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ON A CLASS OF PARABOLIC EQUATIONS WITH VARIABLE DENSITY AND ABSORPTION*

Robert Kersner
Computer and Automation Institute, Hungarian Academy of Sciences
H-1518 Budapest, Hungary

Guillermo Reyes and Alberto Tesei
Dipartimento di Matematica "G. Castelnuovo," Università di Roma
"La Sapienza," P.le A. Moro 5, I-00185 Roma, Italy

(Submitted by: Juan Luis Vazquez)

Abstract. We investigate qualitative properties of solutions to the Cauchy problem for the equation $\rho(x)u_t = (u^m)_{xx} - c_0u^p$, where $m > 1$ and $c_0, p > 0$; the initial data are nonnegative with compact support and the density $\rho(x) > 0$ satisfies suitable decay conditions as $|x| \to \infty$. If $p \geq m$ and $\rho(x)$ decays not faster than $|x|^{-k}$, where $0 < k \leq k^* := 2(p-1)/(p-m)$, the interfaces exist globally in time. On the contrary, if $\rho(x)$ decays faster than $|x|^{-k}$ with $k > k^*$, the interfaces can disappear in finite time. It is also proved that solutions go to zero uniformly as $t \to \infty$, at variance from the case $c_0 = 0$.

1. Introduction

This paper is devoted to investigating support properties of solutions to the Cauchy problem

$$\begin{cases} \rho(x)u_t = (u^m)_{xx} - c_0u^p & \text{in } \mathbb{R} \times (0, \infty) \\ u = u_0 & \text{in } \mathbb{R} \times \{0\}, \end{cases} \tag{1.1}$$

where $m > 1$ and c_0, p are positive constants. The Cauchy data u_0 are nonnegative, continuous and compactly supported. A typical choice for $\rho = \rho(x)$ is

$$\rho(x) = \frac{\bar{\rho}}{(1 + |x|)^k} \tag{1.2}$$

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($\bar{\rho}, k > 0$; precise assumptions concerning ρ are made in Section 2). We refer the reader to [11], [12], [7] and references therein for physical motivations of the model.

Let $u \geq 0$ be any solution to problem (1.1) (solutions are always meant in the weak sense; see Section 2 for a precise definition); its *interfaces* are defined as follows:

$$\zeta^+(t) := \sup\{x : u(x,t) > 0\}, \quad \zeta^-(t) := \inf\{x : u(x,t) > 0\} \qquad (t \geq 0).$$

As is well known, interfaces exist for any time when the density ρ is constant. Moreover, in this case we have the following (see [2], [3], [8]):

- The following estimates are known to hold:

$$\begin{array}{lll} |\zeta^\pm(t)| \leq \mathit{constant} & \text{for} & 1 \leq p < m, \\ |\zeta^\pm(t)| \sim \log t & \text{for} & p = m, \\ |\zeta^\pm(t)| \sim t^\alpha, \quad \alpha > 0 & \text{for} & p > m. \end{array}$$

  Hence we speak of *localization* of the solution if $p < m$, respectively of *positivity* if $p \geq m$.
- If $0 < p < 1$, there is *extinction* of the solution in finite time—namely, there exists $T^* \in (0, \infty)$ such that $u \equiv 0$ in $\mathbb{R} \times (T^*, \infty)$.

Let us mention that the case $\rho = 1$, $c_0 = c_0(x)$ was investigated in [14] and [15].

New interesting phenomena arise when the density depends on the space variable, as was shown in [10] for the case $c_0 = 0$. In this case, if ρ goes to zero sufficiently fast as $|x| \to \infty$, interfaces can disappear in finite time—namely, there possibly exists $\bar{T} \in (0, \infty)$ such that $|\zeta^\pm| \to \infty$ as $t \to \bar{T}^-$ (hence we speak of *blow-up* of interfaces, or of *supp* u). For the choice (1.2) of ρ blow-up occurs if and only if $k > 2$; in fact, for any $k \leq 2$ there exist $a_0, b_0 > 0$ such that

$$|\zeta^\pm(t)| \sim a_0 t^{\frac{1}{2-k}} \quad \text{as} \quad t \to \infty \quad \text{if} \quad k < 2, \tag{1.3}$$

$$|\zeta^\pm(t)| \sim e^{b_0 t} \quad \text{as} \quad t \to \infty \quad \text{if} \quad k = 2, \tag{1.4}$$

(see [10]; see also [5], [6], [7], [13]). The appearence of the *critical value* $k = 2$ for the case $c_0 = 0$ is better understood by observing that a similarity solution to the equation

$$|x|^{-k} u_t = (u^m)_{xx} \tag{1.5}$$

is given by $u(x,t) = f(xt^{-\frac{1}{2-k}})$ if $k \neq 2$, respectively by $u(x,t) = f(x/e^{-\beta t})$ if $k = 2$ ($\beta \in \mathbb{R}$).

Let us also mention a related result, concerning the asymptotic behaviour of solutions to (1.1) in the case $c_0 = 0$: if $\rho \in L^1(\mathbb{R})$, there holds

$$u(\cdot,t) \to \overline{u} = const > 0 \qquad \text{as} \quad t \to \infty, \tag{1.6}$$

the convergence being uniform on compact subsets of $\mathbb{R}$ (see [7] and [12]).

It is the purpose of the present paper to investigate how absorption (i.e., the term $c_0 u^p$ in (1.1), $c_0 > 0$) influences the behaviour of solutions to problem (1.1). In other words, we address the following question: which phenomena in the above picture are structurally stable with respect to the parameter $c_0 \geq 0$?

In the following a fairly complete answer to the above question is provided. As in the case of constant ρ, there is localization of the solution if $p < m$, positivity if $p > m$. In case of positivity, the qualitative behaviour of interfaces turns out to be the same as in the case $c_0 = 0$—yet with a different critical value of k, namely

$$k^* := 2\frac{p-1}{p-m}, \tag{1.7}$$

if ρ satisfies (1.2) or more general conditions in the same spirit (see (2.13) and (2.18) below). On the other hand, the convergence result (1.6) is not structurally stable; in fact, solutions to problem (1.1) go to zero uniformly as $t \to \infty$ for any $c_0 > 0$ (moreover, there is extinction of the solution in finite time if $0 < p < 1$; see Theorem 2.5). Concerning the critical value k^* (which arises for $p > m$), observe in analogy with the case $c_0 = 0$ that the function $u(x,t) = t^\alpha f(x/t^\beta)$ with

$$\alpha := -\frac{2}{(p-m)(k^*-k)} \qquad , \qquad \beta := \frac{1}{k^*-k}, \tag{1.8}$$

is a similarity solution to the equation

$$|x|^{-k} u_t = (u^m)_{xx} - c_0 u^p \tag{1.9}$$

if $k \neq k^*$. If $k = k^*$, the function $u(x,t) = e^{\alpha t} f(x/e^{\beta t})$ is a similarity solution to the above equation for any $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha(p-m) = -2\beta. \tag{1.10}$$

Observe also that $k^* \to 2$ as $p \to \infty$.

In Section 2 we gather some preliminary material and state the main results. In Section 3 we prove localization in the (comparatively easy) range $p < m$, positivity in the range $p > m$. Section 4 is devoted to the range $m \leq p$, $k \leq k^*$; this case is characterized by global existence of the interfaces.

Finally, in Section 5 we concentrate on the blow-up case, which corresponds to the range $k > k^*$.

2. Mathematical framework and results

The basic theory, i.e., well-posedness, comparison and regularity results for initial–boundary value problems related to the equation in (1.1) was studied in [16]. For the reader's convenience, we reproduce below the results which are needed in the sequel.

We consider subsets of $S^T := \mathbb{R} \times (0,T]$ of the following three types:

$$D^T := \{(x,t) \in S^T : \xi_1(t) < x < \xi_2(t)\}, \tag{2.1}$$

$$D_+^T := \{(x,t) \in S^T : \xi_1(t) < x\}, \quad D_-^T := \{(x,t) \in S^T : x < \xi_2(t)\}, \tag{2.2}$$

where $\xi_i' \in Lip[0,T]$ and $\xi_1 < \xi_2$ in $[0,T]$. Let $\psi_i \in C[0,T]$ be nonnegative functions $(i = 1,2)$. By ψ_0, ψ_{0+} and ψ_{0-} we denote nonnegative, bounded and continuous functions defined on $[\xi_1(0), \xi_2(0)]$ (respectively on $[\xi_1(0), +\infty)$ and $(-\infty, \xi_2(0)]$). Moreover, we impose the compatibility conditions

$$\psi_0(\xi_i(0)) = \psi_i(0) \qquad (i = 1,2)$$
$$\psi_{0+}(\xi_1(0)) = \psi_1(0), \quad \psi_{0-}(\xi_2(0)) = \psi_2(0).$$

Definition 2.1. *A* weak bounded solution *to the first boundary value problem*

$$\begin{cases} \rho(x)u_t = (u^m)_{xx} - c_0 u^p & \text{in } D^T \\ u(x,0) = \psi_0(x) & x \in [\xi_1(0), \xi_2(0)] \\ u(\xi_i(t),t) = \psi_i(t) & t \in [0,T] \quad (i = 1,2) \end{cases} \tag{2.3}$$

is any function u nonnegative and continuous on $\overline{D}^T$, such that

$$\begin{aligned} \iint_{D^\tau} \{\rho u \phi_t + u^m\, \phi_{xx} - c_0 u^p\, \phi\} dxdt &= \int_{\xi_1(\tau)}^{\xi_2(\tau)} \rho u(x,\tau)\phi(x,\tau)dx - \int_{\xi_1(0)}^{\xi_2(0)} \rho\psi_0(x)\phi(x,0)dx \\ &+ \int_0^\tau [u^m(\xi_2(t),t)\phi_x(\xi_2(t),t) - u^m(\xi_1(t),t)\phi_x(\xi_1(t),t)]dt \end{aligned}$$

for any $\phi \in C_{x,t}^{2,1}(\overline{D}^T), \phi \geq 0$ such that $\phi(\xi_i(t),t) = 0$ in $[0,T]$ and any $\tau \in [0,T]$. Weak, bounded subsolutions of (2.3) are defined replacing "=" by "≥" in the above equality.

Definition 2.2. *By a* bounded weak solution (subsolution) *to the Cauchy–Dirichlet problem,*

$$\begin{cases} \rho(x)u_t = (u^m)_{xx} - c_0 u^p & \text{in } D_+^T \\ u(x,0) = \psi_{0+}(x), & x \in [\xi_1(0), +\infty) \\ u(\xi_1(t),t) = \psi_1(t), & t \in [0,T], \end{cases} \tag{2.4}$$

we mean any bounded function u_+, nonnegative and continuous on $\overline{D}_+^T$, which is a solution (respectively subsolution) to (2.3) in $D_+^T \cap \{x < r + \xi_1(t), 0 < t \le T\}$ with $\psi_0 := \psi_{0+}|_{[\xi_1(0), r+\xi_1(0)]}$ and $\psi_2 := u_+(r + \xi_1(t), t)$, for every $r > 0$.

In the same way we define weak solutions in regions D_-^T. Finally, we give the following

Definition 2.3. *By a* bounded weak solution (subsolution) *to problem (1.1) we mean any bounded, nonnegative and continuous on $\mathbb{R} \times [0, \infty)$ function u, which is a solution (respectively subsolution) to (2.3) in any rectangle $(-r, r) \times (0, T]$ with $\psi_0 := u_0|_{[-r,r]}$, $\xi_1 := -r$, $\xi_2 := r$, $\psi_1 := u(-r, t)$ and $\psi_2 := u(r, t)$, $r, T > 0$.*

Due to the fact that $s \to s^p$ is only right–Lipschitz continuous for $0 < p < 1$, in [1] a more restrictive concept of supersolution to the problem (2.3) in cylindrical domains was introduced. It was adopted in [16] for more general situations and amounts to the following: we say that v is a *weak supersolution* to any of the problems formulated above if it is a *weak solution* to the problem obtained by adding to the right-hand side of the equation a nonnegative bounded function $h(x,t)$ (see [1] and [16] for details).

Throughout the paper by a solution (respectively by a sub-, supersolution) to any of these problems we always mean a weak bounded solution (sub-, supersolution respectively) in the sense of the definitions given above; moreover, we always refer to nontrivial solutions. ρ, u_0 are always assumed to satisfy the following hypotheses:

$$(A) \quad \begin{cases} (i)\ \rho \in C^3(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad \rho > 0 \text{ in } \mathbb{R}; \\ \\ (ii)\ u_0 \in C(\mathbb{R}), \quad u_0 \ge 0 \ \text{ in } \ \mathbb{R}, \quad \mathit{supp}\ u_0 \text{ compact.} \end{cases}$$

We further assume $u_0(0) > 0$; thus by continuity $u_0 \ge \mu > 0$ in a neighborhood of the origin; there is no loss of generality in doing so, since all assumptions concerning the function ρ (see (2.13) and (2.18)) are invariant under translations and reflections. Moreover, in the following any statement concerning interfaces will be proved only for the right interface.

In [16] the following results have been proved.

Theorem 2.1. *(Well-posedness) If* $m > 1$, $p > 0$, $\rho \in C^3(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\rho > 0$, ξ_i $(i = 1, 2)$, ψ_i $(i = 0, 1, 2)$, $\psi_{0\pm}$ *satisfy the hypotheses stated above and* $u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, *then there exists a unique solution to each of the four above-posed problems.*

Theorem 2.2. *(Comparison) Let* $\underline{u}$ *be a subsolution,* $\overline{u}$ *a supersolution to any of the problems above. Then* $\underline{u} \le \overline{u}$.

Define

$$\mathcal{L}_0 u \equiv -\rho(x) u_t + (u^m)_{xx} - c_0 u^p; \qquad \Gamma_i = \{(\xi_i(t), t), t \in [0, T]\}. \tag{2.5}$$

When dealing with sub- and supersolutions the following simple criteria will be used (see [16] and [B, Lemmas 4.1–4.2]).

Theorem 2.3. *(i) Let* $\bar{u} \in C(\overline{D}^T) \cap C^{0,1}_{x,t}(D^T)$ *be bounded and nonnegative, and let* $\bar{u}^m \in C^{2,0}_{x,t}(D^T)$. *Moreover, let there hold*

$$\begin{cases} -M \le \mathcal{L}_0 \bar{u} \le 0 \quad \text{in} \ \ D^T, \\ \bar{u} \ge \psi_i \text{ on } \Gamma_i \ (i = 1, 2), \quad \bar{u}(x, 0) \ge \psi_0(x) \ \ \text{for} \ \ x \in [\xi_1(0), \xi_2(0)] \end{cases} \tag{2.6}$$

for some $M > 0$. *Then* $\bar{u}$ *is a supersolution to (2.3) in* D^T. *Analogous assertions are true if we replace* D^T *by* D^T_+, D^T_-, S^T.

(ii) Let $\underline{u}$ *satisfy the same assumptions above, yet with (2.6) replaced by*

$$\begin{cases} \mathcal{L}_0 \underline{u} \ge 0 \quad \text{in} \ \ D^T, \\ \underline{u} \le \psi_i \text{ on } \Gamma_i \ (i = 1, 2), \quad \underline{u}(x, 0) \le \psi_0(x) \text{ for } x \in [\xi_1(0), \xi_2(0)]. \end{cases} \tag{2.7}$$

Then $\underline{u}$ *is a subsolution to (2.3) in* D^T. *Analogous assertions are true if we replace* D^T *by* D^T_+, D^T_-, S^T.

Theorem 2.4. *(i) Let* $\bar{u} \in C^\alpha(\overline{D}^T)$, *and let* $\alpha > 1/m$ *be bounded and nonnegative. Let* $\gamma \in C^1[0, T]$ *be such that* $\xi_1(t) < \gamma(t) < \xi_2(t)$ *for every* $t \in [0, T]$. *Let either* $\bar{u} > 0$ *in* $D^T \cap \{x < \gamma(t)\}$ *and* $\bar{u} = 0$ *in* $D^T \cap \{x > \gamma(t)\}$ *or* $\bar{u} = 0$ *in* $D^T \cap \{x < \gamma(t)\}$ *and* $\bar{u} > 0$ *in* $D^T \cap \{x > \gamma(t)\}$. *Let* $\bar{u} \in C^{0,1}_{x,t}(D^T \setminus \gamma)$, $\bar{u}^m \in C^{2,0}_{x,t}(D^T \setminus \gamma)$. *Moreover, let there hold*

$$\begin{cases} -M \le \mathcal{L}_0 \bar{u} \le 0 \qquad \text{in} \ \ D^T \setminus \gamma, \\ \bar{u} \ge \psi_i \text{ on } \Gamma_i \ (i = 1, 2), \quad \bar{u}(x, 0) \ge \psi_0(x) \text{ for } x \in [\xi_1(0), \xi_2(0)] \end{cases} \tag{2.8}$$

for some $M > 0$. *Then* $\bar{u}$ *is a supersolution to (2.3) in* D^T. *Analogous assertions are true if we replace* D^T *by* D^T_+, D^T_-, S^T.

(ii) Let $\underline{u}$ *satisfy the same assumptions above, yet with (2.8) replaced by*

$$\begin{cases} \mathcal{L}_0\underline{u} \geq 0 \qquad \text{in} \;\; D^T \setminus \gamma, \\ \underline{u} \leq \psi_i \text{ on } \Gamma_i \; (i = 1, 2), \quad \underline{u}(x, 0) \leq \psi_0(x) \text{ for } x \in [\xi_1(0), \xi_2(0)]. \end{cases} \tag{2.9}$$

Then $\underline{u}$ *is a subsolution to (2.3) in* D^T. *Analogous assertions are true if we replace* D^T *by* D^T_+, D^T_-, S^T.

Let us now state the results of the present work. Our first result concerns a general decay property of solutions to problem(1.1).

Theorem 2.5. *(Asymptotic behaviour) Let* u *be any solution to problem* (1.1). *Then*

$$\|u(\cdot, t)\|_\infty \to 0 \qquad \text{as} \quad t \to \infty. \tag{2.10}$$

Moreover, if $0 < p < 1$ *there exists* $T^* \in (0, \infty)$ *such that* $u = 0$ *in* $\mathbb{R} \times (T^*, \infty)$.

If $p < m$, solutions are localized for any admissible density ρ, much as in the case $\rho = constant$; this is the content of the following theorem.

Theorem 2.6. *(Localization) Let* $p < m$. *Then for any solution* u *to problem* (1.1) *there exists* $L > 0$ *(depending on* m, p, c_0, u_0*) such that*

$$|\zeta^\pm(t)| \leq L \qquad \text{for any} \quad t \geq 0. \tag{2.11}$$

In constrast with the above situation, if $p > m$ the following positivity result can be proved.

Theorem 2.7. *(Positivity) Let* $p > m$. *Then for any solution* u *to problem (1.1) there exist* $a, b > 0$ *(depending on* $m, p, \|\rho\|_\infty, c_0 and u_0$*) such that*

$$|\zeta^\pm(t)| \geq b[\log(at + 3)]^{1/2} \qquad \text{for any} \quad t \geq 0. \tag{2.12}$$

Now a natural question arises: when positivity prevails, does the support of the solution remain bounded for any positive time? As in the case $c_0 = 0$, the dependence of the support on time is crucially influenced by the decay rate of the density ρ as $|x| \to \infty$. If the exponent k in (1.2) is "small," both interfaces exist for any $t > 0$, as the following two theorems show.

Theorem 2.8. *(Global existence of interfaces:* $p > m$, $k < k^*$*) Let* $p > m$. *Moreover, let* ρ *satisfy the condition*

$$\frac{\rho_1}{(1 + |x|)^k} \leq \rho(x) \leq \rho_0, \qquad (\rho_0, \rho_1 > 0; \; x \in \mathbb{R}) \tag{2.13}$$

where $0 < k < k^* := 2(p-1)/(p-m)$. *Then there exists a positive constant* C_1 *(depending on* m, p, k, ρ_1, c_0 *and* u_0*) such that*

$$|\zeta^{\pm}(t)| \leq C_1 t^{\frac{1}{k^*-k}} \quad \textit{for any} \quad t \geq 0. \tag{2.14}$$

Observe that, since $k^* > 2$, the estimate (2.14) is sharper than the inequality

$$|\zeta^{\pm}(t)| \leq \bar{C}_1 t^{\frac{1}{2-k}} \qquad (k < 2; t \geq 0), \tag{2.15}$$

which holds if $c_0 = 0$ (see [10]). Clearly, this is due to the presence of the absorption term.

Theorem 2.9. *(Global existence of interfaces:* $p > m$, $k = k^*$*) Let* $p > m$. *Moreover, let* ρ *satisfy condition (2.13) with* $k = k^*$. *Then there exist* $C_2, \beta > 0$ *(depending on* m, p, k, ρ_1, c_0 *and* u_0*) such that*

$$|\zeta^{\pm}(t)| \leq C_2 e^{\beta t} \quad \textit{for any} \quad t \geq 0. \tag{2.16}$$

In contrast with the previous situation, we prove below that the interfaces can blow up in finite time if the exponent k is sufficiently large (namely, if $k > k^*$). More precisely, we consider the following class of initial data (satisfying assumption (A)–(ii)):

$$\mathcal{M}_{b,h} := \left\{ u_0 : \frac{m}{m-1} u_0^{m-1}(x) \geq h\big[1 - (|x|/b)\big]_+ \right\} \quad (b, h > 0; x \in \mathbb{R}) \tag{2.17}$$

(here $[r]_+ := \max\{r, 0\}, r \in \mathbb{R}$). Then the following result can be stated.

Theorem 2.10. *(Blow-up of interfaces) Let* $p > m$. *Moreover, let* ρ *satisfy the inequalities*

$$\frac{\rho_1}{(1+|x|)^k} \leq \rho(x) \leq \frac{\rho_2}{(1+|x|)^k}, \qquad (\rho_1, \rho_2 > 0;\ x \in \mathbb{R}), \tag{2.18}$$

with $k > k^*$. *Then for any* $h > 0$ *there exists* $b_0 = b_0(h) > 0$ *(depending on* c_0, ρ_1, ρ_2, k, p, m *and* u_0*) such that, if* $u_0 \in \mathcal{M}_{b,h}$ *with* $b > b_0$, *then*

$$u(x,t) > 0 \ \ \textit{for any} \quad x \in \mathbb{R},\ t > 1. \tag{2.19}$$

Let us finally state the following result, concerning global existence of interfaces in the case $p = m$.

Theorem 2.11. *(Global existence of interfaces:* $p = m$*) Let* $p = m$. *Moreover, let* ρ *satisfy condition (2.13) with* $k > 0$. *Then for any* $\beta > 0$ *there exists* $C_3 > 0$ *(depending on* m, k, ρ_1, c_0 *and* u_0*) such that*

$$|\zeta^{\pm}(t)| \leq C_3 t^{\beta} \qquad \textit{for any} \quad t \geq 0. \tag{2.20}$$

The above results will be proved by comparing solutions to problem (1.1) with suitable explicit sub- and supersolutions (possibly together with a proper splitting of domains). Let us observe that the techniques used for the case $c_0 = 0$—which rely on mass conservation (see [10])—cannot be adapted to the present situation.

3. Localization and positivity

It is expedient for further purposes to deal with the *pressure*

$$v := \frac{m}{m-1} u^{m-1}, \tag{3.1}$$

u being any solution to problem (1.1). It is immediately seen that v satisfies the Cauchy problem

$$\begin{cases} \rho(x) v_t = (m-1) v v_{xx} + v_x^2 - c v^q & \text{in} \quad \mathbb{R} \times (0,\infty) \\ v = v_0 & \text{in} \quad \mathbb{R} \times \{0\}, \end{cases} \tag{3.2}$$

where

$$q := \frac{m+p-2}{m-1} \tag{3.3}$$

and

$$c := c_0 (m-1) \Big(\frac{m-1}{m} \Big)^{\frac{p-1}{m-1}}, \qquad v_0 := \frac{m}{m-1} u_0^{m-1}.$$

In the following we keep the notation introduced in (2.5). We denote by

$$\mathcal{L} v \equiv -\rho(x) v_t + (m-1) v v_{xx} + v_x^2 - c v^q \tag{3.4}$$

the differential operator related to the equation in (3.2). We also set

$$M_0 := \|u_0\|_\infty, \qquad \rho_0 := \|\rho\|_\infty.$$

As for problem (1.1), solutions, sub- and supersolutions to problem (3.2) are meant in the weak sense. Existence, uniqueness and comparison results for problem (3.2) are an immediate consequence of Theorems 2.1–2.2. Regarding Theorems 2.3 and 2.4, they remain valid if we replace $\mathcal{L}_0$ by $\mathcal{L}$, and condition $\bar{u}(\underline{u}) \in C^\alpha(\overline{D}^T)$ by $\bar{v}(\underline{v}) \in Lip(\overline{D}^T)$.

Let us first prove Theorem 2.5.

Proof of Theorem 2.5. Let us distinguish three cases.

(i) $p > 1$: Consider the space-independent auxiliary function

$$w(t) := a(1+t)^{-\frac{1}{p-1}}.$$

An easy calculation shows that

$$\mathcal{L}_0 w = a(1+t)^{-\frac{p}{p-1}}\Big[\frac{\rho(x)}{p-1} - c_0 a^{p-1}\Big].$$

Then, if

$$a > \max\{a_0, M_0\}, \qquad a_0 := \Big(\frac{\rho_0}{c_0(p-1)}\Big)^{\frac{1}{p-1}},$$

we have the inequalities

$$-c_0 a^p \le \mathcal{L}_0 w \le 0 \ \text{ in } \ \mathbb{R}\times(0,\infty),$$
$$w(x,0) \ge u_0(x) \qquad (x\in\mathbb{R}).$$

Then, the conclusion follows by Theorems 2.2 and 2.3, applied in strips S^T with arbitrary $T>0$.
(*ii*) $p=1$: The function $w(t) := ae^{-\frac{t}{b}}$ with $a > M_0$ and $b > \frac{\rho_0}{c_0}$ also meets the requirements of Theorems 2.2 and 2.3 in S^T, as is easily checked.
(*iii*) $p<1$: In this case a supersolution to (1.1) in S^1 is given by

$$w(t) := a[1-bt]_+^{\frac{1}{1-p}},$$

if $a > M_0$ and

$$b < c_0(1-p)a^{p-1}\rho_0^{-1}.$$

Observe that, in this case, $w(x, b^{-1}) \equiv 0$. □

Remark 3.1. Let us observe that Theorem 2.5 is valid for solutions with general (not necessarily compactly supported) initial data.

Let us now prove the localization result stated in Theorem 2.6.
Proof of Theorem 2.6. Consider the time-independent auxiliary function

$$w(x) := a\left[1-\frac{x}{L}\right]_+^{\frac{2}{m-p}},$$

where a and L are positive parameters. Since u_0 has compact support, there exist $a_0 > M_0$ and $L_0 > 0$ such that *supp* $u_0 \cap [0,\infty) \subset [0, L_0]$ and for $a = a_0$, $L = L_0$ there holds

$$u_0(x) \le w(x) \qquad \text{for any } \ x \in [0,\infty), \tag{3.5}$$

$$u(0,t) \le M_0 < a_0 = w(0) \qquad \text{for any } \ t\in[0,\infty). \tag{3.6}$$

Moreover, an easy calculation shows that

$$\mathcal{L}_0 w = a^m\left(\frac{2m(m+p)}{(m-p)^2L^2} - \frac{c_0}{a^{m-p}}\right)\left[1-\frac{x}{L}\right]_+^{\frac{2p}{m-p}}.$$

Choosing $a = a_0$ as above and $L = L_0$ possibly larger gives

$$-c_0 a_0^p \le \mathcal{L}_0 w \le 0 \qquad \text{in} \quad (0,\infty)\times(0,\infty). \tag{3.7}$$

It follows from (3.5), (3.6), (3.7) and Theorems 2.2 and 2.4, applied in $(0,\infty)\times(0,T]$ with arbitrarily large T that

$$u(x,t) \le w(x) \quad \text{for any} \ \ (x,t)\in[0,\infty)\times[0,\infty),$$

with $a = a_0$ and $L = L_0$ chosen above. Since $w = 0$ for $x > L_0$, it follows that *supp* $u(\cdot,t) \subseteq (-\infty, L_0)$ for any $t \ge 0$. Hence the conclusion. □

In order to prove the positivity result in Theorem 2.7, we need the following lemma.

Lemma 3.2. *Let $p \ge m$. Let v be the pressure defined in (3.1), where u denotes any solution to problem (1.1). Then there exist $a_1, C > 0$ (depending on m, p, c, ρ and u_0) such that*

$$v(0,t) \ge C(1+a_1 t)^{-1} \qquad \textit{for any} \ \ t \ge 0. \tag{3.8}$$

Proof. Consider the auxiliary function

$$w(x,t) := (1+at)^{-1}(A^2-x^2) \qquad ((x,t)\in[-A,A]\times[0,+\infty)),$$

where a and A are positive parameters. As already remarked (see Section 2), we can choose $v_0 > 0$ in a neighborhood of the origin; hence there exists $A > 0$ so small that

$$w(x,0) \le v_0(x) \qquad \text{for any} \ \ x\in[-A,A]. \tag{3.9}$$

Moreover, there holds

$$w(\pm A,t) = 0 \le v(\pm A,t) \qquad \text{for any} \ \ t \ge 0. \tag{3.10}$$

A direct computation gives

$$\begin{aligned}\mathcal{L}w = (1+at)^{-2}(A^2-x^2)\Big[a\rho(x) - 2(m-1)\\ - c(1+at)^{-q+2}(A^2-x^2)^{q-1}\Big] + 4(1+at)^{-2}x^2.\end{aligned}$$

Since the assumption $p \ge m$ implies $q \ge 2$ (see (3.3)), it is easily seen that

$$\mathcal{L}w \ge 0 \qquad \text{in} \quad (-A,A)\times(0,\infty), \tag{3.11}$$

if

$$a \ge a_1 := \frac{2(m-1)+cA^{2(q-1)}}{\min_{x\in[-A,A]}\rho(x)}.$$

From (3.9)–(3.11) and Theorems 2.2 and 2.3, applied in $(-A,A)\times(0,T]$ with arbitrary $T > 0$, the conclusion follows (with $C = A^2$). □

Remark 3.3. As for Theorem 2.5, Lemma 3.2 still holds for solutions with general bounded initial data. Observe that, according to this lemma, if $p \geq m$ solutions to problem (1.1) do not vanish in finite time.

Proof of Theorem 2.7. Consider the auxiliary function

$$w(x,t) := (at+3)^{-1} \left[b - x \log^{-1/2}(at+3) \right]_+ \qquad (x>0, t>0), \tag{3.12}$$

where a, b are positive parameters. Set for convenience

$$\theta := at+3, \qquad A := \left[b - x \log^{-1/2} \theta \right]_+ .$$

Consider the sets

$$H := \{(x,t) : x>0, t>0\}, \qquad H_1 := H \cap \{A > 0\}.$$

We shall prove the following claim: There exist $a > 0$ and $b > 0$ such that

$$v \geq w \quad \text{in } \ H, \tag{3.13}$$

where v denotes the pressure (see (3.1)). Then

$$H \cap \mathit{supp}\ u(\cdot,\tau) = H \cap \mathit{supp}\ v(\cdot,\tau) \supseteq H \cap \mathit{supp}\ w(\cdot,\tau)$$

for any $\tau \geq 0$. Since

$$H \cap \mathit{supp}\ w(\cdot,\tau) = \overline{H_1} \cap \{t = \tau\} = [0, b \log^{1/2}(a\tau+3)],$$

the conclusion follows.

To prove the above claim, observe preliminarily that in H_1 there holds

$$\begin{aligned} \mathcal{L}w &= a\rho\theta^{-2}A - \frac{1}{2}ax\rho\theta^{-2}\log^{-3/2}\theta + \theta^{-2}\log^{-1}\theta - c\theta^{-q}A^q \\ &\geq -\frac{1}{2}ax\rho\theta^{-2}\log^{-3/2}\theta + \theta^{-2}\log^{-1}\theta - c\theta^{-q}A^q =: I_1 + I_2 + I_3. \end{aligned}$$

Let us show that by a proper choice of $a, b > 0$ we can ensure

(a) $\quad I_1 + \frac{1}{2}I_2 \geq 0 \quad$ in $\ H_1$,

(b) $\quad \frac{1}{2}I_2 + I_3 \geq 0 \quad$ in $\ H_1$,

so that

$$\mathcal{L}w \geq 0 \quad \text{in} \quad H_1. \tag{3.14}$$

In fact, inequality *(a)* is satisfied if

$$ax\rho(x) \leq \log^{1/2}\theta.$$

Since by definition $0 < x < b\log^{1/2}\theta$ in H_1, the above inequality is satisfied if

$$ab \leq \frac{1}{\rho_0}.$$

On the other hand, inequality *(b)* reads

$$\tfrac{1}{2}\theta^{-2}\log^{-1}\theta \geq c\theta^{-q}A^{q} \qquad \text{in} \quad H_1.$$

Since by definition $A < b$, the above inequality is satisfied if

$$f(\theta) := \theta^{q-2}\log^{-1}\theta > 2cb^{q} \qquad \text{for any} \quad \theta \in [3, +\infty). \tag{3.15}$$

Since $p > m$, we have $q > 2$. Then it is easily checked that the function f is bounded from below by some positive constant $b_0 = b_0(q)$. Hence inequality (3.15) is satisfied if

$$b < (\frac{b_0}{2c})^{1/q}.$$

Let us now consider the values of the function (3.12) on the boundary

$$\partial H \quad = \{(0,t) : t \geq 0\} \cup \{(x,0) : x \geq 0\} =: \gamma_1 \cup \gamma_2.$$

On γ_1 we have

$$w(0,t) = \frac{b}{at+3} \leq \frac{C}{a_1 t + 3} \leq v(0,t)$$

by Lemma 3.2, provided that $a \geq a_1$, $b \leq C$. On γ_2 there holds

$$w(x,0) = \frac{1}{3}\Big[b - \frac{x}{\sqrt{\log 3}}\Big]_+ \leq v_0(x)$$

if b is sufficiently small, since $v_0(0) > 0$. Finally choose $a, b > 0$ such that all the above requirements are satisfied (this is possible, *e.g.* choosing $a = \varepsilon^{-1}$ and $b = \varepsilon^2$ with $\varepsilon > 0$ sufficiently small). Hence inequality (3.14) holds. Since $\mathcal{L}w = 0$ in $H \setminus \overline{H_1}$ and moreover

$$w \leq v \qquad \text{on} \quad \partial H,$$

the result follows from Theorems 2.2 and 2.4, applied to $H \cup \{t \leq T\}$ with arbitrary $T > 0$. □

4. Global existence of the interface

In order to prove Theorems 2.8 and 2.9 we need a preparatory lemma.

Lemma 4.1. *Let $p > m$; let u be any solution to problem (1.1). Then there exists $B > 0$ (depending on m, p, c_0 and u_0) such that*

$$u(x,t) \leq B(1+x)^{-\frac{2}{p-m}} \qquad \text{for any} \;\; x \geq 0, \; t \geq 0. \tag{4.1}$$

Proof. Consider the time-independent auxiliary function

$$w(x) := B(1+x)^{-\frac{2}{p-m}}, \qquad ((x,t) \in [0,\infty) \times [0,\infty)),$$

where $B > 0$ is a parameter. As in the proof of Lemma 3.2, we can choose $B > 0$ so large that

$$u_0(x) \le w(x) \qquad \text{for any} \;\; x \ge 0,$$
$$u(0,t) \le M_0 < B = w(0) \qquad \text{for any} \;\; t \ge 0.$$

On the other hand, there holds

$$\mathcal{L}_0 w = B^m (1+x)^{-\frac{2p}{p-m}} \left\{ \frac{2m(p+m)}{(p-m)^2} - c_0 B^{p-m} \right\},$$

so that

$$-c_0 B^p \le \mathcal{L}_0 w \le 0 \qquad \text{in} \;\; (0,\infty) \times (0,\infty)$$

if B is possibly larger. Then by Theorems 2.2 and 2.3, applied in $(0,\infty) \times (0,T]$, $T > 0$, the conclusion follows. □

Let us now prove Theorem 2.8.

Proof of Theorem 2.8. Consider the auxiliary function

$$w(x,t) := (1+at)^{-\alpha} \left[b^2 - \frac{x^2}{(1+at)^{2\beta}} \right]_+ \qquad (x>0,\; t>0), \tag{4.2}$$

where a, b, α and β are positive parameters. Set for convenience

$$\theta := 1 + at, \qquad A := \left[b^2 - \frac{x^2}{(1+at)^{2\beta}} \right]_+ .$$

Consider the set

$$G := \left\{ (x,t) : t > 0, x > \sqrt{\tfrac{\alpha}{\alpha+\beta}}\, b\theta^\beta \right\},$$

which has nonempty intersection with the set

$$H := \{(x,t) : x > 0,\; t > 0,\; A > 0\};$$

in fact,

$$G \cap H = \left\{ (x,\, t) : t > 0,\; \sqrt{\tfrac{\alpha}{\alpha+\beta}}\, b\theta^\beta < x < b\theta^\beta \right\}. \tag{4.3}$$

We shall prove the following claim: There exist $a, b, \alpha, \beta > 0$ such that

$$v \le w \qquad \text{in} \;\; G, \tag{4.4}$$

where v denotes the pressure defined in (3.1). This implies

$$\mathit{supp}\; u(\cdot,t) \cap [0,\infty) = \mathit{supp}\; v(\cdot,t) \cap [0,\infty) \subseteq [0, b\theta^\beta) \quad \text{for any } t \ge 0. \tag{4.5}$$

In particular, it is possible to choose $\beta = \frac{1}{k^*-k}$ in (4.5); then the conclusion follows.

In $G \cap H$ there holds

$$\begin{aligned}\mathcal{L}w \;&= \alpha a\rho(x)\theta^{-\alpha-1}A - 2\beta a\rho(x)x^2\theta^{-\alpha-2\beta-1} - \\ &-2(m-1)\theta^{-2\alpha-2\beta}A + 4x^2\theta^{-2\alpha-4\beta} - c\theta^{-\alpha q}A^q \\ &\le \alpha a\rho(x)\theta^{-\alpha-1}A - 2\beta a\rho(x)x^2\theta^{-\alpha-2\beta-1} + 4x^2\theta^{-2\alpha-4\beta} \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

It is immediately seen that

$$(a) \qquad I_1 + \tfrac{1}{2}I_2 \le 0 \qquad \text{in } \; G \cap H;$$

in fact,

$$I_1 + \frac{1}{2}I_2 = \frac{a\rho}{\theta^{\alpha+1}}\Big[\alpha b^2 - (\alpha+\beta)\frac{x^2}{\theta^{2\beta}}\Big].$$

In addition,

$$(b) \qquad \tfrac{1}{2}I_2 + I_3 \le 0 \qquad \text{in } \; G \cap H,$$

if the parameters $a, b, \alpha, \beta > 0$ are properly chosen. In fact, inequality *(b)* is equivalent to the following:

$$\rho(x) \ge \frac{4}{\beta a}\theta^{-\alpha-2\beta+1} \qquad \text{in} \quad G \cap H.$$

Due to condition (2.13) and to (4.3), the latter is satisfied if

$$(1 + b\theta^\beta)^{-k} \ge \frac{4}{\rho_1\beta a}\theta^{-\alpha-2\beta+1},$$

or equivalently if

$$\Big(\frac{b\theta^\beta}{1 + b\theta^\beta}\Big)^k \ge \frac{4b^k}{\rho_1\beta a}\theta^{\beta k-\alpha-2\beta+1}. \tag{4.6}$$

Assume

$$(k-2)\beta - \alpha + 1 \le 0; \tag{4.7}$$

then both members of (4.6) are monotonic functions of θ. It follows easily that (4.6) holds for any $\theta \ge 1$ if

$$\Big(\frac{b}{b+1}\Big)^k \ge \frac{4b^k}{\rho_1\beta a},$$

or equivalently if

$$a \ge \frac{4}{\beta\rho_1}(1+b)^k. \tag{4.8}$$

Hence if (4.7) and (4.8) are satisfied, both inequalities (a)–(b) above hold. It follows that

$$-2\beta a\rho_0 b^2 \le \mathcal{L}w \le 0 \qquad \text{in} \quad G \cap H. \tag{4.9}$$

Let us now show that

$$v \le w \qquad \text{on} \quad \partial G, \tag{4.10}$$

if additional requirements concerning a, b, α and β are satisfied. There holds

$$\partial G \;\; = \left\{(x,t) : t \ge 0,\; x = \sqrt{\tfrac{\alpha}{\alpha+\beta}} b\theta^{\beta}\right\} \cup$$

$$\cup \left\{(x,0) : x \ge \sqrt{\tfrac{\alpha}{\alpha+\beta}} b\right\} =: \gamma_1 \cup \gamma_2.$$

Due to Lemma 4.1, we have

$$v|_{\gamma_1}(x,t) \le \frac{mB^{m-1}}{m-1}\Big(1 + \sqrt{\tfrac{\alpha}{\alpha+\beta}} b\theta^{\beta}\Big)^{-\frac{2(m-1)}{p-m}};$$

moreover,

$$w|_{\gamma_1}(x,t) = \frac{\beta}{\alpha+\beta} b^2 \theta^{-\alpha}.$$

Choosing

$$\alpha = \frac{2(m-1)}{p-m}\beta \tag{4.11}$$

we have

$$v|_{\gamma_1} \le w|_{\gamma_1}, \tag{4.12}$$

provided that

$$b \ge \sqrt{\tfrac{m(\alpha+\beta)}{\beta(m-1)} B^{m-1}} =: b_1.$$

Observe that equality (4.11) is consistent with requirement (4.7) since $k < k^*$. In order to satisfy both (4.11) and (4.7) the optimal choice is

$$\alpha = \frac{2(m-1)}{(p-m)(k^* - k)}, \qquad \beta = \frac{1}{k^* - k}. \tag{4.13}$$

Finally, we can ensure

$$v|_{\gamma_2} \le w|_{\gamma_2}, \tag{4.14}$$

if $b \ge b_0$ for some suitable $b_0 > 0$, since v_0 has compact support. Now choose α and β as in (4.13), $b > \max\{b_0, b_1\}$ and a satisfying (4.8); then from (4.9), (4.12), (4.14) and the obvious equality

$$\mathcal{L}w = 0 \qquad \text{in} \quad G \setminus \overline{H},$$

the claim follows by Theorems 2.2 and 2.4, applied in $G \cap \{0 < t \le T\}$, $T > 0$. □

Let us now prove Theorem 2.9. Since the proof is similar to that of Theorem 2.8 above, we omit the details.

Proof of Theorem 2.9. Consider the auxiliary function

$$w(x,t) := e^{-\alpha t}\left[b^2 - x^2 e^{-2\beta t}\right]_+ =: \theta^{-\alpha} A, \tag{4.15}$$

where α, β and b are positive parameters. Define the sets G and H as in the proof of Theorem 2.8, yet with $\theta := e^t$ and $A := [b^2 - x^2 e^{-2\beta t}]$. Again we want to prove that

$$v \le w \qquad \text{in} \quad G, \tag{4.16}$$

if α, β and b are properly chosen.

In the set $G \cap H$ there holds

$$\begin{aligned} \mathcal{L}w \ &= \alpha\rho(x)\theta^{-\alpha}A - 2\beta x^2 \rho(x)\theta^{-\alpha-2\beta} - \\ &-2(m-1)\theta^{-2\alpha-2\beta}A + 4x^2\theta^{-2\alpha-4\beta} - c\theta^{-\alpha q}A^q \\ &\le \alpha\rho(x)\theta^{-\alpha}A - 2\beta x^2\rho(x)\theta^{-\alpha-2\beta} + 4x^2\theta^{-2\alpha-4\beta} =: I_1 + I_2 + I_3. \end{aligned}$$

Again the inequality

$$\textit{(a)} \qquad I_1 + \tfrac{1}{2} I_2 \le 0 \qquad \text{in} \ \ G \cap H$$

is seen to hold. On the other hand, to ensure the inequality

$$\textit{(b)} \qquad \tfrac{1}{2} I_2 + I_3 \le 0 \qquad \text{in} \ \ G \cap H$$

it is sufficient to assume

$$(k-2)\beta - \alpha \le 0, \tag{4.17}$$

$$b \ge 1, \qquad \beta \ge \frac{2^{k+2}}{\rho_1} b^k. \tag{4.18}$$

We conclude that

$$-2\beta\rho_0 b^2 \le \mathcal{L}w \le 0 \qquad \text{in} \ \ G \cap H$$

if (4.17)–(4.18) are satisfied.

Choosing α as in (4.11) gives now

$$(k-2)\beta - \alpha = (k - k^*)\beta = 0,$$

so that (4.17) is satisfied for any β. Arguing as in the proof of Theorem 2.8, we can choose $b \ge 1$ so large that

$$v \le w \qquad \text{on} \quad \partial G.$$

Finally, for fixed b choose β satisfying the second inequality in (4.18). Arguing as in Theorem 2.8 the conclusion follows. □

We conclude this section by discussing Theorem 2.11. For this purpose we need the following lemma, analogous to Lemma 4.1.

Lemma 4.2. *Let $p = m$; let u be any solution to problem (1.1). Then for any $r > 0$ there exists $B > 0$ (depending on r, m, c_0 and u_0) such that*

$$u(x,t) \leq B(1+x)^{-r} \qquad \text{for any} \;\; x \geq 0,\; t \geq 0. \tag{4.19}$$

Proof. For any $k > 0$ the function $u_k(x,t) := u(kx, k^2t)$ solves the problem

$$\begin{cases} \rho_k(x)u_{kt} = (u_k^m)_{xx} - c_{0k}u_k^p & \text{in} \quad \mathbb{R}\times(0,\infty) \\ u_k = u_{0k} \quad \text{in} \quad \mathbb{R}\times\{0\}, \end{cases}$$

where $\rho_k(x) := \rho(kx)$, $u_{0k}(x) := u_0(kx)$ and $c_{0k} := c_0k^2$. Set

$$\mathcal{L}_{0k}u \equiv -\rho_k(x)u_t + (u^m)_{xx} - c_{0k}u^p.$$

Consider the time-independent auxiliary function $w(x) := B_0(1+x)^{-r}$, where $B_0, r > 0$. There holds

$$\mathcal{L}_{0k}w = B_0^m(1+x)^{-rm}\left\{rm(rm+1)(1+x)^{-2} - c_{0k}\right\}.$$

Hence

$$-B_0^m c_{0k} \leq \mathcal{L}_{0k}w \leq 0 \qquad \text{in} \;\; (0,\infty)\times(0,\infty),$$

if

$$k \geq \bar{k} := \left(\frac{rm(rm+1)}{c_0}\right)^{1/2}.$$

As in the proof of Lemma 4.1 , we can choose $B_0 > 0$ so large that

$$u_{\bar{k}}(x) \leq w(x) \qquad \text{for any} \;\; x \geq 0,$$

$$u_{\bar{k}}(0,t) \leq B_0 = w(0) \qquad \text{for any} \;\; t \geq 0.$$

Then by Theorems 2.2 and 2.3,

$$u_{\bar{k}}(x,t) = u(\bar{k}x, \bar{k}^2t) \leq B_0(1+x)^{-r} \qquad \text{for} \quad \text{any} \;\; x \geq 0,\; t \geq 0;$$

hence

$$u(x,t) \leq B_0\left(1+\frac{x}{\bar{k}}\right)^{-r} \leq B(1+x)^{-r}$$

by a proper choice of B. Then the conclusion follows. □

The proof of Theorem 2.11 is now almost verbatim the same as that of Theorem 2.8; by Lemma 4.2 we can choose $r > 0$ so large that $\alpha := r\beta(m-1)$ satisfies inequality (4.7). We leave the details to the reader.

5. Blow-up of the interface

This section is devoted to the proof of Theorem 2.10.

Proof of Theorem 2.10. Consider the auxiliary function

$$w(x,t) := a(1-t)^{\alpha}[b - x(1-t)^{\beta}]_{+} \qquad (x \geq 0,\ t \in [0,1]) \tag{5.1}$$

where b, a, α and β are positive parameters. Set $\theta := 1-t$, $A := [b - x\theta^{\beta}]_{+}$. Consider the half strip

$$Q := \{(x,t) : x > 0, t \in (0,1]\}. \tag{5.2}$$

We shall prove the following claim: There exist $b, a, \alpha, \beta > 0$ such that

$$v \geq w \qquad \text{in} \quad \overline{Q} \tag{5.3}$$

(where v denotes the pressure defined in (3.1)), if $u_0 \in \mathcal{M}_{b,h}$. Then

$$\mathit{supp}\ u(\cdot,t) = \mathit{supp}\ v(\cdot,t) \supseteq [0, b\theta^{-\beta}] \quad \text{for any } t \in [0,1), \tag{5.4}$$

whence the conclusion follows.

To prove the above claim consider the set $H := \{(x,t) \in Q : A > 0\}$. In H there holds

$$\begin{aligned} \tfrac{1}{a}\mathcal{L}w \; &= \alpha\rho(x)\theta^{\alpha-1}A - \beta x\rho(x)\theta^{\alpha+\beta-1} + \\ &+ a\theta^{2\alpha+2\beta} - ca^{q-1}\theta^{\alpha q}A^{q} =: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{5.5}$$

Let us prove that

$$\mathcal{L}w \geq 0 \qquad \text{in} \quad H, \tag{5.6}$$

if the parameters b, a, α and β are properly chosen. To this purpose it is expedient to rewrite (5.5) as follows:

$$\frac{1}{a}\mathcal{L}w = \frac{1}{2}I_1 + I_2 + I_3 + \frac{1}{2}I_1 + I_4. \tag{5.7}$$

Consider the following sets:

$$H_1 = \{(x,t) \in H : \frac{1}{2}I_1 + I_2 \geq 0\}, \tag{5.8}$$

$$H_2 := H \setminus H_1 = \{(x,t) \in H : \frac{\alpha}{\alpha + 2\beta} b\theta^{-\beta} \leq x < b\theta^{-\beta}\}. \tag{5.9}$$

It is immediately seen that inequality (5.6) follows if we prove that

$$\textit{(a)} \qquad \tfrac{1}{2}I_1 + I_4 \geq 0 \qquad \text{in} \quad H,$$

$$\textit{(b)} \qquad I_2 + I_3 \geq 0 \qquad \text{in} \quad H_2$$

under a suitable choice of $b, a, \alpha, \beta > 0$. Observe preliminarily that inequality *(a)* above is equivalent to the following:

$$\alpha\rho(x) \geq 2ca^{q-1}\theta^{\alpha(q-1)+1}A^{q-1}. \tag{5.10}$$

It is easily seen that the above inequality follows from the left inequality in (2.18), from the definition of the set H and from the obvious inequality $A \leq b$, if the following conditions are satisfied:

$$\alpha(q-1) - \beta k + 1 \geq 0, \tag{5.11}$$

$$b \geq 1, \qquad a^{q-1}b^{k+q-1} \leq \frac{\alpha\rho_1}{2^{k+1}c} =: K_1. \tag{5.12}$$

Concerning inequality *(b)*, observe that it is equivalent to the following:

$$\beta x\rho(x) \leq a\theta^{\alpha+\beta+1},$$

which is implied by the right inequality in (2.18) and the definition of the set H_2, provided that

$$\alpha - (k-2)\beta + 1 \leq 0, \tag{5.13}$$

$$ab^{k-1} \geq \beta\rho_2\left(1 + \frac{2\beta}{\alpha}\right)^k =: K_2. \tag{5.14}$$

It is easily seen that the above conditions (5.11)–(5.14) are consistent. Concerning (5.11) and (5.13) it suffices to choose

$$\alpha = \frac{2(m-1)}{p-m}\beta, \quad \beta = \frac{1}{k-k^*}. \tag{5.15}$$

As for (5.12) and (5.14), these requirements imply the compatibility condition

$$K_2^{q-1}b^{-(k-1)(q-1)} \leq K_1 b^{-k-(q-1)}, \tag{5.16}$$

namely

$$b^{\frac{p-m}{m-1}(k-k^*)} \geq \frac{K_2^{q-1}}{K_1}. \tag{5.17}$$

Since by assumption $k > k^*$, both (5.17) and the requirement $b \geq 1$ are satisfied choosing

$$b \geq b_1 := \max\left\{1, \left(\frac{K_2^{q-1}}{K_1}\right)^{\frac{m-1}{(p-m)(k-k^*)}}\right\}; \tag{5.18}$$

then choosing

$$a = \frac{K_2}{b^{k-1}} \tag{5.19}$$

(5.6) follows.

Let us further consider the values of the function w on the boundary of Q. Our purpose is to make sure that the following inequalities are satisfied:

$$w(x,0) \le v_0(x) \qquad \text{for any} \quad x \ge 0, \tag{5.20}$$

$$w(0,t) \le v(0,t) \qquad \text{for any} \quad t \in (0,1). \tag{5.21}$$

Since $u_0 \in \mathcal{M}_{b,h}$ we have

$$w(x,0) = ab\left[1 - \frac{x}{b}\right]_+ \le \frac{m}{m-1} u_0^{m-1}(x) = v_0(x) \tag{5.22}$$

if

$$ab = \frac{K_2}{b^{k-2}} \le h \tag{5.23}$$

(see (5.19)); thus inequality (5.20) follows. As for (5.21), observe that

$$\frac{m}{m-1} u_0^{m-1}(x) \ge h\left[1 - \frac{|x|}{b}\right]_+ \ge h\Big[1 - |x|\Big]_+ \qquad (x \in \mathbb{R}) \tag{5.24}$$

if $u_0 \in \mathcal{M}_{b,h}$ with $b \ge 1$. Denote by $v_{1,h}$ the solution of problem (3.2) with initial data

$$v_0(x) = h\Big[1 - |x|\Big]_+ \qquad\qquad (x \in \mathbb{R}); \tag{5.25}$$

then by Theorem 2.2 and inequality (5.24), there holds

$$v_{1,h} \le v \qquad \text{in} \quad \mathbb{R} \times (0,\infty). \tag{5.26}$$

Hence inequality (5.21) is satisfied if

$$w(0,t) = ab(1-t)^\alpha \le v_{1,h}(0,t) \quad \text{for any} \quad t \in (0,1); \tag{5.27}$$

in turn, this follows if

$$ab = \frac{K_2}{b^{k-2}} \le \min_{t \in [0,1]} v_{1,h}(0,t) =: K_h \tag{5.28}$$

(observe that $K_h > 0$ by Lemma 3.2). Finally, set

$$b_0 = b_0(h) := \max\Big\{ b_1, \big(\frac{K_2}{h}\big)^{\frac{1}{k-2}}, \big(\frac{K_2}{K_h}\big)^{\frac{1}{k-2}} \Big\}. \tag{5.29}$$

For any $b \ge b_0$ and a as in (5.19), inequalities (5.20), (5.21) and (5.6) are satisfied. Moreover, $\mathcal{L}w = 0$ in $Q \setminus \overline{H}$. Then by Theorems 2.2 and 2.4, applied in Q, the conclusion follows. □

Let us finally remark that for densities of the form (1.2), the above results give an almost-complete description of the support properties of solutions. Two questions remain open: positivity for $p = m$ and blow-up of the interface for general initial data. In the general context, it should be noted that the left inequality in the blow-up condition (2.18) is likely to be just a technical restriction imposed by the method of proof.

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