Advances in Differential Equations Volume 7, Number 3, March 2002, Pages 343–364

SHARP L^p–HODGE DECOMPOSITIONS FOR LIPSCHITZ DOMAINS IN $\mathbb{R}^2$

Dorina Mitrea [1]
Department of Mathematics, University of Missouri-Columbia, Mathematical Sciences Building, Columbia, MO 65211

(Submitted by: Y. Giga)

1. Introduction and statement of main results

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$, with outward unit normal $\vec{\nu} = \{\nu_1, \nu_2\}$ and unit tangent $\vec{\tau} = \{-\nu_2, \nu_1\}$ defined almost everywhere with respect to the surface measure $d\sigma$. Recall that a Lipschitz domain is a domain whose boundary is locally given by graphs of Lipschitz functions. For more details see, e.g., [7]. For $1 < p < \infty$, we denote by L^p the space of p-integrable functions (which will be defined either over Ω or $\partial\Omega$), and by $L^p_1(\partial\Omega)$ the space of functions in $L^p(\partial\Omega)$ with tangential gradients in $L^p(\partial\Omega)$. Also, $H^{r,p}(\Omega)$, $r \in \mathbb{R}$, denotes the usual scale of Sobolev spaces on Ω, while $H^{r,p}_0(\Omega)$ denotes the space of distributions in $H^{r,p}(\mathbb{R}^2)$ with support contained in $\bar{\Omega}$; cf. [20].

As usual, $\nabla := \{\partial_1, \partial_2\}$, and for a vector field $\vec{u} = \{u_1, u_2\}$ with locally integrable components in Ω we set

$$\operatorname{div}\vec{u} := \partial_1 u_1 + \partial_2 u_2, \quad \operatorname{rot}\vec{u} := \partial_1 u_2 - \partial_2 u_1, \quad \nabla^t u := \{\partial_2 u, -\partial_1 u\}, \tag{1.1}$$

where the derivatives are considered in the sense of distributions. The spaces of harmonic L^p vector fields with vanishing normal or tangential traces are

$$\mathcal{H}^p_{\tan}(\Omega,\mathbb{R}^2) := \{\vec{u} \in L^p(\Omega,\mathbb{R}^2) : \operatorname{div}\vec{u} = 0,\ \operatorname{rot}\vec{u} = 0 \text{ in } \Omega,\ \vec{\nu}\cdot\vec{u} = 0\}, \tag{1.2}$$

$$\mathcal{H}^p_{\mathrm{nor}}(\Omega,\mathbb{R}^2) := \{\vec{u} \in L^p(\Omega,\mathbb{R}^2) : \operatorname{div}\vec{u} = 0,\ \operatorname{rot}\vec{u} = 0 \text{ in } \Omega,\ \vec{\tau}\cdot\vec{u} = 0\}. \tag{1.3}$$

Theorem 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$. There exists $\varepsilon = \varepsilon(\partial\Omega) \in (0, \frac{1}{4}]$ such that if $1/p_0 := \frac{3}{4} + \varepsilon$ and $1/p_0' := \frac{1}{4} - \varepsilon$, then for*

Accepted for publication: June 2001.
AMS Subject Classifications: 31A10, 58A14, 35J05; 42B20, 45E05.
[1]Supported in part by a University of Missouri Research Board Grant and a Summer Research Fellowship.

each $p_0 < p < p_0'$ the following decompositions hold:

$$L^p(\Omega, \mathbb{R}^2) = \nabla H_0^{1,p}(\Omega) \oplus \nabla^t H^{1,p}(\Omega) \oplus \mathcal{H}^p_{\rm nor}(\Omega, \mathbb{R}^2), \tag{1.4}$$

$$L^p(\Omega, \mathbb{R}^2) = \nabla H^{1,p}(\Omega) \oplus \nabla^t H_0^{1,p}(\Omega) \oplus \mathcal{H}^p_{\rm tan}(\Omega, \mathbb{R}^2), \tag{1.5}$$

where the sums are direct and topological. In fact, for $p_0 < p < p_0'$ we have $\mathcal{H}^p_{\rm nor}(\Omega, \mathbb{R}^2) = \mathcal{H}^2_{\rm nor}(\Omega, \mathbb{R}^2)$ and $\mathcal{H}^p_{\rm tan}(\Omega, \mathbb{R}^2) = \mathcal{H}^2_{\rm tan}(\Omega, \mathbb{R}^2)$. Also, the range $p \in \left[\frac{4}{3}, 4\right]$ for which the decompositions (1.4)–(1.5) *are valid is sharp in the class of Lipschitz domains. However, when $\partial\Omega \in C^1$ one can take $1 < p < \infty$.*

We would like to point out that Theorem 1.1 illustrates the fundamental way in which the smoothness of the underlying domain affects the validity of the Hodge decompositions. Indeed, it has been known for a long time that if $\partial\Omega \in C^\infty$ then the decompositions (1.4)–(1.5) hold for each $1 < p < \infty$. In fact, corresponding Hodge-type decompositions hold true in this latter setting for differential forms of arbitrary degree defined on smooth subdomains of Riemannian manifolds of arbitrary dimension m (cf. [18] and [33]). However, if $\partial\Omega$ is allowed to have irregularities, then the range of indices dramatically changes. The case $|p-2|$ small has been dealt with for Lipschitz domains in Riemannian manifolds (of arbitrary real dimension $m \geq 2$) in [29]. Also, the work in [13] shows that if $m \geq 3$ and $3/2 \leq p \leq 3$, then the Helmholtz decomposition

$$L^p(\Omega, \mathbb{R}^m) = \nabla H^{1,p}(\Omega) \oplus \{\vec{u} \in L^p(\Omega, \mathbb{R}^m) : {\rm div}\, \vec{u} = 0,\ \vec{\nu} \cdot \vec{u} = 0\} \tag{1.6}$$

holds for any Lipschitz domain $\Omega \subset \mathbb{R}^m$. In sharp contrast with the case of smooth domains when any $1 < p < \infty$ will do (cf. [14] and [35]), if $p \notin [3/2, 3]$ then such a decomposition may fail in the class of Lipschitz domains. The aforementioned works leave open the two-dimensional case and the issues of dealing with *Hodge* decompositions instead of the *Helmholtz* decompositions.

Our paper clarifies the *sharp* range of p's for which the Hodge decompositions of $L^p(\Omega, \mathbb{R}^m)$ hold in the situation when $m = 2$ and Ω is an arbitrary Lipschitz domain. This settles a fundamental case of a classical problem whose origins can be traced far back. On the technical side, our approach is constructive and relies on scalar layer potentials, the recent progress in the study of *vector* potential theory from [30] and classical embedding theorems.

These results are also important in physical applications due to the fact that Hodge-type decompositions for the velocity vectors in $\mathbb{R}^2$ have proved to be powerful tools in the treatment of a variety of problems in fluid dynamics. Since most "real-life" models involve domains that are far from being smooth, Theorem 1.1 allows for significant improvements and extensions of such results in mathematical physics.

We mention one example related to Euler flows on rough planar domains. With D standing for a bounded convex domain in $\mathbb{R}^2$, set $V^0(D) := \{\vec{v} \in L^2(D, \mathbb{R}^2) : \text{div}\, \vec{v} = 0,\, \vec{\nu} \cdot \vec{v} = 0\}$. Then the approach developed in [37] augmented with Theorem 1.1 yields that for each $p \in (8/5, 2)$ and $\vec{v} \in V^0(D) \cap H^{1,p}(D)$ there exists a unique $\vec{u} \in L^\infty(\mathbb{R}, H^{1,p}(D)) \cap V^0(D)$ which solves

$$\begin{cases} \partial_t \vec{u} + (\sum_j u_j \partial_j u_i)_i = -\nabla \mathbf{p} \;\text{ in }\; \mathbb{R} \times D, \\ \text{div}\, \vec{u} = 0 \;\text{ in }\; \mathbb{R} \times D, \\ \vec{u}|_{t=0} = \vec{v} \;\text{ in }\; D \end{cases}$$

and, for $r := \frac{2p}{4-p}$, satisfies $\partial_t \vec{u} \in L^\infty(\mathbb{R}, L^r(D))$, $\nabla \mathbf{p} \in L^\infty(\mathbb{R}, L^r(D))$. This improves on a recent result of M. Taylor in chapter III, section 6 of [37] (which, in turn, builds on the earlier work in [10]). With further applications to fluid dynamics in mind, in the light of the present work, it is also natural to make the following

Conjecture 1.2. *For each $\Omega \subset \mathbb{R}^2$ a bounded Lipschitz domain there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that the associated Stokes operator generates an analytic semigroup in $L^p(\Omega, \mathbb{R}^2)$ provided $4/3 - \varepsilon < p < 4 + \varepsilon$.*

Note that, thanks to Theorem 1.1, the range of p's above is precisely that for which the so-called Leray projection is bounded. For the case of smooth domains a result in this spirit has been proved in all dimensions and for any $1 < p < \infty$ in [17].

The portion of our work dealing with Hodge decompositions is part of a broader, ongoing program aimed at developing a two-dimensional potential theory for vector fields in nonsmooth domains. The case of Hodge decompositions for $H^{s,p}$ vector fields is discussed in [28]. In a forthcoming paper we plan to consider the physically relevant case of *unbounded* Lipschitz domains, via layer potential techniques. For C^1 domains with compact boundaries see [35] for a functional analytic approach in the L^p-context; some special cases of domains with noncompact boundaries have also been studied in [36]. Another interesting open problem is that of finding an analogue of the decompositions discussed in [16] for nonsmooth domains.

The organization of the paper is as follows. In Section 3 we state and prove the well-posedness of some vector potential boundary problems. In turn, these are used in the course of the proof of the Theorem 1.1, which is given in Section 4. At the end of this section we also prove other related decompositions of independent interest.

2. Definitions and preliminary results

Throughout the paper, $C = C(\partial\Omega)$ will denote various constants which depend exclusively on the Lipschitz character of Ω, and ε will stand for a *sufficiently* small, positive quantity whose actual value may change from one occurrence to another. For each $0 < |s| < 1$, $1 \leq p, q \leq \infty$, let $B^{p,q}_s(\partial\Omega)$ stand for the usual scale of Besov spaces over $\partial\Omega$ with smoothness s. The latter can be introduced via interpolation and duality. For example, real interpolation gives that $(L^p(\partial\Omega), L^p_1(\partial\Omega))_{\theta,q} = B^{p,q}_\theta(\partial\Omega)$, with $0 < \theta < 1$, $1 < q < \infty$, whereas the spaces $B^{p,q}_s(\partial\Omega)$ with $-1 < s < 0$ are defined from the latter by duality. We will also work with Besov spaces over Ω, in which case, $\big(L^p(\Omega), H^{1,p}(\Omega)\big)_{\theta,q} = B^{p,q}_\theta(\Omega)$. For more complete definitions see [2], [1], and [32]. With Tr standing for the trace operator on $\partial\Omega$, it is known that

$$\mathrm{Tr} : H^{r,p}(\Omega) \longrightarrow B^{p,p}_{r-\frac{1}{p}}(\partial\Omega), \quad \text{for } \tfrac{1}{p} < r < 1 + \tfrac{1}{p},\ 1 < p < \infty. \tag{2.1}$$

It turns out that for $\frac{1}{p} < r < 1 + \frac{1}{p}$, $H^{r,p}_0(\Omega) = \{v \in H^{r,p}(\Omega) : \mathrm{Tr}\, v = 0\}$ (see Proposition 3.3 in [20]). For $1 < p < \infty$ and $\vec{w} \in L^p(\Omega, \mathbb{R}^2)$ with $\mathrm{div}\ \vec{w} \in L^p(\Omega)$ define $\vec{\nu} \cdot \vec{w} \in B^{p,p}_{-\frac{1}{p}}(\partial\Omega)$ by

$$\langle \vec{\nu} \cdot \vec{w}, \varphi \rangle := \langle \mathrm{div}\ \vec{w}, \Phi \rangle + \langle \vec{w}, \nabla\Phi \rangle, \qquad \forall\, \varphi \in B^{q,q}_{\frac{1}{p}}(\partial\Omega), \tag{2.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and $\Phi \in H^{1,q}(\Omega)$ is such that $\mathrm{Tr}\, \Phi = \varphi$. Clearly, one has norm estimates associated with the definition (2.2). Similarly, for $\vec{u} \in L^p(\Omega, \mathbb{R}^2)$ with $\mathrm{rot}\, \vec{u} \in L^p(\Omega)$, we can define $\vec{\tau} \cdot \vec{u} \in B^{p,p}_{-\frac{1}{p}}(\partial\Omega)$ by

$$\langle \vec{\tau} \cdot \vec{u}, \mathrm{Tr}\, \Psi \rangle = \langle \mathrm{rot}\, \vec{u}, \Psi \rangle - \langle \vec{u}, \nabla^t \Psi \rangle, \quad \forall\, \Psi \in H^{1,q}(\Omega), \tag{2.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Once again, natural norm estimates hold. When $\vec{u}$ is regular enough, so that it has a trace on $\partial\Omega$ (in the sense of (2.1)), then the definition given above for $\vec{\tau} \cdot \vec{u}$ coincides with the pointwise product $\vec{\tau} \cdot \mathrm{Tr}(\vec{u})$. A similar comment applies to $\vec{\nu} \cdot \vec{w}$ in (2.2). We conclude this section by recalling some basic results about layer potentials on Lipschitz domains. Let $\Delta := \partial_1^2 + \partial_2^2$ denote the Laplacian in $\mathbb{R}^2$. For $k \in \mathbb{C} \setminus \{0\}$, a fundamental solution of the Helmholtz operator $\Delta + k^2$ is

$$\Gamma_k(x) := -\tfrac{i}{4} H^{(1)}_0(k|x|), \quad x \in \mathbb{R}^2 \setminus \{0\}, \tag{2.4}$$

where $H_0^{(1)}$ is the first Hankel function which has a singularity of the type $\ln|x|$ at $x = 0$. When $k = 0$ we take $\Gamma_0(x) := \frac{1}{2\pi}\ln|x|$. For the rest of the paper we will use the notation $\Gamma_k(x, y) := \Gamma_k(x - y)$, for $x, y \in \mathbb{R}^2$, $x \neq y$. Then one defines the single- and double-layer potential operators by formally setting

$$\mathcal{S}_k f(x) := \int_{\partial\Omega} \Gamma_k(x, y) f(y)\, d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial\Omega, \tag{2.5}$$

$$\mathcal{D}_k f(x) := \int_{\partial\Omega} \frac{\partial\Gamma_k(y, x)}{\partial\nu(y)} f(y)\, d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial\Omega, \tag{2.6}$$

for $f : \partial\Omega \to \mathbb{R}$. Moreover, the Newtonian potential associated with Γ_k is defined by

$$\Pi_k u(x) := \iint_\Omega \Gamma_k(x, y) u(y)\, dy, \quad x \in \Omega, \tag{2.7}$$

for $u : \Omega \to \mathbb{R}$ (on vectors, this acts componentwise). Recall next that the nontangential maximal operator $\mathcal{N}(\cdot)$ acting on a function $u : \Omega \to \mathbb{R}$ is given at each boundary point x by

$$\mathcal{N}(u)(x) := \sup\{|u(y)| : y \in \Omega,\ |x - y| \leq 2\,\mathrm{dist}\,(y, \partial\Omega)\}. \tag{2.8}$$

Finally, $u|_{\partial\Omega}$ will denote the restriction of u to the boundary in the (pointwise) nontangential limit sense (whenever this makes sense).

The main properties of the operators listed above which are relevant for us here are collected in the following theorem.

Theorem 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$, $k \in \mathbb{C}$. Then the following hold.*

(1) *For each $1 < p < \infty$, the single layer satisfies*

$$\|\mathcal{N}(\nabla\mathcal{S}_k f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}, \tag{2.9}$$

uniformly for $f \in L^p(\partial\Omega)$. Also, $\mathcal{S}_k f|_{\partial\Omega_+} = \mathcal{S}_k f|_{\partial\Omega_-} =: S_k f$.

(2) *For each $1 < p < \infty$, the double layer $\mathcal{D}_k$ satisfies*

$$\|\mathcal{N}(\mathcal{D}_k f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}, \tag{2.10}$$

and

$$\mathcal{D}_k f|_{\partial\Omega_\pm} = \left(\pm\tfrac{1}{2}I + K_k\right) f, \tag{2.11}$$

almost everywhere on $\partial\Omega$, for each $f \in L^p(\partial\Omega)$. Hereafter, I will stand for the identity while K_k will denote the principal-value operator

$$K_k f(x) := p.v.\int_{\partial\Omega} \frac{\partial\Gamma_k(y,x)}{\partial\nu(y)} f(y)\,d\sigma(y), \quad \text{for a.e. } x\in\partial\Omega. \tag{2.12}$$

Also, for each $1<p<\infty$,

$$K_k : L^p(\partial\Omega)\to L^p(\partial\Omega), \quad S_k : L^p(\partial\Omega)\to L^p_1(\partial\Omega) \text{ are bounded.} \tag{2.13}$$

(3) *If* K_k^* *is the formal adjoint of* K_k (*and* ∂_ν *denotes the normal derivative*) *then, for each* $1<p<\infty$,

$$\partial_\nu \mathcal{S}_k f|_{\partial\Omega_\pm} = \left(\mp\tfrac{1}{2}I + K_k^*\right) f, \tag{2.14}$$

almost everywhere on $\partial\Omega$, *for each* $f\in L^p(\partial\Omega)$.

(4) *The Newtonian potential*

$$\Pi_k : L^p(\Omega)\to H^{2,p}(\Omega) \quad \text{is bounded for each } 1<p<\infty. \tag{2.15}$$

(5) *Set* $p\vee 2 := max\{p,2\}$. *Then the operators*

$$\mathcal{D}_k : L^p(\partial\Omega)\to B^{p,p\vee 2}_{\frac{1}{p}}(\Omega), \quad \mathcal{D}_k : B^{p,p}_s(\partial\Omega)\to H^{s+\frac{1}{p},p}(\Omega), \tag{2.16}$$

$$\mathcal{S}_k : L^p(\partial\Omega)\to B^{p,p\vee 2}_{1+\frac{1}{p}}(\Omega), \quad \mathcal{S}_k : B^{p,p}_{-s}(\partial\Omega)\to H^{1+\frac{1}{p}-s,p}(\Omega), \tag{2.17}$$

are bounded for each $1<p<\infty$ *and* $0<s<1$.

(6) *There exists* $\varepsilon=\varepsilon(\partial\Omega)>0$ *such that the operators*

$$\pm\tfrac{1}{2}I + K_k : L^p_1(\partial\Omega)\to L^p_1(\partial\Omega), \quad \pm\tfrac{1}{2}I + K_k : L^q(\partial\Omega)\to L^q(\partial\Omega), \tag{2.18}$$

and

$$\pm\tfrac{1}{2}I + K_k^* : L^p(\partial\Omega)\to L^p(\partial\Omega), \quad \pm\tfrac{1}{2}I + K_k^* : L^q_{-1}(\partial\Omega)\to L^q_{-1}(\partial\Omega), \tag{2.19}$$

are isomorphisms for each $1<p<2+\varepsilon$, $1/p+1/q=1$, *and* $k\in i\mathbb{R}_+$ (*hereafter* $\mathbb{R}_+$ *will stand for* $\{a\in\mathbb{R} : a>0\}$). *Moreover, for each* $k\in i\mathbb{R}_+\cup\{0\}$, *the operators*

$$S_k : L^p(\partial\Omega)\to L^p_1(\partial\Omega), \quad S_k : L^q_{-1}(\partial\Omega)\to L^q(\partial\Omega) \tag{2.20}$$

are also isomorphisms for each $1<p<2+\varepsilon$, $1/p+1/q=1$.

(7) *There exists* $\varepsilon=\varepsilon(\partial\Omega)>0$ *such that the operators*

$$\pm\tfrac{1}{2}I + K_k^* : B^{p,p}_{-1/p}(\partial\Omega)\to B^{p,p}_{-1/p}(\partial\Omega) \tag{2.21}$$

are isomorphisms for each $4/3-\varepsilon<p<4+\varepsilon$.

When $\partial\Omega\in C^1$, *then all invertibility results are valid for* $p\in(1,\infty)$.

Proof. (7) is proved in [28]. For (1)–(6) corresponding to the case $k = 0$, we refer the reader to [12], [11], [38], [19], [8] and [31]. The more general situation $k \in \mathbb{C}$ considered here is just a minor variation of these known results. The key observation is that

$$\Gamma_k(x, y) = \Gamma_0(x, y) + C_k + \mathcal{O}\big(|x - y|^2 \ln |x - y|\big) \tag{2.22}$$

as $|x - y| \to 0$, where $C_k \in \mathbb{C}$ is a constant which depends exclusively on k. See, e.g., [6], p. 65; cf. also [15] pp. 59–74. Consequently, the classical layer potentials associated with Γ_0 differ from the ones associated with Γ_k by integral operators whose kernels are only weakly singular. In particular, these residual operators do not affect the index and have no contribution to the jump formulas. With this in mind, the desired conclusions readily follow. □

Occasionally, when $k = 0$ the subscript "zero" is dropped altogether and we simply write Π, $\mathcal{D}$, $\mathcal{S}$, K, S, etc.

We close this section by showing that the spaces (1.2)–(1.3) are invariant with respect to p as long as $4/3 - \varepsilon < p < 4 + \varepsilon$. To state this result, recall that $b_1(\Omega)$ stands for the first Betti number of Ω, i.e., the number of "holes" in Ω.

Proposition 2.2. *Let Ω be an arbitrary bounded Lipschitz domain in $\mathbb{R}^2$. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that the spaces $\mathcal{H}^p_{\rm nor}(\Omega, \mathbb{R}^2)$ and $\mathcal{H}^p_{\rm tan}(\Omega, \mathbb{R}^2)$ are independent of p as long as $4/3-\varepsilon < p < 4+\varepsilon$. In particular, for each such p,*

$$\dim \mathcal{H}^p_{\rm nor}(\Omega, \mathbb{R}^2) = \dim \mathcal{H}^p_{\rm tan}(\Omega, \mathbb{R}^2) = b_1(\Omega). \tag{2.23}$$

If $\partial\Omega \in C^1$ one can take $1 < p < \infty$.

Proof. Let $\varepsilon > 0$ be small enough so that the results stated so far hold true. We will prove that

$$\mathcal{H}^p_{\rm tan}(\Omega, \mathbb{R}^2) = \mathcal{H}^2_{\rm tan}(\Omega, \mathbb{R}^2) \quad \text{for any } \tfrac{4}{3} - \varepsilon < p < 4 + \varepsilon. \tag{2.24}$$

To prove the left-to-right inclusion, let $4/3 - \varepsilon < p < 4 + \varepsilon$, and $\vec{u} \in \mathcal{H}^p_{\rm tan}(\Omega, \mathbb{R}^2)$. Via repeated integrations by parts it follows that

$$\vec{u} = k^2 \Pi_k(\vec{u}) + \nabla^t \mathcal{S}_k(\vec{\tau} \cdot \vec{u}), \quad \text{in } \Omega, \tag{2.25}$$

and taking $\vec{\tau}\cdot$ of (2.25) we have

$$(\tfrac{1}{2} I + K^*_k)(\vec{\tau} \cdot \vec{u}) = k^2\, \vec{\tau} \cdot \Pi_k(\vec{u}). \tag{2.26}$$

Since by (4) in Theorem 2.1 and embedding theorems we get rot $\Pi_k(\vec{u}) \in H^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, it follows (by definition) that $\vec{\tau} \cdot \Pi_k(\vec{u}) \in B^{2,2}_{-1/2}(\partial\Omega)$.

Combining this, the identity (2.26), and (7) in Theorem 2.1, we conclude that $\vec{\tau}\cdot\vec{u}\in B^{2,2}_{-1/2}(\partial\Omega)$. Returning with this information in (2.25) and relying on (2.17), we obtain $\vec{u}\in L^2(\Omega,\mathbb{R}^2)$. This proves the left-to-right inclusion in (2.24).

In addition, for $\vec{u}\in\mathcal{H}^2_{\tan}(\Omega,\mathbb{R}^2)$ and $\vec{\tau}\cdot\Pi_k(\vec{u})=\vec{\tau}\cdot\mathrm{Tr}(\Pi_k(\vec{u}))\in L^2(\partial\Omega,\mathbb{R}^2)$ (by (4) in Theorem 2.1). The latter used in (2.26) in combination with (6) of Theorem 2.1,shows that $\vec{\tau}\cdot\vec{u}\in L^2(\partial\Omega,\mathbb{R}^2)$. Moreover, from (2.25) and (2.17), we get that $\mathcal{H}^2_{\tan}(\Omega,\mathbb{R}^2)\subset B^{2,2}_{1/2}(\Omega,\mathbb{R}^2)\hookrightarrow L^{2+\varepsilon}(\Omega,\mathbb{R}^2)$. This reasoning applied one more time yields

$$\mathcal{H}^2_{\tan}(\Omega,\mathbb{R}^2)\hookrightarrow B^{2+\varepsilon,2}_{\frac{1}{2+\varepsilon}}(\Omega,\mathbb{R}^2)\hookrightarrow L^q(\Omega,\mathbb{R}^2),\quad \forall\, q<4+\varepsilon. \tag{2.27}$$

The right-to-left inclusion in (2.24) is proved. That similar conclusions apply to the case of harmonic vector fields with vanishing tangential components, is a consequence of what we have proved up to this point together with the identity

$$\mathcal{H}^p_{\tan}(\Omega,\mathbb{R}^2)=*\mathcal{H}^p_{\mathrm{nor}}(\Omega,\mathbb{R}^2), \tag{2.28}$$

where the Hodge star-isomorphism is defined by

$$*\vec{u}:=\{-u_2,u_1\},\quad \text{if } \vec{u}=\{u_1,u_2\}. \tag{2.29}$$

Finally, the case when $\partial\Omega\in C^1$ follows along similar lines. The proof is finished.

3. Vector potential theory

In this section we formulate and solve a Poisson-type problem for the *vector* Laplacian. Besides its intrinsic interest, this is going to be a major ingredient in the proof of Hodge decompositions (a task we take up in Section 6).

Theorem 3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$. Then there exists $\varepsilon=\varepsilon(\partial\Omega)>0$ such that, for each $4/3-\varepsilon<p<4+\varepsilon$, the boundary value problem*

$$(BVP_1)\begin{cases}\Delta\vec{u}=\vec{\eta}\in L^p(\Omega,\mathbb{R}^2), \quad \vec{u}\in L^p(\Omega,\mathbb{R}^2),\\ \mathrm{rot}\,\vec{u},\ \mathrm{div}\,\vec{u}\in H^{1,p}(\Omega),\\ \vec{\nu}\cdot\vec{u}=0 \ \mathit{in}\ B^{p,p}_{-\frac{1}{p}}(\partial\Omega),\\ \mathrm{Tr}(\mathrm{rot}\,\vec{u})=0\ \mathit{in}\ B^{p,p}_{1-\frac{1}{p}}(\partial\Omega),\end{cases}$$

has a solution if and only if

$$\vec{\eta}\in\{\mathcal{H}^q_{\tan}(\Omega,\mathbb{R}^2)\}^\circ,\qquad \tfrac{1}{p}+\tfrac{1}{q}=1, \tag{3.1}$$

where $\{\dots\}^\circ$ *denotes the annihilator of* $\{\dots\} \subseteq H^q(\Omega,\mathbb{R}^2)$ *in* $L^p(\Omega,\mathbb{R}^2) = \big(H^q(\Omega,\mathbb{R}^2)\big)^*$.

The space of null solutions for (BVP_1) *is precisely* $\mathcal{H}^2_{\tan}(\Omega,\mathbb{R}^2)$. *Uniqueness is assured by imposing the normalization*

$$\vec{u} \in \{\mathcal{H}^q_{\tan}(\Omega,\mathbb{R}^2)\}^\circ. \tag{3.2}$$

Moreover, if $\partial\Omega \in C^1$ *the above conclusions hold for* $1 < p < \infty$.

Proof. If a solution to (BVP_1) exists, routine integrations by parts show that (3.1) is true. Conversely, suppose (3.1) holds and let $\varepsilon > 0$ be small enough so that the results proved or recalled so far hold. For each $p \in (1,\infty)$ define

$$p_\sharp := \begin{cases} \infty, & \text{if } p \geq 2, \\ \frac{p}{2-p}, & \text{if } 1 < p < 2. \end{cases} \tag{3.3}$$

Also set

$$p^* := \min\{2+\varepsilon, p_\sharp\}. \tag{3.4}$$

It is not hard to see that for any $p \in \left(\frac{4}{3}-\epsilon, 4+\epsilon\right)$ one has

$$p_\sharp > p, \quad p_\sharp > 2-\delta, \tag{3.5}$$

where $\delta := \delta(\epsilon) = 9\epsilon/2 \to 0$, as $\epsilon \to 0$. Since $\vec{\eta} \in L^p(\Omega,\mathbb{R}^2)$, using (2.1) and (4) in Theorem 2.1, we get $\mathrm{Tr}(\mathrm{rot}\,\Pi_0(\vec{\eta})) \in B^{p,p}_{1-\frac{1}{p}}(\partial\Omega) \hookrightarrow L^{p^*}(\partial\Omega)$ (the last inclusion follows from the definition of p^* and standard embedding theorems). The strategy is to look for a solution of (BVP_1) in the form

$$\vec{u} := \Pi_0\vec{\eta} + \vec{w}, \tag{3.6}$$

where, in turn, $\vec{w}$ is a solution of

$$(BVP_2) \begin{cases} \Delta\vec{w} = 0 \text{ in } \Omega, \\ \mathcal{N}(\vec{w}), \mathcal{N}(\mathrm{rot}\,\vec{w}) \in L^{p^*}(\partial\Omega), \\ \vec{\nu}\cdot\vec{w} = -\vec{\nu}\cdot\mathrm{Tr}(\Pi_0\vec{\eta}) \in L^{p^*}(\partial\Omega), \\ (\mathrm{rot}\,\vec{w})|_{\partial\Omega} = -\,\mathrm{Tr}(\mathrm{rot}(\Pi_0\vec{\eta})) \in L^{p^*}(\partial\Omega). \end{cases}$$

Recall that $\mathcal{N}(\cdot)$ is the nontangential maximal operator; see (2.8). From the results in Chapter 5 of [30] we know that (BVP_2) is solvable for small $\varepsilon > 0$ if and only if

$$\vec{\tau}\ \mathrm{Tr}[\mathrm{rot}\,\Pi_0(\vec{\eta})] \in \left\{\vec{h}|_{\partial\Omega} : \vec{h} \in \mathcal{H}^q_{\tan}(\Omega,\mathbb{R}^2)\right\}^\circ, \tag{3.7}$$

$\frac{1}{q}+\frac{1}{p^*} = 1$, where $\{\dots\}^\circ$ denotes the annihilator of $\{\dots\}$ in $L^{p^*}(\partial\Omega,\mathbb{R}^2)$. It should be pointed out that the space whose annihilator is taken in (3.7) is well defined. Indeed, it has been proved in [30] that if $\vec{h} \in \mathcal{H}^q_{\tan}(\Omega)$, $2-\varepsilon <$

$q < 2 + \varepsilon$, then $\mathcal{N}(\vec{h}) \in L^q(\partial\Omega)$, and since $\Delta\vec{h} = \nabla(\operatorname{div}\vec{h}) - \nabla^t(\operatorname{rot}\vec{h}) = 0$, the results in [7] imply that $\vec{h}|_{\partial\Omega}$ exists and belongs to $L^q(\partial\Omega)$.

Now, Proposition 2.2 and straightforward integrations by parts give that (3.7) is equivalent to (3.1), which we assume to be true. Moreover, based on [30], we know that under the assumption (3.7) a solution to (BVP_2) exists and has the form

$$\vec{w} = \mathcal{S}_0(\vec{f}) + \nabla\mathcal{S}_0(g), \tag{3.8}$$

for some $\vec{f} \in L^{p^*}(\partial\Omega, \mathbb{R}^2)$ and $g \in L^{p^*}(\partial\Omega)$. The smoothness conditions required for $\vec{u}$ to qualify as a solution of (BVP_1) readily translate, thanks to (3.6) and Theorem 2.1, into

$$\vec{w} \in L^p(\Omega, \mathbb{R}^2),\ \ \operatorname{rot}\vec{w},\ \ \operatorname{div}\vec{w} \in H^{1,p}(\Omega). \tag{3.9}$$

To this end, observe first that for any $\frac{4}{3} - \varepsilon < p < 4 + \varepsilon$, the embedding

$$B^{p^*, p^*\vee 2}_{\frac{1}{p^*}}(\Omega, \mathbb{R}^2) \hookrightarrow L^p(\Omega, \mathbb{R}^2) \tag{3.10}$$

holds, and hence, $\vec{w} \in L^p(\Omega, \mathbb{R}^2)$. In order to prove that $\operatorname{div}\vec{w} \in L^p(\Omega)$ we employ the integral representation formula:

$$\begin{aligned} \operatorname{div}\vec{w} &= k^2 \operatorname{div}\Pi_k(\vec{w}) - k^2\mathcal{D}_k\left[\left(-\tfrac{1}{2}I + K_k\right)^{-1}(\operatorname{Tr}(\operatorname{div}\Pi_k(\vec{w})))\right] \\ &- \operatorname{div}\mathcal{S}_k[\vec{\tau}\operatorname{Tr}(\operatorname{rot}\Pi_0(\vec{\eta})] + \mathcal{D}_k\left[\left(-\tfrac{1}{2}I + K_k\right)^{-1}(\operatorname{Tr}\operatorname{div}\mathcal{S}_k(\vec{\tau}\operatorname{Tr}(\operatorname{rot}\Pi_0(\vec{\eta}))))\right] \\ &- k^2\mathcal{S}_k(\vec{\nu}\cdot\operatorname{Tr}(\Pi_0(\vec{\eta}))) - k^2\mathcal{D}_k\left[\left(-\tfrac{1}{2}I + K_k\right)^{-1}(S_k(\vec{\nu}\cdot\operatorname{Tr}(\Pi_0(\vec{\eta}))))\right], \end{aligned} \tag{3.11}$$

where $k \in i\mathbb{R}_+$. The proof of (3.11) is modeled upon [30]; so as not to interrupt the flow of the presentation, we sketch the proof of (3.11) at the end of this section. Granted (3.11), we arrive at $\operatorname{div}\vec{w} \in H^{1,p}(\Omega)$ in the following way. For the first term in the right-hand side of (3.11) we use (4) in Theorem 2.1 and the fact that $\vec{w} \in L^p(\Omega)$. Also, by (2.1), we get $\operatorname{Tr}(\operatorname{div}\Pi_k(\vec{w})) \in B^{p,p}_{1-\frac{1}{p}}(\partial\Omega)$; hence, $\left(-\frac{1}{2}I + K_k\right)^{-1}(\operatorname{Tr}(\operatorname{div}\Pi_k(\vec{w}))) \in B^{p,p}_{1-\frac{1}{p}}(\partial\Omega)$ based on (7) in Theorem 2.1. The latter together with (5) in Theorem 2.1 give that the second term in the right-hand side of (3.11) also belongs to $H^{1,p}(\Omega)$. The other terms in (3.11) are treated similarly, and the conclusion that $\operatorname{div}\vec{w} \in H^{1,p}(\Omega)$ follows.

Next, $\Delta(\operatorname{rot}\vec{w}) = 0$ and $(\operatorname{rot}\vec{w})|_{\partial\Omega} = -\operatorname{Tr}(\operatorname{rot}(\Pi_0\vec{\eta})) \in B^{p,p}_{1-\frac{1}{p}}(\partial\Omega)$. The well-posedness of the Dirichlet problem for the Laplacian with boundary trace in $B^{p,p}_{1-\frac{1}{p}}(\partial\Omega)$ is proved in [28]. Using the latter, we conclude that $\operatorname{rot}\vec{w} \in H^{1,p}(\Omega)$.

Summing up, we have shown, modulo the proof of (3.11), that $\vec{u}$ as in (3.6) is a solution of (BVP_1). If $\partial\Omega \in C^1$, then (BVP_2) is solvable for any $p^* \in (1,\infty)$. As a consequence, (BVP_1)–(3.1) has a solution for any $1 < p < \infty$ in this case. The fact that the space of null solutions for (BVP_1) is $\mathcal{H}^2_{\tan}(\Omega,\mathbb{R}^2)$ follows from Proposition 2.2. There remains the proof of (3.11), which we dispose of with the aid of the general identity discussed below. □

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $k \in i\,\mathbb{R}_+$. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that if $\vec{u}$ is a harmonic vector field in Ω with $\mathcal{N}(\vec{u})$ and $\mathcal{N}(\operatorname{rot}\vec{u}) \in L^p(\partial\Omega)$, for some $2-\varepsilon < p < 2+\varepsilon$, then*

$$
\begin{aligned}
\vec{u} &= k^2\Pi_k(\vec{u}) + k^2\nabla^t\mathcal{S}_k\left[\left(\tfrac{1}{2}I + K_k^*\right)^{-1}(\vec{\tau}\cdot\operatorname{Tr}\Pi_k(\vec{u}))\right] \\
&\quad - k^2\mathcal{S}_k\left[\vec{\nu}\left(-\tfrac{1}{2}I + K_k\right)^{-1}(\operatorname{Tr}(\operatorname{div}\Pi_k(\vec{u})))\right] \\
&\quad - k^2\nabla^t\mathcal{S}_k\left[\left(\tfrac{1}{2}I + K_k^*\right)^{-1}\left[\vec{\tau}\cdot S_k\left(\vec{\nu}\left(-\tfrac{1}{2}I + K_k\right)^{-1}(\operatorname{Tr}(\operatorname{div}\Pi_k(\vec{u})))\right)\right]\right] \\
&\quad - \mathcal{S}_k(\vec{\tau}\operatorname{rot}\vec{u}) + \nabla^t\mathcal{S}_k\left[\left(\tfrac{1}{2}I + K_k^*\right)^{-1}(\vec{\tau}\cdot S_k(\vec{\tau}\operatorname{rot}\vec{u}))\right] \\
&\quad + \mathcal{S}_k\left[\vec{\nu}\left(-\tfrac{1}{2}I + K_k\right)^{-1}(\operatorname{div} S_k(\vec{\tau}\operatorname{rot}\vec{u}))\right] \\
&\quad + \nabla^t\mathcal{S}_k\left[\left(\tfrac{1}{2}I + K_k^*\right)^{-1}\left[\vec{\tau}\cdot S_k\left(\vec{\nu}\left(-\tfrac{1}{2}I + K_k\right)^{-1}(\operatorname{div} S_k(\vec{\tau}\operatorname{rot}\vec{u}))\right)\right]\right] \\
&\quad - \nabla\mathcal{S}_k(\vec{\nu}\cdot\vec{u}) + \nabla^t\mathcal{S}_k\left[\left(\tfrac{1}{2}I + K_k^*\right)^{-1}(\vec{\tau}\cdot\nabla S_k(\vec{\nu}\cdot\vec{u}))\right] \\
&\quad - k^2\mathcal{S}_k\left[\vec{\nu}\left(-\tfrac{1}{2}I + K_k\right)^{-1}(S_k(\vec{\nu}\cdot\vec{u}))\right] \\
&\quad - k^2\nabla^t\mathcal{S}_k\left[\left(\tfrac{1}{2}I + K_k^*\right)^{-1}\left[\vec{\tau}\cdot S_k\left(\vec{\nu}\left(-\tfrac{1}{2}I + K_k\right)^{-1}(S_k(\vec{\nu}\cdot\vec{u}))\right)\right]\right].
\end{aligned}
\tag{3.12}
$$

Proof. Based on Theorem 5.1 in [30], there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that, for any $k \in i\,\mathbb{R}_+$, $y_0 \in \Omega$, $\vec{f} = \{f_1, f_2\}$, f_1, f_2 real constants and $2-\varepsilon < p < 2+\varepsilon$, the boundary value problem

$$
(BVP_3)\begin{cases}
(\Delta + k^2)\vec{v} = 0 \text{ in } \Omega, \\
\mathcal{N}(\vec{v}),\ \mathcal{N}(\nabla^t\operatorname{rot}\vec{v}),\ \mathcal{N}(\nabla\operatorname{div}\vec{v}) \in L^p(\partial\Omega), \\
\mathcal{N}(\operatorname{rot}\vec{v}),\ \mathcal{N}(\operatorname{div}\vec{v}) \in L^p(\partial\Omega), \\
\vec{\nu}\cdot\vec{v}|_{\partial\Omega} = -\vec{\nu}\cdot[\Gamma_k(y_0,\cdot)\vec{f}]|_{\partial\Omega}, \\
\operatorname{rot}\vec{v}|_{\partial\Omega} = -\operatorname{rot}(\Gamma_k(y_0,\cdot)\vec{f})|_{\partial\Omega},
\end{cases}
$$

is uniquely solvable. Moreover, the solution has the form

$$\vec{v} = \mathcal{S}_k\vec{c} + \nabla\mathcal{S}_k d, \tag{3.13}$$

where

$$\vec{c} := \vec{\tau}\left(\tfrac{1}{2}I + K_k\right)^{-1}\left[-(\mathrm{rot}(\Gamma_k(y_0,\cdot)\vec{f}))|_{\partial\Omega}\right] \tag{3.14}$$

$$d := \left(-\tfrac{1}{2}I + K_k^*\right)^{-1}\left[-\vec{\nu}\cdot(\Gamma_k(y_0,\cdot)\vec{f})|_{\partial\Omega} - \vec{\nu}\cdot(S_k\vec{c})\right]; \tag{3.15}$$

note that $\vec{c} = \vec{c}(\cdot, y_0)$ and $d = d(\cdot, y_0)$. With ε given above, let p and $\vec{u}$ be as in the hypotheses of Proposition 3.2. If $\frac{1}{q} := 1 - \frac{1}{p}$ there is no loss of generality assuming that $2 - \varepsilon < q < 2 + \varepsilon$. Next we consider $\{\Omega_j\}_j$, $\Omega_j \nearrow \Omega$, a sequence of Lipschitz domains approximating Ω as in [3]. Among other things, $\{\Omega_j\}_j$ have bounded Lipschitz character. Due to the properties of this approximating sequence, Theorem 5.1 in [30] assures that (BVP_3) is solvable for $p \in (2-\varepsilon, 2+\varepsilon)$ if the domain Ω is replaced by Ω_j, $j \in \mathbb{N}$ such that $y_0 \in \Omega_j$. Also, the solution corresponding to Ω_j has the form (3.13)–(3.15). We will use the subscript j to indicate the dependence on Ω_j of all the operators and vectors involved (e.g., the solution of (BVP_3) for Ω_j is $\vec{v}_j = \mathcal{S}_{k,j}\vec{c}_j + \nabla\mathcal{S}_{k,j}d_j$, etc.). Let $y_0 \in \Omega$ be an arbitrary fixed point and q as above. Then there exists $j_0 \in \mathbb{N}$ such that $y_0 \in \Omega_j$ for $j \geq j_0$. For each $j \geq j_0$ define

$$\vec{w}_{k,j}(x) := \Gamma_k(y_0, x)\vec{f} - \mathcal{S}_{k,j}\vec{c}_j - \nabla\mathcal{S}_{k,j}d_j, \quad x \in \Omega_j. \tag{3.16}$$

Relying on the properties of $\vec{w}_{k,j}$ we see that, with δ_{y_0} denoting the Dirac distribution at y_0,

$$(BVP_4)\begin{cases} (\Delta + k^2)\vec{w}_{k,j} = \delta_{y_0}\vec{f} \text{ in } \Omega_j, \\ \mathcal{N}(\vec{w}_{k,j}),\, \mathcal{N}(\mathrm{rot}\,\vec{w}_{k,j}),\, \mathcal{N}(\mathrm{div}\,\vec{w}_{k,j}) \in L^q(\partial\Omega_j), \\ \vec{\nu}_j\cdot\vec{w}_{k,j}|_{\partial\Omega_j} = 0, \\ \mathrm{rot}\,\vec{w}_{k,j}|_{\partial\Omega_j} = 0, \\ \sup\|\mathcal{N}(\vec{w}_{k,j})\|_{L^q(\partial\Omega_j)},\ \sup\|\mathcal{N}(\mathrm{div}\,\vec{w}_{k,j})\|_{L^q(\partial\Omega_j)} < +\infty. \end{cases}$$

The norm estimates in (BVP_4) follow from the properties of the sequence $\{\Omega_j\}_j$ and the fact that the operators $\pm\frac{1}{2}I + K_{k,j}$ and $\pm\frac{1}{2}I + K_{k,j}^*$ are uniformly invertible on $L^p(\partial\Omega_j)$ with respect to j. With these at hand, starting with $\iint_{\Omega_j}\vec{u}\cdot(\Delta + k^2)\vec{w}_{k,j} - \iint_{\Omega_j}(\Delta + k^2)\vec{u}\cdot\vec{w}_{k,j}$, repeated integrations by parts imply that

$$\vec{u}(y_0)\cdot\vec{f} = k^2\iint_{\Omega_j}\vec{u}\cdot\vec{w}_{k,j} + \int_{\partial\Omega_j}\vec{\nu}_j\cdot\vec{u}\,(\mathrm{div}\,\vec{w}_{k,j})|_{\partial\Omega_j}d\sigma_j \\ + \int_{\partial\Omega_j}\vec{\nu}_j\cdot(*\vec{w}_{k,j})(\mathrm{rot}\,\vec{u})|_{\partial\Omega_j}d\sigma_j. \tag{3.17}$$

Next, we use the explicit form of each $\vec{w}_{k,j}$ to rewrite the terms in the right-hand side of (3.17). We start with the double integral, which we write as

$$\iint_{\Omega_j} \vec{u}\cdot\vec{w}_{k,j} = \iint_{\Omega_j} \Gamma_k(y_0,\cdot)\vec{u}\cdot\vec{f} - \iint_{\Omega_j} \vec{u}\cdot\mathcal{S}_{k,j}\vec{c}_j - \iint_{\Omega_j} \vec{u}\cdot\nabla\mathcal{S}_{k,j}d_j$$
$$=: I + II + III. \tag{3.18}$$

Moreover,

$$I = (\Pi_{k,j}\vec{u})(y_0)\cdot\vec{f}, \tag{3.19}$$
$$\begin{aligned} II &= \iint_{\Omega_j}\Big(\int_{\partial\Omega_j}\Gamma_k(x,y)\vec{u}(x)\cdot\vec{c}_j(y)\,d\sigma_j(y)\Big)\,dx \\ &= \int_{\partial\Omega_j}\vec{c}_j(y)\cdot\mathrm{Tr}(\Pi_{k,j}\vec{u})(y)\,d\sigma_j(y) \\ &= \int_{\partial\Omega_j}\vec{\tau}_j(y)\cdot\Pi_{k,j}\vec{u}(y)\left(\tfrac12 I+K_{k,j}\right)^{-1}\Big[-\mathrm{Tr}(\mathrm{rot}(\Gamma_k(y_0,\cdot)\vec{f}))\Big](y)\,d\sigma_j(y) \\ &= -\int_{\partial\Omega_j}\mathrm{Tr}(\mathrm{rot}(\Gamma_k(y_0,y)\vec{f}))\left(\tfrac12 I+K^*_{k,j}\right)^{-1}[\vec{\tau}_j\cdot\Pi_{k,j}\vec{u}]\,(y)\,d\sigma_j(y) \\ &= \nabla^t\mathcal{S}_{k,j}\left(\left(\tfrac12 I+K^*_{k,j}\right)^{-1}(\vec{\tau}_j\cdot\Pi_{k,j}\vec{u})\right)(y_0)\cdot\vec{f}, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} III &= \iint_{\Omega_j}\int_{\partial\Omega_j}\vec{u}(x)\cdot(-\nabla_x\Gamma_k(x,y))d_j(y)\,d\sigma_j(y)\,dx \\ &= \int_{\partial\Omega_j} d_j(y)\,\mathrm{Tr}(\mathrm{div}(\Pi_{k,j}\vec{u}))(y)\,d\sigma_j(y) \\ &= \int_{\partial\Omega_j}\Big[-\vec{\nu}_j(y)\cdot(\Gamma_k(y_0,y)\vec{f}) - \vec{\nu}_j(y)\cdot(S_{k,j}\vec{c}_j)(y)\Big]\times \\ &\qquad\qquad \times\left(-\tfrac12 I+K_{k,j}\right)^{-1}(\mathrm{Tr}(\mathrm{div}\,\Pi_{k,j}\vec{u}))(y)\,d\sigma_j(y) \\ &= -\mathcal{S}_{k,j}\left(\vec{\nu}_j\left(-\tfrac12 I+K_{k,j}\right)^{-1}(\mathrm{Tr}(\mathrm{div}\,\Pi_{k,j}\vec{u}))\right)(y_0)\cdot\vec{f} \\ &\quad -\int_{\partial\Omega_j} S_{k,j}\left(\vec{\nu}_j\left(-\tfrac12 I+K_{k,j}\right)^{-1}(\mathrm{Tr}(\mathrm{div}\,\Pi_{k,j}\vec{u}))\right)(y)\cdot\vec{c}_j(y)\,d\sigma_j(y) \\ &= -\mathcal{S}_{k,j}\left(\vec{\nu}_j\left(-\tfrac12 I+K_{k,j}\right)^{-1}(\mathrm{Tr}(\mathrm{div}\,\Pi_{k,j}\vec{u}))\right)(y_0)\cdot\vec{f} - \nabla^t\mathcal{S}_{k,j} \\ &\quad \times\Big[\left(\tfrac12 I+K^*_{k,j}\right)^{-1}\Big[\vec{\tau}_j\cdot S_{k,j}\big(\vec{\nu}_j\big(-\tfrac12 I+K_{k,j}\big)^{-1}(\mathrm{Tr}(\mathrm{div}\,\Pi_{k,j}\vec{u}))\big)\Big]\Big](y_0)\cdot\vec{f}. \end{aligned} \tag{3.21}$$

The last equality above is seen by working as we did for II.

At this point, passing to the limit $j \to \infty$ (based on (BVP_4) and the properties of the approximating sequence $\{\Omega_j\}_j$; for the convergence properties of $K_{k,j}$ and $K^*_{k,j}$ see [3]) gives that

$$\lim_{j\to\infty}[I + II + III] = \vec{f}\cdot\Big[\Pi_k\vec{u} + \nabla^t\mathcal{S}_k\Big(\big(\tfrac{1}{2}I + K_k^*\big)^{-1}(\vec{\tau}\cdot\Pi_k\vec{u})\Big) \tag{3.22}$$
$$- \mathcal{S}_k\Big(\vec{\nu}\big(-\tfrac{1}{2}I + K_k\big)^{-1}(\mathrm{Tr}(\mathrm{div}\,\Pi_k\vec{u}))\Big)$$
$$- \nabla^t\mathcal{S}_k\Big[\big(\tfrac{1}{2}I + K_k^*\big)^{-1}\Big[\vec{\tau}\cdot S_k\Big(\vec{\nu}\big(-\tfrac{1}{2}I + K_k\big)^{-1}(\mathrm{Tr}(\mathrm{div}\,\Pi_k\vec{u}))\Big)\Big]\Big]\Big](y_0),$$

while for the two boundary integrals in (3.17),

$$\lim_{j\to\infty}\int_{\partial\Omega_j}\vec{\nu}_j\cdot\vec{u}\,(\mathrm{div}\,\vec{w}_{k,j})|_{\partial\Omega_j}d\sigma_j \tag{3.23}$$
$$= \vec{f}\cdot\Big(-\nabla\mathcal{S}_k(\vec{\nu}\cdot\vec{u}) + \nabla^t\mathcal{S}_k\Big[\big(\tfrac{1}{2}I + K_k^*\big)^{-1}(\vec{\tau}\cdot\nabla S_k(\vec{\nu}\cdot\vec{u}))\Big]$$
$$- k^2\mathcal{S}_k\Big[\vec{\nu}\big(-\tfrac{1}{2}I + K_k\big)^{-1}(S_k(\vec{\nu}\cdot\vec{u}))\Big]$$
$$-k^2\nabla^t\mathcal{S}_k\Big[\big(\tfrac{1}{2}I + K_k^*\big)^{-1}\Big[\vec{\tau}\cdot S_k\Big(\vec{\nu}\big(-\tfrac{1}{2}I + K_k\big)^{-1}(S_k(\vec{\nu}\cdot\vec{u}))\Big)\Big]\Big]\Big)(y_0)$$

and

$$\lim_{j\to\infty}\int_{\partial\Omega_j}\vec{\nu}_j\cdot(*\vec{w}_{k,j})(\mathrm{rot}\,\vec{u})|_{\partial\Omega_j}\,d\sigma_j \tag{3.24}$$
$$= \vec{f}\cdot\Big(-\mathcal{S}_k(\vec{\tau}\,\mathrm{rot}\,\vec{u}) + \mathcal{S}_k\Big[\vec{\nu}\big(-\tfrac{1}{2}I + K_k\big)^{-1}(\mathrm{div}\,S_k(\vec{\tau}\,\mathrm{rot}\,\vec{u}))\Big]$$
$$+ \nabla^t\mathcal{S}_k\Big[\big(\tfrac{1}{2}I + K_k^*\big)^{-1}(\vec{\tau}\cdot S_k(\vec{\tau}\,\mathrm{rot}\,\vec{u}))\Big]$$
$$+\nabla^t\mathcal{S}_k\Big[\big(\tfrac{1}{2}I + K_k^*\big)^{-1}\Big[\vec{\tau}\cdot S_k\Big(\vec{\nu}\big(-\tfrac{1}{2}I + K_k\big)^{-1}(\mathrm{div}\,S_k(\vec{\tau}\,\mathrm{rot}\,\vec{u}))\Big)\Big]\Big]\Big)(y_0).$$

Since $\vec{f}$ is constant and y_0 arbitrary in Ω, a combination of (3.17)–(3.24) gives (3.12). With this the proof of Proposition 3.2 is completed. □

Returning for a moment to the proof of (3.11), we write the identity in Proposition 3.2 for the harmonic vector $\vec{w}$, solution of (BVP_2), then take the divergence of both sides; (3.11) follows after some algebra.

4. Green operators and the proof of theorem 1.1

We start by defining suitable Green operators. Fix $\Omega \subset \mathbb{R}^2$ a bounded Lipschitz domain, and let $\varepsilon > 0$ be such that the results proved so far hold.

Also, let $\vec{h}_1, \vec{h}_2, \dots, \vec{h}_{b_1(\Omega)}$ be a basis for $\mathcal{H}^2_{\tan}(\Omega, \mathbb{R}^2)$. Proposition 2.2 assures that this is a basis for $\mathcal{H}^p_{\tan}(\Omega, \mathbb{R}^2)$, for any $\frac{4}{3} - \varepsilon < p < 4 + \varepsilon$. Hence, we can define the projection operator

$$P_{\tan} : L^p(\Omega, \mathbb{R}^2) \to \mathcal{H}^{s,p}_{\tan}(\Omega, \mathbb{R}^2), \quad P_{\tan}(\vec{u}) := \sum_{i=1}^{b_1(\Omega)} \langle \vec{h}_i, \vec{u} \rangle \vec{h}_i. \tag{4.1}$$

For each $\frac{4}{3} - \varepsilon < p < 4 + \varepsilon$ we introduce the Green operator $G_{\tan} : L^p(\Omega, \mathbb{R}^2) \to L^p(\Omega, \mathbb{R}^2)$, defined for $\vec{u} \in L^p(\Omega, \mathbb{R}^2)$ by

$$\begin{aligned} G_{\tan}(\vec{u}) := \ & \text{the unique solution of } (BVP_1)\text{–(3.1)–(3.2)} \\ & \text{for } \vec{\eta} := \vec{u} - P_{\tan}\vec{u}. \end{aligned} \tag{4.2}$$

In a similar way, there exist the projection operator $P_{\text{nor}} : L^p(\Omega, \mathbb{R}^2) \to \mathcal{H}^p_{\text{nor}}(\Omega, \mathbb{R}^2)$, and the Green operator $G_{\text{nor}} : L^p(\Omega, \mathbb{R}^2) \to L^p(\Omega, \mathbb{R}^2)$ defined by

$$\begin{aligned} G_{\text{nor}}(\vec{u}) := \ & \text{the unique solution of } (BVP_5)\text{–(4.4)–(4.5)} \\ & \text{below with } \vec{\eta} := \vec{u} - P_{\text{nor}}(\vec{u}). \end{aligned} \tag{4.3}$$

The boundary value problem alluded to above reads

$$(BVP_5) \begin{cases} \Delta \vec{u} = \vec{\eta} \ \text{ in } \Omega, \quad \vec{u} \in L^p(\Omega, \mathbb{R}^2), \\ \text{rot}\, \vec{u}, \, \text{div}\, \vec{u} \in H^{1,p}(\Omega), \\ \vec{\tau} \cdot \vec{u} = 0 \text{ in } B^{p,p}_{-\frac{1}{p}}(\partial\Omega), \\ \text{Tr}(\text{div}\, \vec{u}) = 0 \text{ in } B^{p,p}_{1-\frac{1}{p}}(\partial\Omega). \end{cases}$$

The fact that (BVP_5) is solvable if and only if

$$\vec{\eta} \in \{\mathcal{H}^q_{\text{nor}}(\Omega, \mathbb{R}^2)\}^\circ, \qquad \tfrac{1}{p} + \tfrac{1}{q} = 1, \tag{4.4}$$

follows from Theorem 3.1, since (BVP_5) is the Hodge adjoint of (BVP_1). In addition, we have that the space of null solutions for (BVP_5) is $\mathcal{H}^2_{\text{nor}}(\Omega, \mathbb{R}^2)$, and we obtain uniqueness in (BVP_5) by imposing the condition

$$\vec{u} \in \{\mathcal{H}^q_{\text{nor}}(\Omega, \mathbb{R}^2)\}^\circ, \qquad \tfrac{1}{p} + \tfrac{1}{q} = 1. \tag{4.5}$$

Before proceeding with the proof of Theorem 1.1, we define one more projection operator which will be useful later in the sequel. More precisely, let $P : L^p(\Omega, \mathbb{R}^2) \to L^p(\Omega, \mathbb{R}^2)$ be defined for $\vec{u} \in L^p(\Omega, \mathbb{R}^2)$ by

$$P\vec{u} := \vec{u} - \nabla(\text{div } G_{\text{nor}}\vec{u}) + \nabla^t(\text{rot } G_{\tan}\vec{u}). \tag{4.6}$$

The fact that P is linear and bounded follows from the properties of $G_{\rm nor}$ and $G_{\rm tan}$. Also, from (4.6) and the properties of the Green operators we can conclude that

$$P\vec{u} \in \mathcal{H}^p(\Omega, \mathbb{R}^2) \text{ for any } \vec{u} \in L^p(\Omega, \mathbb{R}^2), \tag{4.7}$$

where

$$\mathcal{H}^p(\Omega, \mathbb{R}^2) := \{\vec{u} \in L^p(\Omega, \mathbb{R}^2) : \text{div } \vec{u} = \text{rot } \vec{u} = 0 \text{ in } \Omega\}. \tag{4.8}$$

Furthermore, if $\vec{u} \in \mathcal{H}^p(\Omega, \mathbb{R}^2)$, we have that $\text{rot}(G_{\rm tan}\vec{u})$ and $\text{div}(G_{\rm nor}\vec{u})$ are trace-zero harmonic functions in $H^{1,p}(\Omega)$. Invoking the uniqueness in the Dirichlet problem for the Laplacian (see [20] for $\partial\Omega$ connected and [28] for arbitrary topology), we see that $\text{rot}(G_{\rm tan}\vec{u}) = 0$ and $\text{div}(G_{\rm nor}\vec{u}) = 0$ in Ω, granted that $4/3 - \varepsilon < p < 4 + \varepsilon$. Consequently, for p in this range,

$$P\vec{u} = \vec{u} \iff \vec{u} \in \mathcal{H}^p(\Omega, \mathbb{R}^2). \tag{4.9}$$

At the level $p = 2$, repeated integrations by parts show that $\iint_\Omega (\vec{u} - P\vec{u}) \cdot \vec{h} = 0$ for any $\vec{h} \in \mathcal{H}^2(\Omega, \mathbb{R}^2)$, which gives that P is the orthogonal projection of $L^2(\Omega, \mathbb{R}^2)$ onto $\mathcal{H}^2(\Omega, \mathbb{R}^2)$. The latter, plus a simple density argument, implies that

$$\begin{gathered}\text{for } \tfrac{4}{3} - \varepsilon < p < 4 + \varepsilon \text{ the adjoint of } P \text{ acting on } L^p(\Omega, \mathbb{R}^2) \\ \text{is } P \text{ acting on } H^q(\Omega, \mathbb{R}^2),\ \ \tfrac{1}{p} + \tfrac{1}{q} = 1.\end{gathered} \tag{4.10}$$

All the ingredients are now in place for the

Proof of Theorem 1.1. First we show that the decomposition (1.4) holds for any $4/3-\varepsilon < p < 4+\varepsilon$, if $\varepsilon > 0$ is sufficiently small. If $\vec{u} \in L^p(\Omega, \mathbb{R}^2)$, from the properties of the Green operators and the fact that $\Delta = \nabla \,\text{div} - \nabla^t \,\text{rot}$, we have

$$\vec{u} = \nabla(\text{div}(G_{\rm nor}\vec{u})) - \nabla^t(\text{rot}(G_{\rm nor}\vec{u})) + P_{\rm nor}\vec{u}. \tag{4.11}$$

Now, the left-to-right inclusion in (1.4) follows from (4.11), the opposite one being trivial. This takes care of the algebraic aspect of the decomposition.

In order to show that this decomposition is unique, let $4/3 - \varepsilon < p < 4+\varepsilon$, $\alpha \in H_0^{1,p}(\Omega)$, $\beta \in H^{1,p}(\Omega)$, $\vec{\gamma} \in \mathcal{H}^2_{\rm nor}(\Omega, \mathbb{R}^2)$ be such that

$$0 = \nabla\alpha + \nabla^t\beta + \vec{\gamma}. \tag{4.12}$$

Using this and Proposition 2.2, it makes sense to write

$$0 = \langle \nabla\alpha, \vec{\gamma}\rangle + \langle \nabla^t\beta, \vec{\gamma}\rangle + \langle \vec{\gamma}, \vec{\gamma}\rangle. \tag{4.13}$$

Integration by parts gives that the first two terms in (4.13) are zero (recall that $\text{div}\,\vec{\gamma} = 0 = \text{rot}\,\vec{\gamma}$, and $\vec{\tau} \cdot \vec{\gamma} = 0$); thus $\vec{\gamma} = 0$ and, further, $\nabla\alpha = -\nabla^t\beta$.

Next, for $\vec{v} \in L^q(\Omega, \mathbb{R}^2)$, $\frac{1}{p} + \frac{1}{q} = 1$ we use the decomposition we just proved to write

$$\vec{v} = \nabla a + \nabla^t b + \vec{c},$$
$$a \in H_0^{1,q}(\Omega), \quad b \in H^{1,q}(\Omega), \quad \vec{c} \in \mathcal{H}^2_{\mathrm{nor}}(\Omega, \mathbb{R}^2). \tag{4.14}$$

Hence,

$$\langle \nabla \alpha, \vec{v} \rangle = -\langle \nabla^t \beta, \nabla a \rangle + \langle \nabla \alpha, \nabla^t b \rangle - \langle \nabla^t \beta, \vec{c} \rangle. \tag{4.15}$$

Arguing as before, we get that each term in the right-hand side of (4.15) is zero. Since $\vec{v}$ was arbitrary we conclude that $\nabla \alpha = 0$ and $\nabla^t \beta = 0$ in Ω. This takes care of the uniqueness part.

Natural estimates accompanying (1.4) are implicit in (4.11) and what we have proved so far. This completes the proof of (1.4). The decomposition (1.5) can be handled similarly (this time utilizing the operator (4.2)), and we omit the details.

Next we take up the task of proving that the range $\frac{4}{3} \leq p \leq 4$ is sharp in the class of Lipschitz domains. The idea is that the validity of the Hodge decomposition (1.4) entails the well-posedness of the Poisson problem for the Laplacian with homogeneous Dirichlet boundary condition. More specifically, we claim that if (1.4) holds for some $\frac{4}{3} \leq p \leq 4$ then

$$w \in H_0^{1,p}(\Omega), \ \Delta w = f \in H^{-1,p}(\Omega), \tag{4.16}$$

has a unique solution which also satisfies $\|w\|_{H_0^{1,p}(\Omega)} \leq C \|f\|_{H^{-1,p}(\Omega)}$. Indeed, given an arbitrary $f \in H^{-1,p}(\Omega)$ it is not too difficult to construct some $\vec{u} \in L^p(\Omega, \mathbb{R}^2)$ with $\operatorname{div} \vec{u} = f$. Decomposing $\vec{u} = \nabla \alpha + \nabla^t \beta + \vec{\gamma}$ according to (1.4), we see that $w := \alpha$ is the desired solution of (4.16). Uniqueness for (4.16) is immediate from the Proposition 5.17 of [20].

Next, counterexamples to the well-posedness of (4.16) are found in Section 6 of [20]; they correspond to p's not in the interval $[4/3, 4]$. Since the decomposition (1.5) is the Hodge dual of (1.4) (in the sense of (2.29)), it follows that the range $\frac{4}{3} \leq p \leq 4$ is also sharp for (1.5) in the class of Lipschitz domains.

Finally, in the case $\partial\Omega \in C^1$, (1.4) and (1.5) follow along similar lines. The proof of Theorem 1.1 is therefore finished. □

The results we have obtained so far also allow us to prove other important decomposition theorems, some of which we single out below.

Theorem 4.1 (Hodge–Morrey decomposition)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that, for each $\frac{4}{3} - \varepsilon <$*

$p < 4 + \varepsilon$, the following decomposition holds:

$$L^p(\Omega, \mathbb{R}^2) = \nabla H_0^{1,p}(\Omega) \oplus \nabla^t H_0^{1,p}(\Omega) \oplus \mathcal{H}^p(\Omega, \mathbb{R}^2). \tag{4.17}$$

Proof. In order to prove (4.17) we define a suitable Green operator. As before, this is done by first solving a corresponding boundary value problem. In this case, we prove the following.

Theorem 4.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for each $\frac{4}{3} - \varepsilon < p < 4 + \varepsilon$ the boundary value problem*

$$(BVP_6) \begin{cases} \Delta\vec{u} = \vec{\mu} \in L^p(\Omega, \mathbb{R}^2), \quad \vec{u} \in L^p(\Omega, \mathbb{R}^2), \\ \operatorname{rot}\vec{u}, \operatorname{div}\vec{u} \in H^{1,p}(\Omega), \\ \operatorname{Tr}(\operatorname{rot}\vec{u}) = 0 \text{ in } B^{p,p}_{1-\frac{1}{p}}(\partial\Omega), \\ \operatorname{Tr}(\operatorname{div}\vec{u}) = 0 \text{ in } B^{p,p}_{1-\frac{1}{p}}(\partial\Omega), \end{cases}$$

is solvable if and only if

$$\vec{\mu} \in \{\mathcal{H}^q(\Omega, \mathbb{R}^2)\}^\circ, \qquad \tfrac{1}{p} + \tfrac{1}{q} = 1, \tag{4.18}$$

where $\{\dots\}^\circ$ is the annihilator of $\{\dots\}$ in $L^p(\Omega, \mathbb{R}^2)$. The space of null solutions for (BVP_6) is $\mathcal{H}^p(\Omega, \mathbb{R}^2)$, and the condition

$$\vec{u} \in \{\mathcal{H}^q(\Omega, \mathbb{R}^2)\}^\circ \tag{4.19}$$

implies uniqueness for (BVP_6). Moreover, in the case $\partial\Omega \in C^1$, we can take $1 < p < \infty$.

Let us take for granted for the moment Theorem 4.2 and continue with the proof of the decomposition (4.17). With $\varepsilon > 0$ small enough so that Theorem 4.2 holds, fix $\frac{4}{3} - \varepsilon < p < 4 + \varepsilon$. Also, recall the projection operator P defined in (4.6). The Green operator which is relevant in the proof of (4.17) is $G : L^p(\Omega, \mathbb{R}^2) \to L^p(\Omega, \mathbb{R}^2)$ defined for $\vec{u} \in L^p(\Omega, \mathbb{R}^2)$ by

$$\begin{aligned} G(\vec{u}) := {} & \text{ the unique solution of } (BVP_6)\text{–(4.18)–(4.19)} \\ & \text{for } \vec{\mu} := \vec{u} - P\vec{u}. \end{aligned} \tag{4.20}$$

Now, for each $\vec{u} \in L^p(\Omega, \mathbb{R}^2)$ we have

$$\vec{u} = \nabla(\operatorname{div} G(\vec{u})) - \nabla^t(\operatorname{rot} G(\vec{u})) + P(\vec{u}). \tag{4.21}$$

Clearly, this decomposition does the job for the part in (4.17) having to do with existence and estimates. Finally, the uniqueness statement associated with the decomposition (4.17) is easily seen from Proposition 5.17 of [20].

We now turn to the

Proof of Theorem 4.2. The fact that (4.18) is a necessary condition follows from (2.2) and (2.3). Conversely, if (4.18) is true, let $\varepsilon > 0$ be small enough so that the conclusions of Theorem 2.1 and Proposition 2.2 hold, and let p^* be as in (3.4). Then, we look for a solution for (BVP_6) of the form

$$\vec{u} = \Pi_0(\vec{\mu}) + \vec{w}, \tag{4.22}$$

where $\vec{w}$ solves the problem

$$(BVP_7)\begin{cases} \Delta\vec{w} = 0 \text{ in } \Omega, \\ \mathcal{N}(\vec{w}),\, \mathcal{N}(\operatorname{rot}\vec{w}),\, \mathcal{N}(\operatorname{div}\vec{w}) \in L^{p^*}(\partial\Omega), \\ (\operatorname{rot}\vec{w})|_{\partial\Omega} = -\operatorname{Tr}(\operatorname{rot}(\Pi_0\vec{\eta})) \in L^{p^*}(\partial\Omega), \\ (\operatorname{div}\vec{w})|_{\partial\Omega} = -\operatorname{Tr}(\operatorname{div}(\Pi_0\vec{\mu})) \in L^{p^*}(\partial\Omega). \end{cases}$$

Using the results in [30] we know that (BVP_7) is solvable for small $\varepsilon > 0$ if and only if

$$\vec{\tau}\operatorname{Tr}(\operatorname{rot}\Pi_0(\vec{\mu})) + \vec{\nu}\operatorname{Tr}(\operatorname{div}\Pi_0(\vec{\mu})) \tag{4.23}$$

belongs to the annihilator (taken in $L^{p^*}(\partial\Omega, \mathbb{R}^2)$) of the set

$$\{\vec{h}|_{\partial\Omega} : \vec{h} \in C^\infty(\Omega, \mathbb{R}^2),\, \mathcal{N}(\vec{h}) \in L^q(\partial\Omega),\, \operatorname{rot}\vec{h} = 0,\, \operatorname{div}\vec{h} = 0\}, \tag{4.24}$$

where $\frac{1}{p^*} + \frac{1}{q} = 1$. Moreover, as proved in [30], the existing solution is of the form $\vec{w} = \mathcal{S}_0\vec{f} + \nabla\mathcal{S}_0 g$, for some $\vec{f} \in L^{p^*}(\partial\Omega, \mathbb{R}^2)$ and $g \in L^{p^*}(\partial\Omega)$. In our case, (4.18) implies that the aforementioned compatibility condition holds true. Here we remark that for $\vec{h}$ belonging to the set whose trace is considered in (4.24), if we write the identity (2.25) and use (2.17), we see that $\vec{h} \in B^{q,q\vee 2}_{1/q}(\Omega, \mathbb{R}^2)$. In addition, $B^{q,q\vee 2}_{1/q}(\Omega, \mathbb{R}^2) \hookrightarrow L^{p'}(\Omega, \mathbb{R}^2)$, $\frac{1}{p} + \frac{1}{p'} = 1$, which can be checked using Corollary 2 (ii), p. 36 in [32] and the definitions of q and p'. At this point we can conclude that $\vec{h} \in \mathcal{H}^{p'}(\Omega, \mathbb{R}^2)$. Thus, the pairing $\langle\vec{h}, \vec{\mu}\rangle$ is well defined for $\vec{h}$ satisfying the conditions in (4.24). The fact that $\vec{u}$ as in (4.22) is a solution of (BVP_7) follows by employing the same ideas as in the proof of Theorem 3.1; we omit these details.

Clearly $\mathcal{H}^p(\Omega, \mathbb{R}^2)$ is a subspace of the space of null solutions of (BVP_6). For the opposite inclusion, we make use of the uniqueness for the Dirichlet problem for the Laplacian. Finally, we are left with showing that (4.19) guarantees uniqueness once (4.18) is verified. Indeed, if $\vec{u}$ is a null solution of (BVP_6) then, from what we have proved so far, $\vec{u} \in \mathcal{H}^p(\Omega, \mathbb{R}^2)$. Next, fix $\vec{v} \in H^q(\Omega, \mathbb{R}^2)$, $\frac{1}{p} + \frac{1}{q} = 1$, and recall the operator P introduced in (4.6). Thanks to (4.9) and (4.10), we have

$$\langle\vec{u}, \vec{v}\rangle = \langle P\vec{u}, \vec{v}\rangle = \langle\vec{u}, P\vec{v}\rangle = 0,$$

where the last equality is a consequence of $P\vec{v} \in \mathcal{H}^q(\Omega, \mathbb{R}^2)$ plus the fact that $\vec{u}$ satisfies (4.19). Thus, $\vec{u} = 0$, since $\vec{v}$ is arbitrary.

In closing we would like to point out that if $\vec{u}$ is a solution of (BVP_6), then $\vec{u} - P\vec{u}$ is also a solution of (BVP_6), satisfying the additional condition (4.19). □

Theorem 4.3 (Friedrichs decompositions)**.** *Granted the assumptions of Theorem 4.1 we have*

$$\mathcal{H}^p(\Omega, \mathbb{R}^2) = \big(\nabla H^{1,p}(\Omega) \cap \mathcal{H}^p(\Omega, \mathbb{R}^2)\big) \oplus \mathcal{H}^2_{\tan}(\Omega, \mathbb{R}^2), \tag{4.25}$$

$$\mathcal{H}^p(\Omega, \mathbb{R}^2) = \big(\nabla^t H^{1,p}(\Omega) \cap \mathcal{H}^p(\Omega, \mathbb{R}^2)\big) \oplus \mathcal{H}^2_{\text{nor}}(\Omega, \mathbb{R}^2). \tag{4.26}$$

Proof. We will deal only with (4.25), since the proof of (4.26) is similar. The right-to-left inclusion is immediate. Next, if $\vec{u} \in \mathcal{H}^p(\Omega, \mathbb{R}^2)$, from (1.5) we have that $\vec{u} = \nabla a + \nabla^t b + \vec{c}$, $a \in H^{1,p}(\Omega)$, $b \in H^{1,p}_0(\Omega)$, $\vec{c} \in \mathcal{H}^2_{\tan}(\Omega, \mathbb{R}^2)$. Since $\operatorname{div} \vec{u} = 0$ we get $\Delta a = 0$ hence, further, $\nabla a \in \mathcal{H}^p(\Omega, \mathbb{R}^2)$. Also, $\operatorname{rot} \vec{u} = 0$ implies $\Delta b = 0$. Based on the uniqueness for the Dirichlet problem for the Laplacian, we then conclude $b = 0$.

Thus, $\vec{u}$ has the required decomposition. □

We conclude this section by discussing the so-called Helmholtz–Weyl decompositions in the two-dimensional setting. The proof follows more or less directly from Theorem 1.1, and we leave the details to the interested reader.

Theorem 4.4 (Helmholtz–Weyl decompositions)**.** *Under the assumptions of Theorem* 4.1 *we have*

$$L^p(\Omega, \mathbb{R}^2) = \nabla H^{1,p}(\Omega) \oplus \{\vec{u} \in L^p(\Omega, \mathbb{R}^2) : \text{div } \vec{u} = 0,\ \vec{\nu} \cdot \vec{u} = 0\}, \tag{4.27}$$

$$L^p(\Omega, \mathbb{R}^2) = \nabla H^{1,p}_0(\Omega) \oplus \{\vec{u} \in L^p(\Omega, \mathbb{R}^2) : \text{div } \vec{u} = 0\}, \tag{4.28}$$

$$L^p(\Omega, \mathbb{R}^2) = \nabla^t H^{1,p}(\Omega) \oplus \{\vec{u} \in L^p(\Omega, \mathbb{R}^2) : \text{rot } \vec{u} = 0,\ \vec{\tau} \cdot \vec{u} = 0\}, \tag{4.29}$$

$$L^p(\Omega, \mathbb{R}^2) = \nabla^t H^{1,p}_0(\Omega) \oplus \{\vec{u} \in L^p(\Omega, \mathbb{R}^2) : \text{rot } \vec{u} = 0\}, \tag{4.30}$$

where the sums are direct and topological.

The range $4/3 - \varepsilon < p < 4 + \varepsilon$ for which the decompositions (4.27)–(4.30) are valid is sharp in the class of Lipschitz domains. When $\partial\Omega \in C^1$ one can take $1 < p < \infty$.

REFERENCES

[1] C. Bennett and R. Sharpley, "Interpolation of Operators," Academic Press, Boston, 1988.

[2] J. Bergh and J. Löfström, "Interpolation Spaces. An Introduction," Springer–Verlag, Berlin/New York, 1976.

[3] A.P. Calderón, *Boundary value problems for the Laplace equation in Lipschitz domains*, Recent Progress in Fourier Analysis, pp. 33–48, I. Peral and J.-L. Rubio de Francia eds., Elsevier Science Pub., North–Holland, 1985.

[4] R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy definit un opérateur borné sur L^2 pour les courbes Lipschitziennes*, Annals of Math., 116 (1982), 361–388.

[5] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc., 83 (1977), 569–645.

[6] D. Colton and R. Kress, *Inverse acoustic and electromagnetic scattering theory*, Appl. Math. Sci. 93, Springer–Verlag, Berlin/Heidelberg, 1992.

[7] B. Dahlberg, *On estimates of harmonic measure*, Arch. Rat. Mech. Anal., 65 (1977), 275–288.

[8] B. Dahlberg and C. Kenig, *Hardy spaces and the L^p–Neumann problem for Laplace's equation in a Lipschitz domain*, Ann. Math., 125 (1987), 437–465.

[9] B. Dahlberg and C. Kenig, *L^p Estimates for the Three-Dimensional System of Ellastostatics on Lipschitz Domains*, in Lecture Notes Pure Appl. Math., 122 (1990), 621–634.

[10] R. J. DiPerna and A. Majda, *Concentrations in regularizations for 2D incompressible flow*, Comm. Pure Appl. Math., 40 (1987), 301–345.

[11] E. Fabes, *Layer potential methods for boundary value problems on Lipschitz domains*, Potential Theory, Surveys and Problems, J. Král et al. eds., Springer–Verlag Lecture Notes in Math., No. 1344 (1988), 55–80.

[12] E. Fabes, M. Jodeit and N. Rivière, *Potential techniques for boundary value problems on C^1 domains*, Acta Math., 141 (1978), 165–186.

[13] E. Fabes, O. Mendez and M. Mitrea, *Boundary layers on Sobolev–Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains*, J. Funct. Anal., 159 (1998), 323–368.

[14] D. Fujiwara and H. Morimoto, *An L^r-theorem of the Helmholtz decomposition of vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 24 (1977), 685–700.

[15] H. G. Garnir, "Les Problémes aux Limites de la Physique Mathématique," Basel, Birkhäuser, 1958.

[16] R. Griesinger, *Decompositions of L^q and $H_0^{1,q}$ with respect to the operator rot*, Math. Ann., 288 (1990), 245–262.

[17] Y. Giga, *Analyticity of the semigroup generated by the Stokes operator in L_r spaces*, Math. Z., 178 (1981), 297–329.

[18] W. V. D. Hodge, "The Theory and Applications of Harmonic Integrals," Cambridge University Press, 1941.

[19] D. Jerison and C. E. Kenig, *The Neumann problem on Lipschitz domains*, Bull. Amer. Math. Soc., 4 (1981), 203–207.

[20] D. Jerison and C. E. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal., 130 (1995), 161–219.

[21] N. Kalton and M. Mitrea, *Stability results on interpolation scales of quasi-Banach spaces and applications*, Trans. Amer. Math. Soc., 350 (1998), 3903–3922.

[22] C. E. Kenig, *Elliptic boundary value problems on Lipschitz domains*, Beijing lectures in harmonic analysis (Beijing, 1984), 131–183, Ann. of Math. Stud., 12, Princeton Univ. Press, Princeton N.J., 1986.

[23] C. Kenig, "Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems," AMS Regional Conferences Series in Mathematics, Vol. 83, 1994.

[24] V. A. Kozlov, V. G. Maz'ya and J. Rossmann, *Elliptic Boundary Value Problems in Domains with Point Singularities*, Math. Surv. and Mon., Vol. 52, AMS, 1997.

[25] O. Mendez and M. Mitrea, *The Banach envelope of Besov and Triebel–Lizorkin spaces and applications*, J. Four. Anal. Appl., 6 (2000), 503–531.

[26] D. Mitrea, *Layer potential operators and boundary value problems for differential forms on Lipschitz domains*, Ph.D. thesis, University of Minnesota, 1996.

[27] D. Mitrea, *The method of layer potentials for non-smooth domains with arbitrary topology*, Integral Equations and Operator Theory, 29 (1997), 320–338.

[28] D. Mitrea, *Layer potentials and Hodge decompositions in two dimensional Lipschitz domains*, to appear, Math. Ann. (2001).

[29] D. Mitrea and M. Mitrea, *Finite energy solutions of Maxwell's equations and constructive Hodge decompositions on nonsmooth Riemannian manifolds*, J. Funct. Anal., to appear.

[30] D. Mitrea, M. Mitrea and M. Taylor, *Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds*, Memoirs of AMS, Vol. 150, No. 713, 2001.

[31] M. Mitrea and M. Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: Sobolev–Besov space results and the Poisson problem*, J. Funct. Anal., 176 (2000), 1–79.

[32] T. Runst and W. Sickel, "Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Operators," de Gruyter, Berlin, New York, 1996.

[33] G. Schwarz, *Hodge Decomposition—A method for solving boundary-value problems*, Springer–Verlag LNM, No. 1607, 1995.

[34] W. Sickel and H. Triebel, *Hölder inequalities and sharp embeddings in function spaces of $B^s_{p,q}$ and $F^s_{p,q}$ type*, Z. Anal. Anwendungen, 14 (1995), 105–140.

[35] C. G. Simader and H. Sohr, *A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains*, in "Mathematical Problems Relating to the Navier–Stokes Equations," ed. G. P. Galdi, World Scientific Series Adv. Math. Appl. Sci. 11, 1992, pp. 1–35.

[36] H. Sohr and G. Thäter, *Imaginary powers of second order differential operators and L^q-Helmholtz decomposition in the infinite cylinder*, Math. Ann., 311 (1998), 577–602.

[37] M. Taylor, "Pseudodifferential Operators, Paradifferential Operators, and Layer Potentials," Math. Surveys and Monographs, AMS, Providence, RI, 2000.

[38] G. Verchota, *Layer potentials and boundary value problems for Laplace's equation in Lipschitz domains*, J. Funct. Anal., 59 (1984), 572–611.

[39] J. M. Wilson, *A simple proof of atomic decompositions for $H^p(\mathbb{R}^n)$, $0 < p \leq 1$*, Studia Math., 74 (1982), 25–33.