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ENERGY DECAY AND A TRANSMISSION PROBLEM IN ELECTROMAGNETO-ELASTICITY

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Abstract. We consider a transmission problem for a system of electromagneto-elasticity having piecewise constant coefficients in a bounded domain. Under suitable geometric conditions imposed on the domain and the interfaces where the coefficients have a jump discontinuity, results on uniform boundary stabilization are established. Exact boundary controllability is then obtained through Russell's "controllability via stabilizability" principle.

1. Introduction

In this article we consider a dynamic coupled system of electromagneto-elasticity in a bounded domain Ω of $\mathbb{R}^3$ whose boundary $\partial\Omega = S$ is assumed to be smooth and we study a transmission problem associated with it. We will assume a geometric condition on Ω and that in the entire boundary dissipative mechanisms are present. These assumptions will allow us to obtain the uniform exponential decay of the total energy as $t \to +\infty$ for a class of domains which include not only starlike domains but also some other class of nonstarlike regions. Let us describe the model we will consider: The

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bounded domain $\Omega \subseteq \mathbb{R}^3$ is occupied by a multilayered piezo electric body whose motion is governed by the system

$$\begin{aligned} &\rho\, u_{tt} - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} \big(A_{ij} \frac{\partial u}{\partial x_j}\big) + \alpha(x)\, \mathrm{curl}(a(x)E) = 0 \\ &E_t - \mathrm{curl}(b(x)H) - \mathrm{curl}(\alpha(x)u_t) = 0 \\ &H_t + \mathrm{curl}(a(x)E) = 0 \\ &\mathrm{div}\ E = 0 \quad \mathrm{div}\ H = 0 \end{aligned} \tag{1.1}$$

in $\Omega \times (0, +\infty)$. Here $x = (x_1, x_2, x_3) \in \Omega$ and t denotes the time variable. In (1.1) we denote by

$$\begin{aligned} u = (u_1, u_2, u_3) &= \text{ the displacement vector} \\ E = (E_1, E_2, E_3) &= \text{ the electric field} \\ H = (H_1, H_2, H_3) &= \text{ the magnetic field} \\ a(x) &= \text{ the electric permeability} \\ b(x) &= \text{ the magnetic permitivity} \\ \alpha(x) &= \text{ a coupling function} \\ \rho(x) &= \text{ the density at } x. \end{aligned}$$

and the 3×3 matrices $A_{ij} = A_{ij}(x)$ (as well as the functions $a(x)$, $b(x)$ and $\alpha(x)$) will satisfy suitable assumptions given below. This will imply that

$$L = \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} \big(A_{ij} \frac{\partial}{\partial x_j}\big)$$

is an elliptic operator. In the simplest case, when we are considering an isotropic medium, the operator L coincides with

$$L = \mu\Delta + (\lambda + \mu)\nabla\ \mathrm{div}$$

where λ and μ are the Lamé's constants ($\mu > 0,\ \lambda + \mu > 0$).

The coupled system of hyperbolic problem (1.1) is complemented with initial conditions

$$\begin{cases} u(x,0) = f_1(x), & u_t(x,0) = f_2(x) \\ E(x,0) = f_3(x), & H(x,0) = f_4(x) \end{cases} \quad \text{in } \Omega \tag{1.2}$$

and boundary conditions

$$\begin{cases} \sum_{i,j=1}^{3} A_{ij}\,\frac{\partial u}{\partial x_j}\,\eta_i - \alpha\, a(\eta\,\mathbf{x}E) + du = -\beta\, u_t \\ H\mathbf{x}\eta = -\theta\big(\eta\,\mathbf{x}(\eta\,\mathbf{x}E)\big) \end{cases} \tag{1.3}$$

for any $(x,t) \in \partial\Omega \times (0,T)$ where "$\mathbf{x}$" denotes the usual vector product, $\eta = \eta(x)$ denotes the unit outward normal to $\partial\Omega = S$ at x. The functions $d = d(x)$, $\beta = \beta(x)$ and $\theta = \theta(x)$ will satisfy suitable conditions given below and in the simplest case they are just positive constants.

The asymptotic behavior of the solution of problem (1.1), (1.2), (1.3) as $t \to +\infty$ it is of interest in several branches of applied analysis such as exact controllability. Even more interesting is the so called transmission problem associated with model (1.1), (1.2), (1.3). Let us describe what we intend to prove in this article: Let $\Omega \subseteq \mathbb{R}^3$ be as above and consider a sequence $\{B_k\}_{k=1}^n$ of subsets of Ω which are open, connected, with smooth boundary $\partial B_k = S_k$ and such that $\overline{B}_k \subseteq B_{k+1}$ for $1 \le k \le n$. We denote $\Omega_0 = B_1$, $\Omega_k = B_{k+1}\backslash\overline{B}_k$, $1 \le k \le n-1$ and $\Omega_n = \Omega\backslash\overline{B}_n$. We assume to be valid the following conditions.

Hypothesis I. 1) $\rho = \rho(x)$, $a(x)$ and $b(x)$ are piecewise positive constant functions on $\overline{\Omega}$ which lose continuity only on $S_1, S_2, \dots, S_n$.

2) $\alpha = \alpha(x) \in C^1(\overline{\Omega})$ and the functions $d = d(x)$, $\beta = \beta(x)$ and $\theta = \theta(x)$ are real-valued and of class C^1 on $S = \partial\Omega$.

3) $\theta(x) \ge 0$, $\beta(x) \ge 0$ and $d(x) \ge 0$ for all $x \in S$.

4) $A_{ij} = A_{ij}(x)$ are 3×3 matrices given by $A_{ij}(x) = \big[C^{ij}_{kh}(x)\big]_{3\times 3}$, where

$$C^{ij}_{kh}(x) = (1 - \delta_{ih}\,\delta_{ik})a_{ikjh}(x) + \delta_{ik}\,\delta_{jh}\,a_{ihjk}(x)$$

where $\delta_{\ell k} = \begin{cases} 1 \text{ if } \ell = k \\ 0 \text{ if } \ell \ne k \end{cases}$ and $a_{ijkh}(x)$ are Cartesian components of the elastic tensor with the symmetric properties $a_{ijkh} = a_{jikh} = a_{khij}$.

5) The components $a_{ijkh}(x)$ are piecewise constant functions which lose continuity on $S_1, S_2, \dots, S_n$.

6) The matrices $A_{ij}(x)$ satisfy

$$\sum_{i,j=1}^{3} A_{ij}(x)v_j \cdot v_i \ge c_0 \sum_{i=1}^{3} |v_i|^2$$

for some $c_0 > 0$, all $x \in \Omega$ and any vector $v_i = (v_i^1, v_i^2, v_i^3)$ where the dot $\boldsymbol{.}$ denotes the usual inner product in $\mathbb{R}^3$.

Observe that, for an isotropic medium, the constants a_{ijkh} are given by

$$a_{ijkh} = \lambda\, \delta_{ij}\, \delta_{kh} + \mu(\delta_{ik}\, \delta_{jh} + \delta_{ih}\, \delta_{jk})$$

where λ and μ are the Lamé's constants.

Furthermore, from the symmetry of a_{ijkh} it follows that $A^*_{ij} = A_{ji}$ and assumption 6) holds for an isotropic medium with the constant $c_0 = \mu > 0$. In fact, in that case, direct calculation shows that

$$\sum_{i,j=1}^{3} A_{ij}\, v_j \boldsymbol{.}\, v_i = (\lambda + \mu)\Big(\sum_{i=1}^{3} v_i^i\Big)^2 + \mu \sum_{i,j=1}^{3} (v_i^j)^2 \geq \mu \sum_{i=1}^{3} |v_i|^2.$$

Now, we consider the system

$$\begin{cases} \rho^{(k)} u_{tt}^{(k)} - \displaystyle\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\Big(A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j}\Big) + \alpha(x)\,\mathrm{curl}(a^{(k)}(x)\, E^{(k)}) = 0 \\ E_t^{(k)} - \mathrm{curl}(b^{(k)}(x)\, H^{(k)}) - \mathrm{curl}(\alpha(x) u_t^{(k)}) = 0 \\ H_t^{(k)} + \mathrm{curl}(a^{(k)}(x)\, E^{(k)}) = 0 \\ \mathrm{div}\ E^{(k)} = 0 \quad \mathrm{div}\ H^{(k)} = 0 \end{cases} \tag{1.4}$$

in $\Omega_k \times (0,T)$, $k = 0, 1, 2, \ldots, n$. We complement (1.4) with initial conditions

$$\begin{cases} u^{(k)}(x,0) = f_1^{(k)}(x), \qquad u_t^{(k)}(x,0) = f_2^{(k)}(x) \\ E^{(k)}(x,0) = f_3^{(k)}(x), \qquad H^{(k)}(x,0) = f_4^{(k)}(x) \end{cases} \tag{1.5}$$

in Ω_k, $k = 0, 1, \ldots, n$. The boundary conditions on $S \times (0,T)$ are (1.3) where $u^{(n)} = u$, $a^{(n)} = a$ and $A_{ij}^{(n)} = A_{ij}$. Furthermore, we will require the following interface conditions to be satisfied

$$\begin{cases} u^{(k-1)} = u^{(k)} \\ \displaystyle\sum_{i,j=1}^{3} A_{ij}^{(k-1)}\, \frac{\partial u^{(k-1)}}{\partial x_j} \eta_i = \sum_{i,j=1}^{3} A_{ij}^{(k)}\, \frac{\partial u^{(k)}}{\partial x_j}\, \eta_i \\ a^{(k-1)}(\eta\, \mathbf{x} E^{(k-1)}) = a^{(k)}(\eta\, \mathbf{x} E^{(k)}) \\ b^{(k-1)}(\eta\, \mathbf{x} H^{(k-1)}) = b^{(k)}(\eta\, \mathbf{x} H^{(k)}) \end{cases} \tag{1.6}$$

for any $(x,t) \in S_k \times (0,T)$, $k = 1, 2, \ldots, n$. Here, $\eta = \eta(x) = (\eta_1(x), \eta_2(x), \eta_3(x))$ is the unit outward normal vector pointing the exterior of B_k and $a^{(k)}$, $b^{(k)}$, $\rho^{(k)}$, $u^{(k)}$, $E^{(k)}$, $H^{(k)}$ and $f_j^{(k)}$, $1 \leq j \leq 4$ are the restrictions of a, b, ρ, u, E, H and f_j to Ω_k respectively.

Let $\{u, E, H\}$ be the global solution of problem (1.4) satisfying the initial conditions (1.5) the boundary conditions (1.3) and the interface conditions (1.6). We consider the (total) energy $\mathcal{E}(t)$ given by

$$\begin{aligned}\mathcal{E}(t) = \sum_{k=0}^{n} \int_{\Omega_k} \Big\{ \rho^{(k)} |u_t^{(k)}|^2 + \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i} \\ + a^{(k)} |E^{(k)}|^2 + b^{(k)} |H^{(k)}|^2 \Big\} dx + \int_S d |u^{(n)}|^2 \, dS\end{aligned} \tag{1.7}$$

where $a^{(n)} = a$, $b^{(n)} = b$ and $u^{(n)} = u$.

We calculate (formally) the derivative of $\mathcal{E}(t)$, use the equations together with the boundary conditions as well as the interface conditions to obtain that

$$\frac{d\mathcal{E}(t)}{dt} = -2 \int_S \big\{ \beta |u_t^{(n)}|^2 + \theta \, a^{(n)} b^{(n)} |E^{(n)} \times \eta|^2 \big\} dS \leq 0.$$

By assuming suitable geometric conditions on Ω (and S_k) as well as monotonicity assumptions on the coefficients of the system we are able to prove that

$$\mathcal{E}(t) \leq c \, \exp(-\gamma t) \mathcal{E}(0)$$

for any $t > 0$ where γ and c are positive constants.

As an application of the above result, we study the following exact controllability problem: Given a time $T > 0$, the initial distribution $F(x) = (f_1(x), f_2(x), f_3(x), f_4(x))$ and a desired terminal state $G(x) = (g_1(x), g_2(x), g_3(x), g_4(x))$ where F and G belong to an appropriate function space, to find vector-valued functions $\vec{p}(x,t)$ and $\vec{q}(x,t)$ such that the solution of (1.4), (1.5), (1.6) with conditions

$$\begin{cases} \displaystyle\sum_{i,j=1}^{3} A_{ij} \frac{\partial u}{\partial x_j} \eta_i - \alpha \, a(\eta \, \mathbf{x} E) + du = \vec{p}(x,t) \\ \eta \, \mathbf{x} H = \vec{q}(x,t) \end{cases} \tag{1.8}$$

for any $(x,t) \in S \times (0,T)$ satisfies

$$u(x,T) = g_1(x), \; u_t(x,T) = g_2(x), \; E(x,T) = g_3(x), \; H(x,T) = g_4(x). \tag{1.9}$$

Let us mention some related work on the subject: Stabilization and exact boundary controllability for the system of elasticity have been studied by J. Lagnese [12], [13], F. Alabau and V. Komornik [1] and M. Horn [3] among others. In [4] the uniform stabilization and simultaneous controllability was studied for a class of hyperbolic systems which includes the system of elasticity theory. Uniform exponential decay of solutions of Maxwell's equations with boundary dissipation was proved in [5], [6] (including the uniform "simultaneous" stabilization for a pair of Maxwell's equations). Stabilization for the Maxwell system with the Silver-Müller absorbing boundary conditions has been proved by V. Komornik [10] and P. Martinez [19]. In [5], [10] the exact controllability is studied for corresponding initial boundary value problems. The exact controllability problem for the Maxwell system has been studied by D. Russell [24] for a circular cylindrical region, by K. Kime [9] for a spherical region, and by J. Lagnese [14] for a general region. A moment problem approach is used in [9], [24]. In [7], [14] the exact controllability problem has been studied by means of the Hilbert Uniqueness Method introduced by J.-L. Lions [16], [17]. The exact controllability of Maxwell's equations in an inhomogeneous medium was investigated in [7]. Boundary controllability in transmission problems for a class of second order hyperbolic systems has been studied by J. Lagnese [13]. Uniform stabilization and exact control for the Maxwell system in multilayered media were studied in [5]. The question of boundary controllability in transmission problems for the wave equation has been considered by J.-L. Lions [18], and S. Nicaise [20], [21]. In [23] the energy decay of solutions of the system of magneto-elasticity was studied. Let us briefly describe the sections of this paper: Solvability of problem (1.4)–(1.6) with the boundary conditions (1.3) in the required class is shown in Section 2. In order to do that we rewrite the problem in the form

$$\frac{du}{dt} = \mathcal{A}u, \quad u|_{t=0} = f$$

and show that $\mathcal{A}$ and its adjoint $\mathcal{A}^*$ are dissipative. In Section 3 we prove the uniform exponential decay of $\mathcal{E}(t)$ via multipliers. Suitable variable coefficients are considered in these multipliers to take care of the additional boundary terms which appear after the integration of the basic identity. In this part we also need to assume some geometric conditions on Ω and S_k as well as some monotonicity assumptions on the coefficients. In the last section, the controllability problem (1.4), (1.5), (1.6), (1.8)–(1.9) is solved.

We observe that by Hypothesis I, $\rho = \rho(x)$ (as well as $a(x)$, $b(x)$, etc) is a piecewise positive constant function. Therefore by changing notations if necessary in (1.4) we may assume without lost of generality that $\rho \equiv 1$. When studying system (1.1) the restrictions of u, E and H to Ω_k ($k = 0, 1, 2, \ldots, n-1$) will be denoted by $u^{(k)}$, $E^{(k)}$ and $H^{(k)}$ respectively. For $k = n$ we will simple write $u^{(n)} = u$, $E^{(n)} = E$, etc. The same notation will be use for the coefficients α, a, b, etc. At each point x belonging to one of the boundaries $S = \partial\Omega$, $S_1, \dots, S_n$ the unit outward normal vector pointing the exterior will be denote by $\eta = \eta(x)$ and its components by η_i. We use standard notations, for example $H^m(\Omega)$ and $H^r(S)$ will denote the Sobolev spaces of order m and r on Ω and S respectively.

The norm of a vector $v \in \mathbb{R}^3$ will be denote by $|v|$. Observe also that when $\alpha(x) \equiv 0$ then, problem (1.4)–(1.6) with boundary conditions (1.3) decouples into two independent mixed problems of hyperbolic type.

Due to the techniques we use in this article (the multiplier method) to achieve the exponential decay we needed to assume that $d = d(x)$ is bounded by a suitable constant (see (2.34) in Theorem 3.1). It would be natural not to assume this condition but this is a limitation of the method.

The boundary condition $H \,\mathbf{x}\, \eta = -\theta(\eta \,\mathbf{x}(\eta \,\mathbf{x} E))$ in (1.3) for the Maxwell system is known as Leontovich's boundary type condition. In physics text-books is referred to be a condition that a "good conductor" should satisfy. It is been widely used in the literature and in V. Komornik's book [11] pg 120 a nice geometrical meaning of Leontovich's boundary type condition is given in case $\theta(x) \equiv 1$: The tangential component of the magnetic field H is obtained from the tangential component of the electrical field E by a rotation of angle 90° in the positive direction in the tangent plane.

In case $\alpha \equiv 0$ the boundary condition for the displacement vector $u(x,t)$ is essentially a Neumann type condition and when $\alpha > 0$ the term

$$\sum_{i,j=1}^{3} A_{ij} \frac{\partial u}{\partial x_j} \eta_i - \alpha\, (\eta \,\mathbf{x} E)$$

could be interpreted as an stress tensor for the system at the boundary S.

Finally, we would like to remark that the model we are considering in this work is a simplified version of the system of elasticity. In a sense, the system we are discussing "resembles" a pair of decoupled wave equations.

2. Well posedness

We consider the (real) Hilbert space X of quadruples $v = (v_1, v_2, v_3, v_4)$ of three-component vector-valued functions $v_j(x)$ such that

$$v_1^{(k)} \in [H^1(\Omega_k)]^3, \quad v_2^{(k)}, v_3^{(k)}, v_4^{(k)} \in [L^2(\Omega_k)]^3$$

for $k = 0, 1, 2, \dots, n$ and $v_1^{(k-1)}(x) = v_1^{(k)}(x)$ for $x \in S_k$, $k = 1, 2, \dots, n$. Here $v_j^{(k)}(x)$ denotes the restriction of $v_j(x)$ $(1 \le j \le 4)$ on Ω_k. The inner product in X is given by: If $v, w \in X$, then

$$(v,w)_X = \sum_{k=0}^{n} \int_{\Omega_k} \Big\{ v_2^{(k)} \cdot w_2^{(k)} + \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial v_1^{(k)}}{\partial x_j} \cdot \frac{\partial w_1^{(k)}}{\partial x_i} + a^{(k)} v_3^{(k)} \cdot w_3^{(k)} + b^{(k)} v_4^{(k)} \cdot w_4^{(k)} \Big\} dx + \int_S d(x) v_1^{(n)} \cdot w_1^{(n)} \, dS$$

and the norm in X will be denoted by $||v||_X^2 = (v,v)_X$ for any $v \in X$. Let us "built" the domain of $\mathcal{A}$: We consider the Hilbert space Z of pairs $u = (u_1, u_2)$ of three-component vector valued functions, $u_i \in [L^2(\Omega_k)]^3$ with curl $u_i \in [L^2(\Omega_k)]^3$, $k = 0, 1, \ldots, n$ with inner product given by

$$(u,v)_Z = \sum_{k=0}^{n} \int_{\Omega_k} \{ \operatorname{curl} a^{(k)} u_1 \cdot \operatorname{curl} a^{(k)} v_1 + \operatorname{curl} b^{(k)} u_2 \cdot \operatorname{curl} b^{(k)} v_2 + a^{(k)} u_1 \cdot v_1 + b^{(k)} u_2 \cdot v_2 \} dx,$$

where the dot $\cdot$ denotes the usual inner product in $\mathbb{R}^3$. It is well known (see for instance the book of G. Duvaut and J.-L. Lions [2] or B. Kapitonov [6]) that the expressions $\eta \,\mathbf{x} u_1$ and $\eta \,\mathbf{x} u_2$ are well defined on S_k and belong to $[H^{-1/2}(S_k)]^3$ (that is, the dual of $[H^{1/2}(S_k)]^3$) whenever $(u_1, u_2) \in Z$. Moreover, it was shown in [8] (see also [6]) that the expression $u_2 \,\mathbf{x}\, \eta + \theta(x)\, \eta \,\mathbf{x} (\eta \,\mathbf{x} u_1)$ is well defined on S and belongs to $[H^{-1/2}(S)]^3$ provided $(u_1, u_2) \in Z$. This enables us to introduce the closed subspace of Z given by

$$\begin{aligned} V = \big\{ &(u_1, u_2) \in Z \quad \text{such that} \\ &u_2 \,\mathbf{x}\, \eta + \theta(x)\, \eta \,\mathbf{x} (\eta \,\mathbf{x} u_1) = 0 \quad \text{on} \quad S, \\ &a^{(k-1)} \eta \,\mathbf{x} u_1^{(k-1)} = a^{(k)} \eta \,\mathbf{x} u_1^{(k)} \quad \text{on} \quad S_k \\ \text{and } &b^{(k-1)} \eta \,\mathbf{x} u_2^{(k-1)} = b^{(k)} \eta \,\mathbf{x} u_2^{(k)} \quad \text{on} \quad S_k \\ &\text{for } k = 1, 2, \ldots, n \big\}. \end{aligned}$$

We define the unbounded operator $\mathcal{A}\colon \mathcal{D}(\mathcal{A}) \subseteq X \to X$ as follows:

$$\mathcal{A} v = (v_2, Lv_1 - \alpha \operatorname{curl}(a v_3), \operatorname{curl}(b v_4) + \operatorname{curl}(\alpha v_2), -\operatorname{curl}(a v_3))$$

whenever $v \in \mathcal{D}(\mathcal{A})$ where

$$Lv_1 = \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} \big(A_{ij} \frac{\partial v_1}{\partial x_j} \big) \tag{2.1}$$

and the domain of $\mathcal{A}$ is defined as

$$\begin{aligned}\mathcal{D}(\mathcal{A}) = \big\{&v = (v_1, v_2, v_3, v_4) \in X \text{ such that } v_1^{(k)} \in [H^2(\Omega_k)]^3,\\ &v_2^{(k)} \in [H^1(\Omega_k)]^3, \quad (v_3, v_4) \in V,\\ &\sum_{i,j=1}^{3} A_{ij}^{(k-1)} \frac{\partial v_1^{(k-1)}}{\partial x_j}\, \eta_i = \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial v_1^{(k)}}{\partial x_j}\, \eta_i \text{ whenever } x \in S_k\,,\\ &k = 1, 2, \ldots, n \text{ and } \sum_{i,j=1}^{3} A_{ij} \frac{\partial v_1}{\partial x_j}\, \eta_i - \alpha\, a(\eta\, \mathbf{x} v_3) + \beta v_2 + d v_1 = 0\\ &\text{whenever } x \in S\big\}.\end{aligned}$$

Next, we consider the adjoint operator $\mathcal{A}^*$. We can verified that the domain of $\mathcal{A}^*$ coincides with the following subspace

$$\begin{aligned}\mathcal{D}(\mathcal{A}^*) = \big\{&w = (w_1, w_2, w_3, w_4) \in X \text{ such that } w_1^{(k)} \in [H^2(\Omega_k)]^3\\ &w_2^{(k)} \in [H^1(\Omega_k)]^3, \quad (w_3, w_4) \in \widetilde{V},\\ &\sum_{i,j=1}^{3} A_{ij}^{(k-1)} \frac{\partial w_1^{(k-1)}}{\partial x_j}\, \eta_i = \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial w_1^{(k)}}{\partial x_j}\, \eta_i\\ &\text{for } x \in S_k\,, \quad k = 1, 2, 3, \ldots, n \quad \text{and}\\ &\sum_{i,j=1}^{3} A_{ij} \frac{\partial w_1}{\partial x_j}\, \eta_i - \alpha\, a(\eta\, \mathbf{x} w_3) - \beta w_2 + d w_1 = 0 \text{ for } x \in S\big\},\end{aligned}$$

where $\widetilde{V}$ is as the definition of V with $-\theta(x)$ instead of $\theta(x)$. Given $w = (w_1, w_2, w_3, w_4) \in \mathcal{D}(\mathcal{A}^*)$ then, we have that

$$\mathcal{A}^* w = -(w_2, L w_1 - \alpha \text{ curl}(a w_3), \text{curl}(b w_4) + \text{curl}(\alpha w_2), -\text{curl}(a w_3)),$$

where L is given by (2.1).

Lemma 2.1. *Assuming Hypothesis I given in the introduction* (*with* $\rho \equiv 1$) *and the above notations then the operators* $\mathcal{A}$ *and* $\mathcal{A}^*$ *are dissipative, that is*

$$(\mathcal{A} v, v)_X \leq 0 \quad \textit{for any} \quad v \in \mathcal{D}(\mathcal{A}) \tag{2.2}$$

and

$$(\mathcal{A}^* w, w)_X \leq 0 \quad \textit{for any} \quad w \in \mathcal{D}(\mathcal{A}^*) \tag{2.3}$$

Proof. We prove (2.2) for a dense subset of $\mathcal{D}(\mathcal{A})$. In fact, $W = \{v = (v_1, v_2, v_3, v_4) \in \mathcal{D}(\mathcal{A})$ such that $v_1 \in [C^2(\Omega_k)]^3$ $v_2, v_3, v_4 \in [C^1(\Omega_k)]^3$, $k =$

$0, 1, \ldots, n\}$ is dense in $\mathcal{D}(\mathcal{A})$. Let $v \in W$ then taking the inner product of $\mathcal{A}v$ with v in X and using the divergence theorem we obtain

$$
\begin{aligned}
(\mathcal{A}v, v)_X &= \sum_{k=0}^{n} \int_{\Omega_k} \{ L v_1^{(k)} \cdot v_2^{(k)} - \alpha \operatorname{curl}(a^{(k)} v_3^{(k)} \cdot v_2^{(k)}) \\
&+ \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial v_2^{(k)}}{\partial x_j} \cdot \frac{\partial v_1^{(k)}}{\partial x_i} + a^{(k)} \operatorname{curl}(b^{(k)} v_4^{(k)}) \cdot v_3^{(k)} \\
&+ a^{(k)} \operatorname{curl}(\alpha\, v_2^{(k)}) \cdot v_3^{(k)} - b^{(k)} \operatorname{curl}(a^{(k)} v_3^{(k)}) \cdot v_4^{(k)} \} dx + \int_S d(x) v_2^{(n)} \cdot v_1^{(n)}\, dS \\
&= \sum_{k=1}^{n} \int_{S_k} \Big\{ \sum_{i,j=1}^{n} A_{ij}^{(k-1)} \frac{\partial v_1^{(k-1)}}{\partial x_j} \eta_i \cdot v_2^{(k-1)} - \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial v_1^{(k)}}{\partial x_j} \eta_i \cdot v_2^{(k)} \\
&+ \alpha\, v_2^{(k-1)} \cdot a^{(k-1)} (v_3^{(k)} \mathbf{x}\, \eta) - \alpha\, v_2^{(k)} \cdot a^{(k)} (v_3^{(k)} \mathbf{x} \eta) \\
&+ b^{(k-1)} v_4^{(k-1)} \cdot a^{(k-1)} (v_3^{(k-1)} \mathbf{x}\, \eta) - b^{(k)} v_4^{(k)} \cdot a^{(k)} (v_3^{(k)} \mathbf{x}\, \eta) \Big\} dS_k \\
&+ \int_S \Big\{ \sum_{i,j=1}^{3} A_{ij} \frac{\partial v_1}{\partial x_j} \eta_i \cdot v_2 + \alpha\, a\, v_2 \cdot (v_3 \mathbf{x}\, \eta) \\
&+ ab\, v_4 \cdot (v_3 \mathbf{x}\, \eta) + d\, v_2 \cdot v_1 \Big\} dS = \int_S \{ -\beta |v_2|^2 - \theta\, a\, b |v_3 \mathbf{x}\, \eta|^2 \} dS \le 0.
\end{aligned}
$$

In a similar way we can show that $\mathcal{A}^*$ is also a dissipative operator. □

It follows from Lemma 2.1 that the operator $\mathcal{A}$ generates an strongly continuous semigroup of contractions $\{U(t)\}_{t \ge 0}$.

Theorem 2.1. *Let M_1 be the orthogonal complement of the subspace $\{w \in \mathcal{D}(\mathcal{A}^*)$ such that $\mathcal{A}^* w = 0\}$ in X. Assume Hypothesis I given in the introduction (with $\rho \equiv 1$) and let $f = (f_1, f_2, f_3, f_4) \in M_1 \cap \mathcal{D}(\mathcal{A})$, then there exists a unique solution $\{u(x,t), E(x,t), H(x,t)\}$ of problem (1.4), (1.5), (1.6) with boundary conditions (1.3) on $S = \partial\Omega$ such that*

$$
\begin{aligned}
&u \in C(0, +\infty; [H^2(\Omega_k)]^3) \cap C^1(0, +\infty; [H^1(\Omega_k)]^3) \\
&u_{tt} \in C(0, +\infty; [L^2(\Omega_k)]^3) \\
&E, H \in C(0, +\infty; [H^1(\Omega_k)]^3) \cap C^1(0, +\infty; [L^2(\Omega_k)]^3)
\end{aligned}
$$

for all $k = 0, 1, \ldots, n$. Moreover, E and H satisfy

$$E^{(k-1)} \cdot \eta = E^{(k)} \cdot \eta \quad \text{on} \quad S_k$$

$$H^{(k-1)} \cdot \eta = H^{(k)} \cdot \eta \quad on \quad S_k \tag{2.4}$$

$k = 1, 2, \ldots, n$ and we have that

$$||(u, u_t, E, H)||_X \leq ||f||_X .$$

Proof. The first part of Theorem 2.1 it follows from the semigroup theory. It remains to prove (2.4) and clarify why div $E^{(k)} =$ div $H^{(k)} = 0$, $k = 0, 1, \ldots, n$. Observe that the kernel of $\mathcal{A}^*$ is non-empty since it contains elements of the form

$$(0, 0, a^{-1} \operatorname{grad} \varphi_1, b^{-1} \operatorname{grad} \varphi_2)$$

where $\varphi_1, \varphi_2 \in H^2(\Omega) \cap H_0^1(\Omega)$. Also, we observe that elements $v = (v_1, v_2, v_3, v_4)$ belonging to $M_1 \cap \mathcal{D}(\mathcal{A})$ enjoy the property

$$\operatorname{div} v_3^{(k)} = \operatorname{div} v_4^{(k)} = 0, \quad 0, 1, \ldots, n$$

in the sense of distributions. It can be shown (as was done in reference [8]) that whenever $v \in M_1 \cap \mathcal{D}(\mathcal{A})$, then the components v_3 and v_4 do belong to $[H^1(\Omega_k)]^3$, $k = 0, 1, 2, \ldots, n$. If $v \in M_1 \cap \mathcal{D}(\mathcal{A})$ and $w \in \ker(\mathcal{A}^*)$, then

$$\frac{d}{dt}(U(t)v, w)_X = (\mathcal{A}U(t)v, w)_X = (U(t)v, \mathcal{A}^* w)_X = 0.$$

Now, let us prove (2.4). As we mentioned above, elements of the form $w = (0, 0, a^{-1} \operatorname{grad} \varphi, 0)$ belong to the kernel of $\mathcal{A}^*$ whenever $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$. Let $v = (v_1, v_2, v_3, v_4) \in M_1 \cap \mathcal{D}(\mathcal{A})$, then

$$0 = (v, w)_X = \sum_{k=0}^{n} \int_{\Omega_k} v_3^{(k)} \cdot \operatorname{grad} \varphi \, dx = \int_{S_1} v_3^{(0)} \cdot \eta \, \varphi \, dS_1$$
$$- \int_{S_1} v_3^{(1)} \cdot \eta \, \varphi \, dS_1 + \cdots + \int_{S_n} v_3^{(n-1)} \cdot \eta \, \varphi \, dS_n - \int_{S_n} v_3^{(n)} \cdot \eta \, \varphi \, dS_n .$$

Let us choose $\varphi \equiv 0$ on $S_1, S_2, \ldots, S_{k-1}, S_{k+1}, \ldots, S_n$, then

$$\int_{S_k} \{v_3^{(k-1)} \cdot \eta - v_3^{(k)} \cdot \eta\} \varphi \, dS_k = 0$$

which implies that $v_3^{(k-1)} \cdot \eta = v_3^{(k)} \cdot \eta$ on S_k, $k = 1, 2, \ldots, n$. Similarly we can show that $v_4^{(k-1)} \cdot \eta = v_4^{(k)} \cdot \eta$ on S_k. □

Corollary 2.1. *Assume Hypothesis I given in the introduction (with $\rho \equiv 1$). Let $f = (f_1, f_2, f_3, f_4) \in X$, then $U(t)f$ is the weak solution of the problem*

$$\frac{dw}{dt} = \mathcal{A}w, \quad w(0) = f.$$

Proof. Let $f^n = (f_1^n, f_2^n, f_3^n, f_4^n) \in \mathcal{D}(\mathcal{A})$ such that $f^n \to f$ in X as $n \to +\infty$. Then $U(t)f^n$ satisfies the following identity

$$\int_0^T \Big\{ \Big(U(t)f^n, \frac{d\psi}{dt} \Big)_X + (U(t)f^n, \mathcal{A}^*\psi)_X \Big\} dt = -(f^n, \psi(0))_X \tag{2.5}$$

for any $\psi \in L^2(0,T;\mathcal{D}(\mathcal{A}^*))$ such that $\psi_t \in L^2(0,T;X)$ and $\psi(T) = 0$. Passing to the limit in (2.5) as $n \to +\infty$, we obtain

$$\int_0^T \Big\{ \Big(U(t)f, \frac{d\psi}{dt} \Big)_X + (U(t)f, \mathcal{A}^*\psi)_X \Big\} dt = -(f, \psi(0))_X \tag{2.6}$$

which proves Corollary 2.1.

Remark. (2.6) implies that $U(t)$ takes M_1 (defined in Theorem 2.1) into itself. In fact, let $g \in \mathrm{Ker}(\mathcal{A}^*)$ and take $\psi(t) = (T-t)g$, then from (2.6) it follows that

$$\int_0^T (U(t)f, g)_X \, dt = T(f,g)_X$$

which implies that $(U(t)f, g)_X = (f,g)_X \ \forall\, t \geq 0$.

3. Stabilization

The proof of stabilization is based on the theory of multipliers and it is motivated by the invariance of system (1.1) with constant coefficients relative to the one-parameter group of dilations in all variables. The multipliers have to be conveniently modified in such a way that the extra boundary terms appearing in the identities can be controlled by appropriate bounds. Let $\varphi = \varphi(x)$ be an auxiliary (scalar) smooth function on $\overline{\Omega}$ which we will choose later. Let us fix $t_0 > 0$ and consider M_1 defined as

$$M_1 u = (t+t_0)u_t + (\nabla\varphi \cdot \nabla)u + u,$$

where $u = u(x,t) = (u_1, u_3, u_3)$,

$$\nabla = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\Big) \ \text{ and } \ \nabla\varphi \cdot \nabla = \frac{\partial\varphi}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial\varphi}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{\partial\varphi}{\partial x_3}\frac{\partial}{\partial x_3}.$$

We also consider the multipliers

$$M_2 = (t+t_0)E + \nabla\varphi \mathbf{x} H \tag{3.1}$$

$$M_3 = (t+t_0)H - \nabla\varphi \mathbf{x} E - \alpha(\nabla\varphi \cdot \nabla)u - \alpha u. \tag{3.2}$$

We take the inner produce (in $\mathbb{R}^3$) of $M_1 u$, M_2 and M_3 with

$$u_{tt} - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\Big(A_{ij}\,\frac{\partial u}{\partial x_j}\Big) + \alpha\,\mathrm{curl}(A(x)E) = 0,$$

$$E_t - \mathrm{curl}(b(x)H) - \mathrm{curl}(\alpha(x)u_t) = 0 \quad \text{and} \quad H_t + \mathrm{curl}(a(x)E) = 0$$

respectively. Adding the identities we obtain that

$$\frac{\partial}{\partial t}\,D - \mathrm{div}_x\; F - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\,G_i - J = 0, \tag{3.1}$$

where

$$D = (t+t_0)\Big\{|u_t|^2 + \sum_{i,j=1}^{3} A_{ij}\,\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_i} + a|E|^2 + b|H|^2\Big\} \tag{3.3}$$
$$+\,2u_t\cdot[(\nabla\varphi\cdot\nabla)u] + 2\,u\cdot u_t + 2\,\nabla\varphi\cdot(H\mathbf{x}E) - 2\alpha\,H\mathbf{x}u - 2\alpha\,H\cdot(\nabla\varphi\cdot\nabla)u$$

$$F = (2(t+t_0)ab\,H\mathbf{x}E + \nabla\varphi(a|E|^2 + b|H|^2) - 2aE(E\cdot\nabla\varphi) - 2bH(H\cdot\nabla\varphi)$$
$$+\,2(t+t_0)\alpha a\,u_t\mathbf{x}E - 2\alpha H(u_t\cdot\nabla\varphi) \tag{3.4}$$

$$G_i = 2\big[(t+t_0)u_t + (\nabla\varphi\cdot\nabla)u + u\big]\cdot\sum_{i,j=1}^{3} A_{ij}\frac{\partial u}{\partial x_j}$$
$$+\,\frac{\partial\varphi}{\partial x_i}\Big\{|u_t|^2 - \sum_{p,q=1}^{3} A_{pq}\frac{\partial u}{\partial x_q}\cdot\frac{\partial u}{\partial x_p}\Big\} \tag{3.5}$$

and

$$J = (\Delta\varphi - 1)\sum_{i,j=1}^{3} A_{ij}\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_i} - 2\sum_{i,j,p=1}^{3}\frac{\partial^2\varphi}{\partial x_i\partial x_p}A_{ij}\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_p}$$
$$+\,(3-\Delta\varphi)|u_t|^2 + 2\sum_{i,j=1}^{3}\frac{\partial^2\varphi}{\partial x_i\partial x_j}(aE_iE_j + bH_iH_j) \tag{3.6}$$
$$-(\Delta\varphi - 1)(a|E|^2 + b|H|^2) + 2\alpha\,H\cdot[(u_t\cdot\nabla)\nabla\varphi - u_t] + 2(\nabla\varphi\cdot\nabla\alpha)(H\cdot u_t).$$

Observe that if we consider $\alpha \equiv$ constant and we take $\varphi(x) = \frac{1}{2}\,|x-x_0|^2$ for some fixed $x_0 \in \Omega$, then $J \equiv 0$. In this case (3.1) will be a conservation law. However, because of the expressions of F and G_i we can see that we will need (after integration in Ω_k, $k = 0, 1, \ldots, n$) a definite sign of $\frac{\partial\varphi}{\partial\eta}$ · An idea would be to choose $\varphi(x)$ as a "little" perturbation of $\frac{1}{2}\,|x-x_0|^2$ for some $x_0 \in \Omega$. Let us called (3.1) the Fundamental Identity. Let $f = (f_1, f_2, f_3, f_4) \in M_1 \cap \mathcal{D}(\mathcal{A})$ and $\{u, E, H\}$ be the solution of problem (1.4), (1.5), (1.6) with boundary conditions (1.3) on $S = \partial\Omega$ obtained in Theorem 2.1. Integration over $\Omega_k \times (0, T)$ of the Fundamental Identity (3.1) and summation over k implies that

$$(t+t_0)\sum_{k=0}^{n}\int_{\Omega_k}\Big\{|u_t^{(k)}|^2 + \sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\cdot\frac{\partial u^{(k)}}{\partial x_i} + a^{(k)}|E^{(k)}|^2 + b^{(k)}|H^{(k)}|^2\Big\}dx\Big|_{t=0}^{t=T}$$
$$+ 2\sum_{k=0}^{n}\int_{\Omega_k}\Big\{u_t^{(k)}\cdot(\nabla\varphi\cdot\nabla)u^{(k)} + u_t^{(k)}\cdot u^{(k)} + \nabla\varphi\cdot(H^{(k)}\mathbf{x}E^{(k)}) - \alpha\, H^{(k)}\cdot u^{(k)}$$
$$- \alpha(H^{(k)}\cdot(\nabla\varphi\cdot\nabla)u^{(k)}\Big\}dx\Big|_{t=0}^{t=T} = \sum_{k=1}^{n}\int_0^T\int_{S_k}(B_{k-1} - B_k)dS_k\,dt + \int_0^T\int_S B_n\,dS\,dt$$
$$+ \sum_{k=0}^{n}\int_0^T\int_{\Omega_k} J_k(x,t)\,dxdt, \tag{3.2}$$

where

$$B_k = 2\{(t+t_0)u_t^{(k)} + (\nabla\varphi\cdot\nabla)u^{(k)} + u^{(k)}\}\cdot\sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\eta_i$$
$$+ \frac{\partial\varphi}{\partial\eta}\Big\{|u_t^{(k)}|^2 - \sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\cdot\frac{\partial u^{(k)}}{\partial x_i}\Big\} + 2(t+t_0)a^{(k)}b^{(k)}\,\eta\cdot(H^{(k)}\mathbf{x}E^{(k)})$$
$$+ \frac{\partial\varphi}{\partial\eta}\{a^{(k)}|E^{(k)}|^2 + b^{(k)}|H^{(k)}|^2\} - 2a^{(k)}(E^{(k)}\cdot\eta)(E^{(k)}\cdot\nabla\varphi)$$
$$- 2\,b^{(k)}(H^{(k)}\cdot\eta)(H^{(k)}\cdot\nabla\varphi) + 2(t+t_0)\alpha\,a^{(k)}\,\eta\cdot(u_t^{(k)}\mathbf{x}E^{(k)})$$
$$- 2\,\alpha(H^{(k)}\cdot\eta)(u_t^{(k)}\cdot\nabla\varphi) \tag{3.3}$$

and

$$J_k = (\Delta\varphi - 1)\sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\cdot\frac{\partial u^{(k)}}{\partial x_i} - 2\sum_{i,j,p=1}^{3}\frac{\partial^2\varphi}{\partial x_i\partial x_p}A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\cdot\frac{\partial u^{(k)}}{\partial x_p}$$
$$+ (3-\Delta\varphi)|u_t^{(k)}|^2 + 2\sum_{i,j=1}^{3}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\big(a^{(k)}E_i^{(k)}E_j^{(k)} + b^{(k)}H_i^{(k)}H_j^{(k)}\big)$$
$$- (\Delta\varphi - 1)(a^{(k)}|E^{(k)}|^2 + b^{(k)}|H^{(k)}|^2)$$
$$+ 2\alpha\,H^{(k)}\cdot\{(u_t^{(k)}\cdot\nabla)\nabla\varphi - u_t^{(k)}\} + 2(\nabla\varphi\cdot\nabla\alpha)H^{(k)}\cdot u_t^{(k)}. \tag{3.4}$$

Here, $\frac{\partial\varphi}{\partial\eta}$ denotes the normal derivative of φ at $x \in S_k$.

The following Lemma will tell us that we the differences $B_{k-1} - B_k$ in(3.2) will have a "good" sign if we choose φ conveniently and we have a monotonicity conditions on $\{A_{ij}^{(k)}\}$, $\{a^{(k)}\}$ and $\{b^{(k)}\}$.

Lemma 3.1. *Let $f = (f_1, f_2, f_3, f_4) \in M_1 \cap \mathcal{D}(\mathcal{A})$ and let $\{u, E, H\}$ be the solution of problem* (1.4), (1.5), (1.6) *with boundary conditions* (1.3) *on $S = \partial\Omega$ obtained in Theorem* 2.1. *Then, the identity*

$$B_{k-1} - B_k = -\frac{\partial\varphi}{\partial\eta}\Big[\sum_{i,j=1}^{3} A_{ij}^{(k)}\Big(\frac{\partial u^{(k-1)}}{\partial x_j} - \frac{\partial u^{(k)}}{\partial x_j}\Big)\cdot\Big(\frac{\partial u^{(k-1)}}{\partial x_i} - \frac{\partial u^{(k)}}{\partial x_i}\Big)$$

$$+ \sum_{i,j=1}^{3}(A_{ij}^{(k-1)} - A_{ij}^{(k)})\frac{\partial u^{(k-1)}}{\partial x_j}\cdot\frac{\partial u^{(k-1)}}{\partial x_i}$$

$$+ (a^{(k-1)} - a^{(k)})\Big\{|E^{(k-1)}\cdot\eta|^2 + \frac{a^{(k-1)}}{a^{(k)}}|E^{(k-1)}\mathbf{x}\,\eta|^2\Big\}$$

$$+ (b^{(k-1)} - b^{(k)})\Big\{|H^{(k-1)}\cdot\eta|^2 + \frac{b^{(k-1)}}{b^{(k)}}|H^{(k-1)}\mathbf{x}\,\eta|^2\Big\}\Big]$$

holds for $k = 1, 2, \ldots, n$.

Proof. The idea is to use the interface conditions (1.6) and property (2.4) on S_k. In fact

$$B_{k-1} - B_k$$

$$= 2(\nabla\varphi\cdot\nabla)u^{(k-1)}\cdot\sum_{i,j=1}^{3} A_{ij}^{(k-1)}\frac{\partial u^{(k-1)}}{\partial x_j}\eta_i - 2(\nabla\varphi\cdot\nabla)u^{(k)}\cdot\sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\eta_i$$

$$-\frac{\partial\varphi}{\partial\eta}\Big\{\sum_{i,j=1}^{3} A_{ij}^{(k-1)}\frac{\partial u^{(k-1)}}{\partial x_j}\cdot\frac{\partial u^{(k-1)}}{\partial x_i} - \sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\cdot\frac{\partial u^{(k)}}{\partial x_i}\Big\}$$

$$+\frac{\partial\varphi}{\partial\eta}\{a^{(k-1)}|E^{(k-1)}|^2 - a^{(k)}|E^{(k)}|^2 + b^{(k-1)}|H^{(k-1)}|^2 - b^{(k)}|H^{(k)}|^2\}$$

$$-2\,a^{(k-1)}(E^{(k-1)}\cdot\eta)(E^{(k-1)}\cdot\nabla\varphi) + 2\,a^{(k)}(E^{(k)}\cdot\eta)(E^{(k)}\cdot\nabla\varphi)$$

$$-2\,b^{(k-1)}(H^{(k-1)}\cdot\eta)(H^{(k-1)}\cdot\nabla\varphi) + 2\,b^{(k)}(H^{(k)}\cdot\eta)(H^{(k)}\cdot\nabla\varphi). \tag{3.5}$$

Furthermore, using (1.6) and (2.4) we deduce the identities

$$a^{(k)}|E^{(k)}|^2 = a^{(k)}|E^{(k-1)}\cdot\eta|^2 + \frac{(a^{(k-1)})^2}{a^{(k)}}|E^{(k-1)}\mathbf{x}\,\eta|^2$$

$$b^{(k)}|H^{(k)}|^2 = b^{(k)}|H^{(k-1)} \cdot \eta|^2 + \frac{(b^{(k-1)})^2}{b^{(k)}}|H^{(k-1)} \mathbf{x}\, \eta|^2$$

$$a^{(k)}(E^{(k)} \cdot \eta)(E^{(k)} \cdot \nabla\varphi) = a^{(k-1)}(E^{(k-1)} \cdot \eta)(E^{(k-1)} \cdot \nabla\varphi) + \frac{\partial\varphi}{\partial\eta}\,(a^{(k)} - a^{(k-1)})|E^{(k-1)} \cdot \eta|^2 \qquad \text{and}$$

$$b^{(k)}(H^{(k)} \cdot \eta)(H^{(k)} \cdot \nabla\varphi) = b^{(k-1)}(H^{(k-1)} \cdot \eta)(H^{(k-1)} \cdot \nabla\varphi) + \frac{\partial\varphi}{\partial\eta}\,(b^{(k)} - b^{(k-1)})|H^{(k-1)} \cdot \eta|^2. \tag{3.6}$$

Since $u^{(k-1)} - u^{(k)} = 0$ on S_k then we know that

$$\frac{\partial}{\partial x_i}\,(u^{(k-1)} - u^{(k)}) = \eta_i\,\frac{\partial}{\partial\eta}\,(u^{(k-1)} - u^{(k)}) \tag{3.7}$$

on S_k. Consequently, on S_k the identities

$$2(\nabla\varphi \cdot \nabla)u^{(k-1)} \cdot \sum_{i,j=1}^{3} A_{ij}^{(k-1)}\frac{\partial u^{(k-1)}}{\partial x_j}\eta_i - 2(\nabla\varphi \cdot \nabla)u^{(k)} \cdot \sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\eta_i$$

$$= (\nabla\varphi \cdot \nabla)(u^{(k-1)} - u^{(k)}) \cdot \sum_{i,j=1}^{3} A_{ij}^{(k-1)}\frac{\partial u^{(k-1)}}{\partial x_j}\eta_i + (\nabla\varphi \cdot \nabla)(u^{(k-1)} - u^{(k)}) \cdot \sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\eta_i$$

$$= \frac{\partial\varphi}{\partial\eta}\left\{\sum_{i,j=1}^{3} A_{ij}^{(k-1)}\frac{\partial u^{(k-1)}}{\partial x_j} \cdot \frac{\partial}{\partial x_i}\,(u^{(k-1)} - u^{(k)})\right\} + \frac{\partial\varphi}{\partial\eta}\left\{\sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial}{\partial x_i}\,(u^{(k-1)} - u^{(k)})\right\} \tag{3.8}$$

hold. This give us the relation

$$K \equiv 2(\nabla\varphi \cdot \nabla)u^{(k-1)} \cdot \sum_{i,j=1}^{3} A_{ij}^{(k-1)}\frac{\partial u^{(k-1)}}{\partial x_j}\eta_i - 2(\nabla\varphi \cdot \nabla)u^{(k)} \cdot \sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\eta_i$$

$$- \frac{\partial\varphi}{\partial\eta}\sum_{i,j=1}^{3} A_{ij}^{(k-1)}\frac{\partial u^{(k-1)}}{\partial x_j} \cdot \frac{\partial u^{(k-1)}}{\partial x_i} + \frac{\partial\varphi}{\partial\eta}\sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i}$$

$$= \frac{\partial\varphi}{\partial\eta}\Big\{ -\sum_{i,j=1}^{3} A_{ij}^{(k-1)}\frac{\partial u^{(k-1)}}{\partial x_j}\cdot\frac{\partial u^{(k)}}{\partial x_i} + \sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\cdot\frac{\partial u^{(k-1)}}{\partial x_i}\Big\} \tag{3.9}$$

Again, using the interface relations and (3.7) we obtain the identities

$$\begin{aligned}
&\sum_{i,j=1}^{3}(A_{ij}^{(k)} - A_{ij}^{(k-1)})\Big(\frac{\partial u^{(k-1)}}{\partial x_j} - \frac{\partial u^{(k)}}{\partial x_j}\Big)\cdot\frac{\partial u^{(k-1)}}{\partial x_i}\\
&= \sum_{i,j=1}^{3} A_{ij}^{(k)}\Big(\frac{\partial u^{(k-1)}}{\partial\eta} - \frac{\partial u^{(k)}}{\partial\eta}\Big)\eta_j\cdot\frac{\partial u^{(k-1)}}{\partial x_i}\\
&- \sum_{i,j=1}^{3} A_{ij}^{(k-1)}\Big(\frac{\partial u^{(k-1)}}{\partial\eta} - \frac{\partial u^{(k)}}{\partial\eta}\Big)\eta_j\cdot\frac{\partial u^{(k-1)}}{\partial x_i}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i,j=1}^{3} A_{ij}^{(k-1)}\Big(\frac{\partial u^{(k-1)}}{\partial\eta} - \frac{\partial u^{(k)}}{\partial\eta}\Big)\eta_j\cdot\frac{\partial u^{(k-1)}}{\partial x_i}\\
&= \sum_{i,j=1}^{3} A_{ij}^{(k-1)}\frac{\partial u^{(k-1)}}{\partial x_j}\eta_i\cdot\Big(\frac{\partial u^{(k-1)}}{\partial\eta} - \frac{\partial u^{(k)}}{\partial\eta}\Big)\\
&= \sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\eta_i\cdot\Big(\frac{\partial u^{(k-1)}}{\partial\eta} - \frac{\partial u^{(k)}}{\partial\eta}\Big)\\
&= \sum_{i,j=1}^{3} A_{ij}^{(k)}\Big(\frac{\partial u^{(k-1)}}{\partial\eta} - \frac{\partial u^{(k)}}{\partial\eta}\Big)\eta_j\cdot\frac{\partial u^{(k)}}{\partial x_i}.
\end{aligned} \tag{3.10}$$

We observe that the identity

$$\begin{aligned}
&\sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\cdot\frac{\partial u^{(k-1)}}{\partial x_i} - \sum_{i,j=1}^{3} A_{ij}^{(k-1)}\frac{\partial u^{(k-1)}}{\partial x_j}\cdot\frac{\partial u^{(k)}}{\partial x_i}\\
&= \sum_{i,j=1}^{3}(A_{ij}^{(k)} - A_{ij}^{(k-1)})\frac{\partial u^{(k-1)}}{\partial x_j}\cdot\frac{\partial u^{(k-1)}}{\partial x_i}\\
&- \sum_{i,j=1}^{3}(A_{ij}^{(k)} - A_{ij}^{(k-1)})\Big(\frac{\partial u^{(k-1)}}{\partial x_j} - \frac{\partial u^{(k)}}{\partial x_j}\Big)\cdot\frac{\partial u^{(k-1)}}{\partial x_i}
\end{aligned} \tag{3.11}$$

holds. Combining (3.10) with (3.11) we deduce that the left hand-side of (3.9) it is equal to

$$K \equiv -\frac{\partial\varphi}{\partial\eta}\Big[\sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial}{\partial x_j}(u^{(k-1)}-u^{(k)})\cdot\frac{\partial}{\partial x_i}(u^{(k-1)}-u^{(k)}) + \sum_{i,j=1}^{3}(A_{ij}^{(k-1)}-A_{ij}^{(k)})\frac{\partial u^{(k-1)}}{\partial x_j}\cdot\frac{\partial u^{(k-1)}}{\partial x_i}\Big]. \tag{3.12}$$

The conclusion of Lemma 3.1 it follows from (3.5), (3.6) and (3.12). □

Let us choose a convenient function $\varphi(x)$: Let $\Phi(x)$ be the solution of the Neumann problem

$$\begin{cases}\Delta\Phi = 1 \quad \text{in} \quad \Omega\\ \dfrac{\partial\Phi}{\partial\eta} = \dfrac{\text{measure}(\Omega)}{\text{area}(S)} \quad \text{on} \quad \partial\Omega.\end{cases}$$

Thus, $\Phi \in C^2(\Omega)\cap C^1(S)$. Let $\delta > 0$ and $x_0 \in \Omega$ we consider the function

$$\varphi(x) = \delta\,\Phi(x) + \frac{1}{2}\,|x-x_0|^2. \tag{3.13}$$

An easy calculation shows that

$$\frac{\partial\varphi}{\partial\eta} = \delta\,\frac{\text{measure}(\Omega)}{\text{area}(S)} + (x-x_0)\cdot\eta.$$

In order to obtain definite sign for $B_{k-1}-B_k$ we would require that $\frac{\partial\varphi}{\partial\eta} \geq 0$ on S_k, $k = 1, 2, \ldots, n$.

Hypothesis II. Let $R = \max_{\overline{\Omega}}|x-x_0|$, where $x_0 \in \Omega$ will be chosen later, $b_0 = \min_{x\in\overline{\Omega}, k=0,\ldots,n} b^{(k)}(x)$ and

$$\gamma_1 = \max_{x\in\overline{\Omega}, x\neq x_0}\Big|\nabla\alpha\cdot\frac{(x-x_0)}{|x-x_0|}\Big|.$$

We assume the following condition on $\alpha(x)$: $\gamma_1 < \frac{\sqrt{b_0}}{4R}$. Now, we concentrate in estimating the term

$$\sum_{k=0}^{n}\int_0^T\int_{\Omega_k} J_k\,dx\,dt$$

in (3.2) where J_k is given by (3.4).

Lemma 3.2. *Under the assumptions of Lemma* 3.1 *Hypothesis II, Hypothesis I with* $\rho \equiv 1$, $\theta(x) > 0$, $d(x) > 0$, $\beta(x) > 0$ *on* S *and choosing* $\varphi(x)$ *as in* (3.13) *we have that*

$$\sum_{k=0}^{n} \int_0^T \int_{\Omega_k} J_k(x,t)dxdt \le \left(\delta\, c_5 + \frac{4R\gamma_1}{\sqrt{b_0}}\right) \sum_{k=0}^{3} \int_0^T \int_{\Omega_k} \Big\{ |u_t^{(k)}|^2 + \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i} + a^{(k)}|E^{(k)}|^2 + b^{(k)}|H^{(k)}|^2 \Big\} dx\, dt$$

for some positive constant c_5 (*independent of* $u^{(k)}$, $E^{(k)}$, $H^{(k)}$), *where* R, b_0 *and* γ_1 *are as in Hypothesis II.*

Proof. Straightforward calculations using (3.4) and $\varphi(x)$ chosen as in (3.13) lead us to the identity

$$\begin{aligned} J_k &= \delta \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i} - 2\delta \sum_{i,j,p=1}^{3} \frac{\partial^2 \Phi}{\partial x_i \partial x_p} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_p} \\ &- \delta |u_t^{(k)}|^2 + 2\delta \sum_{i,j=1}^{3} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \left(a^{(k)} E_i^{(k)} E_j^{(k)} + b^{(k)} H_i^{(k)} H_j^{(k)} \right) \\ &- \delta \left(a^{(k)} |E^{(k)}|^2 + b^{(k)} |H^{(k)}|^2 \right) + 2\delta\alpha\, H^{(k)} \cdot (u_t^{(k)} \cdot \nabla) \nabla \Phi \\ &+ 2\delta (\nabla \Phi \cdot \nabla \alpha) H^{(k)} \cdot u_t^{(k)} + 2(\nabla \alpha \cdot (x - x_0) H^{(k)} \cdot u_t^{(k)}. \end{aligned} \tag{3.14}$$

For any $\varepsilon > 0$ we have that

$$\begin{aligned} &- 2\delta \sum_{i,j,p=1}^{3} \frac{\partial^2 \Phi}{\partial x_i \partial x_p} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_p} \le \delta\varepsilon \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i} \\ &+ \frac{\delta}{\varepsilon} \sum_{i,j=1}^{3} A_{ij}^{(k)} \Big(\sum_{p=1}^{3} \frac{\partial^2 \Phi}{\partial x_p \partial x_j} \frac{\partial u^{(k)}}{\partial x_p} \Big) \cdot \Big(\sum_{p=1}^{3} \frac{\partial^2 \Phi}{\partial x_p \partial x_i} \frac{\partial u^{(k)}}{\partial x_p} \Big). \end{aligned} \tag{3.15}$$

Let

$$c_2 = \max_{x \in \overline{\Omega}, i,j=1,2,3} ||A_{ij}(x)||, \qquad c_3 = \max_{x \in \overline{\Omega}, i,j=1,2,3} \left| \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \right|$$

and $c_4 = \max_{x\in\Omega} |\nabla\Phi|$. Observe that $c_3 \geq \frac{1}{3}$. With this notations using Cauchy-Schwarz's inequality we obtain that

$$\Big| \sum_{i,j=1}^{3} A_{ij}^{(k)} \Big(\sum_{p=1}^{3} \frac{\partial^2 \Phi}{\partial x_p \partial x_j} \frac{\partial u^{(k)}}{\partial x_p} \Big) \cdot \Big(\sum_{p=1}^{3} \frac{\partial^2 \Phi}{\partial x_p \partial x_i} \frac{\partial u^{(k)}}{\partial x_p} \Big) \Big|$$

$$\leq 27\, c_2 c_3^2 c_0^{-1} \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i}, \tag{3.16}$$

where c_0 is the positive constant in Hypothesis I ((6)).

From (3.15) and (3.16) we obtain that

$$-2\delta \sum^{3} \frac{\partial^2 \Phi}{\partial x_i \partial x_p} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_p} \leq \delta \Big\{ \varepsilon + \frac{27}{\varepsilon c_0}\, c_2 c_3^2 \Big\} \sum_{i,j=1}^{2} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i}. \tag{3.17}$$

Let us bound the last three terms in (3.14). For any positive numbers $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ we have that

$$2\delta\alpha H^{(k)} \cdot (u_t^{(k)} \cdot \nabla)\nabla\Phi + 2\delta(\nabla\Phi \cdot \nabla\alpha) H^{(k)} \cdot u_t^{(k)} + 2(\nabla\alpha \cdot (x - x_0)) H^{(k)} \cdot u_t^{(k)}$$
$$\leq 2\delta\alpha\varepsilon_1 |H^{(k)}|^2 + 18\delta\alpha\varepsilon_1^{-1} c_3^2 |u_t^{(k)}|^2 + 2\delta c_4 |\nabla\alpha| \varepsilon_2^{-1} |u_t^{(k)}|^2 + 2\delta c_4 |\nabla\alpha| \varepsilon_2 |H^{(k)}|^2$$
$$+ 2\varepsilon_2 |\nabla\alpha \cdot (x - x_0)|\, |H^{(k)}|^2 + 2\,\varepsilon_2^{-1} |\nabla\alpha \cdot (x - x_0)|\, |u_t^{(k)}|^2. \tag{3.18}$$

Furthermore,

$$2\delta \sum_{i,j=1}^{3} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \Big(a^{(k)} E_i^{(k)} E_j^{(k)} + b^{(k)} H_i^{(k)} H_j^{(k)} \Big)$$
$$\leq 6\delta\, c_3 \Big(a^{(k)} |E^{(k)}|^2 + b^{(k)} |H^{(k)}|^2 \Big). \tag{3.19}$$

From (3.17), (3.18) and (3.19) we obtain a bound for J_k for any $\varepsilon > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$

$$J_k \leq \delta \Big\{ 1 + \varepsilon + \frac{27}{\varepsilon c_0} c_2 c_3^2 \Big\} \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i}$$
$$+ \delta \{ 18\alpha\varepsilon_1^{-1} c_3^2 + 2\, c_4 |\nabla\alpha| \varepsilon_2^{-1} + 2\,\varepsilon_2^{-1} |\nabla\alpha \cdot (x - x_0)| - 1 \} |u_t^{(k)}|^2$$
$$+ \delta \{ 6c_3 - 1 \} a^{(k)} |E^{(k)}|^2 + \delta \Big\{ 6c_3 - 1 + \frac{2\alpha\varepsilon_1}{b^{(k)}} + \frac{2c_4 |\nabla\alpha| \varepsilon_2}{b^{(k)}} \Big\} b^{(k)} |H^{(k)}|^2$$
$$+ 2\,\varepsilon_2^{-1} |\nabla\alpha \cdot (x - x_0)|\, |u_t^{(k)}|^2 + \frac{2\varepsilon_2}{b^{(k)}} |\nabla\alpha \cdot (x - x_0)| b^{(k)} |H^{(k)}|^2. \tag{3.20}$$

Observe that $|\nabla\alpha \cdot (x - x_0)| \le R\gamma_1$, $c_3 \ge 1/3$ and $\frac{1}{b^{(k)}} \le \frac{1}{b_0}$, therefore by taking ε_1 large enough all coefficients of δ in (3.20) are positive. The maximum of these coefficients let us denote by $c_5 > 0$. The last two terms in the right hand side of (3.20) can be bounded by

$$2\,\varepsilon_2^{-1}\,R\gamma_1|u_t^{(k)}|^2 + \frac{2\varepsilon_2}{b_0}\,R\gamma_1\,b^{(k)}\,|H^{(k)}|^2. \tag{3.21}$$

Now, we choose $\varepsilon_2 = \sqrt{b_0}$ in (3.21) to get the bound

$$\frac{4}{\sqrt{b_0}}\,R\gamma_1\big\{|u_t^{(k)}|^2 + b^{(k)}\,|H^{(k)}|^2\big\}. \tag{3.22}$$

Thus, we deduce from (3.20) and (3.22) that

$$\sum_{k=0}^{n}\int_0^T\int_{\Omega_k} J_k(x,t)dx\,dt \le \left(\delta c_5 + \frac{4}{\sqrt{b_0}}\,R\gamma_1\right)\sum_{k=0}^{n}\int_0^T\int_{\Omega_k}\Bigg\{|u_t^{(k)}|^2 + \sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\cdot\frac{\partial u^{(k)}}{\partial x_i} + a^{(k)}|E^{(k)}|^2 + b^{(k)}|H^{(k)}|^2\Bigg\}dx\,dt.$$

Remark. Since by Hypothesis II we know that $\frac{4R\gamma_1}{\sqrt{b_0}} < 1$, then we can choose constants $\delta > 0$ in Lemma 3.2 so that $\delta\,c_5 + \frac{4R\gamma_1}{\sqrt{b_0}} < 1$, where c_5 is the positive constant we constructed in the proof of Lemma 3.2.

Hypothesis III. There exist a positive constant $\delta_1 > 0$ such that

a) $\delta_1\,\frac{\text{measure}(\Omega)}{\text{area}(S)} + (x - x_0)\cdot\eta \ge 0$ for some point $x_0 \in \Omega$ and all $x \in S_k$, $k = 1, 2, \ldots, n$, where $\eta = \eta(x)$ denotes the unit outward normal to S_k at x.

b) $\delta_1\,\frac{\text{measure}(\Omega)}{\text{area}(S)} + (x - x_0)\cdot\eta > 0$ for all $x \in S$.

c) $\delta_1\,c_5 + \frac{4R\gamma_1}{\sqrt{b_0}} < 1$, where R, γ_1 and b_0 are as in Hypothesis II and c_5 is the positive constant constructed in the proof of Lemma 3.2.

Hypothesis IV. We assume the monotonicity conditions on $\{A_{ij}^{(k)}\}$, $\{a^{(k)}\}$ and $\{b^{(k)}\}$:

$$\sum_{i,j=1}^{3}(A_{ij}^{(k-1)} - A_{ij}^{(k)})v_j\cdot v_i \ge 0 \quad \text{for any} \quad v_i \in \mathbb{R}^3$$
$$a^{(k-1)} \ge a^{(k)} > 0, \quad b^{(k-1)} \ge b^{(k)} > 0$$

for all $k = 1, 2, \ldots, n$.

Now let us get a bound for $\int_0^T \int_S B_n \, dSdt$ in (3.2). Using the boundary conditions (1.3) we can rewrite B_n as

$$\begin{aligned}
B_n = &-\frac{\partial}{\partial t}\{(t+t_0)d|u|^2\} - \frac{\partial}{\partial t}(\beta|u|^2) - \Big\{2\beta(t+t_0) - \frac{\partial\varphi}{\partial\eta}\Big\}|u_t|^2 \\
&- \Big\{2\theta ab(t+t_0) - (b+\theta^2 a)\frac{\partial\varphi}{\partial\eta}\Big\}|E\mathbf{x}\,\eta|^2 - d|u|^2 \\
&- \frac{\partial\varphi}{\partial\eta}\sum_{i,j=1}^{3} A_{ij}\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_i} - \frac{\partial\varphi}{\partial\eta}\{a|E\cdot\eta|^2 + b|H\cdot\eta|^2\} \\
&- 2\beta u_t\cdot(\nabla\varphi\cdot\nabla)u - 2\alpha(u_t\cdot\nabla\varphi)(H\cdot\eta) - 2d\,u\cdot(\nabla\varphi\cdot\nabla)u \\
&- 2a(E\cdot\eta)(E\mathbf{x}\eta)\cdot(\nabla\varphi\mathbf{x}\,\eta) - 2b(H\cdot\eta)(H\mathbf{x}\eta)\cdot(\nabla\varphi\mathbf{x}\eta) \\
&+ 2\alpha a(\nabla\varphi\cdot\nabla)u\cdot(\eta\mathbf{x}E) + 2\alpha a\,u\cdot(\eta\mathbf{x}E) \qquad (3.23)
\end{aligned}$$

for any $x \in S$.

Lemma 3.3. *Under the assumptions of Lemma* 3.2, *and Hypothesis III we can obtain the following bound*

$$\begin{aligned}
\int_0^T \int_S B_n \, dSdt \le &-(t+t_0)\int_S d|u|^2 dS\Big|_{t=0}^{t=T} - \int_S \beta|u|^2 dS\Big|_{t=0}^{t=T} \\
&+ \delta_1 c_5 \int_0^T \int_S d|u|^2 dSdt - \int_0^T \int_S (\gamma_2 - \gamma_3 d)d|u|^2 dSdt, \qquad (3.24)
\end{aligned}$$

where δ_1 and c_5 are as in Hypothesis III and γ_2, γ_3 are positive constants.

Proof. Let ε_j $(1 \le j \le 6)$ be arbitrary positive numbers. Let us bound the terms on the right hand side of (3.23)

$$\begin{aligned}
-2\beta u_t\cdot(\nabla\varphi\cdot\nabla)u &\le 2\beta|u_t|\,|(\nabla\varphi\cdot\nabla)u| \le \beta^2\varepsilon_1^{-1}|\nabla\varphi|\,|u_t|^2 \\
&\quad + \varepsilon_1|\nabla\varphi|(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2) \\
&\le \beta^2\varepsilon_1^{-1}|\nabla\varphi|\,|u_t|^2 + \varepsilon_1|\nabla\varphi|c_0^{-1}\sum_{i,j=1}^{3} A_{ij}\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_i} \qquad (3.25)
\end{aligned}$$

because of Hypothesis I (item (6)). In a similar way we can get a bound for the term

$$-2d\,u\cdot(\nabla\varphi\cdot\nabla)u \le \varepsilon_2^{-1} d^2|\nabla\varphi|\,|u|^2 + \varepsilon_2|\nabla\varphi|c_0^{-1}\sum_{i,j=1}^{3} A_{ij}\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_i}. \quad (3.26)$$

The other terms can be bound in the following way

$$-2b(H\cdot\eta)(H\mathbf{x}\eta)\cdot(\nabla\varphi\mathbf{x}\eta) \le 2b|H\cdot\eta|\,|H\mathbf{x}\eta|\,|\nabla\varphi|$$

$$\leq \varepsilon_4|\nabla\varphi|b|H\,\centerdot\,\eta|^2 + \varepsilon_4^{-1}|\nabla\varphi|b|H\mathbf{x}\eta|^2 \tag{3.27}$$

and

$$\begin{aligned}-2a(E\,\centerdot\,\eta)(E\mathbf{x}\eta)\,\centerdot\,(\nabla\varphi\mathbf{x}\eta) &\leq 2a|E\,\centerdot\,\eta|\,|E\mathbf{x}\eta|\,|\nabla\varphi|\\ &\leq \varepsilon_4|\nabla\varphi|a|E\,\centerdot\,\eta|^2 + \varepsilon_4^{-1}|\nabla\varphi|a|E\mathbf{x}\eta|^2.\end{aligned} \tag{3.28}$$

Similarly

$$\begin{aligned}&-2\alpha(u_t\,\centerdot\,\nabla\varphi)(H\,\centerdot\,\eta) \leq 2|\alpha|\,|\nabla\varphi|\,|u_t|\,|H\,\centerdot\,\eta|\\ &\leq \varepsilon_3|\nabla\varphi|\,|H\,\centerdot\,\eta|^2 + \varepsilon_3^{-1}|\nabla\varphi|\,|\alpha|^2\,|u_t|^2\end{aligned} \tag{3.29}$$

$$\begin{aligned}&2\alpha a(\nabla\varphi\,\centerdot\,\nabla)u\,\centerdot\,(\eta\mathbf{x}E) \leq 2|\alpha|\,a|(\nabla\varphi\,\centerdot\,\nabla)u|\,|\eta\mathbf{x}E|\\ &\leq |\alpha|^2a^2\,\varepsilon_5^{-1}|\nabla\varphi|\,|\eta\mathbf{x}E|^2 + \varepsilon_5|\nabla\varphi|c_0^{-1}\sum_{i,j=1}^{3}A_{ij}\frac{\partial u}{\partial x_j}\,\centerdot\,\frac{\partial u}{\partial x_i}\end{aligned} \tag{3.30}$$

and

$$2\alpha a\,u\,\centerdot\,(\eta\mathbf{x}E) \leq 2|\alpha|\,a|u|\,|\eta\mathbf{x}E| \leq a^2\alpha^2\,\varepsilon_6\,d^{-1}|\eta\mathbf{x}E|^2 + \varepsilon_6^{-1}\,d|u|^2. \tag{3.31}$$

Using estimates (3.25) up to (3.31) we can obtain a bound for B_n (given by (3.24)) as follows:

$$\begin{aligned}B_n \leq &-\frac{\partial}{\partial t}\{(t+t_0)d|u|^2\} - \frac{\partial}{\partial t}(\beta|u|^2) - \Big\{\frac{\partial\varphi}{\partial\eta} - \varepsilon_4|\nabla\varphi|\Big\}a|E\,\centerdot\,\eta|^2\\ &-\Big\{\frac{\partial\varphi}{\partial\eta} - |\nabla\varphi|c_0^{-1}(\varepsilon_1+\varepsilon_2+\varepsilon_5)\Big\}\sum_{i,j=1}^{3}A_{ij}\frac{\partial u}{\partial x_j}\,\centerdot\,\frac{\partial u}{\partial x_i}\\ &-\{1-\varepsilon_6^{-1}-\varepsilon_2^{-1}\,d|\nabla\varphi|\}d|u|^2 - \Big\{\frac{\partial\varphi}{\partial\eta} - \varepsilon_4|\nabla\varphi| - \varepsilon_3|\nabla\varphi|b^{-1}\Big\}b|H\,\centerdot\,\eta|^2\\ &-\Big\{2\beta(t+t_0) - \frac{\partial\varphi}{\partial\eta} - |\nabla\varphi|(\beta^2\varepsilon_1^{-1} + \varepsilon_3^{-1}\,\alpha^2)\Big\}|u_t|^2\\ &-\{2\theta ab(t+t_0) - (b+\theta^2a)|\nabla\varphi| - (\varepsilon_4^{-1}a + \alpha^2a^2\varepsilon_5^{-1})|\nabla\varphi| - a^2\alpha^2\varepsilon_6 d^{-1}\}|E\mathbf{x}\eta|^2\end{aligned} \tag{3.32}$$

for any $x \in S = \partial\Omega$. Let $\delta_0 > 0$ be such that

$$\frac{\partial\varphi}{\partial\eta} \geq \delta_0|\nabla\varphi| \quad \text{for any} \quad x \in S$$

which is possible because $\frac{\partial\varphi}{\partial\eta} > 0$ and S is compact.

Let us choose $\varepsilon_1 = \varepsilon_2 = \varepsilon_5 = \frac{1}{3}\,c_0\delta_0$, $\varepsilon_4 = \frac{1}{2}\,\delta_0$, $\varepsilon_3 = \frac{b}{2}\,\delta_0$ and $\varepsilon_6 = 2$. With this choice the coefficients in (3.32) of $a|E\,\centerdot\,\eta|^2$, $\sum_{i,j=1}^{3}A_{ij}\frac{\partial u}{\partial x_j}\,\centerdot\,\frac{\partial u}{\partial x_i}$ and

$b|H\cdot\eta|^2$ are non-negative. Since $\theta(x)$, $d(x)$ and $\beta(x)$ are strictly positive on S, the quantities

$$A = \max_{x\in S}\left|\frac{1}{2\beta}\left\{\frac{\partial\varphi}{\partial\eta} + |\nabla\varphi|(\beta^2\varepsilon_1^{-1} + \varepsilon_3^{-1}\alpha^2)\right\}\right|$$

and

$$B = \max_{x\in S}\left|\frac{1}{2\theta ab}\{(b+\theta^2 a)|\nabla\varphi| + (\varepsilon_4^{-1}a + \alpha^2 a^2\varepsilon_5^{-1})|\nabla\varphi| - a^2\alpha^2\varepsilon_6 d^{-1}\}\right|$$

are finite, consequently taking $T > 0$ sufficiently large such that $T + t_0 \geq \operatorname{Max}\{A, B\}$, then the coefficients of $|u_t|^2$ and $|E\mathbf{x}\eta|^2$ in (3.32) will be non-negative. Let us study the coefficient of the term $d|u|^2$ in (3.32):

$$\begin{aligned} &-\{1 - \varepsilon_6^{-1} - \varepsilon_2^{-1}\, d|\nabla\varphi|\} = \left\{\frac{1}{2} - \frac{3d}{c_0\delta_0}\,|\nabla\varphi|\right\} \\ &= \delta_1 c_5 - \left\{\frac{1}{2} + \delta_1 c_5 - \frac{3d}{c_0\delta_0}\,|\nabla\varphi|\right\} \leq \delta_1 c_5 - \left\{\frac{1}{2} + \delta_1 c_5 - \frac{3d}{c_0\delta_0}\,\widetilde{c}\right\}, \end{aligned} \tag{3.33}$$

where $\widetilde{c} = \delta_1 \max_{x\in S}|\nabla\Phi| + \max_{x\in S}|x - x_0|$.

From the above considerations and integration (3.32) on $S\times(0,T)$ we obtain the inequality

$$\begin{aligned} \int_0^T\int_S B_n\, dSdt \leq -(t+t_0)\int_S d|u|^2 dS\Big]_{t=0}^{t=T} - \int_S \beta|u|^2 dS\Big]_{t=0}^{t=T} \\ + \delta_2 c_5\int_0^T\int_S d|u|^2 dSdt - \int_0^T\int_S\left\{\frac{1}{2} + \delta_1 c_5 - \frac{3d}{c_0\delta_0}\widetilde{c}\right\} d|u|^2 dS\, dt, \end{aligned}$$

which concludes the proof of Lemma 3.3 with $\gamma_2 = \frac{1}{2} + \delta_1 c_5$ and $\gamma_3 = \frac{3\widetilde{c}}{c_0\delta_0} > 0$.

Theorem 3.1. *Let us assume Hypothesis I, II, III and IV with θ, d, β strictly positive on S and*

$$d \leq \frac{c_0\delta_0}{3\widetilde{c}}\left(\frac{1}{2} + \delta_1 c_5\right) \quad \text{for any} \quad x \in S, \tag{3.34}$$

where the constants δ_0 and $\widetilde{c}$ where chosen in the proof of Lemma 3.3.

Let $f = (f_1, f_2, f_3, f_4) \in M_4 \cap \mathcal{D}(\mathcal{A})$ and $\{u, E, H\}$ be the unique solution of problem (1.4), (1.5), (1.6) with boundary conditions (1.3), obtained in Theorem 2.1. Then there exist positive constants c and w such that

$$\mathcal{E}(t) \leq c\,\exp(-wt)\mathcal{E}(0)$$

for any $t > 0$ where $\mathcal{E}(t)$ is given by (1.7).

Proof. We will use identity (3.2). First, we observe that we need to get a bound for the term

$$I = 2\sum_{k=0}^{n}\int_{\Omega_k}\{u_t^{(k)}\cdot(\nabla\varphi\cdot\nabla)u^{(k)} + u_t^{(k)}\cdot u^{(k)}$$
$$+\nabla\varphi\cdot(H^{(k)}\mathbf{x}E^{(k)}) - \alpha H^{(k)}\cdot u^{(k)} - \alpha H^{(k)}\cdot(\nabla\varphi\cdot\nabla)u^{(k)}\}dx\Big|_{t=0}^{t=T}. \quad (3.35)$$

Each term on the integrand of (3.35) can be bound using Cauchy-Schwarz inequality in the same way we done it in Lemma 3.3. Except that will appear the term $\sum_{k=0}^{n}\int_{\Omega_k}|u^{(k)}|^2dx$. But, since $u^{(k)} \in [H^1(\Omega_k)]^3$ for $k = 0, 1, 2, \dots, n$, then we know that the following inequality

$$\sum_{k=0}^{n}\int_{\Omega_k}|u^{(k)}|^2\,dx \le K_1\Big\{\sum_{k=0}^{n}\int_{\Omega_k}|\nabla u^{(k)}|^2dx + \int_S |u|^2\,dS\Big\}$$

holds for some positive constant $K_1 > 0$. Here $u = u^{(n)}$.

Thus, the term I in (3.35) can be estimate by

$$|I| \le \gamma_4\,\mathcal{E}(T) \le \gamma_4\,\mathcal{E}(0)$$

for some positive constant $\gamma_4 > 0$. It follows from identity (3.2), Lemmas 3.1, 3.2, (3.3) and assumption (3.34) that

$$(T+t_0)\mathcal{E}(T) \le \gamma_5\,\mathcal{E}(0) + \left(\delta_1 c_5 + \frac{4R\gamma_1}{\sqrt{b_0}}\right)\int_0^T \mathcal{E}(t)dt \quad (3.36)$$

for some positive constant $\gamma_5 > 0$ and T large enough as we did in the proof of Lemma 3.3. Let us called by $g(T)$ the right hand side of (3.36). Clearly (3.36) implies that $\frac{g'(T)}{g(T)} \le \frac{1}{T+t_0}(\delta_1 c_5 + \frac{4R\gamma_1}{\sqrt{b_0}})$ which says that $g(T) \le \frac{(T+t_0)^p}{t_0^p}g(0)$ where $0 < p = \delta_1 c_5 + \frac{4R\gamma_1}{\sqrt{b_0}} < 1$. Consequently,

$$\mathcal{E}(T) \le \frac{\gamma_6\,\mathcal{E}(0)}{(T+t_0)^{1-p}} \quad (3.37)$$

with $\gamma_6 = \gamma_5/t_0^p > 0$. We can choose $T > 0$ large enough so that $\gamma_6/(T + t_0)^{1-p}$ is less that one. The semigroup property then implies the conclusion of Theorem 3.1.

4. Exact controllability

In this section we use the result of Theorem 3.1 to prove exact boundary controllability to an arbitrary state of solutions of (1.4), (1.5), (1.6), (1.8).

Theorem 4.1. *Under the assumptions of Theorem* 3.1, *there exists* $\widetilde{T} > 0$ *such that for any* $T > \widetilde{T}$, *given any initial data* $f \in M_1$ *and any terminal state* $g \in M_1$, *there exists a boundary control* $\{\vec{p}(x,t), \vec{q}(x,t)\}$ *belonging to* $[L^2(S \times (0,T))]^3 \times [L^2(S \times (0,T))]^3$ *such that the corresponding solution of* $(1.4), (1.5), (1.6), (1.8)$ *satisfies* (1.9). *Moreover, the inequality*

$$||\vec{p}||^2_{(L^2)^3} + ||\vec{q}||^2_{(L^2)^3} \leq c\big\{||f||^2_X + ||g||^2_X\big\} \tag{4.1}$$

holds.

Proof. With the same notation as in Theorem 3.1 we have that the semigroup $\{U(t)\}$ takes the closed subspace M_1 into itself and have norm less that one whenever $T > 0$ is such that

$$T > \frac{1}{\gamma_6^{\frac{1}{1-p}}} - t_0 = \widetilde{T}. \tag{4.2}$$

Let us consider the following equation in M_1:

$$w - U^*(T)U(T)w = f - U^*(T)g,$$

where $f, g \in M_1$. The operator $F(T) = U^*(T)U(T)$ takes M_1 into itself and $||F(T)|| < 1$ for any $T > \widetilde{T}$. It follows that we can solve (4.2) for $f, g \in M_1$ and

$$||w||_X \leq c\big\{||f||_X + ||g||_X\big\}.$$

Consequently, if we chose $w = (I - F(T))^{-1}(f - U^*(T)g)$ we will have that

$$\begin{aligned}(u_1, u_2, u_3, u_4) &= U(t)w - U^*(T-t)(U(T)w - g)\\ &\equiv (v_1, v_2, v_3, v_4) - (w_1, w_2, w_3, w_4)\end{aligned}$$

is a weak solution of (1.4), (1.5), (1.6), (1.8) with

$$\vec{p}(x,t) = -\beta\, v_2 - \beta\, w_2, \quad \vec{q}(x,t) = \theta\, \eta\mathbf{x}(\eta\mathbf{x}v_3) + \theta\, \eta\mathbf{x}(\eta\mathbf{x}w_3).$$

We observe that

$$(u_1, u_2, u_3, u_4)\big|_{t=T} = g(x)$$

and by the energy identity we obtain (4.1).

5. Final remark and open questions

1) In the case where all coefficients in system (1.1) are positive constants the results of the present article are valid for domains Ω satisfying the condition $(x - x_0) \cdot \eta > -\delta_1 \frac{\text{measure}(\Omega)}{\text{area}(S)}$ for some $x_0 \in \Omega$ and all $x \in S = \partial\Omega$ (see Hypothesis III b)) which are "slightly" more general that star-shaped domains.

2) The results of this article do not include the complete anisotropic case essentially because we are assuming item 6) in Hypothesis I.

3) At this point we do not know if it is possible to obtain the results of this article without Hypothesis III. Perhaps this may be possible adapting the techniques presented in [14] and [3] where this question has been studied for the scalar wave equation and the elastodynamic system. At this time this is an open problem.

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