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ON THE NUMBER OF CLOSED SOLUTIONS FOR POLYNOMIAL ODE'S AND A SPECIAL CASE OF HILBERT'S 16TH PROBLEM

Marta Calanchi and Bernhard Ruf
Dipartimento di Matematica, Università degli Studi
Via Saldini 50, 20133 Milano, Italy

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Abstract. In this paper we prove that the equation $\frac{du}{dt}+\sum_{i=0}^n a_i(t)u^i = f(t)$, $t \in [0,1]$, $u(0) = u(1)$, has for every continuous f at most n solutions provided that n is odd, and the continuous coefficients a_i satisfy $|a_n(t)| \geq \alpha > 0$ and $|a_i(t)| \leq \beta$, $i = 1, \dots, n-1$, with $\beta > 0$ sufficiently small. Furthermore, we show that this result implies that for a restricted subclass of polynomial vector fields of order n in $\mathbb{R}^2$ the maximal number of limit cycles is n. This constitutes a special case of Hilbert's 16th problem.

1. Introduction

In this paper we look for upper bounds on the number of *closed solutions* for the following ordinary differential equation with polynomial nonlinearity:

$$\frac{dx}{dt} + a_n(t)x^n + \cdots + a_1(t)x = f(t)\ ,\ 0 \leq t \leq 1 \tag{1.1}$$

where $a_1, \dots, a_n, f : [0,1] \to \mathbb{R}$ are continuous functions. We call a solution $x(t)$ of (1.1) closed, if $x(0) = x(1)$; if $f, a_1, \dots, a_n$ are 1-periodic, then a closed solution is clearly 1-periodic.

A.L. Neto [12] attributes the following problem to C. Pugh (see also N.G. Lloyd [11]):

Does there exist a number $N(n)$ depending only on the degree n of the polynomial such that (1.1) *has at most N closed solutions* (*if not all solutions defined in* $[0,1]$ *are closed*).

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Pugh's question is motivated by the following results, apparently due to S. Smale and proved in [12]: if not all solutions are closed, then the

$$\textit{Riccati equation:}\quad \frac{dx}{dt} + a_2(t)x^2 + a_1(t)x = f(t)\ ,\ 0 \le t \le 1 \tag{1.2}$$

has at most two closed solutions, i.e., $N(2) = 2$, and the

$$\textit{Abel equation:}\quad \frac{dx}{dt} + a_3(t)x^3 + a_2(t)x^2 + a_1(t)x = f(t)\ ,\ 0 \le t \le 1 \tag{1.3}$$

has at most three closed solutions *provided* that $a_3(t) > 0$, $t \in [0,1]$ (see also Cafagna–Donati [2], Gasull–Llibre [5]).

In [12] A.L. Neto gives a negative answer to Pugh's problem; he gives examples of equations of the form

$$\frac{dx}{dt} = a_3(t)x^3 + a_2(t)x^2\ ,\ 0 \le t \le 1 \tag{1.4}$$

having at least k closed solutions (and not all solutions closed), where k is any given positive integer, and a_2, a_3 are polynomials in t or in $\cos(\pi t)$ and $\sin(\pi t)$. From these examples it follows also that there exist equations of the form

$$\frac{dx}{dt} = x^4 + a_3(t)x^3 + a_2(t)x^2 + a_1(t)x + a_0\ ,\ 0 \le t \le 1 \tag{1.5}$$

which have at least k closed solutions.

In this paper we specify a restricted class of polynomial nonlinearities of degree n for which $N(n) = n$. More precisely, we prove the following result:

Theorem 1.1. *Suppose that n is odd. Assume that $f, a_1, \dots, a_n : [0,1] \to \mathbb{R}$ are continuous, and that $\min_{[0,1]} |a_n(t)| \ge \alpha$, for some $\alpha > 0$. Then there exists some $\beta > 0$ such that if $\max_{[0,1]} |a_i(t)| \le \beta$, $i = 1, \dots, n-1$, then (1.1) has at most n closed solutions, for all right-hand sides $f(t) \in C[0,1]$.*

Remark. We thank a referee for bringing to our attention the recent work of Ilyashenko [8], who proves that for $a_n(t) \equiv 1$ and $|a_i(t)| \le C$, $i = 0, \dots, n-1$, the number of closed solutions of (1.1) is bounded by the number $A(n, C) = 8\exp\{(3C+2)\exp[\frac{3}{2}(2C+3)^n]\}$. Comparing with this result, we note that Theorem 1.1 says that if n is odd and if the bound β on $|a_i(t)|$, $i = 1, \dots, n-1$, is sufficiently small (while *no bound* on the term $a_0(t) = f(t)$ is required), then the above number $A(n, C)$ can be replaced by the (optimal) number n.

As pointed out in [12], the problem of finding bounds for the number of closed solutions of equation (1.1) is related to Hilbert's 16^{th} problem; indeed, it can be viewed as a particular case of it. In the second part of Hilbert's 16^{th} problem one reads

"This is the question as to the maximum number and position of Poincaré's boundary cycles (cycles limites) for a differential equation of the first order degree of the form

$$\frac{dy}{dx} = \frac{Q}{P} \tag{1.6}$$

where P, Q are entire rational integral functions of the nth degree in x, y"; cf. D. Hilbert, [6].

We recall that Poincaré limit cycles are *isolated periodic solutions*, i.e., periodic solutions which have an annulus-like neighborhood free of other periodic solutions in the (x, y)-plane.

Ilyashenko and Yakovenko mention in [9] three versions of this problem:

Individual finiteness problem (or Dulac's problem): equation (1.6) has always at most a finite number of solutions.

Existential Hilbert problem: there exists a number $H(n)$ such that equation (1.6) has at most $H(n)$ solutions, for any polynomial functions $P(x, y)$, $Q(x, y)$ of degree $\leq n$.

Constructive Hilbert problem: give a formula or estimate for $H(n)$.

Up to now, only Dulac's problem has been solved positively, in independent proofs by Yu. Ilyashenko [7] and J. Écalle [4].

We now show that Theorem 1.1 yields $H(n) = n$ (in the Constructive Hilbert problem) for a restricted subclass of polynomials P, Q of odd order n:

Suppose that the polynomial vector field $Y(x, y) = (P(x, y), Q(x, y))$ has a unique singular point, which we may assume to be in $(0, 0)$; i.e.,

$$(P(0, 0), Q(0, 0)) = (0, 0).$$

Then Y can be written in polar coordinates $Y = (Y_r, Y_\theta)$ (see [12]) where

$$Y_r(r, \theta) = \cos\theta \; P(r\cos\theta, r\sin\theta) + \sin\theta \; Q(r\cos\theta, r\sin\theta) \tag{1.7}$$

and

$$Y_\theta(r, \theta) = \frac{1}{r}[\cos\theta \; Q(r\cos\theta, r\sin\theta) - \sin\theta \; P(r\cos\theta, r\sin\theta)]. \tag{1.8}$$

We can then state the following theorem:

Theorem 1.2. *Suppose that $Y = (Y_r, Y_\theta)$ is as above, and that Y_r and Y_θ satisfy for some $k \in \{0, 2, \ldots, n-1\}$, with k even and n odd, the following:*

1) $Y_\theta(r, \theta) = r^k f(\theta)$, *with* $f(\theta) \geq \gamma > 0$.

2) $Y_r(r, \theta) = r^n a_n(\theta) + r^{n-1} a_{n-1}(\theta) + \cdots + r^k a_k(\theta)$ *with*

  2a) $a_n(\theta) \geq \alpha > 0$

2b) $|a_j(\theta)| \le \beta$, $k+1 \le j \le n-1$ *(on* $a_k(\theta)$ *no further restrictions are required).*

Then, if $\beta > 0$ *is sufficiently small, the number* $H(k,n)$ *of limit cycles of the vector field* Y *is bounded by*

$$H(k,n) \le n-k.$$

Proof. Assumptions 1) and 2) imply

$$\frac{dr}{d\theta} = \frac{Y_r}{Y_\theta} = r^{n-k}\tilde{a}_n(\theta) + \cdots + \tilde{a}_k(\theta)$$

where $\tilde{a}_j(\theta) = \frac{a_j(\theta)}{f(\theta)}$. The result now follows by Theorem 1.1, since

$$\tilde{a}_n(\theta) \ge \frac{\alpha}{\max|f(\theta)|} > 0,$$

and provided that $|\tilde{a}_j(\theta)| \le \frac{\beta}{\gamma}$, $k+1 \le j \le n-1$, are sufficiently small. □

Example 1.1. Suppose that the vector field (P,Q) in the plane is given as follows:

$$P(x,y) = -y + x^5 + xy^4 + \sum_{i=1}^{4}(b_i x^i + c_i x y^{i-1}) + d_1 y$$

$$Q(x,y) = x + x^4 y + y^5 + \sum_{i=1}^{4}(b_i y x^{i-1} + c_i y^i) + d_2 x$$

with $|b_i|, |c_i|, |d_i| \le \beta$ sufficiently small. Then some easy calculations yield

$$Y_\theta(r,\theta) = \cos^2(\theta) + \sin^2(\theta) + d_2\cos^2(\theta) - d_1\sin^2(\theta) = f(\theta) \ge 1 - 2\beta > 0$$

and

$$Y_r(r,\theta) = r^5 a_5(\theta) + \sum_{i=1}^{4} r^i a_i(\theta)$$

with $a_5(\theta) = [\cos^4(\theta) + \sin^4(\theta)] \ge \alpha > 0$ and $|a_i(\theta)| \le \beta$, $i = 1, \ldots, 4$.

Hence, the vector field (P,Q) satisfies the hypotheses of Theorem 1.2 with $k=0$, and thus, if $\beta > 0$ is sufficiently small, the vector field (P,Q) has at most five limit cycles.

The situation in the example can be visualized as follows: the vector field $(P_0, Q_0) = (-y, x)$ is the degenerate situation, where *all solutions* are closed (it describes the harmonic oscillator). Adding the term of order five $(x^5 + xy^4, x^4y + y^5)$ all the closed solutions disappear, and the point $(0,0)$ becomes a degenerate singular point (of order 5). Theorem 1.2 says that a perturbation of this situation by (sufficiently small) lower-order terms allows for at most five closed orbits.

We remark that in [13] a similar result to Theorem 1.1 was proved for the partial differential equation on a bounded domain $\Omega \subset \mathbb{R}^n$

$$\begin{aligned} -\Delta u + \sum_{i=1}^{n} a_i(x) u^i = f(x) \ , \ x \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \ \ , \ x \text{ on } \partial\Omega. \end{aligned} \tag{1.9}$$

The proof of Theorem 1.1 follows the same lines as in [13], but differs substantially in the estimates.

2. The reduction method

We now proceed with the proof of Theorem 1.1. We assume throughout the paper that $a_n(t) \geq \alpha > 0$; the case $a_n(t) \leq -\alpha$ is treated analogously.

To study the local behaviour of equation (1.1) we make use of a Lyapunov–Schmidt reduction (cf. [1]). We consider the Banach spaces

$$E = \{u \in C^1[0,1]; u(0) = u(1)\} \ \text{ and } F = C[0,1],$$

with the usual norms. Note that E and F are dense in $L^2(0,1)$. We split the space F into the direct sum $F = \tilde{F} \oplus \mathbb{R}$, where

$$\tilde{F} = \Big\{ u \in F : \int_0^1 u dt = 0 \Big\}.$$

We introduce the projections $Q : F \to \mathbb{R}$, $Qu = \int_0^1 u(t)dt$ and $P : F \to \tilde{F}$, $Pu = u - Qu$, and write $u \in E$ as

$$u = s + y = Qu + Pu.$$

Then u is a closed solution of equation (1.1) if and only if $u = s + y$ solves the following system of equations:

$$\begin{aligned} (P1) \qquad & \dot{y} + Pg(s+y) = \ Pf = f_1 \\ (Q1) \qquad & Qg(s+y) = \ Qf \end{aligned} \tag{2.1}$$

where

$$g(u) = \sum_{j=1}^{n} a_j(t) u^j.$$

We first solve equation $(P1)$ for fixed $s \in \mathbb{R}$:

Proposition 2.1. *For every fixed $s \in \mathbb{R}$ there exists a unique solution $y = y(s) \in \tilde{F}$ of equation $(P1)$.*

Proof. *Existence:* We apply the Leray–Schauder principle, [3], [10], transforming equation (P1) into an equivalent integral equation. Integrating (P1) from 0 to t we have

$$y(t) - y(0) = -\int_0^t g(s+y)d\tau + t\int_0^1 g(s+y)d\tau + \int_0^t f_1 d\tau := K(y). \quad (2.2)$$

Since

$$\int_0^1 y(t)dt = 0$$

we see that

$$-y(0) = \int_0^1 K(y)dt.$$

Thus, we can write equation (P1) as

$$y = K(y) - \int_0^1 K(y)dt. \quad (2.3)$$

Note that a solution $y \in \tilde{F}$ of (2.3) is closed, and that

$$\tilde{K}(y) := K(y) - \int_0^1 K(y)dt$$

maps $\tilde{F}$ into $\tilde{F}$. Furthermore, $\tilde{K}$ is a compact mapping. Thus, it remains to show that there exists an $R > 0$ such that

$$y = \lambda\tilde{K}(y)\ ,\ \ \lambda \in (0,1) \quad (2.4)$$

implies $\|y\|_\infty < R$.

Equation (2.4) is equivalent to

$$\dot{y} + \lambda\Big[g(s+y) - \int_0^1 g(s+y)dt - f_1\Big] = 0. \quad (2.5)$$

Multiplying (2.5) by y and integrating yields

$$\int_0^1 \dot{y}\ y\ dt + \lambda\Big[\sum_{i=1}^{n}\int_0^1 a_i(t)(s+y)^i y\, dt - \int_0^1 f_1 y\, dt\Big] = 0. \quad (2.6)$$

The first term being zero and $\lambda \neq 0$, we have

$$\int_0^1 a_n(t)(s+y)^n y\, dt = -\sum_{i=1}^{n-1}\int_0^1 a_i(t)(s+y)^i y\, dt + \int_0^1 f_1 y\, dt. \quad (2.7)$$

Calculating the various terms, using $a_n(t) \geq \alpha$, $|a_i(t)| \leq c$, $i = 1, \dots, n-1$, $|f_1(t)| \leq c$, and Hölder's inequality, we obtain

$$\begin{aligned} \alpha \int_0^1 y^{n+1} dt &\leq \sum_{j=1}^{n} \int_0^1 |b_j(s,t) y^j| dt \leq b \sum_{j=1}^{n} \int_0^1 |y^j| dt \\ &\leq b \sum_{j=1}^{n} (\int_0^1 y^{n+1} dt)^{\frac{j}{n+1}} \end{aligned} \tag{2.8}$$

where b is a constant which depends only on s and f_1. This inequality clearly implies (recalling that $n+1$ is even) that there exists a constant c, depending only on s and f_1, such that

$$\|y\|_{n+1}^{n+1} = \int_0^1 y^{n+1} dt \leq c \ , \tag{2.9}$$

for all solutions of $(P1)$.

We show next that also $\|y\|_\infty \leq c$, for all solutions y: by equation $(P1)$ we have

$$|\dot{y}| \leq |Pg(s+y)| + |f_1| = |g(s+y) - \int_0^1 g(s+y) dt| + |f_1|.$$

Integrating on $[0, 1]$, using the definition of g and (2.9), we get

$$\begin{aligned} \int_0^1 |\dot{y}| &\leq 2 \sum_{j=1}^{n} \int_0^1 |a_j(t)||(s+y)^j| dt + \int_0^1 |f_1| dt \\ &\leq 2b \sum_{j=1}^{n} (\int_0^1 y^{n+1} dt)^{\frac{j}{n+1}} + c \leq c = c(s, f_1). \end{aligned}$$

Finally, since y has mean value zero, we obtain

$$\|y\|_\infty \leq \|\dot{y}\|_1 \leq c(s, f_1). \tag{2.10}$$

Uniqueness: Let $s \in \mathbb{R}$ be fixed. Assume that y and z are two solutions in $\tilde{E} = E \cap \tilde{F}$ of equation $(P1)$:

$$\begin{aligned} \dot{y} + g(s+y) &= f_1 + \int_0^1 g(s+y) dt \\ \dot{z} + g(s+z) &= f_1 + \int_0^1 g(s+z) dt. \end{aligned}$$

It is not restrictive to assume $z(0) > y(0)$. Since $Qz = Qy = 0$, there exists $t_1 \in (0,1)$ such that $z(t_1) = y(t_1)$ and $\dot{z}(t_1) \leq \dot{y}(t_1)$, and hence by the equations

$$0 \leq \dot{y}(t_1) - \dot{z}(t_1) = -\int_0^1 (g(s+y) - g(s+z))dt.$$

Moreover, there exists $t_2 \in (t_1, 1)$ such that $z(t_2) = y(t_2)$ and $\dot{z}(t_2) \geq \dot{y}(t_2)$, and then

$$0 \geq \dot{y}(t_2) - \dot{z}(t_2) = -\int_0^1 (g(s+y) - g(s+z))dt.$$

Hence we get

$$\int_0^1 g(s+y) = \int_0^1 g(s+z).$$

Thus, y and z satisfy the same Cauchy problem; in particular, from the point t_1 emanates a unique solution, contradicting the assumption. □

Thus, we have proved that for each $s \in \mathbb{R}$ there exists a unique $y = y(s)$, a solution of equation $(P1)$. By the implicit function theorem one can prove the following:

Proposition 2.2. *The function $s \in \mathbb{R} \to y(s) \in \tilde{E}$ is real analytic.*

Proof. (See G. Tarantello [14].) We define the map $G : \tilde{E} \times \mathbb{R} \to \tilde{F}$:

$$G(y, s) = \dot{y} + g(s+y) - \int_0^1 g(s+y)dt - f_1.$$

We claim that G is analytic in $\tilde{E} \times \mathbb{R}$.

For $s_0 \in \mathbb{R}$ let $y_0 = y_0(s_0)$ denote the unique solution of equation $(P1)$; i.e., $G(y_0, s_0) = 0$. Consider the derivative of G with respect to y in the direction $w \in \tilde{E}$:

$$\frac{\partial G}{\partial y}(y_0, s_0)[w] = \dot{w} + g'(s_0 + y_0)w - \int_0^1 g'(s_0 + y_0)w\, dt.$$

$\frac{\partial G}{\partial y}(y_0, s_0)$ is injective; indeed, suppose $w \in \tilde{E}$ is such that

$$\frac{\partial G}{\partial y}(y_0, s_0)[w] = 0;$$

then one shows with an argument similar to that of Proposition 2.1 (uniqueness) that $w = 0$. By a standard application of the *Fredholm alternative* theorem we conclude that $\frac{\partial G}{\partial y}(y_0, s_0)$ defines an invertible operator which maps $\tilde{E} \to \tilde{F}$. Then, by the implicit function theorem, there exists $\epsilon > 0$

and an analytic map $\gamma : (s_0 - \epsilon, s_0 + \epsilon) \to \tilde{E}$ such that $\gamma(s_0) = y_0$ and $G(s, \gamma(s)) = 0$ for all $s \in (s_0 - \epsilon, s_0 + \epsilon)$. By the uniqueness it follows that $y(s) = \gamma(s)$, and hence $y(s)$ is analytic. □

For $s \in \mathbb{R}$, let now $y(s) = y(s, f_1)$ denote the unique solution of equation $(P1)$. We insert this solution into equation $(Q1)$ and obtain an equation in one dimension:

$$\int_0^1 g(s + y(s))dt = \int_0^1 f(t)dt. \tag{2.11}$$

3. The reduced equation

To study equation (2.11), we introduce the following function $\Gamma : \mathbb{R} \to \mathbb{R}$:

$$\Gamma(s) = \int_0^1 g(s + y(s))dt - \int_0^1 f(t)dt. \tag{3.1}$$

In order to show that equation (1.1) has at most n solutions, it is sufficient to show that (3.1) has at most n zeros, in view of Proposition 2.1.

First, we study the local behaviour of Γ. Suppose that $\Gamma(s_0) = 0$. If $\Gamma'(s_0) \neq 0$, then Γ is locally invertible and $\Gamma(s) = 0$ has a unique solution in a neighborhood of s_0.

Suppose now that $\Gamma'(s_0) = 0$; then $u_0 = s_0 + y(s_0)$ is a *singular point* of the mapping $\dot{u} + g(u)$; i.e.,

$$\dot{v} + g'(u_0)v = 0 \ , \text{ for some } v \in E \setminus \{0\}. \tag{3.2}$$

Indeed, differentiating equation $(P1)$ with respect to s, we have

$$\dot{y}_s(s) + P[g'(s + y(s))(1 + y_s(s))] = 0 \ , \ \forall \ s \in \mathbb{R}. \tag{3.3}$$

Since

$$\Gamma'(s) = \int_0^1 g'(s + y_s)(1 + y_s(s))dt \tag{3.4}$$

we get by adding (3.3) and (3.4)

$$\Gamma'(s) = \dot{y}_s + g'(s + y(s))(1 + y_s(s)) = \dot{v} + g'(u)v. \tag{3.5}$$

Equation (3.5) shows that if $\Gamma'(s_0) = 0$, then $v = 1 + y_s(s_0)$ is the unique solution of (3.2) with $\int_0^1 v\,dt = 1$, and vice versa. Indeed, a general solution of (3.2) has the form $ce^{-\int_0^t g'(u_0)d\tau}$. Thus,

$$v = 1 + y_s = \frac{e^{-\int_0^t g'(u_0)d\tau}}{\int_0^1 e^{-\int_0^t g'(u_0)d\tau}dt}.$$

Also, for the corresponding *adjoint equation* we have

$$-\dot{v}^* + g'(u_0)v^* = 0 \ , \quad \text{with} \quad v^* = d\, e^{\int_0^t g'(u_0)d\tau}. \tag{3.6}$$

Choosing

$$d = \int_0^1 e^{-\int_0^t g'(u_0)d\tau} dt,$$

we have $v^* = 1/v$.

We want to show that the stationary points of Γ are degenerate to at most degree $n-1$; that is, we prove

Proposition 3.1. *Suppose that the assumptions of Theorem* 1.1 *hold. If* $\Gamma'(s_0) = 0$, *then* $\Gamma^{(n)}(s_0) \geq C > 0$.

Proof. Differentiating (3.5) with respect to s, we get by induction

$$\begin{aligned}\Gamma^{(k)}(s) = {} & \dot{y}^{(k)} + \sum_{i=1}^n a_i i u^{i-1} y^{(k)} \\ & + \sum_{i=2}^n a_i u^{i-2} \sum_{q \in Q_k(k-1)} p(q) v^{q_1} (y^{(2)})^{q_2} \cdots (y^{(k-1)})^{q_{k-1}} + \cdots + \\ & + \sum_{i=k-1}^n a_i u^{i-(k-1)} \sum_{q \in Q_k(2)} p(q) v^{q_1} (y^{(2)})^{q_2} \\ & + \sum_{i=k}^n a_i u^{i-k} i(i-1)\cdots(i-(k-1)) v^k,\end{aligned} \tag{3.7}$$

where $v = 1 + y_s$,

$$Q_k(m) = \{q = (q_1, \ldots, q_m) \in \{0, 1, \ldots, k\}^m, \sum_{i=1}^m i q_i = k\},$$

and $p(q) \geq 0$ are some integer coefficients.

Multiplying (3.7) by $v^* = 1/v$ and integrating we get, setting $a^* = \int_0^1 v^* dt$ and noting that

$$\int_0^1 \dot{y}^{(k)} v^* dt + \int_0^1 \sum_{i=1}^n a_i i u^{i-1} y^{(k)} v^* dt = \int_0^1 y^{(k)} (-\dot{v}^* + g'(u_0) v^*) dt = 0:$$

$$\Gamma^{(k)}(s) a^* = \sum_{i=2}^n \int_0^1 a_i u^{i-2} \sum_{q \in Q_k(k-1)} p(q) v^{q_1 - 1} \cdots (y^{(k-1)})^{q_{k-1}} dt + \cdots +$$

$$+\sum_{i=k-1}^{n}\int_0^1 a_i u^{i-(k-1)} \sum_{q\in Q_k(2)} p(q) v^{q_1-1}(y^{(2)})^{q_2} dt$$
$$+\sum_{i=k}^{n}\int_0^1 a_i u^{i-k} i(i-1)\cdots(i-(k-1)) v^{k-1} dt. \tag{3.8}$$

In particular, for $k=n$, we have

$$\Gamma^{(n)}(s)a^* = \sum_{i=2}^{n}\int_0^1 a_i u^{i-2} \sum_{q\in Q_n(n-1)} p(q) v^{q_1-1}\cdots(y^{(k-1)})^{q_{k-1}} dt \tag{3.9}$$
$$+\cdots+$$
$$+\sum_{i=n-1}^{n}\int_0^1 a_i u^{i-(n-1)} \sum_{q\in Q_k(2)} p(q) v^{q_1-1}(y^{(2)})^{q_2} dt + \int_0^1 a_n n! v^{n-1} dt.$$

Note that the last term in equation (3.9) can be estimated from below by

$$\int_0^1 a_n n! v^{n-1} dt \geq \alpha n! \int_0^1 v^{n-1} dt \geq \alpha n! (\int_0^1 v dt)^{n-1} = \alpha n!. \tag{3.10}$$

We now estimate the remaining terms of (3.9) from above: a generic term of expression (3.9) (except the last) has the form

$$p(q)\int_0^1 a_i u^{i-j} v^{q_1-1}(y^{(2)})^{q_2}\cdots(y^{(n-1)})^{q_{n-1}} dt \tag{3.11}$$

where

$$\text{either}\quad 2\leq j\leq i\leq n-1 \quad \text{or} \quad 2\leq j< i=n$$

and $n-1\geq q_1\geq 0$, $q_2+\cdots+q_{n-1}\geq 1$.

In Section 4 below the following estimates are proved:

Under the assumptions of Theorem 1.1 *and supposing that* $c_n\beta < 1/2$, *where* $c_n = \frac{n(n-1)}{2}(1+\gamma)$, *with* $\gamma = \max|a_n(t)|$, *one has the following:*

1) $1/2 \leq v(t) \leq 2$, $1/2 \leq v^*(t) \leq 2$, and $a^* = \int_0^1 v^* dt \leq 2$ (cf. Lemma 4.3).

2) $\int_0^1 u^{n-1} dt \leq \frac{\beta}{\alpha}(n-1)$ (cf. Lemma 4.4).

3) $\|y^{(k)}\|_\infty \leq c\,\beta$, $k=1,2,\dots n$ (cf. Lemma 4.5).

With these estimates we now get the following, considering separately three cases:

$2 \le j < i \le n-1$:

$$\begin{aligned} p(q)\int_0^1 & a_i u^{i-j} v^{q_1-1}(y^{(2)})^{q_2}\cdots(y^{(n-1)})^{q_{n-1}}dt \\ &\le p(q)\beta\ \|v\|_\infty^{q_1-1}\|y^{(2)}\|_\infty^{q_2}\cdots\|y^{(n-1)}\|_\infty^{q_{n-1}}\int_0^1 u^{i-j}dt \\ &\le p(q)\beta\ c\beta^{q_2+\cdots+q_{n-1}}\left(\int_0^1 u^{n-1}dt\right)^{\frac{i-j}{n-1}} \le c\ \beta^{2+\frac{1}{n-1}} \le c\ \beta. \end{aligned} \tag{3.12}$$

$2 \le j = i \le n-1$:

$$p(q)\int_0^1 a_i v^{q_1-1}(y^{(2)})^{q_2}\cdots(y^{(n-1)})^{q_{n-1}}dt \tag{3.13}$$

$$\le p(q)\beta\|v\|_\infty^{q_1-1}\|y^{(2)}\|_\infty^{q_2}\cdots\|y^{(n-1)}\|_\infty^{q_{n-1}} \le p(q)\beta c\beta^{q_2+\cdots+q_{n-1}} \le c\beta^2 \le c\beta.$$

$2 \le j < i = n$:

$$\begin{aligned} p(q)\int_0^1 a_n u^{n-j} v^{q_1-1}(y^{(2)})^{q_2}\cdots(y^{n-1})^{q_{n-1}}dt &\le p(q)\gamma c\beta\int_0^1 u^{n-j}dt \\ \le c\beta\Big(\int_0^1 u^{n-1}dt\Big)^{\frac{n-j}{n-1}} \le c\beta^{1+\frac{1}{n-1}} \le c\beta. \end{aligned} \tag{3.14}$$

Thus, joining these estimates, we find for $\Gamma^{(n)}(s)$

$$\Gamma^{(n)}(s) \ge \frac{1}{a^*}(\alpha n! - c(n)\ \beta) \ge \frac{1}{2}(\alpha n! - c(n)\beta). \tag{3.15}$$

Clearly, the last term is positive for $\beta > 0$ sufficiently small, and thus the proposition is proved. □

4. ESTIMATES

In this section we prove the estimates stated above. We assume throughout this section that the hypotheses of Theorem 1.1 are satisfied.

Lemma 4.1. *Assume that $u = s + y(s)$ is a singular point of $\dot{u} + g(u)$. If $0 < \beta < \frac{2\alpha}{n-1}$, then*

$$\int_0^1 u^{n-1} v\, dt \le \frac{\beta}{\alpha}\frac{n-1}{2} < 1. \tag{4.1}$$

Proof. Let $v = 1 + y_s$. Since $u = s + y(s)$ is a singular point, we have

$$\dot{v} + g'(u)v = 0.$$

Integrating we get

$$\int_0^1 g'(u)v\,dt = 0.$$

We isolate the first term:

$$\begin{aligned}\int_0^1 na_n u^{n-1} v dt = -\int_0^1 \Big[&(n-1)a_{n-1}u^{n-2}v + (n-2)a_{n-2}u^{n-3}v + \\ &+ \cdots + 2a_2 v + a_1 v\Big] dt.\end{aligned} \tag{4.2}$$

Using $a_n(t) \geq \alpha > 0$ and Hölder's inequality we obtain

$$\begin{aligned}&\alpha n \int_0^1 u^{n-1} v\,dt \\ &\leq \beta\Big\{(n-1)\int_0^1 |u^{n-2}v| dt + (n-2)\int_0^1 |u^{n-3}v| dt + \cdots + \\ &+ 2\int_0^1 uv\,dt + \int_0^1 v\,dt\Big\} \\ &\leq \beta\Big\{(n-1)\Big(\int_0^1 u^{n-1}v\,dt\Big)^{\frac{n-2}{n-1}}\Big(\int_0^1 v\,dt\Big)^{\frac{1}{n-1}} \\ &+ (n-2)\Big(\int_0^1 u^{n-1}v\,dt\Big)^{\frac{n-3}{n-1}}\Big(\int_0^1 v\,dt\Big)^{\frac{2}{n-1}} + \\ &\quad + \cdots + 2\Big(\int_0^1 u^{n-1}v\,dt\Big)^{\frac{1}{n-1}}\Big(\int_0^1 v\,dt\Big)^{\frac{n-2}{n-1}} + \Big(\int_0^1 v\,dt\Big)\Big\}.\end{aligned} \tag{4.3}$$

Setting $b = \left(\int_0^1 u^{n-1}v\,dt\right)^{\frac{1}{n-1}}$ and using $\int_0^1 v\,dt = 1$ we have

$$\alpha n b^{n-1} \leq \beta\left[(n-1)b^{n-2} + (n-2)b^{n-3} + \cdots + 2b + 1\right]. \tag{4.4}$$

Now, if $b \geq 1$, then

$$n\alpha b^{n-1} \leq \beta\left[(n-1) + (n-2) + \cdots + 3 + 2 + 1\right] b^{n-2};$$

that is, $n\alpha b \leq \beta\frac{n(n-1)}{2}$, which contradicts the assumption $b \geq 1$. Thus necessarily $b < 1$; going back to inequality (4.4) we now get

$$n\alpha b^{n-1} \leq \beta\left[(n-1) + (n-2) + \cdots + 1\right]$$

and hence (4.1). □

Remark 4.1. With the same arguments one shows that for the solution v^* of the adjoint equation (3.6) holds

$$\int_0^1 u^{n-1} v^* dt < \int_0^1 v^* dt.$$

Lemma 4.2. *Let* $\gamma = \max_{[0,1]} |a_n(t)|$. *Then*

$$\|\dot{v}\|_1 \leq \frac{n(n-1)}{2} \beta (1 + \frac{\gamma}{\alpha}) \tag{4.5}$$

$$\|y_s\|_\infty \leq \frac{n(n-1)}{2} \beta (1 + \frac{\gamma}{\alpha}). \tag{4.6}$$

Proof. *Inequality* (4.5): since $\dot{v} + g'(u)v = 0$, one has

$$\int_0^1 |\dot{v}| dt \leq \int_0^1 \left[n a_n u^{n-1} v + (n-1)|a_{n-1} u^{n-2} v| + \cdots + 2|a_2 uv| + |a_1 v| \right] dt.$$

By Hölder's inequality and by Lemma 4.1

$$\begin{aligned} \int_0^1 |\dot{v}| dt &\leq n \int_0^1 |a_n| u^{n-1} v \, dt + (n-1)\beta \Big(\int_0^1 u^{n-1} v \, dt \Big)^{\frac{n-2}{n-1}} \Big(\int_0^1 v \, dt \Big)^{\frac{1}{n-1}} \\ &\quad + \cdots + 2\beta \Big(\int_0^1 u^{n-1} v \, dt \Big)^{\frac{1}{n-1}} \Big(\int_0^1 v \, dt \Big)^{\frac{n-2}{n-1}} + \beta \int_0^1 v \, dt \\ &\leq n \frac{\gamma}{\alpha} \frac{n-1}{2} \beta + \frac{n(n-1)}{2} \beta = \frac{n(n-1)}{2} \beta (\frac{\gamma}{\alpha} + 1). \end{aligned}$$

Inequality (4.6): since $\int_0^1 y_s \, dt = 0$ there exists t_0 such that $y_s(t_0) = 0$; then $y_s(t) = \int_{t_0}^t \dot{v}(t) dt$, and hence

$$\|y_s\|_\infty \leq \|\dot{v}\|_1 \leq \frac{n(n-1)}{2} \beta (1 + \frac{\gamma}{\alpha}).$$

□

Lemma 4.3. *Let* $c_n = \frac{n(n-1)}{2}(1 + \frac{\gamma}{\alpha})$, *and suppose that* $\beta \, c_n \leq 1/2$. *Then*

$$\frac{1}{2} \leq 1 - c_n \beta \leq v(t) \leq 1 + c_n \beta \leq 2 \tag{4.7}$$

$$\frac{1}{2} \leq \frac{1}{1 + c_n \beta} \leq v^* = \frac{1}{v} \leq \frac{1}{1 - c_n \beta} \leq 2. \tag{4.8}$$

Proof. Since $v(t) = 1 + y_s(t)$ and by Lemma 4.2 $\|y_s\|_\infty \leq c_n \beta$, the estimates follow trivially. □

Lemma 4.4. *Suppose that* $\beta\, c_n < 1/2$. *Then*

$$\int_0^1 u^{n-1} dt \le \beta \frac{n-1}{\alpha}.$$

Proof. Since $\|y_s\|_\infty \le c_n \beta$ we have by Lemma 4.1

$$\int_0^1 u^{n-1}(1 - c_n\beta) dt \le \int_0^1 u^{n-1}(1 + y_s) dt \le \frac{n-1}{2\alpha}\beta.$$

Therefore,

$$\int_0^1 u^{n-1} dt \le \frac{n-1}{2} \frac{1}{1 - c_n\beta}\beta \le \frac{n-1}{\alpha}\beta.$$

Lemma 4.5. *Assume that* $u = s + y(s)$ *is a singular point of* $\dot{u} + g(u)$; *i.e.,* $\dot{v} + g'(u)v = 0$ *with* $v = 1 + y_s$. *If* $\beta > 0$ *is sufficiently small, then we have for* $k = 1, \dots, n$

$$\|y^{(k)}\|_\infty \le c\, \beta. \tag{4.9}$$

Proof. We proceed by induction:

$k = 1$: the estimate (4.9) is true by Lemma 4.2, since $y^{(1)} = y_s$.

$k \to k+1$: differentiating equation $(P1)$ of (2.1) $k+1$ times with respect to s one obtains

$$\begin{aligned}
\dot{y}^{(k+1)} = -\Big[& P \sum_{i=1}^{n} a_i i u^{i-1} y^{(k+1)} \\
& + P \sum_{i=2}^{n} a_i u^{i-2} \sum_{q \in Q_{k+1}(k)} p(q) v^{q_1} \cdots (y^{(k)})^{q_k} + \\
& + \cdots + \\
& + P \sum_{i=k}^{n} a_i u^{i-k} \sum_{q \in Q_{k+1}(2)} p(q) v^{q_1} (y^{(2)})^{q_2} + \\
& + P \sum_{i=k+1}^{n} a_i i(i-1)\cdots(i-k) u^{i-(k+1)} v^{k+1} \Big].
\end{aligned} \tag{4.10}$$

Using that

$$|Pz| = |z - \int_0^1 z\, dt| \le |z| + \int_0^1 |z| dt,$$

we obtain by integrating (4.10)

$$
\begin{aligned}
\int_0^1 |\dot y^{(k+1)}|dt \le 2n \int_0^1 |a_n| u^{n-1} |y^{(k+1)}|dt + 2\sum_{i=1}^{n-1} \int_0^1 i|a_i||u^{i-1}||y^{(k+1)}|dt \\
+ 2\sum_{i=2}^{n} \int_0^1 |a_i||u^{i-2}| \sum_{q\in Q_{k+1}(k)} p(q) v^{q_1} \cdots |y^{(k)}|^{q_k} dt \qquad (4.11)\\
+\cdots+ \\
+ 2\sum_{i=k}^{n} \int_0^1 \cdots (i-k+1)|a_i||u^{i-k}| \sum_{q\in Q_{k+1}(2)} p(q) v^{q_1} |y^{(2)}|^{q_2} dt \\
+ 2\sum_{i=k+1}^{n} \int_0^1 |a_i||u^{i-(k+1)}|v^{k+1} dt.
\end{aligned}
$$

We now estimate separately the terms of inequality (4.11):

We begin with the *first line:*

$i = n$: by Lemma 4.4

$$\int_0^1 a_n u^{n-1}|y^{(k+1)}|dt \le \gamma\, c\, \beta\, \|y^{(k+1)}\|_\infty \qquad (4.12)$$

$i = 1, \dots, n-1$:

$$
\begin{aligned}
\int_0^1 |a_i||u^{i-1}||y^{(k+1)}|dt &\le \beta \|y^{(k+1)}\|_\infty \int_0^1 |u^{i-1}|dt \\
&\le \beta\, \|y^{(k+1)}\|_\infty \left(\int_0^1 u^{n-1}dt\right)^{\frac{i-1}{n-1}} \le c\beta \|y^{(k+1)}\|_\infty .
\end{aligned} \qquad (4.13)
$$

We continue with the subsequent terms; these terms are of the general form

$$p(q)\int_0^1 |a_i|\, |u^{i-j}|\, v^{q_1} |y^{(2)}|^{q_2} \cdots |y^{(k)}|^{q_k} dt \qquad (4.14)$$

with $2 \le j = i \le n-1$, respectively $2 \le j < i \le n$, and $0 \le q_1 \le k+1$, $1 \le q_2 + \cdots + q_k$.

We treat separately the two cases:

$2 \le j = i \le n-1$: By the induction hypothesis we have that $\|y^{(m)}\|_\infty \le c\beta$, $m = 1, \dots, k$. Then we get from (4.14)

$$p(q)\int_0^1 |a_i|\, v^{q_1} \cdots |y^{(k)}|^{q_k} dt \le c\, \beta^2. \qquad (4.15)$$

$2 \leq j < i \leq n$: By induction hypothesis we get from (4.14) and by Lemma 4.4

$$p(q) \int_0^1 |a_i|\, |u^{i-j}| v^{q_1} \cdots |y^{(k)}|^{q_k} dt \leq c\, \beta \int_0^1 |u^{i-j}| dt \leq c\, \beta^{\frac{i-j}{n-1}} \beta. \tag{4.16}$$

Therefore we have in all cases

$$\int_0^1 |\dot{y}^{(k+1)}| dt \leq c_1 \beta\, \|y^{(k+1)}\|_\infty + c_2 \beta. \tag{4.17}$$

Since $\int_0^1 y^{(k+1)} dt = 0$ we can estimate

$$\|y^{(k+1)}\|_\infty \leq \|\dot{y}^{(k+1)}\|_1 \leq c_1 \beta\, \|y^{(k+1)}\|_\infty + c_2 \beta. \tag{4.18}$$

Therefore, assuming that $c_1 \beta < 1/2$

$$\|y^{(k+1)}\|_\infty \leq \frac{c_2}{1 - c_1 \beta} \beta \leq 2 c_2 \beta. \tag{4.19}$$

5. Bounds on the number of zeros

In Section 3 it was shown that

$$\text{if} \quad \Gamma'(s_0) = 0\ , \quad \text{then} \quad \Gamma^{(n)}(s_0) > 0.$$

It is easy to see that this implies that $\Gamma(s) = 0$ has *locally* at most n solutions. Indeed, suppose for s near s_0 we can write by Taylor's theorem

$$\Gamma(s) = \sum_{i=2}^{n-1} \Gamma^{(i)}(s_0) \frac{(s - s_0)^i}{i!} + \Gamma^{(n)}(s_0 + \theta) \frac{(s - s_0)^n}{n!}.$$

Since $\Gamma^{(n)}(s_0 + \theta) > 0$ for s near s_0, we conclude that there exists a neighborhood U of s_0 such that $\Gamma(s) = 0$ has at most n solutions in U.

To obtain a *global* result, we employ the following proposition, which is proved in [13]:

Proposition 5.1. *Suppose that $k : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying the following:*

(*i*) *$k'(x) \geq -\delta$, for all $x \in \mathbb{R}$.*

(*ii*) *For any $y \in \mathbb{R}$ with $k'(y) = 0$ holds $|k^{(i)}(y)| \leq \bar{c}$, $i = 2, \ldots, n-1$.*

(*iii*) *Let $I_{\bar{a}} = \{x \in \mathbb{R} : k'(x) < \bar{a}\}$, and suppose that*

$$|k''(x)| \leq \bar{b} \text{ and } k^{(n)}(x) \geq \bar{d} > 0, \ \ \forall\, x \in I_{\bar{a}}.$$

Then, if $\delta > 0$ is sufficiently small (for fixed positive constants $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$), the equation $k(x) = \xi$ has for any $\xi \in \mathbb{R}$ at most n solutions.

Thus, to conclude the proof of Theorem 1.1 it remains to show that the function Γ satisfies the hypotheses of Proposition 5.1.

Lemma 5.1. *Suppose that the assumptions of Theorem* 1.1 *are satisfied. Then the function* $\Gamma(s) = \int_0^1 g(s+y(s))dt - \int_0^1 f dt$ *satisfies the hypotheses of Proposition* 5.1, *for* $\beta > 0$ *sufficiently small.*

Proof. (i) We recall that

$$\Gamma'(s) = \int_0^1 g'(s+y(s))(1+y_s(s))dt = \dot{y}_s + g'(s+y(s))(1+y_s(s)).$$

Multiplying this equation by $1+y_s(s)$ and integrating we obtain

$$\Gamma'(s) = \Gamma'(s)\int_0^1 (1+y_s(s))dt = \int_0^1 g'(s+y(s))(1+y_s(s))^2 dt.$$

We can estimate

$$\begin{aligned}\Gamma'(s) &= \int_0^1 g'(s+y(s))(1+y_s)^2 dt = \int_0^1 g'(u)(1+y_s)^2 dt \\ &= \sum_{i=1}^n \int_0^1 i a_i u^{i-1}(1+y_s)^2 dt \\ &\geq \alpha \int_0^1 n(u)^{n-1}(1+y_s)^2 dt - \beta \sum_{i=1}^{n-1}\int_0^1 i u^{i-1}(1+y_s)^2 dt \\ &\geq \min\{\alpha n r^{n-1} - \beta\sum_{i=1}^{n-1} i|r|^{i-1}\}\int_0^1 (1+y_s)^2 dt \geq -\beta\frac{n(n-1)}{2}\end{aligned}$$

since for $|r| \geq 1$

$$\begin{aligned}\alpha n r^{n-1} - \beta\sum_{i=1}^{n-1} i|r|^{i-1} &= r^{n-1}(\alpha n - \beta(\frac{1}{|r|^{n-1}} + \cdots + \frac{(n-1)}{|r|})) \\ &\geq \alpha n - \beta\frac{n(n-1)}{2}\end{aligned}$$

and for $0 \leq |r| < 1$

$$\alpha n r^{n-1} - \beta\sum_{i=1}^{n-1} i|r|^{i-1} \geq -\beta\frac{n(n-1)}{2}.$$

Since β can be chosen as small as we like, (i) holds.

(ii) The expression for $\Gamma^{(k)}(s)a^*$ is given by (3.8). The terms are estimated as in (3.12) and (3.14), using $a^* \geq \frac{1}{2}$

$$|\Gamma^{(k)}(s)| \leq c\beta^{\frac{2}{n-1}} \frac{1}{a^*} \leq \bar{c}\ ,\ 2 \leq k \leq n-1,\ \text{for any } s \ \text{ with } \Gamma'(s) = 0.$$

(iii) Set $I_{\bar{a}} = \{s \in \mathbb{R} : \Gamma'(s) < \bar{a}\}$ with $\bar{a} > 0$ sufficiently small. One checks that this leads to the following modifications in the estimates of Section 4:

in Lemma 4.1: $\int_0^1 u^{n-1} v\, dt \leq \frac{1}{\alpha}(\beta\frac{n-1}{2} + \frac{\bar{a}}{n})$

in Lemma 4.2: $\|y_s\|_\infty \leq \frac{n(n-1)}{2}\beta(1 + \frac{\gamma}{\alpha}) + \bar{a} = c(\beta + \bar{a})$

in Lemma 4.3: no changes if one sets $c_n = \frac{n(n-1)}{2}(1 + \frac{\gamma}{\alpha}) + \bar{a}$

in Lemma 4.4: $\int_0^1 u^{n-1} dt \leq \frac{2}{\alpha}(\beta\frac{n-1}{2} + \frac{\bar{a}}{n}) = c(\beta + \bar{a})$

in Lemma 4.5: $\|y^{(k)}\|_\infty \leq c(\beta + \bar{a})$; this is obtained by the following changes in the proof:

in line (4.12): $\int_0^1 a_n u^{n-1} |y^{(k+1)}| dt \leq c(\beta + \bar{a}) \|y^{(k+1)}\|_\infty$

in line (4.15): $p(q) \int_0^1 |a_i| v^{q_1} \dots |y^{(k)}|^{q_k} dt \leq c\beta(\beta + \bar{a})$

in line (4.16): $\dots \leq c(\beta + \bar{a})$

which yields

in lines (4.17) and (4.18): $\dots \leq c_1(\beta + \bar{a}) + c_2(\beta + \bar{a})$

Assuming $c_1(\beta + \bar{a}) < \frac{1}{2}$ one finds

in line (4.19): $\|y^{(k+1)}\|_\infty \leq c(\beta + \bar{a})$

One now verifies easily that $|\Gamma''(s)| \leq \bar{b}$ and $\Gamma^{(n)}(s) \geq \bar{d} > 0$ for $s \in I_{\bar{a}}$, if β and $\bar{a}$ are sufficiently small.

Thus, the assumptions of Proposition 5.1 are satisfied, and hence the proof of Theorem 1.1 is complete. □

References

[1] A. Ambrosetti and G. Prodi, "A Primer in Nonlinear Analysis," Cambridge University Press, 1993.

[2] V. Cafagna and F. Donati, *Un rèsultat global de multiplicité pour un problème différentiel non linéaire du premier ordre*, C.R. Acad. Sci. Paris, Ser. I, Math., 300 (1985), 523–526.

[3] K. Deimling, "Nonlinear Functional Analysis," Springer, 1985.

[4] J. Écalle, "Introduction aux Fonctions Analysables et Preuve Constructive de la Conjecture de Dulac," Hermann, Paris, 1992.

[5] A. Gasull and J. Llibre, *Limit cycles of a class of Abel equations*, SIAM J. Math. Anal., 21 (1990), 1235–1244.

[6] D. Hilbert, "Mathematical Problems," Bulletin AMS, 8, 1902.

[7] Yu. Ilyashenko, "Finiteness Theorems for Limit Cycles," Amer. Math. Soc., Providence, RI, 1991.

[8] Yu. Ilyashenko, *Hilbert-type numbers for Abel equations, growth and zeros of holomorphic functions*, Nonlinearity, 13 (2000), 1337–1342.

[9] Yu. Ilyashenko and S. Yakovenko, *Concerning the Hilbert 16th problem*, Amer. Math. Soc. Transl. Ser. 2, 165 (1995), 1–19.

[10] M.A. Krasnosel'ski, "Topological Methods in the Theory of Nonlinear Integral Equations," Pergamon Press, 1963, 123–140.

[11] N.G. Lloyd, *The number of periodic solutions of the equation* $\dot{z} = z^N + p_1(t)z^{N-1} + \cdots + p_N(t)$, Proc. London Math. Soc., 27 (1973), 667–700.

[12] A.L. Neto, *On the number of solutions of the equation* $\frac{dx}{dt} = \sum_{j=0}^{n} a_j(t)x^j, 0 \le t \le 1,$ *for which* $x(0) = x(1)$, Inventiones Math., 59 (1980), 67–76.

[13] B. Ruf, *Bounds on the number of solutions for elliptic problems with polynomial nonlinearities*, J. Diff. Equ., 151 (1999), 111–133.

[14] G. Tarantello, *On the number of solutions for the forced pendulum equation*, J. Diff. Equ., 80 (1989), 79–93.