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NODAL SOLUTIONS TO A CLASS OF NONSTANDARD SUPERLINEAR EQUATIONS ON $\mathbf{R}^{N*}$

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Abstract. We investigate the existence of sign-changing radial solutions for a class of singular equations:

$$-\Delta u(x) + b(|x|)u(x) = |u(x)|^{\theta-1}u(x) + h(|x|) \qquad x \in \mathbb{R}^N$$

where $b(|x|)$ may change sign and behaves like $|x|^{-\alpha}$ at infinity for some $\alpha \in (0,2)$, and $\theta > 1$.

1. Introduction

In this paper we deal with a class of elliptic superlinear problems in $\mathbb{R}^N$ ($N \geq 2$) of the following form:

$$-\Delta u(x) + b(|x|)u(x) = |u(x)|^{\theta-1}u(x) + h(|x|) \tag{P}$$

where $\theta > 1$: our aim is to investigate the existence of radially symmetric solutions with prescribed nodal properties.

The problem of existence and multiplicity of radial solutions to superlinear elliptic equations on $\mathbb{R}^N$ has been widely investigated in literature in the case when the coefficient b is strictly positive; i.e., $b(|x|) \geq b_0 > 0$, at least for large $|x|$. We quote e.g. [2, 3, 5, 6, 7, 10, 12, 13, 16, 17, 18] and references therein for a variety of results and techniques, such as topological, variational, and ODE methods.

In the present work we are interested in the case when $b(|x|)$ vanishes at infinity as $|x|^{-\alpha}$, $\alpha \in (0,2)$. In this situation the equation has an irregular singular point at ∞: as a consequence, a compatibility condition between the exponents θ and α is expected; the reader can compare with [8, 9], where

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the singular linear problem is treated. Moreover we allow b to change sign on a compact set (see assumption (A1)).

To our knowledge, only a few papers deal with nonlinear equations with a singular coefficient; some references can be found in [4] where the authors prove existence and nonexistence results of positive solutions to a class of unforced problems similar to (P).

For our class of singular problems, we shall prove the existence of a double sequence of radial solutions having prescribed nodal properties; as stressed in [6], the situation is structurally different when either $h \equiv 0$ or not. In the first case we are allowed to prescribe the number of nodes of the solutions (Theorem 2.2); on the other hand, when $h \not\equiv 0$ (Theorem 2.3), we can only characterize our solutions by a weaker nodal condition that does not take into account zeroes due to "small" oscillations. Both these results are proved in a suitable variational framework by a sharp extension of the Nehari method [14], along the lines of [19].

2. Assumptions and main results

Throughout the paper we shall make the following assumption on $b(|x|)$:

(A1) $b \in L^\infty$, $\exists 0 < \alpha < 2 : \lim_{r \to +\infty} r^\alpha b(r) = 1$.

Due to the singular behaviour of $b(|x|)$, the natural variational framework to study (P) is the following space:

$$H := \left\{ u : \mathbb{R}^N \to \mathbb{R} : u(x) = u(|x|), \int_{\mathbb{R}^N} \left(|\nabla u(x)|^2 + \frac{1}{1 + |x|^\alpha} u^2(x) \right) dx < +\infty \right\}. \tag{2.1}$$

We will assume that θ is subcritical in the sense of the Sobolev critical exponent (see Section 2.1). When looking for sign-changing solutions to (P), the following nonexistence result (which will be proved in Section 5) shows that the range of the admissible exponents θ has to be bounded also below:

Theorem 2.1. *Let $h \equiv 0$, $0 < \alpha < 2$, $b(r) = r^{-\alpha}$. If*

$$2 < \theta + 1 \leq p_\alpha := 2 + \frac{2\alpha}{N - 1 - \frac{\alpha}{2}}$$

and $u \in H$ solves (P) on $\mathbb{R}^N \setminus B_R$ with $u(R) = 0$, then $u \equiv 0$.

Hence we will assume

(A2) $p_\alpha < \theta + 1 < 2^*$.

We introduce m, the Morse index of the quadratic form

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + b(|x|)u^2)dx,$$

as the number of nonpositive eigenvalues of such a quadratic form; here the eigenvalues are defined as in Lemma 2.2, where we also prove that $m < +\infty$. Under these assumptions and notation we will prove (Section 4)

Theorem 2.2. *Let* $h \equiv 0$ *and assumptions* (A1) and (A2) *hold. Then for any* $k \geq m$ *problem* (P) *has a pair of solutions* u_k^+, $u_k^- \in H$ *with the following properties:*

(i) $u_k^-(0) < 0 < u_k^+(0)$
(ii) $u_k^\pm$ *has exactly* k *simple zeroes.*

When dealing with the forced case, it is natural to expect that solutions with few zeroes may be lost, and it is well known (see [6]) that there are cases when all the possible solutions have infinitely many zeroes. To describe this situation our method leads us to introduce the idea of "essential change of sign" for a radial function u; the precise notion will be given in Definition 3.1, but now, roughly speaking, we can say that the number of changes of sign is at least equal to the number of the essential ones (indeed such a number does not take into account the changes of sign due to small oscillations).

The result we shall prove in Section 4 deals with integrable forcing terms of the following type:

(A3) $h(x) = h(|x|)$, $h \in (L^{\theta+1})'$.

Under these assumptions we have

Theorem 2.3. *Let assumptions* (A1), (A2) *and* (A3) *hold. Then there exists* $k_0 \in \mathbb{N}$ *such that for all* $k \geq k_0$ *problem* (P) *has a pair of solutions* u_k^+, $u_k^- \in H$ *with the following properties:*

(i) $u_k^-(0) < 0 < u_k^+(0)$
(ii) $u_k^\pm$ *has exactly* k *essential changes of sign and* $u_i^\pm \not\equiv u_j^\pm$ *if* $i \neq j$.

2.1. **Preliminaries.** In the following we will always assume that assumptions (A1), (A2) and (A3) hold. The reader can easily check that in the following arguments assumption (A3) can be replaced by

(A3′) $\displaystyle\int_{\mathbb{R}^N} (1 + |x|^\alpha)h(|x|)^2 dx < +\infty$

without any substantial change in the following arguments. We also wish to point out that our results hold for more general nonlinearities than $f(r, s) =$

$|s|^{\theta-1}s$ as in (P). Along the lines of [2], we could prove them for any subcritical f satisfying a monotonicity condition of the following type:

$$\frac{\partial}{\partial s}f(r,s)s^2 \geq \theta f(r,s)s > 0$$

with $\theta > p_\alpha - 1$ as in (A2).

We now come to describe the variational framework where we settle problem (P), giving an equivalent (and more detailed) definition of the space H. First, let X denote the completion of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\Big(\int_{\mathbb{R}^N}(|\nabla u(x)|^2 + b^+(|x|)u^2(x))dx\Big)^{1/2}.$$

Looking for radial solutions of (P), it is natural to introduce the subspace $H \subset X$ of all its radial functions, endowed with the norm

$$\|u\|_H^2 = \int_{\mathbb{R}^N}(|\nabla u(x)|^2 + b^+(|x|)u^2(x))dx = \int_0^{+\infty} r^{N-1}(\dot{u}^2(r)+b^+(r)u^2(r))dr.$$

Note that this definition is equivalent to (2.1). It turns out that H has the following properties:

Lemma 2.1. *For every $R > 0$ there exists $C(R) > 0$ such that*

$$v \in H \implies v^2(r) \leq \frac{C(R)}{r^{N-1-\alpha/2}}\|u\|_H^2 \qquad \forall r > R.$$

Proof. Let $r \geq R$. By elementary computation, we have

$$\begin{aligned} v^2(r) &= -2\int_r^\infty v'(s)v(s)ds \\ &< \frac{2}{r^{N-1-\alpha/2}}\int_r^\infty s^{(N-1)/2}|v'(s)|\cdot s^{(N-1-\alpha)/2}|v(s)|ds \\ &\leq \frac{2}{r^{N-1-\alpha/2}}\Big(\int_r^\infty s^{N-1}|v'(s)|^2 ds\Big)^{\frac{1}{2}}\Big(\int_r^\infty s^{N-1-\alpha}v^2(s)ds\Big)^{\frac{1}{2}} \\ &\leq \frac{C(R)}{r^{N-1-\alpha/2}}\|v\|_H^2, \end{aligned}$$

where the last inequality follows by assumption (A1) and the Poincaré inequality. □

As a consequence, if $N \geq 2$, we have that H is embedded in $L^p(\mathbb{R}^N)$ for all p such that

$$2 + \frac{2\alpha}{N-1-\frac{\alpha}{2}} =: p_\alpha \quad \leq p \quad \leq 2^*$$

where as usual $2^* := 2 + \frac{4}{N-2}$ when $N \geq 3$ and $2^* = +\infty$ when $N = 2$. Indeed, for $p \geq 2$, it holds that:

$$\int_{\{|x|>r\}} |u(x)|^p dx = \int_r^\infty s^{N-1}|u(s)|^p ds \leq \max_{s\geq r}\{s^\alpha |u(s)|^{p-2}\} \int_r^\infty s^{N-1-\alpha} u^2(s) ds$$

$$\leq \frac{C(R)^{(p-2)/2}}{r^{\frac{1}{2}(N-1-\alpha/2)(p-2)-\alpha}} \|u\|_H^{p-2} \int_r^\infty s^{N-1-\alpha} u^2(s) ds \leq \frac{C}{r^{\frac{1}{2}(N-1-\alpha/2)(p-2)-\alpha}} \|u\|_H^p$$

which gives the result for p such that the exponent $(N-1-\alpha/2)(p-2)/2-\alpha$ is not negative; i.e., $p \geq p_\alpha$.

Now a standard argument (see e.g. [20]) proves that the embeddings are compact if the strict inequality occurs; i.e., $p_\alpha < p < 2^*$. Hence one can easily extend the eigenvalue theory for the quadratic form naturally associated to (P):

Lemma 2.2. *Let*

$$\lambda_i := \inf_{\substack{W\subset H\\ \dim W = i}} \sup_{\substack{u\in W\\ u\not\equiv 0}} \frac{\int_{\mathbb{R}^N} \left(|\nabla u|^2 + b(|x|)u^2\right) dx}{\int_{\mathbb{R}^N} \left(|\nabla u|^2 + b^+(|x|)u^2\right) dx}.$$

We have that each λ_i is well defined, simple, achieved by φ_i that changes sign exactly $i-1$ times, and the sequence (λ_i) is strictly increasing to 1 *(in particular $m < +\infty$).*

Proof. We observe that

$$\frac{\int_{\mathbb{R}^N} \left(|\nabla u|^2 + b(|x|)u^2\right) dx}{\int_{\mathbb{R}^N} \left(|\nabla u|^2 + b^+(|x|)u^2\right) dx} = 1 - \frac{\int_{\mathbb{R}^N} b^-(|x|)u^2 dx}{\|u\|_H^2}.$$

Hence by assumption (A1) and standard analysis (in particular by the Sturm comparison lemma) the lemma follows. □

3. Solving the Dirichlet problem on annuli

Let us fix $0 \leq a < b \leq \infty$ and let $[a,b] := \{x \in \mathbb{R}^N : a \leq |x| \leq b\}$ be the annular domain corresponding to the radii a and b. The aim of this section is to find radial solutions for the Dirichlet problem associated to equation (P) on $[a,b]$:

$$\begin{cases} -\Delta u(x) + b(|x|)u(x) = |u(x)|^{\theta-1}u(x) + h(|x|) & x \in [a,b] \\ u(x) = 0 & x \in \partial[a,b]. \end{cases} \qquad (D_h)$$

Solving (D_h) is equivalent to finding critical points of the energy functional naturally associated to the problem

$$J_{[a,b]}(u) = \frac{1}{2}Q_{[a,b]}(u) - \frac{1}{\theta+1}I_{[a,b]}(u) - H_{[a,b]}(u) \qquad (3.1)$$

where

$$\begin{aligned} Q_{[a,b]}(u) &:= \int_{[a,b]} \left(|\nabla u(x)|^2 + b(|x|)u^2(x)\right) dx \\ I_{[a,b]}(u) &:= \int_{[a,b]} |u(x)|^{\theta+1} dx, \quad H_{[a,b]}(u) := \int_{[a,b]} h(|x|)u(x)dx, \end{aligned}$$

defined on a suitable subset of H, say $V(a,b)$, and $C^1(V(a,b),\mathbb{R})$. Precisely, if $0 < a' < b' < +\infty$, we define

$$\begin{aligned} V(0,b') &:= \{u \in H \,:\, u(b') = 0\} \\ V(a',b') &:= \{u \in H \,:\, u(a') = u(b') = 0\} \\ V(a',+\infty) &:= \{u \in H \,:\, u(a') = 0\}. \end{aligned}$$

It is worth noticing that, due to the fact that the coefficient b may change sign on $[a,b]$, the quadratic form $Q_{[a,b]}$ needs not be positively definite. Thus it make sense to introduce the first (possibly negative) eigenvalue of $Q_{[a,b]}$:

$$\lambda_1(a,b) := \inf_{\substack{u \in V(a,b) \\ u \not\equiv 0}} \frac{Q_{[a,b]}(u)}{\|u\|_H^2}.$$

By assumption (A1) λ_1 is well defined, achieved and simple (see the proof of Lemma 2.2). It turns out that

Lemma 3.1. *There exist $\bar{\delta}$ and $\overline{R}$ such that, for all $[a,b]$ with either $|b-a| \le \bar{\delta}$ or $a \ge \overline{R}$, then $\lambda_1(a,b) > 0$.*

Proof. By elementary computations we can prove that the L^2_{rad}-norm of u is dominated by the same norm of the gradient, with coefficient depending on a and $|b-a|$ as follows:

$$\begin{aligned} |u(r)| \le \int_a^b |\dot{u}(s)| ds &\le \frac{1}{a^{(N-1)/2}} \int_a^b s^{(N-1)/2} |\dot{u}(s)| ds \\ &\le \frac{|b-a|^{1/2}}{a^{(N-1)/2}} \Big(\int_a^b s^{N-1} \dot{u}(s)^2 ds \Big)^{\frac{1}{2}}. \end{aligned}$$

By multiplying by r^{N-1} and then integrating for $a < r < b$, we obtain

$$\int_a^b r^{N-1} u^2(r) dr \le \frac{|b-a|}{a^{N-1}} \int_a^b s^{N-1} \dot{u}(s)^2 ds \int_a^b r^{N-1} dr,$$

which gives

$$\int_a^b r^{N-1} \dot{u}^2(r) dr \ge C(a,b) \int_a^b s^{N-1} u(s)^2 ds, \quad C(a,b) := \frac{N a^{N-1}}{|b-a|(b^N - a^N)}.$$

Now fix $\delta > 0$: we can choose $|b-a|$ sufficiently small or a large enough so that $\delta C(a,b) > \|b^-\|_\infty$; this implies

$$\delta \int_a^b r^{N-1}\dot{u}^2(r)dr \geq \int_a^b r^{N-1} b^-(r)u^2(r)dr$$

and provides

$$\int_a^b r^{N-1}(\dot{u}^2(r)dr + b(r)u^2(r))dr \geq (1-\delta)\Big(\int_a^b r^{N-1}(\dot{u}^2(r)dr + b^+(r)u^2(r))dr\Big)$$

for all $u \in V(a,b)$. This finally gives $\lambda_1(a,b) \geq 1-\delta$ as required. □

Remark 3.1. $\lambda_1(a,b)$ is continuous as a function of a and b and decreasing for the inclusion ordering $([a',b'] \subset [a,b] \Rightarrow \lambda_1(a',b') \geq \lambda_1(a,b))$.

The following lemma concerns the behaviour of the optimal constant of the embedding $V(a,b) \subset L^{\theta+1}$ as either $|b-a| \to 0$ or $a \to \infty$:

Lemma 3.2. *For all* $[a,b]$ *there exist* $S(a,b) > 0$ *such that*

$$u \in V(a,b) \implies S(a,b)\|u\|_{\theta+1} \leq \|u\|_H.$$

Moreover, if either $|b-a| \to 0$ *or* $a \to \infty$*, then* $S(a,b) \to \infty$*.*

Proof. Let us define

$$S(a,b) := \inf_{\substack{u\in V(a,b)\\ u\not\equiv 0}} \frac{\|u\|_H}{\|u\|_{\theta+1}}.$$

Let $a > 1$ and $u \in V(a,b)$; taking into account Lemma 2.1 and the fact that $\theta + 1 > p_*$, we can estimate as follows

$$\int_{[a,b]} |u(x)|^{\theta+1}dx = \int_a^b r^{N-1}|u(r)|^{\theta+1}dr$$
$$\leq \max_{a\leq r\leq b}\{r^\alpha |u(r)|^{\theta-1}\} \cdot \int_a^b r^{N-1-\alpha}u^2(r)dr \leq \frac{C}{a^{\frac{1}{2}(N-1-\alpha/2)(\theta-1)-\alpha}}\|u\|_H^{\theta+1}.$$

Thus we obtain $S(a,b)^{\theta+1} \geq C a^{\frac{1}{2}(N-1-\alpha/2)(\theta-1)-\alpha}$; since the exponent of a is positive by condition $\theta+1 > p_*$, we obtain that $S(a,b) \to \infty$ in the case $a \to \infty$.

In order to handle the case of a small annulus $[a,b]$ with $0 < a < b < \infty$, let us consider the constrained minimization leading to the definition of the nonlinear eigenvalue $\lambda(a,b)$, where

$$\lambda(a,b) := \inf_{\substack{u\in V(a,b)\\ u\not\equiv 0}} \frac{\|\nabla u\|_2}{\|u\|_{\theta+1}}$$

(note that $S(a,b) \geq \lambda(a,b)$). It is well known that $\lambda(a,b)$ is strictly positive and it is achieved by a function $\bar{u} \in V(a,b)$ that (suitably rescaled) is a solution of the Dirichlet problem

$$\begin{cases} -\Delta \bar{u}(x) = |\bar{u}(x)|^{\theta-1}\bar{u}(x) & x \in [a,b] \\ \bar{u}(x) = 0 & x \in \partial[a,b]. \end{cases}$$

Now let us fix $0 < A < B < \infty$ such that $|B-A| = 1$ and consider the function $\bar{v} \in V(A,B)$ defined by the identity

$$\bar{u}(x) = |b-a|^{\frac{-2}{\theta-1}}\bar{v}\big(A + \frac{B-A}{b-a}(x-a)\big) \qquad x \in [a,b].$$

By a simple computation it holds that:

$$\int_{[a,b]} |\nabla \bar{u}(x)|^2 dx = |b-a|^{N-2\frac{\theta+1}{\theta-1}} \int_{[A,B]} |\nabla \bar{v}(x)|^2 dx$$
$$\int_{[a,b]} |\bar{u}(x)|^{\theta+1} dx = |b-a|^{N-2\frac{\theta+1}{\theta-1}} \int_{[A,B]} |\bar{v}(x)|^{\theta+1} dx.$$

Thus we obtain

$$S(a,b) \geq \lambda(a,b) = \frac{(\int_{[a,b]} |\nabla \bar{u}(x)|^2 dx)^{\frac{1}{2}}}{(\int_{[a,b]} |\bar{u}(x)|^{\theta+1} dx)^{\frac{1}{\theta+1}}} \geq \lambda(A,B)|b-a|^{\frac{N(\theta-1)-2(\theta+1)}{2(\theta+1)}}$$

where the exponent of $|b-a|$ is negative by the assumption $\theta + 1 < 2^*$, and thus $S(a,b) \to \infty$ if $|b-a| \to 0$. □

Notation. In the following we shall often omit the dependence on $[a,b]$, writing simply V for $V(a,b)$, J for the functional and Q, I, H for the corresponding integrals in (3.1). Moreover in all the computation C will denote any positive constant, independent of $[a,b]$, that we need not specify.

3.1. **The structure of the Nehari set.** Here and in the following we only deal with annuli of the form $[a,b]$ satisfying the assumptions of Lemma 3.1, thus $\lambda_1(a,b) > 0$ and $Q_{[a,b]}$ is positive definite.
Let us consider the so-called Nehari set:

$$\mathcal{N}(a,b) := \{u \in V(a,b) \setminus \{0\} \ : \ \nabla J(u) \cdot u = 0\}.$$

It goes back to the original work by Nehari [14] that, in the unforced case, the set $\mathcal{N} \setminus \{0\}$ is a regular manifold and a natural constraint for the functional. The main goal of this section is that, when either the annulus $[a,b]$ is thin enough or sufficiently far from the origin, then the Nehari set $\mathcal{N}(a,b)$ corresponding to the forced problem (D_h) is the disjoint union of two connected components. A first, possibly irregular component is that of the functions u

which realize the (negative) minimum of J along lines of the form $t \mapsto tu$, $t \geq 0$; the second one is made of the corresponding positive maxima, and indeed it turns out to be a manifold radially homeomorphic to the unit sphere of $V(a,b)$. To achieve these results, we first study the behaviour of the map $t \mapsto J(tu)$.

Lemma 3.3. *There exist $\bar{\delta}$ and $\overline{R}$ such that, for all the annuli of the form $[a,b]$ with either $|b-a| \leq \bar{\delta}$ or $a \geq \overline{R}$, the following property holds: let $u \in V(a,b) \setminus \{0\}$; then the alternative is*

(i) $\exists\, \bar{t}(u) > 0 \ :\ \bar{t}(u)u \in \mathcal{N}$ *and* $J(\bar{t}(u)u) = \max_{t\geq 0} J(tu) > 0$;
(ii) $\exists\ 0 < \underline{t}(u) < \bar{t}(u) \ :\ \bar{t}(u)u \in \mathcal{N}$, $\underline{t}(u)u \in \mathcal{N}$ *and* $J(\underline{t}(u)u) = \min_{t\geq 0} J(tu) < 0$, *while* $J(\bar{t}(u)u) = \max_{t\geq 0} J(tu) > 0$.

Proof. Let $u \in V(a,b) \setminus \{0\}$ and consider the function

$$t \mapsto J(tu) = \frac{t^2}{2} Q(u) - \frac{t^{\theta+1}}{\theta+1} I(u) - tH(u).$$

In order to study the sign of its derivative, we consider the equation

$$tQ(u) - t^\theta I(u) = H(u). \tag{3.2}$$

When $H(u)$ is negative, there exists a unique $\bar{t} > 0$ solution of (3.2), and $t \mapsto J(tu)$ is increasing for $t < \bar{t}$ and decreasing when $t > \bar{t}$, proving assertion (i) Now let

$$m(u) := \max_{t\geq 0}\{tQ(u) - t^\theta I(u)\} \equiv C(\theta) Q(u)^{\frac{\theta}{\theta-1}} / I(u)^{\frac{1}{\theta-1}}.$$

Note that (3.2) has exactly two solutions $0 < \underline{t} < \bar{t}$ if $0 \leq H(u) < m(u)$; moreover $\underline{t} = \underline{t}(u)$ and $\bar{t} = \bar{t}(u)$ satisfy assertion (ii). Thus we need only to prove that $H(u) < m(u)$; since $H(u) \leq C\|h\|_{(\theta+1)'}\|u\|_H$ (where $\|\cdot\|_{(\theta+1)'}$ is the norm in the dual space of $L^{\theta+1}(a,b)$) and $Q(u) \geq \lambda_1 \|u\|_H^2$, it suffices to prove that

$$C\|h\|_{(\theta+1)'}\|u\|_H \leq C\|u\|_H^{\frac{2\theta}{\theta-1}} / \|u\|_{\theta+1}^{\frac{\theta+1}{\theta-1}},$$

which is equivalent to saying

$$\|h\|_{(\theta+1)'} \leq C\|u\|_H^{\frac{\theta+1}{\theta-1}} / \|u\|_{\theta+1}^{\frac{\theta+1}{\theta-1}}.$$

Let us consider the infimum on u of the right-hand side, and note that it coincides with $S(a,b)^{\frac{\theta+1}{\theta-1}}$; by Lemma 3.2 we know that $S(a,b)$ tends to infinity if either $|b-a| \to 0$ or $a \to \infty$, and furthermore $\|h\|_{(\theta+1)'}$ vanishes. Thus the last inequality holds and guarantees that $H(u) < m(u)$ at least for small annuli $[a,b]$ or a large, concluding the proof. □

It turns out that the local minimals along lines are of small norm, as shown by the following lemma:

Lemma 3.4. *If either* $|b-a| \to 0$ *or* $a \to \infty$*, then*

$$\varepsilon(a,b) := \sup\{\|u\|_{\theta+1} \ : \ u \in \mathcal{N}, \ J(u) \le 0\} \to 0.$$

Proof. Since $u \in \mathcal{N}$ it holds that $Q(u) - I(u) - H(u) = 0$; furthermore since $J(u) \le 0$ we obtain that $\frac{\theta+1}{2}Q(u) - I(u) - (\theta+1)H(u) \le 0$. Summing up the two equations we deduce the inequality $\frac{\theta-1}{2}Q(u) \le \theta H(u)$. We continue on the right of the inequality by the Hölder inequality, $|H(u)| \le \|h\|_{(\theta+1)'}\|u\|_{\theta+1} \le \|h\|^2_{(\theta+1)'} + \|u\|^2_{\theta+1}$; on the left we exploit the fact that $\lambda_1(a,b)\|u\|^2 \le Q(u)$, and then the embedding $V(a,b) \subset L^{\theta+1}$ in Lemma 3.2. This leads to

$$C(\lambda_1(a,b)S(a,b) - 1)\|u\|^2_{\theta+1} \le C\|h\|_{(\theta+1)'}.$$

Now we recall Lemma 3.2: as $|b-a| \to 0$ or $a \to \infty$, then $S(a,b) \to \infty$; on the other hand we have $\|h\|_{(\theta+1)'} \to 0$. Thus we obtain the existence of $\alpha(a,b) \to 0$ such that the above inequality is satisfied for all $\|u\|_{\theta+1} \le \alpha(a,b)$. From this the assertion immediately follows. □

Motivated by this result we define

$$M(a,b) := \inf\{\|u\|_{\theta+1} \ : \ u \in \mathcal{N}, \ \|u\|_{\theta+1} > 1\}$$

and we study its behaviour as the annulus becomes either small or far from the origin:

Lemma 3.5. *If either* $|b-a| \to 0$ *or* $a \to \infty$*, then* $M(a,b) \to \infty$.

Proof. Let $u \neq 0$ such that $\nabla J(u) \cdot u = 0$, i.e. $Q(u) - I(u) - H(u) = 0$. Applying the inequality $|H(u)| \le \|h\|^2_{(\theta+1)'} + \|u\|^2_{\theta+1}$ as in the previous lemma, we compute as follows:

$$\lambda_1(a,b)S(a,b)\|u\|^2_{\theta+1} \le Q(u) = I(u) + H(u) \le C\|u\|^{\theta+1}_{\theta+1} + \|h\|^2_{(\theta+1)'} + \|u\|^2_{\theta+1}.$$

Letting $x := \|u\|^2_{\theta+1}$, we have that x must satisfy the inequality

$$Ax \le x^{\frac{\theta+1}{2}} + B, \tag{3.3}$$

where $A = C(\lambda_1(a,b)S(a,b) - 1) \to \infty$ and $B = \|h\|^2_{(\theta+1)'} \to 0$ when either $|b-a| \to 0$ or $a \to \infty$. As before, it follows by an elementary comparison the existence of $\alpha(a,b) \to 0$ and $\beta(a,b) \to \infty$ such that (3.3) is satisfied iff either $x \le \alpha(a,b)$ or $x \ge \beta(a,b)$. Since by assumption $x = \|u\|^2_{\theta+1} > 1$, the second of the two possibilities must hold, and thus $M(a,b) \ge \beta(a,b)$, proving the result. □

Collecting together all the previous results we finally obtain

Lemma 3.6. *There exist $\bar{\delta}$ and $\bar{R}$ such that for all the annuli $[a,b]$ with either $|b-a| \leq \bar{\delta}$ or $a \geq \bar{R}$ the following decomposition holds:*

$$\mathcal{N}(a,b) = \mathcal{M}(a,b) \cup \mathcal{M}_-(a,b),$$

where

$$\begin{aligned}
\mathcal{M}(a,b) &:= \{u \in V(a,b) : \nabla J(u) \cdot u = 0,\ J(u) > 0\} \\
&= \{u \in V(a,b) : \nabla J(u) \cdot u = 0,\ \|u\|_{\theta+1} > 1\} \\
\mathcal{M}_-(a,b) &:= \{u \in V(a,b) : \nabla J(u) \cdot u = 0,\ J(u) < 0\} \\
&= \{u \in V(a,b) : \nabla J(u) \cdot u = 0,\ \|u\|_{\theta+1} < 1\}.
\end{aligned}$$

Moreover, $\mathcal{M}$ is disjoint from $\mathcal{M}_-$; it is a manifold radially homeomorphic to the unit sphere in V, and if $u \in \mathcal{M}$ is a constrained critical point of J on $\mathcal{M}$, then u is a free critical point of J, in the sense that $\nabla J(u) = 0$.

Note that, by combining the assertions in Lemmas 3.3 and 3.5, it holds that

$$M(a,b) \equiv \inf_{u \in \mathcal{M}} \|u\|_{\theta+1}.$$

3.2. **Three variational problems.** In this section we are going to solve the Dirichlet problem (D_h) on $[a,b]$. One might try to minimize $J_{[a,b]}$ on the component $\mathcal{M}(a,b)$ of the Nehari set, and this would immediately provide one solution to the problem. In fact, in order to apply a Nehari-type procedure we need two solutions characterized by "opposite" sign properties. Due to the presence of h, this can not be accomplished by the usual trick of minimizing over $\mathcal{M}$:

$$J^{\pm}(u) := \int_{[a,b]} \big(|\nabla u(x)|^2 + b(|x|)u^2(x) - \frac{1}{\theta+1}|u^{\pm}(x)|^{\theta+1}\big)dx. \tag{3.4}$$

(Here and in the following $u = u^+ - u^-$.) This is not a purely technical obstacle: as already noticed, we cannot expect to find, in any annulus, a completely positive solution and a negative one. As a matter of fact we shall find a first solution u that is "essentially" positive, in the sense that the norm of its negative part u^- is small when compared with the global norm of u itself (analogous arguments will provide a solution "essentially" negative). In order to make this idea clearer we introduce the auxiliary functional

$$g^+(u) := \frac{\|u^-\|_{\theta+1}}{\|u\|_{\theta+1}}$$

—note that $g^+ \in C^1(V,\mathbb{R})$. Let $0 < \varepsilon < 1$ (we shall fix ε in the subsequent Theorem 3.1), and then let us define the cone $\mathcal{C}^+ \subset V$ of the "essentially" positive functions as $\mathcal{C}^+ := \{u \in V \setminus \{0\} \; : \; g^+(u) \leq \varepsilon\}$.

In order to find a solution u for (D_h) lying in $\mathcal{C}^+$, following the line of [2], we define three minimax values as follows:

$$\begin{aligned} d^+(a,b) &:= \inf_{u\in\mathcal{C}^+} \sup_{t>0} J(tu) \\ \varphi^+(a,b) &:= \inf_{\mathcal{M}\cap\mathcal{C}^+} J \\ c^+(a,b) &:= \inf_{\gamma\in\Gamma^+(a,b)} \max_{t\in[0,1]} J(\gamma(t)) \end{aligned} \tag{3.5}$$

where

$$\Gamma^+(a,b) := \{\gamma \in C([0,1],V) : \gamma(0) = 0, J(\gamma(1)) < 0, \;\; \gamma(t) \in \mathcal{C}^+ \; \forall t \in (0,1]\}.$$

We claim that the above values are critical for J at least for suitable choice of ε in the definition of the functional g^+.

Let us start by showing that the three variational problems coincide and admit a solution.

Lemma 3.7. *The functional $J_{[a,b]}$ satisfies the Palais–Smale condition; furthermore for every $u \in V(a,b)\setminus\{0\}$ there exists $R > 0$ such that $J_{[a,b]}(ru) < 0$ if $r > R$.*

The proof of the Palais–Smale condition is standard, with minor changes due to the presence of the forcing term h; it can be found e.g. in Lemma 3.1 of [2]; the second part of the assertion is already contained in Lemma 3.3.

To prove that the three values coincide, it suffices to note that $\varphi^+ = d^+$ by definition of $\mathcal{M}$; furthermore $c^+ \leq d^+$ since for any $u \in \mathcal{C}^+$ the path $\gamma(t) := tRu$ (with R large enough as in Lemma 3.7) belongs to Γ^+; finally $c^+ \geq \varphi^+$ because every $\gamma \in \Gamma^+$ has to cross $\mathcal{M}$ by Lemma 3.6. Moreover these levels becomes larger and larger when the annulus becomes small or sufficiently far from the origin:

Lemma 3.8. *If either $|b-a| \to 0$ or $a \to \infty$ then $\varphi^+(a,b) \to \infty$ uniformly in ε.*

Proof. Let $u \in \mathcal{M}$; since $Q(u) = I(u) + H(u)$ we can eliminate it in the expression of J and then we can estimate as usual:

$$\begin{aligned} J(u) &= \frac{1}{2}Q(u) - \frac{1}{\theta+1}I(u) - H(u) = \frac{1}{2}\frac{\theta-1}{\theta+1}I(u) - \frac{1}{2}H(u) \\ &\geq C\|u\|_{\theta+1}^{\theta+1} - C\|u\|_{\theta+1}^2 - C\|h\|_{(\theta+1)'}^2 \end{aligned}$$

where the constants are independent of a, b and ε. Now the assertion follows by the fact that $\inf_{u\in\mathcal{M}} \|u\|_{\theta+1} = M(a,b)$ and, by Lemma 3.5, $M(a,b)$ explodes when either $|b-a| \to 0$ or $a \to \infty$. □

Note that, if $|b-a|$ is sufficiently small or a is large enough, the proof of the previous lemma shows the existence of $C_1 > 0$ (independent of a, b and ε) such that $\varphi^+(a,b) \geq C_1(M^+(a,b)^{\theta+1} - 1)$ where

$$M^+(a,b) := \inf_{u\in\mathcal{M}\cap\mathcal{C}^+} \|u\|_{\theta+1} \ (\geq M(a,b)).$$

As a matter of fact, an opposite inequality holds too. Indeed, by definition of infimum, let $\bar{u} \in \mathcal{M}\cap C^+$ such that $\|\bar{u}\|_{\theta+1} \leq M^+(a,b)+1$. Computing as in the lemma above we obtain that $\varphi^+(a,b) \leq J(\bar{u}) \leq C\|\bar{u}\|_{\theta+1}^{\theta+1} + C\|\bar{u}\|_{\theta+1}^2 + C$. This provides the existence of $C_2 > 0$ (independent of a, b, ε) such that

$$\varphi^+(a,b) \leq C_2(M^+(a,b)^{\theta+1} + 1) \tag{3.6}$$

at least for small values of $|b-a|$ or a sufficiently large.

It is easy to solve the first variational problem as shown in the following:

Lemma 3.9. *There exists $u \in \mathcal{M}\cap\mathcal{C}^+$ such that $J(u) = \varphi^+$.*

Proof. It follows by a standard argument: let us consider a minimizing sequence $(u_n) \subset \mathcal{M}\cap\mathcal{C}^+$ such that $J(u_n) \to \varphi^+$. Since the Palais–Smale condition holds, u_n strongly converges to a function u; note that u still belongs to $\mathcal{M}\cap\mathcal{C}^+$; thus, by the continuity of J it holds that $J(u) = \varphi^+$. □

We wish to show that each solution u of the minimum problem φ^+ is indeed a critical point for J. This will hold true for suitable choices of ε in the definition of $\mathcal{C}^+$ and thanks to the equivalence between φ^+ and c^+.

Theorem 3.1. (i) *If $h \equiv 0$ (and $\lambda_1(a,b) > 0$) then, $\forall\varepsilon \in (0,1)$, each $u \in \mathcal{M}\cap\mathcal{C}^+$ realizing $J(u) = \varphi^+$ is a critical point of J; i.e., $\nabla J(u) = 0$.*

(ii) *If $h \not\equiv 0$, let there be $|b-a|$ small or a large enough such that the previous results hold. Then $\exists\varepsilon \in (0,1)$, independent of $[a,b]$, such that each $u \in \mathcal{M}\cap\mathcal{C}^+$ realizing $J(u) = \varphi^+$ is a critical point of J.*

Proof. The proof of the claim in the unforced case (i) is standard, since the associated quadratic form $Q_{[a,b]}$ is positive definite. In order to prove (ii), we first observe that the assertion is obvious if u belongs to the interior of $\mathcal{C}^+$, since in that case it is a free critical point by Lemma 3.6. As a matter of fact we are going to prove that, for a suitable choice of ε, u cannot belong to ∂C^+.

Claim: $\exists\varepsilon \in (0,1)$ such that $g^+(u) < \varepsilon$ for all u such that $J(u) = \varphi^+$.

Consider the path $\gamma(t) := tRu$ (R large enough to give $J(Ru) < 0$) and note that $\gamma \in \Gamma^+$ is optimal, in the sense that for all $t \neq 1/R$ it holds that $J(\gamma(t)) < J(\gamma(1/R)) = J(u) = \varphi^+$. Now assume for the sake of contradiction that $g^+(u) = \varepsilon$ and $u \in \partial C^+$. By standard arguments in the theory of constrained critical points the optimality of γ implies the existence of a positive Lagrange multiplier μ such that

$$\nabla J(u) = -\mu \nabla g^+(u).$$

By an easy computation we get $\nabla g^+(u) \cdot (-u^-) = \varepsilon(1 - \varepsilon^{\theta+1})$, which implies

$$\nabla J(-u^-) \cdot (-u^-) \leq 0. \tag{3.7}$$

Now we choose ε in order to obtain the opposite inequality in (3.7). To this aim, note that, since $J(u) = \varphi^+(a,b)$, there exist positive D_1 and D_2 (independent of a, b and ε) such that $D_1\varphi^+(a,b) \leq \|u\|_{\theta+1}^{\theta+1} \leq D_2\varphi^+(a,b)$. Taking into account (3.6) and Lemmas 3.4 and 3.5, we can choose ε in the interval

$$\Big(\frac{\varepsilon^+(a,b)}{(D_1\varphi^+(a,b))^{\frac{1}{\theta+1}}}\ ,\ \frac{M^+(a,b)}{(D_2\varphi^+(a,b))^{\frac{1}{\theta+1}}}\Big)$$

where $\varepsilon^+(a,b) := \sup\{\|u\|_{\theta+1} : u \in \mathcal{M} \cap C^+,\ J(u) \leq 0\} \leq \varepsilon(a,b)$. Note that the choice of ε is independent of the annulus $[a,b]$, due to the fact that the ratio $\frac{M^+(a,b)^{\theta+1}}{\varphi^+(a,b)}$ is bounded below by a positive constant as in (3.6).

Now, since $g^+(u) = \varepsilon$, we have $\|u^-\|_{\theta+1} \in (\varepsilon^+(a,b)\ ,\ M^+(a,b))$, and we can conclude by exploiting a technical property contained in Lemma 3.3, i.e.,

$$v \in \mathcal{C}^+\ :\ \varepsilon^+(a,b) < \|v\|_{\theta+1} < M^+(a,b) \implies \nabla J(v) \cdot v > 0\,. \tag{3.8}$$

Thus we get $\nabla J(-u^-) \cdot (-u^-) > 0$, in contradiction with (3.7), and u cannot belong to the boundary of C^+. □

Lemma 3.10. *$\varphi^\pm(a,b)$ is continuous as a function of a and b.*

Proof. The proof is the same as the proof of Proposition 4.1(d) in [2]. □

Remark 3.2. When $h \equiv 0$, then $\forall \varepsilon > 0$ and $u \in \mathcal{M} \cap C^+$ achieving $\varphi^+(a,b)$, we have $u > 0$ on (a,b), the interior part of the annulus. Indeed, assuming for the sake of contradiction that $u^- \not\equiv 0$, the contribution of $J(u^-)$ is strictly positive; moreover one can easily see that, since they solve the equation except where they are zero, both u^+ and u^- belong to $\mathcal{N} \equiv \mathcal{M}$. Hence $u^+ \in \mathcal{M} \cap C^+$ is such that $J(u^+) < \varphi^+(a,b)$, a contradiction. On the other hand, when $h \not\equiv 0$ and the annulus is not too small then these solutions have no reason to be one sign.

Motivated by the previous remark we introduce the following:

Definition 3.1. We will say that u has an essential change of sign in the annulus $[a,b]$ if there exists $a<c<b$ such that both $u|_{[a,c]}$ achieves $\varphi^+(a,c)$ (respectively $\varphi^-(a,c)$) and $u|_{[c,b]}$ achieves $\varphi^-(c,b)$ (respectively $\varphi^+(c,b)$).

It is easy to see, arguing as in Remark 3.2, that if u achieves $\varphi^+(a,b)$ and $u=\sum_i u_i$, where the u_i's have disjoint supports and they are either positive or negative, then only one of them can belong to $\mathcal{M}$. This justifies the well posedness of the previous definition.

4. Main results

In this section we shall prove the existence Theorems 2.2 and 2.3 as a direct consequence of Theorem 4.1 and 4.2 below. Since the purpose is to find solutions with (essential) changes of sign, we follow the basic idea of Nehari [14]. In order to build a solution with one essential change of sign on $[a,c]$, we try to paste together an essentially positive solution on $[a,b]$, for some $a<b<c$, and an essentially negative solution on $[b,c]$ by minimizing (with respect to b) the function $\varphi^+(a,b)+\varphi^-(b,c)$. To succeed in this procedure it is fundamental the information that, when the left and right derivatives in b are not the same, then the value $\varphi^+(a,b)+\varphi^-(b,c)$ decreases by moving b in the direction of the lower derivative.

Lemma 4.1. *Let* $\psi(r):=\varphi^+(a,r)+\varphi^-(r,b)$ *where* $0\le a<r<b\le+\infty$ *and assume that* $\exists r_0$ *such that* $\psi(r_0)$ *is achieved by* $u_1(|x|)\in\mathcal{M}(a,r_0)\cap\mathcal{C}^+$ *and* $u_2(|x|)\in\mathcal{M}(r_0,b)\cap\mathcal{C}^-$ *solutions of* (D_h). *Moreover let* u_1+u_2 *change sign across* r_0 *(that is,* $u_1(r_0-s_1)\cdot u_2(r_0+s_2)\le 0$ *but not identically* 0 *for every* s_1, s_2 *belonging to some* $[0,\bar{s}]$*). In such a situation, if* $|\dot u_1(r_0-0)|>|\dot u_2(r_0+0)|$ *then* $\exists\bar{h}>0:\forall h\in(0,\bar{h})\ \psi(r_0+h)<\psi(r_0)$.

Proof. Let us consider the ordinary differential equation associated to (P) (as in equation (5.1) below). We can apply to such an equation the argument of [14] (see also [19], Theorem 2.1, Lemma 6.5). The only difference consists in the fact that here $\varphi^\pm$ are defined with one more constraint; namely they are achieved by functions lying in $\mathcal{C}^\pm$. This can easily be overcome observing that u_1 and u_2 lie in the interior of the cones, and since the variation constructed to decrease ψ can be chosen as small as we want, by a continuity argument also $\tilde u_1$ and $\tilde u_2$ (the variated functions that decrease ψ) must lie in the relative $\mathcal{C}^\pm$. □

Now we will consider two different cases, namely either $h\equiv 0$ or $h\not\equiv 0$. Indeed by Remark 3.2 in the unforced case we will be able to establish the precise number of zeroes of the solution. Anyway, both the results will be obtained minimizing a suitable finite-dimensional functional under a constraint; more precisely, we will impose that the zeroes of the solution satisfy all the assumptions in the previous machinery. For the unforced case the minimal number of nodal regions we can expect in the solution is strictly connected to the Morse index of the quadratic form $Q_{[0,+\infty]}(u)$:

Lemma 4.2. *Let m be the Morse index of $Q_{[0,+\infty]}(u)$. Then if $k \geq m+1$ then for every partition $0 < r_1 < \cdots < r_k < +\infty$ there exists at least one i such that $\lambda_1(r_i, r_{i+1}) \geq c(k) > 0$.*

Proof. The proof is exactly the same as in Lemma 5.1 in [5]. □

For the unforced case we introduce the constraint

$$\mathcal{B}_l := \left\{ (r_i) \subset \mathbb{R}^+,\ i = 0, \ldots, k+1 : \begin{array}{l} 0 =: r_0 < r_1 < \cdots < r_{k+1} := +\infty \\ \lambda_1(r_i, r_{i+1}) > l \end{array} \right\},$$

where $k \geq m+1$ and $l > 0$ is small to be fixed. Note that, by Lemma 2.2, $\mathcal{B}_l$ is not empty, indeed it contains, for l sufficiently small, the partition given by the zeroes of the eigenfunction corresponding to the eigenvalue λ_{m+1}. Let

$$\sigma(i) := \begin{cases} + & i \text{ is even} \\ - & i \text{ is odd} \end{cases} .$$

Hence, by Theorem 3.1 part (i), we can associate to each annulus $[r_i, r_{i+1}]$ in $\mathcal{B}_l$ a function $u_i^l \in \mathcal{M} \cap \mathcal{C}^{\sigma(i)}$ that achieves $\varphi^{\sigma(i)}(r_i, r_{i+1})$ (i.e., it solves (P) in the annulus). We have

Theorem 4.1. *Let $h \equiv 0$ and assumptions* (A1) and (A2) *hold. Then for every $k \geq m+1$ there exists $l > 0$ such that the value*

$$c_k := \inf_{\mathcal{B}_l} \sum_{i=0}^{k} \varphi^{\sigma(i)}(r_i, r_{i+1})$$

is achieved by

$$\bar{u} := \sum_{i=0}^{k} u_i^l,$$

a solution of (P) that changes sign exactly k times.

For the proof of Theorem 4.1 we need two technical lemmas (in the following we will assume $h \equiv 0$):

Lemma 4.3. *Let $(a^l, b^l) \to (a^0, b^0)$ as $l \to 0$, and let u^l achieve $\varphi^{\pm}(a^l, b^l)$. If $\lambda_1(a^0, b^0) = 0$ then $\|u^l\|_{C^1} \to 0$.*

Proof. First we prove that if $\lambda_1(a^l, b^l) \to 0$, then $\varphi^+(a^l, b^l) \to 0$ (for φ^- one can easily argue in the same way). Let φ_1^l be the first positive eigenfunction with $\|\varphi_1^l\|_H = 1$. Since $h \equiv 0$ we can explicitly calculate the expression of φ^+ (using its equivalent definition as d^+):

$$\varphi^+(a,b) = \inf_{\substack{u \in \mathcal{C}^+ \\ u \not\equiv 0}} (\frac{1}{2} - \frac{1}{\theta+1}) \frac{(Q_{[a,b]}(u))^{\frac{\theta+1}{\theta-1}}}{(I_{[a,b]}(u))^{\frac{2}{\theta+1}}} \leq C \frac{(Q_{[a,b]}(\varphi_1))^{\frac{\theta+1}{\theta-1}}}{(I_{[a,b]}(\varphi_1))^{\frac{2}{\theta+1}}}$$

$$= C \frac{\lambda_1^{\frac{\theta+1}{\theta-1}}}{(I_{[a,b]}(\varphi_1))^{\frac{2}{\theta+1}}}.$$

Therefore, we have to prove that $I_{[a^l,b^l]}(\varphi_1^l) \not\to 0$. Let for the sake of contradiction $I_{[a^l,b^l]}(\varphi_1^l) \to 0$; since supp $b^-(|x|)$ is bounded, we obtain $\int b^-(|x|)(\varphi_1^l)^2 \to 0$. Thus we have

$$0 \leftarrow \lambda_1(a^l, b^l) = Q_{[a^l,b^l]}(\varphi_1^l) = \|\varphi_1^l\|_H - \int b^-(\varphi_1^l)^2 \to 1,$$

which is a contradiction. Thus $\varphi^+(a^l, b^l) \to 0$. Let u^l achieve $\varphi^+(a^l, b^l)$; we have both

$$\varphi^+ = (\frac{1}{2} - \frac{1}{\theta+1}) Q(u^l) = (\frac{1}{2} - \frac{1}{\theta+1}) I(u^l)$$

and

$$-\Delta u^l + b^+ u^l = |u^l|^{\theta-1} u^l + b^- u^l.$$

Since $\varphi^+ \to 0$ we have $Q(u^l) \to 0$ and $I(u^l) \to 0$, the right-hand side of the previous equation tends to 0 in the dual space of H, $u^l \to 0$ in H^1_{loc} and finally, by standard regularity arguments, $u^l \to 0$ in C^1. □

Lemma 4.4. *Let $(a^l, b^l) \to (a^0, b^0)$ as $l \to 0$, and let u^l achieve $\varphi^\pm(a^l, b^l)$. If $\lambda_1(a^0, b^0) > 0$, then there exists $c > 0$ such that both $|\dot{u}^l(a^l)| \geq c$ and $|\dot{u}^l(b^l)| \geq c$ for all l small enough.*

Proof. We have again

$$\varphi^+(a,b) = \inf_{\substack{u \in \mathcal{C}^+ \\ u \not\equiv 0}} (\frac{1}{2} - \frac{1}{\theta+1}) \frac{(Q_{[a,b]}(u))^{\frac{\theta+1}{\theta-1}}}{(I_{[a,b]}(u))^{\frac{2}{\theta+1}}} \geq C \lambda_1^{\frac{\theta+1}{\theta-1}}(a,b) S^{\frac{2(\theta+1)}{\theta-1}}(a,b),$$

and then $\varphi^+(a^l, b^l) \to \varphi^+(a^0, b^0) \geq C > 0$. If u^l achieves $\varphi^+(a^l, b^l)$ then $u^l \to u^0$ (in H and in C^1), and $J(u^0) \geq \varphi^+(a^0, b^0)$; hence $u^0 \not\equiv 0$, and $\dot{u}^0(a^0)$ and $\dot{u}^0(b^0)$ can not be zero by the Cauchy uniqueness theorem. Since $\dot{u}^l(a^l) \to \dot{u}^0(a^0)$ and $\dot{u}^l(b^l) \to \dot{u}^0(b^0)$ the lemma follows. □

Proof of Theorem 4.1. By Lemmas 3.10 and 3.8 $\varphi^{\sigma(i)}(r_i, r_{i+1})$ is continuous and it explodes as $|r_{i+1} - r_i| \to 0$ or $r_i \to +\infty$. Hence c_k^l is achieved (at least for l sufficiently small) by a partition

$$0 < r_1^l < \cdots < r_k^l < +\infty.$$

Note that, since $l_1 \le l_2$ implies $c_k^{l_1} \le c_k^{l_2}$, we can assume the existence of $t > 0$ independent of l, such that $r_1^l > t$ and $r_k^l < 1/t$. Moreover by Remark 3.2 the minimum $u^l := \sum_{i=0}^k u_i^l$ has exactly k zeroes and each of them provides a change of sign. The only thing we have to prove is that $\dot{u}^l(r_i^k - 0) = \dot{u}^l(r_i^k + 0)$ for every i, and we argue by contradiction. If $\lambda_1(r_{i-1}^l, r_i^l) > l$ and $\lambda_1(r_i^l, r_{i+1}^l) > l$ we easily obtain a contradiction by applying Lemma 4.1.

Claim: there exists a small l such that $\lambda_1(r_i^l, r_{i+1}^l) > l$ for every i.

For the sake of contradiction we assume that for every l there is some annulus of the minimal partition with corresponding first eigenvalue equal to l; on the other hand, by Lemma 4.2, there exists another annulus such that its first eigenvalue is larger than a fixed constant $c(k)$. Letting $l \to 0$ we can assume, at least for some subsequence, that these two intervals are consecutive, say there exists j (independent of l) such that $\lambda_1(r_{j-1}^l, r_j^l) = l$ and $\lambda_1(r_j^l, r_{j+1}^l) \ge c(k)$. By Lemma 4.3 $\dot{u}^l(r_j^l - 0) \to 0$, and by Lemma 4.4 there exists $c > 0$ independent of l such that $|\dot{u}^l(r_j^l + 0)| \ge c$. Therefore, for l sufficiently small, we infer $|\dot{u}^l(r_j^l - 0)| < |\dot{u}^l(r_j^l + 0)|$, and Lemma 4.1 applies, moving r_j^l onto the left-hand side. In this way we obtain another partition belonging to $\mathcal{B}_l$ and strictly decreasing c_k, reaching a contradiction. □

Now we consider the general case. Let $\bar{\delta}$ and $\bar{R}$ satisfy Lemmas 3.1 and 3.6, and $k \in \mathbb{N}$ large to be fixed. In such a situation we define a new constraint

$$\mathcal{B}'_k := \left\{ (r_i) \subset \mathbb{R}^+,\ i = 0, \dots, k+1 : \begin{array}{l} 0 =: r_0 < r_1 < \cdots < r_{k+1} := +\infty \\ r_i \le \bar{R} \Rightarrow |r_{i+1} - r_i| \le \bar{\delta} \end{array} \right\},$$

and when $k > \frac{\bar{R}}{\bar{\delta}}$ we obviously have that $\mathcal{B}'_k$ is not empty.

Theorem 4.2. *Under assumptions* (A1), (A2) *and* (A3) *there exists $\bar{k} \in \mathbb{N}$ such that for every $k \ge \bar{k}$,*

$$c_k := \min_{\mathcal{B}'_k} \sum_{i=0}^{k} \varphi^{\sigma(i)}(r_i, r_{i+1})$$

is achieved by

$$\bar{u} := \sum_{i=0}^{k} u_i,$$

a solution of (P) *that changes sign at least* k *times.*

Proof. As in the proof of Theorem 4.1 the minimum is achieved by some

$$0 < r_1^k < \cdots < r_k^k < +\infty,$$

and the only thing to prove is that $\dot{u}^k(r_i^k - 0) = \dot{u}^k(r_i^k + 0)$; we will argue by contradiction. First consider the case $|r_i^k - r_{i-1}^k| < \bar{\delta}$ and $|r_{i+1}^k - r_i^k| < \bar{\delta}$ if $r_{i-1}^k \le \bar{R}$ and $r_i^k \le \bar{R}$, or any case with $r_{i-1}^k > \bar{R}$. There are two possibilities, either when r_i^k provides a change of sign or not; if it provides a change of sign, one can apply Lemma 4.1 obtaining the usual contradiction. Let us see what happens when there is no change of sign through r_i^k. To fix the ideas we assume that u_{i-1}^k achieves $\varphi^+(r_{i-1}^k, r_i^k)$, u_i^k achieves $\varphi^-(r_i^k, r_{i+1}^k)$, $\dot{u}^k(r_i^k - 0) \ge 0$ and $\dot{u}^k(r_i^k + 0) \le 0$ (but not both zero, otherwise there is nothing to prove). Let $r' < r_i^k$ be the point of change of sign for u_{i-1}^k nearest to r_i^k. We have $u^k|_{[r', r_i^k]} \le 0$, and, since it solves the equation where it is not zero, $u^k|_{[r', r_i^k]} \in \mathcal{N}$; moreover $u^k|_{[r', r_i^k]} \in \mathcal{M}_-$; indeed it must be that $J(u^k|_{[r', r_i^k]}) < 0$ (to prove this assertion one can easily argue as in Remark 3.2). We claim that $u^k|_{[r_{i-1}^k, r']} \in \mathcal{M} \cap \mathcal{C}^+$ and $u^k|_{[r', r_{i+1}^k]} \in \mathcal{M} \cap \mathcal{C}^-$. They surely lie in the relative $\mathcal{N}$, since they solve the equation where they are not zero. Moreover, since $J(u^k|_{[r_{i-1}^k, r']}) > J(u^k|_{[r_{i-1}^k, r_i^k]})$ and $\|u^k|_{[r', r_{i+1}^k]}\|_{\theta+1} > \|u^k|_{[r_i^k, r_{i+1}^k]}\|_{\theta+1}$, by Lemma 3.6 they lie in the relative $\mathcal{M}$. At last, since $u^k|_{[r', r_i^k]} \le 0$, they respectively lie in $\mathcal{C}^+$, $\mathcal{C}^-$, and the claim follows. We obtain

$$\begin{aligned} \varphi^+(r_{i-1}^k, r_i^k) + \varphi^-(r_i^k, r_{i+1}^k) &= J_{[r_{i-1}^k, r_i^k)]}(u^k) + J_{[r_i^k, r_{i+1}^k]}(u^k) \\ &= J_{[r_{i-1}^k, r')]}(u^k) + J_{[r', r_{i+1}^k]}(u^k) > \varphi^+(r_{i-1}^k, r') + \varphi^-(r', r_{i+1}^k), \end{aligned}$$

where the last inequality is strict since $u^k|_{[r', r_{i+1}^k]}$, not solving the equation in r_i^k, can not achieve $\varphi^-(r', r_{i+1}^k)$; thus also in this case we have a contradiction. It remains to prove that for k sufficiently large it must be that $|r_{i+1}^k - r_i^k| < \bar{\delta}$ for every $r_i^k \le \bar{R}$. Assume for the sake of contradiction that for every k there is some annulus $I^k := [r_j^k, r_{j+1}^k]$ with $r_j^k \le \bar{R}$ and $|I^k| = \bar{\delta}$ (note that the number of such annuli must be bounded independently of k). We have $\varphi^\pm(I^k) \le C_1$, independent of k; moreover if we partition I^k into three equal subintervals I_1^k, I_2^k and I_3^k we have (for example) $\varphi^+(I_1^k)+\varphi^-(I_2^k)+\varphi^+(I_3^k) \leq C_2$, and also C_2 does not depend on k. On the other hand when $k \to +\infty$ there must be three consecutive annuli $[r_{i-1}^k, r_i^k]$, $[r_i^k, r_{i+1}^k]$ and $[r_{i+1}^k, r_{i+2}^k]$ (i possibly depending on k) with either $|r_{i+2}^k - r_{i-1}^k| \to 0$ or $r_{i-1}^k \to +\infty$; in both cases this means (for example) $\varphi^+(r_{i-1}^k, r_i^k) + \varphi^-(r_i^k, r_{i+1}^k) \to +\infty$. We choose $\bar{k}$ such that $\forall k \geq \bar{k}$ it holds both that

$$\varphi^+(r_{i-1}^k, r_i^k) + \varphi^-(r_i^k, r_{i+1}^k) > C_2 + 1$$

and that $|r_{i+2}^k - r_{i-1}^k| < \bar{\delta}$ or $r_{i-1}^k > \bar{R}$. For such k's we easily obtain a contradiction: indeed the partition obtained from the extremal by cutting r_i^k, r_{i+1}^k and placing two zeroes inside I^k strictly decreases the value of c_k. □

5. A NONEXISTENCE RESULT

In this final section we will prove Theorem 2.1, using an energy estimate. Let us recall that such a theorem essentially states the optimality of the supercritical assumption $\theta + 1 > p_\alpha$ when searching for nontrivial solutions that do change sign (here and below $h \equiv 0$). To this aim we use some Pucci–Serrin-type estimate (see [15]).

Proof of Theorem 2.1. Let $\theta + 1 \leq p_\alpha$. Assume for the sake of contradiction the existence of a nontrivial solution $u \in H$ of (P) such that $u(R) = 0$; note that, since $h \equiv 0$ and $u \not\equiv 0$, it must hold that $\dot{u}(R) \neq 0$. We introduce the energy associated to the problem (P):

$$E(r) := \frac{1}{2}\dot{u}^2 - \frac{1}{2}r^{-\alpha}u^2 + \frac{1}{\theta+1}|u|^{\theta+1}.$$

By writing the ODE corresponding to (P) in the following form,

$$r^{1-N}\frac{d}{dr}(r^{N-1}\dot{u}) - r^{-\alpha}u + |u|^{\theta-1}u = 0, \tag{5.1}$$

we obtain the derivative of E as

$$\frac{d}{dr}E(r) = -\frac{(N-1)}{r}\dot{u}^2 - \frac{1}{2} - \alpha r^{-\alpha-1}u^2. \tag{5.2}$$

By testing (5.1) with $r^{\beta-1}u$ ($\beta \geq 1$ to be chosen) and then integrating by parts on (R, ∞) we obtain

$$\begin{aligned} &[r^{\beta-1}u\dot{u}]_R^\infty - (\beta - N)\int_R^\infty r^{\beta-2}u\dot{u}\,dr \\ &\qquad - \int_R^\infty r^{\beta-1}\dot{u}^2 dr + \int_R^\infty r^{\beta-1}|u|^{\theta+1}dr = 0. \end{aligned} \tag{5.3}$$

Since $u\dot{u} = \frac{1}{2}\frac{d}{dr}u^2$, integrating by parts again the second term we have

$$\int_R^\infty r^{\beta-1}\dot{u}^2 dr + \int_R^\infty r^{\beta-1-\alpha}u^2 dr = [r^{\beta-1}u\dot{u}]_R^\infty - \frac{1}{2}(\beta - N)[r^{\beta-2}u^2]_R^\infty$$
$$+ \frac{1}{2}(\beta - N)(\beta - 2)\int_R^\infty r^{\beta-3}u^2 dr + \int_R^\infty r^{\beta-1}|u|^{\theta+1}dr. \tag{5.4}$$

Let us now define

$$E_\beta(r) := r^\beta E(r). \tag{5.5}$$

By (5.1) it holds that

$$E_\beta(\infty) - E_\beta(R) = \int_R^\infty \frac{d}{dr}E_\beta(r)dr = \left(\frac{\beta}{2} - N + 1\right)\int_R^\infty r^{\beta-1}\dot{u}^2 dr$$
$$+(\alpha - \beta)\int_R^\infty r^{\beta-1-\alpha}u^2 dr + \frac{\beta}{\theta+1}\int_R^\infty r^{\beta-1}|u|^{\theta+1}dr. \tag{5.6}$$

Now we multiply (5.4) by $\frac{\beta}{\theta+1}$ and then sum up with (5.6):

$$\big(-N + 1 + \frac{\beta}{2} + \frac{\beta}{\theta+1}\big)\int_R^\infty r^{\beta-1}\dot{u}^2 dr + \big(\frac{\alpha}{2} - \frac{\beta}{2} + \frac{\beta}{\theta+1}\big)\int_R^\infty r^{\beta-1-\alpha}u^2 dr$$
$$-\frac{\beta(\beta-N)(\beta-2)}{2(\theta+1)}\int_R^\infty r^{\beta-3}u^2 dr = A_{R,\infty} + E_\beta(\infty) - E_\beta(R),$$

where $A_{R,\infty} = [r^{\beta-1}u\dot{u}]_R^\infty - \frac{1}{2}(\beta - N)[r^{\beta-2}u^2]_R^\infty$. Note that the terms in $A_{R,\infty}$ involving R are zero by the assumption $u(R) = 0$. Let us choose $\beta = N - 1 + \alpha/2$, and hence $2 < \beta < N$. By the fact that $u \in H$ and $\beta < N$ we obtain the existence of a sequence $r_n \to +\infty$ such that $E_\beta(r_n) \to 0$. For the same reason also the terms in $A_{R,\infty}$ involving u and $\dot{u}$ at infinity are zero. Note moreover that the coefficient $-\frac{\beta(\beta-N)(\beta-2)}{2(\theta+1)}$ is positive since $2 < \beta < N$. Thus

$$a\int_R^\infty r^{\beta-1}\dot{u}^2 dr + b\int_R^\infty r^{\beta-1-\alpha}u^2 dr \le -E_\beta(R), \tag{5.7}$$

where

$$a = b = \frac{N - 1 - \frac{\alpha}{2}}{2(\theta+1)} \cdot [p_\alpha - (\theta+1)].$$

Now, since $\theta + 1 \le p_\alpha$, we obtain $a \ge 0$ and $b \ge 0$; furthermore $E(R) > 0$ by assumption. This leads to a contradiction of (5.7). □

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