuously differentiable on I.

For $1 \le q \le p \le \infty$, $\mathcal{M}_q^p$ denotes usual Hörmander space [7] as follows.

Definition 1.5. The space of distributions T in $\mathcal{S}'$ such that $\|T * u\|_p \le C\|u\|_q$ for all $u \in \mathcal{S}$, where C is a constant, is denoted by L_q^p. The set of Fourier transforms $\widehat{T}$ of $T \in L_q^p$ is denoted by $\mathcal{M}_q^p$.

In particular, we write $\mathcal{M}_p = \mathcal{M}_p^p$. For the Hörmander space $\mathcal{M}_q^p$ we have many properties, for example,

(i) $\mathcal{M}_q^p = \mathcal{M}_{p'}^{q'}$, where p' is the dual exponent of p; that is, $1/p+1/p' = 1$.
(ii) $\mathcal{M}_2 = L^\infty$.
(iii) $\mathcal{F}L^1 \hookrightarrow \mathcal{M}_1 \hookrightarrow \mathcal{M}_p$, for all $1 \le p \le \infty$. Here the notation $\hookrightarrow$ means continuous embedding.
(iv) $\mathcal{M}_p$ is a Banach algebra under the pointwise multiplication and addition with $\|\cdot\|_{\mathcal{M}_p}$, where $\|\cdot\|_{\mathcal{M}_q^p} = \sup_{\|\varphi\|_q=1,\ \varphi\in\mathcal{S}} \|\mathcal{F}^{-1}(\cdot\mathcal{F}\varphi)\|_p$.

Our plan in the present paper is as follows. In Section 2 we shall repeat the representation formula of the solution for a linear integro-differential equation with inhomogeneous term in $\mathbf{R}^n$. In Section 3 we shall establish the L^p–L^r estimates and some space–time estimates of the fundamental solution and the inhomogeneous part of the solution for linear integro-differential equation by some harmonic analysis techniques. Section 4 is devoted to the proof of our main results.

2. The representation formula of solution for linear integro-differential equation

When $n = 1$ and $f = 0$, Y. Fujita [3, 4] obtained the representation of the solution of (1.1) using two different ways. In this section, we generalize his results in the case $n \ge 2$.

We consider the following Cauchy problem for the integro-differential equation

$$\begin{cases} \dfrac{\partial u(t)}{\partial t} = \displaystyle\int_0^t R_\alpha(t-s)\Delta u(s)ds + \int_0^t R_\alpha(t-s)f(\cdot,s)ds, \\ \qquad\qquad u = u(x,t), \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}^+, \\ u(0) = \varphi, \quad \varphi = \varphi(x). \end{cases} \tag{2.1}$$

We have the following Duhamel-type formula for (2.1) in $\mathbf{R}^n$.

Proposition 2.1. *Let* $\varphi \in \mathcal{S}(\mathbf{R}^n)$*,* $f \in L^1([0,\infty);\mathcal{S}(\mathbf{R}^n))$ *and* $0 < \alpha < 1$*. Then,* (2.1) *admits a unique solution* $u \in C^{1+\alpha}([0,\infty);\mathcal{S}(\mathbf{R}^n))$ *such that*

$$\begin{aligned} u(x,t) &= \big(\widetilde{E}_{1+\alpha} * \varphi\big)(x,t) \\ &\quad + \Big(\int_0^t \widetilde{E}_{1+\alpha}(\cdot,t-s) * \int_0^s R_\alpha(s-\tau) f(\cdot,\tau) d\tau\, ds\Big)(x) \\ &= \big(\widetilde{E}_{1+\alpha} * \varphi\big)(x,t) + \int_0^t (t-\tau)^\alpha \big(\widetilde{E}_{1+\alpha,1+\alpha}(\cdot,t-\tau) * f(\cdot,\tau)\big)(x) d\tau, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \widetilde{E}_{1+\alpha}(x,t) &= \big(\frac{1}{2\pi}\big)^{n/2} \mathcal{F}^{-1}\big(E_{1+\alpha}(-|\cdot|^2 t^{1+\alpha})\big)(x) \\ &= \big(\frac{1}{2\pi}\big)^n \int_{\mathbf{R}^n} e^{ix\xi} E_{1+\alpha}(-|\xi|^2 t^{1+\alpha}) d\xi, \end{aligned}$$

and

$$\begin{aligned} \widetilde{E}_{1+\alpha,1+\alpha}(x,t) &= \big(\frac{1}{2\pi}\big)^{n/2} \mathcal{F}^{-1}\big(E_{1+\alpha,1+\alpha}(-|\cdot|^2 t^{1+\alpha})\big)(x) \\ &= \big(\frac{1}{2\pi}\big)^n \int_{\mathbf{R}^n} e^{ix\xi} E_{1+\alpha,1+\alpha}(-|\xi|^2 t^{1+\alpha}) d\xi. \end{aligned}$$

Remark 2.2. (i) Integro-differential equation (2.1) corresponds to a fractional-order partial differential equation

$$\begin{cases} \dfrac{\partial^{1+\alpha} u}{\partial t^{1+\alpha}} = \Delta u + f(\cdot,t), \quad u = u(x,t), \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}^+, \\ u(0) = \varphi, \quad \varphi = \varphi(x), \quad x \in \mathbf{R}^n, \end{cases} \tag{2.3}$$

formally. If we take $\alpha = 0$, (2.3) is just as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(\cdot,t), \quad u = u(x,t), \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}^+, \\ u(0) = \varphi, \quad \varphi = \varphi(x), \quad x \in \mathbf{R}^n. \end{cases} \tag{2.4}$$

In view of (2.2) and $E_1(-|\xi|^2 t) = e^{-|\xi|^2 t}$, we can easily see that the solution of (2.4) is

$$\begin{aligned} u(x,t) &= \big(\mathcal{F}^{-1}(e^{-t|\cdot|^2}) * \varphi\big)(x) + \int_0^t \big(\mathcal{F}^{-1}(e^{-(t-s)|\cdot|^2}) * f(\cdot,s)\big)(x) ds \\ &= (G(\cdot,t) * \varphi)(x) + \int_0^t \big(G(\cdot,t-s) * f(\cdot,s)\big)(x) ds, \end{aligned}$$

where $G(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$.

(ii) If we take $\alpha = 1$ in (2.3) and add the initial condition about u_t as ψ, the equation is just as follows:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \Delta u + f(\cdot,t), \quad u = u(x,t), \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}^+, \\ u(0) = \varphi, \quad \varphi = \varphi(x), \quad x \in \mathbf{R}^n, \\ u_t(0) = \psi, \quad \psi = \psi(x), \quad x \in \mathbf{R}^n. \end{cases} \tag{2.5}$$

In this case, the representation formula of the solution u changes as follows:

$$\begin{aligned} u(x,t) &= \big(\widetilde{E}_2 * \varphi\big)(x,t) + \int_0^t \big(\widetilde{E}_2 * \psi\big)(x,s)ds \\ &\quad + \int_0^t \Big(\widetilde{E}_2(\cdot,t-s) * \int_0^s R_1(s-\tau) f(\cdot,\tau)\Big)(x) d\tau ds \\ &= \big(\widetilde{E}_2 * \varphi\big)(x,t) + \int_0^t \big(\widetilde{E}_2 * \psi\big)(x,s)ds \\ &\quad + \int_0^t (t-\tau)^\alpha \big(\widetilde{E}_{2,2}(\cdot,t-\tau) * f(\cdot,\tau)\big)(x) d\tau. \end{aligned}$$

Using the formulas

$$E_2(-|\xi|^2 t^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (|\xi| t)^{2k}}{\Gamma(2k+1)} = \cos(|\xi| t)$$

and

$$E_{2,2}(-|\xi|^2 t^2) = \frac{1}{|\xi| t} \sum_{k=0}^{\infty} \frac{(-1)^k (|\xi| t)^{2k+1}}{\Gamma(2k+2)} = \frac{1}{|\xi| t} \sin(|\xi| t),$$

one easily finds that the solution of (2.5) is

$$\begin{aligned} u(x,t) &= \big(\widetilde{E}_2 * \varphi\big)(x,t) + \int_0^t \big(\widetilde{E}_2 * \psi\big)(x,s)ds \\ &\quad + \int_0^t (t-s) \big(\widetilde{E}_{2,2}(\cdot,t-s) * f(\cdot,s)\big)(x) ds \\ &= \big(\mathcal{F}^{-1}(\cos(|\cdot| t)) * \varphi\big)(x) + \Big(\mathcal{F}^{-1}\Big(\frac{\sin(|\cdot| t)}{|\cdot|}\Big) * \psi\Big)(x) \\ &\quad + \int_0^t \Big(\mathcal{F}^{-1}\Big(\frac{\sin(|\cdot|(t-\tau))}{|\cdot|}\Big) * f(\cdot,\tau)\Big)(x) d\tau. \end{aligned}$$

Proof of Proposition 2.1. First, we take the Laplace transform of both sides of (2.1) with respect to $t \in \mathbf{R}^+$. Remarking that

$$\int_0^\infty e^{-ts}\frac{du}{dt}dt = -u\big|_{t=0} + s\int_0^\infty e^{-ts}u\,dt,$$

and

$$\begin{aligned}
&\int_0^\infty \Big(\int_0^t e^{-ts}\frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)}v(r)dr\Big)dt = \int_0^\infty \Big(\int_r^\infty e^{-ts}\frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)}dt\Big)v(r)dr\\
&= \int_0^\infty \frac{1}{\Gamma(\alpha)}\Big(\int_0^\infty e^{-(\tau-r)s}\tau^{\alpha-1}d\tau\Big)v(r)dr\\
&= \int_0^\infty \frac{e^{-rs}}{\Gamma(\alpha)}\Big(\int_0^\infty e^{-\tau s}\tau^{\alpha-1}d\tau\Big)v(r)dr\\
&= \int_0^\infty \frac{e^{-rs}}{\Gamma(\alpha)}s^\alpha\Big(\int_0^\infty e^{-\sigma}\sigma^{\alpha-1}d\sigma\Big)v(r)dr = \int_0^\infty e^{-rs}s^{-\alpha}v(r)dr,
\end{aligned}$$

we obtain

$$s\mathcal{L}u - \varphi = s^{-\alpha}\Delta\mathcal{L}u + \mathcal{L}\int_0^t R_\alpha(t-r)f(x,r)dr, \tag{2.6}$$

where s is the dual variable of t. Now taking the Fourier transform of both sides of (2.6) with respect to $x \in \mathbf{R}^n$, it is easy to see that

$$s\mathcal{L}\hat{u} - \hat{\varphi} = -s^{-\alpha}|\xi|^2\mathcal{L}\hat{u} + \mathcal{L}\int_0^t R_\alpha(t-r)\hat{f}(\xi,r)dr;$$

that is,

$$\mathcal{L}\hat{u} = \frac{s^\alpha}{s^{1+\alpha}+|\xi|^2}\Big(\hat{\varphi} + \mathcal{L}\int_0^t R_\alpha(t-r)\hat{f}(\xi,r)dr\Big). \tag{2.7}$$

Noting that Mittag–Leffler's function $E_\alpha(z)$ satisfies

$$\int_0^\infty e^{-t}E_\alpha(t^\alpha z)dt = \frac{1}{1-z}, \qquad \forall\alpha \geq 0$$

(see [13, 14]), we have

$$\int_0^\infty e^{-t}E_{1+\alpha}(t^{1+\alpha}z)dt = \frac{1}{1-z}, \quad \forall\alpha \geq 0. \tag{2.8}$$

Now we consider

$$\mathcal{L}E_{1+\alpha}(t^{1+\alpha}z) = \int_0^\infty e^{-ts}E_{1+\alpha}(t^{1+\alpha}z)dt = \frac{1}{s}\int_0^\infty e^{-\tau}E_{1+\alpha}(\tau^{1+\alpha}\frac{z}{s^{1+\alpha}})d\tau. \tag{2.9}$$

Hence (2.8) and (2.9) imply

$$\mathcal{L}E_{1+\alpha}(t^{1+\alpha}z) = \frac{s^\alpha}{s^{1+\alpha} - z}.$$

If we take $z = -|\xi|^2$, we have

$$\int_0^\infty e^{-ts} E_{1+\alpha}(-|\xi|^2 t^{1+\alpha}) dt = \frac{s^\alpha}{s^{1+\alpha} + |\xi|^2},$$

hence we have

$$\mathcal{L}^{-1}\Big(\frac{s^\alpha}{s^{1+\alpha} + |\xi|^2}\Big) = E_{1+\alpha}(-|\xi|^2 t^{1+\alpha}). \tag{2.10}$$

We take the inverse Laplace transform of both sides of (2.7) to obtain

$$\hat{u} = E_{1+\alpha}(-|\xi|^2 t^{1+\alpha})\hat{\varphi} + \int_0^t E_{1+\alpha}(-|\xi|^2 (t-r)^{1+\alpha}) \int_0^r R_\alpha(r-\tau)\hat{f}(\xi,\tau) d\tau dr$$

by (2.10). Hence we have

$$u = \big(\widetilde{E}_{1+\alpha} * \varphi\big)(x,t) + \int_0^t \Big(\widetilde{E}_{1+\alpha}(\cdot, t-r) * \int_0^r R_\alpha(r-\tau) f(\cdot,\tau)\Big)(x) d\tau dr.$$

On the other hand, noting that

$$E_{1+\alpha}(-|\xi|^2 t^{1+\alpha}) = \sum_{k=0}^\infty \frac{(-1)^k |\xi|^{2k} t^{(1+\alpha)k}}{\Gamma((1+\alpha)k+1)} \tag{2.11}$$

and

$$\begin{aligned}
&\int_\tau^s R_\alpha(s-\tau)(t-s)^{(1+\alpha)k} ds = \frac{1}{\Gamma(\alpha)} \int_\tau^t (s-\tau)^{\alpha-1}(t-s)^{(1+\alpha)k} ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (t-\tau)^{\alpha+(1+\alpha)k} \nu^{\alpha-1}(1-\nu)^{(1+\alpha)k} d\nu \qquad (2.12)\\
&= \frac{1}{\Gamma(\alpha)} (t-\tau)^{\alpha+(1+\alpha)k} B(\alpha, (1+\alpha)k+1) \\
&= \frac{\Gamma((1+\alpha)k+1)}{\Gamma((1+\alpha)k+1+\alpha)} (t-\tau)^{(1+\alpha)k+\alpha},
\end{aligned}$$

we have

$$\begin{aligned}
&\int_0^t \Big(\widetilde{E}_{1+\alpha}(\cdot, t-r) * \int_0^r R_\alpha(r-\tau) f(\cdot,\tau)\Big)(x) d\tau dr \\
&= \int_0^t \int_\tau^t \big(\widetilde{E}_{1+\alpha}(\cdot, t-r) R_\alpha(r-\tau) * f(\cdot,\tau)\big)(x) dr d\tau
\end{aligned}$$

$$= \int_0^t (t-\tau)^\alpha \big(\widetilde{E}_{1+\alpha,1+\alpha}(\cdot,t-\tau) * f(\cdot,\tau)\big)(x)d\tau,$$

by (2.11) and (2.12).

Now we consider the regularity of the solution with respect to t. Since we can easily see that

$$\int_0^t \Big(\widetilde{E}_{1+\alpha}(\cdot,t-r) * \int_0^r R_\alpha(r-\tau)f(\cdot,\tau)\Big)(x)d\tau dr$$
$$= \int_0^t (t-\tau)^\alpha \big(\widetilde{E}_{1+\alpha,1+\alpha}(\cdot,t-\tau) * f(\cdot,\tau)\big)(x)d\tau \in C^{1+\alpha}([0,\infty);\mathcal{S}),$$

we need only to show that $\widetilde{E}_{1+\alpha} * \varphi \in C^{1+\alpha}([0,\infty);\mathcal{S}(\mathbf{R}^n))$ for $0<\alpha<1$. In fact, we see

$$\int_0^t R_{1+\alpha}(t-s)s^{(1+\alpha)(k-1)}ds = \frac{1}{\Gamma(1+\alpha)}\int_0^t (t-s)^\alpha s^{(1+\alpha)(k-1)}ds$$
$$= \frac{t^{(1+\alpha)k}}{\Gamma(1+\alpha)}\int_0^1 (1-s)^\alpha s^{(1+\alpha)(k-1)}ds$$
$$= \frac{t^{(1+\alpha)k}}{\Gamma(1+\alpha)}B(1+\alpha,(1+\alpha)(k-1)+1) = \frac{\Gamma((1+\alpha)(k-1)+1)}{\Gamma((1+\alpha)k+1)}t^{(1+\alpha)k};$$

this means that

$$\frac{d^{1+\alpha}}{dt^{1+\alpha}}\Big(\frac{t^{(1+\alpha)k}}{\Gamma((1+\alpha)k+1)}\Big) = \frac{t^{(1+\alpha)(k-1)}}{\Gamma((1+\alpha)(k-1)+1)}.$$

Hence, using Lebesgue's dominated convergence theorem, we have

$$\frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}E_{1+\alpha}(-|\xi|^2t^{1+\alpha}) = \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}\sum_{k=0}^\infty \frac{t^{(1+\alpha)k}(-|\xi|^2)^k}{\Gamma((1+\alpha)k+1)}$$
$$= \sum_{k=0}^\infty \frac{t^{(1+\alpha)(k-1)}(-|\xi|^2)^k}{\Gamma((1+\alpha)(k-1)+1)} = -|\xi|^2E_{1+\alpha}(-|\xi|^2t^{1+\alpha});$$

this implies $\widetilde{E}_{1+\alpha} * \varphi \in C^{1+\alpha}(I;\mathcal{S}(\mathbf{R}^n))$ for $\varphi \in \mathcal{S}(\mathbf{R}^n)$. Uniqueness is an immediate result of Gronwall's inequality; see [9] for details. Therefore we have completed the proof of Proposition 2.1. □

3. The space–time estimates of the solution for a linear integro-differential equation

According to the discussion of Section 2, we know that the following Cauchy problem of a linear integro-differential equation,

$$\begin{aligned}
&\frac{\partial u(t)}{\partial t} = \int_0^t R_\alpha(t-s)\Delta u(s)\,ds + \int_0^t R_\alpha(t-s)f(\cdot,s)ds,\\
&\qquad\qquad u = u(x,t),\ (x,t)\in \mathbf{R}^n\times\mathbf{R}^+,\\
&u(0)=\varphi,\quad \varphi=\varphi(x),\quad x\in\mathbf{R}^n,
\end{aligned}$$

has a unique solution

$$\begin{aligned}
u(x,t) &= \big(\widetilde{E}_{1+\alpha}*\varphi\big)(x,t) + \int_0^t \Big(\widetilde{E}_{1+\alpha}(\cdot,t-s)*\int_0^s R_\alpha(s-\tau)f(\cdot,\tau)\Big)(x)d\tau ds\\
&= (\widetilde{E}_{1+\alpha}*\varphi)(x,t) + \int_0^t (t-\tau)^\alpha\big(\widetilde{E}_{1+\alpha,1+\alpha}(\cdot,t-\tau)*f(\cdot,\tau)\big)(x)d\tau\\
&\overset{\triangle}{=} \big(\widetilde{E}_{1+\alpha}*\varphi\big)(x,t) + Gf. \qquad (3.1)
\end{aligned}$$

Lemma 3.1. *Let $1<p<\infty$ and $0<\alpha<1$. Then $E_{1+\alpha}(-|\cdot|^2)\in\mathcal{M}_p$.*

Proof. Let

$$a_{1+\alpha}(\xi)\equiv|\xi|^{\frac{2}{1+\alpha}}e^{\frac{i\pi}{1+\alpha}},\quad b_{1+\alpha}(\xi)\equiv|\xi|^{\frac{2}{1+\alpha}}e^{-\frac{i\pi}{1+\alpha}},$$

and

$$f_{1+\alpha}(\xi)\equiv\begin{cases}-\dfrac{\sin(\alpha\pi)}{\pi}\displaystyle\int_0^\infty \frac{|\xi|^2 s^\alpha e^{-s}}{s^{2+2\alpha}-2|\xi|^2 s^{1+\alpha}\cos(\alpha\pi)+|\xi|^4}ds, & |\xi|\neq 0,\\ 1-\frac{2}{1+\alpha}, & |\xi|=0,\end{cases}$$

then Mittag–Leffler's function $E_{1+\alpha}(-|\xi|^2)$ has the integral representation

$$E_{1+\alpha}(-|\xi|^2)=\frac{1}{1+\alpha}\exp\{a_{1+\alpha}(\xi)\}+\frac{1}{1+\alpha}\exp\{b_{1+\alpha}(\xi)\}+f_{1+\alpha}(\xi), \quad (3.2)$$

by a complex-analysis method, and $E_{1+\alpha}(-|\xi|^2)$ is a continuous function with respect to $|\xi|\in[0,\infty)$; see [3] and [12] for details. On the other hand, it is easy to see

$$\begin{aligned}
\lim_{|\xi|\to\infty}|\xi|^2 f_{1+\alpha}(\xi) &= -\frac{\sin(\alpha\pi)}{\pi}\lim_{|\xi|\to\infty}\int_0^\infty \frac{|\xi|^4 s^\alpha e^{-s}}{s^{2+2\alpha}-2|\xi|^2 s^{1+\alpha}\cos(\alpha\pi)+|\xi|^4}ds\\
&= -\frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-s}s^\alpha ds = -\frac{\Gamma(1+\alpha)}{\pi}\sin(\alpha\pi),
\end{aligned}$$

so

$$|E_{1+\alpha}(-|\xi|^2)| \le C(1+|\xi|^2)^{-1} \text{ for all } \xi \in \mathbf{R}^n. \tag{3.3}$$

We claim that

$$\sum_{|\beta|\le[\frac{n}{2}]+1} \int_{\frac{R}{2}<|\xi|<2R} \left|R^{|\beta|}\partial_\xi^\beta E_{1+\alpha}(-|\xi|^2)\right|^2 d\xi \le CR^n \tag{3.4}$$

for some constant C, independent of R. In fact, when $R \le 1$, the above estimate is obvious for $R < 1$, since the integrated function in (3.4) is bounded from the definition of Mittag–Leffler's function (1.6) and the volume of the integral domain has order R^n. On the other hand, for $R > 1$, noting that

$$\mathrm{Re}\left(e^{\pm\frac{i\pi}{1+\alpha}}\right) = \cos\left(\frac{\pi}{1+\alpha}\right) < 0, \quad \text{for } \ 0 < \alpha < 1,$$

we need only the estimate of the term containing $|f_{1+\alpha}|^2$. And we know

$$|\partial_\xi^\beta f_{1+\alpha}(\xi)| \le C_\beta |\xi|^{-2-|\beta|}$$

from the integral form of the definition of $f_{1+\alpha}$, we can estimate

$$\begin{aligned} &\sum_{|\beta|\le[\frac{n}{2}]+1} \int_{\frac{R}{2}<|\xi|<2R} \left|R^{|\beta|}\partial_\xi^\beta f_{1+\alpha}(\xi)\right|^2 d\xi \\ &\le C \sum_{|\beta|\le[\frac{n}{2}]+1} \int_{\frac{R}{2}}^{2R} R^{2|\beta|} r^{-4-2|\beta|} r^{n-1} dr \le CR^n. \end{aligned}$$

So, we obtain (3.4). By Mihlin–Hörmander's theorem [8, 16, 20] ([8, Theorem 7.9.5]), we have $E_{1+\alpha}(-|\cdot|^2) \in \mathcal{M}_p$. □

Lemma 3.2. (L^p–L^r estimates) *Let $1 < p < \infty$ and $n/2 < r \le p$. Then, for any $\varphi \in L^p(\mathbf{R}^n)$, we have*

$$\|\widetilde{E}_{1+\alpha}(\cdot, t) * \varphi\|_p \le C\|\varphi\|_p, \ \text{for any } t, \tag{3.5}$$

and for any $\varphi \in L^r(\mathbf{R}^n)$, we have

$$\|\widetilde{E}_{1+\alpha}(\cdot, t) * \varphi\|_p \le Ct^{-\frac{n(1+\alpha)}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|\varphi\|_r, \tag{3.6}$$

where C is a constant independent of t and φ.

Proof. Since $\|\rho(\lambda\cdot)\|_{\mathcal{M}_p} = \|\rho(\cdot)\|_{\mathcal{M}_p}$ for any $\lambda > 0$, (3.5) is an immediate result of Lemma 3.1. Now we prove (3.6). One verifies

$$\begin{aligned} &\|\widetilde{E}_{1+\alpha}(\cdot, t) * \varphi\|_\infty = (2\pi)^{-n}\left\|\mathcal{F}^{-1}\left(E_{1+\alpha}(-|\cdot|^2 t^{1+\alpha})\hat{\varphi}\right)\right\|_\infty \qquad (3.7) \\ &\le \int_{\mathbf{R}^n} |E_{1+\alpha}(-|\xi|^2 t^{1+\alpha})\hat{\varphi}(\xi)| d\xi \le \|\hat{\varphi}\|_{r'}\left(\int_{\mathbf{R}^n} |E_{1+\alpha}(-|\xi|^2 t^{1+\alpha})|^r d\xi\right)^{\frac{1}{r}} \end{aligned}$$

$$\leq C|t|^{-\frac{n(1+\alpha)}{2r}}\|\varphi\|_r\Big(\int_{\mathbf{R}^n}|E_{1+\alpha}(-|\xi|^2)|^r d\xi\Big)^{\frac{1}{r}} \leq C|t|^{-\frac{n(1+\alpha)}{2r}}\|\varphi\|_r,$$

where we used the condition $r > n/2$, which assures $E_{1+\alpha}(-|\cdot|^2) \in L^r(\mathbf{R}^n)$ by (3.3). If $n/2 < r \leq \max(2, p)$, using (3.5) and (3.7), it is easy to see that

$$\begin{aligned}\|\widetilde{E}_{1+\alpha}(\cdot,t)*\varphi\|_p &\leq \|\widetilde{E}_{1+\alpha}(\cdot,t)*\varphi\|_\infty^{1-\frac{r}{p}}\|\widetilde{E}_{1+\alpha}(\cdot,t)*\varphi\|_r^{\frac{r}{p}}\\ &\leq Ct^{-\frac{n(1+\alpha)}{2r}(1-\frac{r}{p})}\|\varphi\|_r = Ct^{-\frac{n(1+\alpha)}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|\varphi\|_r;\end{aligned}$$

that is,

$$\|E_{1+\alpha}(-|\cdot|^2t^{1+\alpha})\|_{\mathcal{M}_r^p} \leq Ct^{-\frac{n(1+\alpha)}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}.$$

In the case that $p \geq r \geq 2$, noting that $\frac{1}{r}-\frac{1}{p} = \frac{1}{p'}-\frac{1}{r'}$ and $p' \leq r' \leq 2$, we have

$$\|E_{1+\alpha}(-|\cdot|^2t^{1+\alpha})\|_{\mathcal{M}_r^p} = \|E_{1+\alpha}(-|\cdot|^2t^{1+\alpha})\|_{\mathcal{M}_{p'}^{r'}} \leq C|t|^{-\frac{n(1+\alpha)}{2}\left(\frac{1}{r}-\frac{1}{p}\right)},$$

by $\mathcal{M}_p^r = \mathcal{M}_{r'}^{p'}$. Then, we obtain (3.6) for any $p \geq r$ by the definition of Hörmander space. □

Due to Giga's idea [5], we have the following time–space estimates for the free part of the solution of an integro-differential equation.

Proposition 3.3. *Let (p, q, r) be any admissible triplet such that $r > n/2$ and $\varphi \in L^r(\mathbf{R}^n)$; then*

$$\widetilde{E}_{1+\alpha}*\varphi \in C_b(I;L^r)\cap L^q(I;L^p), \tag{3.8}$$

and

$$\|\widetilde{E}_{1+\alpha}*\varphi\|_{L^q(I;L^p)} \leq C\|\varphi\|_r, \tag{3.9}$$

where $I = [0, T)$ or $I = [0, \infty)$ and C is a constant independent of I.

Proof. $\widetilde{E}_{1+\alpha}*\varphi \in C_b(I;L^r)$ is an immediate result from (3.5) and the continuity of $\widetilde{E}_{1+\alpha}$ with respect to t. So, we consider only the case $p > r$. We put

$$U(t)\varphi \equiv \|\widetilde{E}_{1+\alpha}(\cdot,t)*\varphi\|_p,$$

and regard U as a (nonlinear) operator from the space of measurable functions on $\mathbf{R}^n$ to the one for $t \in [0,\infty)$: $U : \varphi \mapsto U(\cdot)\varphi$. Obviously, U is a subadditive operator; that is,

$$U(t)(\varphi_1+\varphi_2) \leq U(t)\varphi_1 + U(t)\varphi_2.$$

(3.6) implies $U(t)\varphi \leq Ct^{-\frac{1}{q}}\|\varphi\|_r$, so we obtain

$$m\{t : |U(t)\varphi| > \tau\} \leq m\{t|Ct^{-\frac{1}{q}} : \varphi\|_r > \tau\}$$

$$= m\Big\{t : t < \Big(\frac{C\|\varphi\|_r}{\tau}\Big)^q\Big\} \le \Big(\frac{C\|\varphi\|_r}{\tau}\Big)^q,$$

where $m(A)$ is a Lebesgue measure of A in $[0,\infty)$. This means that U is a weak type (r, q) operator. On the other hand, (3.5) implies

$$\sup_t U(t)\varphi \le C\|\varphi\|_p;$$

this means that U is also a weak type (p,∞) operator. Now for any admissible triplet (p,q,r), we take $\epsilon > 0$ sufficiently small and take another admissible triplet $(p,q_\epsilon,r-\epsilon)$ such that

$$\frac{1}{q_\epsilon} = \frac{n(1+\alpha)}{2}\Big(\frac{1}{r-\epsilon} - \frac{1}{p}\Big), \quad 1 < r-\epsilon < p < \begin{cases} \frac{n(1+\alpha)(r-\epsilon)}{n(1+\alpha)-2}, & n \ge 2, \\ \infty, & n = 1, \end{cases}$$

and $n/2 < r$. (This is possible by virtue of the definition of admissible triplet.) Remark that $q_\epsilon < q < \infty$, and $r - \epsilon < r < p$. From above argument, U is a weak type (p,∞) and $(r-\epsilon, q_\epsilon)$ operator. Hence, the generalized Marcinkiewicz's interpolation theorem [15, 19] implies U is a strong (r,q)-type operator, that is,

$$\|\widetilde{E}_{1+\alpha} * \varphi\|_{L^q(0,T;L^p)} \le C\|\varphi\|_r,$$

so we complete the proof of Proposition 3.3. □

Proposition 3.4. (i) *Let (p,q,r) be any admissible triplet and σ satisfy*

$$\max(n(1+\sigma), n(1+\alpha)\sigma) < 2p \quad \text{and} \quad n\sigma < 2r. \tag{3.10}$$

Then, for any $f \in L^{\frac{q}{1+\sigma}}(0,T;L^{\frac{p}{1+\sigma}})$, $Gf \in L^q(0,T;L^p)$ and

$$\|Gf\|_{L^q(0,T;L^p)} \le CT^{(1+\alpha)(1-\frac{n\sigma}{2r})}\|f\|_{L^{\frac{q}{1+\sigma}}\left(0,T;L^{\frac{p}{1+\sigma}}\right)},$$

where C is a constant independent of f and T.

(ii) *Let (p_1,q_1,r) be any admissible triplet and σ satisfy*

$$2r(1+\sigma) > n\sigma(1+\alpha), \quad n\sigma < 2r \quad \text{and} \quad \frac{\sigma}{r(1+\sigma)} + \frac{1}{p_1} < \frac{2}{n}. \tag{3.11}$$

Then, we can take p and q as (p,q,r) form an admissible triplet, such that for any $f \in L^{\frac{q}{1+\sigma}}\left(0,T;L^{\frac{p}{1+\sigma}}\right)$,

$$\|Gf\|_{L^{q_1}(0,T;L^{p_1})} \le CT^{1+\alpha-\frac{n(1+\alpha)\sigma}{2r}}\|f\|^{\frac{\sigma}{1+\sigma}}_{L^{\frac{q}{1+\sigma}}\left(0,T;L^{\frac{p}{1+\sigma}}\right)}\big\||f|^{\frac{1}{1+\sigma}}\big\|_{L^{q_1}(0,T;L^{p_1})}.$$

Proof. By virtue of the condition $n(1+\sigma) < 2p$, we can apply (3.6) as $r = p/(1+\sigma)$, and we have

$$\|Gf\|_{L^q(0,T;L^p)} = \Big\| \int_0^t \widetilde{E}_{1+\alpha}(\cdot, t-s) * \int_0^s R_\alpha(s-\tau) f(\cdot,\tau) d\tau ds \Big\|_{L^q(0,T;L^p)}$$

$$\le \Big\| \int_0^t \Big\| \widetilde{E}_{1+\alpha}(\cdot,t-s) * \int_0^s R_\alpha(s-\tau) f(\cdot,\tau)d\tau \Big\|_p ds \Big\|_{L^q(0,T)}$$

$$\le C \Big\| \int_0^t |t-s|^{-\frac{n(1+\alpha)}{2}\left(\frac{1+\sigma}{p} - \frac{1}{p}\right)} \Big\| \int_0^s R_\alpha(s-\tau) f(\cdot,\tau) d\tau \Big\|_{\frac{p}{1+\sigma}} ds \Big\|_{L^q(0,T)}$$

$$\le C \Big\| \int_0^t |t-s|^{-\frac{n(1+\alpha)\sigma}{2p}} \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \|f(\cdot,\tau)\|_{\frac{p}{1+\sigma}} d\tau ds \Big\|_{L^q(0,T)}$$

$$= C \Big\| \int_0^t \int_\tau^t |t-s|^{-\frac{n(1+\alpha)\sigma}{2p}} (s-\tau)^{\alpha-1} ds \cdot \|f(\cdot,\tau)\|_{\frac{p}{1+\sigma}} d\tau \Big\|_{L^q(0,T)}$$

$$= C \Big\| \int_0^t |t-\tau|^{-\frac{n(1+\alpha)\sigma}{2p}+\alpha} \|f(\cdot,\tau)\|_{\frac{p}{1+\sigma}} d\tau \Big\|_{L^q(0,T)}.$$

Let $\chi(t) \equiv t^{\alpha - n(1+\alpha)\sigma/(2p)}$, if $0 < t < T$, and $\equiv 0$ otherwise. Remarking that $\chi \in L^\rho(\mathbf{R})$ for $1/\rho > 1 - \sigma/q$, which is assured by the condition $2r > n\sigma$, by using Young's inequality, we have

$$C \Big\| \int_0^t |t-\tau|^{-\frac{n(1+\alpha)\sigma}{2p}+\alpha} \|f(x,\tau)\|_{\frac{p}{1+\sigma}} d\tau \Big\|_{L^q(0,T)}$$
$$\le CT^{1+\alpha-\frac{n(1+\alpha)\sigma}{2r}} \|f\|_{L^{\frac{q}{1+\sigma}}(0,T;L^{\frac{p}{1+\sigma}})}.$$

This implies (i).

We now prove (ii). Let $p = r(1+\sigma) - \epsilon$ for some small $\epsilon > 0$ and take q such that (p,q,r) becomes an admissible triplet. Then, by using (3.6) as $1/r_2 = \sigma/p + 1/p_1$, we have

$$\|Gf\|_{L^{q_1}(0,T;L^{p_1})}$$
$$\le C \Big\| \int_0^t \Big\| \tilde{E}_{1+\alpha}(\cdot,t-s) * \int_0^s (s-\tau)^{\alpha-1} f(\cdot,\tau) d\tau \Big\|_{p_1} ds \Big\|_{L^{q_1}(0,T)}$$
$$\le C \Big\| \int_0^t |t-s|^{-\frac{n(1+\alpha)\sigma}{2}\left(\frac{1}{r_2} - \frac{1}{p_1}\right)} \Big\| \int_0^s (s-\tau)^{\alpha-1} f(\cdot,\tau) d\tau \Big\|_{r_2} ds \Big\|_{L^{q_1}(0,T)}$$
$$\le C \Big\| \int_0^t |t-s|^{-\frac{n(1+\alpha)\sigma}{2p}} \int_0^s (s-\tau)^{\alpha-1} \|f(\cdot,\tau)\|_{r_2} ds \Big\|_{L^{q_1}(0,T)}$$
$$\le C \Big\| \int_0^t |t-\tau|^{\alpha - \frac{n(1+\alpha)\sigma}{2p}} \|f(\cdot,\tau)\|_{r_2} d\tau \Big\|_{L^{q_1}(0,T)}$$

$$\leq C\Big\| \int_0^t |t-\tau|^{\alpha - \frac{n(1+\alpha)\sigma}{2p}} \|f(\cdot,\tau)\|_{\frac{p}{1+\sigma}}^{\frac{\sigma}{1+\sigma}} \big\||f(\cdot,\tau)|^{\frac{1}{1+\sigma}}\big\|_{p_1} d\tau \Big\|_{L^{q_1}(0,T)}.$$

Now, remarking that the relation $1 + 1/q_1 = (1 - \sigma/q) + \sigma/q + 1/q_1$ holds, we can apply Young's and Hölder's inequalities to the right-hand side of the above expression, and we obtain the result. Here, we need the conditions

$$\frac{1}{r_2} < \frac{2}{n} \quad \text{and} \quad \frac{n(1+\alpha)\sigma}{2p} - \alpha < 1 - \frac{\sigma}{q},$$

which are guaranteed by (3.11). □

Remark 3.5. In a way similar to the proof of (i), we also obtain

$$\|\widetilde{E}_{1+\alpha,1+\alpha}(\cdot,t) * \varphi\|_p \leq Ct^{-\frac{n(1+\alpha)}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} \|\varphi\|_r,$$

if $\frac{1}{r} - \frac{1}{p} < \frac{2}{n(1+\alpha)}$. We will also use this estimate in next section.

4. The proof of Theorem 1.3

We use the idea of [5, 12] to prove Theorem 1.3. It is well known that (1.1) is equivalent to the following integral equation:

$$\begin{aligned} u(\cdot,t) &= \widetilde{E}_{1+\alpha}(\cdot,t) * \varphi + \int_0^t \widetilde{E}_{1+\alpha}(\cdot,t-s) * \int_0^s R_\alpha(s-\tau) f(u(\tau)) d\tau\, ds \\ &= \widetilde{E}_{1+\alpha}(\cdot,t) * \varphi + \int_0^t (t-\tau)^\alpha \widetilde{E}_{1+\alpha,1+\alpha}(\cdot,t-\tau) * f(u(\tau)) d\tau \\ &\triangleq \widetilde{E}_{1+\alpha}(\cdot,t) * \varphi + \mathcal{J}u. \end{aligned} \tag{4.1}$$

Now we introduce some notation and estimate the nonlinear term. For any admissible triplet (p,q,r), we define

$$X_{p,q}(0,T) = \big\{u \in C((0,T];L^p) : \|u\|_{X_{p,q}(0,T)} = \sup_{0<t\leq T} t^{\frac{1}{q}} \|u(t)\|_p < \infty\big\},$$

and $Y_{p,q}(0,T) = L^q(0,T;L^p)$. We now prove some necessary nonlinear estimates in $X_{p,q}(0,T)$ and $Y_{p,q}(0,T)$.

Lemma 4.1. *Let (p,q,r) be any admissible triplet satisfying* (3.10), *and f satisfy* (1.2). *For any $u,v \in X_{p,q}(0,T)$ or $Y_{p,q}(0,T)$, we have $\mathcal{J}u$, $\mathcal{J}v \in X_{p,q}(0,T)$ or $Y_{p,q}(0,T)$ respectively, and*

$$\|\mathcal{J}u\|_{X_{p,q}(0,T)} \leq C\left[T^{1+\alpha}\|u\|_{X_{p,q}(0,T)} + T^{(1+\alpha)(1-\frac{n\sigma}{2r})}\|u\|^{1+\sigma}_{X_{p,q}(0,T)}\right], \tag{4.2}$$

$$\begin{aligned} \|\mathcal{J}u - \mathcal{J}v\|_{X_{p,q}(0,T)} \leq C&\left[T^{1+\alpha} + T^{(1+\alpha)(1-\frac{n\sigma}{2r})}(\|u\|^\sigma_{X_{p,q}(0,T)} + \|v\|^\sigma_{X_{p,q}(0,T)})\right] \\ &\times \|u-v\|_{X_{p,q}(0,T)}, \end{aligned} \tag{4.3}$$

$$\|\mathcal{J}u\|_{Y_{p,q}(0,T)} \le C\left[T^{1+\alpha}\|u\|_{Y_{p,q}(0,T)} + T^{(1+\alpha)(1-\frac{n\sigma}{2r})}\|u\|^{1+\sigma}_{Y_{p,q}(0,T)}\right], \qquad (4.4)$$

and

$$\|\mathcal{J}u - \mathcal{J}v\|_{Y_{p,q}(0,T)} \le C\left[T^{1+\alpha} + T^{(1+\alpha)(1-\frac{n\sigma}{2r})}(\|u\|^{\sigma}_{Y_{p,q}(0,T)} + \|v\|^{\sigma}_{Y_{p,q}(0,T)})\right] \times \|u-v\|_{Y_{p,q}(0,T)}. \qquad (4.5)$$

Proof. It is easy to see that (4.4) and (4.5) are immediate results of Proposition 3.4, so we prove only (4.2) and (4.3). In a way similar to the proof of Proposition 3.4, we have

$$\begin{aligned}
\|\mathcal{J}u\|_{X_{p,q}(0,T)} &\le C\bigg[\sup_{0\le t<T}|t|^{\frac{1}{q}}\int_0^t |t-s|^{\alpha}\|u(s)\|_p ds \\
&\quad + \sup_{0\le t<T}|t|^{\frac{1}{q}}\int_0^t |t-s|^{\alpha-\frac{n(1+\alpha)\sigma}{2p}}\|u(s)\|_p^{1+\sigma}ds\bigg] \\
&\le C\bigg[|t|^{\frac{1}{q}}\int_0^t |t-s|^{\alpha}s^{-\frac{1}{q}}ds\cdot\|u\|_{X_{p,q}(0,T)} \qquad 4.6) \\
&\quad + |t|^{\frac{1}{q}}\int_0^t |t-s|^{\alpha-\frac{n(1+\alpha)\sigma}{2p}}s^{-\frac{1+\sigma}{q}}ds\cdot\|u\|^{1+\sigma}_{X_{p,q}(0,T)}\bigg] \\
&\le C\bigg[|t|^{1+\alpha}\int_0^1 |1-s|^{\alpha}s^{-\frac{1}{q}}ds\cdot\|u\|_{X_{p,q}(0,T)} \\
&\quad + |t|^{(1+\alpha)(1-\frac{n\sigma}{2r})}\int_0^1 |1-s|^{\alpha-\frac{n(1+\alpha)\sigma}{2p}}s^{-\frac{1+\sigma}{q}}ds\cdot\|u\|^{1+\sigma}_{X_{p,q}(0,T)}\bigg],
\end{aligned}$$

by (3.5), (3.6) and the definition of $X_{p,q}(0,T)$. Note that $\frac{n(1+\alpha)\sigma}{2p} - \alpha < 1$ and $0 < \frac{1+\sigma}{q} < 1$, so (4.6) implies (4.2). In a way similar to that leading to (4.2), we easily obtain the estimate (4.3). □

The proof of Theorem 1.3. Fix $p \ge r$ and take q such that (p,q,r) form an admissible triplet. It is easy to see that the right-hand side of integral equation (4.1)

$$\mathcal{T}u \overset{\triangle}{=} \widetilde{E}_{1+\alpha} * \varphi + \mathcal{J}u$$

defines a mapping $\mathcal{T}$ from $X_{p,q}(I)$ to itself by Lemma 4.1. Now we take $R > 0$ as $C\|\varphi\|_r = R/2$, and $T > 0$ as $C(T^{1+\alpha} + T^{(1+\alpha)(1-n\sigma/(2r))}R^{\sigma}) = 1/4$. Then $\mathcal{T}$ is a contraction mapping on

$$X^R_{p,q}(0,T) = \{u \in X_{p,q}(0,T) : \|u\|_{X_{p,q}(0,T)} \le R\}.$$

In fact, in view of Lemma 4.1, we have

$$\|\mathcal{T}u\|_{X_{p,q}(0,T)} \le C\|\varphi\|_r + CT^{1+\alpha}(2C\|\varphi\|_r)$$

$$+ CT^{(1+\alpha)(1-\frac{n\sigma}{2r})}(2C\|\varphi\|_r)^\sigma 2C\|\varphi\|_r \le R$$

and

$$\begin{aligned}\|\mathcal{T}u - \mathcal{T}v\|_{X_{p,q}(0,T)} &\le (CT^{1+\alpha} + CT^{(1+\alpha)(1-\frac{n\sigma}{2r})})\|u-v\|_{X_{p,q}(0,T)}\\ &\le \frac{1}{2}\|u-v\|_{X_{p,q}(0,T)},\end{aligned}$$

for any $u, v \in X^R_{p,q}(0,T)$. So there exists a unique solution u of (4.1) on $(0,T)$ such that $u \in X^R_{p,q}(0,T)$ by the Banach contraction mapping principle. The above estimates are valid for any $p \ge r$; in particular, we can take $p = r$, $u \in C([0,T]; L^r)$ and $\|u(T)\|_r \le R$. Remarking that T is determined only by $\|\varphi\|_r$, independent of p, we can prolong this solution u on the maximal interval $[0, T^*)$, and this T^* is independent of p. In the case $T^* < \infty$, take $0 < t_0 < T^*$, so $\|u(t_0)\|_p < \infty$. By the above argument, we can prolong this solution u at least on $[t_0, t_1]$ such that $C(t_1 - t_0)^{1+\alpha} + C(t_1 - t_0)^{(1+\alpha)(1-n\sigma/(2p))} = 1/2$ and $t_1 - t_0 \le 1$. These inequalities mean $\|u(t_0)\|_p^\sigma \ge (t_1 - t_0)^{(1+\alpha)(1-n\sigma/(2p))}$, so we obtain the blow-up rate estimate (1.5).

Next, we will show (1.4). Fix $\varphi \in L^r$; for any $\varepsilon > 0$, we can take $\psi \in L^r \cap L^p$ such that $\|\varphi - \psi\|_r \le \varepsilon$. Let u and v be the solutions of $u = \widetilde{E}_{1+\alpha} * \varphi + \mathcal{J}u$ and $v = \widetilde{E}_{1+\alpha} * \psi + \mathcal{J}v$ respectively. Then, we have

$$\begin{aligned}&\sup_{0<t\le T} t^{1/q}\|u(t) - v(t)\|_p\\ &\le \sup_{0<t\le T} t^{1/q}\|\widetilde{E}_{1+\alpha} * (\varphi - \psi)\|_p + \sup_{0<t\le T} t^{1/q}\|\mathcal{J}u - \mathcal{J}v\|_p\\ &\le C\|\varphi - \psi\|_r + C(T^{1+\alpha} + T^{(1+\alpha)(1-\frac{n\sigma}{2r})})\\ &\times (\|u\|^\sigma_{X_{p,q}(0,T)} \vee \|v\|^\sigma_{X_{p,q}(0,T)})\|u - v\|_{X_{p,q}(0,T)},\end{aligned}$$

so taking $T > 0$ such as $C(T^{1+\alpha} + T^{(1+\alpha)(1-\frac{n\sigma}{2r})})R^\sigma \le 1/2$, we have

$$\sup_{0<t\le T} t^{1/q}\|u(t) - v(t)\|_p \le C\varepsilon.$$

On the other hand, since

$$\begin{aligned}&\sup_{0<t\le T} t^{1/q}\|v(t)\|_p \le \sup_{0<t\le T}\|(\widetilde{E}_{1+\alpha} * \psi)(t)\|_p + \|\mathcal{J}v\|_{X_{p,q}(0,T)}\\ &\le CT^{1/q}\|\psi\|_p + C\big(T^{1+\alpha}\|v\|_{X_{p,q}(0,T)} + T^{(1+\alpha)(1-\frac{n\sigma}{2r})}\|v\|^{1+\sigma}_{X_{p,q}(0,T)}\big)\\ &\le CT^{1/q}\|\psi\|_p + C\big(T^{1+\alpha} + T^{(1+\alpha)(1-\frac{n\sigma}{2r})}R^\sigma\big)\|v\|_{X_{p,q}(0,T)},\end{aligned}$$

if we take $T > 0$ sufficiently small such that $C\big(T^{1+\alpha} + T^{(1+\alpha)(1-\frac{n\sigma}{2r})}R^\sigma\big) \leq 1/2$, the above estimate shows

$$\sup_{0<t\leq T} t^{1/q}\|v(t)\|_p \leq T^{1/q}\|\psi\|_p \to 0, \quad \text{as} \quad T \to 0.$$

Combining these two estimates, we obtain (1.4).

Finally, we prove (d). By using (4.4) and (4.5) instead of (4.2) and (4.3), we can construct the solution $\tilde{u}$ of (1.1) such that $\tilde{u} \in L^q(0,T;L^p(\mathbf{R}^n)) \cap C(0,T;L^r(\mathbf{R}^n))$ for some $T > 0$, for any (p,q) such as (p,q,r) form an admissible triplet. The uniqueness results in $C(0,T;L^r)$, $u = \tilde{u}$ and $u \in L^q(0,T;L^p(\mathbf{R}^n))$ for any $0 < T < T^*$. This completes the proof. □

Remark. If the nonlinear function f satisfies $f'(u) \leq C|u|^\sigma$ for $\sigma > \max\{1+\alpha, 2/n\}$, we can obtain following small data globalness result. That is, for any $\phi \in L^r(\mathbf{R}^n)$, $r = n\sigma/2$ such that $\|\phi\|_r$ is sufficiently small, there exists a time global solution $u \in X_{p,q}$ of (1.1), where p satisfies $p > n(1+\sigma)/2$ and (p,q,r) make admissible triplet.

Acknowledgment. The second author is grateful to Prof. H. Wada and Prof. M. Morimoto for their enlightening discussions, and to Prof. K. Uchiyama for his kind helps and valuable suggestions. The first author is supported by the Ministry of Education, Science, Sports, and Culture of Japan under the Grant-in-Aid for Encouragement of Young Scientists, No.10740076, and the second one is supported by the NSF of China and STEC grant of Sophia University.

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