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GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR FOR A VISCOUS, HEAT-CONDUCTIVE, ONE-DIMENSIONAL REAL GAS WITH FIXED AND CONSTANT TEMPERATURE BOUNDARY CONDITIONS*

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Abstract. This paper is concerned with the global existence and asymptotic behaviour, as time tends to infinity, of solutions to the system for a nonlinear viscous, heat-conductive, one-dimensional real gas. Our results show that the global solution approaches to the solution in the H^1 norm to the corresponding stationary problem, as time tends to infinity.

1. Introduction

This paper is concerned with global existence, uniqueness and asymptotic behaviour, as time tends to infinity, of solution to the system for nonlinear viscous, heat-conductive, one-dimensional real gas. The referential (Lagrangian) form of the conservation laws of mass, momentum, and energy for a one-dimensional gas with the reference density $\rho_0 = 1$ is

$$u_t - v_x = 0, \tag{1.1}$$

$$v_t - \sigma_x = 0, \tag{1.2}$$

$$(e + \frac{v^2}{2})_t - (\sigma v)_x + Q_x = 0, \tag{1.3}$$

and the second law of thermodynamics is expressed by the Clausius–Duhem inequality

$$\eta_t + (\frac{Q}{\theta})_x \geq 0. \tag{1.4}$$

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Here subscripts indicate partial differentiations, u, v, σ, e, Q, η and θ denote the specific volume, velocity, stress, internal energy, heat flux, specific entropy and temperature, respectively. Note that u, θ and e may take only positive values. We consider the system (1.1)–(1.3) in the region $\{0 \leq x \leq 1, t \geq 0\}$ under the initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x) \quad \text{on} \quad [0,1], \tag{1.5}$$

and the boundary conditions of the form

$$v(0,t) = v(1,t) = 0, \qquad \theta(0,t) = \theta(1,t) = T_0 \tag{1.6}$$

where $T_0 > 0$ is a constant, a reference temperature.

For the case of an ideal gas, i.e.,

$$e = c_v\theta, \quad \sigma = -R\tfrac{\theta}{u} + \mu\tfrac{v_x}{u}, \quad Q = -k\tfrac{\theta_x}{u}, \tag{1.7}$$

with suitable positive constants c_v, R, μ and k, Kazhikhov [1], Kazhikhov and Shelukhin [2], Kawashima and Nishida [3], and Nagasawa[4–6] established the existence of a global solution to the system (1.1)–(1.3). As is known, the constitutive equations of a real gas are well approximated within moderate ranges of u and θ by the model of an ideal gas. However, under very high temperatures and densities, (1.7) becomes inadequate. Thus a more realistic model than (1.7) would be a linearly viscous gas (or Newtonian fluid)

$$\sigma(u,\theta,v_x) = -p(u,\theta) + \frac{\mu(u,\theta)}{u}v_x, \tag{1.8}$$

satisfying Fourier's law of heat flux

$$Q(u,\theta,\theta_x) = -\frac{k(u,\theta)}{u}\theta_x, \tag{1.9}$$

whose internal energy e and pressure p are coupled by the standard thermodynamical relation

$$e_u(u,\theta) = -p(u,\theta) + \theta p_\theta(u,\theta) \tag{1.10}$$

to be consistent with (1.4).

In this paper we assume that e, p, σ and k are twice continuously differentiable on $0 < u < +\infty$ and $0 \leq \theta < +\infty$, and we also assume that there are exponents q and r satisfying one of the following conditions:

$$0 \leq r \leq 1/3,\ 1/3 < q, \tag{1.11}$$

$$1/3 < r < 4/7,\ (2r+1)/5 < q, \tag{1.12}$$

$$4/7 \leq r \leq 1,\ (5r+1)/9 < q, \tag{1.13}$$

$$1 < r \leq 13/3,\ (9r+1)/15 < q, \tag{1.14}$$

$$13/3 < r,\ (11r+3)/19 < q, \tag{1.15}$$

and concerning the growth of the temperature we require that there be positive constants $\nu,\ p_1,\ p_2,\ k_0$ and for any $\underline{u} > 0$ that there be positive constants $N(\underline{u}),\ p_3(\underline{u}),\ p_4(\underline{u})$ and $k_1(\underline{u})$ such that for any $u \geq \underline{u}$ and $\theta \geq 0$ the following conditions hold:

$$0 \leq e(u,0),\ \nu(1+\theta^r) \leq e_\theta(u,\theta) \leq N(\underline{u})(1+\theta^r), \tag{1.16}$$

$$0 < p_1 \leq up(u,\theta) \leq p_2(1+\theta^{r+1}), \tag{1.17}$$

$$\begin{aligned} &-p_3(\underline{u})[l+(1-l)\theta+\theta^{r+1}] \leq p_u(u,\theta) \\ &\qquad \leq -p_4(\underline{u})[l+(1-l)\theta+\theta^{r+1}), \quad l = 0 \,\text{or}\, 1, \end{aligned} \tag{1.18}$$

$$|p_\theta(u,\theta)| \leq p_4(\underline{u})(1+\theta^r), \tag{1.19}$$

$$k_0(1+\theta^q) \leq k(u,\theta) \leq k_1(\underline{u})(1+\theta^q), \tag{1.20}$$

$$|k_u(u,\theta)| + |k_{uu}(u,\theta)| \leq k_1(\underline{u})(1+\theta^q). \tag{1.21}$$

For the viscosity $\mu(u,\theta)$, we assume that

$$\mu(u,\theta) = \mu_0 \tag{1.22}$$

with $\mu_0 > 0$ being a constant.

The purposes of this paper are

(1) to establish the global existence, uniqueness and asymptotic behaviour of solutions to problem (1.1)–(1.3), (1.5)–(1.6);

(2) to investigate how the exponents q and r affect the global existence and asymptotic behavior of solutions to problem (1.1)–(1.3), (1.5)–(1.6).

We shall use the familiar notation $H^{n+\alpha}$, $H_T^{n+\alpha}$ and $B_T^{n+\alpha}$ for a nonnegative integer n and $\alpha \in (0,1),\ T > 0$, whose definitions are the same as those in [4]. In general and without danger of confusion we will use the same symbol to denote state functions as well as their values along a thermodynamic process, e.g., $p(u,\theta)$ and $p(u(x,t),\theta(x,t))$. L^p, $1 \leq p \leq \infty$, $H^1 = W^{1,2}$ and $H_0^1{=}W_0^{1,2}$ denote the usual Sobolev spaces on (0,1); $\|\cdot\|_B$ denotes the norm in the space B, $\|\cdot\| := \|\cdot\|_{L^2}$. Analogously, ∂_t or $\frac{d}{dt}$ or a subscript t and ∂_x or a subscript x denote the derivative with respect to t and x in the distribution sense, respectively. $\bar{f}(t) = \int_0^1 f(x,t)dx,\ f^*(x,t) = f(x,t) - \bar{f}(t)$, $(f(x,t),g(x,t)) = \int_0^1 f(x,t)g(x,t)dx$. The letters C and C_i will denote universal constants depending only on the initial data.

We are now in a position to state our main theorem.

Theorem 1.1. *In addition to the assumptions* (1.11)–(1.22), *we assume that for* $\alpha \in (0,1)$ *the initial data satisfy that* $(u_0(x),v_0(x),\theta_0(x)) \in H^{1+\alpha} \times$

$H^{2+\alpha} \times H^{2+\alpha}$ with $u_0(x) > 0$ and $\theta_0(x) > 0$ for any $x \in [0,1]$, and that the compatibility conditions hold. Then problem (1.1)–(1.3), (1.5)–(1.6) *admits a unique global solution $(u(t), v(t), \theta(t)) \in B_T^{1+\alpha} \times H_T^{2+\alpha} \times H_T^{2+\alpha}$ for any $0 < T < +\infty$. Moreover, as $t \to +\infty$, we have*

$$\|u - \overline{u_0}\|_{H^1} \to 0, \quad \|v\|_{H^1} \to 0, \quad \|v\|_{L^\infty} \to 0, \tag{1.23}$$

$$\|\theta_x\| \to 0, \quad \|\theta - T_0\|_{H^1} \to 0, \quad \|\theta - T_0\|_{L^\infty} \to 0, \tag{1.24}$$

$$\|\sigma^*(t)\| \to 0, \ \|p^*(t)\| \to 0, \tag{1.25}$$

$$\|p(u,\theta) - p(\overline{u_0}, T_0)\|_{H^1} \to 0, \quad \|\sigma(u,\theta) - p(\overline{u_0}, T_0)\| \to 0, \tag{1.26}$$

and there exist positive constants C_1, C_2 and C_3 such that for all $t \geq C_1$, it holds that

$$\|u(t) - \overline{u_0}\|_{H^1} + \|v(t)\|_{H^1} + \|\theta(t) - T_0\|_{H^1} \leq C_2 \exp(-C_3 t) \tag{1.27}$$

where $(\overline{u_0}, 0, T_0)$ is the unique solution to the corresponding stationary problem to problem (1.1)–(1.3), (1.5)–(1.6).

There are two main difficulties in proving our results. The first arises from the higher-order nonlinearities of θ in the system (1.1)–(1.3). In order to overcome this one, we make full use of Lemma 2.4 and interpolation techniques to reduce the higher order of θ. The second is that in order to study the asymptotic behavior we have to establish the uniform estimates depending only on the initial data, but independent of any length of time.

The main contribution concerning the results in this paper is as follows:

(i) We establish both the global existence and asymptotic behaviour of solutions for the cases of (1.11)–(1.15) with the restriction $q < r+1$ on their right-hand side, which have not been studied before in the literature. Our results on asymptotic behavior also imply those in [12] where the exponents satisfy $r \in [0,1]$ and $r + 1 \leq q$ (refer to Figure 1.1).

The main contributions concerning the methods in this paper are as follows:

(ii) We use a different method from that in [12] to prove the results on asymptotic behaviour.

(iii) We bound the norm of u, v and θ as well as their derivatives in terms of an expression of the form $(1+\sup_{0\leq s\leq t} \|\theta(s)\|_{L^\infty})^\Lambda$ with Λ being a positive constant depending only on the exponents q and r.

(iv) We make full use of both Lemma 2.4 and delicate interpolation techniques to reduce the higher order of θ.

Before proving our results, let us first recall the related results in the literature. Kazhikhov [1] and Kazhikhov and Shelukhin [2] established the global existence of a generalized solution to the system of (1.1)–(1.3) for an ideal gas. Dafermos [9], Dafermos and Hsiao [8], Jiang [11] and the author [21, 24] proved the global existence (asymptotic behaviour in [21, 24]) of a classical solution to the system of (1.1)–(1.3) for a class of solid-like materials. Under the assumptions (1.16), (1.19)–(1.21) and

$$-\frac{p_2[l+(1-l)\theta+\theta^{1+r}]}{u^2}\le p_u(u,\theta)\le -\frac{p_1[l+(1-l)\theta+\theta^{1+r}]}{u^2},\quad l=0 \text{ or } l=1, \tag{1.28}$$

$$0\le p(u,\theta),\quad p(u,\theta)\to 0 \quad \text{as}\quad u\to+\infty,\ r\in[0,1],\ q\ge r+1, \tag{1.29}$$

Jiang [12] also established the results on asymptotic behavior of global classical solution for the system (1.1)–(1.3) with the boundary conditions (1.6). Under the assumptions (1.16), and

$$-p_2(1+\theta^{1+r})u^{-2}\le p_u(u,\theta)\le -p_1(1+\theta^{1+r})u^{-2}, \tag{1.30}$$

$$|p_\theta(u,\theta)|\le p_3(\underline{u})u^{-1/2}(1+\theta^r),\quad up(u,\theta)\le p_4(1+\theta^{1+r}), \tag{1.31}$$

$$0<p(u,\theta)\le N(\underline{u})(1+\theta^{r+1}), \tag{1.32}$$

$$k_0(1+\theta^q)\le k(u,\theta)\le k_2(1+\theta^q), \tag{1.33}$$

$$|k_u(u,\theta)|+|k_{uu}(u,\theta)|\le k_2(1+\theta^q), \tag{1.34}$$

and for some constants $\gamma<2$ and $\eta_0>0$,

$$\eta(u,\theta)\le((Mu)^\gamma+\eta_0)e(u,\theta) \tag{1.35}$$

with $Mu:=\int_1^u\frac{\mu(\xi)}{\xi}d\xi$, $r\in[0,1]$ and $q\ge 2r+2$, Kawhol [7] succeeded in globally solving the system (1.1)–(1.3) with the boundary conditions (1.6) or

$$Q(0,t)=Q(1,t)=0,\qquad \sigma(0,t)=\sigma(1,t)=0 \tag{1.36}$$

or

$$v(0,t)=v(1,t)=0,\qquad Q(0,t)=Q(1,t)=0. \tag{1.37}$$

Under assumptions (1.16), (1.28) (for (1.38)–(1.39)), (1.20)–(1.21), (1.27) and

$$|p_\theta(u,\theta)|\le p_3(\underline{u})u^{-1}(1+\theta^r), \tag{1.38}$$

$$up(u,\theta)\le p_4(1+\theta^{r+1}),\quad p_u(u,T_0)\le 0,\quad \text{for}\quad (1.6), \tag{1.39}$$

$$|p_\theta(u,\theta)|\le N(\underline{u})(1+\theta^r), \tag{1.40}$$

$$0<\mu_0\le\mu(u)\le\mu_1,\quad \text{for}\quad (1.38)\text{–}(1.39), \tag{1.41}$$

$$\mu(u)=\mu_0,\qquad \text{for}\qquad (1.6), \tag{1.42}$$

with the exponents $r \in [0,1]$, $q \geq r+1$, Jiang [10] also established the global existence for the boundary conditions of (1.6) or (1.37) or

$$Q(0,t) = Q(1,t) = 0, \quad \sigma(0,t) = v(0,t), \quad \sigma(1,t) = -v(1,t) \tag{1.43}$$

or

$$\theta(0,t) = \theta(1,t) = T_0, \qquad \sigma(0,t) = v(0,t), \qquad \sigma(1,t) = -v(1,t) \tag{1.44}$$

with a fixed constant $T_0 > 0$. Here the boundary conditions $\sigma(0,t) = v(0,t)$ and $\sigma(1,t) = -v(1,t)$ indicate that the endpoints of the interval [0,1] are connected to some sort of dashpot.

Now let us compare our results with those in [7], [10] and [12]. It is obvious that our assumotion (1.18) is weaker than (1.25) and (1.28) in [7], [10] and [12]. Our assumption (1.19) is weaker than (1.31) in [7] and (1.38) in [10], while our assumptions (1.20) and (1.21) are weaker than (1.33) and (1.34) in [7], respectively.

In [12] the asymptotic behavior was obtained for the case of $r \in [0,1]$, $q \geq r+1$. The case of $q = r = 0$ was studied by the author in [22] and the cases of (1.11)–(1.15) with the restriction $q < r+1$ on their right-hand sides, were not studied before in the literature. In this paper we establish the results on both global existence and asymptotic behaviour for the cases mentioned above. Moreover, we also discuss the case which improves the results in [12]. The earlier paper [23] by the author established the global existence and asymptotic behaviour of solution to the the system (1.1)–(1.3) with the boundary conditions (1.37) under the basically same assumptions as (1.16)–(1.22) and when the exponents q and r satisfy $0 \leq r$, $r+1 \leq q < (5r+3)/2$. Therefore, the results in this paper improve those in [23]. With the basically same assumptions on the exponents q and r as those in (1.11)–(1.15), the author [25] also established the global existence and asymptotic behaviour of a solution to the system (1.1)–(1.3) with the boundary conditions (1.37). Figure 1.1 describes the ranges of the exponents q and r in [10] and [12] and in this paper (denoted by (1), (2) and (3), respectively) in the (r,q)-plane.

In this direction, we would also like to mention that Racke and Zheng [16] investigated the global existence, uniqueness and asymptotic behaviour of weak solutions to the model in shape memory alloys also with the stress-free boundary condition at least at one end of the rod. Shen, Zheng and Zhu [15] also studied the global existence, uniqueness and asymptotic behaviour of a weak solution to the model in shape memory alloys with the boundary conditions (1.37), but with different constitutive relations from our assumptions (1.16)–(1.22). Sprekels, Zheng and Zhu [14] obtained the results on asymptotic behaviour of solutions to the Landau–Ginzburg model in shape memory alloys, for which Sprekels and Zheng [17] established the existence of maximal attractor. We also refer to the works in [18] and [20].

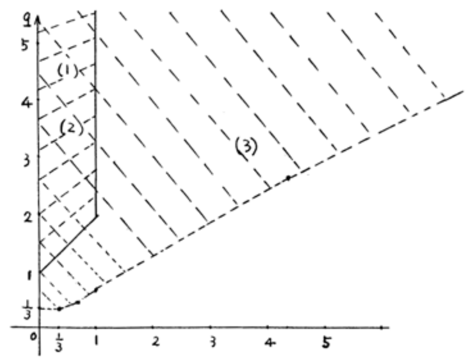

Figure 1.1

Remark 1.1. Theorem 1.1 is also valid under the assumptions in [12], i.e., (1.16), (1.19)–(1.21) and (1.28)–(1.29). Moreover, (1.11)–(1.15) imply $0 \leq r$, $r+1 \leq q$, so the results in Theorem 1.1 improve those in [12].

We organize our present paper as follows. In Section 2, we shall first get uniform a priori estimates of a solution which result in global existence with the help of the local existence of a solution. In Section 3, we shall sketch the proof of the results on asymptotic behaviour since it is the same as that in [23].

2. Uniform a priori estimates

The global existence of solution in Theorem 1.1 is based on a priori estimates that can be used to continue a local solution globally in time. The existence and uniqueness of local solutions (with positive u and θ) can be obtained by linearization of problem (1.1)–(1.3) and (1.5)–(1.6), and by use of the Banach contraction mapping theorem (cf. [19]).

Theorem 2.1. *Let (u, v, θ) be a smooth solution as described in Theorem 1.1; then we have for any $T > 0$,*

$$|||u|||_T^{(1+\alpha)} + ||v||_T^{(2+\alpha)} + |||\theta|||_T^{(2+\alpha)} \leq C, \tag{2.1}$$

and $0 < C^{-1} \leq u(x,t) \leq C,\ \ 0 < \theta(x,t) \leq C,\ \ \forall t \in [0,1] \times [0, +\infty)$.

The proofs of Theorem 1.1 and Theorem 2.1 are divided into a series of lemmas.

Lemma 2.1.

$$\theta(x,t) > 0, \quad \forall (x,t) \in [0,1] \times [0,+\infty), \tag{2.2}$$

$$\int_0^1 u(x,t)dx = \int_0^1 u_0(x)dx = \bar{u}_0, \quad \forall t > 0. \tag{2.3}$$

Proof. Inequality (2.2) is a consequence of the maximum principle (see [26]) applied to the following equation, which is equivalent to (1.3),

$$e_\theta(u,\theta)\theta_t + \theta p_\theta(u,\theta)v_x - \mu_0 \frac{v_x^2}{u} = (\frac{k(u,\theta)\theta_x}{u})_x, \tag{2.4}$$

by considering the positivity of θ_0 and (1.16), (1.21) and (1.22). Equality (2.3) is a direct result of (1.1) and (1.6). □

Lemma 2.2. $\forall t > 0$,

$$\int_0^1 [(\theta/T_0 - \log(\theta/T_0) - 1) + \theta^{1+r} + v^2]dx + \int_0^t \int_0^1 [\frac{v_x^2}{u\theta} + \frac{(1+\theta^q)\theta_x^2}{u\theta^2}]dx\,ds \le C_4. \tag{2.5}$$

Proof. Let $\Psi(u,\theta) = e(u,\theta) - \theta\eta(u,\theta)$ be the Helmholtz free-energy function. Then

$$-\Psi_\theta(u,\theta) = \eta(u,\theta), \ \Psi_u(u,\theta) = \sigma(u,\theta,0) \equiv -p(u,\theta). \tag{2.6}$$

Denote

$$E(u,\theta) := \Psi(u,\theta) - \Psi(1,T_0) - \Psi_u(1,T_0)(u-1) - \Psi_\theta(u,\theta)(\theta - T_0). \tag{2.7}$$

Using (1.1), (1.2), (1.8), (2.4) and (2.6), noting that $e_\theta(u,\theta) = -\theta\Psi_{\theta\theta}(u,\theta)$, we deduce after a direct calculation that

$$\begin{aligned}\partial_t[E(u,\theta) + \frac{v^2}{2}] + T_0[\frac{\mu_0 v_x^2}{u\theta} + \frac{k(u,\theta)\theta_x^2}{u\theta^2}]\\ = (\sigma v)_x + p(1,T_0)v_x + [\frac{(\theta - T_0)k(u,\theta)\theta_x}{u\theta}]_x.\end{aligned} \tag{2.8}$$

Integrating (2.8) over Q_t and using (1.6) lead to

$$\begin{aligned}&\int_0^1 (E(u,\theta) + \frac{v^2}{2})(x,t)dx + T_0 \int_0^t \int_0^1 (\frac{\mu_0 v_x^2}{u\theta} + \frac{k(u,\theta)\theta_x^2}{u\theta^2})dx\,ds\\ &= \int_0^1 (E(u_0,\theta_0) + \frac{v_0^2}{2})dx.\end{aligned} \tag{2.9}$$

In view of (1.18), we have $\Psi_{uu}(u,T_0) = -p_u(u,T_0) > 0$ for $u > 0$. Therefore it follows from the Taylor theorem and (1.18) that

$$E(u,\theta) - \Psi(u,\theta) + \Psi(u,T_0) + (\theta - T_0)\Psi_\theta(u,\theta)$$

$$= \Psi(u, T_0) - \Psi(1, T_0) - \Psi_u(1, T_0)(u-1)$$
$$= (u-1)^2 \int_0^1 (1-\xi)\Psi_{uu}(1+\xi(u-1), T_0)d\xi \geq 0. \tag{2.10}$$

Thus,

$$E(u,\theta) \geq \Psi(u,\theta) - \Psi(u,T_0) - (\theta - T_0)\Psi_\theta(u,\theta)$$
$$= -(T_0-\theta)^2 \int_0^1 (1-\tau)\Psi_{\theta\theta}(u, \theta + \tau(T_0-\theta))d\tau$$
$$\geq \nu(T_0-\theta)^2 \int_0^1 \frac{(1-\tau)\{1+[\theta+\tau(T_0-\theta)]^r\}}{\theta+\tau(T_0-\theta)} d\tau;$$

i.e.,

$$E(u,\theta) \geq \begin{cases} \nu T_0(\theta/T_0 - \log(\theta/T_0) - 1) + \frac{\nu T_0(T_0^r - \theta^r)}{r} - \frac{\nu T_0(T_0^{r+1} - \theta^{r+1})}{r+1}, \\ \qquad \text{for } r > 0, \\ 2\nu T_0(\theta/T_0 - \log(\theta/T_0) - 1), \quad \text{for } r = 0. \end{cases}$$
$$\geq \nu T_0(\theta/T_0 - \log(\theta/T_0) - 1) + C_5\theta^{r+1} - C_6 \tag{2.11}$$

which, combined with (1.16) and (2.9), yields (2.5). □

Lemma 2.3.

$$0 < C^{-1} \leq u(x,t) \leq C, \quad \forall (x,t) \in [0,1] \times [0, +\infty). \tag{2.12}$$

Proof. See, e.g., [23]. □

Lemma 2.4.

$$C - CV(t) \leq \theta^{2m_1}(x,t) \leq C + CV(t), \quad \forall (x,t) \in [0,1] \times [0,+\infty), \tag{2.13}$$

with $0 \leq m_1 \leq m = (q+r+1)/2$ *and* $V(t) = \int_0^1 \frac{(1+\theta^q)\theta_x^2}{\theta^2} dx$ *satisfying* $\int_0^\infty V(t)dt < \infty$.

Proof. See, e.g., [25]. □

Lemma 2.5. *Suppose that a nonnegative function* $y(t)$ *is absolutely continuous on* $[0,T]$ *and satisfies, for almost all* $t \in [0,T]$, *the inequality*

$$y(t) \leq g(t) + \int_0^t f(s)y(s)ds \tag{2.14}$$

with nonnegative functions $g(t)$ and $f(t)$ *being summable on* $[0,T]$. *Then*

$$y(t) \leq g(t) + \int_0^t f(s)g(s)\exp(\int_s^t f(\tau)d\tau)ds. \tag{2.15}$$

In addition, if $g(t)$ is monotone increasing on $[0,+\infty)$ and $\int_0^{+\infty} f(s)ds < +\infty$, then

$$y(t) \le Cg(t) \tag{2.16}$$

with $C > 0$ being a constant independent of $t > 0$.

Proof. See, e.g., [25]. □

In the following, we set $A(t) = 1 + \sup_{0\le s\le t} \|\theta(s)\|_{L^\infty}$.

Lemma 2.6.

$$\int_0^t \|v\|^2_{L^\infty}\, ds \;\le\; C, \quad \forall \quad t > 0, \tag{2.17}$$

$$\int_0^t \int_0^1 (1+\theta)^{2m} v^2 dx\, ds \;\le\; C, \quad \forall \quad t > 0, \tag{2.18}$$

$$\|u_x\|^2 + \int_0^t \int_0^1 (1+\theta^{1+r}) u_x^2\, dx\, ds \;\le\; CA(t)^\beta, \quad \forall \quad t > 0, \tag{2.19}$$

with $\beta = \max(r + 1 - q, 0)$.

Proof. It follows from (1.6), Lemma 2.2 and Lemma 2.4 that

$$\int_0^t \|v\|^2_{L^\infty}\, ds \le \int_0^t \int_0^1 \frac{v_x^2}{\theta}\, dx \int_0^1 \theta\, dxds \le C \int_0^t \int_0^1 \frac{v_x^2}{\theta}\, dx\, ds \le C,$$

and

$$\int_0^t \int_0^1 (1+\theta)^{2m} v^2 dx\, ds \le C \int_0^t \int_0^1 v^2 dx\, ds + C\int_0^t V(s)\|v\|^2 ds \le C.$$

On the other hand, equation (1.2) can be rewritten as

$$(v - \mu_0 \frac{u_x}{u})_t + p_u(u,\theta)u_x = -p_\theta(u,\theta)\theta_x. \tag{2.20}$$

Multiply (2.20) by $v - \mu_0 \frac{u_x}{u}$, then integrate the result over Q_t , use Lemma 2.4 and (2.18) and note the facts

$$\begin{aligned}\int_0^t \int_0^1 \frac{\theta^2(1+\theta^r)^2}{1+\theta^q} v^2 dx\, ds &\le CA(t)^{\max(1+r-2q,0)} \int_0^t \int_0^1 (1+\theta)^{2m} v^2 dx\, ds \\ &\le CA(t)^{\max(1+r-2q,0)},\end{aligned}$$

$$\begin{aligned}\int_0^t \int_0^1 \frac{(1+\theta^r)^2}{1+\theta^{1+r}} \theta_x^2 dx\, ds &\le C \int_0^t \int_0^1 (1+\theta)^{r-1}\theta_x^2 dx\, ds \\ &\le CA(t)^\beta \int_0^t V(s)ds \le CA(t)^\beta,\end{aligned}$$

$$\int_0^t\int_0^1 u_x^2 dx\,ds \le C\int_0^t V(s)\|u_x\|^2 ds + C\int_0^t\int_0^1 \theta^{r+1}u_x^2 dx\,ds, \tag{2.21}$$

to get

$$\begin{aligned} &\|v-\mu_0\frac{u_x}{u}\|^2 + \int_0^t\int_0^1 [l+(1-l)\theta+\theta^{r+1}]u_x^2 dx\,ds \\ \le\ & C + C\int_0^t\int_0^1 [(1+\theta^{r+1})|u_x v| + (1+\theta^r)|\theta_x(v-\mu_0\frac{u_x}{u})|]dx\,ds \\ \le\ & C + C\int_0^t\int_0^1 (1+\theta^{1+r})(\epsilon u_x^2 + Cv^2)dx\,ds + C\int_0^t V(s)ds \\ +\ & C\int_0^t\int_0^1 \frac{\theta^2(1+\theta^r)^2}{1+\theta^q}v^2 dx\,ds + C\epsilon\int_0^t\int_0^1 (1+\theta^{1+r})u_x^2 dx\,ds \\ +\ & C\int_0^t\int_0^1 \frac{(1+\theta^r)^2\theta_x^2}{1+\theta^{1+r}} dx\,ds \\ \le\ & CA(t)^\beta + C\epsilon\int_0^t\int_0^1 (1+\theta^{1+r})u_x^2 dx\,ds + C\int_0^t \|v\|_{L^\infty}^2 ds \\ \le\ & CA(t)^\beta + C\epsilon\int_0^t V(s)\|u_x\|^2 ds + C\epsilon\int_0^t\int_0^1 \theta^{r+1}u_x^2 dx\,ds. \end{aligned} \tag{2.22}$$

Thus for small $\epsilon > 0$ and by Lemma 2.5, we have

$$\|u_x\|^2 + \int_0^t\int_0^1 [l+(1-l)\theta+\theta^{r+1}]u_x^2\,dx\,ds \le CA(t)^\beta$$

which (for $l = 1$) with (2.15) (for $l = 0$) imply (2.19). □

Lemma 2.7.

$$\int_0^t\int_0^1 (1+\theta)^{2m}u_x^2 dx\,ds \le CA(t)^\beta, \quad \forall \quad t > 0. \tag{2.23}$$

Proof. The proof is the same as that of (2.18). □

Lemma 2.8.

$$\int_0^t \|v_x\|^2 ds \le CA(t)^{\beta/2}, \quad \forall \quad t > 0, \tag{2.24}$$

$$\int_0^t \|v_{xx}\|^2 ds \le CA(t)^{\beta_1}, \quad \forall \quad t > 0, \tag{2.25}$$

$$\|v_x\|^2 + \int_0^t \|v_t\|^2 ds \le CA(t)^{\beta_4}, \quad \forall \quad t > 0, \tag{2.26}$$

$$\int_0^t \|v_x\|_{L^\infty}^2 ds \quad \le \quad CA(t)^{\beta_3}, \quad \forall \quad t > 0, \tag{2.27}$$

where $\beta_1 = \max(5\beta/2, \beta_4) = 5(1+r-q)/2$, $\beta_4 = \max(2r+2-q, 0)$ and $\beta_3 = \beta/4 + \beta_1/2$.

Proof. Multiplying (1.2) by v, v_{xx} and v_t, respectively, then integrating the resultant over Q_t and using Lemmas 2.1–2.6, we have

$$\begin{aligned}
&\|v\|^2 + \int_0^t \|v_x\|^2 ds \le C + C\int_0^t\int_0^1 [(1+\theta^{1+r})|u_x v| + (1+\theta^r)|\theta_x v|]dx\,ds \\
&\le C + C(\int_0^t\int_0^1 (1+\theta)^{r+1} u_x^2 dx\,ds)^{1/2} (\int_0^t\int_0^1 (1+\theta)^{r+1} v^2 dx\,ds)^{1/2} \qquad (2.28)\\
&+ C(\int_0^t V(s)ds)^{1/2} (\int_0^t\int_0^1 \frac{(1+\theta^r)^2\theta^2}{1+\theta^q} v^2 dx\,ds)^{1/2} \le CA(t)^{\max(r+1-2q,0)/2} \\
&+ C(\int_0^t\int_0^1 (1+\theta)^{2m} u_x^2 dx\,ds)^{1/2} (\int_0^t\int_0^1 (1+\theta)^{2m} v^2 dx\,ds)^{1/2} \le CA(t)^{\beta/2}, \\
&\|v_x\|^2 + \int_0^t \|v_{xx}\|^2 ds \\
&\le C + C\int_0^t\int_0^1 [|u_x v_x v_{xx}| + (1+\theta^{1+r})|u_x v_{xx}| + (1+\theta^r)|\theta_x v_{xx}|]dx\,ds \\
&\le C + \frac{1}{4}\int_0^t \|v_{xx}\|^2 ds + C\int_0^t\int_0^1 [v_x^2 u_x^2 + (1+\theta^{1+r})^2 u_x^2 + (1+\theta)^{2r}\theta_x^2]dx\,ds \\
&\le C + \frac{1}{4}\int_0^t \|v_{xx}\|^2 ds + CA(t)^\beta \int_0^t \|v_x\|_{L^\infty}^2 ds \\
&+ CA(t)^\beta \int_0^t\int_0^1 (1+\theta)^{2m} u_x^2\, dx\,ds + CA(t)^{\beta_4}\int_0^t V(s)ds \\
&\le CA(t)^{2\beta} + \frac{1}{4}\int_0^t \|v_{xx}\|^2 ds + CA(t)^{\beta_4} \\
&+ CA(t)^\beta (\int_0^t \|v_x\|^2 ds)^{1/2} (\int_0^t \|v_{xx}\|^2 ds)^{1/2} \\
&\le CA(t)^{5\beta/2} + CA(t)^{\beta_4} + \frac{1}{2}\int_0^t \|v_{xx}\|^2 ds \le CA(t)^{\beta_1} + \frac{1}{2}\int_0^t \|v_{xx}\|^2 ds;
\end{aligned}$$

i.e.,

$$\|v_x\|^2 + \int_0^t \|v_{xx}\|^2 ds \le CA(t)^{\beta_1} \tag{2.29}$$

and

$$\begin{aligned}
&\|v_x\|^2+\int_0^t\|v_t\|^2ds\le C+C\int_0^t[\|p_x\|^2+\int_0^1\frac{|v_x|^3}{u^2}dx]ds\\
\le\ & C+C\int_0^t\int_0^1[(1+\theta)^{2r+2}u_x^2+(1+\theta)^{2r}\theta_x^2+|v_x|^3]dx\,ds\\
\le\ & CA(t)^{2\beta}+CA(t)^{\beta_4}\int_0^t V(s)ds+\int_0^t\|v_x\|^{5/2}\|v_{xx}\|^{1/2}ds\\
\le\ & CA(t)^{\max(2\beta,\beta_4)}+C\sup_{0\le s\le t}\|v_x\|(\int_0^t\|v_x\|^2ds)^{3/4}(\int_0^t\|v_{xx}\|^2ds)^{1/4}\\
\le\ & CA(t)^{\beta_4}+\frac{1}{2}\sup_{0\le s\le t}\|v_x\|^2,
\end{aligned}$$

which implies (2.26). □

Lemma 2.9.

$$\int_0^t\int_0^1(1+\theta)^{2m}v_x^2\,dx\,ds \le CA(t)^{\beta_4},\quad\forall\quad t>0,\tag{2.30}$$

$$\int_0^t\int_0^1(1+\theta)^{2m+1}v_x^2\,dx\,ds \le CA(t)^{\beta_5},\quad\forall\quad t>0,\tag{2.31}$$

$$\int_0^t\int_0^1(1+\theta)^{q+1}|v_x|^3dx\,ds \le CA(t)^{\beta_6},\quad\forall\quad t>0,\tag{2.32}$$

$$\int_0^t\int_0^1(1+\theta)^{q-r}v_x^4\,dx\,ds \le CA(t)^{\beta_7},\quad\forall\quad t>0,\tag{2.33}$$

where

$$\begin{aligned}
\beta_5 &= \min(1+\beta_4,2m+1+\beta/2),\\
\beta_6 &= \min[q_1+(5\beta_4+\beta_1)/4,q+1+\beta_4/2+3\beta/8+\beta_1/4],\\
\beta_7 &= \min[q_2+3\beta_4/2+\beta_1/2,\max(q-r,0)+\beta/4+\beta_4+\beta_1/2],\\
q_1 &= \min[(q+1-3r)/4,0],q_2=\max[(q-3r-1)/2,0].
\end{aligned}$$

Proof. It follows from Lemmas 2.1–2.8 that

$$\begin{aligned}
\int_0^t\int_0^1(1+\theta)^{2m}v_x^2dx\,ds &\le C\int_0^t\int_0^1 v_x^2dx\,ds+C\int_0^t\int_0^1 V(s)v_x^2dx\,ds\\
&\le CA(t)^{\beta/2}+CA(t)^{\beta_4}\le CA(t)^{\beta_4}
\end{aligned}$$

which implies

$$\int_0^t\int_0^1(1+\theta)^{2m+1}v_x^2\,dx\,ds\le CA(t)^{1+\beta_4}. \tag{2.34}$$

On the other hand, (2.24) gives

$$\int_0^t\int_0^1(1+\theta)^{2m+1}v_x^2\,dx\,ds\le CA(t)^{2m+1+\beta/2}$$

which, along with (2.34), leads to (2.31).

The interpolation inequality and Lemmas 2.1–2.8 imply

$$\begin{aligned}
&\int_0^t\int_0^1(1+\theta)^{q+1}|v_x|^3dx\,ds\le CA(t)^{q_1}\int_0^t\int_0^1(1+\theta)^{3m/2}|v_x|^3dx\,ds\\
\le\ & CA(t)^{q_1}[\int_0^t\int_0^1|v_x|^3dx\,ds+\int_0^t\int_0^1V^{3/4}(s)|v_x|^3dx\,ds]\\
\le\ & CA(t)^{q_1}[\sup_{0\le s\le t}\|v_x\|(\int_0^t\|v_x\|^2ds)^{3/4}(\int_0^t\|v_{xx}\|^2ds)^{1/4}\\
&+\sup_{0\le s\le t}\|v_x\|^{5/2}(\int_0^t\|v_{xx}\|^2ds)^{1/4}]\le CA(t)^{q_1+5\beta_4/4+\beta_1/4}
\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}
\int_0^t\int_0^1(1+\theta)^{q+1}|v_x|^3dx\,ds\ &\le\ CA(t)^{q+1}\int_0^t\|v_x\|^{5/2}\|v_{xx}\|^{1/2}ds\\
&\le\ CA(t)^{q+1+\beta_4/2+3\beta/8+\beta_1/4}
\end{aligned} \tag{2.36}$$

which, together with (2.35), implies (2.32). Similarly,

$$\begin{aligned}
&\int_0^t\int_0^1(1+\theta)^{q-r}v_x^4dx\,ds\le CA(t)^{q_2}\int_0^t\int_0^1(1+\theta)^mv_x^4dx\,ds \qquad (2.37)\\
&\le CA(t)^{q_2}\int_0^t[\|v_x\|^3\|v_{xx}\|+V^{1/2}(s)\|v_x\|^3\|v_{xx}\|]ds\\
&\le CA(t)^{q_2}[\sup_{0\le s\le t}\|v_x\|^2(\int_0^t\|v_x\|^2ds)^{1/2}(\int_0^t\|v_{xx}\|^2ds)^{1/2}\\
&+\sup_{0\le s\le t}\|v_x\|^3(\int_0^tV(s)ds)^{1/2}(\int_0^t\|v_{xx}\|^2ds)^{1/2}]\le CA(t)^{q_2+3\beta_4/2+\beta_1/2}.
\end{aligned}$$

But we can easily see that

$$\int_0^t\int_0^1(1+\theta)^{q-r}v_x^4\,dx\,ds\le CA(t)^{\max(q-r,0)}\int_0^t\int_0^1v_x^4\,dx\,ds$$

$$\leq \quad CA(t)^{\max(q-r,0)} \sup_{0\leq s\leq t} \|v_x\|^2 (\int_0^t \|v_x\|^2 ds)^{1/2} (\int_0^t \|v_{xx}\|^2 ds)^{1/2}$$
$$\leq \quad CA(t)^{\max(q-r,0)+\beta_4+\beta/4+\beta_1/2}$$

which, combined with (2.37), yields (2.33). □

Lemma 2.10.

$$\|\theta^{1+r}\|^2 + \int_0^t \int_0^1 [\frac{(T_0-\theta)^2(1+\theta)^{q+r}\theta_x^2}{\theta^2} + \theta^{q+r-1}\theta_x^2] dx\, ds \leq CA(t)^{\beta_8}, \ \forall t>0, \tag{2.38}$$

$$\int_0^t \int_0^1 (1+\theta)^{q+r}\theta_x^2\, dx\, ds \leq CA(t)^{\beta_{10}}, \quad \forall \quad t>0, \tag{2.39}$$

where

$$\begin{aligned}\beta_9 &= \min[\max(2r+1-2q,0), \max(3r+3-2q,0)/2],\\ \beta_{10} &= \max(\beta_8, r),\\ \beta_8 &= \max[3\beta/2, \beta_3, \beta_9, \beta+1, \max((3r+3-2q)/2, 0)].\end{aligned}$$

Proof. Let $E_1(u,\theta) = E(u,\theta) + C_6$. Thus we know from (2.11) that

$$E_1(u,\theta) \geq C_5\theta^{r+1} > 0.$$

First, we shall prove $|E_1| \leq C(1+\theta^{r+1})$. In fact, it follows from (2.10) and (2.11) that

$$\begin{aligned}E_1 &= C_6 + E = C_6 + (u-1)^2 \int_0^1 (1-\xi)\Psi_{uu}(1+\xi(u-1), T_0) d\xi\\ &- (T_0-\theta)^2 \int_0^1 (1-\tau)\Psi_{\theta\theta}(u, \theta+\tau(T_0-\theta)) d\tau\\ &\leq C_6 - (u-1)^2 \int_0^1 (1-\xi) p_u(1+\xi(u-1), T_0) d\xi\\ &+ N(\underline{u})(T_0-\theta)^2 \int_0^1 \frac{(1-\tau)\{1+[\theta+\tau(T_0-\theta)]^r\}}{\theta+\tau(T_0-\theta)} d\tau\\ &\leq \begin{cases} C+N(\underline{u})T_0[\frac{\theta}{T_0} - \log(\frac{\theta}{T_0}) - 1] + \frac{N(\underline{u})T_0(T_0^r-\theta^r)}{r} - \frac{N(\underline{u})(T_0^{1+r}-\theta^{1+r})}{r+1}, & \text{for } r>0,\\ C+2N(\underline{u})T_0[\theta/T_0 - \log(\theta/T_0) - 1], & \text{for } r=0.\end{cases}\\ &\leq C(1+\theta^{1+r}).\end{aligned}$$

Second, (2.8) can be rewritten as

$$\partial_t E_1 + T_0(\frac{\mu_0 v_x^2}{u\theta} + \frac{k(u,\theta)\theta_x^2}{u\theta^2}) = \sigma v_x + p(1,T_0)v_x + [\frac{(\theta-T_0)k(u,\theta)\theta_x}{u\theta}]_x. \tag{2.40}$$

Multiplying (2.40) by E_1, integrating the resultant over Q_t, using (2.6)–(2.7), integrating by parts, and noting that $e_\theta(u,\theta) = -\theta\Psi_{\theta\theta}(u,\theta)$ and

$$E_{1x} = E_x = (p(1,T_0) - p(u,\theta))u_x + p_\theta(u,\theta)(\theta - T_0)u_x + e_\theta(u,\theta)\frac{(\theta - T_0)\theta_x}{\theta},$$

we get

$$\begin{aligned}
&\|\theta^{r+1}\|^2 + \int_0^t \int_0^1 [\theta^{q+r-1}\theta_x^2 + \frac{(\theta - T_0)^2(1+\theta)^{q+r}\theta_x^2}{\theta^2}]dx\,ds \\
\leq\ & C + C\int_0^t \int_0^1 [|v(E_1 p)_x| + \mu_0 \frac{v_x^2|E_1|}{u} + |p(1,T_0)vE_{1x}| \\
&+ \frac{|(\theta - T_0)k(u,\theta)\theta_x|}{\theta u}|(p(1,T_0) - p(u,\theta))u_x + p_\theta(u,\theta)(\theta - T_0)u_x|]dx\,ds \\
\leq\ & C + C\int_0^t \int_0^1 [(1+\theta)^{2r+2}|vu_x| + (1+\theta)^{2r+1}|v\theta_x| \\
&+ \frac{(1+\theta)^{2r+1}|(\theta - T_0)v\theta_x|}{\theta}]dx\,ds + C\int_0^t \|v_x\|_{L^\infty}^2 \int_0^1 (1+\theta^{1+r})dx\,ds \\
&+ C\int_0^t \int_0^1 \frac{(1+\theta)^{q+r+1}|(\theta - T_0)u_x\theta_x|}{\theta}dx\,ds \qquad (2.41)
\end{aligned}$$

where

$$\begin{aligned}
&\int_0^t \int_0^1 (1+\theta)^{2r+2}|vu_x|dx\,ds \\
&\leq CA(t)^\beta \Big(\int_0^t \int_0^1 (1+\theta)^{2m}v^2 dx\,ds\Big)^{1/2}\Big(\int_0^t \int_0^1 (1+\theta)^{2m}u_x^2 dx\,ds\Big)^{1/2} \\
&\leq CA(t)^{3\beta/2}, \qquad (2.42)
\end{aligned}$$

$$\begin{aligned}
&\int_0^t \int_0^1 (1+\theta)^{2r+1}|v\theta_x|dx\,ds \\
&\leq \Big(\int_0^t V(s)ds\Big)^{1/2}\Big(\int_0^t \int_0^1 \frac{\theta^2(1+\theta)^{4r+2}}{1+\theta^q}v^2 dx\,ds\Big)^{1/2} \qquad (2.43) \\
&\leq CA(t)^{\max(3r+3-2q,0)/2}\Big(\int_0^t \int_0^1 (1+\theta)^{2m}v^2 dx\,ds\Big)^{\frac{1}{2}} \leq CA(t)^{\max(3r+3-2q,0)/2}
\end{aligned}$$

and

$$\int_0^t \int_0^1 \frac{(1+\theta)^{2r+1}|(\theta - T_0)v\theta_x|}{\theta}dx\,ds$$

$$\begin{aligned}
&\le \frac{1}{8}\int_0^t\int_0^1 \frac{(T_0-\theta)^2(1+\theta)^{q+r}\theta_x^2}{\theta^2}dx\,ds + C\int_0^t\int_0^1 (1+\theta)^{3r+2-q}v^2 dx\,ds \\
&\le \frac{1}{8}\int_0^t\int_0^1 \frac{(T_0-\theta)^2(1+\theta)^{q+r}\theta_x^2}{\theta^2}dx\,ds + CA(t)^{\max(2r+1-2q,0)}. \qquad (2.44)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^t\int_0^1 \frac{(1+\theta)^{2r+1}|(\theta-T_0)v\theta_x|}{\theta}dx\,ds \\
&\le (\int_0^t V(s)ds)^{1/2}(\int_0^t\int_0^1 (1+\theta)^{4r+4-q}v^2 dx\,ds)^{1/2} \\
&\le CA(t)^{\max(3r+3-2q,0)/2}(\int_0^t\int_0^1 (1+\theta)^{2m}v^2 dx\,ds)^{\frac{1}{2}} \le CA(t)^{\max(3r+3-2q,0)/2}
\end{aligned}$$

which, combined with (2.44), leads to

$$\begin{aligned}
&\int_0^t\int_0^1 \frac{(1+\theta)^{2r+1}|(\theta-T_0)v\theta_x|}{\theta}dx\,ds \\
&\le \frac{1}{8}\int_0^t\int_0^1 \frac{(T_0-\theta)^2(1+\theta)^{q+r}\theta_x^2}{\theta^2}dx\,ds + CA(t)^{\beta_9}. \qquad (2.45)
\end{aligned}$$

By Lemma 2.2 and Lemmas 2.6–2.7, we have

$$\int_0^t \|v_x\|_{L^\infty}^2 \int_0^1 (1+\theta^{r+1})dxds \le CA(t)^{\beta_3} \qquad (2.46)$$

and

$$\begin{aligned}
&\int_0^t\int_0^1 \frac{(1+\theta)^{q+r+1}|(\theta-T_0)u_x\theta_x|}{\theta}dx\,ds \\
&\le \frac{1}{8}\int_0^t\int_0^1 \frac{(T_0-\theta)^2(1+\theta)^{q+r}\theta_x^2}{\theta^2}dx\,ds + C\int_0^t\int_0^1 (1+\theta)^{2m+1}u_x^2 dx\,ds \\
&\le \frac{1}{8}\int_0^t\int_0^1 \frac{(T_0-\theta)^2(1+\theta)^{q+r}\theta_x^2}{\theta^2}dx\,ds + CA(t)^{\beta+1}. \qquad (2.47)
\end{aligned}$$

Therefore, it follows from (2.40)–(2.47) that

$$\begin{aligned}
&\|\theta^{r+1}\|^2 + \int_0^t\int_0^1 [\theta^{q+r-1}\theta_x^2 + \frac{(T_0-\theta)^2(1+\theta)^{q+r}\theta_x^2}{\theta^2}]dx\,ds \\
&\le C[A(t)^{3\beta/2} + A(t)^{\max(3r+3-2q,0)/2} + A(t)^{\beta_9} + A(t)^{\beta_3} + A(t)^{\beta+1}] \le CA(t)^{\beta_8}.
\end{aligned}$$

Let $\widetilde{\theta} = \theta - T_0 \log \theta$. Then

$$\widetilde{\theta}_x = \frac{(\theta - T_0)\theta_x}{\theta}, \qquad \theta_x = \widetilde{\theta}_x + T_0 \frac{\theta_x}{\theta}. \tag{2.48}$$

Thus we have from (2.48) that

$$\int_0^t \int_0^1 (1+\theta)^{q+r} \theta_x^2 dx\, ds \le C \int_0^t \int_0^1 [(1+\theta)^{q+r} \widetilde{\theta}_x^2 + \frac{(1+\theta)^{q+r}\theta_x^2}{\theta^2}] dx\, ds$$
$$\le C \int_0^t \int_0^1 \frac{(T_0 - \theta)^2 (1+\theta)^{q+r} \theta_x^2}{\theta^2} dx\, ds + CA(t)^r \le CA(t)^{\beta_{10}}.$$

Lemma 2.11.

$$\int_0^1 (1+\theta)^{2q} \theta_x^2 dx + \int_0^t \int_0^1 (1+\theta)^{q+r} \theta_t^2 dx\, ds \le CA(t)^{\beta_{19}}, \quad \forall\ t > 0, \tag{2.49}$$

where

$\beta_{11} = \max(3q + 2 - r, 0)/2 + (\beta_1 + \beta_{10})/2,$
$\beta_{12} = \min[\beta_{11}, (3q + 4 + \beta_1)/2],$
$\beta_{13} = \max[2\max(q - r, 0) + 2\beta + \beta_{10}, \max(q - r, 0) + \beta + (\beta_{10} + \beta_5)/2,$
$\quad \max(q - r, 0) + \beta + (\beta_{10} + \beta_7)/2],$
$\beta_{14} = \max[\max(q - r, 0) + \beta + q + 2, 2\max(q - r, 0) + 2\beta + r + 2,$
$\quad \max(q - r, 0) + (\beta_5 + r + 2)/2 + \beta, \max(q - r, 0) + (\beta_7 + r + 2)/2 + \beta],$
$\beta_{15} = \min[\beta_{13}, \beta_{14}],$
$\beta_{16} = \max[(\beta_3 + \max(q - r, 0) + \beta_{10})/2, (2\beta_3 + \beta_{10})/3,$
$\quad \beta_3/2 + (\beta_{10} + \beta_5)/4, \beta_3/2 + (\beta_{10} + \beta_7)/4],$
$\beta_{17} = \max[(\beta_3 + q + 2)/2, (2\beta_3 + r + 2)/3, \beta_3/2 + (\beta_5 + r + 2)/4,$
$\quad \beta_3/2 + (\beta_7 + r + 2)/4],$
$\beta_{18} = \min[\beta_{16}, \beta_{17}],$
$\beta_{19} = \max[\beta_5, \beta_6, \beta_7, \beta_{12}, \beta_{15}, \beta_{18}].$

Proof. Let

$$H(x,t) = H(u,\theta) = \int_0^\theta \frac{k(u,\xi)}{u} d\xi,$$

$$X(t) = \int_0^t \int_0^1 (1+\theta)^{q+r} \theta_t^2\, dx\, ds, \qquad Y(t) = \int_0^1 (1+\theta)^{2q} \theta_x^2\, dx.$$

Then it is easy to verify that

$$H_t = H_u v_x + \frac{k\theta_t}{u}, \qquad H_{xt} = [\frac{k\theta_x}{u}]_t + H_u v_{xx} + H_{uu} v_x u_x + (\frac{k}{u})_u u_x \theta_t.$$

Multiplying (2.4) by H_t and integrating the resultant over Q_t lead to

$$\int_0^t \int_0^1 (e_\theta \theta_t + \theta p_\theta v_x - \frac{\mu_0 v_x^2}{u}) H_t \, dx \, ds + \int_0^t \int_0^1 \frac{k\theta_x H_{tx}}{u} dx \, ds - \int_0^t (\frac{k\theta_x H_t}{u})(1,s) ds + \int_0^t (\frac{k\theta_x H_t}{u})(0,s) ds = 0. \qquad (2.50)$$

We can deduce from (1.20) and (1.21) that

$$|H_u| + |H_{uu}| \le C(1+\theta)^{q+1}. \qquad (2.51)$$

Now we estimate each term in (2.50) by using (1.16), (1.19)–(1.21), (2.51) and Lemmas 2.1–2.10.

$$\int_0^t \int_0^1 e_\theta \theta_t H_t \, dx \, ds \ge C_0 X(t) - CA(t)^{\beta_5}, \qquad (2.52)$$

$$\begin{aligned} & |\int_0^t \int_0^1 (\theta p_\theta v_x - \frac{\mu_0 v_x}{u}) H_t \, dx \, ds| \\ \le \; & C \int_0^t \int_0^1 [(1+\theta)^{q+r+2} v_x^2 + (1+\theta)^{q+1} |v_x|^3 \\ & + (1+\theta)^{q+r+1} |v_x \theta_t| + (1+\theta^q) v_x^2 |\theta_t|] dx \, ds \\ \le \; & \frac{C_0}{8} X(t) + CA(t)^{\beta_5} + CA(t)^{\beta_6} + CA(t)^{\beta_7}, \end{aligned} \qquad (2.53)$$

$$\int_0^t \int_0^1 \frac{k\theta_x}{u} (\frac{k\theta_x}{u})_t \, dx \, ds \ge CY(t) - C, \qquad (2.54)$$

$$\begin{aligned} & |\int_0^t \int_0^1 \frac{k\theta_x}{u} (H_u v_{xx} + H_{uu} v_x u_x) dx \, ds| \\ \le \; & C \int_0^t \int_0^1 [(1+\theta)^{2q+1} |\theta_x| (|v_{xx}| + |v_x u_x|)] dx \, ds \\ \le \; & C (\int_0^t \int_0^1 (1+\theta)^{4q+2} \theta_x^2 \, dx \, ds)^{1/2} [(\int_0^t \|v_{xx}\|^2 ds)^{1/2} + (\int_0^t \|v_x u_x\|^2 ds)^{1/2}] \\ \le \; & CA(t)^{\max(3q+2-r,0)/2} (\int_0^t \int_0^1 (1+\theta)^{q+r} \theta_x^2 \, dx \, ds)^{1/2} \\ & [A(t)^{\beta_1/2} + \sup_{0 \le s \le t} \|u_x\| (\int_0^t \|v_x\|_{L^\infty}^2 ds)^{1/2}] \end{aligned}$$

$$\leq \quad CA(t)^{\max(3q+2-r,0)/2+(\beta_1+\beta_{10})/2} + CA(t)^{\max(3q+2-r,0)/2+(\beta_3+\beta+\beta_{10})/2}$$
$$\leq \quad CA(t)^{\beta_{11}} \tag{2.55}$$

where $\beta_1 \geq \beta_3 + \beta$. Similarly,

$$|\int_0^t\int_0^1 \frac{k\theta_x}{u}(H_u v_{xx} + H_{uu}v_x u_x)dx\,ds|$$
$$\leq C(\int_0^t\int_0^1(1+\theta)^{4q+2}\theta_x^2\,dx\,ds)^{1/2}[(\int_0^t\|v_{xx}\|^2 ds)^{1/2} + (\int_0^t\|v_x u_x\|^2 ds)^{1/2}]$$
$$\leq CA(t)^{(3q+4)/2}(\int_0^t V(s)ds)^{1/2}[A(t)^{\beta_1/2} + A(t)^{(\beta_3+\beta)/2}] \ \leq CA(t)^{(3q+4+\beta_1)/2}$$

which, together with (2.55), results in

$$|\int_0^t\int_0^1 \frac{k\theta_x}{u}(H_u v_{xx} + H_{uu}v_x u_x)dx\,ds| \leq CA(t)^{\beta_{12}}. \tag{2.56}$$

By Lemmas 2.1–2.10, we get

$$|\int_0^t\int_0^1 \frac{k\theta_x}{u}(\frac{k}{u})_u u_x\theta_t\,dx\,ds| \leq C\int_0^t\int_0^1 |\frac{k\theta_x}{u}|(1+\theta)^q|u_x\theta_t|dx\,ds$$
$$\leq \quad \frac{C_0}{8}X(t) + C\int_0^t\int_0^1(\frac{k\theta_x}{u})^2(1+\theta)^{q-r}u_x^2\,dx\,ds$$
$$\leq \quad \frac{C_0}{8}X(t) + CA(t)^{\max(q-r,0)+\beta}\int_0^t\|\frac{k\theta_x}{u}\|^2_{L^\infty}ds$$
$$\leq \quad \frac{C_0}{8}X(t) + CA(t)^{\max(q-r,0)+\beta}\int_0^t[\|\frac{k\theta_x}{u}\|^2 + \int_0^1|\frac{k\theta_x}{u}(\frac{k\theta_x}{u})_x|dx]ds$$
$$\leq \quad \frac{C_0}{8}X(t) + CA(t)^{\max(q-r,0)+\beta}[A(t)^{\max(q-r,0)}\int_0^t\int_0^1(1+\theta)^{q+r}\theta_x^2 dx\,ds$$
$$+(\int_0^t\int_0^1(1+\theta)^{q+r}\theta_x^2\,dx\,ds)^{1/2}(\int_0^t\int_0^1(1+\theta)^{q-r}|(\frac{k\theta_x}{u})_x|^2 dx\,ds)^{1/2}]$$
$$\leq \quad \frac{C_0}{8}X(t) + CA(t)^{2\max(q-r,0)+\beta+\beta_{10}} + CA(t)^{\max(q-r,0)+\beta+\beta_{10}/2}$$
$$\{X(t) + \int_0^t\int_0^1[(1+\theta)^{q+r+2}v_x^2 + (1+\theta)^{q-r}v_x^4]dx\,ds\}^{1/2}$$
$$\leq \quad \frac{C_0}{4}X(t) + CA(t)^{2\max(q-r,0)+2\beta+\beta_{10}}$$
$$+CA(t)^{\max(q-r,0)+\beta+(\beta_5+\beta_{10})/2} + CA(t)^{\max(q-r,0)+\beta+(\beta_7+\beta_{10}/2)}$$

$$\leq \quad \frac{C_0}{4}X(t) + CA(t)^{\beta_{13}}. \tag{2.57}$$

We can also deduce that

$$|\int_0^t\int_0^1 \frac{k\theta_x}{u}(\frac{k}{u})_u u_x\theta_t\,dx\,ds| \leq \frac{C_0}{8}X(t) + C\int_0^t\int_0^1 (\frac{k\theta_x}{u})^2(1+\theta)^{q-r}u_x^2 dx\,ds$$
$$\leq \frac{C_0}{8}X(t) + CA(t)^{\max(q-r,0)+\beta}$$
$$\times[\int_0^t\int_0^1 (1+\theta)^{2q}\theta_x^2\,dx\,ds + \int_0^t\int_0^1 |(1+\theta)^q\theta_x(\frac{k\theta_x}{u})_x|dx\,ds]$$
$$\leq \frac{C_0}{8}X(t) + CA(t)^{\max(q-r,0)+\beta}$$
$$\times[A(t)^{q+2} + (\int_0^t V(s)ds)^{1/2}(\int_0^t\int_0^1 \theta^2(1+\theta)^q|(\frac{k\theta_x}{u})_x|^2 dx\,ds)^{1/2}]$$
$$\leq \frac{C_0}{8}X(t) + CA(t)^{\max(q-r,0)+q+\beta+2}$$
$$+CA(t)^{\max(q-r,0)+\beta+(r+2)/2}(\int_0^t\int_0^1 (1+\theta)^{q-r}|(\frac{k\theta_x}{u})_x|^2 dx\,ds)^{1/2}$$
$$\leq \frac{C_0}{4}X(t) + CA(t)^{\max(q-r,0)+\beta+q+2} + CA(t)^{2\max(q-r,0)+2\beta+r+2}$$
$$+CA(t)^{\max(q-r,0)+(\beta_5+r+2)/2+\beta} + CA(t)^{\max(q-r,0)+(\beta_7+r+2)/2+\beta}$$
$$\leq \frac{C_0}{4}X(t) + CA(t)^{\beta_{14}}$$

which, along with (2.57), implies

$$|\int_0^t\int_0^1 \frac{k\theta_x}{u}(\frac{k}{u})_u u_x\theta_t\,dx\,ds| \leq \frac{C_0}{4}X(t) + CA(t)^{\beta_{15}}. \tag{2.58}$$

For $\eta = 0$ or 1, we have from (1.20) and (1.21)

$$|H_t(\eta,t)| = |(H_u v_x)(\eta,t)| = |\int_0^{T_0}(\frac{k(u(\eta,t),\xi)}{u(\eta,t)})_u\,d\xi v_x(\eta,t)|$$
$$\leq C|v_x(\eta,t)| \leq C\|v_x\|_{L^\infty}.$$

By Lemmas 2.1–2.10 and Young's inequality, we deduce that

$$|\int_0^t (\frac{k\theta_x H_t}{u})(\eta,s)ds| \leq C(\int_0^t \|v_x\|_{L^\infty}^2 ds)^{1/2}(\int_0^t \|\frac{k\theta_x}{u}\|_{L^\infty}^2 ds)^{1/2}$$
$$\leq CA(t)^{\beta_3/2}\{\int_0^t[(\int_0^1 \frac{k\theta_x}{u}dx)^2 + \int_0^1 |\frac{k\theta_x}{u}(\frac{k\theta_x}{u})_x|dx]ds\}^{1/2}$$
$$\leq CA(t)^{\beta_3/2}\{A(t)^{\max(q-r,0)}\int_0^t\int_0^1 (1+\theta)^{q+r}\theta_x^2\,dx\,ds$$

$$
\begin{aligned}
&+(\int_0^t\int_0^1(1+\theta)^{q+r}\theta_x^2\,dx\,ds)^{1/2}[\int_0^t\int_0^1(1+\theta)^{q-r}|(\frac{k\theta_x}{u})_x|^2dx\,ds]^{1/2}\}^{1/2}\\
&\le CA(t)^{(\max(q-r,0)+\beta_3+\beta_{10})/2}+CA(t)^{\beta_3/2+\beta_{10}/4}\\
&\times\{X(t)+\int_0^t\int_0^1[(1+\theta)^{q+r+2}v_x^2+(1+\theta)^{q-r}v_x^4]dx\,ds\}^{1/4}\\
&\le\frac{C_0}{8}X(t)+CA(t)^{(\beta_3+\beta_{10}+\max(q-r,0))/2}+CA(t)^{(2\beta_3+\beta_{10})/3} \qquad (2.59)\\
&+CA(t)^{\beta_3/2+(\beta_{10}+\beta_5)/4}+CA(t)^{\beta_3/2+(\beta_{10}+\beta_7)/4}\le\frac{C_0}{8}X(t)+CA(t)^{\beta_{16}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|\int_0^t(\frac{k\theta_xH_t}{u})(\eta,s)ds|\\
&\le CA(t)^{\beta_3/2}\{\int_0^t[(\int_0^1\frac{k\theta_x}{u}dx)^2+\int_0^1|\frac{k\theta_x}{u}(\frac{k\theta_x}{u})_x|dx]ds\}^{1/2}\\
&\le CA(t)^{\beta_3/2}[A(t)^{q+2}\int_0^tV(s)ds+(\int_0^tV(s)ds)^{1/2}\\
&\times(\int_0^t\int_0^1\theta^2(1+\theta)^q|(\frac{k\theta_x}{u})_x|^2dx\,ds)^{1/2}]^{1/2}\\
&\le CA(t)^{(\beta_3+q+2)/2}+CA(t)^{\beta_3/2+(r+2)/4}[\int_0^t\int_0^1(1+\theta)^{q-r}|(\frac{k\theta_x}{u})_x|^2dx\,ds]^{1/4}\\
&\le CA(t)^{(\beta_3+q+2)/2}+CA(t)^{\beta_3/2+(r+2)/4}\\
&\times\{X(t)+\int_0^t\int_0^1[(1+\theta)^{q+r+2}v_x^2+(1+\theta)^{q-r}v_x^4]dx\,ds\}^{1/4}\\
&\le\frac{C_0}{8}X(t)+CA(t)^{(\beta_3+q+2)/2}+CA(t)^{(2\beta_3+r+2)/3} \qquad (2.60)\\
&+CA(t)^{\beta_3/2+(\beta_5+r+2)/4}+CA(t)^{\beta_3/2+(\beta_7+r+2)/4}\le\frac{C_0}{8}X(t)+CA(t)^{\beta_{17}}.
\end{aligned}
$$

Thus (2.59) and (2.60) give

$$
|\int_0^t(\frac{k\theta_xH_t}{u})(\eta,s)ds|\le\frac{C_0}{8}X(t)+CA(t)^{\beta_{18}}. \qquad (2.61)
$$

Therefore it follows from (2.50)–(2.61) that

$$
\begin{aligned}
X(t)+Y(t)\le{}& CA(t)^{\beta_5}+CA(t)^{\beta_6}+CA(t)^{\beta_7}+CA(t)^{\beta_{12}}\\
&+CA(t)^{\beta_{15}}+CA(t)^{\beta_{18}}\le CA(t)^{\beta_{19}}.
\end{aligned}
$$

Lemma 2.12. $\forall\ t > 0$,

$$\|\theta\|_{L^\infty} \le C, \tag{2.62}$$

$$\int_0^1 [\theta_x^2 + u_x^2 + v_x^2]dx + \int_0^t \int_0^1 [u_x^2 + \theta_x^2 + \theta_t^2 + v_t^2 + v_x^2 + v_{xx}^2]dx\,ds \le C. \tag{2.63}$$

Proof. It follows from the Nirenberg inequality and Lemmas 2.10–2.11 that

$$\begin{aligned}\|\theta\|_{L^\infty}^{q+(r+3)/2} &\le C + C\int_0^1 |\theta^{q+(r+1)/2}\theta_x|dx \\ &\le C + CY^{1/2}(t)(\int_0^1 \theta^{r+1}dx)^{1/2} \le C + CY^{1/2}(t) \le CA(t)^{\beta_{19}/2}\end{aligned}$$

and

$$\|\theta\|_{L^\infty}^{q+r+2} \le C + CY^{1/2}(t)\|\theta^{r+1}\| \le CA(t)^{(\beta_8+\beta_{19})/2}.$$

Noting that a long, complicated calculation shows (1.11)–(1.15) imply $\beta_{19} < 2q + r + 3$ or $\beta_8 + \beta_{19} < 2q + 2r + 4$, by the Young inequality, we get (2.62). Inequality (2.63) is the direct consequence of Lemmas 2.1–2.10. □

3. Asymptotic behaviour

In this section, we shall sketch the proof of the results on asymptotic behaviour by use of a useful lemma by Shen and Zheng [13].

Lemma 3.1.

$$\int_0^t (\|p^*\|^2 + \|\sigma^*\|^2)ds \le C, \quad \forall \quad t > 0, \tag{3.1}$$

$$\frac{d}{dt}\|p^*\|^2 \le C(\|v_t\|^2 + \|\theta_t\|^2 + 1), \quad \forall \quad t > 0, \tag{3.2}$$

$$\frac{d}{dt}\|\sigma^*\|^2 \le C(\|v_t\|^2 + \|\theta_t\|^2 + 1), \quad \forall \quad t > 0. \tag{3.3}$$

Proof. See, e.g., [23] and [25]. □

Lemma 3.2.

$$\frac{d}{dt}\|u_x\|^2 \le \|v_{xx}\|^2 + \|u_x\|^2, \quad \forall \quad t > 0, \tag{3.4}$$

$$\frac{d}{dt}\|\theta_x\|^2 + C_7\int_0^1 (1+\theta)^{q-r}\theta_{xx}^2\,dx \le C(\|v_{xx}\|^2 + 1), \quad \forall \quad t > 0, \tag{3.5}$$

$$\|\theta_x\|^2 + \int_0^t \int_0^1 (1+\theta)^{q-r}\theta_{xx}^2\,dx\,ds \le C, \quad \forall \quad t > 0. \tag{3.6}$$

Proof. See, e.g., [23] and [25]. □

Lemma 3.3. (Shen and Zheng [13]) *Suppose that y and h are nonnegative functions on $[0, +\infty), y'$ is locally integrable, and y and h satisfy*

$$\forall t > 0 : y'(t) \leq A_1 y^2(t) + A_2 + h(t), \tag{3.7}$$

$$\forall T > 0 : \int_0^T y(s)ds \leq A_3, \int_0^T h(s)ds \leq A_4, \tag{3.8}$$

with A_1, A_2, A_3 and A_4 being positive constants independent of t and T. Then for any $r > 0$

$$\forall t \geq 0 : y(t+r) \leq (\frac{A_3}{r} + A_2 r + A_4)e^{A_1 A_2}. \tag{3.9}$$

Moreover,

$$\lim_{t\to+\infty} y(t) = 0. \tag{3.10}$$

Lemma 3.4. *As $t \to +\infty$, we have*

$$\|u - \overline{u_0}\|_{H^1} \to 0, \quad \|u_x\| \to 0, \quad \|u - \overline{u_0}\| \to 0, \tag{3.11}$$

$$\|v\|_{H^1} \to 0, \quad \|v_x\| \to 0, \tag{3.12}$$

$$\|\theta - T_0\|_{H^1} \to 0, \quad \|\theta_x\| \to 0, \|\theta - T_0\|_{L^\infty} \to 0, \tag{3.13}$$

$$\|p^*\| \to 0, \quad \|\sigma^*\| \to 0, \tag{3.14}$$

$$\|p(u,\theta) - p(\overline{u_0}, T_0)\|_{H^1} \to 0, \quad \|\sigma(u,\theta) - p(\overline{u_0}, T_0)\| \to 0 \tag{3.15}$$

where $(\overline{u_0}, 0, T_0)$ is the solution to the stationary problem corresponding to problem (1.1)–(1.3), (1.5)–(1.6). *Moreover, there exist positive constants C_1, C_2 and C_3 such that for all $t \geq C_1$, it holds that*

$$\|v(t)\|_{H^1} + \|\theta(t) - T_0\|_{H^1} + \|u(t) - \overline{u_0}\|_{H^1} \leq C_2 \exp(-C_3 t). \tag{3.16}$$

Proof. Estimates (3.11), (3.13) and (3.14) are the direct results by applying Lemmas 2.3–2.12 and Lemma 3.3. It is obvious that as $t \to +\infty$,

$$\|(\frac{v_x}{u})^*\|^2 \leq C(\|\sigma^*\|^2 + \|p^*\|^2) \to 0$$

and

$$\|v_x\| \leq C\|\frac{v_x}{u}\| \leq C(\|(\frac{v_x}{u})^*\| + \|\int_0^1 \frac{v u_x}{u^2}dx\|) \leq C(\|(\frac{v_x}{u})^*\| + \|u_x\|) \to 0$$

as $t \to +\infty$. Thus, $\|v\|_{H^1} \leq C\|v_x\| \to 0$.

The mean value theorem and (3.11)–(3.13) yield estimate (3.15). We easily verify that $(\bar{u}_0, 0, T_0)$ is the solution to the corresponding stationary problem (1.1)–(1.3), (1.5)–(1.6).

Now since $(u(t)-\bar{u}_0, v(t), \theta(t)-T_0)$ is small in the H^1 norm for sufficiently large t, we can deduce the desired estimate (3.16) by the same method as that in [27] (see Theorem 4.1).

Proofs of Theorem 1.1 and Theorem 2.1. The proof of Theorem 1.1 follows from Lemmas 2.1–2.12. Lemmas 2.5–2.12 yield (2.1) by the standard argument (see [3–4]), from which the proof of Theorem 2.1 also follows.

Remark 2.1. It follows from the proofs of Lemmas 2.1–2.12 and Theorem 1.1 that all the constants in Lemmas 2.1–2.12 depend only on the H^1 norm of the initial data (u_0, v_0, θ_0). Thus the following results of global existence, uniqueness and the same results of the asymptotic behaviour as Theorem 1.1 hold: If $(u_0, v_0, \theta_0) \in H^1 \times H_0^1 \times H^1$, problem (1.1)–(1.3), (1.5)–(1.6) admits a unique generalized solution $(u(t), v(t), \theta(t))$ in the sense that $u \in L^\infty(0, +\infty; H^1)$, $u'(t) \in L^\infty(0, +\infty; L^2)$ and $(v, \theta) \in L^\infty(0, +\infty; H^1) \cap L^2(0, +\infty; H^2) \cap H^1(0, +\infty; L^2)$.

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