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MULTIPLICITY RESULTS IN A BALL FOR p-LAPLACE EQUATION WITH POSITIVE NONLINEARITY

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Abstract. We consider the equation $-\Delta_p u = u^\alpha + u^q$ where $0 \leq q < p-1 < \alpha \leq p^* - 1$ in the ball $B_R(0) \subset \mathbb{R}^N$, $N \geq 2$. Here, $p^* = Np/(N-p)$. We show the existence of at least two positive solutions to the above equation for small enough balls when $\alpha = p^* - 1$ and $q > 0$. Further if $p \in (1, 2)$ and $\alpha \leq p^* - 1$, we show the existence of exactly two positive solutions for small enough balls when $q > 0$, and at most two solutions when $q = 0$. This we do by the asymptotic analysis of the corresponding Emden-Fowler equation.

1. Introduction

For $R > 0$, let $B_R(0)$ denote the open ball of radius R centered at the origin in $\mathbb{R}^N$, $N \geq 2$. Let $p \in (1, N)$ and define $\nu = \frac{N-p}{p-1}$, $k = p(\frac{N-1}{N-p})$, $l = \frac{k-1}{p-1}$. The critical exponent is $p^* = \frac{Np}{N-p}$ which we can rewrite as $p^* = \frac{p}{p-1}(k-1) = pl$. For real numbers α, q such that $0 \leq q < p - 1 < \alpha \leq p^* - 1$, we consider the following problem:

$$\left.\begin{array}{rcl} -\Delta_p u & = & u^\alpha + u^q \\ u & > & 0 \end{array}\right\} \text{ in } B_R(0), \qquad u = 0 \text{ on } \partial B_R(0). \tag{1.1}$$

Here, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the usual p-Laplace operator. We are interested in the question of existence of multiple solutions to (1.1) and, if possible, to get exact multiplicity results. From the Gidas-Ni- Nirenberg type symmetry results shown in Theorem D of [8], any solution of (1.1) is radial about the origin and strictly radially decreasing about the origin.

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Therefore, it is sufficient to study the equivalent (in our case!) Ordinary Differential Equation (ODE) version of (1.1) given below:

$$(P_R) \qquad \begin{cases} -\left(r^{N-1}|u'|^{p-2}u'\right)' = r^{N-1}(u^\alpha + u^q) \\ u > 0 \end{cases} \quad r \in (0,R), \qquad u'(0) = 0,\ u(R) = 0.$$

In fact, we will prove the following two theorems subsequently:

Theorem I. *Let $\alpha = p^* - 1$ and $0 < q < p - 1$. Then there exists a real number $R_* > 0$ such that for all $R \in (0, R_*)$ the problem (P_R) admits at least two solutions and no solution for any $R > R_*$.*

Theorem II. *Let $p \in (1,2)$. Then*

(i) *for any numbers q, α such that $0 < q < p - 1 < \alpha \le p^* - 1$, we can find $R_{**} \in (0, R_*)$ such that (P_R) admits exactly two solutions for all $R \in (0, R_{**})$.*

(ii) *If $q = 0$ and $p - 1 < \alpha \le p^* - 1$, then we may find $R_{**} > 0$ such that there exist at most two solutions of (P_R) for all $R \in (0, R_{**})$.*

We now provide a brief account of what is known about problems of the type (P_R). In the pioneering work of Ambrosetti-Brezis-Cerami [3], it was shown for the first time that a combination of convex and concave nonlinearities can give rise to multiple solutions on a general domain in $\mathbb{R}^N$. More precisely, they consider the following problem for $\lambda > 0$, $\Omega \subset \mathbb{R}^N (N \ge 2)$ a bounded domain:

$$\left.\begin{aligned} -\Delta u &= u^\alpha + \lambda u^q \\ u &> 0 \end{aligned}\right\} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

where $0 < q < 1 < \alpha \le 2^* - 1$. They established the following facts (see Theorems 2.1 and 2.3) in [3]:

There exists a real number $\Lambda > 0$ such that

(i) *for any $\lambda > \Lambda$, (1.2) has no solution,*

(ii) *for any $\lambda \in (0, \Lambda)$, (1.2) has at least two solutions.*

Subsequently, in the important work of Ambrosetti-Azorero-Peral [2], the above results were extended to the case of the p-Laplace equation in a ball in $\mathbb{R}^N, N \ge 2$. The equation now is:

$$(P_\lambda) \qquad \begin{cases} -\left(r^{N-1}|u'|^{p-2}u'\right)' = r^{N-1}(u^\alpha + \lambda u^q) \\ u > 0 \end{cases} \quad r \in (0,1), \qquad u'(0) = 0,\ u(1) = 0,$$

where $0 < q < p - 1 < \alpha \le p^* - 1$. Then the following results were shown in [2] (see Lemma 3.6, Theorem 3.10 and Theorem 3.12):

Let $\alpha < p^* - 1$. There exists a real number $\Lambda > 0$ such that

(i) *for all $\lambda > \Lambda, (P_\lambda)$ does not admit any solution,*

(ii) *for all $\lambda \in (0, \Lambda)$ there exist at least two solutions,*

(iii) *let v_0 be the unique solution of (P_0). Then there exists a bifurcation branch (i.e., a continuum) of positive solutions of (P_λ) emanating from $(0, 0)$ which contains v_0 for $\lambda = 0$.*

We remark that in [2], the restriction to the ball is required only to ensure a priori estimate for solutions (see Lemma 3.8). Otherwise, the methods in [2] are applicable to general domains in $\mathbb{R}^N$. The situation when (P_λ) is posed on a general bounded domain in $\mathbb{R}^N$ and when the exponent $\alpha = p^* - 1$ was studied by Azorero-Peral[6]. They proved the following theorem:

Let $\alpha = p^* - 1$. Then there exists a constant $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, (P_λ) posed on a smooth bounded domain in $\mathbb{R}^N$ admits at least two positive solutions provided either one of the following conditions holds:

(i) $\frac{2N}{N+2} < p < 3,\ 0 < q < p - 1.$

(ii) $p \geq 3, p - 1 > q > q_0 = p^* - \frac{2}{p-1} - 1.$

Remark. In the above theorem, $p - 1 > q_0$ implies the following restriction for the dimension: $N > p(1 + \frac{p(p-1)}{2})$. Also $\frac{2N}{N+2} < p$ is equivalent to $p^* > 2$. Further we note that the above theorem does not ensure multiplicity for all λ on a maximal interval.

Suppose u is a solution of (P_λ). We do the following transformation

$$r \longrightarrow \lambda^{-\frac{(\alpha-p+1)}{p(\alpha-q)}} r, \quad w(r) = \lambda^{-\frac{1}{\alpha-q}} u\big(\lambda^{-\frac{(\alpha-p+1)}{p(\alpha-q)}} r\big), \quad r < \lambda^{\frac{\alpha-p+1}{p(\alpha-q)}}. \tag{1.3}$$

Then, it is easy to see that w solves the problem (P_R) with $R = \lambda^{\frac{\alpha-p+1}{p(\alpha-q)}}$ and conversely. Thus the results of [2] and [6] imply:

(i) *If $\alpha < p^* - 1$, we can find a real number $R_* > 0$ such that the problem (P_R) admits no solution for all $R > R_*$ and at least two solutions for all $R \in (0, R_*)$.*

(ii) *If $\alpha = p^* - 1$, there exists $R_0 > 0$ such that the problem (P_R) admits at least two solutions for all $R \in (0, R_0)$ provided either $\frac{2N}{N+2} < p < 3,\ 0 < q < p - 1$, or $p \geq 3, p - 1 > q > q_0 = p^* - \frac{2}{p-1} - 1$.*

From the results quoted above it is clear that the question of multiplicity of solutions to (P_R) for R in a maximal interval and when $\alpha = p^* - 1, 0 < q < p - 1$ for any $p > 1$ has been left open. We address this question completely in Theorem I where we use asymptotic analysis of solutions to the "Emden-Fowler" equation obtained from (P_R). This asymptotic analysis is done for the case of the Laplace operator in the beautiful work of [5].

Regarding exact multiplicity results for (P_R), in the case of Laplace equation i.e., when $p = 2$, the following result was obtained in [1] (see theorem 1.1):

Let $0 < q < 1 < \alpha \leq 2^* - 1$. *Then there exists* $R_{**} < R_*$ *such that* (P_R) *admits exactly two solutions for all* $R \in (0, R_{**})$. *If* $q = 0, (P_R)$ *admits at most two solutions for all* $R \in (0, R_{**})$.

In Theorem II, we extend the above exact multiplicity result to the case of p-Laplace operator when $p \in (1, 2)$.

Let u be a solution of (P_R). We make the Emden-Fowler transformation:

$$t = (\frac{\nu}{r})^{\nu},\ y(t) = u(r). \tag{1.4}$$

Then, it is easy to check that y satisfies the Emden-Fowler equation:

$$(P_T) \qquad \begin{cases} -\left(|y'|^{p-2}y'\right)' = t^{-k}(y^{\alpha} + y^{q}) \\ y > 0 \end{cases} \Big\}\ t \in (T, \infty) \\ \qquad\qquad y(T) = 0,\ y'(\infty) = 0,$$

where $T = (\frac{\nu}{R})^{\nu}$. To study the multiplicity of solutions to (P_R) it is natural to consider the following version of (P_T) which is parametrized by a parameter $\gamma > 0$:

$$(P_\gamma) \qquad -\left(|y'|^{p-2}y'\right)' = t^{-k}(y^{\alpha} + y^{q}), \quad \lim_{t \longrightarrow \infty} y(t) = \gamma,\ y'(\infty) = 0.$$

The above "initial value" problem with the initial data prescribed at infinity admits a unique solution thanks to Proposition A4 in [10]. Let $y(t, \gamma)$ denote any solution of (P_γ). Define $T(\gamma) = \inf\{t > 0;\ y(., \gamma) > 0 \text{ in } (t, \infty)\}$. Then it is easy to see that $T(\gamma)$ is the first zero of $y(t, \gamma)$ as t is decreased from infinity. From continuous dependence on initial values we note that $T(\gamma)$ is a continuous function on $(0, \infty)$. It is also easy to see by integration that $y(., \gamma)$ is a strictly concave function in $(T(\gamma), \infty)$.

2. Preliminary lemmas

In this section we prove some lemmas that are necessary for the proof of Theorems I and II. We also remark that the proofs of Lemmas 2.1-2.4 are inspired by the analysis for $p = 2$ in Atkinson- Peletier [5]. For Lemmas 2.1-2.4 we fix $\alpha = p^* - 1$. Let $f(s) = s^{p^*-1} + s^q, q \in (0, p - 1)$ and define for $\gamma > 0$,

$$z(t, \gamma) = \gamma t\Big(t^{l-1} + \gamma^{-1}\big(\frac{f(\gamma)}{k-1}\big)^{\frac{1}{p-1}}\Big)^{-\frac{1}{l-1}},\ t \geq 0. \tag{2.1}$$

Then, it is easy to see that

$$sf'(s) - (p^* - 1)f(s) < 0,\ \ \forall\ s > 0. \tag{2.2}$$

Lemma 2.1. *Let $y(t,\gamma)$ be a solution of (P_γ). Then $y(t,\gamma) < z(t,\gamma)$ for $T(\gamma) < t < \infty$.*

Proof. Let y be any solution of (P_γ). Define

$$H(t) = t(y')^p - y(y')^{p-1} + \frac{1}{k-1} t^{1-k} y f(y).$$

Computing $H'(t)$ and using (2.2) we get,

$$H'(t) = \frac{t^{1-k}y'}{k-1}\{yf'(y) - (pl-1)f(y)\} < 0 \ \text{ on } (T(\gamma),\infty). \tag{2.3}$$

Integrating the ODE in (P_γ) from t to ∞ , for a fixed $\gamma > 0$, we get

$$(y')^{p-1}(t) = \int_t^\infty s^{-k} f(y(s)) ds.$$

This implies

$$(y')^{p-1}(t) = O(t^{-k+1}) \ \text{ as } \ t \to \infty.$$

That is,

$$(y')(t) = O(t^{-(\frac{k-1}{p-1})}) = O(t^{-l}) \ \text{ as } \ t \to \infty.$$

Therefore, for fixed $\gamma > 0$,

$$H(t) = O(t^{1-pl}) + O(t^{-l(p-1)}) + O(t^{1-k}) \ \text{ as } t \to \infty.$$

Since $l > 1$, we obtain that $1 - pl < -l(p-1)$. Hence,

$$H(t) = O(t^{1-pl}) + O(t^{1-k}) \ \text{ as } t \to \infty.$$

Therefore, $H(\infty) = 0$. Since by (2.3), H is strictly decreasing in $(T(\gamma),\infty)$, it follows that $H(t) > 0 \ \ \forall t \in [T(\gamma),\infty)$. We now note that

$$((y')^{p-1}t^{k-1}y^{1-k})' = -(k-1)t^{k-2}y^{-k}H(t) < 0 \ \ \forall t \in [T(\gamma),\infty). \tag{2.4}$$

Therefore,

$$(y')^{p-1}(t)t^{k-1}y^{1-k}(t) > \lim_{s\to\infty}(y'(s))^{p-1}s^{k-1}y(s)^{1-k} = \gamma^{1-k}\lim_{s\to\infty}\frac{(y'(s))^{p-1}}{s^{1-k}}.$$

By L'Hospital's rule the above limit is evaluated as,

$$\lim_{s\to\infty}\frac{(y'(s))^{p-1}}{s^{1-k}} = \lim_{s\to\infty}\frac{((y'(s))^{p-1})'}{(1-k)s^{-k}} = \frac{1}{k-1}f(\gamma).$$

Therefore,

$$(y')^{p-1}(t)t^{k-1}y^{1-k}(t) > \frac{\gamma^{1-k}}{k-1}f(\gamma).$$

From the above inequality, we rearrange the terms and note that $l = \frac{k-1}{p-1}$ to get,

$$y'y^{-l} > (\gamma t)^{-l}\Big(\frac{f(\gamma)}{k-1}\Big)^{\frac{1}{p-1}}, \quad t \in (T(\gamma), \infty).$$

Integrating the above inequality from t to ∞ we get,

$$\frac{\gamma^{1-l}}{1-l} - \frac{y^{1-l}(t)}{1-l} > \Big(\int_t^\infty s^{-l} ds\Big)\gamma^{-l}\Big(\frac{f(\gamma)}{k-1}\Big)^{\frac{1}{p-1}}.$$

Rearranging the above inequality, we obtain,

$$y(t) < t\gamma\Big[t^{l-1} + \gamma^{-1}\Big(\frac{f(\gamma)}{k-1}\Big)^{\frac{1}{p-1}}\Big]^{-\frac{1}{l-1}}, \ T(\gamma) \le t < \infty.$$

This proves the lemma. □

Lemma 2.2. $T(\gamma) > 0, \ \ \forall \gamma > 0.$

Proof. By the definition of $T(\gamma)$, $T(\gamma) \ge 0$. We assume that $T(\gamma) = 0$ for some $\gamma > 0$ and derive a contradiction. Fix one such γ. Note that, in view of the ODE in (P_γ), necessarily $y(0) = 0$. By Lemma 2.1, we have $y(t) < z(t)$ on $(0, \infty)$. Further, we may find $C > 0$ so that $z(t) \le Ct$ on $[0, \infty)$. Also since $y'(\infty) = 0$, we may choose C larger so that additionally, $y' \le C$ on $[0, \infty)$. Then, as $t \to 0$,

$$\begin{aligned} H(t) &= t(y')^p - y(y')^{p-1} + \frac{1}{k-1} t^{1-k} y f(y) \\ &= O(t) + O(t) + O(t^{1-k})O(t)O(t^q) = O(t) + O(t^{2-k+q}). \end{aligned}$$

If $2 - k + q > 0$, we get $\lim_{t \to 0} H(t) = 0$. Since $H(\infty) = 0$ and $H'(t) < 0$ on $[0, \infty)$, we get a contradiction.

Suppose now $2 - k + q \le 0$. We may find a constant $c > 0$ so that $y(t) \ge ct \ \ \forall t > 0$ small. Therefore, for all $t > 0$ small enough,

$$t^{-k} f(y(t)) \ge \lambda t^{-k} c^q t^q = \lambda c^q t^{q-k}.$$

Now using the ODE in (P_γ) we get,

$$(p-1)(y')^{p-2} y'' = -t^{-k} f(y) \le -\lambda c^q t^{q-k} \tag{2.5}$$

for all $t > 0$ small. Now, let η_1, η_2 be two positive constants such that $\eta_1 \le y'(t) \le \eta_2$ for all $t > 0$ small enough. From (2.5), there exists a constant $\theta > 0$ such that

$$y''(t) \le -\Big[\Big(\frac{\lambda c^q}{p-1}\Big)\theta\Big] t^{q-k} = -\mu t^{q-k} \text{ (say!)} \tag{2.6}$$

for all $t > 0$ small enough. Fix $t_0 > 0$ small enough and let $t \in (0, t_0)$. Integrating (2.6) between t and t_0,

$$y'(t_0) - y'(t) \leq -\frac{\mu t_0^{q-k+1}}{q-k+1} + \frac{\mu t^{q-k+1}}{q-k+1}.$$

From the above inequality, for some constant A, we obtain,

$$y'(t) \geq \frac{-\mu t^{q-k+1}}{(q-k+1)} + A.$$

Integrating the above inequality once more we get,

$$y(t_0) - y(t) \geq \begin{cases} \frac{-\mu}{(q-k+1)(q-k+2)}(t_0^{q-k+2} - t^{q-k+2}) + A(t - t_0) & \text{if } q-k+2 < 0, \\ \frac{-\mu}{(q-k+1)} \log(\frac{t_0}{t}) + A(t - t_0) & \text{if } q-k+2 = 0. \end{cases}$$

Therefore, for some constant B,

$$y(t) \leq \begin{cases} \frac{-\mu t^{q-k+2}}{(q-k+1)(q-k+2)} + B & \text{if } q-k+2 < 0, \\ \frac{\mu}{(q-k+1)} \log\left(\frac{t_0}{t}\right) + B & \text{if } q-k+2 = 0. \end{cases}$$

Therefore, $\lim_{t\to 0} y(t) = -\infty$, a contradiction. This contradiction proves the lemma. □

Define $S(\gamma)$ by

$$S^{l-1}(\gamma) = \gamma^{-1}\left(\frac{f(\gamma)}{k-1}\right)^{\frac{1}{p-1}}.$$

Lemma 2.3. *For each fixed $\epsilon > 0$, we may find a positive function d_ϵ such that $d_\epsilon(\gamma) \to 0$ as $\gamma \to \infty$ and the following inequality is true:*

$$y(t,\gamma) \geq (1 - d_\epsilon(\gamma))z(t,\gamma) \quad \textit{for} \quad t \geq \max\{T(\gamma), \epsilon S(\gamma)\}.$$

Proof. Integrating the ODE in (P_γ) we obtain,

$$(y')^{p-1}(t) = \int_t^\infty s^{-k} y^{p^*-1}(s)ds + \int_t^\infty s^{-k} y^q(s)ds.$$

Since by Lemma 2.1, $y(t) < z(t)$ for $t \geq T(\gamma)$, we get,

$$(y')^{p-1}(t) \leq \int_t^\infty s^{-k} z^{p^*-1}(s)ds + \int_t^\infty s^{-k} z^q(s)ds = I_1(t) + I_2(t).$$

Now a simple computation gives that, $z(\epsilon S(\gamma), \gamma) = \eta(\epsilon)\gamma$, where $\eta(\epsilon) = \epsilon(1 + \epsilon^{l-1})^{-\frac{1}{l-1}}$. Since $t \geq \epsilon S(\gamma)$ and z is increasing, we get

$$I_1(t) \geq (\eta(\epsilon)\gamma)^{p^*-q-1} \int_t^\infty s^{-k} z^q(s)ds = (\eta(\epsilon)\gamma)^{p^*-q-1} I_2(t).$$

Therefore,

$$\big(y'(t)\big)^{p-1} \le \Big(1 + (\eta(\epsilon)\gamma)^{q-p^*+1}\Big) \int_0^\infty s^{-k} z^{p^*-1}(s)ds. \tag{2.7}$$

We note that z satisfies the differential equation

$$-\big((z')^{p-1}(t)\big)' = t^{-k}\gamma^{-p^*+1} f(\gamma) z^{p^*-1}(t).$$

Integrating the above differential equation from t to ∞ , we obtain

$$(z')^{p-1}(t) = \Big(1 + \gamma^{q-p^*+1}\Big) \int_t^\infty s^{-k} z^{p^*-1}(s)ds. \tag{2.8}$$

Therefore, letting

$$\theta(c,\gamma) = \frac{1}{(1 + (c\gamma)^{q-p^*+1})},$$

we obtain from (2.7) and (2.8) that $\theta(\eta(\epsilon),\gamma)(y')^{p-1} - \theta(1,\gamma)(z')^{p-1} \le 0$. Therefore,

$$y'(t) \le \Big[\frac{\theta(1,\gamma)}{\theta(\eta(\epsilon),\gamma)}\Big]^{\frac{1}{p-1}} z'(t), \quad \forall t \ge \max\{\epsilon S(\gamma), T(\gamma)\}.$$

Integrating the above inequality once more between t and ∞,

$$\gamma - y(t) \le \left[\frac{\theta(1,\gamma)}{\theta(\eta(\epsilon),\gamma)}\right]^{\frac{1}{p-1}} (\gamma - z(t)). \tag{2.9}$$

Now we note that $[\frac{\theta(1,\gamma)}{\theta(\eta(\epsilon),\gamma)}]^{\frac{1}{p-1}} < 1$ for ϵ small enough. Let $d_\epsilon(\gamma) = 1 - [\frac{\theta(1,\gamma)}{\theta(\eta(\epsilon),\gamma)}]^{\frac{1}{p-1}}$. Then (2.9) implies $y(t) \ge (1 - d_\epsilon(\gamma))z(t) + d_\epsilon(\gamma)\gamma \ge (1 - d_\epsilon(\gamma))z(t)$. It is now easy to see that $d_\epsilon(\gamma) \to 0$ as $\gamma \to \infty$ for fixed $\epsilon > 0$ which proves the lemma. □

Lemma 2.4. *Let $\sigma = \frac{p}{p-1}$ and suppose for some sequence $\gamma \to \infty$, we have $T(\gamma) < a\gamma^\sigma$ for any $a > 0$. Then the following estimates hold for some positive constants k_1, K, L which depend on k and q only:*

(a) *$y(t,\gamma) < k_1\gamma^{1-\sigma} t$ for $t > T(\gamma)$.*
(b) *$y(2T(\gamma),\gamma) = o(\gamma)$ as $\gamma \to \infty$.*
(c) *$y(\gamma^\sigma,\gamma) > K\gamma$ for all large γ.*
(d) *$K\gamma^{1-\sigma} < y'(2T(\gamma),\gamma) < L\gamma^{1-\sigma}$ for all large γ.*
(e) *$K\gamma^{1-\sigma}(t - T(\gamma)) \le y(t,\gamma) \le L\gamma^{1-\sigma}(t - T(\gamma))$ for $2T(\gamma) \le t \le \gamma^\sigma$ and γ sufficiently large.*

Proof. **(a)** From Lemma 2.1, we know that $y(t,\gamma) < z(t,\gamma)$, for $t \ge T(\gamma)$ and the definition of $z(t,\gamma)$ gives that $z(t,\gamma) \le k_1 t\gamma^{1-\sigma}$ for some constant k_1.

(b) From (a), we get $y(2T(\gamma),\gamma) < 2k_1(\gamma^{-\sigma}T(\gamma))\gamma = o(\gamma)$.

(c) We note that $\gamma^{p^*-1} \le f(\gamma)$. From the definition of $z(t,\gamma)$ we have, for some $c>0$,

$$z(\gamma^\sigma,\gamma)=\gamma^{\sigma+1}\Big(\gamma^{\sigma(l-1)}+\gamma^{-1}\big(\frac{f(\gamma)}{k-1}\big)^{\frac{1}{p-1}}\Big)^{-\frac{1}{l-1}}\ge\gamma\Big(1+c\gamma^{\sigma(1-l)+\frac{p^*-1}{p-1}-1}\Big)^{-\frac{1}{l-1}}.$$

We note that $-\sigma(l-1)+\frac{p^*-1}{p-1}-1=0$. Therefore, $z(\gamma^\sigma,\gamma)\ge\gamma(1+c)^{-\frac{1}{l-1}}$. Since $z(\epsilon S(\gamma),\gamma)=O(\epsilon)\gamma$, we may choose ϵ small so that $\gamma^\sigma\ge\epsilon S(\gamma)$. By Lemma 2.3, we have

$$y(\gamma^\sigma,\gamma)\ge(1-d_\epsilon(\gamma))z(\gamma^\sigma,\gamma)\ge K\gamma,$$

for some $K>0$ and all large γ.

(d) Choose γ large so that $2T(\gamma)<\gamma^\sigma$. Then by strict concavity of y, the following two inequalities hold:

$$y'(2T(\gamma))<\frac{y(2T(\gamma))-y(T(\gamma))}{T(\gamma)},\quad y'(2T(\gamma))>\frac{y(\gamma^\sigma)-y(2T(\gamma))}{\gamma^\sigma-2T}.$$

From the first inequality we obtain, using (a),

$$y'(2T(\gamma))<\frac{y(2T(\gamma))}{T(\gamma)}<k_1\gamma^{1-\sigma}.$$

Using (b) and (c) in the second inequality we obtain,

$$y'(2T(\gamma))>\frac{K\gamma-o(\gamma)}{\gamma^\sigma-2T}$$

Since, by assumption, $T(\gamma)=o(\gamma^\sigma)$, we obtain that for some $K>0$

$$y'(2T(\gamma),\gamma)>K\gamma^{1-\sigma}.$$

(e) By (a), we get

$$y(t,\gamma)\le\frac{k_1}{\epsilon}t\epsilon\gamma^{1-\sigma}\le(\frac{k_1}{\epsilon})\gamma^{1-\sigma}(t-T(\gamma))\ \ \forall t\in[2T(\gamma),\gamma^\sigma],$$

if ϵ is chosen small enough. This proves the right inequality in (e). Again by concavity of y, we get

$$\frac{y(t)-y(T(\gamma))}{t-T(\gamma)}\ge\frac{y(\gamma^\sigma)-y(T(\gamma))}{\gamma^\sigma-T(\gamma)}.$$

That is, by (c),

$$y(t)\ge y(\gamma^\sigma)\Big[\frac{t-T(\gamma)}{\gamma^\sigma-T(\gamma)}\Big]>\frac{K\gamma(t-T(\gamma))}{\gamma^\sigma-T}.$$

Since ,by assumption, $T(\gamma)=o(\gamma^\sigma)$, the left inequality in (e) follows. □

We now show that the solutions of (P_R) are ordered.

Lemma 2.5. *Let* $0 \le q \le p-1 < \alpha \le p^*-1$ *and let* $f(u) = u^\alpha + u^q$*. Suppose* u_1, u_2 *are any two solutions of* (P_R) *with* $u_1(0) < u_2(0)$*. Then* $u_1 < u_2$ *in* $[0, R)$.

Proof. Let $\xi \in (0, R]$. We write the Pohožaev identity for the problem (P_R) (see [9], equation (2.5)):

$$\Big(\frac{pN}{\alpha+1} - (N-p)\Big)\int_0^r t^{N-1}|u'|^p + \frac{pN(\alpha-q)}{(\alpha+1)(q+1)}\int_0^r t^{N-1}u^{q+1}$$
$$= \frac{p}{\alpha+1}r^N u^{\alpha+1}(r) + \frac{p}{q+1}r^N u^{q+1}(r) + (p-1)r^N|u'|^p(r)$$
$$+ \frac{Np}{\alpha+1}r^{N-1}|u'|^{p-2}u'(r)u(r). \tag{2.10}$$

Suppose the conclusion of the lemma is not true. Then there are two possibilities: There exists $r_0 \in (0, R)$ such that

case (1) $u_1(r_0) = u_2(r_0)$ and $u_1 \le u_2$ in $(0, R)$

case (2) $\frac{u_1}{u_2}(r_0) > 1$.

If case (1) holds, we see that $\frac{u_1}{u_2}$ achieves the global maximum at r_0 and hence, $(\frac{u_1}{u_2})'(r_0) = 0$. That is, $u_1'(r_0) = u_2'(r_0)$. Since $u_1(r_0) = u_2(r_0)$, we get a contradiction by appealing to the uniqueness theory of ODEs.

Suppose case (2) holds. Let ξ_0 be the first local maximum of $\frac{u_1}{u_2}$ with $(\frac{u_1}{u_2})(\xi_0) > 1$, if it exists, in $(0, R)$. Otherwise, let $\xi_0 = R$. Let $t_0 = \frac{u_1(\xi_0)}{u_2(\xi_0)}$. If $\xi_0 = R$, by L'Hospital's rule, we let $t_0 = \frac{u_1'(R)}{u_2'(R)}$. If $\xi_0 < R$, clearly, $(\frac{u_1}{u_2})'(\xi_0) = 0$. Therefore, we have

$$1 < t_0 = \frac{u_1(\xi_0)}{u_2(\xi_0)} = \frac{u_1'(\xi_0)}{u_2'(\xi_0)}, \quad u_1(r) < t_0 u_2(r) \ \text{ for all } \ r \in [0, \xi_0). \tag{2.11}$$

Claim 1: $\frac{u_1'(r)}{u_2'(r)} < t_0$ for all $r \in [0, \xi_0)$.

Let $r_1 < \xi_0$ be such that $u_1(r_1) = u_2(r_1)$. Since $\xi_0 > r_1$ is the first local maximum of $\frac{u_1}{u_2}$, we have $(\frac{u_1}{u_2})' \ge 0$ in $[r_1, \xi_0)$. This gives

$$\frac{u_1'(r)}{u_2'(r)} \le \frac{u_1(r)}{u_2(r)} \ \text{ for all } \ r \in [r_1, \xi_0).$$

Thus, from (2.11), we have

$$\frac{u_1'(r)}{u_2'(r)} < t_0 \ \text{ for all } r \in [r_1, \xi_0).$$

Thus it remains to prove the claim only when $r \in [0, r_1)$. Integrating the ODE in (P_R), we obtain,

$$-|u'|^{p-2}(r)u'(r) = r^{1-N} \int_0^r t^{N-1}(u^\alpha(t) + u^q(t))dt \ .$$

Therefore, for any $r \in (0, r_1)$,

$$\frac{|u_1'|^{p-2}(r)u_1'(r)}{|u_2'|^{p-2}(r)u_2'(r)} = \frac{\int_0^r t^{N-1}(u_1^\alpha(t) + u_1^q(t))dt}{\int_0^r t^{N-1}(u_2^\alpha(t) + u_1^q(t))dt} < 1 \tag{2.12}$$

since $u_1(r) < u_2(r)\ \forall\ r \in [0, r_1)$. Now note that the map $t \longrightarrow |t|^{p-2}t$ is strictly increasing. Therefore, we obtain, by noting that $u_1', u_2' < 0$, that $\frac{u_1'(r)}{u_2'(r)} < 1$ for all $r \in (0, r_1)$. When $r = 0$ we apply L'Hospital's rule to (2.12) to conclude

$$\lim_{r\to 0} \frac{|u_1'|^{p-2}(r)u_1'(r)}{|u_2'|^{p-2}(r)u_2'(r)} = \frac{u_1^\alpha(0) + u_1^q(0)}{u_2^\alpha(0) + u_2^q(0)} < 1.$$

This proves Claim 1.

Claim 2: $\xi_0 < R$. Suppose $\xi_0 = R$. Then from (2.10) with $u = u_i, i = 1, 2$, we obtain,

$$\Big(\frac{pN}{\alpha+1} - (N-p)\Big) \int_0^R t^{N-1}|u_i'|^p + \frac{pN(\alpha-q)}{(\alpha+1)(q+1)} \int_0^R t^{N-1}u_i^{q+1} = (p-1)R^N|u_i'|^p(R).$$

Therefore,

$$\Big(\frac{pN}{\alpha+1} - (N-p)\Big) \int_0^R t^{N-1}(|u_1'|^p - t_0^p|u_2'|^p) + \frac{pN(\alpha-q)}{(\alpha+1)(q+1)} \int_0^R t^{N-1}(u_1^{q+1} - t_0^p u_2^{q+1})$$
$$= (p-1)R^N(|u_1'|^p(R) - t_0^p|u_2'|^p(R)). \tag{2.13}$$

From (2.11), we get $|u_1'|^p(R) = t_0^p|u_2'|^p(R)$, and $u_1^{q+1}(t) < t_0^{q+1}u_2^{q+1}(t) < t_0^p u_2^{q+1}(t)$ for all $t \in [0, R)$. Also from Claim 1, we get $|u_1'|^p(t) < t_0^p|u_2'(t)|^p$ for all $t \in [0, R)$. Since $\alpha \le \frac{Np}{N-p} - 1$, we can ensure $\frac{pN}{\alpha+1} - (N-p) \ge 0$. Thus (2.13) gives a contradiction if we assume $\xi_0 = R$. This proves Claim 2.

Since $q < p-1$, we have for $r \in (0, \xi_0)$,

$$\left(u_1^{q+1} - t_0^p u_2^{q+1}\right)'(r) = (q+1)(u_1'u_1^q - t_0^p u_2'u_2^q)(r) > (q+1)u_2^q(r)t_0^{p-1}(u_1' - t_0u_2')(r).$$

Since $u_1' - t_0u_2' > 0$ by Claim 1, we obtain from the above equation that $u_1^{q+1} - t_0^p u_2^{q+1}$ is strictly increasing in $(0, \xi_0)$. Thus for any $r \in (0, \xi_0)$,

$$u_1^{q+1}(r) - t_0^p u_2^{q+1}(r) < u_1^{q+1}(\xi_0) - t_0^p u_2^{q+1}(\xi_0). \tag{2.14}$$

Since $\xi_0 < R$ and ξ_0 is local maximum of $\frac{u_1}{u_2}$, we have

$$\Big(\frac{u_1}{u_2}\Big)''(\xi_0) \le 0.$$

Using (2.11), the above inequality gives,

$$u_1''(\xi_0) - t_0 u_2''(\xi_0) \le 0. \tag{2.15}$$

From (2.10) and using (2.11) we get,

$$\begin{aligned}
&\left(\frac{pN}{\alpha+1} - (N-p)\right)\int_0^{\xi_0} t^{N-1}(|u_1'|^p - t_0^p|u_2'|^p)\\
&+ \frac{pN(\alpha-q)}{(\alpha+1)(q+1)}\int_0^{\xi_0} t^{N-1}(u_1^{q+1} - t_0^p u_2^{q+1})\\
&= \frac{p}{\alpha+1}\xi_0^N\left(u_1^{\alpha+1}(\xi_0) - t_0^p u_2^{\alpha+1}(\xi_0)\right) + \frac{p\xi_0^N}{q+1}\left(u_1^{q+1}(\xi_0) - t_0^p u_2^{q+1}(\xi_0)\right).
\end{aligned}$$

We get, using (2.14) in the above equation,

$$\begin{aligned}
&\left(\frac{pN}{\alpha+1} - (N-p)\right)\int_0^{\xi_0} t^{N-1}(|u_1'|^p - t_0^p|u_2'|^p)\\
&> \frac{pN(q-\alpha)}{(\alpha+1)(q+1)}\left(u_1^{q+1}(\xi_0) - t_0^p u_2^{q+1}(\xi_0)\right)\int_0^{\xi_0} t^{N-1}dt\\
&+ \frac{p\xi_0^N}{\alpha+1}\left(u_1^{\alpha+1}(\xi_0) - t_0^p u_2^{\alpha+1}(\xi_0)\right) + \frac{p\xi_0^N}{q+1}\left(u_1^{q+1}(\xi_0) - t_0^p u_2^{q+1}(\xi_0)\right).
\end{aligned}$$

Since $|u_1'|^p - t_0^p|u_2'|^p < 0$ in $(0,\xi_0)$ and $\frac{pN}{\alpha+1} \ge N-p$, we obtain

$$\left(u_1^{\alpha+1}(\xi_0) - t_0^p u_2^{\alpha+1}(\xi_0)\right) + \left(u_1^{q+1}(\xi_0) - t_0^p u_2^{q+1}(\xi_0)\right) < 0.$$

That is, using (2.11),

$$(t_0^{\alpha+1} - t_0^p)u_2^{\alpha+1}(\xi_0) + (t_0^{q+1} - t_0^p)u_2^{q+1}(\xi_0) < 0. \tag{2.16}$$

From the differential equation satisfied by u_i, $i = 1, 2$, we have,

$$\begin{aligned}
&(p-1)\xi_0^{N-1}\Big[|u_1'|^{p-2}(\xi_0)u_1''(\xi_0) - t_0^{p-1}|u_2'|^{p-2}(\xi_0)u_2''(\xi_0)\Big]\\
&+ (N-1)\xi_0^{N-2}\Big[|u_1'|^{p-2}(\xi_0)u_1'(\xi_0) - t_0^{p-1}|u_2'|^{p-2}(\xi_0)u_2'(\xi_0)\Big]\\
&= -\xi_0^{N-1}\left[(u_1^\alpha(\xi_0) + u_1^q(\xi_0)) - t_0^{p-1}\left(u_2^\alpha(\xi_0) + u_2^q(\xi_0)\right)\right].
\end{aligned}$$

Using (2.11) and (2.15), we get

$$\begin{aligned}
0 &\ge (p-1)\xi_0^{N-1}|u_2'|^{p-2}(\xi_0)t_0^{p-2}\left[u_1''(\xi_0) - t_0 u_2''(\xi_0)\right]\\
&= -\xi_0^{N-1}\left[(t_0^\alpha - t_0^{p-1})u_2^\alpha(\xi_0) + (t_0^q - t_0^{p-1})u_2^q(\xi_0)\right].
\end{aligned}$$

That is,

$$(t_0^\alpha - t_0^{p-1})u_2^\alpha(\xi_0) + (t_0^q - t_0^{p-1})u_2^q(\xi_0) \geq 0, \tag{2.17}$$

which contradicts (2.16) and this proves the lemma. □

Remark. In the case of $q \geq p-1$, a similar ordering lemma is proved by Erbe-Tang [9] (see Corollaries 4.3 and 4.4) .

Definition 2.6. A nontrivial solution u of (P_R) (resp. (P_λ)) is called a minimal solution if for any other solution v of (P_R) (resp. (P_λ)), we have $v \geq u$ in $(0, R)$ (resp. in $(0,1)$).

Remark. By the Ordering Lemma 2.5, a minimal solution of (P_R) (resp. (P_λ)), if it exists, has the smallest L^∞ norm among all the solutions of (P_R) (resp. (P_λ)).

Now we consider problem (P_λ) for $0 < q < p-1 < \alpha \leq p^* - 1$ and $\lambda > 0$. From the works of [2] and [7] we know that there exists $\Lambda > 0$ such that for all $\lambda \in (0, \Lambda)$, (P_λ) admits at least one solution. We now have the following:

Lemma 2.7. For any $\lambda \in (0, \Lambda)$, (P_λ) admits a minimal solution u_λ. Moreover, as $\lambda \to 0$, $u_\lambda(0) \sim \lambda^{\frac{1}{p-q-1}}$.

Proof. Let $\lambda \in (0, \Lambda)$ and u be any solution of (P_λ). Let z_λ be the unique (ensured by Theorem A1 in the Appendix) solution of

$$(P_\lambda^*) \qquad \begin{cases} \left.\begin{array}{rcl} -(r^{N-1}|z'|^{p-2}z')' &=& \lambda r^{N-1} z^q \\ z &>& 0 \end{array}\right\} \text{ in } (0,1), \\ z'(0) = 0, z(1) = 0. \end{cases}$$

Then it is easy to see that u is a super-solution of (P_λ^*) and by the Weak Comparison Theorem proved in the Appendix, we obtain $u \geq z_\lambda$ in $(0,1)$. But, letting $w = \lambda^{\frac{-1}{p-q-1}} z_\lambda$, we verify that w solves

$$\left.\begin{array}{rcl} -(r^{N-1}|w'|^{p-2}w')' &=& r^{N-1} w^q \\ w &>& 0 \end{array}\right\} \text{ in } (0,1), \quad w'(0) = 0, w(1) = 0.$$

Hence, $u \geq \lambda^{\frac{1}{p-q-1}} w$. From this inequality it follows that (P_λ) admits a minimal solution u_λ for all $\lambda \in (0, \Lambda)$ with $u_\lambda(0) \geq \lambda^{\frac{1}{p-q-1}} w(0)$.

Let e be the solution to the problem

$$\left.\begin{array}{rcl} -(r^{N-1}|u'|^{p-2}u')' &=& r^{N-1} \\ u &>& 0 \end{array}\right\} \text{ in } (0,1), \quad u'(0) = 0, u(1) = 0.$$

Let $\theta = \sup_{x\in[0,1]} e(x)$ and let $M_\lambda = \mu\lambda^{\frac{1}{p-q-1}}$, where μ will be chosen later. Then we have,

$$(M_\lambda e)^\alpha + \lambda(M_\lambda e)^q \le \theta^\alpha M_\lambda^\alpha + \lambda\theta^q M_\lambda^q = M_\lambda^{p-1}\left[\theta^\alpha\mu^{\alpha-p+1}\lambda^{\frac{\alpha-p+1}{p-q-1}} + \theta^q\mu^{q-p+1}\right].$$

Now, fix μ large enough so that $\theta^q\mu^{q-p+1} \le \frac{1}{4}$. Once fixing μ, we now choose $\lambda_0 > 0$ small enough so that

$$\theta^\alpha\mu^{\alpha-p+1}\lambda^{\frac{\alpha-p+1}{p-q-1}} \le \frac{1}{4} \quad \forall\lambda \in (0, \lambda_0).$$

Thus, with this choice of μ and λ_0,

$$(M_\lambda e)^\alpha + \lambda\,(M_\lambda e)^q \le M_\lambda^{p-1}$$

and hence $M_\lambda e$ is a super-solution of (P_λ) for all such μ and λ. Since u_λ is a minimal solution we have immediately that $u_\lambda \le M_\lambda e$. This implies that $u_\lambda(0) \le \theta\mu\lambda^{\frac{1}{p-q-1}}$. This proves the lemma. □

3. Proof of Theorem I

We let $\alpha = p^* - 1$ and y any solution of (P_γ) .Our strategy for proving Theorem I will be to obtain good asymptotics for the first zero $T(\gamma)$ in the limits: $\gamma \to 0$ and $\gamma \to \infty$. In fact, we will show that $T(\gamma) \to \infty$ in these two limits. As we know already from Lemma 2.2 that $T(\gamma) > 0$ for all $\gamma > 0$, the proof follows.

Lemma 3.1. *There exists $c > 0$ such that $T(\gamma) > c\gamma^{\frac{(p-q-1)}{(p-1)(k-q-1)}}$ as $\gamma \to \infty$.*

Proof. Let γ be a sequence such that $\gamma \to \infty$. Recall that $\sigma = \frac{p}{p-1}$. If $T(\gamma) > a\gamma^\sigma$ for all large γ and some $a > 0$, then we are done. Therefore, let us assume $T(\gamma) < a\gamma^\sigma$ for all $a > 0$ and all γ large. Integrating the ODE in (P_γ) from $2T$ to γ^σ, we get

$$(y')^{p-1}(2T(\gamma)) > \int_{2T}^{\gamma^\sigma} s^{-k}y^q(s)ds.$$

By (d), (e) of Lemma 2.4, we get

$$\begin{aligned}\left(L\gamma^{1-\sigma}\right)^{p-1} &> \int_{2T}^{\gamma^\sigma} s^{-k}\left(K\gamma^{1-\sigma}\right)^q(s-T)^q ds\\ &= K^q\gamma^{(1-\sigma)q}T^{q-k+1}\int_2^{\gamma^\sigma/T} s^{-k}(s-1)^q ds.\end{aligned}$$

Now,

$$\int_2^{\gamma^\sigma/T} s^{-k}(s-1)^q \to \tilde{C}, \quad \text{as} \quad \gamma \to \infty$$

for some positive constant $\tilde{C}$. Therefore, as $\gamma \to \infty$, we may find $c > 0$ so that

$$T(\gamma) > c\gamma^{\frac{(p-q-1)}{(k-q-1)(p-1)}}.$$

This proves the lemma. □

Lemma 3.2. $T(\gamma) \sim \gamma^{-\frac{(p-q-1)}{k-p}}$ *as* $\gamma \to 0$.

Proof. Let u_λ be the minimal solution of (P_λ). Let $\gamma = u_\lambda(0)$. From Lemma 2.7 it follows that $\gamma \to 0$ as $\lambda \to 0$. Let $y_\lambda(t,\gamma) = u_\lambda(r)$, where t is given by the Emden-Fowler transformation: $t = \left(\frac{\nu}{r}\right)^\nu$. Then y_λ solves (P_γ) with $f(s) = s^{p^*-1} + \lambda s^q$ as the nonlinearity on the right hand side. Also the first zero of y_λ, denoted by T_λ, satisfies $T_\lambda(\gamma) = T_0 = \nu^\nu$ for all $\lambda > 0$. We now define y by the equation

$$y_\lambda(t,\gamma) = c^{-\beta} y(c^{\overline{\beta}} t, c^\beta \gamma),$$

where we choose $\overline{\beta}$, $\beta > 0$ such that $\beta(p-p^*) + \overline{\beta}(k-p) = 0$ and let $c = \lambda^{\frac{-1}{\beta(p-q-1)+\overline{\beta}(k-p)}}$. Then a straight forward computation gives that y solves the equation:

$$-\left((y')^{p-1}\right)'(c^{\overline{\beta}}t, c^\beta\gamma) = (c^{\overline{\beta}}t)^{-k}\left[y^{p^*-1}(c^{\overline{\beta}}t, c^\beta\gamma) + y^q(c^{\overline{\beta}}t, c^\beta\gamma)\right].$$

Now since $y_\lambda(T_0,\gamma) = 0\ \ \forall\lambda$ small, we get,

$$0 = y_\lambda(T_0,\gamma) = c^{-\beta} y(c^{\overline{\beta}} T_0, c^\beta\gamma).$$

This implies that $c^{\overline{\beta}} T_0$ is the first zero of $y(t, c^\beta\gamma)$. That is,

$$T(c^\beta\gamma) = c^{\overline{\beta}} T_0. \tag{3.1}$$

Let now $\tilde{\gamma}$ be a sequence of positive numbers with $\tilde{\gamma} \to 0$. Define a new sequence γ of positive numbers by letting $\gamma = c^{-\beta}\tilde{\gamma}$. Since by Lemma 2.7 we have $\gamma \sim \lambda^{\frac{1}{p-q-1}}$, it follows that $\tilde{\gamma} \sim \lambda^{\frac{\overline{\beta}(k-p)}{[\beta(p-q-1)+\overline{\beta}(k-p)](p-q-1)}}$. Therefore, from (3.1) we obtain,

$$T(\tilde{\gamma}) = T_0 \lambda^{-\frac{\overline{\beta}}{\beta(p-q-1)+\overline{\beta}(k-p)}} \sim \tilde{\gamma}^{-\frac{(p-q-1)}{(k-p)}}$$

which proves the lemma. □

We can now give the

Proof of Theorem I. Define $T^* = \inf_{\gamma\in(0,\infty)} T(\gamma)$. From Lemmas 2.2 , 3.1 and 3.2 we obtain that $T^* > 0$. Define R^* by the equation $T^* = (\frac{\nu}{R^*})^\nu$. Then clearly (P_R) does not admit a solution for all $R > R^*$ and at least two solutions for all $R < R^*$. This proves the theorem. □

4. Proof of Theorem II

First, we establish a few necessary lemmas. Throughout this section, we assume $0 \le q < p-1 < \alpha \le p^*-1$ and $f(s) = s^\alpha + s^q$. Fix $\beta > 0$. Consider the following initial value problem for $\gamma > 0$:

$$\left.\begin{array}{c} -\left(r^{N-1}|w'|^{p-2}w'\right)' = r^{N-1}\left(|w+\beta|^\alpha + |w+\beta|^q\right), \quad r \in (0,\infty), \\ w(0) = \gamma, \quad w'(0) = 0. \end{array}\right\} \qquad (4.1)$$

Let $w(.,\gamma)$ be any solution of the above problem. Define

$$R(\gamma) = \sup\{r > 0\ ;\ \ w(s,\gamma) > 0 \ \ \forall s \in (0,r)\}.$$

Then, clearly $R(\gamma)$ is the first zero of $w(.,\gamma)$.

Lemma 4.1. *There exists $\gamma_0 > 0$ such that for all $\gamma \in (0,\gamma_0)$ we have $R(\gamma)$ is finite and $R(\gamma) \to 0$ as $\gamma \to 0$.*

Proof. Let $f_\beta(s) = |s+\beta|^\alpha + |s+\beta|^q$. For $\epsilon > 0$ small enough, consider the problem

$$\left.\begin{array}{rcl} -\left(r^{N-1}|u'|^{p-2}u'\right)' &=& \epsilon r^{N-1} f_\beta(u) \\ u &>& 0 \end{array}\right\} \ r \in (0,1), \qquad (4.2)$$
$$u'(0) = 0,\ u(1) = 0.$$

Let e denote the unique solution of

$$\left.\begin{array}{rcl} -\left(r^{N-1}|u'|^{p-2}u'\right)' &=& r^{N-1} \\ u &>& 0 \end{array}\right\} \ r \in (0,1),$$
$$u'(0) = 0,\ u(1) = 0.$$

Let ϕ denote the first eigenfunction of the p-Laplace operator with zero Dirichlet condition on $B_1(0)$. Then it is well known from the results of [4] that ϕ is a positive radial function on $B_1(0)$. It is now an easy matter to check that if we choose μ large enough and ϵ and δ small enough, the functions $\overline{u} = \mu\epsilon^{\frac{1}{p-1}}e$ and $\underline{u} = \delta\phi$ are respectively super and sub solutions for the problem (4.2) with $\overline{u} \ge \underline{u}$ in $B_1(0)$. Thus, (4.2) admits a solution u_ϵ for all small $\epsilon > 0$ such that $\underline{u} \le u_\epsilon \le \overline{u}$. This means that $\|u_\epsilon\|_{L^\infty(0,1)} \to 0$ as $\epsilon \to 0$. Define now $v_\epsilon(r) = u_\epsilon(\epsilon^{-\frac{1}{p}}r)$. Then v_ϵ solves (4.1) with $\gamma_\epsilon = \|v_\epsilon\|_{L^\infty(0,1)} = \|u_\epsilon\|_{L^\infty(0,1)}$. Also, the first zero of v_ϵ is given by $R(\gamma_\epsilon) = \epsilon^{\frac{1}{p}}$. This proves the lemma. □

Lemma 4.2. *Let $q > 0$. Let $T(\gamma)$ denote the first zero of the solution $y(.,\gamma)$ of (P_γ). Then, $\limsup_{\gamma\to 0,\infty} T(\gamma) = \infty$.*

Proof. In the case of $\alpha = p^*-1$, Lemmas 3.1 and 3.2 give the proof. Let us consider the case of $\alpha < p^*-1$. From the bifurcation theorem of [2] (see

Theorem 3.12), there exists a sequence $\lambda_n \to 0$ and two sequences of solutions $\{u_{\lambda_n}\}, \{\underline{u}_{\lambda_n}\}$ of (P_{λ_n}) such that $u_{\lambda_n} \to v_0$ uniformly in $(0,1)$, where v_0 is the unique nontrivial solution of (P_0) and $\underline{u}_{\lambda_n}$ are the minimal solutions of (P_{λ_n}) such that $\underline{u}_{\lambda_n} \to 0$ uniformly in $(0,1)$. We also recall from Lemma 2.7 that $\underline{u}_{\lambda_n} \sim \lambda_n^{\frac{1}{p-q-1}}$ as $\lambda_n \to 0$. Now using the transformation (1.3), we obtain corresponding solutions w_n and $\underline{w}_n$ of (P_{R_n}) with $R_n = \lambda_n^{\frac{\alpha-p+1}{p(\alpha-q)}}$. Clearly $R_n \to 0$ as $\lambda_n \to 0$. Finally, using the Emden-Fowler transformation (1.4), we obtain correspondingly, solutions y_n and $\underline{y}_n$ of (P_{T_n}), where $T_n = (\frac{\nu}{R_n})^\nu$. We denote $\gamma_n = y_n(\infty) = w_n(0)$, $\underline{\gamma}_n = \underline{y}_n(\infty) = \underline{w}_n(0) \sim \lambda_n^{\frac{\alpha-p+1}{(p-q-1)(\alpha-q)}}$. Clearly, $T(\gamma_n) = T(\underline{\gamma}_n) = T_n \to \infty$ as $n \to \infty$. Also, $\gamma_n = w_n(0) = \lambda_n^{-\frac{1}{\alpha-q}} u_{\lambda_n}(0) \to \infty$ as $n \to \infty$ (recall: $u_{\lambda_n}(0) \to v_0(0)$) and $\underline{\gamma}_n \to 0$ as $n \to \infty$. This proves the lemma. □

Lemma 4.3. *Let v_R denote any solution other than the minimal solution of (P_R). Then, $\lim_{R\to 0} v_R(0) = \infty$.*

Proof. Let $\gamma = v_R(0)$. Then $R(\gamma) = R \to 0$ implies that either $\gamma \to 0$ or $\gamma \to \infty$. From the results of [2] and Lemma 2.7, we know that for all $\lambda > 0$ small enough, (P_λ) possesses a minimal solution u_λ with $u_\lambda(0) \sim \lambda^{\frac{1}{p-q-1}}$. Using the transformation (1.3) and the ODE uniqueness theory, we obtain that for all small enough γ, any solution u_R of (P_R) for any R, with $u_R(0) = \gamma$, is in fact a minimal solution. Therefore, if $\gamma = v_R(0) \to 0$, we can conclude that v_R is the minimal solution of (P_R) which gives a contradiction. This proves the lemma. □

We can now give the

Proof of Theorem II. Define $\beta_0 = (\frac{q(1-q)}{\alpha(\alpha-1)})^{\frac{1}{\alpha-q}}$ if $q < 1$ and $\beta_0 = 0$ otherwise. Let $f_{\beta_0}(t) = (t+\beta_0)^\alpha + (t+\beta_0)^q$. Then, we note that f_{β_0} is strictly convex for $t > 0$. Then, by Lemma 4.1, for $R_1 > 0$ small enough we may find $\gamma_0 > 0$ so that (4.1) with $\beta = \beta_0$ admits a solution u with $u(0) \le \gamma_0$ for all $R < R_1$. Now, thanks to Theorem I and the results of [2], by shrinking R_1 further if necessary, we obtain that problem (P_R) admits at least two solutions for all $R < R_1$. Also, by Lemma 4.3 we may further shrink R_1, if necessary, to conclude that for all $R < R_1$ if u is any solution of (P_R) other than the minimal solution, then $u(0) \ge \beta_0 + 2\gamma_0$.

Now, suppose for some $\tilde{R} < R_1$, $(P_{\tilde{R}})$ admits three solutions u_1, u_2, u_3 with $u_1(0) < u_2(0) < u_3(0)$. Here u_1 is the minimal solution. By the "ordering result" in Lemma 2.5, we have $u_1 < u_2 < u_3$ in $[0, \tilde{R})$. Since $u_2(0) \ge \beta_0 + 2\gamma_0$, we may obtain $R_2 < \tilde{R}$ such that $u_2(R_2) = \beta_0$. Let $v_i = u_i - \beta_0$ for i=2,3. Then v_2 is a solution of (4.1) with $v_2(R_2) = 0$ and $v_3 > v_2$ in $[0, R_2]$. Since $v_2(0) \geq 2\gamma_0$ and $R_2 < R_1$, we may find (by Lemma 4.1) a solution v_1 of (4.1) with $\beta = \beta_0$ and $v_1(R_2) = 0$ and $v_1(0) \leq \gamma_0$. Since $R(\gamma) \to 0$ as $\gamma \to 0$, we may assume that $v_1 < v_2$ in $[0, R_2)$. Thus, we have $v_1 < v_2 < v_3$ in $[0, R_2)$ and for i=1,2,3,

$$\begin{cases} \left.\begin{array}{rcl} -\left(r^{N-1}|v_i'|^{p-2}v_i'\right)' &=& r^{N-1}f_{\beta_0}(v_i) \\ v_i &>& 0 \end{array}\right\} r \in (0, R_2), \\ v_1(R_2) = v_2(R_2) = 0, v_3(R_2) > 0, v_i'(0) = 0. \end{cases}$$

We perform the Emden-Fowler transform $t = (\frac{\nu}{r})^\nu$, $y_i(t) = v_i(r)$, $i = 1, 2, 3$. Let $T = (\frac{\nu}{R_2})^\nu$. Then we have,

$$y_1 < y_2 < y_3 \text{ in } [T, \infty), \tag{4.3}$$

and for i=1,2,3, y_i satisfy the equation

$$\begin{array}{c} \left.\begin{array}{rcl} -\left(|y_i'|^{p-2}y_i'\right)' &=& t^{-k}f_{\beta_0}(y_i) \\ y_i &>& 0 \end{array}\right\} t \in (T, \infty), \\ y_1(T) = y_2(T) = 0,\ y_3(T) > 0, y_i'(\infty) = 0. \end{array} \tag{4.4}$$

From (4.3) and integrating the equation in (4.4) we obtain that

$$0 < y_1' < y_2' < y_3' \text{ in } [T, \infty).$$

From (4.4) we obtain the following two equations:

$$\int_T^\infty \left[(y_3')^{p-1} - (y_2')^{p-1}\right](y_2 - y_1)' = \int_T^\infty t^{-k}\left[f_{\beta_0}(y_3) - f_{\beta_0}(y_2)\right](y_2 - y_1),$$

$$\int_T^\infty \left[(y_2')^{p-1} - (y_1')^{p-1}\right](y_3 - y_2)' + \left[(y_2')^{p-1}(T) - (y_1')^{p-1}(T)\right](y_3 - y_2)(T)$$
$$= \int_T^\infty t^{-k}\left[f_{\beta_0}(y_2) - f_{\beta_0}(y_1)\right](y_3 - y_2).$$

Subtracting the second equation from the first we get,

$$\begin{aligned} &\int_T^\infty (y_2 - y_1)'(y_3 - y_2)'\Big[\frac{(y_3')^{p-1} - (y_2')^{p-1}}{(y_3 - y_2)'} - \frac{(y_2')^{p-1} - (y_1')^{p-1}}{(y_2 - y_1)'}\Big] \\ &- \left[(y_2')^{p-1}(T) - (y_1')^{p-1}(T)\right](y_3 - y_2)(T) \\ &= \int_T^\infty t^{-k}(y_2 - y_1)(y_3 - y_2)\Big[\frac{f_{\beta_0}(y_3) - f_{\beta_0}(y_2)}{y_3 - y_2} - \frac{f_{\beta_0}(y_2) - f_{\beta_0}(y_1)}{y_2 - y_1}\Big]. \end{aligned} \tag{4.5}$$

Since f_{β_0} is strictly convex for $t > 0$ and $0 < y_1 < y_2 < y_3$ on $[T, \infty)$, we conclude that the right hand side of (4.5) is positive. On the other hand by our assumption that $p \in (1, 2)$, the map $t \longrightarrow t^{p-1}$ is strictly concave. Since

$0 < y_1' < y_2' < y_3'$ on $[T, \infty)$, the integral on the left hand side is negative and so is the second term. This gives a contradiction. This contradiction shows that there exists $R_{**} < R_*$ such that (P_R) admits at most two solutions for all $R \in (0, R_{**})$. Now part (i) follows by recalling that the results of [2] and Theorem I ensure at least two solutions of (P_R) for all $R \in (0, R_*)$ when $0 < q < p-1 < \alpha \le p^* - 1$. This proves the theorem. □

5. Appendix

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. Let $p \in (1, \infty)$. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $f(0) = 0$ and $f(s)s^{1-p}$ is non-increasing in $(0, \infty)$. Using the ideas in Marcus-Shafrir [12] we prove the following weak comparison theorem.

Theorem A1. *Let $u_1, u_2 \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ with $u_1 > 0$ and $u_2 > 0$ in Ω satisfies*

$$-\Delta_p u_1 \le f(u_1), \tag{5.1}$$

$$-\Delta_p u_2 \ge f(u_2). \tag{5.2}$$

Then, $u_1 \le u_2$ on $\partial\Omega$ implies $u_1 \le u_2$ in Ω.

Proof. Consider the functions

$$w_1 = \frac{(u_1^p - u_2^p)_+}{u_1^{p-1}}, \quad w_2 = \frac{(u_1^p - u_2^p)_+}{u_2^{p-1}}.$$

Then $w_1, w_2 \in W_0^{1,p}(\Omega)$. Define $E = \{x \in \Omega : u_1(x) > u_2(x)\}$. Multiplying (5.1) with w_1 and (5.2) with w_2 we get,

$$\int_E |\nabla u_1|^{p-2} \nabla u_1 . \nabla w_1 \le \int_E f(u_1) w_1, \tag{5.3}$$

$$\int_E |\nabla u_2|^{p-2} \nabla u_2 . \nabla w_2 \ge \int_E f(u_2) w_2. \tag{5.4}$$

Subtracting (5.4) from (5.3) we get,

$$\int_E |\nabla u_1|^{p-2} \nabla u_1 . \nabla w_1 - |\nabla u_2|^{p-2} \nabla u_2 . \nabla w_2 \le \int_E \Big[\frac{f(u_1)}{u_1^{p-1}} - \frac{f(u_2)}{u_2^{p-1}}\Big](u_1^p - u_2^p).$$

Using arguments similar to those in [11] we get, for $p \ge 2$,

$$L.H.S. \ge \frac{1}{2^{p-1} - 1} \int_E \Big(\frac{1}{u_1^p + u_2^p}\Big) |u_1 \nabla u_2 - u_2 \nabla u_1|^p \ge 0,$$

and if $p \in (1, 2)$,

$$L.H.S. \ge \frac{3p(p-1)}{16} \int_E \Big(\frac{1}{u_1^p} + \frac{1}{u_2^p}\Big) \frac{(u_1 \nabla u_2 - u_2 \nabla u_1)^2}{(u_1 \nabla u_2 + u_2 \nabla u_1)^{2-p}} \ge 0.$$

But $R.H.S. \leq 0$ since $f(s)s^{1-p}$ is non-increasing. Therefore, if E is non-empty, on each component E_i of E we may find k_i such that $u_1 = k_i u_2$ on E_i. But $u_1 = u_2$ on $\partial E_i \cap \Omega$, which implies $u_1 = u_2$ on E_i, a contradiction to the definition of E_i. Therefore, $E = \varnothing$ which proves the theorem. □

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