Advances in Differential Equations Volume 7, Number 3, March 2002, Pages 319–342

THE MINIMUM FREE ENERGY FOR A CLASS OF COMPRESSIBLE VISCOELASTIC FLUIDS

M. Fabrizio and G. Gentili[1]
Dipartimento di Matematica, Università di Bologna
P.zza di p.ta S. Donato 5, 40127 Bologna, Italy

J.M. Golden
School of Mathematics and Statistics, Dublin Institute of Technology
Kevin Street, Dublin 8, Ireland

(Submitted by: Viorel Barbu)

Abstract. A general expression is derived for the isothermal minimum free energy of a compressible viscoelastic fluid with linear dependence on the history of strain and nonlinear dependence on the density. This involves the solution of a Wiener–Hopf integral equation, for which an existence and uniqueness theorem is proved. The factorization of a particular material tensor, which is fundamental to the methodology, can be carried out explicitly in this case.

1. Introduction

Recently, explicit formulae have been given for the maximum recoverable work from a specified viscoelastic state, under isothermal conditions, for a scalar [6] and a general tensor [4] constitutive relation. These formulae also represent the minimum free energy associated with a given viscoelastic state, by virtue of general theorems identifying this quantity with the maximum recoverable work [7].

The aim of the present work is to derive an expression for the minimum free energy of a compressible viscoelastic fluid with linear dependence on the history of strain. Essentially the same techniques apply, though there are significant differences with respect to the earlier work [6, 4]. Firstly, there is the equilibrium pressure term, with a nonlinear dependence on the density, in the constitutive relations; and secondly, there is the fact that the constitutive relations are those for an isotropic material. This is of course

Accepted for publication: May 2001.
AMS Subject Classifications: 76A10, 74A15, 45E10, 30E20.
[1]Deceased 3rd December, 2000.

a special case of the full anisotropic tensorial treatment in [4]. However, it is particularly interesting in that a factorization, which is fundamental to the methodology, can be carried out as explicitly as in the scalar case [6] for general viscoelastic response. In the full anisotropic case treated in [4], it can be proved that the required factorization exists, but no general method for determining the factors has yet been given; though of course, this can be done for specific material responses.

In order to obtain the process which provides the maximum recoverable work, it is necessary to solve an integral equation of the Wiener–Hopf type. An existence and uniqueness theorem is proved for this integral equation, using conditions which follow from thermodynamics.

In Section 2, constitutive equations are given for the class of compressible viscoelastic fluids with linear dependence on strain history. Also, thermodynamic states and processes are defined, and the notion of equivalent states is introduced. The subclass of compressible fluids considered in this work is defined in Section 3 while in Section 4, certain thermodynamic concepts and results are presented, with application to the particular types of material under consideration. In Section 5, the crucial factorization is carried out, while in the following section, the Wiener–Hopf integral equation for the process yielding the maximum recoverable work is derived and shown to have a unique solution. In Section 7, an explicit formula for the minimum free energy is constructed and discussed; and a function on the equivalence class of states is presented.

Certain notational usages are defined and basic formulae listed in an Appendix.

2. Compressible viscoelastic fluids

The state σ of a compressible viscoelastic fluid can be described [9, 13] by means of the mass density $\rho(\mathbf{x}, t)$ and the history of strain

$$\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\top})$$

($\mathbf{u}$ is the displacement) relative to its present value, i.e., by means of the couple $(\rho, \mathbf{E}_r^t)$ where the relative strain history $\mathbf{E}_r^t$ is defined by

$$\mathbf{E}_r^t(\mathbf{x}, s) = \mathbf{E}^t(\mathbf{x}, s) - \mathbf{E}(\mathbf{x}, t), \quad s \in \mathcal{R}^{++}$$

using the notation $\mathbf{E}^t(\mathbf{x}, s) = \mathbf{E}(\mathbf{x}, t - s)$ for the strain history. The dependence on the spatial variable $\mathbf{x}$ will henceforth be omitted.

A fluid is necessarily isotropic. We will also assume for simplicity that it is homogeneous. The constitutive equation for the stress is given by

$$\mathbf{T}(\rho, \mathbf{E}_r^t) = -p(\rho)\mathbf{I} + \tilde{\mathbf{T}}(\rho, \mathbf{E}_r^t) \tag{2.1}$$

where $\mathbf{I}$ is the identity second-order tensor and p denotes the pressure. The quantity $\tilde{\mathbf{T}}$ is referred to as the extra stress and is given by (see (A.2))

$$\tilde{\mathbf{T}}(\rho, \mathbf{E}_r^t) = \int_0^\infty \lambda_1(\rho, s) E_r^t(s)\, ds\, \mathbf{I} + 2\int_0^\infty \mu_1(\rho, s)\mathbf{E}_r^t(s)\, ds \tag{2.2}$$

where, for each fixed ρ, the memory kernels λ_1 and μ_1 have the property

$$\lambda_1(\rho, \cdot), \mu_1(\rho, \cdot) \in (L^1 \cap L^2)(\mathcal{R}^+)\,, \qquad \forall \rho > 0. \tag{2.3}$$

Let $\mathbf{E}^\dagger$ denote a constant history; i.e., $\mathbf{E}^t(s) = \mathbf{E}^\dagger\ \forall s \in \mathcal{R}^{++}$. If $\mathbf{E}^t = \mathbf{E}^\dagger$, the relative strain history $\mathbf{E}_r^t = \mathbf{0}^\dagger$ where $\mathbf{0}^\dagger$ is the zero strain history. In this case, the extra stress is zero:

$$\tilde{\mathbf{T}}(\rho, \mathbf{0}^\dagger) = \mathbf{0}. \tag{2.4}$$

Moreover, the *relaxation property* ensures that, for a static continuation $\mathbf{E}_{r\,c}^{t+\tau}$, defined as

$$\mathbf{E}_{r\,c}^{t+\tau}(s) = \begin{cases} \mathbf{0} & \text{for } s \le \tau \\ \mathbf{E}_r^t(s-\tau) & \text{for } s > \tau \end{cases}\,, \tag{2.5}$$

the extra stress vanishes as τ diverges, viz.

$$\lim_{\tau\to\infty} \tilde{\mathbf{T}}(\rho, \mathbf{E}_{r\,c}^{t+\tau}) = \tilde{\mathbf{T}}(\rho, \mathbf{0}^\dagger) = \mathbf{0}. \tag{2.6}$$

Given the state $\sigma(t) = (\rho(t), \mathbf{E}_r^t)$ the stress is uniquely determined by (2.1–2.2).

A process P of duration d_p will be described by a function $\mathbf{D}^P : [0, d_p) \to$ *Sym* where $\mathbf{D}^P(\tau) = \dot{\mathbf{E}}_P(\tau)$, the derivative of a strain tensor specified over a time segment of duration d_P. For any process $\mathbf{D}^P$ the evolution of $\sigma(t+\tau) = (\rho(t+\tau), \mathbf{E}_r^{t+\tau})$, $\tau \in [0, d_p)$, is determined as the solution of the differential equations given by

$$\frac{d}{d\tau}\mathbf{E}_r^{t+\tau}(s) = \mathbf{D}^P(\tau - s) - \mathbf{D}^P(\tau)\,, \quad 0 < s < \tau \tag{2.7}$$

and the balance of mass[2]

$$\frac{d}{d\tau}\rho(t+\tau) = -\rho(t+\tau)\nabla\cdot\mathbf{v}(t+\tau) = -\rho(t+\tau)D^P(\tau)\,. \tag{2.8}$$

[2]Note that $D^P = tr(\dot{\mathbf{E}})$.

The effect of a given process on a particular strain history is described in more detail in [4]. Note that the solution of (2.8) is given by

$$\rho(t+\tau) = \rho(t)\, \exp\Big[-\int_0^\tau D^P(s)\, ds \Big]. \tag{2.9}$$

Henceforth, Π will denote the set of all admissible process $\mathbf{D}^P$ of finite duration whereas the set of all admissible states will be denoted by Σ: [3]

$$\Sigma = \{\sigma = (\rho, \mathbf{E}_r^t) : |\mathbf{T}(\rho, \mathbf{E}_{r\,c}^{t+\tau})| < \infty \ ,\ \forall \tau \geq 0\} \ . \tag{2.10}$$

Actually two different states $\sigma_1 = (\rho_1(t), \mathbf{E}_{1r}^t)$ and $\sigma_1 = (\rho_2(t), \mathbf{E}_{2r}^t)$ may yield the same stress $\mathbf{T}$. We recall the following definition [15]:

Definition 2.1. Two states $\sigma_1(t) = (\rho_1(t), \mathbf{E}_{1r}^t)$ and $\sigma_2(t) = (\rho_2(t), \mathbf{E}_{2r}^t)$ are said to be equivalent if, for every process $\mathbf{D} : [0, d_p) \to Sym$, the subsequent states, $\sigma_1(t+\tau)$ and $\sigma_2(t+\tau)$, $\tau \in [0, d_p)$, obtained by (2.7–2.8), satisfy

$$\mathbf{T}(\rho_1(t+\tau), \mathbf{E}_{1r}^{t+\tau}) = \mathbf{T}(\rho_2(t+\tau), \mathbf{E}_{2r}^{t+\tau})\ , \quad \forall \tau \in [0, d_p). \tag{2.11}$$

Such a definition introduces an equivalence relation, whose equivalence classes are named the minimal states $\sigma_{(m)}$ of the material. In other words, if σ_1 and σ_2 are equivalent in the sense of Definition 2.1, they represent the same state $\sigma_{(m)}$.

Thus the space of the minimal states $\Sigma_{(m)}$ is the space of the equivalent classes of Σ induced by Definition 2.1.

Observe that, by virtue of decomposition (A.2), the constitutive equations (2.1–2.2) may be rewritten as

$$\mathbf{T}(\rho, \mathbf{E}_r^t) = -p(\rho)\mathbf{I} + \int_0^\infty \kappa_1(\rho, s) E_r^t(s)\, ds\, \mathbf{I} + 2\int_0^\infty \mu_1(\rho, s) \overset{\circ}{\mathbf{E}}{}_r^t(s)\, ds \tag{2.12}$$

where $\kappa_1 = \lambda_1 + \frac{2}{3}\mu_1$. Equation (2.12) may be written in compact form as

$$\mathbf{T}(\rho, \mathbf{E}_r^t) = -p(\rho)\mathbf{I} + \int_0^\infty \mathbb{G}_1(\rho, s) \mathbf{E}_r^t(s)\, ds \tag{2.13}$$

where the relaxation function $\mathbb{G}_1$ is a fourth-order tensor-valued function $\mathbb{G}_1 : \mathcal{R}^+ \times \mathcal{R}^+ \to \mathcal{D}(Sym)$. The first element of $\mathbb{G}_1(\rho, s)$ in the expansion (A.6) is given by $\kappa_1(\rho, s)$ whereas the other nonvanishing (i.e., diagonal) elements are equal to $2\mu_1(\rho, s)$.

[3] In the sequel we make use of a slightly more restrictive definition, in order to render Σ compatible with linear thermodynamic theory.

3. A particular class of viscoelastic fluids

In this paper we consider a class of viscoelastic fluids whose memory kernels μ_1, λ_1 assume the form

$$\mu_1(\rho, s) = \rho\mu'(s); \qquad \lambda_1(\rho, s) = \rho\lambda'(s)$$

so that the constitutive equations for the stress are given by (2.1–2.2), where the extra stress $\tilde{\mathbf{T}}$, described by (2.2), assumes the form

$$\tilde{\mathbf{T}}(\rho, \mathbf{E}_r^t) = \rho\mathbf{V}(\mathbf{E}_r^t) = \rho\int_0^\infty \kappa'(s)E_r^t(s)\,ds\,\mathbf{I} + 2\rho\int_0^\infty \mu'(s)\overset{\circ}{\mathbf{E}}{}_r^t(s)\,ds \qquad (3.1)$$

where $k'(s) = \lambda'(s) + \frac{2}{3}\mu'(s)$, or using the compact representation (2.13)

$$\tilde{\mathbf{T}}(\rho, \mathbf{E}_r^t) = \rho\mathbf{V}(\mathbf{E}_r^t) = \rho\int_0^\infty \mathbb{G}'(s)\mathbf{E}_r^t(s)\,ds \qquad (3.2)$$

where $\mathbb{G}'(s) \in \mathcal{D}(Sym)$ with the first element in the expansion (A.6) given by $\kappa'(s)$ and the other nonvanishing elements by $2\mu'(s)$.

Condition (2.3) now reads

$$\lambda', \mu' \in (L^1 \cap L^2)(\mathcal{R}^+)\,. \qquad (3.3)$$

Moreover, we assume that

$$\lambda'(0), \mu'(0) \in \mathcal{R}^{--} \qquad (3.4)$$

and

$$\lambda, \mu \in (L^1 \cap L^2)(\mathcal{R}^+)\,. \qquad (3.5)$$

where $\lambda(t) = -\int_t^\infty \lambda'(s)\,ds$ and $\mu(t) = -\int_t^\infty \mu'(s)\,ds$.

Observe that the model for weakly compressible fluids studied in [5] is included in the class of constitutive equations described by (2.1–3.2). In fact, linearization of (2.12) or more directly of (2.1) and (3.1) yield the linearized constitutive equation (2.9) of [5].

For materials of type (3.2), under the assumption that any finite density ρ yields a finite pressure $p(\rho)$, it is easy to check that the space of admissible states Σ, given by (2.10), may be written as $\Sigma = \mathcal{R}^+ \times \Gamma$ where

$$\Gamma = \left\{\mathbf{E}_r^t : \left|\int_0^\infty \mathbb{G}'(s+\tau)\mathbf{E}_r^t(s)\,ds\right| < \infty\,,\ \forall\tau \geq 0\right\}.$$

Moreover, the state space $\Sigma_{(m)}$ can be characterized by means of the following property:

Theorem 3.1. *For a viscoelastic fluid of type* (3.1), *two states* $\sigma_1 = (\rho_1, \mathbf{E}^t_{1r})$ *and* $\sigma_2 = (\rho_2, \mathbf{E}^t_{2r})$ *are equivalent in the sense of Definition* 2.1 *if and only if*

$$\rho_1(t) = \rho_2(t), \ \int_0^\infty \mu'(s+\tau)\mathring{\mathbf{E}}^t_r(s)\,ds = 0, \ \int_0^\infty \kappa'(s+\tau)E^t_r(s)\,ds = 0, \ \forall\tau \ge 0 \tag{3.6}$$

where $\mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2$.

Proof. Obviously if (3.6) is satisfied, then (2.11) holds. Suppose now that (2.11) holds for any process. Noting that

$$\begin{aligned}\tilde{\mathbf{T}}(\rho, \mathbf{E}^{t+\tau}_r) &= \rho(t+\tau)\int_0^\infty \mathbb{G}'(s)\mathbf{E}^{t+\tau}_r(s)\,ds \qquad (3.7)\\ &= \rho(t+\tau)\int_0^\infty \mathbb{G}'(u+\tau)\mathbf{E}^t_r(u)\,du + \rho(t+\tau)\int_{-\tau}^0 \mathbb{G}'(u+\tau)\mathbf{E}^t_r(u)\,du\end{aligned}$$

where $\mathbf{E}^t_r$ in the last integral is determined by the process, according to (2.7) where $\mathbf{E}^t_r(u) = \mathbf{E}^{t+\tau}_r(\tau+u)$, we find that (2.11) gives

$$\begin{aligned}&-(p(\rho_1(t+\tau)) - p(\rho_2(t+\tau)))\mathbf{I} + \rho_1(t+\tau)\int_0^\infty \mathbb{G}'(u+\tau)\mathbf{E}^t_{1r}(u)\,du\\ &-\rho_2(t+\tau)\int_0^\infty \mathbb{G}'(u+\tau)\mathbf{E}^t_{2r}(u)\,du \qquad (3.8)\\ &+(\rho_1(t+\tau) - \rho_2(t+\tau))\int_{-\tau}^0 \mathbb{G}'(u+\tau)\mathbf{E}^t_r(u)\,du = 0\end{aligned}$$

for any process. Observe that $\rho_1(t+\tau)$ and $\rho_2(t+\tau)$ are determined by the scalar part of the process, according to (2.9). For each choice of the scalar part D_P, we can vary the trace-free part of the process arbitrarily, which affects only the final integral in (3.8). Thus, we must have

$$\rho_1(t+\tau) = \rho_2(t+\tau), \quad \tau \in \mathcal{R}^{++} \tag{3.9}$$

or

$$\rho_1(t)\,\exp\Big[-\int_0^\tau D^P(s)\,ds\Big] = \rho_2(t)\,\exp\Big[-\int_0^\tau D^P(s)\,ds\Big], \quad \tau \in \mathcal{R}^{++}. \tag{3.10}$$

It follows that

$$\rho_1(t) = \rho_2(t). \tag{3.11}$$

Also, $(3.6)_2$ and $(3.6)_3$ follow immediately from (3.8) and (3.9), on recognizing that the scalar and trace-free parts must vanish separately. □

Denoting by Γ_0 the set of all the histories $\mathbf{E}_r^t \in \Gamma$ satisfying $(3.6)_2$ and $(3.6)_3$, and by Γ/Γ_0 the usual quotient space, Theorem 3.1 implies that the minimal state of a linear viscoelastic material is an element of

$$\Sigma_{(m)} := \mathcal{R}^+ \times (\Gamma/\Gamma_0). \tag{3.12}$$

We also view a process as a function $P : \Sigma \to \Sigma$ which associates with an initial state $\sigma^i \in \Sigma$ a final state $P\sigma^i = \sigma^f \in \Sigma$. Such an evolution is governed by the differential equations (2.7–2.8). Considering P as a function $P : \Gamma \to \Gamma$ associating with any initial history $\gamma^i \in \Gamma$, a final history $P\gamma^i = \gamma^f \in \Gamma$, the evolution related to P is governed only by (2.7).

4. Thermodynamics

In this paper we confine our attention to isothermal processes, so that the second law of thermodynamics reduces to the dissipation principle stating that the work on a cycle is nonnegative; viz.,

$$W(\sigma, P) = \oint_0^{d_p} \frac{1}{\rho}\, \mathbf{T}(\rho, \mathbf{E}_r^t) \cdot \mathbf{D}(t)\, dt \geq 0 \tag{4.1}$$

for a process P, starting from a state σ, taken to be at time $t = 0$, and such that $P\sigma = \sigma$.

Observe that, in view of (2.8), for a material of the type (2.1),

$$\oint_0^{d_p} \frac{p(\rho)}{\rho}\, D(t)\, dt = -\oint_0^{d_p} \frac{p(\rho)}{\rho^2}\, \dot{\rho}\, dt = 0 \tag{4.2}$$

on a cycle, so that (4.1) reduces to

$$W(\sigma, P) = \oint_0^{d_p} \frac{1}{\rho}\, \tilde{\mathbf{T}}(\rho, \mathbf{E}_r^t) \cdot \mathbf{D}(t)\, dt \geq 0\ . \tag{4.3}$$

A set $\mathcal{S} \subset \Sigma$ is said to be *invariant* if for every $\sigma_1 \in \mathcal{S}$, and $P \in \Pi$, the state $\sigma = P\sigma_1 \in \mathcal{S}$.

Definition 4.1. A function $\psi : \mathcal{S} \to \mathcal{R}^+$ is a *free energy density*[4] if

i. the domain $\mathcal{S}$ is invariant,

ii. for any pair $\sigma_1, \sigma_2 \in \mathcal{S}$ and $P \in \Pi$, such that $P\sigma_1 = \sigma_2$ we have

$$\psi(\sigma_2) - \psi(\sigma_1) \leq W(\sigma_1, P). \tag{4.4}$$

[4]Henceforth the free energy density will be referred to simply as the free energy.

In Day [2, 3], Coleman and Owen [1], and Fabrizio, Giorgi and Morro [7], the existence of a free energy was proved as a consequence of the dissipation principle (4.1).

A state $\sigma \in \Sigma$ is referred to as *attainable* from all of Σ if, for any initial state σ^i, there exists a process $P \in \Pi$ such that $P\sigma^i = \sigma$. A simple material system is attainable if any state σ is attainable from every other state $\sigma' \in \Sigma$.

However, for a simple material with fading memory, not all states are attainable. In particular, cyclic processes constitute a very narrow class of processes related to a very narrow set of states. For this reason the dissipation principle is not restrictive enough, and we need a stronger formulation of the second law.

To this aim we denote by

$$\mathcal{W}(\sigma) := \{W(\sigma, P) : P \in \Pi\} \tag{4.5}$$

the set of the works done by all possible processes $P \in \Pi$ acting on a given state σ. The following principle shall be adopted:

Strong Dissipation Principle: The set $\mathcal{W}(\sigma)$ is bounded from below for all $\sigma \in \Sigma$. Furthermore, there is a state $\sigma^\dagger$, which we refer to as the zero state, such that

$$\inf \left\{ W(\sigma^\dagger, P) : P \in \Pi \right\} = 0. \tag{4.6}$$

The strong dissipation principle requires that we redefine the set of all admissible states Σ, modifying (2.10) as follows:

$$\Sigma^T = \{\sigma = (\rho, \mathbf{E}_r^t) : |\mathbf{T}(\rho, \mathbf{E}_{r\,c}^{t+\tau})| < \infty \ , \ \forall \tau \geq 0\}$$

$$\Sigma := \left\{ \sigma \in \Sigma^T : \inf \mathcal{W}(\sigma) > -\infty \right\}. \tag{4.7}$$

If $(4.7)_2$ were not true, we would have a contradiction with the second law. In fact, $-W(\sigma, P)$ is the work yielded by the material. If it were unbounded from above, as P varies, we could extract infinite energy from the material, and then generate a perpetual motion.

For a material of type (2.1–2.2), the zero state is $\sigma^\dagger = (\rho_0, \mathbf{0}^\dagger)$, where ρ_0 is the equilibrium mass density and $\mathbf{0}^\dagger$ is the zero history introduced before (2.4).

Definition 4.2. A functional ψ_m is the *minimum free energy* if

i) ψ_m is a free energy in the sense of Definition 4.1 with domain $\mathcal{S} = \Sigma$,

ii) the zero state $\sigma^\dagger \in \Sigma$ is such that $\psi_m(\sigma^\dagger) = 0$, and

iii) for any free energy $\psi : \mathcal{S} \to \mathcal{R}^+$ such that $\sigma^\dagger \in \mathcal{S}$, and $\psi(\sigma^\dagger) = 0$, we have

$$\psi(\sigma) \geq \psi_m(\sigma), \quad \text{for all } \sigma \in \mathcal{S}. \tag{4.8}$$

Theorem 4.1. *The functional* $\psi_m(\sigma) := -\inf \mathcal{W}(\sigma)$ *is the minimum free energy.*

The proof of this theorem has been given in [7] and will be omitted.

It is always possible to represent the minimum free energy as a function of the minimal state σ_m. In fact, if two equivalent states σ_1 and σ_2, continued with the same process P, yield the same stress, then they also yield the same work, giving $W(\sigma_1, P) = W(\sigma_2, P)$. Therefore $W(\sigma, P) = \hat{W}(\sigma_{(m)}, P)$ so that $\mathcal{W}(\sigma) = \hat{\mathcal{W}}(\sigma_m)$ and $\inf \mathcal{W}(\sigma) = \inf \hat{\mathcal{W}}(\sigma_m)$. As a consequence

$$\psi_m(\sigma) = \hat{\psi}_m(\sigma_m) \tag{4.9}$$

Hence the minimum free energy is independent of the definition of state that is used.

We conclude the section by proving an important property of the free energy of a material described by (2.1).

Theorem 4.2. *For materials described by* (3.1), *every free energy may be written as the sum of two terms*

$$\psi(\sigma) = \phi(\rho) + \varphi(\gamma) \tag{4.10}$$

where

$$\phi(\rho) = \int_{\rho_0}^{\rho} \frac{1}{\xi^2}\, p(\xi)\, d\xi, \tag{4.11}$$

ρ_0 *being the equilibrium mass density and* $\varphi : \mathcal{S}_\Gamma \to \mathcal{R}$ *is defined on a set* $\mathcal{S}_\Gamma$ *that is* Γ*-invariant* (*namely, if* $\gamma \in \mathcal{S}_\Gamma$ *then* $P\gamma \in \mathcal{S}_\Gamma$ *for every* $P \in \Pi$) *and satisfies*

$$\varphi(\gamma_2) - \varphi(\gamma_1) \leq \int_0^{d_p} \mathbf{V}(\mathbf{E}_r^t) \cdot \mathbf{D}(t)\, dt \tag{4.12}$$

where $P\gamma_1 = \gamma_2$ *and we have dropped the superscript* P *on* $\mathbf{D}$. *Moreover, if* $\psi(\sigma^\dagger) = 0$, *then*

$$\varphi(\mathbf{0}^\dagger) = 0. \tag{4.13}$$

Proof. The solution of (2.8) is given by (2.9). Given a state $\sigma(t) = (\rho(t), \gamma(t)) \in \mathcal{S}$ and a process P of duration d, we have that $P\sigma(t) \in \mathcal{S}$ if $(\rho(t+d), \gamma(t+d)) \in \mathcal{S}$, where $\rho(t+d)$ is given by (2.9) with $\tau = d$, and $\gamma(t+d) = P\gamma(t)$, determined by the solution of (2.7). Therefore $\mathcal{S}$ is invariant if and only if $\mathcal{S} = \mathcal{R}^+ \times \mathcal{S}_\Gamma$ where $\mathcal{S}_\Gamma$ is Γ-invariant.

Moreover, given a process P of duration d such that $P\sigma_1 = \sigma_2$, taking σ_1 to be the state at $t = 0$, inequality (4.4) reads

$$\psi(\sigma_2) - \psi(\sigma_1) \leq \int_0^d \frac{1}{\rho}\mathbf{T}\cdot\mathbf{D}\,dt = -\int_0^d \frac{p(\rho(t))}{\rho(t)}\,D(t)\,dt + \int_0^d \left[\int_0^\infty \kappa'(s)E_r^t(s)\,ds\,D(t) + 2\int_0^\infty \mu'(s)\mathring{\mathbf{E}}_r^t(s)\cdot\mathring{\mathbf{D}}(t)\,ds\right]dt. \tag{4.14}$$

By virtue of the balance of mass (2.8), we have

$$-\int_0^d \frac{p(\rho(t))}{\rho(t)}\,D(t)\,dt = \int_{\rho_1}^{\rho_2} \frac{p(\rho)}{\rho^2}\,d\rho = \phi(\rho_2) - \phi(\rho_1) \tag{4.15}$$

where $\phi(\rho)$ is given by (4.11). Then (4.14) becomes

$$\psi(\sigma_2) - \psi(\sigma_1) \leq \phi(\rho_2) - \phi(\rho_1) + \int_0^d \left[\int_0^\infty \kappa'(s)E_r^t(s)\,ds\,D(t) + 2\int_0^\infty \mu'(s)\mathring{\mathbf{E}}_r^t(s)\cdot\mathring{\mathbf{D}}(t)\,ds\right]dt. \tag{4.16}$$

Thus the function φ, defined by $\varphi = \psi - \phi$, satisfies

$$\varphi(\gamma_2) - \varphi(\gamma_1) \leq \int_0^d \left[\int_0^\infty \kappa'(s)E_r^t(s)\,ds\,D(t) + 2\int_0^\infty \mu'(s)\mathring{\mathbf{E}}_r^t(s)\cdot\mathring{\mathbf{D}}(t)\,ds\right]dt. \tag{4.17}$$

By virtue of (3.1), the right-hand sides of (4.12) and (4.17) are equal.

Finally, since $\psi(\sigma^\dagger) = \phi(\rho_0) + \varphi(\mathbf{0}^\dagger)$, and $\phi(\rho_0) = 0$, then $\psi(\sigma^\dagger) = 0$ if and only if $\varphi(\mathbf{0}^\dagger) = 0$. $\square$

Henceforth the right-hand side of (4.12) will be termed the Γ-work and denoted by

$$W(\gamma, P) = \int_0^{d_p} \frac{1}{\rho}\,\tilde{\mathbf{T}}(\rho, \mathbf{E}_r^t)\cdot\mathbf{D}(t)\,dt = \int_0^{d_p} \mathbf{V}(\mathbf{E}_r^t)\cdot\mathbf{D}(t)\,dt \tag{4.18}$$

whereas $\mathcal{W}_\Gamma(\gamma)$ will denote the set of all the Γ-works starting from $\gamma \in \Gamma$, viz. $\mathcal{W}_\Gamma(\gamma) = \{W(\gamma, P) : P \in \Pi\}$. It is easy to check that Theorems 4.1 and 4.2 imply that the minimum free energy can be written as

$$\psi_m(\sigma) = \phi(\rho) + \varphi_m(\gamma) \tag{4.19}$$

with φ_m given by

$$\varphi_m(\gamma) := -\inf \mathcal{W}_\Gamma(\gamma). \tag{4.20}$$

The right-hand side of (4.20) represents the maximum recoverable Γ-work.

5. Thermodynamic restrictions and factorization

Before determining the "optimal" process maximizing the recoverable Γ-work, we recall some properties of the relaxation function $\mathbb{G}$ required by thermodynamics.

In particular, as a consequence of the dissipation principle, for materials described by constitutive equations (2.1–3.2) the memory kernels κ' and μ' must satisfy [10]

$$\frac{1}{\omega}\mu'_s(\omega) < 0 \,, \qquad \frac{1}{\omega}\kappa'_s(\omega) < 0 \,, \qquad \forall\omega \in \mathcal{R} \,, \tag{5.1}$$

using the notation defined by (A.8). Since

$$\mathbb{G}_c(\omega) = -\frac{1}{\omega}\mathbb{G}'_s(\omega), \tag{5.2}$$

the thermodynamic restrictions (5.1) ensure that $\mathbb{G}_c(\omega)$ is positive definite for every $\omega \in \mathcal{R}$. Moreover $\mathbb{G}_c(\omega)$ vanishes as ω^{-2} as ω tends to infinity. In fact it is easy to check that, if $\mathbb{G}''$ is integrable, we have

$$\lim_{\omega\to\infty} \omega^2\mathbb{G}_c(\omega) = -\mathbb{G}'(0) \tag{5.3}$$

where $\mathbb{G}'(0)$ is negative definite by virtue of (3.4).

The above thermodynamic properties ensure that the function $\mathbb{G}_c(\omega)$ may be factorized by [4, 11]

$$\mathbb{G}_c(\omega) = \mathbb{G}_{(+)}(\omega)\mathbb{G}_{(-)}(\omega) \tag{5.4}$$

where the singularities of $\mathbb{G}_{(\pm)}$ on the complex plane, and the zeros of its determinant, are all in $\Omega^{(\pm)}$ respectively.

In fact, since $\mathbb{G}_c(\omega) \in \mathcal{D}(Sym)$, we can easily extend the result given in [6] for scalar functions. Thus, putting $\mathbb{G}_h = (-\mathbb{G}'(0))^{1/2}$, the function $\mathbb{K} : \mathcal{R} \to \mathcal{D}(Sym)$ defined by

$$\mathbb{K}(\omega) := \log\left[(1+\omega^2)\mathbb{G}_h^{-2}\mathbb{G}_c(\omega)\right] \tag{5.5}$$

is an analytic matrix-valued function on $\mathcal{R}$, vanishing as ω^{-2} for large ω. Then the Plemelj formulae [14] allow us to write $\mathbb{K}(\omega)$ as

$$\mathbb{K}(\omega) = \mathbb{M}_{(+)}(\omega) - \mathbb{M}_{(-)}(\omega) \tag{5.6}$$

where

$$\mathbb{M}_{(\pm)}(\omega) := \lim_{\alpha\to 0^\pm} \mathbb{M}(\omega + i\alpha); \;\; \mathbb{M}(z) := \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathbb{K}(\omega)}{\omega - z}\, d\omega, \;\; z \in \Omega\setminus\mathcal{R}. \tag{5.7}$$

Therefore, $\mathbb{G}_c$ can be factorized in the form (5.4) where

$$\mathbb{G}_{(+)}(\omega) := \frac{1}{\omega - i}\mathbb{G}_h e^{-\mathbb{M}_{(-)}(\omega)}, \quad \mathbb{G}_{(-)}(\omega) := \frac{1}{\omega + i}\mathbb{G}_h e^{\mathbb{M}_{(+)}(\omega)} . \tag{5.8}$$

Observe that the property $\mathbb{G}(s) \in \mathcal{D}(Sym)$ is crucial in that it ensures that all the matrices involved in this section are diagonal, so they commute. Moreover, factorization of $\mathbb{G}_c$ is equivalent to the factorization of its components. In other words, the first component of $\mathbb{G}_{(\pm)}$ is given by $\kappa_{(\pm)}$ and the other diagonal elements are $\sqrt{2}\mu_{(\pm)}$, where

$$\kappa_{(\pm)}(\omega) = \frac{1}{\omega \mp i}\kappa_h e^{\mp k_{(\mp)}(\omega)}, \quad \mu_{(\pm)}(\omega) = \frac{1}{\omega \mp i}\mu_h e^{\mp m_{(\mp)}(\omega)} . \tag{5.9}$$

Since $\kappa_h = (-\kappa'(0))^{1/2}$ and $\mu_h = (-\mu'(0))^{1/2}$,

$$k_{(\pm)}(\omega) = \lim_{\alpha \to 0^\pm} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log[(1+\omega'^2)\kappa_h^{-2}\kappa_c(\omega')]}{\omega' - (\omega + i\alpha)}\, d\omega' \tag{5.10}$$

and

$$m_{(\pm)}(\omega) = \lim_{\alpha \to 0^\pm} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log[(1+\omega'^2)\mu_h^{-2}\mu_c(\omega')]}{\omega' - (\omega + i\alpha)}\, d\omega'. \tag{5.11}$$

6. Maximum recoverable Γ-work

In this section we prove that the problem of finding the “optimal” process maximizing the recoverable Γ-work has one and only one solution by virtue of the thermodynamic properties of the relaxation function.

From here on, we denote by $\mathbb{G}(|s|)$ the extension of $\mathbb{G}(s)$ to an even function on $\mathcal{R}$, and we suppose any process $P \in \Pi$ to be defined over all $[0,\infty)$, by means of the trivial extension

$$P(t) = \begin{cases} P(t) & \text{for } t \in [0, d_P) \\ 0 & \text{for } t \in [d_P, \infty). \end{cases} \tag{6.1}$$

Now let us consider the work $W(\gamma_{0,}P)$, where $\gamma_0 = \mathbf{E}^0$ is the strain history evaluated at $t = 0$, and $P \in \Pi$ is a process such that $P(t) = \mathbf{D}(t),\ \ t\in[0, d_p)$. Changing variables and integrating by parts, it is easy to check that the extra stress (3.1) may be written as

$$\frac{\tilde{\mathbf{T}}}{\rho} = \mathbf{V}(\mathbf{E}_r^t) = \int_0^t \mathbb{G}(s)\dot{\mathbf{E}}^t(s)\, ds - \mathbf{I}_0(t, \mathbf{E}_r^0), \tag{6.2}$$

where

$$\mathbf{I}_0(t, \mathbf{E}_r^0) = -\int_0^\infty \mathbb{G}'(t+\tau)\mathbf{E}_r^0(\tau)\, d\tau. \tag{6.3}$$

Moreover, it follows from (6.1) that there exists the limit $\mathbf{E}(\infty) = \lim_{t\to+\infty} \mathbf{E}(t)$. We have from (4.18) that

$$\begin{aligned} W(\gamma, P) &= \int_0^\infty \left(\int_0^t \mathbb{G}(s)\dot{\mathbf{E}}^t(s)\, ds - \mathbf{I}_0(t, \mathbf{E}_r^0) \right) \cdot \mathbf{D}(t)\, dt \qquad (6.4) \\ &= \frac{1}{2}\int_0^\infty \int_0^\infty \mathbb{G}(|t-\tau|)\mathbf{D}(t)\cdot \mathbf{D}(\tau) d\tau dt - \int_0^\infty \mathbf{I}_0(t, \mathbf{E}_r^0)\cdot \mathbf{D}(t)\, dt. \end{aligned}$$

To obtain the maximum recoverable Γ-work from state $\gamma_0 = \mathbf{E}_r^0$, we consider the supremum of $-W(\gamma_0, P)$ with respect to the set of functions given by

$$\mathbf{E}(t) = \mathbf{E}^{(m)}(t) + \varepsilon \mathbf{e}(t) \ \ , \quad t \in \mathcal{R}^+,$$

where ε is a real parameter, $\mathbf{e}$ is an arbitrary smooth function such that $\mathbf{e}(0) = 0$ and $\mathbf{E}^{(m)}$ is the "optimal" process minimizing $\mathcal{W}(\gamma_0, P)$. Then we have

$$\begin{aligned} \frac{d}{d\varepsilon}\left[-W(\gamma_0, P)\right]_{\varepsilon=0} = -\int_0^\infty \int_0^\infty \mathbb{G}(|t-\tau|)\mathbf{D}^{(m)}(t)\cdot \dot{\mathbf{e}}(\tau)\, d\tau\, dt + \\ + \int_0^\infty \mathbf{I}_0(t, \mathbf{E}_r^0)\cdot \dot{\mathbf{e}}(t)\, dt = 0. \end{aligned} \tag{6.5}$$

Since $\dot{\mathbf{e}}(t)$ is arbitrary, it follows that

$$\int_0^\infty \mathbb{G}(|t-\tau|)\mathbf{D}^{(m)}(\tau)\, d\tau \ = \mathbf{I}_0(t, \mathbf{E}_r^0) \ \ t \in \mathcal{R}^+. \tag{6.6}$$

Equation (6.6) is a Wiener–Hopf equation, the solution of which maximizes the recoverable Γ-work. Since

$$\varphi_m(\gamma_0) = -\inf\left\{W(\gamma_0, P),\ \forall\ P \in \Pi\right\}$$

we have from (6.4) and (6.6)

$$\varphi_m(\mathbf{E}_r^0) = \frac{1}{2}\int_0^\infty \int_0^\infty \mathbb{G}(|t-\tau|)\mathbf{D}^{(m)}(t)\cdot \mathbf{D}^{(m)}(\tau) d\tau dt \tag{6.7}$$

where $\mathbf{D}^{(m)}$ is now the solution of the equation (6.6). For this reason it is important to prove the existence and uniqueness of the solution of the Wiener–Hopf equation (6.6). We denote with $\mathcal{G}$ the completion of the set $\tilde{\mathcal{G}}$ defined as

$$\tilde{\mathcal{G}} = \left\{ \mathbf{D} : [0,\infty) \to Sym : \int_0^\infty \int_0^\infty \mathbb{G}(|t-\tau|)\mathbf{D}(t)\cdot \mathbf{D}(\tau) d\tau dt < \infty \right\} \tag{6.8}$$

with respect to the norm $\|\cdot\|_{\mathcal{G}}$ defined by

$$\|\cdot\|_{\mathcal{G}} = \int_0^\infty \int_0^\infty \mathbb{G}(|t-\tau|)\mathbf{D}(t)\cdot\mathbf{D}(\tau)d\tau dt. \tag{6.9}$$

The thermodynamic restrictions (5.1) imply that the kernel $\mathbb{G}(|t|)$ is positive definite, as may be seen from the frequency domain representation of (6.9) [8]. Then we can introduce an inner product on $\mathcal{G}$ defined by

$$(\mathbf{D}_1\cdot\mathbf{D}_2) = \int_0^\infty \int_0^\infty \mathbb{G}(|t-\tau|)\mathbf{D}_1(t)\cdot\mathbf{D}_2(\tau)d\tau dt$$

which makes $\mathcal{G}$ a Hilbert space. The set of processes Π is a subset of $\mathcal{G}$.

Remark 6.1. By means of the norm of $\mathcal{G}$, it is possible to provide the set of the processes Π with a topology. In particular, the closure of Π using the norm (6.9) is the Hilbert space $\mathcal{G}$.

The equation (6.6) can be written as

$$\mathcal{A}\mathbf{D} = \mathbf{I}_0 \tag{6.10}$$

where $\mathcal{A}$ is an operator from $\mathcal{G}$ to its dual $\mathcal{G}'$. It is bounded and coercive. Then, from the Lax–Milgram theorem, we can give the following:

Theorem 6.1. *For any $\mathbf{I}_0 \in \mathcal{G}'$, equation* (6.6) *has a unique solution $\mathbf{D}\in\mathcal{G}$ such that*

$$\|\mathbf{D}\|_{\mathcal{G}} \le K\,\|\mathbf{I}_0\|_{\mathcal{G}'}\,.$$

In other words, there exists an isomorphism between $\mathcal{G}$ and $\mathcal{G}'$. Moreover, we have from Definition 3.1 the following:

Proposition 6.1. *Two histories $\mathbf{E}_1^0$ and $\mathbf{E}_2^0$ correspond to two equivalent states in the sense of* (3.6_2–3.6_3) *if and only if*

$$\mathbf{I}_0(t,\mathbf{E}_1^0) = \mathbf{I}_0(t,\mathbf{E}_2^0)\quad \forall t\in\mathcal{R}^+. \tag{6.11}$$

Proof. If (6.11) holds for any $t\in\mathcal{R}^+$ and $\mathbf{E}^0 = \mathbf{E}_1^0 - \mathbf{E}_2^0$, then $\mathbf{I}_0(t,\mathbf{E}_1^0) = 0$ for any $t\in\mathcal{R}^+$. In view of definition (6.3) we have

$$\int_0^\infty \mathbb{G}'(t+\tau)\mathbf{E}_r^0(\tau)\,d\tau = 0 \quad \forall t\in\mathcal{R}^+;$$

namely, (3.6_2–3.6_3) hold. The converse is trivial.

Remark 6.2. Proposition 6.1 yields a bijective map between $\mathcal{G}'$ and the quotient space $\Gamma_{(m)} = \Gamma/\Gamma_0$. In other words it is possible to identify any class of equivalent histories with a function $\mathbf{I}_0$.

This result allows us to represent the minimum free energy as a function defined on the set $\Gamma_{(m)}$ of equivalent histories.

7. Construction of the minimum free energy

Rewriting the Wiener–Hopf equation (6.6) at any time t (rather than $t = 0$), we obtain

$$\int_0^\infty \mathbb{G}(|\tau - s|)\mathbf{D}^{(m)}(s)\,ds \; = \mathbf{I}_0(\tau, \mathbf{E}_r^t)\;,\; \tau > 0 \tag{7.1}$$

with

$$\mathbf{I}_0(\tau, \mathbf{E}_r^t) = -\int_0^\infty \dot{\mathbb{G}}(\tau + s)\mathbf{E}_r^t(s)\,ds\;, \qquad \tau \geq 0 \tag{7.2}$$

and where $\mathbf{D}^{(m)}$ is the optimal process acting on γ. The maximum recoverable work gives the minimum free energy $\psi_m(\rho, \mathbf{E}_r^t) = \phi(\rho) + \varphi_m(\mathbf{E}_r^t)$ with ϕ defined by (4.11) and φ_m, the maximum recoverable Γ-work, given by

$$\varphi_m(\mathbf{E}_r^t) = \frac{1}{2}\int_0^\infty \int_0^\infty \mathbb{G}(|\tau - s|)\mathbf{D}^{(m)}(\tau)\cdot \mathbf{D}^{(m)}(s)d\tau ds. \tag{7.3}$$

Wiener–Hopf equations of the first kind are not solvable in the general case. Nevertheless the thermodynamic properties of the integral kernel $\mathbb{G}$ allow us to determine the solution $\mathbf{D}^{(m)}$ of (7.1).

Let us introduce a function $\mathbf{r} : \mathcal{R}^- \to Sym$, defined as

$$\mathbf{r}(\tau) = \int_0^\infty \mathbb{G}(|\tau - s|)\mathbf{D}^{(m)}(s)\,ds \quad \tau \in \mathcal{R}^-$$

which, when added to (7.1), yields

$$\int_{-\infty}^\infty \mathbb{G}(|\tau - s|)\mathbf{D}^{(m)}(s)\,ds = \mathbf{I}_0(\tau, \mathbf{E}_r^t) + \mathbf{r}(\tau), \quad \tau \in \mathcal{R}, \tag{7.4}$$

where $supp(\mathbf{D}^{(m)}) \subseteq \mathcal{R}^+$, $supp(\mathbf{I}(\cdot, \mathbf{E}_r^t)) \subseteq \mathcal{R}^+$ and $supp(\mathbf{r}) \subseteq \mathcal{R}^-$. The Fourier transform of (7.4) gives

$$2\mathbb{G}_c(\omega)\mathbf{D}_+^{(m)}(\omega) = \mathbf{I}_+^t(\omega) + \mathbf{r}_-(\omega), \tag{7.5}$$

where

$$\mathbf{I}_+^t(\omega) = \mathbf{I}_+(\omega, \mathbf{E}_r^t) = \int_0^\infty \mathbf{I}_0(\tau, \mathbf{E}_r^t)e^{-i\omega\tau}d\tau\,. \tag{7.6}$$

The factorization (5.4–5.8) allows us to rewrite (7.5) as follows:

$$\mathbb{G}_{(+)}(\omega)\mathbf{D}_+^{(m)}(\omega) = \frac{1}{2}\mathbb{G}_{(-)}^{-1}(\omega)\mathbf{I}_+^t(\omega) + \frac{1}{2}\mathbb{G}_{(-)}^{-1}(\omega)\mathbf{r}_-(\omega)$$

where the fact that $\det \mathbb{G}_{(-)}(\omega) \neq 0, \forall \omega \in \mathcal{R}$ has been used. Applying the Plemelj formulae, the quantity $\frac{1}{2}\mathbb{G}^{-1}_{(-)}(\omega)\mathbf{I}^t_+(\omega)$ may be rewritten as

$$\frac{1}{2}\mathbb{G}^{-1}_{(-)}(\omega)\mathbf{I}^t_+(\omega) = \mathbf{p}^t_{(-)}(\omega) - \mathbf{p}^t_{(+)}(\omega) \tag{7.7}$$

where $\mathbf{p}^t_{(\pm)}(z)$ are analytic respectively for $z \in \Omega^\mp$ and are defined by

$$\begin{aligned} \mathbf{p}^t(z) \quad &:= \quad \frac{1}{4\pi i}\int_{-\infty}^{\infty} \frac{\mathbb{G}^{-1}_{(-)}(\omega)\mathbf{I}^t_+(\omega)}{\omega - z} d\omega, \quad z \in \Omega \setminus \mathcal{R}, \\ \mathbf{p}^t_{(\pm)}(\omega) \quad &:= \quad \lim_{\alpha \to 0^\mp} \mathbf{p}^t(\omega + i\alpha). \end{aligned} \tag{7.8}$$

Therefore, we obtain

$$\mathbb{G}_{(+)}(\omega)\mathbf{D}^{(m)}_+(\omega) = -\mathbf{p}^t_{(+)}(\omega) + \mathbf{p}^t_{(-)}(\omega) + \frac{1}{2}\mathbb{G}^{-1}_{(-)}(\omega)\mathbf{r}_-(\omega). \tag{7.9}$$

Observe that the quantities $\mathbb{G}_{(+)}(z)\mathbf{D}^{(m)}_+(z)$ and $\mathbf{p}^t_{(+)}(z)$ are analytic for $z \in \Omega^-$, whereas $\mathbf{p}^t_{(-)}(z)$ and $\mathbb{G}^{-1}_{(-)}(\omega)\mathbf{r}_-(\omega)$ are analytic for $z \in \Omega^+$. Therefore, the quantity

$$\mathbf{L}(\omega) = \mathbb{G}_{(+)}(\omega)\mathbf{D}^{(m)}_+(\omega) + \mathbf{p}^t_{(+)}(\omega) = \mathbf{p}^t_{(-)}(\omega) + \frac{1}{2}\mathbb{G}^{-1}_{(-)}(\omega)\mathbf{r}_-(\omega)$$

is analytic on the whole complex plane. We have that $\mathbf{p}^t_{(+)}(\omega) = O\left(\frac{1}{\omega}\right)$ as $|\omega| \to \infty$ and $\lim_{\omega\to\infty} \omega\mathbb{G}_{(+)}(\omega) = \mathbb{G}_h$ by virtue of (5.3), (5.4) and the definition of $\mathbb{G}_h$ before (5.5). Thus, $\mathbf{L}$ vanishes at infinity, which implies that it must be zero everywhere so that

$$\mathbf{D}^{(m)}_+(\omega) = -\mathbb{G}_{(+)}(\omega)^{-1}\mathbf{p}^t_{(+)}(\omega), \quad \mathbf{p}^t_{(-)}(\omega) = -\frac{1}{2}\mathbb{G}^{-1}_{(-)}(\omega)\mathbf{r}_-(\omega). \tag{7.10}$$

Remark 7.1. It follows that

$$\lim_{\omega\to\infty} \mathbf{D}^{(m)}_+(\omega) \neq \mathbf{0}\ ,$$

so that $\mathbf{D}^{(m)}(\tau)$ has an initial delta-function type singularity as $\tau \to 0^+$. Thus the optimal continuation $\mathbf{E}^{(m)}$, where $\mathbf{D}^{(m)} = \dot{\mathbf{E}}^{(m)}$, has an initial discontinuity as $\tau \to 0^+$ so that $\mathbf{E}(0^+) \neq \mathbf{E}(t)$.

Remark 7.2. Since $\det \mathbb{G}_{(+)}(0) \neq 0$, it follows that

$$\mathbf{E}^{(m)}(\infty) - \mathbf{E}^{(m)}(0^-) = \int_{0^-}^{\infty} \mathbf{D}^{(m)}(\tau)d\tau = \mathbf{D}^{(m)}_+(0) = -\mathbb{G}_{(+)}(0)^{-1}\mathbf{p}^t_{(+)}(0),$$

where we have emphasized that the integral includes the discontinuity. Therefore, the optimal continuation tends to the finite limit

$$\lim_{\tau\to\infty}\mathbf{E}^{(m)}(\tau)=\mathbf{E}^{(m)}(\infty)=\mathbf{E}(t)-\mathbb{G}_{(+)}(0)^{-1}\mathbf{p}^t_{(+)}(0).$$

The substitution of (7.10) into (7.3) yields

$$\begin{aligned}\varphi_m(\mathbf{E}^t_r)&=\frac{1}{2}\int_0^\infty\int_0^\infty\mathbb{G}(|\tau-s|)\mathbf{D}^{(m)}(s)\cdot\mathbf{D}^{(m)}(\tau)\,ds\,d\tau \qquad (7.11)\\ &=\frac{1}{2\pi}\int_{-\infty}^\infty\mathbb{G}_c(\omega)\mathbf{D}^{(m)}_+(\omega)\cdot\overline{\mathbf{D}^{(m)}_+(\omega)}\,d\omega=\frac{1}{2\pi}\int_{-\infty}^\infty|\mathbf{p}^t_{(+)}(\omega)|^2\,d\omega.\end{aligned}$$

Therefore, the minimum free energy takes the form

$$\psi_m(\rho,\mathbf{E}^t_r)=\phi(\rho)+\frac{1}{2\pi}\int_{-\infty}^\infty|\mathbf{p}^t_{(+)}(\omega)|^2\,d\omega. \qquad (7.12)$$

Actually, in view of equations (7.1) and (7.3), Proposition 6.1 and Remark 6.2, it is clear that φ_m is a function of the element γ_m of $\Gamma_{(m)}$, namely

$$\hat{\varphi}_m(\gamma_m)=\varphi_m(\mathbf{E}^t_r),$$

and hence $\mathbf{p}^t_{(+)}$ provides an explicit representation of the equivalence class γ_m, as explicitly shown by the following theorem.

Theorem 7.1. *For every viscoelastic material with a symmetric relaxation function, a given strain history $\mathbf{E}^t_r$ is equivalent to the zero history $\mathbf{0}^\dagger$ in the sense of* $(3.6)_2$ *and* $(3.6)_3$ *if and only if the quantity $\mathbf{p}^t_{(+)}$, related to $\mathbf{E}^t_r$ by* (7.6) *and* (7.7), *is such that*

$$\mathbf{p}^t_{(+)}(\omega)=0\,,\quad\forall\omega\in\mathcal{R}. \qquad (7.13)$$

Before proving the theorem we need the following lemma:

Lemma 7.1. *The quantity $\mathbf{p}^t_{(+)}$, related to $\mathbf{E}^t_r$ by* (7.7) *and* (7.8), *is also given by*

$$\mathbf{p}^t_{(+)}(\omega)=\lim_{z\to\omega^-}\frac{1}{2\pi i}\int_{-\infty}^\infty\frac{i\omega'\mathbb{G}_{(+)}(\omega')\overline{\mathbf{E}^t_{r+}(\omega')}}{\omega'-z}d\omega'. \qquad (7.14)$$

Proof. We identify $\mathbf{E}^t_r$ with its causal extension (viz. $\mathbf{E}^t_r(\tau)=0$ for $\tau<0$) and consider the odd extension $\dot{\mathbb{G}}^{(d)}$ of $\dot{\mathbb{G}}$, viz.

$$\dot{\mathbb{G}}^{(d)}(\xi)=\begin{cases}\dot{\mathbb{G}}(\xi) & \text{for } \xi\ge 0\\ -\dot{\mathbb{G}}(-\xi) & \text{for } \xi<0\end{cases}\,,\qquad\text{so that } \dot{\mathbb{G}}^{(d)}_F(\omega)=-2i\dot{\mathbb{G}}_s(\omega).$$

Then (7.2) can be rewritten as

$$\mathbf{I}(\tau, \mathbf{E}_r^t) = -\int_{-\infty}^{\infty} \dot{\mathbb{G}}^{(d)}(\tau + s)\mathbf{E}_r^t(s)\, ds\ , \qquad \tau \geq 0. \tag{7.15}$$

Moreover, let $\mathbf{I}^{(n)}(\cdot, \mathbf{E}_r^t) : \mathcal{R}^- \to Sym$ be defined as

$$\mathbf{I}^{(n)}(\tau, \mathbf{E}_r^t) = -\int_{-\infty}^{\infty} \dot{\mathbb{G}}^{(d)}(\tau + s)\mathbf{E}_r^t(s)\, ds\ , \qquad \tau < 0\ , \tag{7.16}$$

and let us consider the extension $\mathbf{I}^{(R)}(\cdot, \mathbf{E}_r^t) : \mathcal{R} \to Sym$ of $\mathbf{I}(\tau, \mathbf{E}_r^t)$ to the real line as follows:

$$\mathbf{I}^{(R)}(\tau, \mathbf{E}_r^t) = -\int_{-\infty}^{\infty} \dot{\mathbb{G}}^{(d)}(\tau + s)\mathbf{E}_r^t(s)\, ds = \begin{cases} \mathbf{I}(\tau, \mathbf{E}_r^t) & \text{for } \tau \geq 0 \\ \mathbf{I}^{(n)}(\tau, \mathbf{E}_r^t) & \text{for } \tau < 0 \end{cases}. \tag{7.17}$$

Introducing the quantity $\mathbf{E}_{r,n}^t(s) = \mathbf{E}_r^t(-s)$, $s \leq 0$ with the extension $\mathbf{E}_{r,n}^t(s) = 0$ for $s > 0$, we have

$$\mathbf{I}^{(R)}(\tau, \mathbf{E}_r^t) = -\int_{-\infty}^{\infty} \dot{\mathbb{G}}^{(d)}(\tau - s)\mathbf{E}_{r,n}^t(s)\, ds\ ,$$

so that

$$\mathbf{I}_F^{(R)}(\omega, \mathbf{E}_r^t) = 2i\dot{\mathbb{G}}_s(\omega)\mathbf{E}_{r,n-}^t(\omega) = 2i\dot{\mathbb{G}}_s(\omega)\overline{\mathbf{E}_{r+}^t(\omega)}. \tag{7.18}$$

By virtue of (7.17), we have the property

$$\mathbf{I}_F^{(R)}(\omega, \mathbf{E}_r^t) = \mathbf{I}_+(\omega, \mathbf{E}_r^t) + \mathbf{I}_-^{(n)}(\omega, \mathbf{E}_r^t)$$

using the notation of (A.8), so that

$$\frac{1}{2}\mathbb{G}_{(-)}^{-1}(\omega)\mathbf{I}_F^{(R)}(\omega, \mathbf{E}_r^t) = \frac{1}{2}\mathbb{G}_{(-)}^{-1}(\omega)\mathbf{I}_+(\omega, \mathbf{E}_r^t) + \frac{1}{2}\mathbb{G}_{(-)}^{-1}(\omega)\mathbf{I}_-^{(n)}(\omega, \mathbf{E}_r^t).$$

By virtue of (7.7), we have

$$\frac{1}{2}\mathbb{G}_{(-)}^{-1}(\omega)\mathbf{I}_F^{(R)}(\omega, \mathbf{E}_r^t) = \mathbf{p}_{(-)}^t(\omega) - \mathbf{p}_{(+)}^t(\omega) + \frac{1}{2}\mathbb{G}_{(-)}^{-1}(\omega)\mathbf{I}_-^{(n)}(\omega, \mathbf{E}_r^t).$$

On the other hand, the quantity $\frac{1}{2}\mathbb{G}_{(-)}^{-1}(\omega)\mathbf{I}_F^{(R)}(\omega, \mathbf{E}_r^t)$ may be written, using the Plemelj formulae, as follows:

$$\frac{1}{2}\mathbb{G}_{(-)}^{-1}(\omega)\mathbf{I}_F^{(R)}(\omega, \mathbf{E}_r^t) = \mathbf{p'}_{(-)}^t(\omega) - \mathbf{p'}_{(+)}^t(\omega), \tag{7.19}$$

where $\mathbf{p'}_{(\pm)}^t(z)$ are analytic in $\Omega^{\mp}$ respectively and are defined analogously to (7.8). Thus, we have

$$\mathbf{p'}_{(-)}^t(\omega) - \mathbf{p'}_{(+)}^t(\omega) = \mathbf{p}_{(-)}^t(\omega) - \mathbf{p}_{(+)}^t(\omega) + \frac{1}{2}\mathbb{G}_{(-)}^{-1}(\omega)\mathbf{I}_-^{(n)}(\omega, \mathbf{E}_r^t)\ .$$

Observe that the quantity

$$\mathbf{J}'(\omega) = \mathbf{p}^t_{(+)}(\omega) - \mathbf{p}'^t_{(+)}(\omega) = \mathbf{p}^t_{(-)}(\omega) - \mathbf{p}'^t_{(-)}(\omega) + \frac{1}{2}\mathbb{G}^{-1}_{(-)}(\omega)\mathbf{I}^{(n)}_{-}(\omega, \mathbf{E}^t_r) \tag{7.20}$$

is analytic on Ω^- by virtue of the first relation of (7.20) and analytic on Ω^+ by virtue of the rightmost relation of (7.20). Also, $\mathbf{J}'(\omega)$ goes to zero for large ω so that it vanishes everywhere and

$$\mathbf{p}^t_{(+)}(\omega) = \mathbf{p}'^t_{(+)}(\omega)\ . \tag{7.21}$$

It follows from (7.18) and (5.2) that

$$\frac{1}{2}\mathbb{G}^{-1}_{(-)}(\omega)\mathbf{I}^{(R)}_F(\omega, \mathbf{E}^t_r) = i\mathbb{G}^{-1}_{(-)}(\omega)\dot{\mathbb{G}}_s(\omega)\overline{\mathbf{E}^t_{r+}(\omega)} = -i\omega\mathbb{G}_{(+)}(\omega)\overline{\mathbf{E}^t_{r+}(\omega)}. \tag{7.22}$$

Equalities (7.19), (7.22) and (7.21) imply (7.14). □

Proof of Theorem 7.1. Observe that $(3.6)_2$ and $(3.6)_3$ are equivalent to the statement that $\mathbf{I}(\tau, \mathbf{E}^t_r) = 0 \quad \forall \tau \geq 0$. Thus the theorem states that

$$\mathbf{I}(\tau, \mathbf{E}^t_r) = 0 \quad \forall \tau \geq 0 \qquad \Longleftrightarrow \qquad \mathbf{p}^t_{(+)}(\omega) = 0 \quad \forall \omega \in \mathcal{R}. \tag{7.23}$$

To this aim, observe that (7.2) can be rewritten as

$$\mathbf{I}(\tau, \mathbf{E}^t_r) = -\int_\tau^\infty \dot{\mathbb{G}}(s)\mathbf{E}^t_r(s-\tau)\, ds\ , \quad \tau \geq 0.$$

A causal extension of $\mathbf{E}^t_r$ and an odd extension of $\dot{\mathbb{G}}$ provide the following representation of $\mathbf{I}(\cdot, \mathbf{E}^t_r)$ in the frequency domain, on application of Plancherel's theorem (A.11):

$$\mathbf{I}(\tau, \mathbf{E}^t_r) = \frac{1}{\pi}\int_{-\infty}^\infty i\dot{\mathbb{G}}_s(\omega)\overline{\mathbf{E}^t_{r+}(\omega)}e^{i\omega\tau}\, d\omega = -\frac{1}{\pi}\int_{-\infty}^\infty i\omega\mathbb{G}_c(\omega)\overline{\mathbf{E}^t_{r+}(\omega)}e^{i\omega\tau}\, d\omega,$$

for $\tau \geq 0$. Moreover, representation (7.14) ensures that the Plemelj formula for $i\omega\mathbb{G}_{(+)}(\omega)\overline{\mathbf{E}^t_{r+}(\omega)}$ may be given by

$$i\omega\mathbb{G}_{(+)}(\omega)\overline{\mathbf{E}^t_{r+}(\omega)} = \mathbf{p}^t_{(-)}(\omega) - \mathbf{p}''^t_{(+)}(\omega) \tag{7.24}$$

where $\mathbf{p}''^t_{(-)}(\omega)$ is a function analytic on Ω^+. Then

$$\begin{aligned}\mathbf{I}(\tau, \mathbf{E}^t_r) &= \frac{1}{\pi}\int_{-\infty}^\infty \mathbb{G}_{(-)}(\omega)\left(\mathbf{p}^t_{(+)}(\omega) - \mathbf{p}''^t_{(-)}(\omega)\right)e^{i\omega\tau}\, d\omega \\ &= \frac{1}{\pi}\int_{-\infty}^\infty \mathbb{G}_{(-)}(\omega)\mathbf{p}^t_{(+)}(\omega)\, e^{i\omega\tau}\, d\omega, \quad \tau \geq 0\ ,\end{aligned} \tag{7.25}$$

the second equality holding since $\mathbb{G}_{(-)}(\omega)$, $\mathbf{p}''^t_{(-)}(\omega)$, and $e^{i\omega\tau}$, $\tau \in \mathcal{R}^+$, are analytic in the half plane Ω^+.

Now, $\mathbf{p}^t_{(+)}(\omega) = 0 \ \forall\omega \in \mathcal{R}$ implies that $\mathbf{I}(\tau, \mathbf{E}^t_r) = 0 \ \forall\tau \geq 0$ by virtue of representation (7.25). On the other hand, (7.6) and (7.8) imply that if $\mathbf{I}(\tau, \mathbf{E}^t_r) = 0 \ \forall\tau \geq 0$, then $\mathbf{p}^t_{(+)}(\omega) = 0 \ \forall\omega \in \mathcal{R}$. □

As a consequence of expression (7.11) and Theorem 7.1, we have that $\hat{\varphi}_m$ provides a norm in $\Gamma_{(m)}$, namely $\|\gamma_m\|^2 = \hat{\varphi}_m(\gamma_m)$. Thus, the minimum free energy ψ_m induces a norm in the space of the minimal states $\Sigma_{(m)}$. In fact, if $\sigma_m = (\rho, \gamma_m)$ and $\hat{\psi}(\sigma_m) = \psi(\rho, \mathbf{E}^t_r)$, then equations (7.11–7.12) yield

$$\|\sigma_m\|^2 = \hat{\psi}(\sigma_m) = \phi(\rho) + \hat{\varphi}_m(\gamma_m) \ .$$

8. Concluding remarks

The results obtained above may be seen to be entirely consistent with those in [6, 4] for a linear viscoelastic solid. The factor $\mathbb{G}_{(\pm)}(\omega)$ in (5.4) corresponds to $\mathbf{H}_{\pm}(\omega)/\omega$ in [4]. Also, $-i\mathbf{p}^t_{+}(\omega)$, given by (7.14), is the complex conjugate of the quantity $\mathbf{p}^t(\omega)$ introduced in section 9 of [4].

The explicit results for materials with relaxation functions given by a sum of exponentials presented in [6, 4] are equally applicable in the present case, on taking account of the notational equivalences specified in the last paragraph and on putting $\mathbb{G}(\infty) = 0$.

In obtaining these explicit results in [4], it is assumed that all fourth-order tensors are simultaneously diagonalizable; and if the results are to be used, it must be possible to find the diagonal forms explicitly. Observe that in the present work, the diagonal form for the relaxation tensor is achieved without difficulty by (2.12), and the factors $\mathbb{G}_{\pm}$ are diagonal, so the no assumptions are necessary.

The results presented here apply also to isotropic linear viscoelastic solids, with the minor modification of introducing a nonzero $\mathbb{G}(\infty)$.

Appendix: Notation and basic formulae

The space of symmetric second-order tensors acting on $\mathcal{R}^3$ is denoted by Sym and is isomorphic to $\mathcal{R}^6$. Operating on Sym is the space of fourth-order tensors $Lin(Sym)$. We shall often write a second-order tensor $\mathbf{A}$ in terms of its trace A and its trace-free part $\mathring{\mathbf{A}}$,

$$A = tr(\mathbf{A}), \qquad \mathring{\mathbf{A}} = \mathbf{A} - \frac{1}{3}A\mathbf{I} \ , \tag{A.1}$$

where $\mathbf{I}$ is the identity tensor in Sym.

We introduce a scalar product on Sym as follows: if $\mathbf{A}, \mathbf{B} \in Sym$ then $\mathbf{A}\cdot\mathbf{B} = tr(\mathbf{AB})$. The associated norm is defined as $|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A}$. Since $\mathbf{I}$ and $\overset{\circ}{\mathbf{A}}$ are orthogonal, i.e., $tr(\mathbf{I}\overset{\circ}{\mathbf{A}}) = 0$, for every $\mathbf{A} \in Sym$, the decomposition

$$\mathbf{A} = \frac{1}{3}A\,\mathbf{I} + \overset{\circ}{\mathbf{A}} \tag{A.2}$$

is unique; namely, for any tensor $\mathbf{A} \in Sym$ there exist a unique scalar A and a unique trace-free tensor $\overset{\circ}{\mathbf{A}} \in Sym$ satisfying (A.2). As a consequence, the decomposition (A.2) allows us to introduce an orthonormal basis of Sym

$$\mathbf{N}_1, \ldots, \mathbf{N}_6 \; : \; tr(\mathbf{N}_h\mathbf{N}_k) = \delta_{hk}, \; \mathbf{N}_1 = \frac{1}{\sqrt{3}}\mathbf{I}, \quad \mathbf{N}_i = \overset{\circ}{\mathbf{N}}_h\,, \; h = 2, \ldots, 6, \tag{A.3}$$

δ_{hk} being the Kronecker symbol.

Henceforth we treat each tensor $\mathbf{A} \in Sym$ as a vector in $\mathcal{R}^6$ whose first component is $\frac{1}{\sqrt{3}}A = \frac{1}{\sqrt{3}}tr(\mathbf{A})$. Observe that if

$$\mathbf{A} = \sum_{i=1}^{6} A_i\mathbf{N}_i\,, \qquad \mathbf{B} = \sum_{i=1}^{6} B_i\mathbf{N}_i\,, \tag{A.4}$$

then

$$\mathbf{A}\cdot\mathbf{B} = tr(\mathbf{AB}) = \sum_{i=1}^{6} A_iB_i.$$

Consequently [12], any fourth-order tensor $\mathbb{K} \in Lin(Sym)$ will be identified with an element of $Lin(\mathcal{R}^6)$ by the representation

$$\mathbb{K} = \sum_{i,j=1}^{6} K_{ij}\mathbf{N}_i \otimes \mathbf{N}_j \tag{A.5}$$

and $\mathbb{K}^\top$ means the transpose of $\mathbb{K}$ as an element of $Lin(\mathcal{R}^6)$. A norm $|\mathbb{K}|$ of $\mathbb{K} \in Lin(Sym)$ may be defined by

$$|\mathbb{K}|^2 = tr\left(\mathbb{K}\mathbb{K}^\top\right) = \sum_{i,j=1}^{6} K_{ij}K_{ji}\,.$$

Note that the operation of $\mathbf{N}_i \otimes \mathbf{N}_j \in Lin(Sym)$ on an element $\mathbf{A} \in Sym$ is defined as $(\mathbf{N}_j \cdot \mathbf{A})\mathbf{N}_i$. In the sequel we deal with the space $\mathcal{D}(Sym)$ of fourth-order tensors whose representation in $Lin(\mathcal{R}^6)$ is a diagonal matrix, viz.

$$\mathbb{K} = \sum_{i,i=1}^{6} K_i \delta_{ij} \mathbf{N}_i \otimes \mathbf{N}_j \,, \tag{A.6}$$

and also with complex-valued tensors. Then, denoting by Ω the complex plane, $Sym(\Omega)$ and $\mathcal{D}(Sym(\Omega))$ denote respectively the tensors described by (A.4) and (A.6) with $A_i, B_i, K_i \in \Omega$, and the norms $|\mathbf{A}|$ and $|\mathbb{K}|$ of $\mathbf{A} \in Sym(\Omega)$ and $\mathbb{K} \in \mathcal{D}(Sym(\Omega))$ are given respectively by

$$|\mathbf{A}|^2 = \left(\mathbf{A} \cdot \overline{\mathbf{A}}\,\right), \qquad |\mathbb{K}|^2 = tr\,(\mathbb{K}\mathbb{K}^*) = \sum_{i=1}^{6} |K_i|^2 \,, \tag{A.7}$$

where the overhead bar indicates complex conjugate and $\mathbb{K}^* = \overline{\mathbb{K}}^\top$ is the hermitian conjugate.

The symbols $\mathcal{R}, \mathcal{R}^+$ and $\mathcal{R}^{++}$ denote the reals, the nonnegative reals and the strictly positive reals, respectively, while $\mathcal{R}^-$ and $\mathcal{R}^{--}$ denote the nonpositive and strictly negative reals.

For every function $f : \mathcal{R} \to \mathcal{V}$, where $\mathcal{V}$ is any finite-dimensional vector space, in particular in the present context Sym or $\mathcal{D}(Sym)$, let f_F, denote its *Fourier transform,* viz. $f_F(\omega) = \int_{-\infty}^{\infty} f(s) e^{-i\omega s} ds$. Also, we define

$$\begin{aligned} f_+(\omega) &= \int_0^\infty f(s) e^{-i\omega s} ds, \qquad & f_-(\omega) &= \int_{-\infty}^0 f(s) e^{-i\omega s} ds \\ f_s(\omega) &= \int_0^\infty f(s) \sin \omega s \, ds, \qquad & f_c(\omega) &= \int_0^\infty f(s) \cos \omega s \, ds. \end{aligned} \tag{A.8}$$

The relations defining f_F and (A.8) are to be understood as applying to each component of the tensor quantities involved. Some constraint must be placed on these components to ensure that the Fourier transforms exist. Unless otherwise stated, it is assumed that all components of tensors in the time domain belong to $L^1(R) \cap L^2(R)$ (or $L^1(R^\pm) \cap L^2(R^\pm)$ in the case of $f_\pm$) so that in the frequency domain, they belong to $L^2(R)$ (or $L^2(R^\pm)$) [17, 16].

For $f : \mathcal{R}^+ \to \mathcal{V}$ we can always extend the domain of f to $\mathcal{R}$, by considering its *causal* extension, viz.

$$f(s) = \begin{cases} f(s) & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases} \tag{A.9}$$

in which case

$$f_F(\omega) = f_+(\omega) = f_c(\omega) - i f_s(\omega). \tag{A.10}$$

Plancherel's theorem for the Fourier transform [17, 16] gives that, for a specified product of tensors (denoted here by a dot product) $g \ : \mathcal{R} \to \mathcal{V}_1$ and $h \ : \mathcal{R} \to \mathcal{V}_2$ with real components belonging to $L^2(R)$, we have

$$\int_{-\infty}^{\infty} g(\xi) \cdot h(\xi) d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_F(\omega) \cdot \bar{h}_F(\omega) d\omega. \tag{A.11}$$

The complex ω plane, denoted earlier by Ω, will play an important role in our discussions. We define the following sets:

$$\Omega^+ = \{\zeta \in \Omega : \Im_m \zeta \geq 0\} \ , \quad \Omega^{(+)} = \{\zeta \in \Omega : \Im_m \zeta > 0\} \, . \tag{A.12}$$

Analogous meanings are assigned to Ω^- and $\Omega^{(-)}$.

The quantities $f_\pm$ defined by (A.8) are analytic in $\Omega^{(\mp)}$ respectively. This analyticity is extended by assumption to $\Omega^\mp$. The function f_+ may be defined by (A.8) and analytic on a portion of Ω^+ if for example f decays exponentially at large times. Over that portion of Ω^+ for which the integral definition is meaningless, f_+ is defined by analytic continuation.

Acknowledgments. This work has been supported by Italian M.U.R.S.T. through the 40th project "Mathematical Methods for Materials Science" and by a grant under the Enterprise Ireland International Collaboration Programme (2000).

References

[1] B.D. Coleman and D. R. Owen, *A mathematical foundation for thermodynamics*, Arch. Rational Mech. Anal., 54 (1974), 1–104.

[2] W.A. Day, *Reversibility, recoverable work and free energy in linear viscoelasticity*, Quart. J. Mech. Appl. Math., 23 (1970), 1–15.

[3] W.A. Day, *The thermodynamics of materials with memory,* in "Materials with Memory," ed. D. Graffi, Liguori, Naples, 1979.

[4] L. Deseri, G. Gentili, and J.M. Golden, *An explicit formula for the minimum free energy in linear viscoelasricity*, J. of Elasticity, 54 (1999), 141–185.

[5] G. Gentili, *Free energy and domain of dependence for weakly compressible viscoelastic fluids*, Diff. and Integral Equations, 10 (1997), 757–776.

[6] J.M. Golden, *Free energies in the frequency domain: the scalar case*, Quart. Appl. Math., 58 (2000), 127–150.

[7] M. Fabrizio, C. Giorgi, and A. Morro, *Free energies and dissipation properties for systems with memory*, Arch. Rational Mech. Anal., 125 (1994), 341–373.

[8] M. Fabrizio and J.M. Golden, *Maximum and minimum free energies for a linear viscoelastic material*, Quart. Appl. Math. (to appear).

[9] M. Fabrizio and B. Lazzari, *On asymptotic stability for linear viscoelastic fluids*, Diff. and Int. Eq., 6 (1993), 491–505.

[10] M. Fabrizio and A. Morro, "Mathematical Problems in Linear Viscoelasticity," SIAM, Philadelphia, 1992.

[11] I.C. Gohberg and M.G. Kreĭn, *Systems of integral equations on a half-line with kernels depending on the difference of arguments*, Am. Math. Soc. Transl. Ser. 2, 14 (1960), 217–287.
[12] P.R. Halmos, "Finite-Dimensional Vector Spaces," Springer-Verlag, New York, 1972
[13] D.D. Joseph, "Fluid Dynamics of Viscoelastic Liquids," Applied Mathematical Sciences, Springer-Verlag, 1990.
[14] N.I. Muskhelishvili, "Singular Integral Equations," Noordhoff, Groningen, 1953.
[15] W. Noll, *A new mathematical theory of simple materials*, Arch. Rational Mech. Anal., 48 (1972), 1–50.
[16] I.N. Sneddon, "The Use of Integral Transforms," McGraw-Hill, New York, 1972.
[17] E.C. Titchmarsh, "Introduction to the Theory of Fourier Integrals," Clarendon, Oxford, 1937.