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POSITIVITY, CHANGE OF SIGN AND BUCKLING EIGENVALUES IN A ONE-DIMENSIONAL FOURTH ORDER MODEL PROBLEM

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Abstract. We study a one dimensional Dirichlet problem of fourth order and a corresponding "buckling eigenvalue problem" under Dirichlet boundary conditions. These problems may serve as model problems for "Orr-Sommerfeld" like boundary and eigenvalue problems. It turns out that eigenvalue curves in appropriate parameter domains look completely different than for the same equation under so called Navier boundary conditions. Further emphasis is laid on positivity properties, and also here, fundamental differences with Navier conditions arise: It may e.g. happen that one has infinitely many linearly independent positive eigenfunctions. Connections and analogies with the clamped plate boundary value problem on families of deformed domains are discussed.

1. Introduction

When the eigenvalue problem arising in the linearized stability problem for the parallel plane Couette flow is transformed by means of the so called poloidal-toroidal decomposition of the velocity field (see [4, p. 156]), among others a nonselfadjoint eigenvalue problem of the following "Orr-Sommerfeld"-type arises:

$$Lu + \lambda\Delta u := \Delta^2 u + \sum_{i=1}^{3} a_i(\,.\,)\frac{\partial}{\partial x_i}\Delta u + \lambda\Delta u = 0. \tag{1.1}$$

The coefficients $a_i(\,.\,)$ are real valued sufficiently smooth functions. This eigenvalue problem has to be studied in a layer-like domain $\Omega \subset \mathbb{R}^3$ under suitable boundary conditions, and one wants to show stability by proving

$$\mathrm{Re}\ \lambda > 0 \tag{1.2}$$

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for all eigenvalues $\lambda \in \mathbb{C}$ of (1.1). "Rigid" boundary conditions in the Couette problem lead to Dirichlet conditions

$$u = \nabla u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

while "stress free" boundary conditions yield Navier conditions

$$u = \Delta u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.4}$$

In related (strongly) non-selfadjoint second order problems

$$-\Delta u - \sum a_i \frac{\partial}{\partial x_i} u = \lambda u \quad \text{in } \Omega, \ u|\partial\Omega = 0,$$

the desired property (1.2) is shown with help of real analytic methods, namely maximum principles and a Krein-Rutman type argument, which yield a positive first eigenfunction. This may serve as comparison function in the corresponding nonstationary problem. In what follows we would like to ask whether one may expect that these methods may be extended to treat also problems like (1.1). So, we may restrict to real parameters $\lambda \in \mathbb{R}$. As far as problem (1.1) under *Navier boundary conditions* is concerned, the method just explained can be perfectly extended. For example, (u, λ) is a solution of

$$\begin{cases} \Delta^2 u + \sum_{i=1}^{n} a_i(\,.\,)\frac{\partial}{\partial x_i}\Delta u + \lambda \Delta u = 0, \ u \not\equiv 0, & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

if and only if (v, λ) solves

$$\begin{cases} -\Delta v - \sum_{i=1}^{n} a_i(\,.\,)\frac{\partial v}{\partial x_i} = \lambda v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $v = -\Delta u$; and u can be reconstructed from v as solution of $-\Delta u = v$ in Ω, $u = 0$ on $\partial\Omega$. In particular, we have existence, uniqueness and positivity of suitably normalized first eigenfunctions. Similarly one proves a comparison principle for solutions u of

$$\begin{cases} \Delta^2 u + \sum_{i=1}^{n} a_i(\,.\,)\frac{\partial}{\partial x_i}\Delta u + \lambda \Delta u = f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

For $\lambda < \lambda_{1,\text{Navier}}$ one has that $f \geq 0$ implies $u \geq 0$. Here $\lambda_{1,\text{Navier}}$ is the (positive) first eigenvalue of (1.5). With help of these tools, based on comparison and positivity arguments, it is a not too difficult task to treat the original stability problem.

The situation changes completely, when the Navier boundary conditions (1.4) are replaced with Dirichlet boundary conditions (1.3). It is true that the underlying stability problem has been solved in a special geometric situation [11] by means of extremely difficult explicit and numerical calculations, based on "elementary functions". For a more extensive treatment, see also [3]. But with regard to a more elegant proof, to possible extensions and to a better understanding, it would be nevertheless interesting, whether similar results as for (1.5) and (1.7) may be obtained also under Dirichlet conditions (1.3), at least in geometrically very simple domains like balls or layers with additional periodic boundary conditions with respect to the unbounded directions. Although there is an encouraging partial result, see Proposition 1 below, an exhaustive discussion of these problems with respect to positivity under Dirichlet conditions seems out of reach (at least for us). To get an idea, which kind of results may be expected, we study the one-dimensional model problem

$$\begin{cases} u'''' + a\,u''' + \lambda\,u'' = f \text{ in } (-1,1), \\ u(-1) = u'(-1) = u(1) = u'(1) = 0, \end{cases} \tag{1.8}$$

and the corresponding eigenvalue problem

$$\begin{cases} u'''' + a\,u''' = \lambda\,(-u''), \qquad u \not\equiv 0, \quad \text{in } (-1,1), \\ u(-1) = u'(-1) = u(1) = u'(1) = 0. \end{cases} \tag{1.9}$$

We call the boundary value problem (1.8) (or more precisely: its solution operator) *positivity preserving*, if positive data always yield positive solutions, i.e. $f \geq 0 \Rightarrow u \geq 0$. For $\lambda \leq 0$ we can prove that (1.8) is positivity preserving, see Proposition 1 below, while for the discussion of $\lambda > 0$ we have to combine explicit calculations, a positivity criterion of Schröder [12, 13, 14] and support from the graphical package of MAPLE to obtain a complete description of the positivity or, more general, sign properties of (1.8) and (1.9).

Our results concerning the connections between positivity preserving properties and eigenvalue problems may be summarized as follows: In particular for $a = 0$ and also for $|a|$ small, the problem behaves similarly under Dirichlet boundary conditions as under Navier conditions. On the other hand, for $|a|$ large, there are striking differences: if we keep a fixed and increase λ, before reaching the first eigenvalue (which is on the same eigenvalue-curve

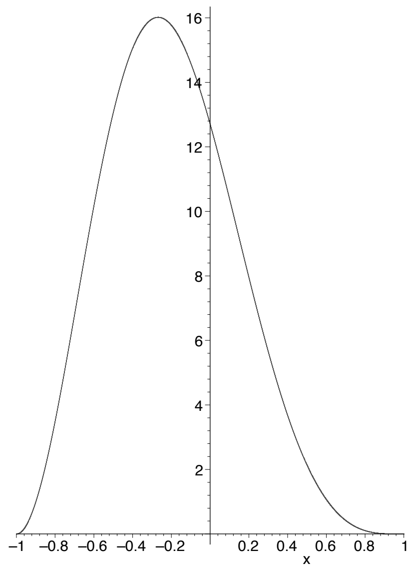

FIGURE 1. Sign changing first eigenfunction for $a = 2.11$

in the a-λ-plane as the third, fifth or seventh etc. eigenvalue for $a = 0$ resp., see Figure 5), first change of sign and also further oscillations of the Green function for (1.8) occur.

Further the first eigenfunction need not be of fixed sign, even not in the case, when the first eigenvalue is simple. As an example, the first eigenfunction for $a = 2.11$ is plotted in Figures 1 and 2.

Not only for the sake of curiosity we would like to already mention that for $a = 0$ the even eigenfunctions are all of fixed sign

$$u_{\lambda_{2k-1}} = 1 - (-1)^k \cos\left(\sqrt{\lambda_{2k-1}}x\right), \qquad \lambda_{2k-1} = k^2\,\pi^2, \tag{1.10}$$

while the odd ones

$$\begin{aligned} &u_{\lambda_{2k}} = x \sin\left(\sqrt{\lambda_{2k}}\right) - \sin\left(\sqrt{\lambda_{2k}}x\right), \\ &\lambda_{2k} \text{ the } k\text{-th positive solution of } \sqrt{\lambda} = \tan\sqrt{\lambda}, \end{aligned} \tag{1.11}$$

have the expected number of $(2k-1)$ nodes in $(-1,1)$.

The existence of infinitely many positive eigenfunctions for (1.9) with $a = 0$ shows that this problem is *not selfadjoint* as an eigenvalue problem, although the differential operator itself is symmetric under appropriate boundary conditions and my be defined as selfadjoint operator.

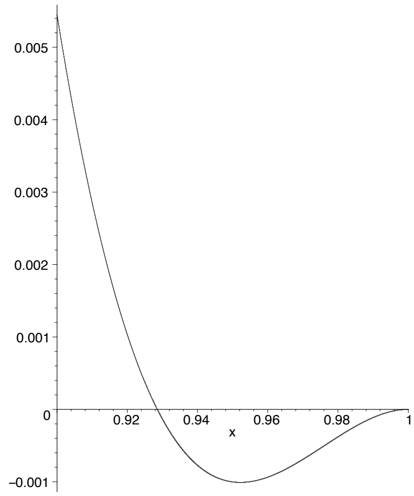

FIGURE 2. Looked at more closely

Different parameter dependent buckling eigenvalue problems in variational form have e.g. been studied in [9], see also the references therein.

2. A POSITIVE RESULT

Proposition 1. *For $\lambda \leq 0$ and $a \in \mathbb{R}$ the solution operator for* (1.8) *is strongly positivity preserving. That means that for every solution u of*

$$\begin{cases} u'''' + a\,u''' + \lambda\,u'' = f \ \textit{in} \ (-1,1), \\ u(-1) = u'(-1) = u(1) = u'(1) = 0, \end{cases}$$

$0 \not\equiv f \geq 0$ *implies* $u > 0$.

Proof. We define $v := u''$, then v is a subsolution of

$$v'' + a\,v' + \lambda v \geq 0.$$

We will exploit a weak maximum principle for v in the following form: Assume there are $x_1, x_2 \in [-1,1]$, $x_1 < x_2$ with $v(x_1) = v(x_2) = 0$, then $v \leq 0$ in $[x_1, x_2]$. We distinguish four cases according to the sign of v at the boundary of $(-1,1)$.

Case 1: $v(-1) \leq 0$ and $v(1) \leq 0$. Then by the maximum principle, $v \leq 0$ in $[-1,1]$. Hence u is concave and can satisfy the boundary conditions only if $u \equiv 0$. A contradiction; this case cannot occur.

Case 2: $v(-1) > 0$ and $v(1) \leq 0$. There exists $x_1 \in (-1, 1]$ such that $v > 0$ on $[-1, x_1)$ and $v \leq 0$ on $[x_1, 1]$. The strict convexity of u and $u(-1) = u'(-1) = 0$ imply that $u(x_1) > 0$. On the other hand, looking at $[x_1, 1]$, we find also that $u(x_1) \leq 0$. Again a contradiction; this case cannot occur, too.

Case 3: $v(-1) \leq 0$ and $v(1) > 0$. By considering $x \mapsto u(-x)$, this reduces to Case 2 and cannot occur neither.

Case 4: $v(-1) > 0$ and $v(1) > 0$. As u cannot be convex everywhere, there exist $-1 < x_1 < x_2 < 1$ such that $v > 0$ on $[-1, x_1) \cup (x_2, 1]$ and $v \leq 0$ on $[x_1, x_2]$. The boundary data of u imply that $u > 0$ on $(-1, x_1) \cup (x_2, 1)$. Concavity of u on $[x_1, x_2]$ finally yields also $u > 0$ on $[x_1, x_2]$. □

Remarks. 1) The preceding proof remains valid without any change, if we consider continuous functions $a, \lambda : [-1, 1] \to \mathbb{R}$, where $\lambda \leq 0$.
2) The preceding proof is strictly one dimensional. In dimensions $n \geq 2$ it seems that even in the special case $a = \lambda = 0$ nobody has yet been able to take advantage of the seemingly quadratic form of the biharmonic operator Δ^2 under Dirichlet conditions in the context of positivity.

From now on we want to investigate the case $\lambda > 0$. As the positivity behaviour for (1.8) is rather subtle we restrict to the case of constant coefficients and first even to $a = 0$. It will turn out that the positivity and, more generally, sign properties *in the latter case* are similar as one would expect by analogy from the second order theory, except the sign of the respective even eigenfunctions already mentioned in the introduction.

Definition 1. Let $\lambda_k = \lambda_k(a)|_{a=0}$, $0 < \lambda_1 < \lambda_2 < \ldots$ denote the eigenvalues of $u'''' = \lambda\,(-u'')$ in $(-1, 1)$, $u(-1) = u'(-1) = u(1) = u'(1) = 0$.

These eigenvalues are zeros of

$$\sin\sqrt{\lambda}\big(\sin\sqrt{\lambda} - \sqrt{\lambda}\cos\sqrt{\lambda}\big) = 0; \tag{2.1}$$

the respective (suitably) normalized eigenfunctions are given in (1.10) and (1.11) above.

Corollary 1. *For $\lambda < \lambda_1 = \lambda_1(a)|_{a=0}$, the solution operator of the problem*

$$\begin{cases} u'''' + \lambda\, u'' = f \ \text{ in } (-1, 1) \\ u(-1) = u'(-1) = u(1) = u'(1) = 0, \end{cases} \tag{2.2}$$

is positivity preserving, i.e. $f \geq 0$ always implies $u \geq 0$.

Proof. By means of the previous proposition and a positivity result of [14, p. 147] it suffices to show that for $0 < \lambda < \lambda_1$, all the eigenvalue problems

$$u'''' = \lambda(-u'') \text{ in } (-1,1), \quad u(-1) = u'(-1) = u(1) = u'(1) = 0, \tag{2.3}$$

$$u'''' = \lambda(-u'') \text{ in } (-1,1), \quad u(-1) = u(1) = u'(1) = u''(1) = 0, \tag{2.4}$$

$$u'''' = \lambda(-u'') \text{ in } (-1,1), \quad u(-1) = u'(-1) = u''(-1) = u'(1) = 0, \tag{2.5}$$

only have the trivial solution $u(x) \equiv 0$. For (2.3) this is obvious by definition of λ_1.

For $\lambda > 0$, the general solution of the differential equation $u'''' = \lambda(-u'')$ is:

$$u(x) = \alpha + \beta\, x + \gamma \, \cos\left(\sqrt{\lambda} x\right) + \delta \sin\left(\sqrt{\lambda} x\right).$$

The determinant, which has to vanish in order that (2.4) has a nontrivial solution, is

$$\det \begin{pmatrix} 1 & -1 & \cos\sqrt{\lambda} & -\sin\sqrt{\lambda} \\ 1 & 1 & \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \\ 0 & 1 & -\sqrt{\lambda}\sin\sqrt{\lambda} & \sqrt{\lambda}\cos\sqrt{\lambda} \\ 0 & 0 & -\lambda\cos\sqrt{\lambda} & -\lambda\sin\sqrt{\lambda} \end{pmatrix} = \lambda\left(2\sqrt{\lambda} - \sin\left(2\sqrt{\lambda}\right)\right).$$

The latter term, however, is always positive for $\lambda > 0$. A very similar matrix with the same determinant shows that also (2.5) only has the trivial solution. □

Remark. This proof is strictly one dimensional, too, even if Proposition 1 could be shown in higher dimensions. The positivity criterion of Schröder has not yet been generalized to $n \geq 2$, although under suitable and sufficiently strong assumptions such a generalization seems very likely to hold true.

Also for regular values $\lambda > \lambda_1$ the behaviour of the Green function G_λ corresponding to problem (2.2) is as expected. This Green function can be calculated explicitly:

$$\begin{aligned} G_\lambda(x,y) = {} & \frac{1}{2\lambda}|x-y| - \frac{1}{2\lambda\sqrt{\lambda}}\sin\left(\sqrt{\lambda}|x-y|\right) - \frac{\cos\sqrt{\lambda} + \sqrt{\lambda}\sin\sqrt{\lambda}}{2\lambda\sqrt{\lambda}\sin\sqrt{\lambda}} \\ & + \frac{\cos\left(\sqrt{\lambda}x\right)}{2\lambda\sqrt{\lambda}\sin\sqrt{\lambda}} + \frac{\cos\left(\sqrt{\lambda}y\right)}{2\lambda\sqrt{\lambda}\sin\sqrt{\lambda}} + \frac{\cos\sqrt{\lambda}}{2\sqrt{\lambda}\left(\sqrt{\lambda}\cos\sqrt{\lambda} - \sin\sqrt{\lambda}\right)}\, xy \\ & - \frac{y\sin\left(\sqrt{\lambda}x\right)}{2\lambda\left(\sqrt{\lambda}\cos\sqrt{\lambda} - \sin\sqrt{\lambda}\right)} - \frac{x\sin\left(\sqrt{\lambda}y\right)}{2\lambda\left(\sqrt{\lambda}\cos\sqrt{\lambda} - \sin\sqrt{\lambda}\right)} \end{aligned} \tag{2.6}$$

$$- \frac{\cos\sqrt{\lambda}}{2\lambda\sqrt{\lambda}\sin\sqrt{\lambda}} \cos\left(\sqrt{\lambda}x\right)\cos\left(\sqrt{\lambda}y\right)$$
$$+ \frac{\cos\sqrt{\lambda}+\sqrt{\lambda}\sin\sqrt{\lambda}}{2\lambda\sqrt{\lambda}\left(\sqrt{\lambda}\cos\sqrt{\lambda}-\sin\sqrt{\lambda}\right)} \sin\left(\sqrt{\lambda}x\right)\sin\left(\sqrt{\lambda}y\right).$$

A feature, which this Green function shares with Green functions for many different examples, is the oscillatory behaviour beyond the first eigenvalue:

Theorem 1. *The Green function for* (2.2) *changes sign for all regular values of* λ *with* $\lambda > \lambda_1$.

In particular, we do *not* have a uniform anti-maximum-principle. That means that there is *no* $\varepsilon > 0$ such that for $\lambda \in (\lambda_1, \lambda_1 + \varepsilon)$ the solution operator for (1.8) is sign *reversing*, i.e. $G_\lambda < 0$. Cf. [2, 7].

Proof. First we have to show that for every regular $\lambda > \lambda_1$, the Green function has (also) positive values. For this purpose we consider

$$f_\lambda(x) := G_\lambda(x,x)$$

and calculate its derivatives:

$$f_\lambda'(x) = \frac{\left(\cos\left(\sqrt{\lambda}x\right)-\cos\sqrt{\lambda}\right)\left(\sin\left(\sqrt{\lambda}x\right)-x\sin\sqrt{\lambda}\right)}{\sqrt{\lambda}\sin\sqrt{\lambda}\left(\sqrt{\lambda}\cos\sqrt{\lambda}-\sin\sqrt{\lambda}\right)};$$

$$f_\lambda''(x) = \frac{-\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)\left(\sin\left(\sqrt{\lambda}x\right)-x\sin\sqrt{\lambda}\right)}{\sqrt{\lambda}\sin\sqrt{\lambda}\left(\sqrt{\lambda}\cos\sqrt{\lambda}-\sin\sqrt{\lambda}\right)} + \frac{\left(\cos\left(\sqrt{\lambda}x\right)-\cos\sqrt{\lambda}\right)\left(\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)-\sin\sqrt{\lambda}\right)}{\sqrt{\lambda}\sin\sqrt{\lambda}\left(\sqrt{\lambda}\cos\sqrt{\lambda}-\sin\sqrt{\lambda}\right)};$$

$$f_\lambda'''(x) = \frac{-\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)\left(\sin\left(\sqrt{\lambda}x\right)-x\sin\sqrt{\lambda}\right)}{\sin\sqrt{\lambda}\left(\sqrt{\lambda}\cos\sqrt{\lambda}-\sin\sqrt{\lambda}\right)} - \frac{\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)\left(\cos\left(\sqrt{\lambda}x\right)-\cos\sqrt{\lambda}\right)}{\sin\sqrt{\lambda}\left(\sqrt{\lambda}\cos\sqrt{\lambda}-\sin\sqrt{\lambda}\right)} - 2\frac{\sin\left(\sqrt{\lambda}x\right)\left(\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)-\sin\sqrt{\lambda}\right)}{\sin\sqrt{\lambda}\left(\sqrt{\lambda}\cos\sqrt{\lambda}-\sin\sqrt{\lambda}\right)}.$$

We conclude $f_\lambda(1) = f_\lambda'(1) = f_\lambda''(1) = 0$ and $f_\lambda'''(1) = -2$. Hence for any regular $\lambda > \lambda_1$ there exists an $\varepsilon > 0$ such that for $x \in (1-\varepsilon, 1)$, we have

$$G_\lambda(x,x) > 0.$$

It remains to show that for any regular $\lambda > \lambda_1$ there are also $(x, y) \in (-1, 1) \times (-1, 1)$ with $G_\lambda(x, y) < 0$.

Case 1: $\sqrt{\lambda} \in ((2k-1)\pi, 2k\pi)$, $k \in \mathbb{N}$. Here we consider

$$G_\lambda(0,0) = \frac{1 - \cos\sqrt{\lambda}}{\lambda\sqrt{\lambda}\sin\sqrt{\lambda}} - \frac{1}{2\lambda}.$$

This value is obviously negative, as $\sin\sqrt{\lambda} < 0$.

Case 2: $\sqrt{\lambda} \in \left(2k\pi, \left(2k + \frac{1}{2}\right)\pi\right)$, $k \in \mathbb{N}$. Here, again we consider

$$G_\lambda(0,0) = \frac{1 - \cos\sqrt{\lambda}}{\lambda\sqrt{\lambda}\sin\sqrt{\lambda}} - \frac{1}{2\lambda},$$

but now we have to argue a bit more carefully. We have $\cos\sqrt{\lambda} > 0$, $\sin\sqrt{\lambda} > 0$, $1 \le |\cos\sqrt{\lambda}| + |\sin\sqrt{\lambda}| = \cos\sqrt{\lambda} + \sin\sqrt{\lambda}$, and hence

$$G_\lambda(0,0) \le \frac{1}{\lambda\sqrt{\lambda}} - \frac{1}{2\lambda} = \frac{2 - \sqrt{\lambda}}{2\lambda\sqrt{\lambda}} < 0.$$

Case 3: $\sqrt{\lambda} \in \left(\left(2k + \frac{1}{2}\right)\pi, (2k+1)\pi\right)$, $k \in \mathbb{N}$. Now we consider

$$\frac{\partial^2}{\partial y^2} G_\lambda(x,1) = \frac{\cos\left(\sqrt{\lambda}x\right) - \cos\sqrt{\lambda}}{2\sqrt{\lambda}\sin\sqrt{\lambda}} + \frac{x\sin\sqrt{\lambda} - \sin\left(\sqrt{\lambda}x\right)}{2\left(\sqrt{\lambda}\cos\sqrt{\lambda} - \sin\sqrt{\lambda}\right)}$$

and choose

$$x_\lambda = \frac{2\pi - \sqrt{\lambda}}{\sqrt{\lambda}} \in (-1, 0).$$

For this choice we obtain

$$\frac{\partial^2}{\partial y^2} G_\lambda(x_\lambda, 1) = \frac{\pi\sin\sqrt{\lambda}}{\sqrt{\lambda}(\sqrt{\lambda}\cos\sqrt{\lambda} - \sin\sqrt{\lambda})} < 0,$$

because $\sin\sqrt{\lambda} > 0$ and $\cos\sqrt{\lambda} < 0$. Consequently, for y close to 1, it follows that $G_\lambda(x_\lambda, y) < 0$. □

A full discussion of the very complicated explicit expression for the Green function G_λ would be too difficult and require too much effort. Instead we give a few plots, which illustrate, how the number of oscillations increases, whenever λ has crossed an eigenvalue, see Figures 3 and 4.

3. The eigenvalue problem with a nonsymmetric term

In the preceding section we were mainly concerned with the symmetric eigenvalue and resolvent problem (2.2). It turned out, that except the sign of the even eigenfunctions, this problem seems to resemble in many respects the second order resolvent problem

$$-u'' = \lambda u + f \text{ in } (-1,1), \qquad u(-1) = u(1) = 0.$$

When turning to the nonsymmetric problem (1.8) with $a \in \mathbb{R} \setminus \{0\}$, the situation becomes completely different, at least when $|a|$ is large. While for the second order eigenvalue problem

$$-u'' - a\,u' = \lambda u \text{ in } (-1,1), \qquad u(-1) = u(1) = 0,$$

the eigenvalue curves are given by the following family of parabolas

$$\lambda_k = \frac{k^2}{4} + \frac{a^2}{4}, \tag{3.1}$$

we find for the eigenvalue curves of the fourth order "buckling" eigenvalue problem

$$\begin{cases} u'''' + au''' = \lambda(-u''), \qquad u \not\equiv 0, \qquad \text{in } (-1,1) \\ u(-1) = u'(-1) = u(1) = u'(1) = 0, \end{cases} \tag{3.2}$$

in the a-λ-plane the following result:

Theorem 2. *The connected components of the eigenvalue curves in the a-λ-plane corresponding to the buckling eigenvalue problem* (3.2) *are compact.*

Eigenvalue curves are the set $\{(a,\lambda) : \text{ (3.2) has a solution}\}$.

Proof. First we show that for any $a \in \mathbb{R}$ and any eigenvalue $\lambda \in \mathbb{R}$ of (3.2) there holds

$$\lambda > \frac{a^2}{4}. \tag{3.3}$$

Indeed, for $\lambda \le \frac{a^2}{4}$ and $-1 \le x_1 < x_2 \le 1$, we have positivity of the Green function for

$$Lv = -\big(v'' + a\,v' + \lambda v\big) \quad \text{in} \quad (x_1,x_2), \qquad v(x_1) = v(x_2) = 0,$$

cf. (3.1). And that was the only point in the proof of Proposition 1, where we used $\lambda \le 0$. Hence one has strong positivity preserving for the solution operator for (1.8), if the coefficients a and λ are constant and satisfy $\lambda \le \frac{a^2}{4}$ and in particular no nontrivial solution of the homogeneous equation.

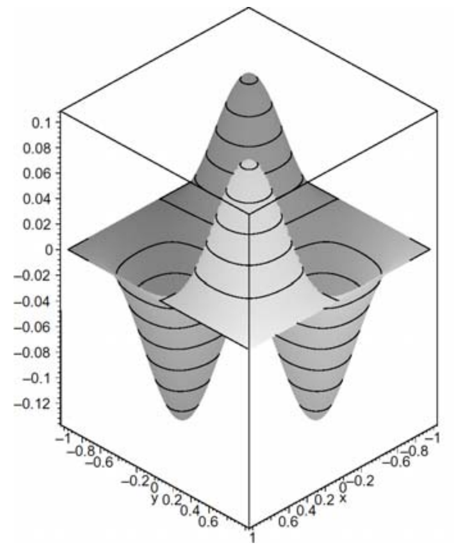

FIGURE 3. G_λ for $\sqrt{\lambda} = 4.4$, between $\sqrt{\lambda_1} = \pi$ and $\sqrt{\lambda_2} \approx 4.49$

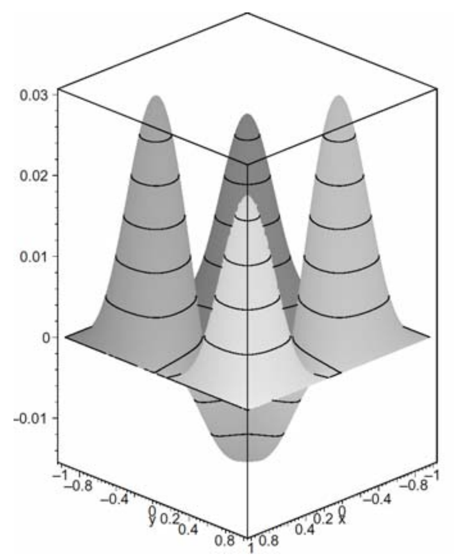

FIGURE 4. G_λ for $\sqrt{\lambda} = 4.4$, between $\sqrt{\lambda_1} = \pi$ and $\sqrt{\lambda_2} \approx 4.49$

So, in what follows, we may restrict ourselves to the case $\lambda > \frac{a^2}{4}$. Here the general solution of the *differential equation*

$$u'''' + au''' + \lambda u'' = 0$$

is given by

$$u = \alpha + \beta x + \gamma e^{-ax/2} \cos\left(\sqrt{\lambda - \tfrac{a^2}{4}}x\right) + \delta e^{-ax/2} \sin\left(\sqrt{\lambda - \tfrac{a^2}{4}}x\right).$$

In order to find eigenvalues, we have to investigate, whether there is a non-trivial choice of parameters $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4 \setminus \{0\}$, such that also the boundary conditions

$$u(-1) = u'(-1) = u(1) = u'(1) = 0$$

are satisfied. This gives us the task to find the zeroes

$$f(a, \lambda) = 0 \tag{3.4}$$

of the following determinant:

$$f : \left\{(a, \lambda) \in \mathbb{R}^2 : \ \lambda > \tfrac{a^2}{4}\right\} \to \mathbb{R},$$

$$\begin{aligned} f(a, \lambda) &= \lambda \cos\left(\sqrt{\lambda - \tfrac{a^2}{4}}\right) \sin\left(\sqrt{\lambda - \tfrac{a^2}{4}}\right) \\ &\quad - \left(\sqrt{\lambda - \tfrac{a^2}{4}}\right)\left(\sinh^2\left(\tfrac{a}{2}\right) + \sin^2\left(\sqrt{\lambda - \tfrac{a^2}{4}}\right)\right) \\ &= \frac{1}{2}\sqrt{\lambda - \tfrac{a^2}{4}}\left\{2\lambda \frac{\sin\sqrt{4\lambda - a^2}}{\sqrt{4\lambda - a^2}} + \cos\sqrt{4\lambda - a^2} - \cosh(a)\right\}. \end{aligned} \tag{3.5}$$

We will show first that for any $k \in \mathbb{N}$

$$f(a, k^2\pi^2) < 0 \quad \text{for} \quad a \in (-2k\pi, 2k\pi) \setminus \{0\}. \tag{3.6}$$

For this purpose we consider for $k \in \mathbb{N}$, $|a| < 2k\pi$:

$$g_k(a) := \frac{2\, f(a, k^2\pi^2)}{\sqrt{k^2\pi^2 - \frac{a^2}{4}}} = 2k^2\pi^2 \frac{\sin\sqrt{4k^2\pi^2 - a^2}}{\sqrt{4k^2\pi^2 - a^2}} + \cos\sqrt{4k^2\pi^2 - a^2} - \cosh(a).$$

For $a \neq 0$ we always have $\cos(\,.\,) - \cosh(a) < 0$, and for $a^2 \leq (4k-1)\pi^2$, there holds:

$$\sqrt{4k^2\pi^2 - a^2} \in [(2k-1)\pi, 2k\pi]\,.$$

Hence the first term of $g_k(a)$ is here nonpositive, too, and it remains to consider $(4k-1)\pi^2 < a^2 < 4k^2\pi^2$. Here we have:

$$g_k(a) \leq 2k^2\pi^2 + 1 - \frac{1}{2}e^{|a|} \leq 2k^2\pi^2 + 1 - \frac{1}{2} - \frac{1}{2}|a| - \frac{1}{4}a^2 - \frac{1}{48}a^4$$

$$\leq 2k^2\pi^2 + \frac{1}{2} - \frac{1}{2}\sqrt{7}\pi - \left(k - \frac{1}{4}\right)\pi^2 - \frac{\pi^2}{48}(4k-1)^2\pi^2$$
$$< 2k^2\pi^2 - \left(k - \frac{1}{4}\right)\pi^2 - \frac{\pi^2}{96}\left(4 - \frac{1}{k}\right)^2 \left(2k^2\pi^2\right).$$

If we assume further that $k \geq 2$, we may conclude

$$\frac{\pi^2}{96}\left(4 - \frac{1}{k}\right)^2 \geq \frac{49\pi^2}{192} > 1$$

and hence $g_k(a) < 0$.

If $k = 1$, for $3\pi^2 < a^2 < 4\pi^2$ we obtain from the estimate above:

$$g_1(a) < \left(2 - \frac{3}{4} - \frac{3\pi^2}{16}\right)\pi^2 < 0.$$

We show now that $f(\lambda, a) < 0$ also for $|a|$ sufficiently small and $\lambda < k^2\pi^2$ and close to $k^2\pi^2$. In this situation the first term in curly brackets in (3.5) is negative as well as the sum of the second and third term.

That means that any connected component of the eigenvalue curves

$$\{(a, \lambda) : \ f(a, \lambda) = 0\}$$

can be found in a suitable slice ($k \in \mathbb{N}$):

$$\{(a, \lambda) : \ \lambda > \frac{a^2}{4}, \ k^2\pi^2 < \lambda < (k+1)^2\pi^2\}. \qquad \square$$

A MAPLE-plot gives a more detailed impression of the shape of the components of these eigenvalue curves, see Figure 5.

In particular, the intervals $\{0\} \times (\lambda_{2k}, \lambda_{2k+1})$, $k \in \mathbb{N}$, where the Green function is also oscillating, can be connected in the a-λ-plane with the subregion $\{(a, \lambda) : \ \lambda \leq 0\}$ of positivity without intersecting any eigenvalue curve. At the first glance this is somehow unexpected, and it means that the mechanisms, which lead to the sign changing and oscillatory behaviour of the Green function in the above mentioned intervals depends on the path, with which one connects these intervals with the halfplane $\{(a, \lambda) : \ \lambda \leq 0\}$. We will look at this phenomenon more closely in the following section.

A further interesting point is also that positive and sign changing eigenfunctions (corresponding to λ_{2k-1} and λ_{2k}) are on the same connected component of the eigenvalue curves. At which point (a, λ), starting with the positive eigenfunction for $(0, \lambda_{2k-1})$, can change of sign be observed for the first time? Which special feature characterizes the corresponding eigenfunction? Also these questions and their connections to the problems mentioned just before will be addressed in the next section.

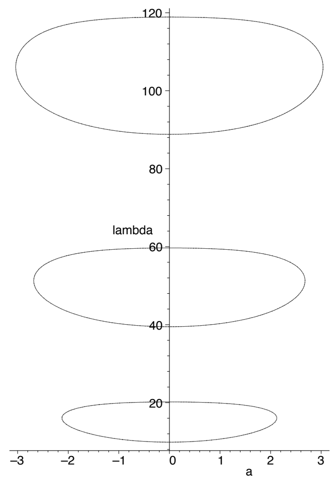

FIGURE 5. Eigenvalue curves

4. Transition from positivity to sign change, behaviour of the resolvent

In Section 2 the sign behaviour of the resolvent in the selfadjoint case $a = 0$ seemed to be quite similar as for related second order problems: positivity for $\lambda < \lambda_1$ and sign change for $\lambda > \lambda_1$. And the number of oscillations of the Green function G_λ increases, whenever one passes across a further eigenvalue.

But after we have taken into account the skew symmetric term $a\,u'''$ and discussed the shape of the eigenvalue curves in the a-λ-plane in the preceding section, this point of view seems no longer adequate. The intervals $(\lambda_{2k}(a)|a = 0, \lambda_{2k+1}(a)|a = 0)$ on the λ-axis can now be reached from the point $(0, 0)$ and hence from the region of positivity by curves in the a-λ-plane without passing across an eigenvalue curve. That means that in our situation the transition from positivity to sign change must occur by a completely different mechanism.

It will turn out that *further* eigenvalue problems (and the corresponding eigenvalue curves in the a-λ-plane) have to be considered under *different* boundary conditions. As in the proof of Corollary 1 we again refer to a positivity criterion of J. Schröder, but now, we need the version for nonselfadjoint operators:

Lemma 1. **(**[13, Theorem 10]**)**. *For $(a, \lambda) \in \mathbb{R}^2$ we consider the boundary value problem*

$$Lu := u'''' + au''' + \lambda u'' = f \quad in \quad (-1, 1), \tag{4.1}$$

$$u(-1) = u'(-1) = u(1) = u'(1) = 0. \tag{4.2}$$

Suppose that some $(a_0, \lambda_0) \in \mathbb{R}^2$ can be connected with $(0, 0)$ by a smooth curve such that for every (a, λ) on this curve, the eigenvalue problem

$$L\,u = 0 \tag{4.3}$$

under either Dirichlet boundary conditions (4.2) *or one of the unsymmetric boundary conditions*

$$u(-1) = u'(-1) = u''(-1) = u(1) = 0 \tag{4.4}$$

or

$$u(-1) = u(1) = u'(1) = u''(1) = 0 \tag{4.5}$$

only has the trivial solution.

Then the boundary value problem (4.1), (4.2) *is positivity preserving for (a_0, λ_0) and any (a, λ) on the connecting curve, i.e., here one has:*

$$f \geq 0 \Rightarrow u \geq 0.$$

To see that this lemma fits into the original formulation of [13, Theorem 10], one has to consider $\tilde{L}u = (\tilde{a}u'')'' - (\tilde{b}u')' + \beta u'$ with $\tilde{a} = \exp\left(\frac{a}{2}x\right)$, $\tilde{b} = (\frac{a^2}{4} - \lambda)\exp(\frac{a}{2}x)$ and $\beta = \tilde{b}' = \frac{a}{2}(\frac{a^2}{4} - \lambda)\exp(\frac{a}{2}x)$. This transformation was introduced in [12].

On any curve connecting $(0, 0)$ and any point in

$$\{0\} \times (\lambda_{2k}(a)|a = 0, \quad \lambda_{2k+1}(a)|a = 0)\,,$$

which does not intersect the eigenvalue curves of the preceding section, we know by Theorem 1, that transition from positivity to sign change occurs. Such curves exist indeed according to the proof of Theorem 2. Lemma 1 shows (after some lengthy, tedious but nevertheless elementary calculations) that this transition must occur in a point of intersection with the set

$$\Big\{(a, \lambda) \in \mathbb{R}^2 : \ \lambda > \frac{a^2}{4} \text{ and } g(a, \lambda) \cdot g(-a, \lambda) = 0\Big\}, \tag{4.6}$$

where

$$g(a, \lambda) = \left(2\lambda - a^2\right)\sin\sqrt{4\lambda - a^2} + a\sqrt{4\lambda - a^2}\cos\sqrt{4\lambda - a^2}$$
$$- (2\lambda + a)\sqrt{4\lambda - a^2}\exp(-a).$$

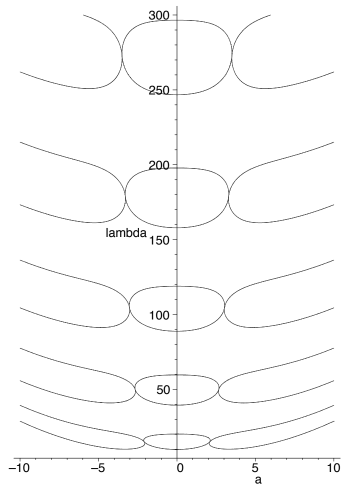

FIGURE 6. Eigenvalue curves and curves of sign change

The nodal set (4.6) is plotted in Figure 6 together with the eigenvalue curves $\{(a, \lambda) : \ f(a, \lambda) = 0\}$; the latter can already be seen in Figure 5. Here f is defined in (3.5).

Actually we think that it happens precisely that the number of oscillations of the Green function increases, whenever a curve in the a-λ-plane crosses this nodal set in the direction of increasing λ.

This picture as well as some MAPLE experiments with eigenfunctions, which can be determined explicitly – up to finding zeroes of a transcendental equation –, suggest the following conjecture:

Conjecture 1.

- *Any component of the "sign change curves" $\{g(a, \lambda) = 0\}$ intersects exactly one component of the "eigenvalue curves" $\{f(a, \lambda) = 0\}$ tangentially in precisely one point. This tangent is not parallel to the λ-axis.*
- *When moving on the k-th component of the "eigenvalue curve" $\{f(a, \lambda) = 0\}$ and starting in $(0, \lambda_{2k-1}(a)|a = 0)$, the corresponding eigenfunction of (3.2) is of fixed sign precisely until one reaches the intersection point just mentioned. In this intersection point, the corresponding eigenfunction has a boundary degeneracy $u''(1) = 0$ or $u''(-1) = 0$ resp. If $k \geq 2$, it seems that the skew term takes the local*

minima of the eigenfunctions up, so that they become even strictly positive on this part of the eigenvalue curve.

- *When proceeding further on this component of the "eigenvalue curve" to* $(0, \lambda_{2k}(a)|a = 0)$, *– in the case* $k \geq 2$ *– additional nodes of the respective eigenfunctions arise in the* interior *of the interval* $(-1, 1)$*. That may happen by local minima of these eigenfunctions, which were originally positive, becoming smaller and finally negative.*

5. Transition to sign change in a family of two dimensional ellipses

Quite similar positivity and sign change phenomena arise in a two dimensional problem, which looks at the first glance rather different from (1.1) and our model problem (1.8). Already Boggio [1] and Hadamard [8] were interested in positivity properties of the clamped plate boundary value problem

$$\begin{cases} \Delta^2 u = f \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \\ u|\partial\Omega = \nabla u|\partial\Omega = 0. \end{cases} \tag{5.1}$$

As before we call (5.1) *positivity preserving*, if always $f \geq 0 \Rightarrow u \geq 0$. (More precisely: it's the solution operator which preserves positivity.) Like Boggio and Hadamard we are interested in, for which $\Omega \subset \mathbb{R}^2$ the clamped plate boundary value problem (5.1) is indeed positivity preserving.

It is known that the answer is affirmative, if Ω is the unit disk B (see [1, p. 126]), and that it is negative if Ω is e.g. an Ellipse E_1 with half axes 1 and ≈ 1.6, see [5]. For a more extensive historical survey and bibliography we refer to [6]. However, if we connect $E_0 := B$ and E_1 by the smooth family of ellipses

$$E_t := \left\{(x_1, x_2) : \ x^2 + \left(\frac{y}{1 + 0.6\,t}\right)^2\right\},$$

we will have transition from positivity to sign change. We will show that (5.1) can be transformed into a problem, which indeed resembles the one-dimensional model problem (1.8) so much that we would like to suggest:

> The transition from positivity to sign change in the clamped plate boundary value problem (5.1), when deforming the disk B via the family E_t into the ellipse E_1, is caused by the same mechanisms as in the problem (1.8), when passing with a curve in the a-λ-plane across the "sign change curves" $\{g(a, \lambda) = 0\}$ in (4.6).

We feel that positivity criteria like in [12, 13, 14] should also hold for the clamped plate boundary value problem e.g. on families of ellipses $(E_t)_{t\in(0,1)}$ and think that the following explanations give strong support to it. Let $h_t : B \to E_t$ be biholomorphic mappings according to the Riemann mapping theorem. The explicit description is somehow delicate (see [15]). However, in [6, 10] it was shown (also in a much more general setting) that these mappings can be chosen such that for any k

$$\|h_t - h_{t_0}\|_{C^k(\bar{B})} \to 0 \quad \text{as} \quad t \to t_0.$$

We keep t fixed and omit that index for a moment. We assume that u solves (5.1) in E_t. We pull this solution back to the unit disk

$$v : B \to \mathbb{R}, \qquad v(x) := (u \circ h)(x),$$

and find the following boundary value problem for v:

$$\begin{cases} \left(\dfrac{1}{|h'|^2}\Delta\right)^2 v = f \circ h & \text{in } B, \\ v = \nabla v = 0 & \text{on } \partial B. \end{cases}$$

We determine the explicit form of this operator; thereby we use the complex notation $x = x_1 + ix_2$, $\frac{\partial}{\partial x} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$, $h' = \frac{\partial}{\partial x}h$, etc.

$$\begin{aligned} Lv &:= |h'|^4\big(\frac{1}{|h'|^2}\Delta\big)^2 v = |h'|^2\Delta\big(\frac{1}{|h'|^2}\Delta v\big) = 4|h'|^2\frac{\partial}{\partial x}\frac{\partial}{\partial \bar{x}}\big(\frac{1}{|h'|^2}\Delta v\big) \\ &= 4|h'|^2\frac{\partial}{\partial x}\big(-\frac{h'\,\overline{h''}}{(h'\overline{h'})^2}\Delta v + \frac{1}{h'\overline{h'}}\frac{\partial}{\partial \bar{x}}\Delta v\big) \\ &= \Delta^2 v - \frac{2}{|h'|^2}\big(h'\overline{h''}\big((\Delta v)_{x_1} - i(\Delta v)_{x_2}\big) + \overline{h'}h''\big((\Delta v)_{x_1} + i(\Delta v)_{x_2}\big)\big) \\ &+ 4\frac{|h''|^2}{|h'|^2}\Delta v \\ &= \Delta^2 v - 4\frac{\operatorname{Re}\big(h'\overline{h''}\big)}{|h'|^2}(\Delta v)_{x_1} - 4\frac{\operatorname{Im}\big(h'\overline{h''}\big)}{|h'|^2}(\Delta v)_{x_2} + 4\frac{|h''|^2}{|h'|^2}\Delta v. \end{aligned}$$

Hence, when pulling back the clamped plate boundary value problem (5.1) from the ellipses E_t to the unit disk B with help of the holomorphic mappings h_t, we come up with the following operators under homogeneous Dirichlet boundary conditions:

$$\begin{cases} L_t v := \Delta^2 v + \sum_{j=1}^2 a_{t,j}(x)\,(\Delta v)_{x_j} + \lambda_t(x)\Delta v & \text{in } B, \\ v = \nabla v = 0 & \text{on } \partial B. \end{cases} \tag{5.2}$$

Here, the coefficients are given by:

$$a_{t,1}(x) = -4\frac{\mathrm{Re}\,(h_t'\overline{h_t''})}{|h_t'|^2}, \qquad a_{t,2}(x) = -4\frac{\mathrm{Im}\,(h_t'\overline{h_t''})}{|h_t'|^2}, \qquad \lambda_t(x) = 4\frac{|h_t''|^2}{|h_t'|^2}.$$

This differential operator has indeed the same form as in (1.1) and hence we feel that our considerations concerning the model problem (1.8) should be applicable. We expect the modulus of the coefficients $a_{t,j}$, λ_t to grow and to tend to infinity as $t \to \infty$, if we extend our family of ellipses to arbitrarily eccentric ones.

That means that we are here in a situation parallel to Sect. 4. Although there the reasoning is strictly one dimensional, we think that similar mechanisms also apply and similar positivity criteria should hold true also in the two (and higher) dimensional problems (1.1), (5.1), (5.2). In particular, somehow unexpectedly, in (5.2) transition from positivity to sign change will occur without passing any kind of "eigenvalue curves".

To conclude and to sum up the "key message" of the present paper: In problems like (1.1), the difference between Dirichlet conditions (1.3) and Navier conditions (1.4) is not only small and of technical nature but substantial, and many fundamental properties do not carry over from Navier to Dirichlet conditions. Instead, new and (in our opinion) completely unexpected phenomena arise.

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